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A fast finite difference scheme for the time-space fractional diffusion equation ⋆

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Abstract: This paper proposes a fast numerical scheme for time-space fractional diffusion equation in two spatial dimensions, where the temporal fractional partial derivative is in the Caputo sense of order $\alpha \in (0,1)$ and the spatial fractional derivative is given by fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$ in two space dimensions with $\beta \in (1,2)$. The temporal Caputo derivative is discretized by a fast evaluation based on the L2-1_σ formula. The spatial discretization adopts fractional centered difference formula in two dimensions. The proposed fully discrete scheme is proved to be stable and convergent with 2nd order accuracy both in time and space. The feasibility, stability, and convergence of the proposed scheme are demonstrated by the numerical example.

Keywords: Time-space fractional diffusion equation, Caputo derivative, Fractional Laplacian, Fast finite difference scheme

1. INTRODUCTION

Fractional partial differential equations have caught the eyes of researchers in numerous fields, since the hereditary and nonlocality of fractional operators are adequate instruments interpreting history and long-range interaction. In most of the existing fractional partial differential models, the temporal derivative is governed by the Caputo(fractional) derivative for $\varphi(t) \in AC^n[0,T]$,

$$ {}_C\mathrm{D}_{0,t}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\varphi^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s $$

with $n-1 < \alpha < n \in \mathbb{Z}^+$. The spatial derivative in two or higher dimensions can be given by fractional Laplacian defined by a singular or hyper-singular integral,

$$ (-\Delta)^{\frac{\beta}{2}} \psi(\mathbf{x}) = c_\beta \, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{\psi(\mathbf{x}) - \psi(\mathbf{z})}{|\mathbf{x}-\mathbf{z}|^{d+\beta}} \mathrm{d}\mathbf{z}, \qquad (1) $$

with

$$ c_\beta = \frac{2^\beta \Gamma(d/2 + \beta/2)}{\pi^{d/2} |\Gamma(-\beta/2)|}, $$

d being the number of dimensions, and P.V. standing for the Cauchy principal value. See Jiao et al. (2021), Li and Zeng (2015), and Wang et al. (2022) for more information. One sufficient condition for existence of the fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}\psi(\mathbf{x})$ is that $\psi(\mathbf{x})$ belongs to the following Schwartz space (see Li and Cai (2019))

$$ \mathcal{S}(\mathbb{R}^d) = \Big\{ \psi \in C^\infty(\mathbb{R}^d) : \sup_{\mathbf{x}\in\mathbb{R}^d} (1+|\mathbf{x}|)^N \sum_{k=0}^{N} |D^k \psi(\mathbf{x})| < +\infty, \ N = 0,1,2,\cdots \Big\}. $$

It is well-known that nonlocality and singularity of fractional operators increase the computation burden. As a result, developing fast algorithms for fractional differential equations is important. In this paper, we numerically study the two-dimensional fractional diffusion equation of the form

$$ \begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u(x,y,t) + (-\Delta)^{\frac{\beta}{2}} u(x,y,t) = f(x,y,t), \\ \qquad\qquad\qquad\qquad (x,y) \in \Omega, \ t \in (0,T], \\ u(x,y,t) = 0, \qquad (x,y) \in \mathbb{R}^2 \backslash \Omega, \ t \in (0,T], \\ u(x,y,0) = u_0(x,y), \qquad (x,y) \in \Omega, \end{cases} \qquad (2) $$

with $\alpha \in (0,1)$, $\beta \in (1,2)$, $\Omega = (-L,L)^2 \subset \mathbb{R}^2$ being a bounded domain, $u_0(x,y)$ being a given function, and $f(x,y,t)$ being a source term. Here we apply the fast L2-1_σ formula to evaluate the Caputo derivative and use the fractional centred finite difference formula to approximate the fractional Laplacian.

This paper is organized as follows: Sec. 2 briefly introduces the fast L2-1_σ formula for Caputo derivative and the fractional centred finite difference formula for fractional Laplacian, along with their properties. Sec. 3 constructs the fully discrete scheme for Eq. (2) and show corresponding numerical analysis. Sec. 4 gives the numerical example to verify feasibility, stability, and convergence of the proposed numerical scheme.

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2. PRELIMINARIES

2.1 Fast evaluation based on L2-1$_\sigma$ formula

To deal with the discretization in time, we briefly introduce a fast evaluation based on L2-1$_\sigma$ formula.

