Homework 3

Josh Horejs

Portland State University

ECE 315 – Fall 2020

Problem 1

With a given LTI system response $h[n]$.

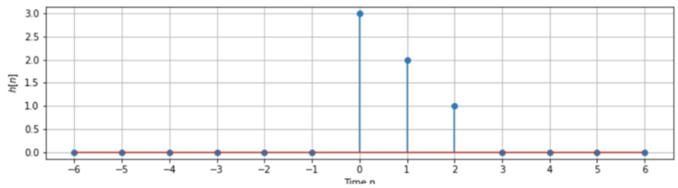


Figure 1 System impulse response h[n]

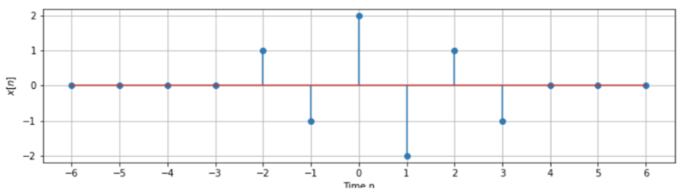


Figure 2 input signal x[n]

a) Find and plot the signal $h[m + 5]$ as a function of the integer m by hand.
b) Find and plot the signal $h[5 - m]$ as a function of the integer m by hand.

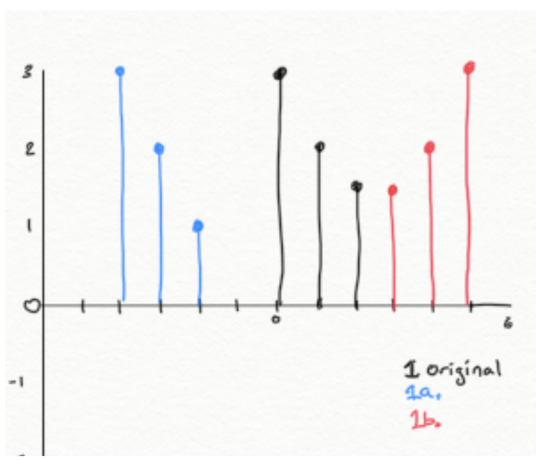


Figure 3 Hand drawn plot of part 1 a and b

c) Calculate the output $y[n]$ of the system in response to the input signal $x[n]$ using convolution by hand.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

$$y[-3] = 0$$

$$y[-2] = x[-2]h[0]$$
$$= 1 \bullet 3$$
$$= 3$$

$$y[-1] = x[-1]h[0] + x[-2]h[1]$$
$$= -3 + 2$$
$$= -1$$

$$y[0] = x[0]h[0] + x[-1]h[1] + x[-2]h[2]$$
$$= 6 - 2 + 1$$
$$= 5$$

$$y[1] = x[1]h[0] + x[0]h[1] + x[-1]h[2]$$
$$= -6 + 4 - 1$$
$$= -3$$

$$y[2] = x[2]h[0] + x[1]h[1] + x[0]h[2]$$
$$= 3 - 4 + 2$$
$$= 1$$

$$y[3] = x[3]h[0] + x[2]h[1] + x[1]h[2]$$
$$= -3 + 2 - 2$$
$$= -3$$

$$y[4] = x[4]h[0] + x[3]h[1] + x[2]h[2]$$
$$= 0 - 2 + 1$$
$$= -1$$

$$y[5] = x[5]h[0] + x[4]h[1] + x[3]h[2]$$
$$= 0 + 0 - 1$$
$$= -1$$

$$y[6] = 0$$

d) Verify your solution using the MATLAB function conv. Include your code and its output in your answer.

