Computing Temporal Order without an Analytical Solution

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July 28, 2015

It is possible to estimate the observed temporal (and spatial) order of accuracy without using an analytical solution. The order of accuracy is defined as the exponent of the leading order term of the truncation error. It respresents the rate at which the error between the exact and computed solutions is reduced with consistent refinement in time (or space).

Using a general Taylor series expansion for a quantity f, we have for a p-th order scheme

$$f_1 = \tilde{f} + a_p \Delta t^p + O(\Delta t^{p+1}) \tag{1}$$

The same formula may be applied for two more refinement levels

$$f_2 = \tilde{f} + a_p (r\Delta t)^p + O(r^{p+1}\Delta t^{p+1}) \tag{2}$$

$$f_3 = \tilde{f} + a_p (r^2\Delta t)^p + O(r^{2p+2}\Delta t^{p+1}) \tag{3}$$

where $r = \frac{\Delta t_{\text{fine}}}{\Delta t_{\text{coarse}}} < 1$ is the temporal (grid) refinement ratio, f_i is the computed solution (finite precision computer) corresponding to the i^{th} refinement level, and $\tilde{f}$ is the exact solution to the continuous PDE. If the step size is small enough and the higher order terms are negligible, it is possible to estimate the observed order of accuracy as

$$\frac{f_3 - f_2}{f_2 - f_1} = \frac{a_p r^p \Delta t^p (r^p - 1)}{a_p \Delta t^p (r^p - 1)} = r^p \tag{4}$$

or

$$p = \frac{\ln\left(\frac{f_3 - f_2}{f_2 - f_1}\right)}{\ln r} \tag{5}$$

This approach work very well when certain conditions are met. According to [1], these conditions are

1. All discrete solutions are in the asymptotic range
2. Meshes have a uniform (Cartesian) spacing over the domain
3. Coarse and fine meshes are related through systematic refinement

4. The solutions are smooth

5. The other sources of numerical error are small

In practice, one does the following for computing the observed temporal order of accuracy

1. Choose a very fine grid resolution Δx. This resolution has to be small enough so that the spatial errors are smaller than the temporal errors.

2. Pick the largest timestep size Δt that allows for a stable solution, e.g. choose Δt for a CFL of 1 or 0.95

3. Specify the number of refinement levels. You will need at least three temporal refinements to compute a single estimate for the order of accuracy. In general, if N is the number of refinement levels, then one can compute $N-2$ estimates for the observed order of accuracy

4. Compute the following solutions: $f_1 = f(\Delta t)$, $f_2 = f(\frac{1}{2}\Delta t)$, $f_3 = f(\frac{1}{4}\Delta t)$,... For f_1 take two timesteps, for f_2 take four timesteps, for f_3 take eight timesteps... Because Richardson's extrapolation is used, one must examine the solutions at the same state in time

5. Compute the observed orders: $p_1 = \frac{\ln\left(\frac{f_3-f_2}{f_2-f_1}\right)}{\ln 0.5}$, $p_2 = \frac{\ln\left(\frac{f_4-f_3}{f_3-f_2}\right)}{\ln 0.5}$,...

Using this procedure, one can design a general script for their code that takes an input file, generates additional input files to compute the various solutions f_i, executes the code with the generated input files, and finally, collects relevant data and computes the observed order.

References

[1] W. L. Oberkampf and C. J. Roy. *Verification and validation in scientific computing*. Cambridge University Press, 2010.