$\mathbb{Z}$-Δ_2-FLATNESS CONSTANT
CASE 1 OF TRIANGLE COMPUTATIONS

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Abstract. Details for bounding the width of the triangles from case 3 of [1, Table 1]. `case_03.pl` contains the polymake code and `case_03.py` contains the python code.

Let e_1, e_2 be the standard basis of $\mathbb{Z}^2$. Consider the locking points $(A, B, C) = (e_1+e_2, e_2, -2e_1-e_2)$ and let $P' \subset \mathbb{R}^2$ be their convex hull. Let $P \subset \mathbb{R}^2$ be a triangle circumscribed around those three locking points. Recall that $\mathbf{0}$ and e_1 are assumed to be contained in the interior of P. We consider the width directions $e_2^*, e_1^* - e_2^*$. The slopes m_{XY}, m_{YZ} and m_{ZX} of the facets of P through $\{X, Y\}$, $\{Y, Z\}$ and

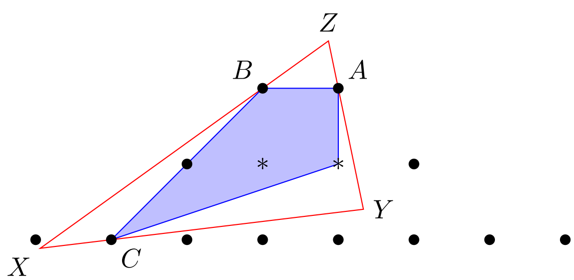

Figure 1. A triangle P (in red) with locking points $(A, B, C) = (e_1 + e_2, e_2, -2e_1 - e_2)$.

$\{Z, X\}$ respectively can be expressed in terms of λ, μ, and ν (use `case_03.py`):

$$m_{XY} = \frac{-2 + 2\lambda + 2\mu}{-2 + 2\lambda + 3\mu}, \qquad m_{YZ} = \frac{-2 + 2\mu}{-3 + 3\mu + \nu}, \qquad m_{ZX} = \frac{2\lambda}{1 + 2\lambda - \nu}.$$

Since $P' \subset P$, we have $m_{XY} \leq \frac{1}{3}$, $m_{YZ} \leq 0$, and $0 \leq m_{ZX} \leq 1$. Since $-e_2$ is not in the interior of P, we have $m_{XY} \geq 0$. Similarly, since $-3e_1 - e_2$ is not in the interior of P, we have $m_{ZX} \geq \frac{2}{3}$. Hence the slopes of P satisfy:

$$0 \leq m_{XY} \leq \frac{1}{3}, \qquad m_{YZ} \leq 0, \qquad \frac{2}{3} \leq m_{ZX} \leq 1.$$

Arithmetic manipulations of these inequalities yield constraints on the parameters λ, μ, and ν which define the polytope of admissible parameters

$$Q = \{(\lambda, \mu, \nu) \in [0,1]^3 \colon \lambda + \mu - 1 \geq 0, \lambda + \nu - 1 \geq 0, -4\lambda - 3\mu + 4 \geq 0, \\ 2\lambda - \nu + 1 \geq 0, 3\mu + \nu - 3 \geq 0\}.$$

We now determine the widths of P in the directions e_2^*, and $e_1^* - e_2^*$ (use `case_03.py`). On Q, these are achieved at $Z - X$ and $Y - X$ respectively:

$$\mathrm{width}_{e_2^*}(P) = e_2^*(Z - X) = \frac{2\lambda}{\delta}$$
$$\mathrm{width}_{e_1^*-e_2^*}(P) = (e_1^* - e_2^*)(Y - X) = \frac{\mu}{\delta}.$$

We thus obtain

$$\begin{aligned}\mathrm{width}(P) &\leq \min\left\{\mathrm{width}_{e_2^*}(P), \mathrm{width}_{e_1^*-e_2^*}(P)\right\}\\ &= \frac{\min\{2\lambda, \mu\}}{\delta} =: \frac{f(\lambda,\mu,\nu)}{\delta}.\end{aligned}$$

By using `case_03.pl` and Mathematica, we get that

$$\max_{(\lambda,\mu,\nu)\in Q} \frac{f(\lambda,\mu,\nu)}{\delta} = \frac{10}{3},$$

and the maximum is achieved exactly at $(\lambda,\mu,\nu) = (\frac{2}{5}, \frac{4}{5}, \frac{3}{5})$. Hence there is a unique maximiser, namely the triangle with vertices given by

$$\frac{1}{3}\begin{pmatrix} -12 & 3 & 3 \\ -5 & 5 & 0 \end{pmatrix}.$$

Notice that the width maximiser is **not** admissible since e_1 lies on the boundary.

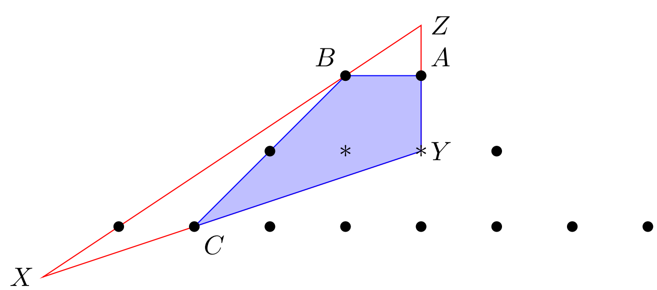

FIGURE 2. The width maximiser (in red) with locking points $(A, B, C) = (e_1 + e_2, e_2, -2e_1 - e_2)$.

Thus the width of admissible inclusion-maximal $\mathbb{Z}$-Δ_2-free triangles with locking points (A, B, C) approaches $\frac{10}{3}$ but never reaches it.

References

[1] G. Codenotti, T. Hall, J. Hofscheier, *Generalised flatness constants: a framework applied in dimension* 2, preprint, arxiv.