Let $t_k = k\tau$ with $k = 0, 1, \cdots, N \in \mathbb{Z}^+$ and define $t_{k+\sigma} = t_k + \sigma\tau$ with $\sigma = 1 - \frac{\alpha}{2}$, where $\tau = T/N$ is the temporal stepsize. On the basis of this temporal division, Jiang et al. (2017) claimed the following result.

Lemma 1. Given $\alpha \in (0, 1)$, $\varepsilon > 0$, $\hat{\tau} = \sigma\tau > 0$, and $T > 0$, there exist a positive integer $N_{\exp}^{(\alpha)}$, and positive numbers $s_l^{(\alpha)}$, $w_l^{(\alpha)} (1 \leqslant l \leqslant N_{\exp}^{(\alpha)})$ such that

$$\left| t^\alpha - \sum_{l=1}^{N_{\exp}^{(\alpha)}} w_l^{(\alpha)} \mathrm{e}^{-s_l^{(\alpha)} t} \right| \leqslant \varepsilon, \ t \in [\hat{\tau}, T],$$

where the number $N_{\exp}^{(\alpha)}$ satisfies

$$N_{\exp}^{(\alpha)} = O\Bigg(\log \frac{1}{\varepsilon} \left(\log\log \frac{1}{\varepsilon} + \log \frac{T}{\hat{\tau}} \right) + \log \frac{1}{\hat{\tau}} \left(\log\log \frac{1}{\varepsilon} + \log \frac{T}{\hat{\tau}} \right) \Bigg).$$

In this setting, if $\varphi(t) \in C^3[0, T]$, the Caputo derivative ${}_C\mathrm{D}_{0,t}^\alpha \varphi(t)$ at $t = t_{k+\sigma}$ $(1 \leqslant k \leqslant N)$ can be evaluated by the following fast L2-1$_\sigma$ approximation which is proposed by Yan et al. (2017),

$$\begin{aligned} &{}_C\mathrm{D}_{0,t}^\alpha \varphi(t)\big|_{t=t_{k-1+\sigma}} \\ =&\frac{1}{\Gamma(1-\alpha)} \left[\sum_{l=1}^{N_{\exp}^{(\alpha)}} w_l^{(\alpha)} F_l^{k+1} + \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{1-\alpha} (\varphi(t_k) - \varphi(t_{k-1})) \right] \\ &+ \mathcal{O}(\tau^{3-\alpha}) \\ \triangleq& {}^{\mathcal{F}}\Delta_{0,t}^\alpha \ \varphi(t_{k-1+\sigma}) + \mathcal{O}(\tau^{3-\alpha}). \end{aligned} \tag{3}$$

Here F_l^k can be computed recursively as follows,

$$\begin{cases} F_l^1 = 0, 1 \leqslant l \leqslant N_{\exp}^{(\alpha)}, \\ F_l^k = \mathrm{e}^{-s_l^{(\alpha)}\tau} F_l^{k-1} + A_l \left[\varphi(t_{k-1}) - \varphi(t_{k-2}) \right], \\ \qquad + B_l \left[\varphi(t_k) - \varphi(t_{k-1}) \right], 1 \leqslant l \leqslant N_{\exp}^{(\alpha)}, \ k \geqslant 2, \end{cases} \tag{4}$$

where

$$A_l = \int_0^1 \left(\frac{3}{2} - \xi \right) \mathrm{e}^{-s_l^{(\alpha)}(\sigma+1-\xi)\tau} \mathrm{d}\xi,$$

$$B_l = \int_0^1 \left(\xi - \frac{1}{2} \right) \mathrm{e}^{-s_l^{(\alpha)}(\sigma+1-\xi)\tau} \mathrm{d}\xi.$$

The fast L2-1$_\sigma$ approximation can also be rewritten in the following form,

$${}^{\mathcal{F}}\Delta_{0,t}^\alpha \ \varphi(t_{k-1+\sigma}) = \sum_{i=1}^{k} d_{k-i,k}^{(\alpha)} \left[\varphi(t_i) - \varphi(t_{i-1}) \right], \tag{5}$$

where

$$d_{0,1}^{(\alpha)} = \frac{\sigma^{1-\alpha}\tau^{-\alpha}}{\Gamma(2-\alpha)}, \tag{6}$$

and for $k \geqslant 2$,

$$\begin{cases} d_{0,k}^{(\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}^{(\alpha)}} w_l^{(\alpha)} B_l + d_{0,1}^{(\alpha)}, \\ d_{i,k}^{(\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}^{(\alpha)}} w_l^{(\alpha)} \left(\mathrm{e}^{-s_l^{(\alpha)} t_{i-1}} A_l + \mathrm{e}^{-s_l^{(\alpha)} t_i} B_l \right), \\ \qquad\qquad 1 \leqslant i \leqslant k-2, \\ d_{k-1,k}^{(\alpha)} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\sum_{l=1}^{N_{\exp}^{(\alpha)}} w_l^{(\alpha)} \mathrm{e}^{-s_l^{(\alpha)} t_{k-2}} A_l. \end{cases} \tag{7}$$

Properties of the coefficients $d_{i,k}^{(\alpha)}$ can be concluded as follows.

