Contents

1 Complex numbers

1. You can use **Euler's relation**, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:

   $z = re^{i\theta}$

   where $r = |z|$ and $\theta = \arg z$.

2. For any two complex numbers $z_1 = r_1e^{i\theta_1}$ $z_2 = r_2e^{i\theta_2}$,

   - $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
   - $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

3. **De Moivre's theorum:**

   For any integar n, $(r(\cos\theta + i\sin\theta))^n = r^n\cos n\theta + i\sin n\theta)$

4. - $z + \frac{1}{z} = 2\cos\theta$
   - $z - \frac{1}{z} = 2i\sin\theta$
   - $z^n + \frac{1}{z^n} = 2\cos n\theta$
   - $z^n - \frac{1}{z^n} = 2i\sin n\theta$

5. For $w, z \in \mathbb{C}$,

   - $\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \ldots + w^{n-1} = \frac{w(z^n - 1)}{z - 1}$
   - $\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \ldots = \frac{w}{1 - z}, |z| < 1$

6. If z and w are non-zero complex numbers and n is a positive integar, then the equation $z^n = w$ has n distinct solutions

7. For any complex number $z = r(cos\theta + i\sin\theta)$, you can write

   $z = r(cos(\theta + 2k) + i\sin(\theta + 2k))$

   where k is any integar

8. In general, the solutions to $z^n = 1$ are $z = \cos(\frac{2k}{n}) + i\sin(\frac{2k}{n}) = e^{\frac{2ik}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.

   If n is a positive integar, then there is an nth root of unity $w = e^{\frac{2i}{n}}$ such that:

   - The nth roots of unity are $1, w, w^2, \ldots, w^{n-1}$
   - $1, w, w^2, \ldots, w^{n-1}$ form the vertices of a regular n-gon
   - $1, w, w^2, \ldots, w^{n-1} = 0$

9. The nth roots of any complex number s lie on the verticies of a regular n-gon with its center at the origin

10. If z_1 is one root of the equation $z^n = s$, and $1, w, w^2, \ldots, w^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1w, z_1w^2, \ldots, z_1w^{n-1}$.

2 Series

1. If the general term, u_n of a series can be expressed in the form $\mathrm{f}(r) - \mathrm{f}(r+1)$

   Then $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (\mathrm{f}(r) - \mathrm{f}(r+1))$.

   so $u_1 = \mathrm{f}(1) - \mathrm{f}(2)$
   $u_2 = \mathrm{f}(2) - \mathrm{f}(3)$
   $u_3 = \mathrm{f}(3) - \mathrm{f}(4)$
   $\vdots$
   $u_n = \mathrm{f}(n) - \mathrm{f}(n+1)$

   Then adding $\sum_{r=1}^{n} u^r = \mathrm{f}(1) - \mathrm{f}(n+1)$

2. The **Maclaurin series** of a function $\mathrm{f}(x)$ is given by

   - $\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \cdots + \frac{\mathrm{f}^{(r)}(0)}{r!}x^r + \ldots$

   The series is valid provided that $\mathrm{f}(0), \mathrm{f}'(0), \mathrm{f}''(0), \ldots, \mathrm{f}^{(r)}(0), \ldots$ all have finite values.

3. The following Maclaurin series are given in the formulae booklet:

   $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \ldots$ for all x

   $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r+1}\frac{x^r}{r}$ $-1 < x \leq 1$

   $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$ for all x

   $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$ for all x

   $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots$ $-1 \leq x \leq 1$

3 Methods in calculus

1. The integral $\int_a^b f(x)dx$ is **improper** if either:
   - one or both of the limits is infinite
   - $f(x)$ is undefined at $x = a$, $x = b$ or another point in the interval $[a, b]$.

2. The **mean value** of the function $f(x)$ overt the interval $[a, b]$, is given by

   $$\boldsymbol{\frac{1}{b-a}\int_a^b f(x)dx}$$

3. If the function $f(x)$ has mean value of $\overline{f}$ over the interval $[a, b]$, and k is a real constant, then:
   - $f(x) + k$ has mean value $\overline{f} + k$ over the interval $[a, b]$
   - $kf(x)$ has mean value $k\overline{f}$ over the interval $[a, b]$
   - $-f(x)$ has mean value $-\overline{f}$ over the interval $[a, b]$.

4. - $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
   - $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$
   - $fracddx(\arctan x) = \frac{1}{1+x^2}$

5. - $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(\frac{x}{a}) + c$ $a > 0, |x| < a$
   - $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + c$

4 Volumes of revolution

1. The volume of revolution formed when $y = f(x)$ is rotated through 2π radians about the x-axis between $x = a$ and $x = b$ is given by

$$\textbf{Volume} = \boldsymbol{\pi \int_a^b y^2 dx}$$

2. The volume of revolutino formed when $x = f(y)$ is rotated through 2π radians about the y-axis between $y = a$ and $y = b$ is given by

$$\text{Volume} = \boldsymbol{\pi \int_a^b x^2 dy}$$

3. The volume of revolution formed when the parametric curve with equations $x = f(t)$ and $y = g(t)$ is rotated through 2π radians about the x-axis and $x = a$ and $x = b$ is given by

$$\text{Volume} = \boldsymbol{\pi \int_{x=a}^{x=b} y^2 dx = \pi \int_{t=q}^{t=p} y^2 \frac{dx}{dt} dt}$$

