Contents

1 **Algabraic methods** **1**

2 **Functions and graphs** **2**

3 **Sequences and series** **2**

4 **Binomeal expansion** **3**

5 **Radians** **3**

6 **Trigonometric functions** **4**

7 **Trigonometry and modeling** **5**

8 **Parametric equations** **5**

9 **Differentiation** **5**

10 **Numerical methods** **6**

11 **Integration** **6**

12 **Vectors** **7**

1 Algabraic methods

1. To prove a statement by contradiction you start by assuming it is **not true**. You can then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption or a contradiction of fact you know to be true). You can conclude that you assumption was incorrect, and the origional statement **was true**.
2. A rational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integars.
3. To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.
4. To devide two fractions, multiply the first fraction by the reciprocal of the second fraction.
5. To add or subtract two fractions, find a common denominator.
6. A single fraction with two distinct linear factors in the denominator can be split into two seperate fractions with linear denominators. This is called splitting it into **partial fractions**:
   $$\frac{5}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$
7. The method of partial fractions can also be used when therem are more then two distinct linear factors in the denominator:
   $$\frac{7}{(x-2)(x+6)(x+3)} = \frac{A}{x-2} + \frac{B}{x+6} + \frac{C}{x+3}$$
8. A single fraction with a repeated linear factor in the denominator can be split into two or more sperate fractions:
   $$\frac{2x+9}{(x-5)(x+3)^2} = \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$
9. An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.
10. You can either use:
    - angebraic devision
    - or the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$
11. to convert an improper fraction into a mixed fraction.

2 Functions and graphs

1. A modulus function is, in general a function of type $y = |f(x)|$.
   - When $f(x) \geq 0, |f(x)| = f(x)$
   - When $f(x) < 0, |f(x)| = -f(x)$
2. To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then refledt the section of the graph below the x -axis in the x-axis.
3. A mapping is a **function** if every input has a distinct output. Functions can either be **one to one** or **many to one**.
4. $fg(x)$ means apply g first, then apply f, $fg(x) = f(g(x))$
5. Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x)$ and $f^{-1}f(x) = x$.
6. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.
7. The domain of $f(x)$ is the range $f^{-1}(x)$.
8. The range of $f(x)$ is the domain $f^{-1}(x)$.
9. To sketch the graph of $y = |f(x)|$
   - Sketch the of $y = f(x)$
   - Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x -axis
   - Delete the parts below the x-axis
10. To sketch the graph of $y = f(|x|)$
    - Sketch the graph of $y = f(x)$ for $x \geq 0$
    - Reflect this in the y-axis
11. $f(x + a)$ is a horizontal translation of $-a$.
12. $f(x) + a$ is a vertical translation of $+a$
13. $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$.
14. $af(x)$ is a vertical stretch of scale factor a.
15. $f(-x)$ reflects $f(x)$ in the y-axis.
16. $-f(x)$ reflects $f(x)$ in the x -axis.

3 Sequences and series

1. In an **arithmetic sequence**, the difference between consecutive terms is constant.
2. The formulae for the n^{th} term of an arithmetic sequence is $u_n = a + (n-1)d$, where a is the first term and d is the common difference.
3. An arithmatic series is the sum of the terms of an arithmatic sequence.

   The sum of the first n terms of an arithmatic sequence is given by $S_n = \frac{n}{2}(2a + (n-1)d)$, where a is the first term and d is the common difference.

   You can also write this formulae as $S_n \frac{n}{2}(a + l)$, where l is the last term.
4. A **geometric sequence** has a **common ratio** between consequtive terms.
5. The formulae for the n^{th} term of a geometric sequence is $u_n = a^{n-1}$, where a is the first term and r is the common ratio.
6. The sum of the first n terms of a gemetric sequence is given by $S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$
7. $S_m = \frac{a(r^n - 1)}{r - 1}, r \neq 1$

where a is the first term and r is the common ratio.

