ECE 316 HW 2

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Author Note

I worked with David D and Nick L and Luke Tutino today: We just discussed the fourier trates and a lot of the lecture notes. We didn’t really help with specific problems. We just worked independently and if we had a question then we’d ask and someone would tell us where to find it in the notes or book.

Problem 1

- **Explain, in your own words, why we would want to use the Laplace transform.**
  The concise answer would be that we use Laplace for systems in the real plane. This is useful for multiple reasons but specifically we use it to describe relatively complicated systems in algebraic forms.

- **When can we employ it?**
  In cases of systems, or when a fourier transform is not useful. Specifically: LTI, BIBO
- **Why is it beneficial?**

We are able to solve complex problems with algebra among many other things.

Problem 2

Starting with the definition of the Laplace transform, work through the math to obtain transforms for the following signals. Do not use the Laplace transform ``look-up" table. Remember to specify region of convergence (ROC) in terms of σ. Show all work when determining the transform and ROC.

$$\mathcal{L}(x) \coloneqq X(t) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

Pick 2 of the following 3 to work through:

1. $x(t) = (t^2 + 3t + e^t)u(t)$

$$\mathcal{L}(x) \coloneqq X(t) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{(\sigma+j\omega)t}dt$$

2. $x(t) = (e^{3t} + 4e^{-t})u(t)$

$$\mathcal{L}(x) \coloneqq X(t) = \int_{-\infty}^{\infty} (e^{3t} + 4e^{-t})u(t)e^{-st}dt$$

$$= \int_{0}^{\infty} (e^{3t} + 4e^{-t})e^{-st}dt\,, \qquad considering\ t\ only$$

$$= \int_{0}^{\infty} (e^{3t} + 4e^{-t})e^{-st}dt$$

$$= \int_{0}^{\infty} 4e^{-(\sigma+j\omega)t} + e^{3t-(\sigma+j\omega)t}dt\,, \qquad to\ find\ p's\ \&\ z's$$

$$= \int_{0}^{\infty} 4e^{-t(j\omega+(\sigma+1))} + e^{-t((\sigma-3)+i\omega)}dt\,, \qquad trivial\ steps\ calc.$$

$$= \int_{0}^{\infty} 4e^{-st-t} + e^{3t-st}dt$$

$$= \frac{1}{3-s}e^{-st+3t} + \frac{4}{-s-1}e^{-st-t}\bigg|_0^{\infty}\,, \qquad \lim_{t\to\infty} X(t) = 0\ \ and\ t = 0, e = 1$$

$$= \frac{4}{s+1} + \frac{1}{s-3}$$

$$s+1=0, \qquad s-3=0, \qquad pole's: s = -1\ \&\ 3 \therefore ROC: \sigma > 3$$

3. $x(t) = \cos(9\pi t)\, u(t-3)$

$$\cos(x) = \frac{e^{jx}}{2} + \frac{e^{-jx}}{2}$$

$$\mathcal{L}(x) \coloneqq X(t) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2}\left(e^{j(9\pi t)} + e^{-j(9\pi t)}\right)u(t-3)\right)e^{-st}dt\,, \qquad only\ considering\ t > 3$$

$$= \int_{3}^{\infty} \left(\frac{1}{2}\left(e^{j(9\pi t)} + e^{-j(9\pi t)}\right)1\right)e^{-st}dt$$

$$= \frac{1}{2}\int_{3}^{\infty} \left(e^{-ts-9j\pi t} + e^{-ts+9j\pi t}\right)dt\,, \qquad Calculator$$

$$= \frac{e^{-t(s-9\,i\,\pi)}}{2\,(-s\ +\ 9\,i\,\pi)} - \frac{e^{-t(s+\,9\,i\,\pi\,)}}{2\,(s\ +\ 9\,i\,\pi)}\Bigg|_{3}^{\infty}$$

$$= \lim_{t\to\infty} 0 - \left(\frac{e^{-s\,3\,+\,9\,i\,\pi\,3}}{2\,(-s\ +\ 9\,i\,\pi)} - \frac{e^{-s\,3\,-\,9\,i\,\pi\,3}}{2\,(s\ +\ 9\,i\,\pi)}\right)$$

$$= -\frac{e^{3s}s}{s^2 + 81\pi^2}$$

$Poles\ at$:

