Modeling a Rocket's Flight with Differential Equations

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March 2024

**Introduction**

Mankind yearns to explore more about the Earth. We've explored every continent, noting geographical features, and dived into oceans, curious to find more. Our desire to explore has also taken us into the sky as we send projectiles and rockets into flight. Our endeavor to explore the sky can be explained through fundamental physics, and we've created a simplified scenario that can help us understand how differential equations apply to a rocket's flight.

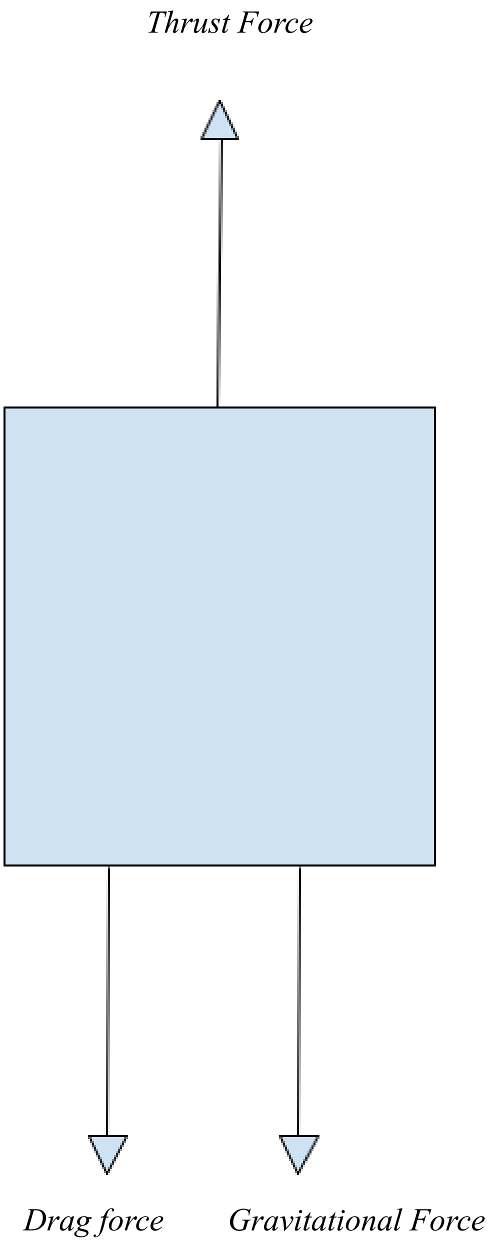
In the scenario, we have a rocket with 100 liters of fuel (with a mass density of 0.98kg/l). The fuel weighs 98 kg, and the rocket alone weighs 400kg. As the rocket burns fuel at a rate of 3 liters per second, the rocket has an upward thrust force of 5900 newtons with a drag force of 2 v(t). In this report, we will find the maximum height of the rocket.

To find the maximum height of the rocket, we will derive the "burn period" and model the rocket with a free-body diagram during this period. From this model, we will create differential equations that we can use Euler's method to solve for the rocket's altitude at given times. Our given times will be the rocket's maximum altitude during the burn period, after the burn period, and finally the maximum height from the model.

**Analysis**

First, we must find the "burn period" of the rocket or the amount of time the rocket spends burning fuel. Since the rocket has 100 liters of fuel and burns the fuel at a rate of 3 liters per second, the rocket will go through all the fuel in $\frac{100}{3}$ (33.33) seconds.

Now, we will model the forces upon the rocket over the burn period by using a free-body diagram. There will be the force of thrust, force of gravity, and drag force modeled on our rocket.



During the rocket's motion during the burn period, we can describe it using Newton's second law of motion, where $F_{net}$ is the net force acting on the rocket, *m* is the mass of the rocket, and *a* is its acceleration.

$$F = ma$$

The rocket's mass as a function of time, *m(t)*, decreases due to fuel consumption. Since the rocket has 100 liters of fuel, its initial mass is $400 + 100(0.98) = 498$ kg. The rate of change of mass, $\frac{dm}{dt}$, is $-3(.98) = -2.94$ kg/s due to the fuel being used up.

