Contents

1 Algabraic expressions

1. You can use the laws of indicies to simplify power of the **same base**.
   - $a^m \times a^n = a^{m=n}$
   - $a^m \div a^n = a^{m-n}$
   - $(a^m)^n = a^{mn}$
   - $(ab)^n = a^n b^n$
2. Factorising is the opposite of expanding brackets
3. $x^2 - y^2 = (x+y)(x-y)$
4. A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.
5. You can use the laws of indicies with any rational power.
   - $a^{\frac{1}{m}} = \sqrt[m]{a}$
   - $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
   - $a^{-m} = \dfrac{1}{a^m}$
   - $a^0 = 1$
6. You can manipulate surds using these rules:
   - $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
   - $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$
7. The rules to rationalize denominators are:
   - Fractions in the form $\dfrac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
   - Fractions in the form $\dfrac{1}{a+\sqrt{b}}$, multiply the numeerator and denominator by $a - \sqrt{b}$
   - Fractions in the form $\dfrac{1}{a-\sqrt{b}}$, multiply the numeerator and denominator by $a + \sqrt{b}$

2 Quadratics

1. To solve a quadatic equation by factorising:
   - Write the equation in the form $ax^2 + bx + c = 0$
   - Factorize the left-hand side
   - Set each factor equal to zero and solve to find the value(s) of x
2. The solutions to the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. $x^2 + bx = (x + \dfrac{b}{2})^2 - (\dfrac{b}{2})^2$
4. $ax^2 + bx + c = a(x + \dfrac{b}{2a})^2 + (c - \dfrac{b^2}{4a})$
5. The set of possible inputs for a function is called the **domain**.

   The set of possible outputs of a function is called the **range**.
6. The **roots** of a function are the values of x for which $f(x) = 0$
7. You can find the corrdinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.
8. For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:
   - If $b^2 - 4ac > 0$ then a quadratic function has two distinct roots.
   - If $b^2 - 4ac = 0$ then a quadratic function has one distinct root.
   - If $b^2 - 4ac < 0$ then a quadratic function has no real roots.
9. Quadratics can be used to model real-life situations.

3 Equations and inequalities

1. Linear simultanious equations can be solved using elimination or substitution.
2. Simultanious equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the utions are paired correctly.
3. The solutions of a pair of simultanious equations represent the points of intersection of the graphs.
4. For a pair of simultanious equations the produce a quadratic equation of the form $ax^2 + bx + c = 0$:

   $b^2 - 4ac > 0$ two real solutions

   $b^2 - 4ac = 0$ one real solution

   $b^2 - 4ac < 0$ no real solutions
5. The solution of an innequality is the set of real numbers x that make the inequality true.
6. To solve the quadratic innequality
   - Rearrange so the right-hand side of the innequality if 0
   - Solve the corresponding quadratic equation to find the critical values
   - Sketch the graph of the quadratic function
   - Use your sketch to find the required set of values.
7. The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.

   The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.
8. $y < f(x)$ represents the points on the corrdinate grid below the curve $y = f(x)$.

   $y > f(x)$ represents the points on the corrdinate grid above the curve $y = f(x)$.
9. If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region represented by a dotted line.

   If $y \geq f(x)$ or $y \leq f(x)$ then the curve $y = f(x)$ is not included in the region represented by a solid line.

4 Graphs and transformations

1. If p is a root of the function $f(x)$, then the graph of $y = f(x)$ touches or crosses the x-axis at the point$(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(x) at the points of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.
4. The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asumptoes are also translated.
7. The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor a in the vertical direction.
8. The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -f(x)$ is a reflection of the graph $y = f(x)$ in the x-axis.
10. The graph of $y = f(-x)$ is a reflection of the graph $y = f(x)$ in the y-axis.

5 Straight line graphs

1. The gradient m of the line goining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) can be calculated using the formula

   $$m = \frac{y_2 - y_2}{x_2 - x_1}$$

2.
   - The equation of a straight line can be written in the form

     $$y = mx + c$$

     where m is the gradient and $(0, c)$ is the y-intercept.
   - The equation of a straight line can also be written in the form

     $$ax + by + c = 0,$$

     where a, b and c are integars.
3. The equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) can be written as $y - y_1 = m(x - x_1)$.
4. Parallel lines have the same gradient.
5. If a line has a gradient m, a line purpendicular to it has a gradient of $-\frac{1}{m}$
6. If two lines are perpendicular, the product of their gradient is -1.
7. You can find the distance d between (x_1, y_1) and (x_2, y_2) by using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
8. The point of intersection of two lines can be found using simultanious equations.
9. Two quantities are in direct proportion when they increase at the same rate.