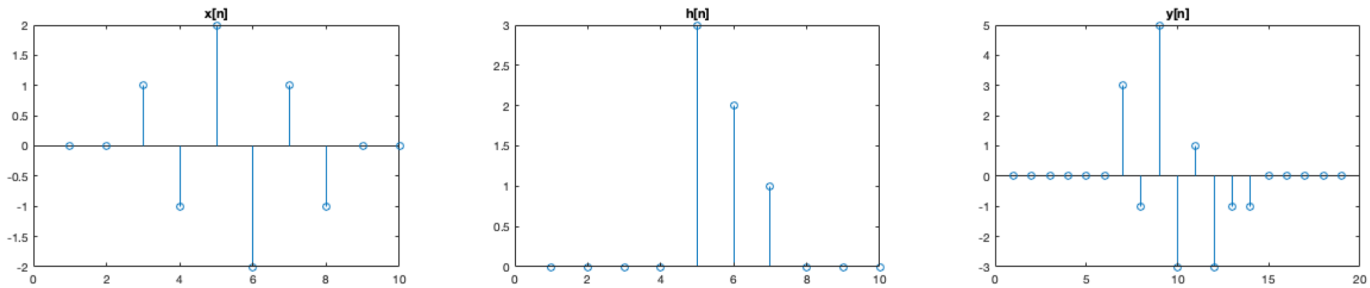


Figure 4 MATLAB plots of x[n], h[n] and their convolution y[n].

```
x = [0 0 1 -1 2 -2 1 -1 0 0];
h = [0 0 0 0 3 2 1 0 0 0 ];
y = conv(x,h)
subplot(131);
stem(x);
title('x[n]');
subplot(132);
stem(h);
title('h[n]');
subplot(133)
stem(y);
title('y[n]');
```

Figure 5 Input MATLAB code. Note: I was unable to figure out a efficient way to plot the required points over the appropriate n points.

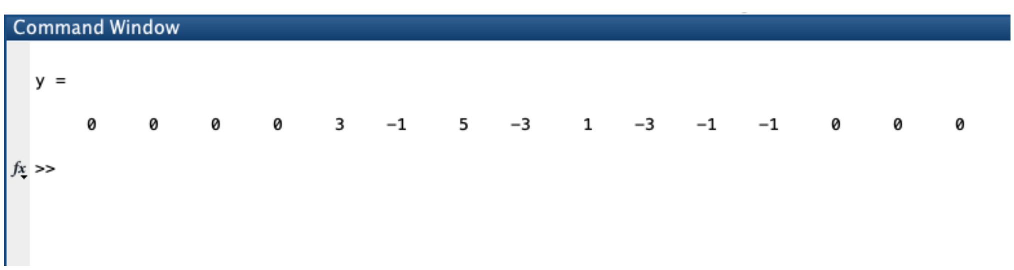


Figure 6 Output of the above MATLAB code.

Problem 2

Consider the discrete-time LTI system.

$$1_0 y[n] + 2_1 y[n-1] + 2_2 y[n-2] = \; 3_0 x[n] - x_1[n-1] + 2_2 x[n-2].$$

$$N = M = 2$$

The coefficients are:

$$a_0 = 1, \qquad a_1 = 2, \qquad a_2 = 2$$
$$b_0 = 3, \qquad b_1 = -1, \qquad b_2 = 3$$

a) **Find the general solution** $y_h[n]$ to the homogeneous equation

$$y_h[n] + 2y_h[n-1] + 2y_h[n-2] \; = \; 0$$

From the given formula the homogeneous equation yielded the following

$$ay[n] - \beta y[n-1] + \gamma y[n-2] = 0$$
$$(az^2 - \beta z + \gamma)z^{n-2} = 0$$
$$az^2 - \beta z + \gamma = 0$$
$$z = z \pm$$
$$= \frac{\beta \pm \sqrt{\beta^2 - 4a\gamma}}{2a}$$

With the given equation the general solution is found using their magnitudes

$$a = 1, \qquad \beta = -2, \qquad \gamma = 2$$

$$z = \frac{-2 \pm \sqrt{-2^2 - 4(2)1}}{2}$$
$$= -1 \pm \sqrt{-1}$$

The characteristic equation is:

$$0 = r^2 + 2r + 2$$
$$= (r + 1 - j)_1 (r + 1 + j)_2$$
$$\therefore$$
$$\boldsymbol{r_1 = -1 + j}, \qquad \boldsymbol{r_2 = -1 - j}$$

b) **Find initial values** $\hat{h}[0]$ and $\hat{h}[1]$ for the impulse response $\hat{h}[n]$,where

$$\hat{h}[n] + 2\hat{h}[n-1] + 2\hat{h}[n-2] = \delta[n]$$

$$\hat{h}[n] = \delta[n] - 2\hat{h}[n-1] - 2\hat{h}[n-2]$$

Applying lecture 8 slide 28 equation 36.