8. A geometric series is a convergent if and only if $|r| < 1$, where r is the common ratio.

   The **sum to infinity** of a convergent geometric series is given by $S_\infty = \dfrac{a}{1-r}$

9. The Greek capital letter 'sigma' is used to signify a sum. You can write it as Σ. You can write limits on the top and bottom to show which terms you are summing.

10. A recurrencerelation of the form $u_{n+1} = f(u_n)$ defines each term of a sequence as a function of the previous term.

11. A sequence is **increasing** $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

    A sequence is **decreasing** $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

    A sequence is **periodic** if the terms repeat, in a cycle. For a periodic sequence ther is an integar k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the **order** of the sequence.

4 Binomeal expansion

1. This form of the binomeal expansion can be applied to negative or fractional values for n to obtain an infinate series:

   $$(1+x)^n = 1 + nx + \frac{n(n+1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + ... + \frac{n(n-1)...(n-r+1)x^r}{r!} + ...$$

   The expansion is valid when $|x| < 1, n \in \mathbb{R}$.

2. The expansion of $(1+bx)^n$, where n, is a negative or a fraction, is valid for $|bx| < 1$, or $|x| < \dfrac{1}{|b|}$

3. The expansion of $(a+bx)^2$, where n is negative or a fraction, is valid for $|\dfrac{b}{a}x| < 1$, or $|x| < |\dfrac{a}{b}|$.

5 Radians

1.
   - 2π radians $= 360°$
   - π radians $= 180°$
   - 1 radians $= \dfrac{180°}{\pi}$

2.
   - $30° = \dfrac{\pi}{6}$ radians
   - $45° = \dfrac{\pi}{4}$ radians
   - $60° = \dfrac{\pi}{3}$ radians
   - $90° = \dfrac{\pi}{2}$ radians
   - $180° = \pi$ radians
   - $360° = 2\pi$ radians

3. You need to learn the exact values of trigonometric ratios of these angles measured in radians.
   - $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$
   - $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$
   - $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$
   - $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$
   - $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$
   - $\tan\dfrac{\pi}{3} = \sqrt{3}$
   - $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
   - $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
   - $\tan\dfrac{\pi}{4} = 1$

4. You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angles made with the x-axis, θ.
   - $\sin(\pi - \theta) = \sin\theta$
   - $\sin(\pi + \theta) = -\sin\theta$
   - $\sin(2\pi - \theta) = -\sin\theta$
   - $\cos(\pi - \theta) = -\cos\theta$
   - $\cos(\pi + \theta) = -\cos\theta$
   - $\cos(2\pi - \theta) = \cos\theta$
   - $\tan(\pi - \theta) = -\tan\theta$
   - $\tan(\pi + \theta) = \tan\theta$
   - $\tan(2\pi - \theta) = -\tan\theta$

5. To find the arc length l of a sector of a circle use the formulae $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.

6. to find the area A of a sector of a circle use the formulae $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.

7. The area of a segment in a circle of radius r is

   $A = \frac{1}{2}r^2 (\theta - \sin\theta)$

8. Where θ is small and measured in radians:

   - $\sin\theta \approx \theta$
   - $\tan\theta \approx \theta$
   - $\cos\theta \approx 1 - \frac{\theta^2}{2}$

6 Trigonometric functions

1. - $\sec x = \frac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)
   - $\csc x = \frac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
   - $\cot x = \frac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)
   - $\cot x = \frac{\cos x}{\sin x}$

2. The graph of $y = \sec x$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$

   - The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90^\circ, 270^\circ, 450^\circ, ...$ or any odd multiple of 90°.
   - In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$ or any odd multiple of $\frac{\pi}{2}$
   - The range of $y = \sec x$ is $y \leq -1$ or $y \geq 1$.

3. The graph of $y = \csc x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$

   - The domain of $y = \csc x$ is $x \in \mathbb{R}$, $x \neq 0^\circ, 180^\circ, 360^\circ, ...$ or any odd multiple of 180°.
   - In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, ...$ or any odd multiple of π
   - The range of $y = \sec x$ is $y \leq -1$ or $y \geq 1$.