Lemma 2. For $\alpha \in (0, 1)$, $\sigma = 1 - \frac{\alpha}{2}$, and $\varepsilon < \frac{2(1-\sigma)}{\sigma(7\sigma-1)(1+\sigma)^\alpha} \tau^{-\alpha}$, the coefficients $d_{i,k}^{(\alpha)}$ given by Eqs. (6) and (7) satisfy:

(1) $d_{0,k}^{(\alpha)} > d_{1,k}^{(\alpha)} > d_{2,k}^{(\alpha)} > \cdots > d_{k-1,k}^{(\alpha)} > 0$.
(2) $(2\sigma - 1) d_{0,k}^{(\alpha)} - \sigma d_{1,k}^{(\alpha)} > 0$.
(3) $\dfrac{1}{d_{k-1,k}^{(\alpha)}} \leqslant 2 t_k^\alpha \Gamma(1-\alpha)$.

Remark 3. It is claimed in that Yan et al. (2017) that compared with the classical L2-1$_\sigma$ formula, the fast L2-1$_\sigma$ formula in Eq. (3) reduces the overall computational cost from $\mathcal{O}(MN^2)$ to $\mathcal{O}(MN \log^2 N)$ and the overall storage from $\mathcal{O}(MN)$ to $\mathcal{O}(M \log^2 N)$, where M is the total number of grid points and N represents the total number of time steps.

2.2 Fractional centered difference formula for fractional Laplacian

Let $h = \frac{2L}{M}$ with $M \in \mathbb{Z}^+$, $x_j = -L + jh$ with $0 \leqslant j \leqslant M$, and $y_k = -L + kh$ with $0 \leqslant k \leqslant M$. Denote $\widetilde{\Omega}_h = \{(x_j, y_k) \mid 0 \leqslant j, k \leqslant M\}$, $\Omega_h = \widetilde{\Omega}_h \cap \Omega$, and $\partial\Omega_h = \widetilde{\Omega}_h \cap \partial\Omega$.

Define the spaces of grid functions in two dimensions as $S_h^\circ = \{w \mid w = \{w_{jk}\}, w_{jk} = 0 \text{ with } (x_j, y_k) \in \partial\Omega_h\}$ and $\widetilde{S}_h = \{w \mid w = \{w_{jk}\}, (x_j, y_k) \in \Omega_h\}$. For any $w, v \in S_h^\circ$, the discrete inner product and the associated norm are defined as

$$(w, v)_h = h^2 \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} w_{jk} v_{jk}, \quad \|w\|_h^2 = (w, w)_h.$$

Set $L_h^2 = \left\{ w \mid w = \{w_{jk}\}, \|w\|_h^2 < +\infty \right\}$. For $w \in L_h^2$, define the semi-discrete Fourier transform $\widehat{w} : [-\frac{\pi}{h}, \frac{\pi}{h}]^2 \to \mathbb{C}$ as

$$\widehat{w}(\eta_1, \eta_2) = h^2 \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} w_{jk} e^{-\mathbf{i}(\eta_1 jh + \eta_2 kh)},$$

and the inverse semi-discrete Fourier transform as

$$w_{jk} = \frac{1}{4\pi^2} \int_{-\pi/h}^{\pi/h} \int_{-\pi/h}^{\pi/h} \widehat{w}(\eta_1, \eta_2) e^{-\mathbf{i}(\eta_1 jh + \eta_2 kh)} \mathrm{d}\eta_1 \mathrm{d}\eta_2$$

with $\mathbf{i}$ being the imaginary unit. The Parseval's identity implies that the continuous definition of the inner product takes the form

$$(w,v)_h = \frac{1}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/h}^{\pi/h} \widehat{w}(\eta_1,\eta_2)\widehat{v}(\eta_1,\eta_2)\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

with the norm given by

$$\|w\|_h^2 = \frac{1}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/h}^{\pi/h} |\widehat{w}(\eta_1,\eta_2)|^2\,\mathrm{d}\eta_1\mathrm{d}\eta_2.$$