And $\hat{h}[n] = 0$ for $n < 0$.

$$\hat{h}[n] = \frac{1}{a_0}\left(\delta[n] - \sum_{k=0}^{N} a_k y[n-k]\right)$$
$$= \delta[n] - \sum_{k=0}^{2} a_k y[n-k]$$

$$\hat{h}[0] = \delta[0] - 2\hat{h}[0-1] - 2\hat{h}[0-2]$$
$$= \mathbf{1}$$
$$\hat{h}[1] = \delta[1] - 2\hat{h}[0] - 2\hat{h}[-1]$$
$$= 0 - 2(1) - 0$$
$$= -\mathbf{2}$$

c) Use the initial values calculated in the previous part to determine the **values of the undetermined coefficients** in the solution $y[n] = y_h[n]u[n]$ to the problem

$$y[n] + 2y[n-1] + 2y[n-2] = x[n]$$

$$c_1 + c_2 = 1, \qquad (-1+j)c_1 + (-1-j)c_2 = -2$$

$$c_1 = 1 - c_2 = -2 - \frac{1-(-1-j)c_2}{-1+j} = -3 + c_2 - \frac{1-(-1-j)c_2}{-1+j}$$

$$c_1 = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{i}}{\mathbf{2}}, \qquad c_2 = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{i}}{\mathbf{2}}$$

$$\therefore$$

$$\hat{h}[n] = (r_1 c_1 + r_2 c_2)u[n]$$

$$= \left(\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{i}}{\mathbf{2}}\right)(-\mathbf{1}+\boldsymbol{j})^n + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{i}}{\mathbf{2}}\right)(-\mathbf{1}-\boldsymbol{j})^n\right)u[n]$$

d) Use the solution calculated in the previous part to construct the impulse response for the original difference equation
$$y[n] + 2y[n-1] + 2y[n-2] = 3x[n] - x[n-1] + 2x[n-2].$$

From the process above we see the equality:

$$\begin{aligned}\hat{h}[n] &= (r_1 c_1 + r_2 c_2)u[n] \\ &= \left(\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{i}}{\mathbf{2}}\right)(-\mathbf{1}+\boldsymbol{j})^n + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{i}}{\mathbf{2}}\right)(-\mathbf{1}-\boldsymbol{j})^n\right)u[n]\end{aligned}$$

Applying these results and creating the impulse response for the original difference equation as follows:

$$y[n] + 2y[n-1] + 2y[n-2] = 3x[n] - x[n-1] + 2x[n-2]$$

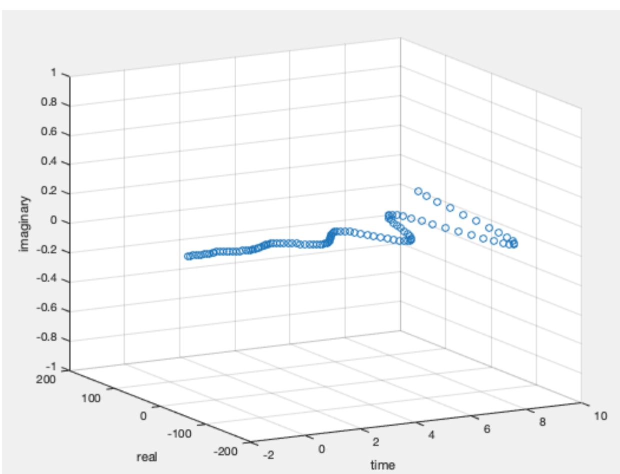
$$\hat{h}[n] = 3\hat{h}[n] - \hat{h}[n-1] + 2\hat{h}[n-2]$$

$$= \mathbf{3}\left(\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}+\boldsymbol{j})^{\boldsymbol{n}} + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}-\boldsymbol{j})^{\boldsymbol{n}}\right)\boldsymbol{u}[\boldsymbol{n}]$$

