Contents

n#+LATEX$_{\text{CLASS}}$: article

1 Complex numbers

1. $i = \sqrt{-1}$ and $i^2 = -1$
2. An **imaginary number** is a number in the form bi, where $b \in \mathbb{R}$
3. A **complex number** is written in the form $\mathbf{a + bi}$, where $a, b \in \mathbb{R}$
4. Complex number can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.
5. You can multiply a real number by a complex number by multiplying out the brackets in the usual way.
6. If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct roots, neither of which is real.
7. For any comlex number $z = a + bi$, the **complex conjugate** of the number is defined as $z^* = a - bi$
8. For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.
9. If the roots of a quadratic equation are α, β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.
10. If $z(x)$ is a polynomial with real coefficients, and z_1 is a root of $f(z) = 0$, then z_1^* is also a root of $f(z) = 0$.
11. An equation in the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.
12. For a cubic equation with real coefficients, either:
    - all three roots are real, or
    - one root is real and the other two roots form a complex conjugate pair.
13. An equation in the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.
14. For a quatic equation with real coefficients, either:
    - all four roots are real, or
    - two roots are real and the other two roots form a complex pair, or
    - two roots form a complex pair and the other two roots also form a complex conjugate pair.

2 Argand diagrams

1. You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the \$y\$-axis is called the **imaginary axis**. The complex number $z = x + iy$ is represented on the diagramby the point $P(x, y)$, where x and y are Cartesian coordinates.

2. The complex number $z = x + iy$ can be represented as the vector\$ $\begin{pmatrix} x \\ y \end{pmatrix}$ \$ on an Argand diagram.

3. The **modulus** of a complex number $|z|$, is the distance from the origin to that on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$

4. The **argument** of a complex number, arg z, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x+iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$

5. Let α be the positive acute angle made with the real axis by the line joining the origin and z.

   - If z lies in the first quadrant then arg $z = \alpha$.
   - If z lies in the second quadrant then arg $z = \pi - \alpha$.
   - If z lies in the third quadrant then arg $z = -(\pi - \alpha)$.
   - If z lies in the fourth quadrant then arg $z = -\alpha$.

6. For a complex number z with $|z| = r$ and arg $z = \theta$, the modulus-argument form of z is $z = r(\cos\theta + i\sin\theta)$

7. For any two complex numbers z_1 and z_2.

   - $|z_1 z_2| = |z_1||z_2|$
   - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
   - \$ |z1 $\frac{}{z_2| = \frac{|z_1|}{|z_2|}}$ \$
   - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

8. For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.

9. Given $z_1 = x_1 + iy_1$, the locus of points z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + iy_1)| = r$ is a cricle with center $x_1,\ y_1$ and radius r.

10. Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the purpendicular bisector of the line segment joining z_1 and z_2.

11. Given $z_1 = x_1 + y_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.

3 Series

1. To find the sum of a series of constant terms you can use the formulae $\sum_{r=1}^{n} 1 = n$.

2. The formulae for the sum of the first n natural numbers is $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

3. To find the sum of a series that does not start at $r = 1$, use $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$

4. You can rearrange expressions involving sigma notation.

   - $\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$
   - $\sum_{r=1}^{n} (f(r) + g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

5. The formulae for the sum of squares of the first n natural numbers is

- $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n\left(n+1\right)\left(2n+1\right)$

6. The formulae for the sum of cubes of the first n natural numbers is

- $\displaystyle\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2\left(n+1\right)^2$

4 Roots of polynomials

1. If α and β are roots of the equation $ax^2+bx+c=0$, then:

- $\alpha+\beta=-\dfrac{b}{a}$
- $\alpha\beta=\dfrac{c}{a}$

2. If α, β and γ are roots of the equation $ax^3+bx^2+cx+d=0$, then:

- $\alpha+\beta+\gamma=\Sigma\alpha=-\dfrac{b}{a}$
- $\alpha\beta+\beta\gamma+\gamma\alpha=\Sigma\alpha\beta=\dfrac{c}{a}$
- $\alpha\beta\gamma=-\dfrac{d}{a}$

