Assignment 9

Reading Assignment:

1. Chapter 10: Expectations and Bounds.

Problems:

1. Let X be a random variable that takes the values 1, 2, and 3, with the following probabilities:

$$\Pr(X=1)=\frac{1}{2}, \qquad \Pr(X=2)=\frac{1}{4}, \qquad \Pr(X=3)=\frac{1}{4}.$$

   Find the moment generating function of X and use it to obtain the first three moments, $\mathrm{E}[X]$, $\mathrm{E}\left[X^2\right]$, $\mathrm{E}\left[X^3\right]$.

2. Find the PDF of the continuous random variable X associated with the transform (moment generating function)

$$M(s)=\frac{1}{3}\cdot\frac{2}{2-s}+\frac{2}{3}\cdot\frac{3}{3-s}.$$

3. Suppose that X is a standard normal random variable.

   (a) Calculate $\mathrm{E}\left[X^3\right]$ and $\mathrm{E}\left[X^4\right]$.

   (b) Define a new random variable Y such that

$$Y=a+bX+cX^2.$$

   Find the correlation coefficient $\rho(X,Y)$.

4. **The Chernoff bound.** The Chernoff bound is a powerful tool that relies on the transform associated with a random variable, and provides bounds on the probabilities of certain tail events.

   (a) Show that the inequality

$$\Pr(X\geq a)\leq e^{-sa}M(s)$$

   holds for every a and every $s\geq 0$, where $M(s)=\mathrm{E}\left[e^{sX}\right]$ is the transform associated with the random variable X.

   (b) Show that the inequality

$$\Pr(X\leq a)\leq e^{-sa}M(s)$$

   holds for every a and every $s\geq 0$.

   (c) Show that the inequality

$$\Pr(X\geq a)\leq e^{-\phi(a)}$$

   holds for every a, where

$$\phi(a)=\max_{s\geq 0}\left(sa-\ln M(s)\right).$$

(d) Show that if $a > \mathrm{E}[X]$, then $\phi(a) > 0$.

(e) Apply the result of part (c) to obtain a bound for $\Pr(X \geq a)$, for the case where X is a standard normal random variable and $a > 0$.

(f) Let $X_1, X_2, \ldots$ be independent random variables with the same distribution as X. Show that for any $a > \mathrm{E}[X]$, we have

$$\Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) \leq e^{-n\phi(a)},$$

so that the probability that the sample mean exceeds the mean by a certain amount decreases exponentially with n.

5. **Jensen inequality.** A twice differentiable real-value function f of a single variable is called *convex* if its second derivative $(d^2 f/dx^2)(x)$ is nonnegative for all x in its domain of definition.

(a) Show that the functions $f(x) = e^{\alpha x}$, $f(x) = -\ln x$, and $f(x) = x^4$ are all convex.

(b) Show that if f is twice differentiable and convex, then the first order Taylor approximation of f is an underestimate of the function, that is,

$$f(a) + (x - a)\frac{df}{dx}(a) \leq f(x),$$

for every a and x.

(c) Show that if f has the property in part (b), and if X is a random variable, then

$$f(\mathrm{E}[X]) \leq \mathrm{E}[f(X)].$$

6. Let Z be a standard normal random variable, and for a fixed x, set

$$X = \begin{cases} Z & \text{if } Z > x \\ 0 & \text{otherwise} \end{cases}.$$

Show that $\mathrm{E}[X] = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.