$$s^2 + 81\pi^2 = 0$$

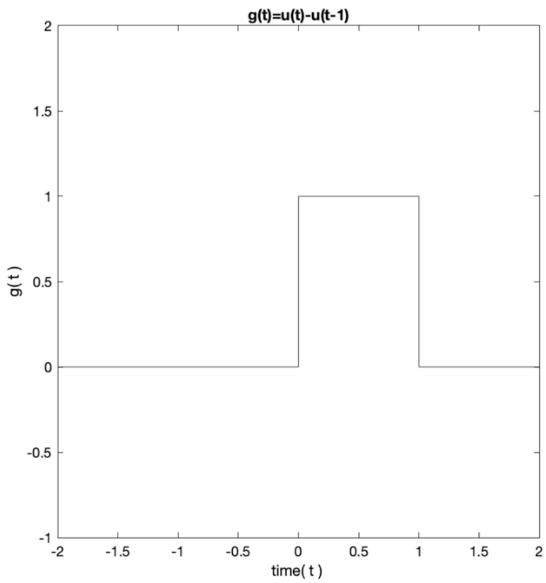
$$s = 0 \pm 9\pi j$$

ROC: $\sigma > 0$

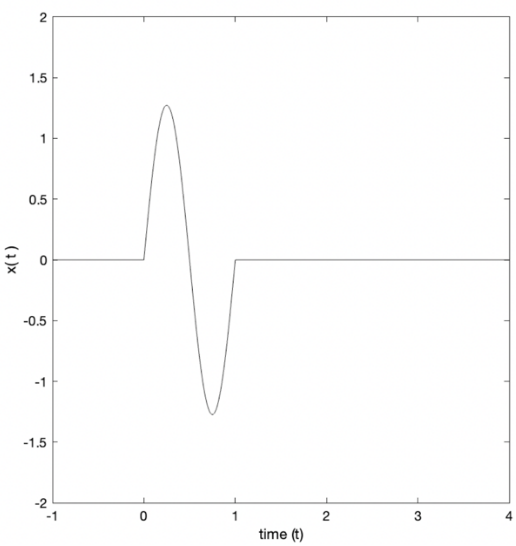
Problem 3

Exercise 8.5din Roberts. Think of this as finding the response of an LTI system when you have impulse response $h(t) = 8\,cos(2\pi t)u(t)$ and excitation $g(t) = [u(t) - u(t-1)]$.

1. Drawing out the plot of u(t)−u(t−1) can be helpful for visualizing.



2. Plot the end result in Matlab, save it as an image, and include this your answer.



$$
\begin{aligned}
x(t) &= 8cos(2\pi t)u(t) * (u(t) - u(t-1)) \\
&= \int_{\infty} 8cos(2\pi t)u(t))\ (u(t) - u(t-1))dt \\
&= (u(t) - u(t-1)) \int_{\infty} (8cos(2\pi t)u(t))\ dt \\
&= (u(t) - u(t-1))\frac{4}{\pi}sin(2\pi t) \\
&= u(t)\frac{4}{\pi}sin(2\pi t) - u(t-1)\frac{4}{\pi}sin(2\pi t
\end{aligned}
$$

Problem 4
Exercise 8.7c in Roberts. Use partial fraction expansion, and the Laplace transform pair table in your book (Table 8.1)

$$X(s) = \frac{5}{s^2 + 6s + 73}, \qquad \sigma > -3$$

I cheated a bit and took the inverse laplace of X(s) then took the laplace of that to see what Matlab came up with.

$$X(s) = \frac{5}{s^2 + 6s + 73} = \frac{5}{(s+3)^2 + 8^2}$$

Laplace is:

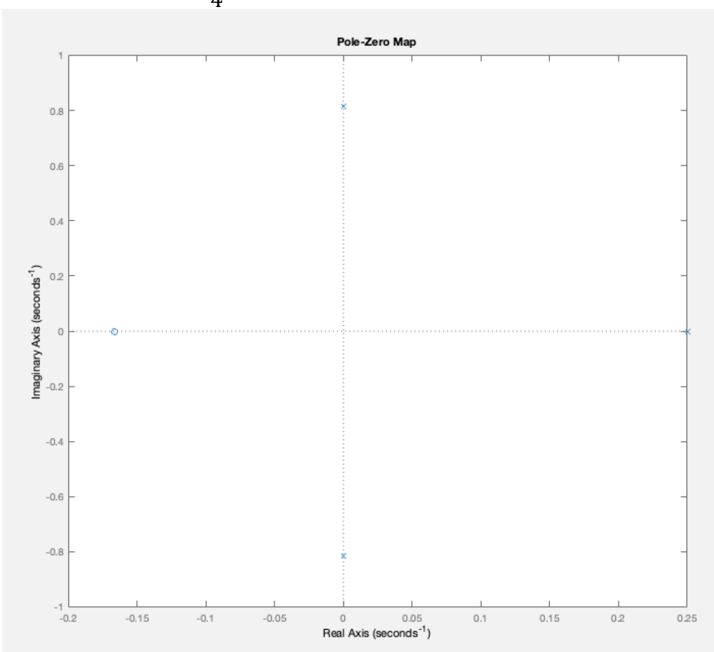
$$\frac{5}{8}\sin(8t)e^{-3t}$$

Problem 5

Suppose we have a system with right-sided impulse response h(t) for which the Laplace transform yields the following transfer function