$$-\left(\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}+\boldsymbol{j})^{\boldsymbol{n}-\mathbf{1}} + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}-\boldsymbol{j})^{\boldsymbol{n}-\mathbf{1}}\right)\boldsymbol{u}[\boldsymbol{n}-\mathbf{1}]$$

$$+\,\mathbf{2}\left(\left(\frac{\mathbf{1}}{\mathbf{2}} + \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}+\boldsymbol{j})^{\boldsymbol{n}-\mathbf{2}} + \left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\boldsymbol{j}}{\mathbf{2}}\right)(-\mathbf{1}-\boldsymbol{j})^{\boldsymbol{n}-\mathbf{2}}\right)\boldsymbol{u}[\boldsymbol{n}-\mathbf{2}]$$

e) Use MATLAB to plot the impulse response of the original system. If the impulse response is complex, plot its real and imaginary parts and plot it as a stem plot in the 3 dimensions of its real and imaginary parts and time n.



The function is purely real.

```
clc
close all
clear all

syms n;
n = -1:.1:10;
a = 1/2+i/2;
b = 1/2-i/2;
c = i-1;
d = -1-i;

h =
3.*(a.*c.^(n)+b.*d.^(n)).*heavisi
de(n)-(a.*c.^(n-1)+b.*d.^(n-
1)).*heaviside(n-1)+2.*(a.*c.^(n-
2)+b.*d.^(n-2)).*heaviside(n-2)
stem3(n,real(h),imag(h));
xlabel('time');
ylabel('real');
zlabel('imaginary');
grid on
```

Problem 3

Consider the discrete-time LTI system

$$y[n] + 2y[n-1] + 2y[n-2] = 3x[n] - x[n-1] + 2x[n-2]$$

As with problem 2, the coefficients are:

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2$$
$$b_0 = 3, \quad b_1 = -1, \quad b_2 = 2$$

a) Find the transfer function for the system analytically.

$$H(z) = \frac{\sum_{k=0}^{3} b_k\, z^{-k}}{\sum_{k=0}^{3} a_k z^{-k}}$$

$$\therefore$$

$$\frac{3z^{-0} + (-1)z^{-1} + 2z^{-2}}{1z^0 + 2z^{-1} + 2z^{-2}} = \frac{3 - \frac{1}{z} + \frac{2}{z^2}}{1 + \frac{2}{z} + \frac{2}{z^2}}$$

$$= \frac{3 - z^{-1} + 2z^{-2}}{1 + 2z^{-1} + 2z^{-2}}$$

$$= \frac{3z^2 - z^1 + 2}{z^2 + 2z^1 + 2}$$

b) Find the frequency response for the system analytically and use MATLAB to plot the magnitude and phase of your result as functions of angular frequency Ω for $0 \leq \Omega \leq \pi$.

$$\frac{3z^2 - z^1 + 2}{z^2 + 2z^1 + 2}, \quad \underset{f\ ,}{\Longrightarrow} \quad \frac{3e^{2\Omega j} - e^{\Omega j} + 2}{e^{2\Omega j} + 2e^{\Omega j} + 2}$$

Graph below.

c) Use the MATLAB function `freqz` to determine and plot the magnitude and phase of the frequency response for $0 \leq \Omega \leq \pi$.