3. If α, β, γ \$$\delta$ \$and are roots of the equation $ax^4+bx^3+cx^2+dx+e=0$, then:

- $\alpha+\beta+\gamma+\delta=\Sigma\alpha=-\dfrac{b}{a}$
- $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\Sigma\alpha\beta=\dfrac{c}{a}$
- $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=\Sigma\alpha\beta\gamma=-\dfrac{d}{a}$
- $\alpha\beta\gamma\delta=\dfrac{e}{a}$

4. The rules for **reciprocals**:

- Quadratic: $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=\dfrac{\alpha+\beta}{\alpha\beta}$
- Cubic: $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
- Quartic: $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}+\dfrac{1}{\delta}=\dfrac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}$

5. The rules for **products of powers**:

- Quadratic: $\alpha^n\times\beta^n=\left(\alpha\beta\right)^n$
- Cubic: $\alpha^n\times\beta^n\times\gamma^n=\left(\alpha\beta\gamma\right)^n$
- Quartic: $\alpha^n\times\beta^n\times\gamma^n+\delta^n=\left(\alpha\beta\gamma\delta\right)^n$

6. The rules for **sums of squares**:

- Quadratic: $\alpha^2+\beta^2=\left(\alpha+\beta\right)^2-2\alpha\beta$
- Cubic: $\alpha^2+\beta^2+\gamma^2=\left(\alpha+\beta+\gamma\right)^2-2\left(\alpha\beta+\beta\gamma+\gamma\alpha\right)$
- Quartic: $\alpha^2+\beta^2+\gamma^2+\delta^2=\left(\alpha+\beta+\gamma+\delta\right)^2-2\left(\alpha\beta+\alpha\gamma+\alpha\gamma+\beta\gamma+\beta\delta+\gamma\delta\right)$

7. The rules for **sums of cubes**:

- Quadratic: $\alpha^3+\beta^3=\left(\alpha+\beta\right)^3-3\alpha\beta\left(\alpha+\beta\right)$
- Cubic: $\alpha^3+\beta^3+\gamma^3=\left(\alpha+\beta+\gamma\right)^3-3\left(\alpha+\beta+\gamma\right)\left(\alpha\beta+\beta\gamma+\alpha\gamma\right)+3\alpha\beta\gamma$

5 Volumes of revolution

1. The volumes of revolution formed when $y = (f)x$ is rotated abou the x-axis between $x = a$ and $x = b$ is given by

    - **Volume** $= \pi \int_a^b y^2 dx$

2. The volumes of revolution formed when $y = (f)x$ is rotated abouthe y-axis between $y = a$ and $y = b$ is given by

    - **Volume** $= \pi \int_a^b x^2 dy$

3. A cylinder of height h and radius r has volume $\pi r^2 h$

4. A cone of height h and base radius r has volume $\frac{1}{3}\pi r^2 h$

6 Matrices

1. A **square matrix** is one where the number of rows and columns are the same.

2. A zero matrix is one in which all of the numbers are zero. The zero matrix is 0.

3. An identity matrix is a square matrix in which the numbers in the leading diagonal (starting top left) are 1 and all the rest are 0. Identity matrices are denoted by $\mathbf{I}_k$, where k describes the size. The 3 x 3 identity matrix is $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. To add or subtract matrices, you add or subtract the corresponging elements in each matrix. You can onlny add or subtrac matrices that are the same size.

5. To multiply a matrix by a scalar, you multiply every element in the matrix by the scalar.

6. - Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix. In this case the first is said to be multiplicatively conformable with the second.
    - To find the product of two multiplicatively conformable matrices, you multiply the elements in each row in the left-hand matrix by the corresponding elements in each column in the right-hand matrix, thena dd the results together.

