Signals & Systems

Homework 4

ECE 315 - Fall 2020

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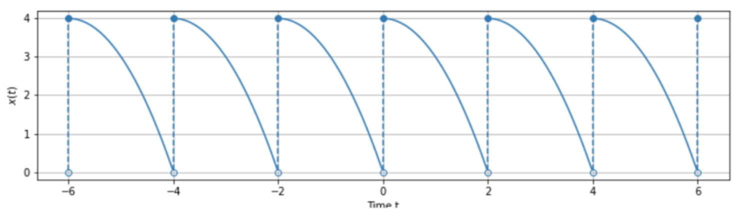
Problem 1

Consider the periodic continuous-time signal

$$x(t) = 4 - (t - 2n)^2, \qquad for: 2n \leq t < 2n + 2$$

For all integers n.



Part (a)

Determine the continuous-time Fourier series (CTFS) coefficients for x(t).

Step 1: determine the fundamental period T_0 and choose the period to evaluate.

$$T_0 = 2, \qquad interval: 0 \leq t < 2, \qquad therefore: n_{interval} = 0$$

$$x(t) = 4 - \left(t - 2(0)\right)^2 = 4 - t^2, \qquad for: 0 \leq t < 2$$

Step 2:

$$\boldsymbol{c_x}[k] = \frac{1}{T_0}\int_{T_0} x(t)e^{-\frac{j2\pi kt}{2}}dt$$

$$= \frac{1}{2}\int_0^2 (4 - t^2)\, e^{-j\pi kt}dt$$

For $k = 0$

$$c_k[0] = \frac{1}{2}\int_0^2 (4 - t^2)\, e^{-0}dt = \frac{8}{3}$$

For $k \neq 0$

$$c_x[k] = \frac{1}{2}\int_0^2 (4 - t^2)\, e^{-j\pi k t} dt$$

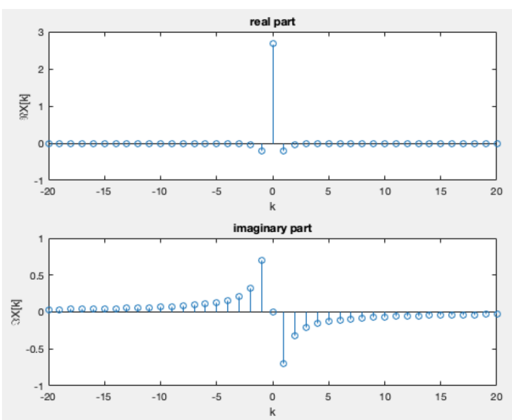
$$= \frac{-2i\pi^2 k^2 + 2i^2 \pi k e^{-2i\pi k} - i e^{-2i\pi k} - i}{\pi^3 k^3}$$

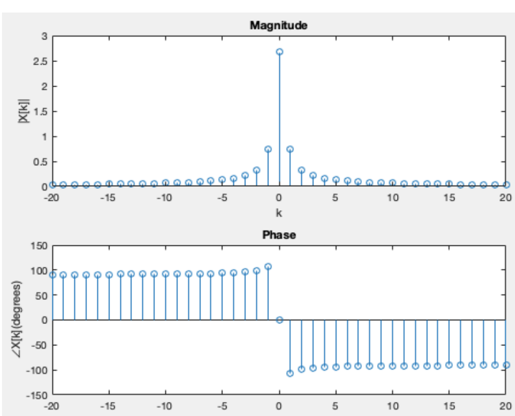
The coefficients of the CTFS are:

$$\boldsymbol{c_x[k]} = \begin{cases} \boldsymbol{\dfrac{8}{3}}, & \boldsymbol{k = 0} \\ \boldsymbol{-2i\pi^5 k^5 - 2\pi^4 k^4 e^{-2i\pi k} - i\pi^3 k^3 e^{-2i\pi k} - i\pi^3 k^3}, & \boldsymbol{k \neq 0} \end{cases}$$

Part (b)

Use MATLAB to plot the real and imaginary parts and the magnitude and phase of the CTFS coefficients for harmonic numbers $k = -20, -19, \ldots, 20$. Include your MATLAB code as part of your solution.



