Derivation of the Intermediate-Stage Simulation Times for the Third-Order, Strong Stability Preserving Runge-Kutta Method

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Abstract

In this document, I derive the proper intermediate stage time for the third-order, Strong Stability Preserving Runge-Kutta (SSP-RK3) scheme. This is done by first casting the low-storage formulation of Gottlieb et al. (2001) in terms of the traditional Runge-Kutta (RK) explicit formulation. Then, the consistency condition on the right-hand-side (RHS) coefficients of the traditional RK framework is used to derived the appropriate time that should be used in the intermediate stages of the SSP-RK3 formulation.

1 Derivation

Consider the solution of the following ordinary differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = F(t,\phi) \tag{1}$$

The SSP-RK3 scheme is given by

$$\begin{aligned} \phi^{(1)} &= \phi^n + \Delta t\, F(\phi^n) \\ \phi^{(2)} &= \tfrac{3}{4}\phi^n + \tfrac{1}{4}\phi^{(1)} + \tfrac{1}{4}\Delta t\, F[\phi^{(1)}] \\ \phi^{n+1} &= \tfrac{1}{3}\phi^n + \tfrac{2}{3}\phi^{(2)} + \tfrac{2}{3}\Delta t\, F[\phi^{(2)}] \end{aligned} \tag{2}$$

The original paper by Gottlieb et al. (2001) does not indicate how the time t varies within the intermediate RK-stages. Here, we derive what these times should be. This can be done by casting equation (2) into the traditional Runge-Kutta form

$$\phi^{n+1} = \phi^n + \sum_{i=1}^{s} b_i k_i, \tag{3}$$

where

$$\begin{aligned} k_1 =& \Delta t F(t^n, \phi^n) \\ k_2 =& \Delta t F(t^n + c_2\Delta t, \phi^n + a_{21}k_1) \\ k_3 =& \Delta t F(t^n + c_3\Delta t, \phi^n + a_{31}k_1 + a_{32}k_2) \\ &\vdots \\ k_s =& \Delta t F(t^n + c_s\Delta t, \phi^n + \sum_{j=1}^{s-1} a_{sj}k_j) \end{aligned} \tag{4}$$

The equations given in equation (4) are consistent if, and only if, for a given stage m

$$\sum_{j=1}^{m-1} a_{mj} = c_m \tag{5}$$

Given that the a_{ij} coefficients are known from Gottlieb et al. (2001), one can use equation (5) to derive the proper SSP time at the intermediate RK stages.

We first start by casting equation (2) into the traditional form equation (4). We start by setting

$$k_1 \equiv \Delta t F(t^n, \phi^n) \tag{6}$$

We then expand $\phi^{(2)}$ in terms of ϕ^n

$$\phi^{(2)} = \phi^n + \frac{1}{4}k_1 + \frac{1}{4}\Delta t F[\phi^n + k_1] \tag{7}$$

We then set the last term in the previous equation to k_2

$$k_2 \equiv \Delta t F(t^n + c_2\Delta t, \phi^n + k_1) \tag{8}$$

Using equation (5) yields

$$c_2 = 1 \tag{9}$$

Finally, we expand ϕ^{n+1} from equation (2) in terms of ϕ^n. One recovers

$$\phi^{n+1} = \phi^n + \frac{1}{6}k_1 + \frac{1}{6}k_2 + \frac{2}{3}\Delta t F[\phi^{(2)}] = \phi^n + \frac{1}{6}k_1 + \frac{1}{6}k_2 + \frac{2}{3}\Delta t F[\phi^n + \frac{1}{4}k_1 + \frac{1}{4}k_2] \tag{10}$$

Again, we set the last term in the previous equation as k_3. In other words

$$k_3 \equiv \Delta t F[t^n + c_3\Delta t, \phi^n + \frac{1}{4}k_1 + \frac{1}{4}k_2] \tag{11}$$

Finally, using the consistency condition equation (5), we calculate c_3 as

$$c_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{12}$$

In summary, the SSP-RK3 scheme can be written in traditional form as

$$\begin{aligned} k_1 &= \Delta t F(t^n, \phi^n) \\ k_2 &= \Delta t F(t^n + \Delta t, \phi^n + k_1) \\ k_3 &= \Delta t F(t^n + \frac{1}{2}\Delta t, \phi^n + \frac{1}{4}k_1 + \frac{1}{4}k_2) \end{aligned} \tag{13}$$

along with

$$\phi^{n+1} = \phi^n + \frac{1}{6}(k_1 + k_2 + 4k_3) \tag{14}$$

In terms of the original Gottlieb et al. (2001) form, we have

$$\begin{aligned} \phi^{(1)} &= \phi^n + \Delta t\, F(t^n, \phi^n) \\ \phi^{(2)} &= \tfrac{3}{4}\phi^n + \tfrac{1}{4}\phi^{(1)} + \tfrac{1}{4}\Delta t\, F[t^n + \Delta t, \phi^{(1)}] \\ \phi^{n+1} &= \tfrac{1}{3}\phi^n + \tfrac{2}{3}\phi^{(2)} + \tfrac{2}{3}\Delta t\, F[t^n + \tfrac{1}{2}\Delta t, \phi^{(2)}] \end{aligned} \tag{15}$$

2 Example

Consider the solution of

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = t; \quad \phi(0) = 0 \tag{16}$$

The exact solution is

$$\phi(t) = \frac{1}{2}t^2 \tag{17}$$

Here, we examine the numerical solution of this governing equation at $t = 1$s. The exact solution is $\phi(1) = \frac{1}{2}$.

Wasatch Implementation

Wasatch implements the fomulation shown in the previous section. Here, we carry out a single SSP-RK3 timestep to check whether the formulation is correct or not. Let $\Delta t = 1$s, then

$$\begin{aligned}
\phi^{(1)} =& \phi^0 + \Delta t F(t^0) = 0 + 1 \times t^0 = 0 \\
\phi^{(2)} =& \frac{3}{4}\phi^0 + \frac{1}{4}\phi^{(1)} + \frac{1}{4}\Delta t\, F[t^0 + \Delta t, \phi^{(1)}] = 0 + 0 + \frac{1}{4} \times \Delta t \times (\Delta t) = \frac{1}{4} \\
\phi^1 =& \frac{1}{3}\phi^0 + \frac{2}{3}\phi^{(2)} + \frac{2}{3}\Delta t\, F[t^0 + \frac{1}{2}\Delta t, \phi^{(2)}] = 0 + \frac{2}{3} \times \frac{1}{4} + \frac{2}{3} \times \Delta t \times (\frac{1}{2}\Delta t) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}
\end{aligned} \tag{18}$$

Clearly, this formulation captures the exact solution at $t = 1$s.

References

Gottlieb, S., Shu, C., and Tadmor, E. (2001). Strong stability-preserving high-order time discretization methods. *SIAM review*, 43(1):89–112.