7. For a 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of \$\\$ad - bc$.

8. - If det $\mathbf{M} = 0 then * M$ is a **singular** matrix
    - If det $\mathbf{M} \neq 0$ then **M\$ is a *non-singular** matrix.

9. You find the determinant \$3 $\times$ 3\$matrix by 3×3 2×2 determinatns using the formula:

    $$\text{a} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \text{a} \begin{vmatrix} e & f \\ h & i \end{vmatrix} \text{-b} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \text{c} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

10. The **minor** of an element $3x3$ matrix is the determinant of \$2x2\$matrix that remains after the row and column containing that element have been crossed out.

11. The **inverse** of a matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

12. If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{M}^{-1} = \frac{1}{\text{det}\mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

13. If $\mathbf{A}$ and $\mathbf{B}$ are non-singular matrices, then $(\mathbf{AB})^{-1}) = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

14. The **transpose** of a matrix is found by interchanging the rows and the columns.

15. Finding the inverse \$3x3\$matrix $\mathbf{A}$ usually consists of the following 5 steps.

    (a) Find the determinant of $\mathbf{A}$, det $\mathbf{A}$.

    (b) Form the matrix of the minors of $\mathbf{A}$, $\mathbf{M}$.
    In forming the matrix $\mathbf{M}$, each of the nine elements of the matrix $\mathbf{A}$ is replaced by its minor.

    (c) From the matrix of minors, form the matrix of **cofactors**, $\mathbf{C}$, by changing signs of some elements of the matrix of minors acording to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

You leave the elements of the matrix of minors corresponding to the + signs in this pattern unchanged. You change the signs of the elements corresponding to the - signs.

(d) Write down $\mathbf{C}^T$, of the matrix of cofactors.

(e) The inverse of the matrix $\mathbf{A}$ is given by the formula

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^T$$

7 Linear transformations

1.
   - Linear transformations always map the origin onto itself.
   - Any linear transformation can be represented by a matrix.
2. The $T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ can be represented by $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$.
3. A reflection y-axis is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Points y-axis are invarient points, and the lines $x = 0$ and $y = k$ for any value k are invarient lines.
4. A reflection x-axis is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Points x-axis are invarient points, and the lines $y = 0$ and $x = k$ for any value k are invarient lines.
5. A reflection on $y = x$ is represented by the matrix $\begin{pmatrix} p0 & 1 \\ 1 & 0 \end{pmatrix}$. Points on $y = x$ are invarient points, and the lines $y = x$ and $y = -x + k$ for any value of k are invarient lines.
6. A reflection on $y = -x$ is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Points on $y = -x$ are invarient points, and the lines $y = -x$ and $y = x + k$ for any value of k are invarient lines.
7. The matrix representing a θanticlockwise about the $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The only invariant point is $(0, 0)$. For $\theta \neq 180°$, there are no invariant $\theta = 180°$, any line passing through the origin is an invariant line.
8. A transformation represented by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is a stretch of scale factor a parallel to the x-axis and a stretch of $b$parallel y-axis.

   In the $a = b$, the transformation is an enlargement with a.
9. For any stretch of the above form, the $x-$ and y-axis are invariant lines and the origin is an invariant point.
10. For a stretch paralllel to the x-axis, points y-axis are invariant points, and any line parallel x-axis is an invariant line.
11. For a stretch paralllel to the y-axis only, points x-axis are invariant points, and any line parallel y-axis is an invariant line.
12. For a linear transformation represented by matrix $\mathbf{M}$, det $\mathbf{M}$ represents the scale factor for the change in area. This is sometimes called the **area scale factor**.
13. The matrix $\mathbf{PQ}$ represents Q, with the matrix $\mathbf{Q}$, followed by P, with $\mathbf{P}$.
14. A reflection in the plane $x = 0$is represented by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
15. A reflection in the plane $y = 0$is represented by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
16. A reflection in the plane $z = 0$is represented by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

17. A rotation, angle θ, anticlockwise about the x-axis is represented by matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

18. A rotation, angle θ, anticlockwise about the y-axis is represented by matrix $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

19. A rotation, angle θ, anticlockwise about the z-axis is represented by matrix $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

20. The transformation described by the matrix $\mathbf{A}^{-1}$has the effect of reversing the transformation described by the matrix $\mathbf{A}$.