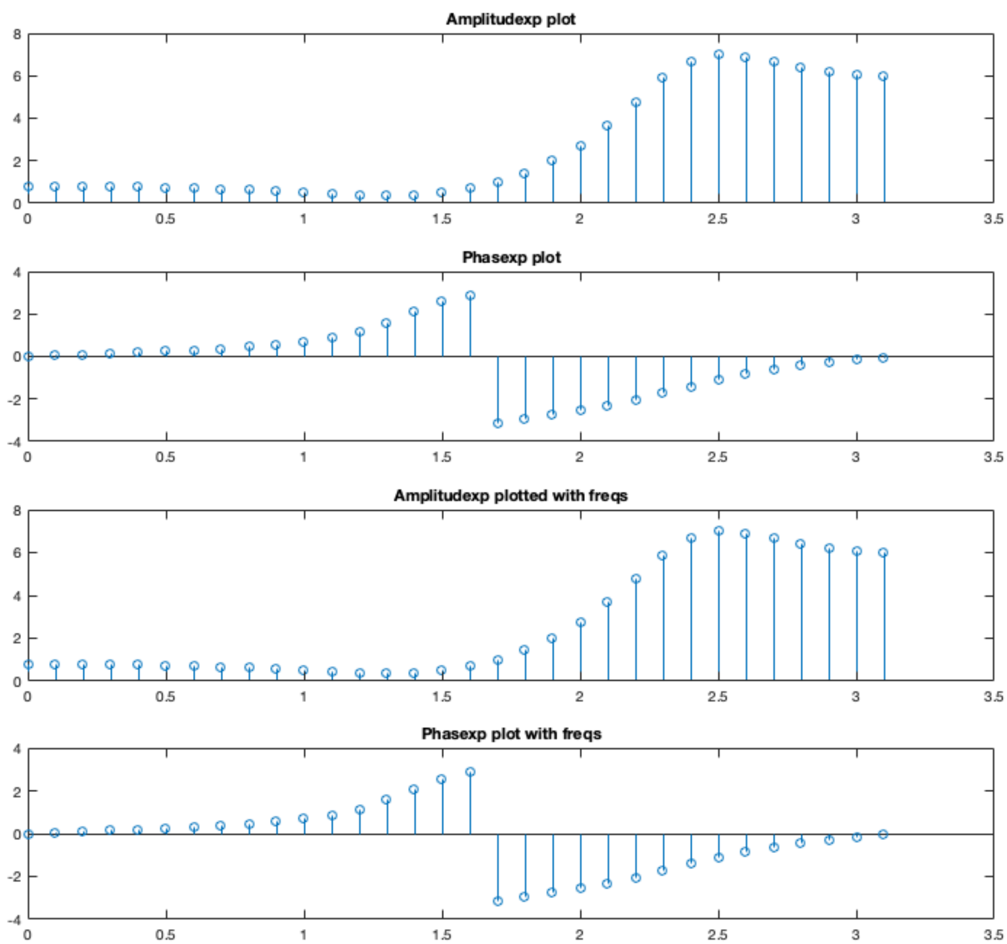
MATLAB code for parts b and c:

```
b = [3 -1 2];
a = [1 2 2];
g = 0:.1:pi;
n = 3.*exp(g.*1i.*2)-exp(g.*1i)+2;
d = exp(g.*1i.*2)+2.*exp(g.*1i)+2;
H = n ./ d;
h = freqz(b,a,g);

subplot(4,1,1), stem(g, abs(H)), title('Amplitudexp plot');
subplot(4,1,2), stem(g, angle(H)), title('Phasexp plot');

subplot(4,1,3), stem(g, abs(h)), title('Amplitudexp plotted with freqs');
subplot(4,1,4), stem(g, angle(h)), title('Phasexp plot with freqs');
```

Graph for parts b and c:



Problem 4

Suppose:

$$h(t) = \begin{cases} 2 - t, & 1 \le t < 2 \\ 0, & else \end{cases}$$

Where $h(t)$ is the impulse response for a continuous-time LTI system and that the system is stimulated with the signal.

$$x(t) = \begin{cases} t(2 - t), & 0 < t < 2 \\ 0, & else \end{cases}$$

a) Plot the signal $x(t)$ by hand.
b) Find and plot the signal $h(\tau + 5/2)$ as a function of τ by hand.
c) Find and plot the signal $h(5/2 - \tau)$ as a function of τ by hand.

