Assignment 10

Reading Assignment:

1. Chapter 11: Multiple Continuous Random Variables.

Problems:

1. Let X be a random variable with PDF
$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$
and let A be the event $\{X \geq 2\}$.
   (a) Find $\mathrm{E}[X]$, $\Pr(A)$, $f_{X|A}(x)$, and $\mathrm{E}[X|A]$.
   (b) Let $Y = X^2$. Find $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$.

2. We start with a stick of length ℓ. We break it at a point which is chosen according to a uniform distribution and keep the piece, of length X, that contains the left end of the stick. We then repeat the same process on the piece that we were left with, and let Y be the length of the remaining piece after breaking for the second time.
   (a) Find the joint PDF of X and Y.
   (b) Find the marginal PDF of Y.
   (c) Use the PDF of Y to evaluate $\mathrm{E}[Y]$.
   (d) Evaluate $\mathrm{E}[Y]$, by exploiting the relation $Y = X \cdot (Y/X)$.

3. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF
$$f_P(p) = \begin{cases} pe^p, & p \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$
A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.
   (a) Find the probability that a coin toss results in heads.
   (b) Given that a coin toss resulted in heads, find the conditional PDF of P.
   (c) Given that a first coin toss resulted in heads, find the conditional probability of heads on the next toss.

4. Let X and Y be the Cartesian coordinates of a randomly chosen point (according to a uniform PDF) in the triangle with vertices at $(0, 1)$, $(0, -1)$, and $(1, 0)$. Find the CDF and the PDF of $|X - Y|$.

5. If X and Y are independent uniform $(0, 1)$ random variables, show that

$$E\left[|X - Y|^{\alpha}\right] = \frac{2}{(\alpha+1)(\alpha+2)} \quad \text{for } \alpha > 0.$$

6. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables having uniform distributions over $(0, 1)$, find

   (a) $\mathrm{E}[\max(X_1, \ldots, X_n)]$;

   (b) $\mathrm{E}[\min(X_1, \ldots, X_n)]$.