For arbitrary positive constant s, the fractional Sobolev semi-norm is given by

$$|w|_{H_h^s}^2 = \frac{1}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/h}^{\pi/h} \left(\eta_1^2+\eta_2^2\right)^s |\widehat{w}(\eta_1,\eta_2)|^2\,\mathrm{d}\eta_1\mathrm{d}\eta_2.$$

Based on the above settings, Hao et al. (2021) proposed the two-dimensional fractional centered difference operator

$$(-\Delta_h)^{\frac{\beta}{2}}\psi(x,y) = \frac{1}{h^\beta}\sum_{j,k\in\mathbb{Z}} a_{j,k}^{(\beta)}\psi(x+jh,y+kh), \quad (8)$$

with

$$a_{j,k}^{(\beta)} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[4\sin^2(\frac{\eta_1}{2})+4\sin^2(\frac{\eta_2}{2})\right]^{\frac{\beta}{2}} \cdot \mathrm{e}^{-\mathbf{i}(\eta_1 j+\eta_2 k)}\,\mathrm{d}\eta_1\mathrm{d}\eta_2. \quad (9)$$

Here the coefficients $a_{j,k}^{(\beta)}$ can be numerically computed by the built-in function "fft2" in Matlab. Throughout the numerical simulations in this paper, we calculate $a_{j,k}^{(\beta)}$ as the way given by Hao et al. (2021).

For properties of the fractional central finite difference operator, we have the following two lemmas from Hao et al. (2021) and Cai et al. (2024).

Lemma 4. Let $\psi(x,y)\in W^{\frac{\beta}{2},2}(\mathbb{R}^2)$ with

$$W^{s,2}(\mathbb{R}^2) = \left\{\psi \,\middle|\, \psi\in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2}\left(1+|\eta|^{2s}\right)|\widehat{\psi}(\eta_1,\eta_2)|^2\mathrm{d}\eta_1\mathrm{d}\eta_2 < +\infty\right\},$$

where $|\eta| = \sqrt{\eta_1^2+\eta_2^2}$. The fractional Laplacian defined in Eq. (1) can be approximated by the fractional centered difference operator, i.e.,

$$(-\Delta)^{\frac{\beta}{2}}\psi(x,y) = (-\Delta_h)^{\frac{\beta}{2}}\psi(x,y) + R_L(x,y), \quad 0<\beta\leqslant 2,$$

where the truncation error satisfies

$$|R_L(x,y)| \leqslant Ch^2\int_{\mathbb{R}^2}(1+|\eta|)^{\beta+2}\left|\widehat{\psi}(\eta_1,\eta_2)\right|\mathrm{d}\eta_1\mathrm{d}\eta_2 = Ch^2, \quad (10)$$

with C being a positive constant independent of h.

Lemma 5. Let $\psi\in S_h^\circ$. If $\psi\in H_h^{\frac{\beta}{2}}(\mathbb{R}^2)$ with $\beta\in(1,2)$, there holds

$$C\left(\frac{2}{\pi}\right)^\beta \frac{2^{\frac{\beta-2}{2}}}{4\pi^2}\|\psi\|_h^2 \leqslant \left((-\Delta_h)^{\frac{\beta}{2}}\psi,\psi\right)_h \leqslant 2^{\frac{\beta}{2}}\frac{\pi^\beta}{h^\beta}\|\psi\|_h^2,$$

where C is a positive constant independent of h.

3. FULLY DISCRETE SCHEMES

In this section, we consider constructing a numerical scheme for Eq. (2) by using the fast L2-1_σ formula and the fractional centred difference formula.

In order to apply the fast L2-1_σ formula which evaluates the temporal Caputo derivative at the non-integer point $t=t_{k+\sigma}$ via a linear combination of the values at integer points t_i, it is necessary to pre-process Eq. (2) using the following assertion.

Lemma 6. Assume that $\varphi(t)\in C^2[0,T]$, $0<\alpha<1$, and $\gamma\in(0,1)$. Then

$$\varphi(t_{k+\gamma}) = (1-\gamma)\varphi(t_k)+\gamma\varphi(t_{k+1})+\mathcal{O}(\tau^2),\ 0\leqslant k\leqslant N-1.$$

Proof. The above equality can be readily derived via Taylor's expansion and so is omitted here.

