$\mathbb{Z}$-Δ_2-FLATNESS CONSTANT
CASE 2 OF TRIANGLE COMPUTATIONS

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ABSTRACT. Details for bounding the width of the triangles from case 2 of [1, Table 1]. `case_02.pl` contains the polymake code and `case_02.py` contains the python code.

Let e_1, e_2 be the standard basis of $\mathbb{Z}^2$. Consider the locking points $(A, B, C) = (e_1+e_2, e_2, -e_1-e_2)$ and let $P' \subset \mathbb{R}^2$ be their convex hull. Let $P \subset \mathbb{R}^2$ be a triangle circumscribed around those three points. Recall that $\mathbf{0}$ and e_1 are assumed to be contained in the interior of P. We consider the width directions $e_1^*, e_2^*, e_1^* - e_2^*$. The

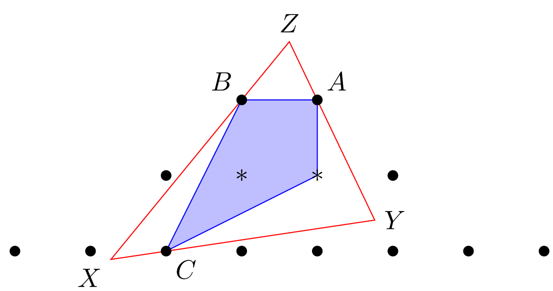

FIGURE 1. A triangle P (in red) with locking points $(A, B, C) = (e_1 + e_2, e_2, -e_1 - e_2)$.

slopes m_{XY}, m_{YZ} and m_{ZX} of the facets of P through $\{X, Y\}$, $\{Y, Z\}$ and $\{Z, X\}$ respectively can be expressed in terms of λ, μ, and ν (use `case_02.py`):

$$m_{XY} = \frac{-2 + 2\lambda + 2\mu}{-1 + \lambda + 2\mu}, \qquad m_{YZ} = \frac{-2 + 2\mu}{-2 + 2\mu + \nu}, \qquad m_{ZX} = \frac{2\lambda}{1 + \lambda - \nu}.$$

Since $P' \subset P$, we have $m_{XY} \leq \frac{1}{2}$, $m_{YZ} \leq 0$, and $0 \leq m_{ZX} \leq 2$. Since $-e_2$ is not in the interior of P, we have $m_{XY} \geq 0$. Similarly, since $-2e_1 - e_2$ is not in the interior of P, we have $m_{XY} \geq 1$. Hence the slopes of P satisfy:

$$0 \leq m_{XY} \leq \frac{1}{2}, \qquad m_{YZ} \leq 0, \qquad 1 \leq m_{ZX} \leq 2.$$

Arithmetic manipulations of these inequalities yield constraints on the parameters λ, μ, and ν which define the polytope of admissible parameters

$$Q = \{(\lambda, \mu, \nu) \in [0, 1]^3 \colon -1 + \lambda + \mu \geq 0, -1 + \lambda + \nu \geq 0, 1 + \lambda - \nu \geq 0, \\ -2 + 2\mu + \nu \geq 0, 3 - 3\lambda - 2\mu \geq 0\}.$$

We now determine the widths of P in the directions e_1^*, e_2^*, and $e_1^* - e_2^*$ (use `case_02.py`). On Q, these are achieved at $Z - X$, $Y - X$, and $Z - Y$ respectively:

$$\begin{aligned}
\operatorname{width}_{e_1^*}(P) &= e_1^*(Y - X) = \frac{-1 + \lambda + 2\mu}{\delta} \\
\operatorname{width}_{e_2^*}(P) &= e_2^*(Z - X) = \frac{2\lambda}{\delta} \\
\operatorname{width}_{e_1^* - e_2^*}(P) &= (e_1^* - e_2^*)(Y - Z) = \frac{\nu}{\delta}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\operatorname{width}(P) &\leq \min\bigl\{\operatorname{width}_{e_1^*}(P), \operatorname{width}_{e_2^*}(P), \operatorname{width}_{e_1^* - e_2^*}(P)\bigr\} \\
&= \frac{\min\{-1 + \lambda + 2\mu, 2\lambda, \nu\}}{\delta} =: \frac{f(\lambda, \mu, \nu)}{\delta}.
\end{aligned}$$

By using `case_02.pl` and Mathematica, we get that

$$\max_{(\lambda,\mu,\nu) \in Q} \frac{f(\lambda, \mu, \nu)}{\delta} = \frac{2}{\sqrt{7} - 2} \approx 3.09716754,$$

and the maximum is achieved exactly at $(\lambda, \mu, \nu) = \left(\frac{1}{\sqrt{7}}, \frac{1}{2} + \frac{1}{2\sqrt{7}}, \frac{2}{\sqrt{7}}\right)$ Hence there is a unique maximiser, namely the triangle with vertices given by

$$\frac{1}{3\sqrt{7}} \begin{pmatrix} -6 - 3\sqrt{7} & 8 + \sqrt{7} & 1 + 2\sqrt{7} \\ -9 & -2 - \sqrt{7} & 5 + 4\sqrt{7} \end{pmatrix}$$

Notice that the maximiser is an admissible triangle with locking points (A, B, C) and interior lattice points $\mathbf{0}$ and e_1.

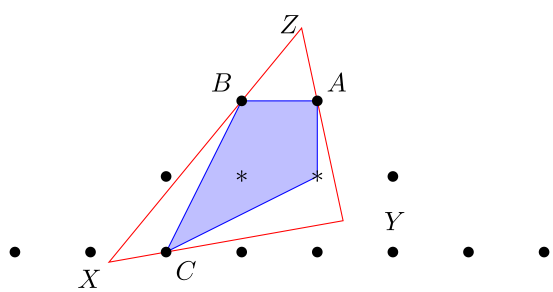

FIGURE 2. The width maximiser (in red) with locking points $(A, B, C) = (e_1 + e_2, e_2, -e_1 - e_2)$.

REFERENCES

[1] G. Codenotti, T. Hall, J. Hofscheier, *Generalised flatness constants: a framework applied in dimension* 2, preprint, arxiv.