4. The volume of revolution formed by rotation the same curve through 2π radians about the y-axis between $y = a$ and $y = b$ is given by

$$\text{Volume} = \boldsymbol{\pi \int_{y=a}^{x=b} x^2 dy = \pi \int_{t=q}^{t=p} x^2 \frac{dy}{dt} dt}$$

5 Polar coordinates

1. For a point P with polar coordinates (r, θ) and Cartesian coordinates (x, y),
   - $r \cos\theta = x$ and $r \sin\theta = y$
   - $r^2 = x^2 + y^2$, $\theta = \arctan(\frac{y}{x})$

   Care must be taken to ensure that θ is in the correct quadrent.
2. - $r = a$ is a circle with centre O and radius a.
   - $\theta = \alpha$ is a half-line through O and making an angle α with the initial line.
   - $r = a\theta$ is a spiral starting at O.
3. The **area of a sector** bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula

$$\text{Area} = \boldsymbol{\frac{1}{2} \int_\alpha^\beta r^2 d\theta}$$

4. - To find a tangent parallel to the initial line set $\frac{dy}{d\theta} = 0$.
   - To find a tangent perpendicular to the initial line set $\frac{dx}{d\theta} = 0$.

6 Hyperbolic functions

1. - Hyperbloic sin (or **sinh**) is defined as $\sinh \equiv \frac{e^x - e^- x}{2}$, $x \in \mathbb{R}$
   - Hyperbloic cos (or **cosh**) is defined as $\cosh \equiv \frac{e^x + e^- x}{2}$, $x \in \mathbb{R}$
   - Hyperbloic tan (or **tanh**) is defined as $\tanh \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$, $x \in \mathbb{R}$
2. - The graph of $y = \sinh x$:

     For any value a, $\sinh(-a) = -\sinh a$.
   - The graph of $y = \cosh x$:

     For any value a, $\cosh(-a) = \cosh a$.
3. The table shows the inverse hyperbolic functions, with domains restricted where necessary.

Hyperbolic function	**Inverse hyperbolic function**
$y = \sinh x$	$y = x$
$y = \cosh x$, $x \geq 0$	$y = arcoshx$,$x \geq 1$
$y = \tanh x$	$y = artanhx$, $\lvert x \rvert < 1$

4.
- $x = \ln(x + \sqrt{(x^2+1)})$
- $x = \ln(x + \sqrt{x^2-1}),\ x \geq 1$
- $x = \frac{1}{2}\ln(\frac{1+x}{1-x}),\ |x| < 1$

5. $\cosh^2 A - \sinh^2 A \equiv 1$

6.
- $\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$
- $\cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

7.
- $\frac{d}{dx}(\sinh x) = \cosh x$
- $\frac{d}{dx}(\cosh x) = \sinh x$
- $\frac{d}{dx}(tanhx) =^2 x$
- $\frac{d}{dx}(x) = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx}(x) = \frac{1}{\sqrt{x^2-11}}$
- $\frac{d}{dx}(x) = \frac{1}{1-x^2}$

8.
- $\int \sinh x\, dx = \cosh x + c$
- $\int \cosh x\, dx = \sinh x + c$

9. $\int \tanh x\, dx = \ln \cosh x + c$

10.
- $\int \frac{1}{\sqrt{a^2+x^2}}\, dx = (\frac{x}{a}) + c$
- $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = (\frac{x}{a}) + c,\ x > a$

7 Methods in differential equations

1. You can solve a first-order differential equation of the form $\boldsymbol{\frac{dy}{dx} + P(x)y = Q(x)}$ by multiplying every term by the **integrating factor** $e^{\int P(x)\, dx}$

2. The natures of the roots α and β of the **auxiliary equation** determine the **general solution** to the second-order differential equation $\boldsymbol{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0}$.

   You need to consider three different cases:

   - **Case 1: $\boldsymbol{b^2 > 4ac}$**
   - The auxilary equation has two real roots α and β ($\alpha \neq \beta$). The general solution will be of the form $y = Ae^{\alpha x} + Be^{\beta x}$ where A and B are arbitrary constants.
   - **Case 2: $\boldsymbol{b^2 = 4ac}$**
   - The auxilary equation has repeated root α. The general solution will be of the form $y = (A+B)e^{\alpha x}$ where A and B are arbitrary constants.
   - **Case 3: $\boldsymbol{b^2 < 4ac}$**
   - The auxilary equation has two complex conjugate roots α and β equal to $p \pm qi$. The general solution will be of the form $y = e^{px}(A\cos qx + B\sin qx)$ where A and B are arbitrary constants.

3. A **particular integral** is a function which satisfies the origional differential equation.

4. To find the general solution to the differential equation $\boldsymbol{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)}$.

   - Solve the corresponding homogeneous equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0$ to find the complementary function, C.F.
   - Choose an appropriate form for the particular integral, P.I. and substitute into the orgional equation to find the values of any coefficients.
   - The general solution is $y = \text{C.F.} + \text{P.I.}$

8 Modelling with differential equations

1. **Simple harmonic motion** (S.H.M.) is motion in which the acceleration of a particle P is always towards a fixed point O ont the line of motion P. The acceleration is proportional to the displacement of P from O.

2. $\ddot{x} = v\frac{dv}{dx}$

3. For a particle moving the **damped harmonic motion**

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

where x is the displacement from a fixed point at time t, and k and ω^2 are positive constants.

4. For a particle moving the **forced harmonic motion**

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$$

where x is the displacement from a fixed point at time t, and k and ω^2 are positive constatns.

5. You can solve coupled first-order linear differential equations by eliminating one of the dependent variables to form a second-order differential equation.