In view of Lemma 6, Eq. (2) can be approximated by the following systems,

$$\begin{cases} {}_C\mathrm{D}_{0,t}^\alpha u(x,y,t)\big|_{t_{k+\sigma}} + (1-\sigma)(-\Delta)^{\frac{\beta}{2}}u(x,y,t_k) \\ +\sigma(-\Delta)^{\frac{\beta}{2}}u(x,y,t_{k+1}) = f(x,y,t_{k+\sigma})+\mathcal{O}(\tau^2), \\ \qquad\qquad (x,y)\in\Omega, 0\leqslant k\leqslant N-1, \\ u(x,y,t)=0, \qquad (x,y)\in\mathbb{R}^2\backslash\Omega, t\in(0,T], \\ u(x,y,0)=u_0(x,y), \qquad (x,y)\in\Omega, \end{cases} \quad (11)$$

Once the approximation in Lemma 6 is applied, the following lemma which is useful for numerical analysis of the coming algorithm can be derived.

Lemma 7. Let $\alpha\in(0,1)$, $\sigma = 1-\frac{\alpha}{2}$, and the coefficients $d_{i,k}^{(\alpha)}$ be defined by Eqs. (6) and (7). For $w^0,w^1,\cdots,w^k\in\mathcal{S}_h^\circ$, the following inequality holds,

$$\begin{aligned}&\sum_{i=0}^{k-1} d_{i,k}^{(\alpha)}\left(w^{k-i}-w^{k-i-1},\sigma w^k+(1-\sigma)w^{k-1}\right)_h \\ \geqslant&\frac{1}{2}\sum_{i=1}^{k-1} d_{i,k}^{(\alpha)}\left(\|w^{k-i}\|_h^2-\|w^{k-i-1}\|_h^2\right).\end{aligned} \quad (12)$$

Proof. The proof is similar to that in Cai et al. (2024) and is omitted here.

Now we establish a numerical scheme for Eq. (2) and show the corresponding numerical analysis. Applying the aforementioned fast L2-1_σ formula and fractional centred difference formula, Eq. (11) which approximates Eq. (2) with 2nd order accuracy in time can be discretized as follows,

$$\begin{cases} \sum_{i=0}^{n-1} d_{i,n}^{(\alpha)}\left(U_{jk}^{n-i}-U_{jk}^{n-i-1}\right) \\ +(-\Delta_h)^{\frac{\beta}{2}}\left(\sigma U_{jk}^n+(1-\sigma)U_{jk}^{n-1}\right) = f_{jk}^{n-1+\sigma}, \\ \qquad\qquad 1\leqslant n\leqslant N,\ 1\leqslant j,k\leqslant M-1, \\ U_{jk}^0 = u_0(x_j,y_k), \qquad 1\leqslant j,k\leqslant M-1, \\ U_{jk}^n = 0, \qquad (x_j,y_k)\in\partial\Omega_h, 0\leqslant n\leqslant N. \end{cases} \quad (13)$$

Here U_{jk}^n denotes the approximation of $u(x,y,t)$ at (x_j,y_k,t_n). This finite difference system can be written into the following matrix form,

$$\begin{aligned}&\sum_{i=0}^{n-1} d_{i,n}^{(\alpha)}\left(\mathbf{U}^{n-i}-\mathbf{U}^{n-i-1}\right)+\frac{1}{h^\beta}\mathbf{A}(\sigma\mathbf{U}^n+(1-\sigma)\mathbf{U}^{n-1}) \\ =&\mathbf{F}^{n-1+\sigma},\ 1\leqslant n\leqslant N,\end{aligned} \quad (14)$$

where

$$\mathbf{F}^{n-1+\sigma} = (f_{11}^{n-1+\sigma},\cdots,f_{(M-1)1}^{n-1+\sigma}, \cdots,f_{1(M-1)}^{n-1+\sigma},\cdots,f_{(M-1)(M-1)}^{n-1+\sigma})^{\mathrm{T}},$$

and $\mathbf{U}^n = \left(\mathbf{U}_1^n, \mathbf{U}_2^n, \cdots, \mathbf{U}_{M-1}^n\right)$ with

$$\mathbf{U}_j^n = \left(U_{1j}^n, U_{2j}^n, \cdots, U_{(M-1)j}^n\right)^{\mathrm{T}}.$$

The coefficient matrix $\mathbf{A}$ is a real block symmetric matrix with Toeplitz blocks, say,

$$\mathbf{A} = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{M-3} & A_{M-2} \\ A_1 & A_0 & A_1 & \cdots & A_{M-4} & A_{M-3} \\ A_2 & A_1 & A_0 & \cdots & A_{M-5} & A_{M-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{M-3} & A_{M-4} & A_{M-5} & \cdots & A_0 & A_1 \\ A_{M-2} & A_{M-3} & A_{M-4} & \cdots & A_1 & A_0 \end{pmatrix}, \quad (15)$$