   The graph of these quantities is a straight line through the origin.
10. A mathematical model is an attempt to represent a real-life situation using mathematical concepts. It is often necessary to make assumptions about the real-life problems in order to create a model.

6 Circles

1. The midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$.
2. The perpendicular bisector of a line segmen AB is the straight line that is perpendicular to AB and passes through the midpoit AB.

   If the gradient of AB is m then the gradient if its perpendicular bisector l, will be $-\frac{1}{m}$.
3. The equation of a circle with centre $(0, 0)$ and radius r is $x^2 + y^2 = r^2$.
4. The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.
5. The equation of a circle can be given in the form: $x^2 + y^2 + 2fx + 2gy + c = 0$

   The circle has centre $(-f - g)$ and radius $\sqrt{f^2 + g^2 - c}$
6. A straight line can intersect a circle once, by touching the circle, or twice. Not all straight lines will intersect a given circle.
7. A tangent to a circle is perpendicular to the radius of the circle at the point of intersection
8. The purpendicular bisector of chord will go through the centre of a circle.
9. - If $\angle PQR = 90°$ then R lies on the circle with diameter PQ.
   - The angle in a semicircle is always a right angle.
10. To find the centre of a circle given by any three points:
    - Find the equations of the purpendicular bisectors of two different chords.
    - Find the coordinates of intersection of the perpendicular bisectors.

7 Alagabraic methods

1. When simplifying an algabraic farction, factorise the numerator and denominator where possible and then cancel common fractions.
2. You can use long division to divide a polynomial by $(x \pm p)$, where p is a constant.
3. The **factor theorum** states that is $f(x)$ is a polynomial then:
   - If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
   - If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$
4. You can prove a mathematical statement is true by **deduction**. This means starting from knwon factors or definitions, then using logical steps to reach the desired conclusion.
5. In a mathematical proof you must
   - State any known information or assumptions you are using
   - Sate every step of your proof clearly
   - Make sure that every step follows logically from the previous step
   - Make sure you have covered all possible cases
   - Write a statement of proof at the end of your working
6. To prove an identity you should
   - Start with the expression on one side of the identity
   - Manipulate that expression algebraically until it matches the other side
   - Show every step of your algebraic working
7. You can prove a mathematical statement is true by **exhaustion**. This means breaking that statement into smaller cases and proving each cases proving each case seperately.
8. You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one examample that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.

8 The binomeal expansion

1. Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.
2. The $(n+1)^{\text{th}}$ row of Pascal's triangle gives the coeficients of the expansion of $(a+b)^n$.
3. $n! = n \times (n-1) \times (n-2) \times ...\ x \times 3 \times 2 \times 1$.
4. You can use the factorial notation and your calculator to find entries in Pascal's triangle quickly.
   - The number of ways of choosing r items from a group n times is written as ${}^nC_r$ or $\binom{n}{r}$ $:^n C_r = \binom{n}{r} = \dfrac{n!}{r!n-r!}$
   - The r^{th} entry is the n^{th} row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.
5. The binomeal expansion is:

   $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ...\ \binom{n}{r}a^{n-r}b^r + ...\ b^n (n \in \mathbb{N})$

   where $\binom{n}{r} =^n C_r = \dfrac{n!}{r!n-r!}$
6. In the expansion of $(a+b)^n$ the general term in given by $\binom{n}{r}a^{n-r}b^r$.
7. The first few terms in the binomeal expansion can be used to find an approximate value for a complicated function.

9 Trigonometric ratios

1. This versionof the cosine rule is used to find a missing side if you know two sides and the angle between them:

   $a^2 = b^2 + c^2 - 2bc\cos A$
2. This version of the cosine rule is used to find an angle if you know all three sides:

   $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
3. This version of the sine rule is used to find the length of a missing side:

   $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
4. This version of the sine rule is used to find a missing angle:

   $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$
5. The sine rule sometimes produces two possible solutions for a missing angle:

   $\sin\theta = \sin(180° - \theta)$
6. Area of a triangle $= \dfrac{1}{2}ab\sin C$
7. The graphs of sine, cosine and tangent are **periodic**. They repeat themselves after a certain interval.
   - The graph of $y = \sin\theta$: repeats every $360°$ and crosses the x-axis at $..., -180°, 0, 180°, 360°, ...$
     has a miximum value of 1 and a minimum value of -1.
   - The graph of $y = \cos\theta$: repeats every $360°$ and crosses the x-axis at $..., -90°, 90°, 270°, 450°, ...$
     has a miximum value of 1 and a minimum value of -1.
   - The graph of $y = \tan\theta$: repeats every $180°$ and crosses the x-axis at $..., -180°, 0, 180°, 360°, ...$
     has no miximum or maximum value
     has vertical asymptotes at $x = -90°, x = 90°, x = 270°, ...$