$$H(s) = \frac{6s+1}{(3s^2+2)(4s-1)}$$

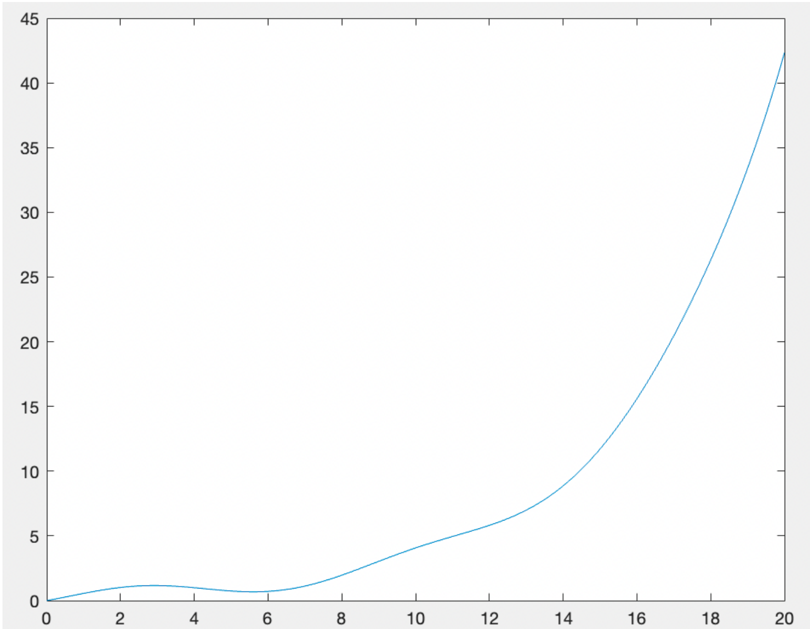
1. Where are the poles and zeros? Draw the pole-zero diagram.
   I used matlab for this see my code below.
   Zeros: $6s + 1 = 0,\ s = -\frac{1}{6}$
   Poles: $\pm.8165i, \frac{1}{4}$



2. Comment on the general expected form of the system response to an arbitrary set of initial conditions. Does the system have components which grow exponentially? Components which decay? Are there oscillatory components? Explain your reasoning.

As we see here there are poles on the right side of the real plane which means we are looking at a exponential growth continuous time system. The parts in the imaginary plane produce oscillatory elements to the system and the ¼ pole will cause a slight decay but after 4s quickly the system will exceed that component and grow.
As we can see by the inverse laplace transform the function is exponentially growing which will go to infinity relatively quick, making it a more exponential than oscillatory.

$$\frac{2e^{\frac{t}{4}}}{7} - \frac{\left(2cos\left(\frac{\sqrt{2}\sqrt{3}t}{3}\right)\right)}{7} + \frac{\left(3\sqrt{2}\sqrt{3}sin\left(\frac{(\sqrt{2}\sqrt{3}t)}{3}\right)\right)}{14}$$



This is the inverse laplace of the transfer function. The behavior of the function is sinusoidal but more importantly the function will grow exponentially very fast.

3. Can we find the continuous time Fourier transform directly from H(s)? Explain your reasoning.

Because we have poles outside one plane the Fourier transform will not be possible in this case.

Problem 6

Exercise 8.16b in Roberts

Using the unilateral Laplace transform, solve the differential equation for $t \geq 0$.

$$x"(t) - 2x'(t) + 4x(t) = u(t), \qquad x(0\hat{}-) = 0, \qquad \frac{d}{dt}\big(x(t)\big)_{t=0^-} = 4$$

Using Example 8.13:
For times greater than t>0.