with

$$A_j = \begin{pmatrix} a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} & a_{2,j}^{(\beta)} & \cdots & a_{M-3,j}^{(\beta)} & a_{M-2,j}^{(\beta)} \\ a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} & \cdots & a_{M-4,j}^{(\beta)} & a_{M-3,j}^{(\beta)} \\ a_{2,j}^{(\beta)} & a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} & \cdots & a_{M-5,j}^{(\beta)} & a_{M-4,j}^{(\beta)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{M-3,j}^{(\beta)} & a_{M-4,j}^{(\beta)} & a_{M-5,j}^{(\beta)} & \cdots & a_{0,j}^{(\beta)} & a_{1,j}^{(\beta)} \\ a_{M-2,j}^{(\beta)} & a_{M-3,j}^{(\beta)} & a_{M-4,j}^{(\beta)} & \cdots & a_{1,j}^{(\beta)} & a_{0,j}^{(\beta)} \end{pmatrix}.$$

It follows from Lemma 5 that the matrix $\mathbf{A}$ is a real symmetric positive definite matrix.

For the fully discrete scheme (13), we have the following stability analysis and error estimate.

Theorem 8. Let $\alpha \in (0,1)$, $\beta \in (1,2)$, and $\sigma = 1 - \frac{\alpha}{2}$. The finite difference scheme (13) is unconditionally stable and the numerical solution satisfies the following priori estimate for $1 \leqslant n \leqslant N$,

$$\|\mathbf{U}^n\|_h \leqslant \|\mathbf{U}^0\|_h + \frac{8\pi^{2+\beta}\Gamma(1-\alpha)}{C2^{\frac{3\beta}{2}}} \max_{1\leqslant k\leqslant n} \left\{t_k^\alpha \|\mathbf{F}^{k-1+\sigma}\|_h^2\right\}.$$

Here C is a positive constant.

Proof. Taking the two-dimensional discrete inner product of Eq. (14) with $\sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}$ gives

$$\begin{aligned} &\left(\sum_{i=0}^{n-1} d_{i,n}^{(\alpha)} \left(\mathbf{U}^{n-i} - \mathbf{U}^{n-i-1}\right), \sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}\right)_h \\ =& -\left(\frac{1}{h^\beta}\mathbf{A}(\sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}), \sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}\right)_h \\ &+ \left(\mathbf{F}^{n-1+\sigma}, \sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}\right)_h, \ 1 \leqslant n \leqslant N. \end{aligned} \quad (16)$$

Applying Lemmas 5 and 7, and Cauchy-Schwarz inequality to Eq. (16) yields

$$\begin{aligned} &\frac{1}{2}\sum_{i=0}^{n-1} d_{i,n}^{(\alpha)} \left(\|\mathbf{U}^{n-i}\|_h^2 - \|\mathbf{U}^{n-i-1}\|_h^2\right) \\ \leqslant& -C\left(\frac{2}{\pi}\right)^\beta \frac{2^{\frac{\beta-2}{2}}}{4\pi^2} \|\sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}\|_h^2 \\ &+ C\left(\frac{2}{\pi}\right)^\beta \frac{2^{\frac{\beta-2}{2}}}{4\pi^2} \|\sigma\mathbf{U}^n + (1-\sigma)\mathbf{U}^{n-1}\|_h^2 \\ &+ \frac{2\pi^{2+\beta}}{C2^{\frac{3\beta}{2}}} \|\mathbf{F}^{n-1+\sigma}\|_h^2 \\ =& \frac{2\pi^{2+\beta}}{C2^{\frac{3\beta}{2}}} \|\mathbf{F}^{n-1+\sigma}\|_h^2. \end{aligned} \quad (17)$$

From Lemma 2, we know

$$\frac{1}{d_{k-1,k}^{(\alpha)}} \leqslant 2t_k^\alpha \Gamma(1-\alpha).$$

Then Eq. (17) can be rewritten as

$$\begin{aligned} &d_{0,n}^{(\alpha)} \|\mathbf{U}^n\|_h^2 \\ \leqslant& \sum_{i=1}^{n-1} (d_{i+1,n}^{(\alpha)} - d_{i,n}^{(\alpha)}) \|\mathbf{U}^{n-i}\|_h^2 + d_{n-1,n}^{(\alpha)} \|\mathbf{U}^0\|_h^2 \\ &+ \frac{4\pi^{2+\beta}}{C2^{\frac{3\beta}{2}}} \|\mathbf{F}^{n-1+\sigma}\|_h^2 \\ \leqslant& \sum_{i=1}^{n-1} (d_{i+1,n}^{(\alpha)} - d_{i,n}^{(\alpha)}) \|\mathbf{U}^{n-i}\|_h^2 + d_{n-1,n}^{(\alpha)} \Big[\|\mathbf{U}^0\|_h^2 \\ &+ \frac{8\pi^{2+\beta} t_n^\alpha \Gamma(1-\alpha)}{C2^{\frac{3\beta}{2}}} \|\mathbf{F}^{n-1+\sigma}\|_h^2\Big]. \end{aligned}$$

