2. De la sección 1.6.6 resuelva los ejercicios 2, 5 y 6.

- Demuestre

a) $\cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha)$.

Partiendo de $z = \cos\theta + i\sin\theta$, podemos escribir $z = e^{\theta i}$, de este modo: $z = e^{3\theta i} = (e^{\theta i})^3 = (\cos\theta + i\sin\theta)^3$

$= \cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta$

$+ i^3\sin^3\theta$, igualando componentes y simplificando:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$,

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$, se demuestran ambos casos.

5 Encuentre todas las raices de las siguientes expresiones

a) $\sqrt{2i}$

Usando la fórmula de Moivre

$$\sqrt[n]{z} = r^{1/n}\left(\cos\left(\frac{\theta + 2K\pi}{n}\right) + i\sin\left(\frac{\theta+2K\pi}{n}\right)\right),$$

$$z = \sqrt{2i},$$

$$z^2 = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right),$$

$$\sqrt[2]{z} = \sqrt{2}\left(\cos\left(\frac{\pi/2 + 2K\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{2}+2K\pi}{2}\right)\right),$$

Para $K=0$

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i,$$

Para $K=1$

$$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1-i.$$

b) $\sqrt{1-\sqrt{3}i}$

$$z^2 = 1 - \sqrt{3}i \quad , \quad r = \sqrt{1^2+3} = 2 \quad , \quad \theta = \arctan\left(\sqrt{3}\right) + 2\pi = \frac{7\pi}{3}$$

$$z^{1/2} = \sqrt{2}\left(\cos\frac{\frac{7\pi}{3}+2K\pi}{2}\right) + i\sin\left(\frac{\frac{7\pi}{3}+2K\pi}{n}\right)$$

Para $K=0$

$$z_1 = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

Para $K=1$

$$z_2 = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

c) $(-1)^{1/3}$

$z^3 = -1 \quad , \quad r = 1 \quad , \quad \theta = \pi .$

$\Rightarrow \quad z^{1/3} = \cos\left(\frac{\pi + 2K\pi}{3}\right) + i\sin\left(\frac{\pi + 2K\pi}{3}\right) \quad . \quad K = 0, 1, 2.$

$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \quad (K = 0)$

$z_2 = \cos\pi + i\sin\pi = -1 \quad (K = 1).$

$z_3 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i \quad (K = 2).$

d) $(8)^{1/6}$

$z^6 = 8 \quad , \quad \theta = 0 \quad , \quad r = 8$

$z^{1/6} = \sqrt[6]{8}\left(\cos\left(\frac{2K\pi}{6}\right) + i\sin\left(\frac{2K\pi}{6}\right)\right) \quad . \quad K = 0, \ldots 5.$

$z_1 = \sqrt[6]{8} \quad (K = 0)$

$z_2 = \frac{\sqrt[6]{8}}{2} + \frac{\sqrt[6]{8}\sqrt{3}}{2} i \quad (K = 1)$

$z_3 = \sqrt[6]{8}\cos\left(\frac{2\pi}{3}\right) + \sqrt[6]{8}\sin\left(\frac{2\pi}{3}\right) i = -\frac{\sqrt[6]{8}}{2} + \frac{\sqrt[6]{8}\sqrt{3}}{2} i$

$z_4 = \sqrt[6]{8}\cos(\pi) + \sqrt[6]{8}\, i\sin(\pi) = -\sqrt[6]{8} \quad (K = 3)$

$z_5 = \sqrt[6]{8}\cos\left(\frac{4\pi}{3}\right) + i\sqrt[6]{8}\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt[6]{8}}{2} - \frac{\sqrt[6]{8}\sqrt{3}}{2} i \, (K = 4)$

$z_6 = \sqrt[6]{8}\cos\left(\frac{5\pi}{3}\right) + i\sqrt[6]{8}\sin\left(\frac{5\pi}{3}\right) = \frac{\sqrt[6]{8}}{2} - i\frac{\sqrt[6]{8}\sqrt{6}}{2} \quad (K = 5)$

e) $\sqrt[4]{-8 - 8\sqrt{3}i}.$

$z^4 = -8 - 8\sqrt{3}i \quad , \quad r = \sqrt{8^2 + 8^2(3)} = 16 \quad , \quad \theta = \frac{4\pi}{3}$

$z^{1/4} = 2\left(\cos\left(\frac{\frac{4\pi}{3} + 2K\pi}{4}\right) + i\sin\left(\frac{\frac{4\pi}{3} + 2K\pi}{4}\right)\right)$

$$z_1 = 2\cos\frac{\pi}{3} + 2\sin\frac{\pi}{3} i = 1 + \frac{\sqrt{3}}{2} i \quad (K=0)$$

$$z_2 = 2\cos\left(\frac{\frac{4\pi}{3}+2\pi}{4}\right) + i2\sin\left(\frac{\frac{4\pi}{3}+2\pi}{4}\right) = -\sqrt{3} + i \quad (K=1)$$

$$z_3 = 2\cos\left(\frac{\frac{4\pi}{3}+4\pi}{4}\right) + i2\sin\left(\frac{\frac{4\pi}{3}+4\pi}{4}\right) = -1 - \sqrt{3} i \quad (K=2)$$

$$z_4 = 2\cos\left(\frac{\frac{4\pi}{3}+6\pi}{4}\right) + i2\sin\left(\frac{\frac{4\pi}{3}+6\pi}{4}\right) = \sqrt{3} - i \quad (K=3)$$

6. Demuestre que:

a) $\text{Log}(-ie) = 1 - \frac{\pi}{2} i$

$r = |-ie| = |-i||e| = e \quad , \quad \theta = -\frac{\pi}{2}$

$\text{Log}(-ie) = \ln(e) + i\left(-\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} i$ ▪

b) $\text{Log}(1-i) = \frac{\ln 2}{2} - \frac{\pi i}{4}$

$r = \sqrt{2} \quad , \quad \theta = -\frac{\pi}{4}$

$\text{Log}(1-i) = \ln(\sqrt{2}) + i\left(-\frac{\pi}{4}\right) = \frac{1}{2}\ln(2) - \frac{\pi}{4} i$ ▪

c) $\text{Log}(e) = 1 + 2\pi n i$

$r = e \quad , \quad \theta = 0$

$\text{Log}(e) = \ln(e) + i(0 + 2\pi n) = 1 + 2\pi n i \ , \ n \in \mathbb{Z}$ ▪

d) $\text{Log}(i) = \left(2n + \frac{1}{2}\right)\pi i$

$r = 1 \quad , \quad \theta = \frac{\pi}{2}$

$\text{Log}(i) = \ln(1) + i\left(\frac{\pi}{2} + 2\pi n\right) = 0 + i\frac{\pi}{2} + \underline{2\pi n i}$

$= \left(\frac{1}{2} + 2n\right)\pi i$ ▪