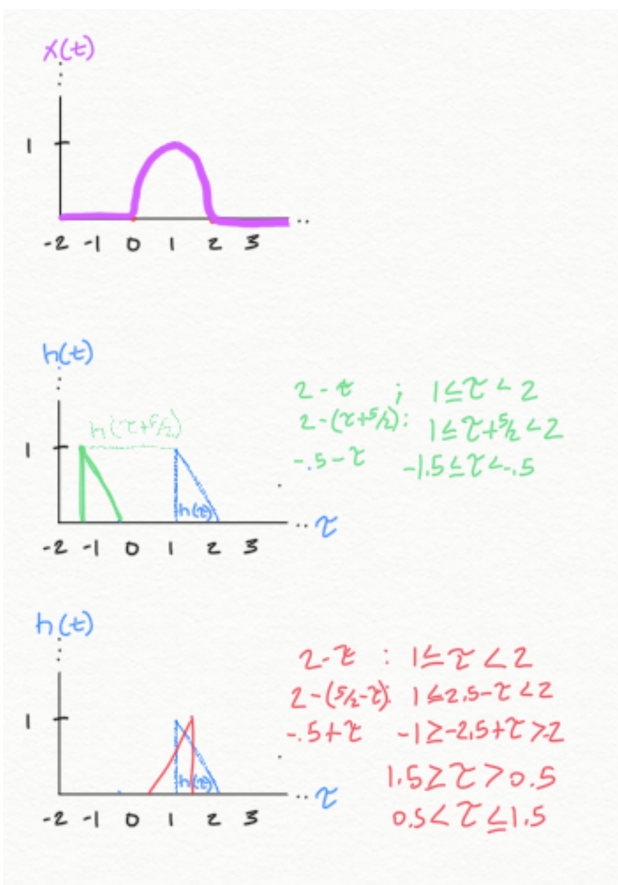


Figure 7 Plot of parts a-c

For $0 < t < 1$

$$\begin{aligned} y_a(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^t (2\tau-\tau^2)(2-\tau)d\tau \\ &= \int_0^t 4\tau - 4\tau^2 + \tau^3 d\tau \\ &= 2\tau^2 - \frac{4}{3}\tau^3 + \frac{1}{4}\tau^4\Big|_0^t \\ &= 2t^2 - \frac{4}{3}t^3 + \frac{1}{4}t^4\Big|_0^1 \end{aligned}$$

For $1 < t < 2$

$$\begin{aligned} y_b(t) &= \int_{t-2}^{t} 4\tau - 4\tau^2 + \tau^3 d\tau \\ &= 2\tau^2 - \frac{4}{3}\tau^3 + \frac{1}{4}\tau^4\Big|_{t-2}^t \\ &= 2t^2 - \frac{4}{3}t^3 + \frac{1}{4}t^4 - \left(2(t-2)^2 - \frac{4}{3}(t-2)^3 + \frac{1}{4}(t-2)^4\right)\Big|_1^2 \end{aligned}$$

For $2 < t < 3$

$$\begin{aligned} y_c(t) &= \int_{t-2}^{2} 4\tau - 4\tau^2 + \tau^3 d\tau \\ &= 2t^2 - \frac{4}{3}t^3 + \frac{1}{4}t^4\Big|_{t-2}^2 \\ &= 12 - \frac{32}{3} - \left(2(t-2)^2 - \frac{4}{3}(t-2)^3 + \frac{1}{4}(t-2)^4\right)\Big|_2^3 \end{aligned}$$

$\therefore$

$$\boldsymbol{y(t)} = \begin{cases} \boldsymbol{y_a}, & \boldsymbol{0 < t < 1} \\ \boldsymbol{y_b}, & \boldsymbol{1 < t < 2} \\ \boldsymbol{y_c}, & \boldsymbol{2 < t < 3} \\ \boldsymbol{0}, & \boldsymbol{else} \end{cases}$$

d) Use MATLAB to plot the convolution $y(t)$.

```
t = -1:.1:3;
x = zeros(1,length(t));
h = zeros(1,length(t));
for i=1:numel(t)
    if t(i)<0 || t(i)>2
        x(i)=0;
    else
        x(i) = 2.*(t(i))-
(t(i)).^2;
    end
    if t(i)<=1 || t(i)>2
        h(i) = 0;
    else
        h(i) = 2-t(i);
    end
end
y = conv(x,h,'same');
plot(t,h,'r',t,x,'b',t,y,'g')
```