$$x"(t) - 2x'(t) + 4x(t) = 1, \qquad laplace\ transform\ both\ sides$$

$$s^2X(s) - sx(0^-) - \frac{d}{dt}\big(x(t)\big)_{t=0^-} - 2sX(s) + 2x(0^-) + 4X(s) = \frac{1}{s}$$

$$X(s)(s^2 - 2s + 4) = \frac{1}{s} + sx(0^-) + \frac{d}{dt}\big(x(t)\big)_{t=0^-} - 2x(0^-)$$

$$X(s) = \frac{s}{s}\frac{\frac{1}{s} + sx(0^-) + \frac{d}{dt}\big(x(t)\big)_{t=0^-} - 2x(0^-)}{s^2 - 2s + 4}$$

$$= \frac{1 + s^2x(0^-) + s\frac{d}{dt}\big(x(t)\big)_{t=0^-} - 2sx(0^-)}{s^3 - 2s^2 + 4s}$$

$$= \frac{1 + 0 + 4s - 0}{s^3 - 2s^2 + 4s}$$

$$= \frac{4s + 1}{s^3 - 2s^2 + 4s}$$

Matlab ilaplace see code:

$$\frac{1}{4}\left(1 - e^t\left(\cos\big(\sqrt{3}t\big) - \frac{17}{\sqrt{3}}\sin\big(\sqrt{3}t\big)\right)\right)u(t)$$

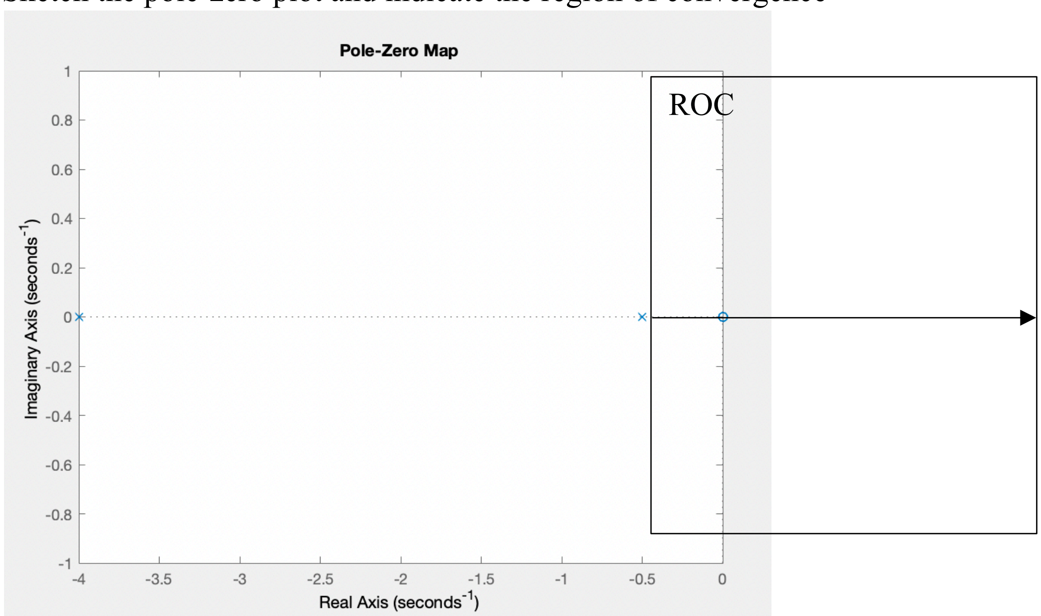
Problem 7

Let h(t) be a right-sided impulse response of a system, and it's Laplace transform given by

$$H(s) = \frac{7s^2}{(2s+1)(s+4)} = \frac{7s^2}{2s^2+9s+4}$$

1. Find the poles and zeros of the system
   See Matlab code below.
   Z's: 0
   P's: -4, -1/2

2. Sketch the pole-zero plot and indicate the region of convergence



   The region of convergence will be to the right of -1/2 since that is where the right most pole is.
3. Is the system stable? Justify your answer

The poles are to the left of the imaginary plane the exponential will decay. This is due to the absence a frequency response(im=0) and the transient responses is $\sigma < \infty$.

Problem 8

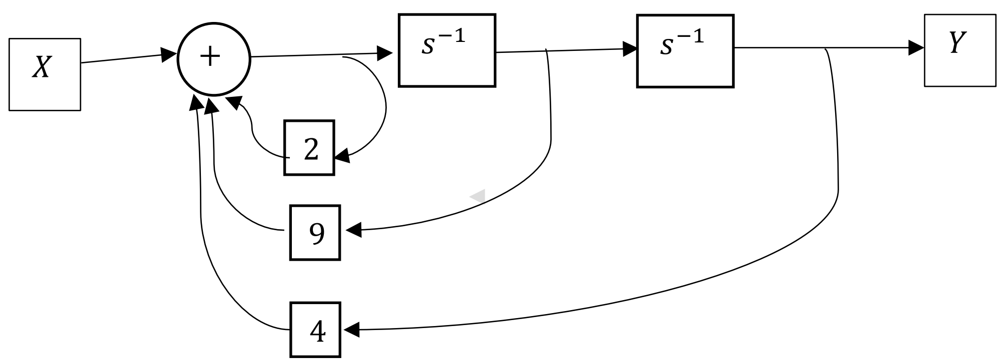
Draw the Direct Form II (aka "control canonical form") realization for the transfer function H(s) from Problem 7. First draw each component (the feedforward and the feedback), then draw the full system realization.