8 Proof by induction

1. You can use **proof by induction** to prove that a general statement is true for all positive integers.

2. Proof by mathematical induction usally consists of the following four steps:

   - **Basis:** Show the general statement is true for $n = 1$.
   - **Assumption:** Assume that the general statement is true for $n = k$.
   - **Inductive:** Show the general statement is true for $n = k + 1$.
   - **Conclution:** State that the general statement is then true of all positive integars, n.

9 Vectors

1. A vector equation of a straight line passing through the point A with position vector $\mathbf{a}$, and parallel to the vector $\mathbf{b}$, is

   - $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

   where λ is a scalar parameter.

2. A vector equation of a straight line passing through the points D, with position vectors $\mathbf{c}$ and $\mathbf{d}$ respectively, is

   - $\mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$ **-** $\mathbf{c}$

   where λ is a scalar parameter.

3. $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can given in Cartesian form as:

   - $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$
     Each of these three expressions is λ.

4. The vector equation of a plane is

   $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:
   - $\mathbf{r}$ is the position vector of a general point in the plane
   - $\mathbf{a}$ is the position of a point in the plane
   - $\mathbf{b}$ and $\mathbf{c}$ are non-parallel, non-zero vectors in the plane
   - λ and μ are scalars

5. A Cartesian equation of a plane in three dimensions can be written in $ax + by + cz = d$ where a, b, c and d are constants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal to the vector to the plane.

6. The **scalar product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{a.b}$ (say '$\mathbf{a}$ dot $\mathbf{b}$'), and defined as

   $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

   θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

7. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of the $\cos(\angle AOB) = \frac{textbfa.b}{|\mathbf{a}||\mathbf{b}|}$

8. The non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are purpendicular if and only if $\mathbf{a.d} = 0$.

9. If $\mathbf{a}$ and $\mathbf{b}$ $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}$. In particular, $\mathbf{a.a} = |\mathbf{a}|^2$.

10. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a.b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

11. The acute angle θ between two intersecting straight lines is given by

$$\cos\theta = |\frac{\mathbf{a.d}}{|\mathbf{a}||\mathbf{b}|}|$$

where $\mathbf{a}$ and $\mathbf{b}$ are direction vecors of the lines.

12. The scalar product form of the equation of a plane is $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in the plane with a position vector $\mathbf{a}$.

13. The acute angle θ between the lines with equation $\mathbf{r} = \mathbf{a}\lambda\mathbf{b}$ and the plane with equation $\mathbf{r.n} = k$ is given by the formula

$$\sin\theta = |\frac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}|$$

14. The acute angle θ between the plane with equation $\mathbf{r.n}_1 = k_1$ and the plane with equation $\mathbf{r.n}_2 = k_2$ is given by the formula

$$\cos\theta = |\frac{\mathbf{n}_1\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}|$$

15. Two lines are **skew** if they are not parallel and they do not intersect.

16. For any two non-intersecting lines l_1 and l_2 there is a unique line segment AB such that A lies on l_1 and B lies on l_2 and AB is perpendicular to both lines.

17. The perpendicular from a point P to a lines l is a line drawn from P at right angles to l.

18. The perpendicular from a point P to a plane Π is a line drawn from P parallel to the normal vector $\mathbf{n}$.

19. k is the length of the perpendicular from the origin to the plane Π, where the equation of the plane Π is writtenin the form $\mathbf{r}.\hat{\mathbf{n}} = k$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to Π.

    The perpendicular distance from the point with coordinates (α, β, γ) and the plane with equatino $ab + by + cz = d$ is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$