4. The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$

   - The domain of $y = \csc x$ is $x \in \mathbb{R}$, $x \neq 0^\circ, 180^\circ, 360^\circ, ...$ or any odd multiple of 180°.
   - In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, ...$ or any odd multiple of π
   - The range of $y =\in \mathbb{R}$.

5. $\sec x = k$ and $\csc x = k$ have no solutions for $-1 < k < 1$.

6. You can use the identity $\sin^2 x + cos^2 x \equiv 1$ to prove the following identities:

   - $1 + tan^2 x \equiv \sec^2 x$
   - $1 + cot^2 x \equiv \csc^2 x$

7. The inverse function of $\sin x$ is called $\arcsin x$.

   - The domain if $y = \arcsin x$ is $-1 \leq x \leq 1$
   - The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$ or $-90^\circ \leq \arcsin x \leq 90^\circ$

8. The inverse function of $\cos x$ is $\arccos x$.

   - The domain if $y = \arccos x$ is $-1 \leq x \leq 1$
   - The range of $y = \arccos x$ is $0 \leq \arccos x \leq \pi$ or $0^\circ \leq \arccos x \leq 180^\circ$

9. The inverse function of $\tan x$ is called $\arctan x$.

   - The domain if $y = \arctan x$ is $x \in \mathbb{R}$
   - The range of $y = \arctan x$ is $-\frac{\pi}{2} \leq \arctan x \leq \frac{\pi}{2}$ or $-90^\circ \leq \arctan x \leq 90^\circ$

7 Trigonometry and modeling

1. The **addition** (or compounded-angle) formulae are:
   - $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$
   - $\cos(A+B) \equiv \cos A \cos B + \sin A \sin B$
   - $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
   - $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$
   - $\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$
   - $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
2. The **double-angle** formulae are:
   - $\sin 2A \equiv 2 \sin A \cos A$
   - $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
   - $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$
3. For positive values of a and b,
   - $a \sin x \pm b \cos x$ can be expressed in the form $R \sin(x \pm \alpha)$
   - $a \cos x \pm b \sin x$ can be expressed in the form $R \sin(x \mp \alpha)$

   with $R > 0$ and $0 < \alpha < 90^\circ$ $\left(\text{or} \dfrac{\pi}{2}\right)$

   where $R \cos \alpha = a$ and $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$

8 Parametric equations

9 Differentiation

1. For small angles, measured in radians:
   - $\sin x \approx x$
   - $\cos x \approx 1 - \dfrac{1}{2}x^2$
2.
   - If $y = \sin kx$, then $\dfrac{dy}{dx} = k \cos kx$
   - If $y = \cos kx$, then $\dfrac{dy}{dx} = -k \sin kx$
3.
   - If $y = e^{kx}$, then $\dfrac{dy}{dx} = ke^{kx}$
   - If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$
4. If $y = a^{kx}$, where k is a real constant and $a > 0$, then $\dfrac{dy}{dx} = a^{kx} k \ln a$
5. The **chain rule** is: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

   where y is a function of u and u is another function of x.
6. The chain rule enables you to differentiate a function of a function. In general,
   - if $y = (f(x))^n$ then $\dfrac{dy}{dx} = n(f(x))^{n-1} f'(x)$
   - if $y = f(g(x))$ then $\dfrac{dy}{dx} = f'(g(x))\, g'(x)$
7. $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$
8. The **product rule**:
   - If $y = uv$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, where u and v are functions of x.
   - If $f(x) = g(x)\,h(x)$ then $f'(x) = g(x)\,h'(x)\,h(x)\,g'(x)$
9. The **quotent rule**:

- If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where u and v are functions of x.
- If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{h(x)\,g'(x) - g(x)\,h'(x)}{(h(x))^2}$
- If $y = \tan kx$, then $\frac{dy}{dx} = k\sec^2 kx$
- If $y = \csc kx$, then $\frac{dy}{dx} = -k\csc kx \cot kx$
- If $y = \sec kx$, then $\frac{dy}{dx} = k\sec kx \tan kx$
- If $y = \cot kx$, then $\frac{dy}{dx} = -k\csc^2 kx$

10. If x and y are given as functions of a parameter, t : $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

11. - $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$
    - $\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$
    - $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$

12. - The function $f(x)$ is **concave** on a given interval if and only if $f'(x) \leq 0$ for every value of x in that interval.
    - The function $f(x)$ is **convex** on a given interval if and inly if $f'(x) \geq 0$ for every value of x in that inerval.

13. A **point of inflection** if a point at which $f''(x)$ changes sign.

14. You can use the chain rule to connect rates of change in situations involving more than two variables.

10 Numerical methods

1. If the function $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has at least one root, x, which satisfies $a < x < b$.

2. To solve an equation of the form $f(x) = 0$ by an iterative method, rearrange $f(x) = 0$ into the form $x = g(x)$ and use the iterative formulae $x_{n+1} = g(x_n)$.

3. The Newton-Raphson formulae for approximating the roots of a funtion $f(x)$ is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

11 Integration

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\quad$ $\int e^x dx = e^x + c$ $\quad$ $\int \frac{1}{x} dx = \ln|x| + c$

   $\int \cos x dx = \sin x + c$ $\quad$ $\int \sin x dx = -\cos x + c$ $\quad$ $\int \sec^x dx = \tan x + c$

   $\int \csc x \cot x dx = -\csc x + c$ $\quad$ $\int \csc^x dx = -\cot x + c$ $\quad$ $\int \sec x \tan x dx = \sec x + c$

2. $\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$

3. Trigonometric identities can be used to integrate expressions. This allows an expression that cannto be integrated to be replaced by an identical expression that can be integrated.

4. To integrate expressions of the form $\int k\frac{f'(x)}{f(x)}dx$, try $\ln|f(x)|$ and differentiate to check, and then adjust any constant.

5. Sometimes an expression of the form $\int kf'(x)(f(x))^n\,dx$, try $(f(x))^{n+1}$ and differentiate to check and then adjust any constant.

6. Sometimes you can simplify an integral by changing to variable. This process is similar to using the chain rule in differentiation and called **integration by substitution**.

7. The **integration by parts** formulae is given by: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

8. Partial farctions can be used to integrate algebraic fractions.

9. The area bounded by two curves can be found using integration:

   Area of $R = \int_a^b \left(f\left(x\right) - g\left(x\right)\right) dx = \int_a^b f\left(x\right) dx - \int_a^b g\left(x\right) dx$

10. The **trapezium rule** is:

    $$\int_a^b y dx \approx \frac{1}{2} h \left(y_0 + 2\left(y_1 + y_2 ... + y_{n-1}\right) + y_n\right)$$

    where $h = \frac{b-a}{n}$ and $y_i = f\left(a + ih\right)$

11. When $\frac{dy}{dx} = f\left(x\right) g\left(y\right)$ you can write

    $$\int \frac{1}{g\left(y\right)} dy = \int f\left(x\right) dx$$

12 Vectors

1. The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$

2. The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2 + \left(z_1 - z_2\right)^2}$

3. The unit vectors along the x-, y- and z-axes are denoted as **i**, **j** and **k** respectively.

   $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

   Any 3D vector can be written in column form as $p\mathbf{i} + q\mathbf{j} + r\mathbf{k} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

4. If the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes an angle θ, with the positive x-axis then $\cos\theta_x = \frac{x}{|\mathbf{a}|}$ and similarly for the angles θ_y and θ_z.

5. If **a**, **b** and **c** are vectors in three dimensions which do not all lie in the same plane then you can compare their coefficients on both sides of an equation.