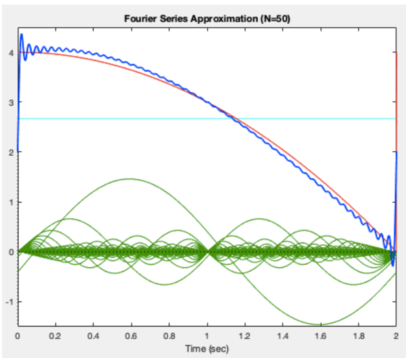


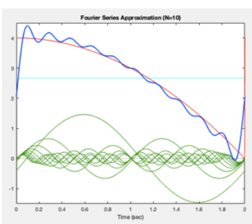

```
N=20;
k=-N:N;
X=(pi.*k).^(-3).*((-
2.*1i.*(pi.*k).^2)+(2.*1i.^2.*(pi.
*k).*exp(-2.*1i.*pi.*k))-
(1i.*exp(-2.*1i.*pi.*k))-(1i));
X(N+1)=8/3;
subplot(211);
stem(k,real(X));
xlabel('k');
ylabel('\Re{X[k]}');
title('real part');
subplot(212);
stem(k,imag(X));
xlabel('k');
ylabel('\Im{X[k]}');
title('imaginary part');
figure
subplot(211);
stem(k,abs(X));
xlabel('k');
ylabel('|X[k]|');
title('Magnitude');
subplot(212);
stem(k,angle(X).*(180/pi));
xlabel('k');
ylabel('\angle{X[k]}(degrees)');
title('Phase');
```

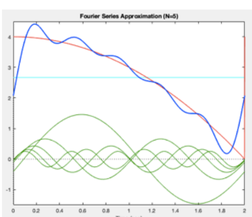
Part (c)

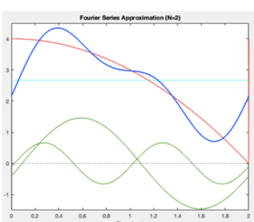
Use MATLAB to form the partial sums $x_N(t)$ of the CTFS and plot the partial sums with the signal x(t) for N = 1,2,5,10,50 and for $0 \le t < 2$. Include your MATLAB code as part of your solution.

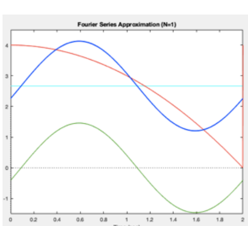
Partial summation equation for a CTFS is: $x_N(t) = \sum_{k=-N}^{N} c_k[k]e^{j2\pi kt/T}$







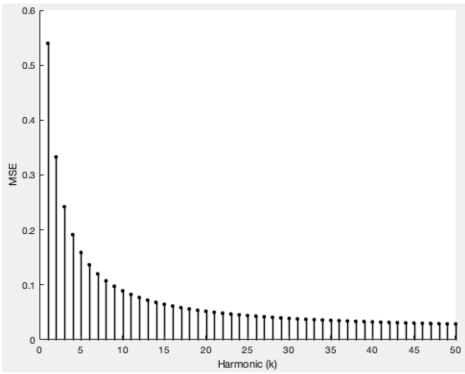




```
close all;
clear all;
clc;

T  = 2;
w0 = 2*pi/T;
Ts = 0.002;
t  = (-0:Ts:2)';
tm = mod(t,T);

x  = (4-tm.^2).*(tm>=0 & tm<2);

N = [1, 2, 5, 10, 50];

MSE = zeros(length(max(N)), 1);

for cnt = 1:length (N)

    figure

    xh = 8/3;  % DC Offset c_0
    t2 = [-0 2];
    h = plot(t2, xh*[1 1], 'b');
    set(h, 'Color', [0 1 1]);
    hold on;

    for cnt2=1:N(cnt),

        k  = cnt2;
        ck = (pi.*k).^(-3).*((-
2.*1i.*(pi.*k).^2)+(2.*1i.^2.*(pi.*k).*exp(-
2.*1i.*pi.*k))-(1i.*exp(-2.*1i.*pi.*k))-(1i));
        xk = 2*abs(ck)*cos(k*w0*t + angle(ck));
        xh = xh + xk;

        t2 = (-4:0.02:4);
        xk = 2*abs(ck)*cos(k*w0*t2 + angle(ck));

        h = plot(t2, xk, 'r');
        set (h, 'Color', [0 0.6 0]);
        MSE(cnt2) = sum((x-xh).^2)*Ts/T;

    end

    h = plot(t, x, 'r', t, xh, 'b', [-2 2], [0 0],
'k:', [0 0], [-2 2], 'k:');
    hold off;

    set(h(1), 'LineWidth', 1.2);
    set(h(2), 'LineWidth', 1.7);
    xlabel('Time (sec)');
    st = sprintf('Fourier Series Approximation
(N=%d)', N(cnt));
    title(st);
    axis([-0 2 -1.50 4.5]);

end
```

