$\mathbb{Z}$-Δ_2-FLATNESS CONSTANT
CASE 1 OF TRIANGLE COMPUTATIONS

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Abstract. Details for bounding the width of the triangles from case 1 of [1, Table 1]. `case_01.pl` contains the polymake code and `case_01.py` contains the python code.

Let e_1, e_2 be the standard basis of $\mathbb{Z}^2$. Consider the locking points $(A, B, C) = (e_1 + e_2, -e_2, e_2)$ and let $P' \subset \mathbb{R}^2$ be their convex hull. Let $P \subset \mathbb{R}^2$ be a triangle circumscribed around those three locking points. Recall that $\mathbf{0}$ and e_1 are assumed to be contained in the interior of P. We consider the width directions $e_1^*, e_2^*, e_1^* - e_2^*$. The slopes m_{XY}, m_{YZ} and m_{ZX} of the facets of P through $\{X, Y\}$, $\{Y, Z\}$ and

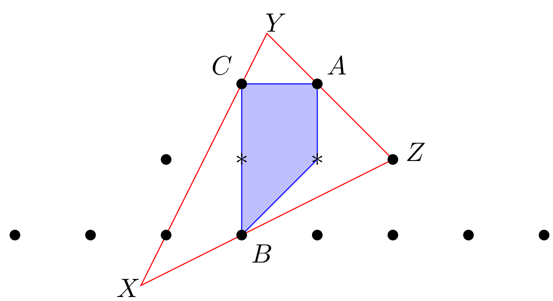

Figure 1. A triangle P (in red) with locking points $(A, B, C) = (e_1 + e_2, -e_2, e_2)$.

$\{Z, X\}$ respectively can be expressed in terms of λ, μ, and ν (use `case_01.py`):

$$m_{XY} = \frac{2 - 2\lambda}{\mu}, \qquad m_{YZ} = \frac{2\nu}{-1 + \mu + \nu}, \qquad m_{ZX} = \frac{2 - 2\lambda - 2\nu}{1 - \nu}.$$

Since $P' \subset P$, we have $m_{XY} \geq 0$, $m_{YZ} \leq 0$, and $m_{ZX} \leq 1$. Since $e_1 - e_2$ is not in the interior of P, we have $m_{ZX} \geq 0$. Similarly, since $-e_1 - e_2$ is not in the interior of P, we have $m_{XY} \geq 2$. Hence the slopes of P satisfy:

$$m_{XY} \geq 2, \qquad m_{YZ} \leq 0, \qquad 0 \leq m_{ZX} \leq 1.$$

Arithmetic manipulations of these inequalities yield constraints on the parameters λ, μ, and ν which define the polytope of admissible parameters

$$Q = \{(\lambda, \mu, \nu) \in [0,1]^3 \colon 1 - \lambda - \mu \geq 0, 1 - \mu - \nu \geq 0, 1 - \lambda - \nu \geq 0, \\ -1 + 2\lambda + \nu \geq 0\}.$$

We now determine the widths of P in the directions e_1^*, e_2^*, and $e_1^* - e_2^*$ (use `case_01.py`). On Q, these are achieved at $Z - X$, $Y - X$, and $Z - Y$ respectively:

$$\begin{aligned}\mathrm{width}_{e_1^*}(P) &= e_1^*(Z - X) = \frac{1-\nu}{\delta}\\ \mathrm{width}_{e_2^*}(P) &= e_2^*(Y - X) = \frac{2-2\lambda}{\delta}\\ \mathrm{width}_{e_1^*-e_2^*}(P) &= (e_1^* - e_2^*)(Z - Y) = \frac{1-\mu+\nu}{\delta}.\end{aligned}$$

We thus obtain

$$\begin{aligned}\mathrm{width}(P) &\le \min\bigl\{\mathrm{width}_{e_1^*}(P), \mathrm{width}_{e_2^*}(P), \mathrm{width}_{e_1^*-e_2^*}(P)\bigr\}\\ &= \frac{\min\{1-\nu, 2-2\lambda, 1-\mu+\nu\}}{\delta} =: \frac{f(\lambda,\mu,\nu)}{\delta}.\end{aligned}$$

By using `case_01.pl` and Mathematica, we get that

$$\max_{(\lambda,\mu,\nu)\in Q} \frac{f(\lambda,\mu,\nu)}{\delta} = \frac{10}{3},$$

and the maximum is achieved exactly at $(\lambda, \mu, \nu) = (\frac{3}{5}, \frac{2}{5}, \frac{1}{5})$. Hence there is a unique maximiser, namely the triangle with vertices given by

$$\frac{1}{3}\begin{pmatrix} -4 & 1 & 6 \\ -5 & 5 & 0 \end{pmatrix}.$$

Notice that the maximiser is an admissible triangle with locking points (A, B, C)

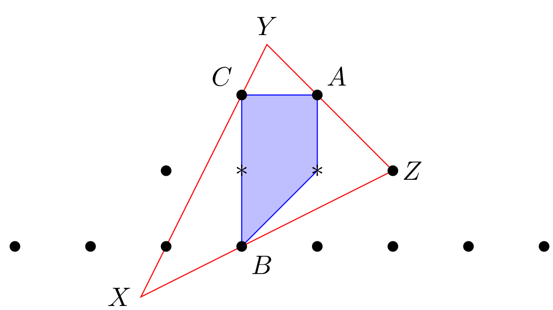

FIGURE 2. The width maximiser (in red) with locking points $(A, B, C) = (e_1 + e_2, -e_2, e_2)$.

and interior lattice points $\mathbf{0}$ and e_1.

References

[1] G. Codenotti, T. Hall, J. Hofscheier, *Generalised flatness constants: a framework applied in dimension* 2, preprint, arxiv.