Exam 1

February 16, 2015

"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work."

Signature:

Name:

Axioms of Probability: A *probability law* specifies the likelihood of any event related to an experiment. Formally, a probability law assigns to every event A a number $\Pr(A)$, called the *probability of event* A, such that the following axioms are satisfied.

1. **(Nonnegativity)** $\Pr(A) \geq 0$, for every event A.
2. **(Normalization)** The probability of the sample space Ω is equal to one, $\Pr(\Omega) = 1$.
3. **(Countable Additivity)** If A and B are disjoint events with $A \cap B = \emptyset$, then the probability of their union satisfies
$$\Pr(A \cup B) = \Pr(A) + \Pr(B).$$
More generally, if $A_1, A_2, \ldots$ is a sequence of disjoint events and $\bigcup_{k=1}^{\infty} A_k$ is itself an admissible event then
$$\Pr\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \Pr(A_k).$$

Total Probability Theorem: Let $A_1, A_2, \ldots, A_n$ be a collection of events that forms a partition of the sample space Ω. Suppose that $\Pr(A_k) > 0$ for all k. Then, for any event B, we can write

$$\begin{aligned}\Pr(B) &= \Pr(A_1 \cap B) + \Pr(A_2 \cap B) + \cdots + \Pr(A_n \cap B) \\ &= \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2) + \cdots + \Pr(A_n)\Pr(B|A_n).\end{aligned}$$

Bayes' Rule: Let $A_1, A_2, \ldots, A_n$ be a collection of events that forms a partition of the sample space Ω. Suppose that $\Pr(A_k) > 0$ for all k. Then, for any event B such that $\Pr(B) > 0$, we can write
$$\Pr(A_i|B) = \frac{\Pr(A_i)\Pr(B|A_i)}{\Pr(B)} = \frac{\Pr(A_i)\Pr(B|A_i)}{\sum_{k=1}^{n} \Pr(A_k)\Pr(B|A_k)}.$$

Chain Rule of Probability: Let $A_1, A_2, \ldots, A_n$ be a collection of events, with a non-vanishing intersection. The probability of events A_1 through A_n taking place at the same time is given by

$$\Pr\left(\bigcap_{k=1}^{n} A_k\right) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2)\cdots\Pr\left(A_n \middle| \bigcap_{k=1}^{n-1} A_k\right).$$

Problems:

1. True or False:

   (a) **1 pt** – If two events A and B are independent, then they are necessarily disjoint with $A \cap B = \emptyset$.

   (b) **1 pt** – Let $A_1, A_2, \ldots, A_n$ be a collection of events, then

   $$\Pr\left(\bigcap_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \Pr(A_k).$$

   (c) **1 pt** – A jar contains 8 red balls and 4 green balls. Two balls are drawn from this jar without replacement. Let B_1 denote the color of the first ball; and B_2, the color of the second ball. In this problem, B_2 is more likely to be green because the first ball is more likely to be red and, once a red ball is out, the probability that the second ball is green becomes larger.

2. Assign ONE of the choices below to each statement: event, outcome, sample space, combination, permutation, partition, power set, Bayes' rule, De Morgan's law, total probability theorem, conditional probability, independence, conditional independence, distinct and mutually exclusive, collectively exhaustive.

   (a) **0.5 pt** – The collection of all subsets of S.

   (b) **0.5 pt** – What property of the elements of a sample space ensures that the outcome of an experiment is unique.

   (c) **0.5 pt** – This result relates the conditional probability of A given B to the conditional probability of B given A.

   (d) **0.5 pt** – This property asserts that $\Pr(A_1 \cap A_2|B) = \Pr(A_1|B)\Pr(A_2|B)$.

3. Two friends from Texas A&M University, David and Julia, recently joined AgDroid, a stratup that focuses on mobile technologies. Including these two new employees, the development team now features 12 engineers. The company is going through some reorganization and it must divide its workforce into 3 teams, each containing four engineers. These teams are labeled A, B, C. The 12 employees are assigned at random to the team, with each outcome being equally probable.

   (a) **1 pt** – What is the probability that David gets assigned to Team A?

   (b) **1 pt** – What is the probability that both David and Julia get assigned to Team A?

   (c) **1 pt** – What is the probability that David and Julia end up on the same team?

   (d) **1 pt** – How many ways are there to assign the employees to the 3 teams?

   (e) **1 pt** – What is the probability that David and Julia end up on the same team, conditioned on the fact that neither of them is in Team C?

4. Suppose that a fair die is rolled repetitively until at least one odd number and one even number are observed. The total number of rolls N is recorded as the outcome of this random experiment. Note that the sample space for this experiment is $\Omega = \{1, 2, \ldots\}$.

   (a) **0.5 pt** – What is the probability that a one is observed on the first roll?

   (b) **0.5 pt** – What is the probability that an even number is observed on the first roll?

   (c) **0.5 pt** – What is the minimum number of rolls for which the experiment can stop?

   (d) **1 pt** – What is the probability of having to roll the die exactly n times before the experiment terminates?

   (e) **1 pt** – What is the probability that the number of rolls N is odd?

   (f) **1.5 pt** – What is the probability that a six is observed before the experiment terminates?

5. A wind farm operator is trying to assess the cost associated with each turbine. On a stormy day, a turbine breaks down with probability 0.2. On a non-stormy day, the probability that a turbine breaks down is equal to 0.1. Weather analysis reveals that the probability of a day being stormy is 0.4, independently of other days. When wind turbines are co-located, they break down in a conditionally independent manner given the weather.

   (a) **1 pt** – If a day is selected at random, what is the probability that the wind turbine breaks down?

   (b) **1 pt** – On a particular day, the operator finds out that the wind turbine broke down. What is the probability that it is stormy on that day?

   (c) **1.5 pt** – Suppose that the operator acquires two turbines that located in a same geographical area. If the first turbine breaks down on a particular day, what is the conditional probability that the second one also breaks down on that day?

   (d) **1.5 pt** – Suppose the operator has five turbine that are co-located. On a particular day, three out of five turbines break down. What is the probability that it is stormy on that day?