$$H = \frac{Y}{X} = \frac{Y_{1_{internal}}}{X_{input}} \frac{Y_{output}}{Y_{1_{internal}}} = H_1 H_2$$

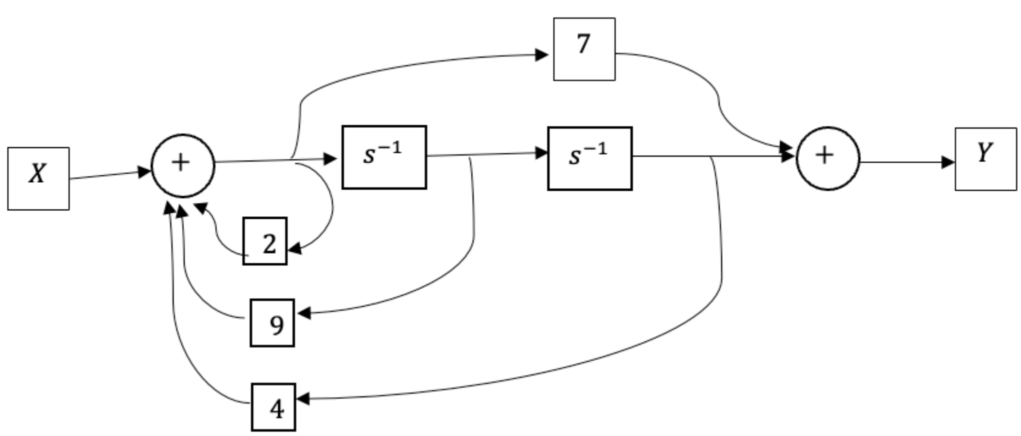
$$H_1 = \frac{Y_{1_{internal}}}{X_{input}} = \frac{1}{2_{a_0} + 9s_{a_1}^{-1} + 4s_{a_2}^{-2}}, \qquad H_2 = \frac{Y_{output}}{Y_{1_{internal}}} = \frac{7}{1}$$

Feedforward:

Feedback:



Full System realization:



Matlab:

```
%% Problem 2.2
syms x t s m
f = exp(3*t)+4*exp(-t)*heaviside(t)
L = laplace(f)
iL= ilaplace(L)


%% Problem 2.3
syms x t s
g = t*9*pi;
h = cos(g);
u = heaviside(t-3);
f = (exp(g*1i)/2+exp(-g*1i)/2)*u;
q = int(f*1,3,inf);
L = laplace(h*u)
iL= ilaplace(L)



%% Problem 3
syms x t s
h = 8*cos(2*pi*t);
g = heaviside(t)-heaviside(t-1);

x = int(h);
y = x*g;
%fplot(t,y);

%% Problem 4
syms a b s
X = 1/((s+3)^2+64)%5/(s^2+6*s+73);
iL = ilaplace(X);
L = laplace(5*sin(8*t)*exp(-3*t)/8);

partfrac(X,s)

%% Problem 5
% for factoring with complex values
t= 0:.1:20;
y= (2*exp(t/4))/7 - (2*cos((2^(1/2)*3^(1/2)*t)/3))/7 +
(3*2^(1/2)*3^(1/2)*sin((2^(1/2)*3^(1/2)*t)/3))/14
b = [6 1];
a = [12 -3 8 -2]
H = tf(b, a)
[r,p,k] = residue(b,a)
iL = ilaplace((6*s+1)/(12*s^3 - 3*s^2 + 8*s - 2))

% Pole zero plot
H = tf([6 1],[12 -3 8 -2]);
pzmap(H)
plot(t,y)

%% Problem 6
a = 4*s+1;
b = s^3-2*s^2+4*s;
N = ilaplace(a/b);

 %% Problem 7
c = expand((2*s+1)*(s+4));
a = [7 0 0];
b = [2 9 4];
```

```
% Pole zero plot
H = tf(a,b);
%print out poles and zeros
[p,z]=pzmap(H)
%plot poles and zeros
pzmap(H)

%% Problem 8
clc;close all;
Y = 7;                          %7s^2
X = (2+9*s^(-1)+4*s^(-2)); %2s^2+9s+4
H = Y/X;
H1 = 1/(X);
H2 = Y;
```