10 Trigonometric identities and equations

1. For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:
   - $\cos\theta = x = x$-coordinate of P
   - $\sin\theta = y = y$-coordinate of P
   - $\tan\theta = \dfrac{y}{x} =$ gradient of OP
2. You can use quadrants to determine where each of the trigonometric ratios is positive or negative.

   For an angle θ in the second quadrant, only $\sin\theta is positive.$

   For an angle θ in the first quadrant, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.

   For an angle θ in the third quadrant, only $\tan\theta$ is positive.

   For an angle θ in the fourth quadrant, only $\cos\theta$ is positive.
3. You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.

   $\sin(180° - \theta) = \sin\theta$
   $\cos(180° - \theta) = -\cos\theta$
   $\tan(180° - \theta) = -\tan\theta$

   $\sin(180° + \theta) = -\sin\theta$
   $\cos(180° + \theta) = -\cos\theta$
   $\tan(180° + \theta) = \tan\theta$

   $\sin(360° - \theta) = -\sin\theta$
   $\cos(360° - \theta) = \cos\theta$
   $\tan(360° - \theta) = -\tan\theta$
4. The trigonometric ratios of 30°, 45° and 60° have exact forms, given below:

   $\sin 30° = \dfrac{1}{2}$
   $\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
   $\sin 60° = \dfrac{\sqrt{3}}{2}$

   $\cos 30° = \dfrac{\sqrt{3}}{2}$
   $\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
   $\cos 60° = \dfrac{1}{2}$

   $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$
   $\tan 45° = 1$
   $\tan 60° = \sqrt{3}$
5. For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$
6. For all values of θ, such that $\cos \neq 0$, $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$
7.
   - Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leq k \leq 1$
   - Solutions to $\tan\theta = p$ exist for all valeus of p.
8. When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principle angle**.
9. You calculator will give principle values in the following ranges:
   - $\cos^{-1}$ in the range $0 \leq \theta \leq 180°$
   - $\sin^{-1}$ in the range $0 \leq \theta \leq 90°$
   - $\tan^{-1}$ in the range $0 \leq \theta \leq 90°$

11 Vectors

1. If $\overrightarrow{PQ} = \overrightarrow{RS}$ then the line segments PQ and RS are equal in length and are parallel.
2. $\overrightarrow{AB} = -\overrightarrow{BA}$ as the line segment AB is equal in length, parallel and in the opposite direction to BA.
3. **Triangle law for vector addition**: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

   If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $a + b = c$
4. Subtracting a vector is equivalent to 'adding the negative vector': $a - b = a + (-b)$
5. Adding the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ gives the zero vector O: $\overrightarrow{PQ} + \overrightarrow{QP} = 0$.

6. Any vector parallel to the vector **a** may be written as $\lambda\mathbf{a}$ where λ is a non-zero scalar.

7. To multiply a column vector by a scalar, multiply each component by the scalar: $\lambda\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}\lambda p\\\lambda q\end{pmatrix}$

8. To add two column vectors, add the x-components and the y-components $\begin{pmatrix}p\\q\end{pmatrix} + \begin{pmatrix}r\\s\end{pmatrix}\begin{pmatrix}p+r\\q+s\end{pmatrix}$

9. A unit vector is a vector of length 1. The unit vectors along the x- and y-axes are usually denoted by **i** and **j** respectively. $\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$ $\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$

10. For any two-dimensional vector: $\begin{pmatrix}p\\q\end{pmatrix} = p\mathbf{i} + q\mathbf{j}$

11. For the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix}x\\y\end{pmatrix}$, the magnitude of the vector is given by: $|\mathbf{a}| = \sqrt{x^2 + y^2}$

12. A unit vector in the direction of **a** is $\dfrac{\mathbf{a}}{|\mathbf{a}|}$

13. In general, a point P with coordinates (p, q) has position vector:

    $$\overrightarrow{OP} = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix}p\\q\end{pmatrix}$$

14. $\overrightarrow{OP} = \overrightarrow{OB} - \overrightarrow{OA}$, where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B respectively.