Based on the above inequality, it can be obtained by using the mathematical induction that

$$\|\mathbf{U}^n\|_h \leqslant \|\mathbf{U}^0\|_h + \frac{8\pi^{2+\beta}\Gamma(1-\alpha)}{C2^{\frac{3\beta}{2}}} \max_{1\leqslant k\leqslant n} \left\{t_k^\alpha \|\mathbf{F}^{k-1+\sigma}\|_h^2\right\}$$

holds for arbitrary $n = 1, \cdots, N$. The proof is thus completed.

Theorem 9. Assume that $\alpha \in (0,1)$, $\beta \in (1,2)$, $\sigma = 1 - \frac{\alpha}{2}$, and $u(x,y,t) \in C^3([0,T], W^{\frac{\beta}{2},2}(\Omega))$ is the solution to Eq. (2). Let u_{jk}^n be the exact solution to Eq. (2) at (x_j, y_k, t_n) and U_{jk}^n be the numerical solution determined by scheme (13). Then the numerical error given by

$$e_{jk}^n = U_{jk}^n - u_{jk}^n, \ 0 \leqslant j,k \leqslant M, \ 0 \leqslant n \leqslant N$$

has the following estimate when the temporal stepsize τ and the spatial stepsize h are sufficiently small,

$$\|e^n\|_h \leqslant C\left(\tau^2 + h^2\right), \ 0 \leqslant n \leqslant N, \quad (18)$$

with C being a positive constant independent of h and τ.

Proof. It follows from Eqs. (11) and (13) that

$$\begin{cases} \sum_{i=0}^{n-1} d_{i,n}^{(\alpha)} \left(e_{jk}^{n-i} - e_{jk}^{n-i-1}\right) \\ + (-\Delta_h)^{\frac{\beta}{2}} \left(\sigma e_{jk}^n - (1-\sigma) e_{jk}^{n-1}\right) = R_{jk}^{n-1+\sigma}, \\ \qquad 0 \leqslant n \leqslant N-1, 1 \leqslant j,k \leqslant M-1, \\ e_{jk}^0 = 0, \qquad 1 \leqslant j,k \leqslant M-1, \\ e_{jk}^n = 0, \qquad (x_j, y_k) \in \partial\Omega_h, 0 \leqslant n \leqslant N. \end{cases} \quad (19)$$

By Eq. (3), Lemma 4, and Lemma 6, it is evident that for the truncation error $R_{jk}^{n-1+\sigma}$ with $1 \leqslant n \leqslant N$ and $1 \leqslant j,k \leqslant M-1$, there holds

$$\left|R_{jk}^{n-1+\sigma}\right| \leqslant C(\tau^2 + h^2). \quad (20)$$

In view of Theorem 8, we have

$$\begin{aligned} \|e^n\|_h^2 &\leqslant \|e^0\|_h^2 + \frac{8\pi^{2+\beta}\Gamma(1-\alpha)}{C2^{\frac{3\beta}{2}}} \max_{1\leqslant k\leqslant n} \left\{t_k^\alpha \|R^{k-1+\sigma}\|_h^2\right\} \\ &\leqslant \frac{8\pi^{2+\beta}\Gamma(1-\alpha)}{C2^{\frac{3\beta}{2}}} T^\alpha \left(C\left(\tau^2 + h^2\right)\right)^2, \ 1 \leqslant n \leqslant N, \end{aligned}$$

which yields that

$$\|e^n\|_h \leqslant C\left(\tau^2 + h^2\right), \ 1 \leqslant n \leqslant N.$$

This finishes the proof.

4. NUMERICAL EXPERIMENT

Example 10. Consider the following fractional diffusion equation in two space dimensions with $\alpha \in (0,1)$ and $\beta \in (1,2)$,

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u(x,y,t) + (-\Delta)^{\frac{\beta}{2}} u(x,y,t) = f(x,y,t), \\ \qquad\qquad\qquad\qquad (x,y) \in \Omega, t \in (0,1], \\ u(x,y,t) = 0, \qquad (x,y) \in \mathbb{R}^2 \backslash \Omega, t \in (0,1], \\ u(x,y,0) = 0, \qquad (x,y) \in \Omega, \end{cases}$$

where $\Omega = (-1,1)^2$ and the exact solution is set as $u(x,y,t) = t^{3+\alpha}(1-x^2)^4(1-y^2)^4$. The source term f is not explicitly known and we use very fine mesh to compute it.