Figure 9 Code using piecewise int

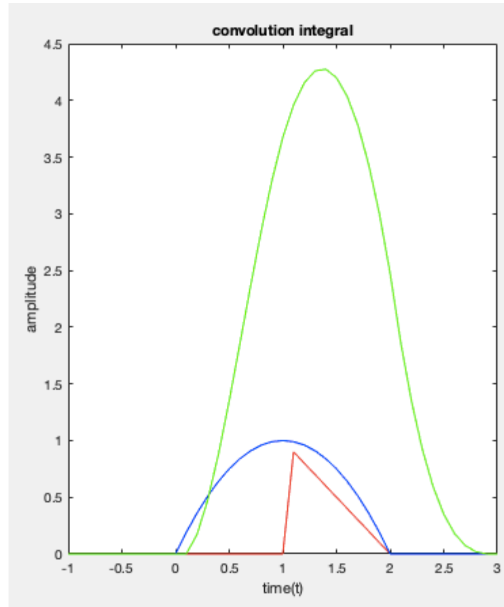


Figure 8 Convolution of h(t) and x(t)

Problem 5

Consider the continuous-time LTI system

$$2y(t) - \frac{2}{3}y'(t) + \frac{5}{6}y''(\text{t})=\text{x(t)+1/2 x'(t)-3/5x''}(t)$$

a) Find the transfer function for the system analytically.

$$H(t) = \frac{\text{x(t)+1/2 x'(t)-3/5x}(t)}{2y(t) - \frac{2}{3}y'(t) + \frac{5}{6}y''(t)}$$
$$= \frac{\mathbf{1 + \frac{1}{2}s - \frac{3}{5}s^2}}{\mathbf{2 - \frac{2}{3}s + \frac{5}{6}s^2}}$$

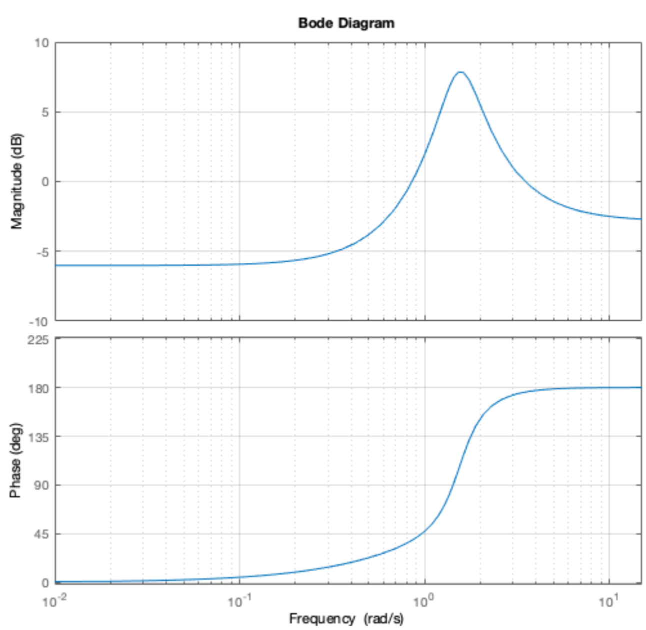
b) Find the frequency response for the system analytically and use MATLAB to plot the magnitude and phase of your result as functions of angular frequency ω for $0 \leq \omega \leq 15$.

$$\frac{1 + \frac{1}{2}s - \frac{3}{5}s^2}{2 - \frac{2}{3}s + \frac{5}{6}s^2}, \qquad where\ s = jw, \qquad \frac{\mathbf{1 + \frac{1}{2}jw - \frac{3}{5}jw^2}}{\mathbf{2 - \frac{2}{3}jw + \frac{5}{6}jw^2}}$$

Code:

```
H = tf([-3/5, 1/2, 1],[5/6, -2/3, 2]);
bode(H,{0,15});
```

Plot:



c) Use the MATLAB function `freqs` to determine and plot the magnitude and phase of the frequency response for $0 \leq \omega \leq 15$.

1. **Code:**

```
freqs([-3/5, 1/2, 1],[5/6, -2/3, 2],0:.1:15);
```

2. **Plot:**