Therefore, the equation can be expressed as:

$$F = T - mg - 2v(t)$$

$$ma = T - mg = 2v(t)$$

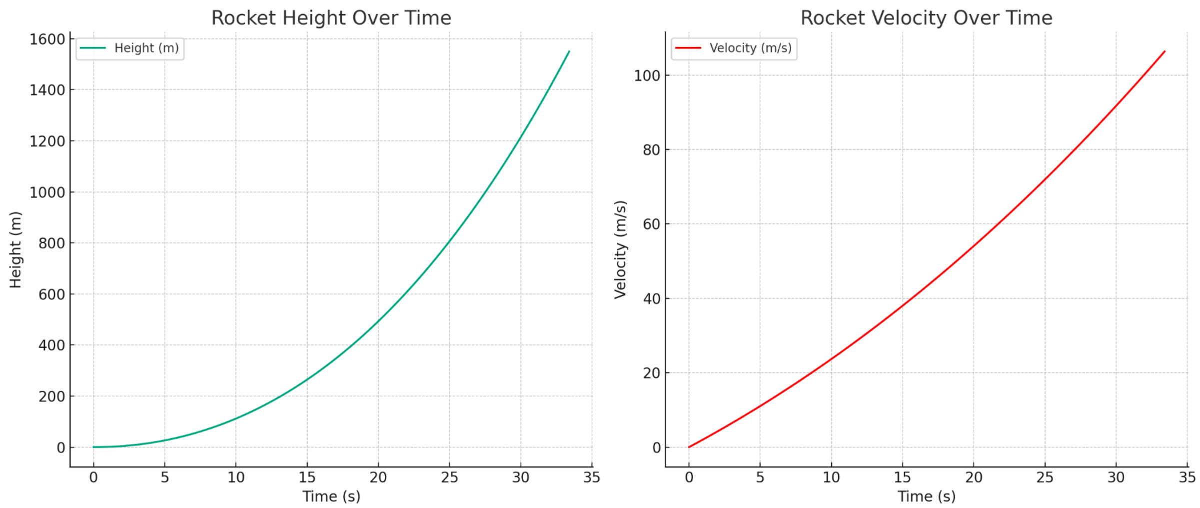
$$a = dv/dt \frac{T}{m(t)} - g - \frac{2v(t)}{m(t)}$$

To approximate our solution of the differential equation, we will use Euler's method. The method can be modeled as follows:

$$v_{next} = v + a \bullet \Delta t$$

$$h_{next} = h + v \bullet \Delta t$$

Using Python, we set an appropriate time step $\Delta t$ and calculate the rocket's altitude and velocity over the burn period until it reaches its highest point.




The plots above illustrate the rocket's altitude and velocity over the burn period using Euler's method.

The left plot shows the rocket's altitude over time, indicating a continuous increase in altitude during the fuel burn period. The right plot displays the rocket's velocity over time, which initially increases as the rocket accelerates upwards. The velocity's rate of increase slows down over time due to the decreasing mass of the rocket and the increasing effect of aerodynamic resistance.

Now we need to model what happens to the rocket after the burn period by modeling the flight without thrust. After the burn period, the thrust is no longer acting on the rocket, so the forces we need to consider are only gravity and aerodynamic resistance *(2v(t))*.

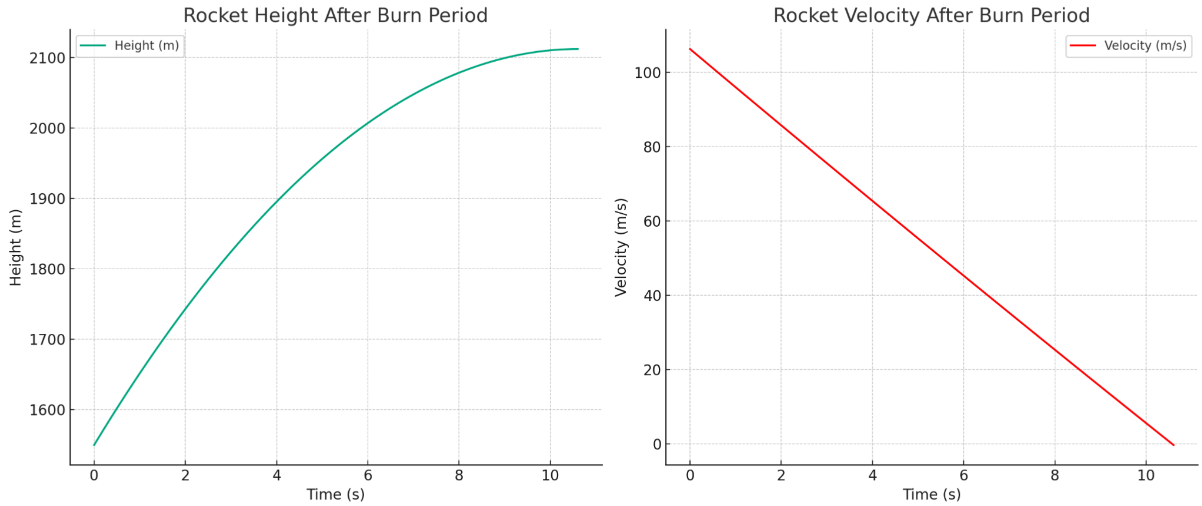
The differential equation can be simplified to:

$$ma = -mg - 2v(t)$$

$$a = -g - \frac{2v(t)}{m}$$

Since the rocket’s mass does not change after the burn period, we can consider it constant for this part of the flight. The initial conditions at the start of this phase are the velocity and altitude of the rocket at the end of the burn period, calculated previously.

We will use Euler's method to include the post-burn phase until the rocket reaches its apex, and then continue until it starts descending.




The plots above illustrate the rocket's altitude and velocity after the fuel burn period until it reaches its apex, based on the post-burn dynamics.

The graph on the left shows the rocket's altitude over time after the burn period. The altitude continues to increase for a while, even after the fuel has been depleted, due to the initial upward velocity. The graph on the right displays the rocket's velocity over time after the burn period. The velocity decreases because of gravity and drag force. It eventually reaches zero, marking the maximum height of the flight.

The maximum altitude reached during this phase is the maximum height of the rocket's flight. The maximum height reached by the rocket during its flight is approximately 2112.0 meters.

We checked our work using the ‘solve_ivp’ function from SciPy’s integration Python module, which utilizes a form of the Runge-Kutta method to solve ordinary differential equations. Our estimated, calculated answer was off by 1% compared to the Python module, giving us confidence that our answer was sufficiently accurate to our model.

**Conclusion**

In this project, we modeled a rocket's flight by applying differential equations and Euler's method from class. Our approach involved creating a representative formula of the rocket's dynamics through Newton's second law and aerodynamic principles and applying Euler's method to solve the differential equations. This allowed us to solve for the rocket’s altitude and velocity which is a maximum altitude of approximately 2112 meters.

However, our model for representing the rocket’s motion was slightly inaccurate due to the limitations of Euler’s method. More advanced methods of solving differential equations such as the Runge-Katta would offer more precise simulations by reducing errors and better handling variable air densities at different altitudes. Although some limitations were found through our technique of representing the rocket’s trajectory, our use of differential equations and Euler’s method in order to represent the real-world applications of these concepts and use of code in order to expedite finding solutions shows the practicality of math concepts learned in class.

## References


*Rocket Flight Path - Digital Commons @ USF*, digitalcommons.usf.edu/cgi/viewcontent.cgi?article=4858&context=ujmm. Accessed 29 Mar. 2024.

Winkel, Brian. "1-128-Rocketflight-Modelingscenario." *QUBES*, qubeshub.org/community/groups/simiode/publications?id=3469&v=1. Accessed 29 Mar. 2024.


## Appendix

```
# code for euler simulation

import numpy as np
import matplotlib.pyplot as plt

g = 9.8
T = 5900
fuel_burn_rate = 3 * 0.98
initial_fuel_mass = 100 * 0.98
rocket_mass_without_fuel = 400
initial_mass = rocket_mass_without_fuel + initial_fuel_mass
burn_time = initial_fuel_mass / fuel_burn_rate

dt = 0.1

time = np.arange(0, burn_time + dt, dt)
velocity = np.zeros_like(time)
height = np.zeros_like(time)
mass = initial_mass - fuel_burn_rate * time

for i in range(1, len(time)):
    a = (T - mass[i] * g - 2 * velocity[i-1]) / mass[i]
    velocity[i] = velocity[i-1] + a * dt
    height[i] = height[i-1] + velocity[i-1] * dt

plt.figure(figsize=(14, 6))

plt.subplot(1, 2, 1)
plt.plot(time, height, label='Height (m)')
plt.xlabel('Time (s)')
plt.ylabel('Height (m)')
plt.title('Rocket Height Over Time')
plt.legend()
```

```
plt.subplot(1, 2, 2)
plt.plot(time, velocity, label='Velocity (m/s)', color='red')
plt.xlabel('Time (s)')
plt.ylabel('Velocity (m/s)')
plt.title('Rocket Velocity Over Time')
plt.legend()

plt.tight_layout()
plt.show()
```