In the present simulation, we evaluate the source term by $f \approx f_h = {}_C\mathrm{D}_{0,t}^{\alpha} u(x,y,t) + (-\Delta_h)^{\frac{\beta}{2}} u(x,y,t)$ with $h = 2^{-8}$. The numerical error in the spatial direction is defined as

$$E(h) = \sqrt{h^2 \sum_{j=0}^{M} \sum_{k=0}^{M} \left| U_{jk}^N(h,\tau) - U_{2j,2k}^N(h/2,\tau) \right|^2},$$

when τ is small enough. The numerical error in the temporal direction is given by

$$F(\tau) = \sqrt{h^2 \sum_{j=0}^{M} \sum_{k=0}^{M} \left| U_{jk}^N(h,\tau) - U_{jk}^{2N}(h,\tau/2) \right|^2},$$

if h is small enough.

Tables 1 and 2 display the numerical error and convergence order for the finite difference scheme (13). We can observe that the numerical results are stable and of 2nd order accuracy in both time and space.

Table 1. Spatial convergence of scheme (13) with $\tau = 1/2^7$

β	h	$\alpha = 0.3$ $E(h)$	Order	$\alpha = 0.6$ $E(h)$	Order	$\alpha = 0.9$ $E(h)$	Order
	$1/2^2$	-	-	-	-	-	-
	$1/2^3$	1.40e-2	-	1.30e-2	-	1.15e-2	-
1.3	$1/2^4$	3.31e-3	2.08	3.08e-3	2.08	2.75e-3	2.07
	$1/2^5$	8.17e-4	2.02	7.61e-4	2.02	6.81e-4	2.02
	$1/2^6$	2.04e-4	2.00	1.90e-4	2.00	1.70e-4	2.00
	$1/2^2$	-	-	-	-	-	-
	$1/2^3$	1.70e-2	-	1.61e-2	-	1.47e-2	-
1.5	$1/2^4$	3.99e-3	2.10	3.77e-3	2.09	3.46e-3	2.09
	$1/2^5$	9.82e-4	2.02	9.30e-4	2.02	8.54e-4	2.02
	$1/2^6$	2.45e-4	2.01	2.32e-4	2.01	2.13e-4	2.01
	$1/2^2$	-	-	-	-	-	-
	$1/2^3$	2.02e-2	-	1.94e-2	-	1.81e-2	-
1.7	$1/2^4$	4.67e-3	2.11	4.48e-3	2.11	4.20e-3	2.11
	$1/2^5$	1.15e-3	2.03	1.10e-3	2.03	1.03e-3	2.02
	$1/2^6$	2.86e-4	2.01	2.74e-4	2.01	2.57e-4	2.01

Table 2. Temporal convergence of scheme (13) with $h = 1/2^6$

β	τ	$\alpha = 0.3$ $F(\tau)$	Order	$\alpha = 0.6$ $F(\tau)$	Order	$\alpha = 0.9$ $F(\tau)$	Order
	$1/2^5$	-	-	-	-	-	-
	$1/2^6$	1.04e-4	-	2.59e-4	-	3.94e-4	-
1.3	$1/2^7$	2.59e-5	2.00	6.46e-5	2.01	9.77e-5	2.01
	$1/2^8$	6.49e-6	2.00	1.61e-5	2.01	2.42e-5	2.01
	$1/2^9$	1.62e-6	2.00	4.00e-6	2.01	6.00e-6	2.01
	$1/2^5$	-	-	-	-	-	-
	$1/2^6$	1.08e-4	-	2.72e-4	-	4.14e-4	-
1.5	$1/2^7$	2.71e-5	2.00	6.78e-5	2.00	1.03e-4	2.01
	$1/2^8$	6.77e-6	2.00	1.69e-5	2.00	2.55e-5	2.01
	$1/2^9$	1.69e-6	2.00	4.21e-6	2.00	6.33e-6	2.01
	$1/2^5$	-	-	-	-	-	-
	$1/2^6$	1.12e-4	-	2.83e-4	-	4.32e-4	-
1.7	$1/2^7$	2.81e-5	2.00	7.07e-5	2.00	1.08e-4	2.01
	$1/2^8$	7.04e-6	2.00	1.76e-5	2.00	2.67e-5	2.01
	$1/2^9$	1.76e-6	2.00	4.40e-6	2.00	6.64e-6	2.01

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