15. If the point P divides the line segment AB in the ratio $\lambda : \mu$, then

    $$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}\\ &= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}(\overrightarrow{OB} - \overrightarrow{OA})\end{aligned}$$

16. If **a** and **b** are two non-parallel vectors and $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b}$ then $p = r$ and $q = s$

12 Differentation

1. The **gradient** of a **curve** at a given point is defined as the gardient of the **tangent** to the curve at that point

2. The **gradient function**, or **derivative**, of the curve $y = f(x)$ is written as $f'(x)$ or $\dfrac{dy}{dx}$

    $$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

    The gradient function can be used to find the gradient of the curve for any value of x.

3. For all real values of n, and for a constant a:
    - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$
    - If $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$
    - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$
    - If $y = ax^n$ then $\dfrac{dy}{dx} = anx^{n-1}$

4. For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by

    $$\frac{dy}{dx} = 2ax + b$$

5. If $y = f(x) \pm g(x)$, then $\dfrac{dy}{dx} = f'(x) \pm g'(x)$.

6. The tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

    $$y - f(a) = f'(a)(x - a)$$

7. The normal to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

    - $y - f(a) = \dfrac{1}{f'(a)}(x - a)$

8. - The function $f(x)$ is **increasing** on the interval $[a, b]$ if $f'(x) \geq 0$ for all values of x such that $a < x < b$.
   - The function $f(x)$ is **decreasing** on the interval $[a, b]$ if $f'(x) \leq 0$ for all values of x such that $a < x < b$.

9. Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\dfrac{d^2y}{dx^2}$

10. Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a **stationary point**. For small positive value: h:

Type of stationary point	$f'(x-h)$	$f'(x)$	$f'(x+h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
2***Point of inflection**	Negative	0	Positive
	Positive	0	Positive

11. If a function $f(x)$ has a stationary point when $x = a$, then:
    - if $f''(a) > 0$ the point is a local minimum
    - if $f''(a) < 0$ the point is a local maximum.

    If $f''(a) = 0$, the point could be called a minimum, local maximum or a point of inflection.

    You will need to look at points on either side to determine its nature.

13 Integration

1. If $\dfrac{dy}{dx} = x^n$, then $y = \dfrac{1}{n+1}x^{n+1}, +c$, $n \neq -1$.

   Using function notation, if $f'(x) = x^n$, then $f(x) = \dfrac{1}{n+1}x^{n+1}, +c$, $n \neq -1$

2. If $\dfrac{dy}{dx} = kx^n$, then $y = \dfrac{k}{n+1}x^{n+1}, +c$, $n \neq -1$.

   Using function notation, if $f'(x) = kx^n$, then $f(x) = \dfrac{k}{n+1}x^{n+1}, +c$, $n \neq -1$.

   When integrating polynomials, apply the rule of integration seperately to each term.

3. $\int f'(x)dx = f(x) + c$

4. $\int (f(x) + g(x)dx) = \int f(x)dx + \int (g)dx$

5. To find the constant of integration, c
   - Integrate the function
   - Substitute the values (x, y) of a point on the curve, or the value of the function given point $f(x) = k$ into the integrated function
   - Solve the equation to find c

6. If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definate integral is defined as $\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$

7. The are between a positive curve, the x axis and the lines $x = a$ and $x = b$ is given by Area $= \int_a^b ydx$ where $y = f(x)$ is the equation of the curve.

8. When the area bounded by a curve and the x-axis is below the x-axis, $\int ydx$ gives a negative answer.

9. You can use definate integration together with areas of trapiziums and triangles to find more complicated areas of graphs.

14 Exponentials and logarithms

1. For all real values of x:
   - If $f(x) = e^x$ then $f'(x) = e^x$
   - If $y = e^x$ then $\frac{dy}{dx} = e^x$
2. For all real values of x and any constant k:
   - If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
   - If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$
3. $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$
4. **The laws of logarithms**:
   - $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
   - $\log_a x - \log_a y = \log_a(\frac{x}{y})$ (the division law)
   - $\log_a x^k = k \log_a x$ (the power law)
5. You should also learn to recognise the following special cases:
   - $\log_a(\frac{1}{x}) = \log_a(x^{-1}) = -\log_a x$ (the powerlaw when $k = -1$)
   - $\log_a a = 1$ $(a > 0, a \neq 1)$
   - $\log_a 1 = 0$ $(a > 0, a \neq 1)$
6. Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$
7. The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$
8. $e^{\ln x} = \ln(e^x) = x$
9. If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient $\log b$ and vertical intercept $\log a$.