Exam 2

March 25, 2015

"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work."

Signature:

Name:

- Let X and Y be discrete random variables. If x is a possible value for X and y is a possible value for Y, the **joint probability mass function** (PMF) of X and Y evaluated at (x, y) is defined by $p_{X,Y}(x, y) = \Pr(X = x, Y = y)$. We can compute the marginal PMFs of X and Y from the joint PMF $p_{X,Y}(\cdot, \cdot)$ by using the formulas

$$p_X(x) = \sum_{y \in Y(\Omega)} p_{X,Y}(x, y); \qquad p_Y(y) = \sum_{x \in X(\Omega)} p_{X,Y}(x, y).$$

- Let X and Y be two discrete random variables associated with a same experiment. The **conditional probability mass function** of Y given $X = x$, which we write $p_{Y|X}(\cdot|\cdot)$, is defined by

$$p_{Y|X}(y|x) = \Pr(Y = y|X = x) = \frac{p_{X,Y}(x, y)}{p_X(x)},$$

  provided that $p_X(x) \neq 0$.

- If X is a discrete random variable, the expected value of $g(X)$ is defined by

$$\mathrm{E}\left[g(X)\right] = \sum_{x \in X(\Omega)} g(x) p_X(x).$$

  If X is a continuous random variable, the expected value of $g(X)$ is defined by

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

- The **conditional expectation** of Y given $X = x$ is simply the expectation of Y with respect to the conditional PMF $p_{Y|X}(\cdot|x)$,

$$\mathrm{E}[Y|X = x] = \sum_{y \in Y(\Omega)} y p_{Y|X}(y|x).$$

  This conditional expectation can be viewed as a function of x,

$$h(x) = \mathrm{E}[Y|X = x].$$

  It is therefore mathematically accurate and sometimes desirable to talk about the random variable $h(X) = \mathrm{E}[Y|X]$, which is also called the conditional expectation of Y given X.

Problems:

1. Short Answers:

   (a) **1 pt** – Suppose that X and Y are two random variables, each with finite mean. In general, when is the expectation of the sum of X and Y equal to the sum of their respective expectations?

   (b) **1 pt** – Write two ways to compute the variance of discrete random variable X.

   (c) **1 pt** – Consider a Bernoulli random variables with PMF

$$p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1. \end{cases}$$

   where $p \in [0, 1]$. Find the value(s) of p that maximize(s) the variance of X.

2. Assign one of the choices to each of the following definitions: Domain, Codomain, Image, Preimage, Sample Space, Subset, Mean, Variance, Moments, Ordinary generating function, Joint probability mass function, Marginal probability mass functions, Conditional probability mass function, Conditional expectation, Independence, Discrete convolution.

   (a) **0.5 pt** – This descriptive quantity is always nonnegative and provides a measure of the dispersion of X around its expected value.

   (b) **0.5 pt** – This function is defined by $G_X(z) = \mathrm{E}\left[z^X\right] = \sum_{k=0}^{\infty} z^k p_X(k)$.

   (c) **0.5 pt** – Suppose that X and Y are integer-valued random variables. Let $p_X(\cdot)$ and $p_Y(\cdot)$ be their respective PMFs. The PMF of $S = X + Y$ is obtained by applying this operation to $p_X(\cdot)$ and $p_Y(\cdot)$.

   (d) **0.5 pt** – Suppose that $p_X(x) > 0$, this quantity is given by

$$\mathrm{E}[Y|X = x] = \sum_{y \in Y(\Omega)} y p_{Y|X}(y|x).$$

3. Let X be a continuous random variable, and assume that the cumulative distribution function (CDF) of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ Cx + \frac{x^2}{6} & 0 \le x \le 2 \\ 1 & x > 2. \end{cases}$$

   (a) **1 pt** – What is the value of the constant C?

   (b) **1 pt** – Find the probability density function (PDF) of X.

   (c) **1 pt** – Sketch the functions $f_X(\cdot)$ and $F_X(\cdot)$ from 0 to 2.

   (d) **1 pt** – Compute the mean of X.

   (e) **1 pt** – Compute the variance of X.

4. Xie and Yoram enter a five-person challenge. At the end of the challenge, the participants are ranked one through five. Furthermore, all possible permutations of the five players are equally likely. Let X denote the rank of Xie, and Y represent the rank of Yoram. Note that ties are not permitted.

   (a) **1 pt** – Find the joint PMF of X and Y. Use the table similar to the one below to display your answer.

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$
$x=1$					
$x=2$					
$x=3$					
$x=4$					
$x=5$					

   .

   (b) **1 pt** – Find the marginal PMF of Y.

   (c) **1 pt** – Find the conditional PMF of X given $Y=2$.

   (d) **1 pt** – What is the expected value of X given $Y=2$?

   (e) **1 pt** – Recall that a conditional expectation can also be viewed as a random variable $U = \mathrm{E}[X|Y]$. Find the PMF of this random variable when X and Y are as defined above.

5. A computer saves a block of data onto a blank CD. When a *zero* is written, the reflective medium is left unchanged. However, to write a *one*, maximum laser power is employed to heat the material and make it lose its reflectivity. Assume that a block of 7 bits is written to the CD. Every bit is either a *zero* or a *one* with equal probability, and they are independent of one another. We use $B_1, B_2, \ldots, B_7$ to denote the value of the original bits.

   (a) **1 pt** – Let N be the total number of *ones* within the set $\{B_1, B_2, \ldots, B_7\}$. Find the PMF distribution of random variable N.

   (b) **1 pt** – Find the expected value of N.

   Unfortunately, the laser beam employed to write data onto the CD is inconsistent. The temperature it provides fluctuates, and this introduces errors. Since the medium is left untouched for *zeros*, the defective laser does not produce errors for the corresponding bits. However, when writing a *one*, the device introduces an error with probability $1/50$, independently of other bits. Again, suppose that seven bits are written to the CD and then read from the CD. Let M be the number of bits that are read incorrectly.

   (c) **1 pt** – Find the conditional distribution of M given $N=3$.

   (d) **2 pt** – Find $\mathrm{E}[M]$, the expected number of errors within a block.