$\mathbb{Z}$-Δ_2-FLATNESS CONSTANT
CASE 1 OF TRIANGLE COMPUTATIONS

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Abstract. Details for bounding the width of the triangles from case 4 of [1, Table 1]. `case_04.pl` contains the polymake code and `case_04.py` contains the python code.

Let e_1, e_2 be the standard basis of $\mathbb{Z}^2$. Consider the locking points $(A, B, C) = (e_1 + e_2, -e_2, -3e_1 - e_2)$ and let $P' \subset \mathbb{R}^2$ be their convex hull. Let $P \subset \mathbb{R}^2$ be a triangle circumscribed around those three locking points. Recall that $\mathbf{0}$ and e_1 are assumed to be contained in the interior of P. We consider the width directions e_2^*. The slopes m_{XY}, m_{YZ} and m_{ZX} of the facets of P through $\{X, Y\}$, $\{Y, Z\}$ and

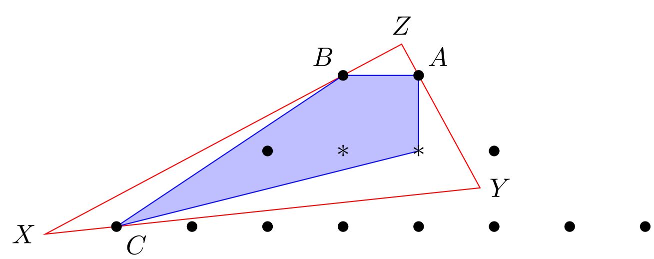

Figure 1. A triangle P (in red) with locking points $(A, B, C) = (e_1 + e_2, -e_2, e_2)$.

$\{Z, X\}$ respectively can be expressed in terms of λ, μ, and ν (use `case_01.py`):

$$m_{XY} = \frac{-2 + 2\lambda + 2\mu}{-3 + 3\lambda + 4\mu}, \qquad m_{YZ} = \frac{-2 + 2\mu}{-4 + 4\mu + \nu}, \qquad m_{ZX} = \frac{2\lambda}{1 + 3\lambda - \nu}.$$

Since $P' \subset P$, we have $m_{XY} \le \frac{1}{4}$, $m_{YZ} \le 0$, and $0 \le m_{ZX} \le \frac{2}{3}$. Since $-e_2$ is not in the interior of P, we have $m_{XY} \ge 0$. Similarly, since $-4e_1 - e_2$ is not in the interior of P, we have $m_{ZX} \ge \frac{1}{2}$. Hence the slopes of P satisfy:

$$0 \le m_{XY} \le \frac{1}{4}, \qquad m_{YZ} \le 0, \qquad \frac{1}{2} \le m_{ZX} \le \frac{2}{3}.$$

Arithmetic manipulations of these inequalities yield constraints on the parameters λ, μ, and ν which define the polytope of admissible parameters

$$Q = \{(\lambda, \mu, \nu) \in [0,1]^3 : 0 \le \lambda + \mu - 1, 0 \le \lambda + \nu - 1, 0 \le -5\lambda - 4\mu + 5,$$
$$0 \le 3\lambda - \nu + 1, 0 \le 4\mu + \nu - 4\}.$$

We now determine the widths of P in the directions e_2^* (use `case_04.py`). On Q, these are achieved at $Z - X$, $Y - X$, and $Z - Y$ respectively:

$$\mathrm{width}_{e_2^*}(P) = e_2^*(Z - X) = \frac{2\lambda}{\delta}$$

We thus obtain

$$\mathrm{width}(P) \leq \mathrm{width}_{e_2^*}(P) = \frac{2\lambda}{\delta} =: \frac{f(\lambda, \mu, \nu)}{\delta}.$$

By using `case_04.pl` and Mathematica, we get that

$$\max_{(\lambda,\mu,\nu)\in Q} \frac{f(\lambda, \mu, \nu)}{\delta} = 3,$$

and the maximum is achieved exactly at $(\lambda, \mu, \nu) = (\frac{1}{3}, \frac{5}{6}, \frac{2}{3})$. Hence there is a unique maximiser, namely the triangle with vertices given by

$$\frac{1}{2}\begin{pmatrix} -10 & 2 & 2 \\ -3 & 3 & 0 \end{pmatrix}.$$

Notice that the width maximiser is **not** admissible since e_1 is contained in the

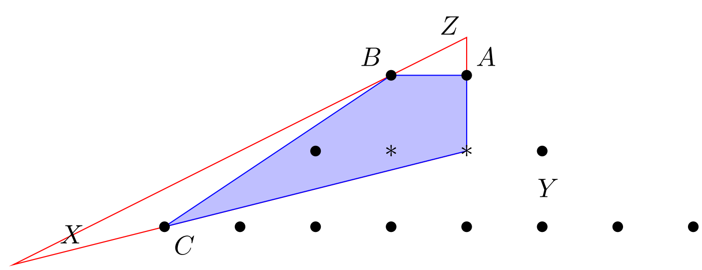

FIGURE 2. The width maximiser (in red) with locking points $(A, B, C) = (e_1 + e_2, e_2, -3e_1 - e_2)$.

boundary. Hence the widths of admissible (i.e., $\mathbf{0}$ and e_1 contained in the interior) inclusion-maximal $\mathbb{Z}$-Δ_2-free triangles with locking points (A, B, C) approach 3, but never reach it.

REFERENCES

[1] G. Codenotti, T. Hall, J. Hofscheier, *Generalised flatness constants: a framework applied in dimension* 2, preprint, arxiv.