Part (d)

Use MATLAB to plot the MSE of the partial sums you calculated previously as a function of N for N = 1,2,⋯ ,50.




```
figure
h = stem(1:max(N), MSE, 'k');
set(h(1), 'Marker', '.');
set(h(1), 'MarkerSize', 11);
set(h(1), 'LineWidth', 1.4);
xlabel('Harmonic (k)');
ylabel('MSE');
saveas(gcf, 'mse.png')
box off;
```

Problem 2

Part Consider the system

$$2y'''(t) - y''(t) + 3y'(t) - 6y(t) = x(t).$$

1. (a) Use the Fourier properties to determine the harmonic response for the system.

   From the tables we know the relationship: $\frac{dx(t)}{dt} \overset{\mathcal{F}s}{\Longleftrightarrow} j2\pi k \bullet C\_[k]$

$$\begin{aligned} F &= H[s] \\ &= \frac{c_y[s]}{c_x[s]} \\ &= \frac{1}{2s^3 - s^2 + 3s - 6} \end{aligned}$$

2. (b) Determine the frequency response of the system. How is the harmonic response related to the frequency response?

$$H(j\omega) = \frac{1}{\left(2\left(\frac{2\pi k}{T_0}\right)\right)^3 - \left(\frac{2\pi k}{T_0}\right)^2 + \left(3\left(\frac{2\pi k}{T_0}\right)\right) - (6)}, \qquad \textit{frequency response}$$

$$= \frac{1}{\left(2(jk\omega)\right)^3 - (jk\omega)^2 + (3(jk\omega)) - (6)}$$

$$H(k) = \frac{1}{\left(2(jk\omega)\right)^3 - (jk\omega)^2 + (3(jk\omega)) - (6)}, \qquad \textit{Harmonic response}$$

$$H(j\omega) = H(kj\omega),$$

The CTFS coefficients comprise a "discrete-frequency" function of harmonically-related frequencies numbered by k and they represent a periodic signal using a countably infinite number of frequencies.

3. (c) Use the result from part (a) and the Fourier series pairs to deter- mine the response of the system to the signal

$$\omega = 2, \qquad T_0 = 4, \qquad M = 1$$

$$x(t) = sinc\left(\frac{t}{2}\right) * \delta_4(t) \Leftrightarrow \sum_{k=-\infty}^{\infty} \frac{1}{2} rect(k/2)\delta[k] e^{j2\pi kt/4}$$

$$y(t) = \frac{1}{2}\sum_{k=-\infty}^{\infty} \frac{rect\left(\frac{k}{2}\right)}{\left(2\left(\frac{2\pi k}{4}\right)\right)^3 - \left(\frac{2\pi k}{4}\right)^2 + \left(3\left(\frac{2\pi k}{4}\right)\right) - (6)} \delta[k] e^{\frac{j2\pi kt}{4}}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty} \frac{rect\left(\frac{k}{2}\right)}{\left(2\left(\frac{\pi k}{2}\right)\right)^3 - \left(\frac{\pi k}{2}\right)^2 + \left(3\left(\frac{\pi k}{2}\right)\right) - (6)} e^{j2\pi kt/4}$$

Problem 3

Consider the continuous-time signal:

$$x(t) = u(t)te^{-t/2}$$

Part (a)

Derive the continuous-time Fourier transform (CTFT) of x(t) as a function of the angular frequency ω, as listed in Table 6.3 on p. 260 in the textbook.

$$te^{-at}u(t) \Leftrightarrow \frac{n!}{(j\omega + a)^{n+1}}$$

$$\therefore X(j\omega) = \frac{1}{\left(\frac{1}{2} + j\omega\right)^2}$$

Part (b)

Determine the real and imaginary parts and the magnitude and phase of the CTFT by hand and use MATLAB to plot them as functions of the angular frequency ω for −10 ≤ ω ≤ 10. Include your MATLAB code as part of your solution.

$$x(jw) = \frac{1}{\left(\frac{1}{2} + j\omega\right)^2}$$

$$= \frac{1}{\frac{1}{4} - \omega^2 + j\omega}_{rationalize}$$

$$= \frac{\frac{1}{4} - \omega^2 - j\omega}{\left(\frac{1}{4} - \omega^2 + j\omega\right)^2}$$

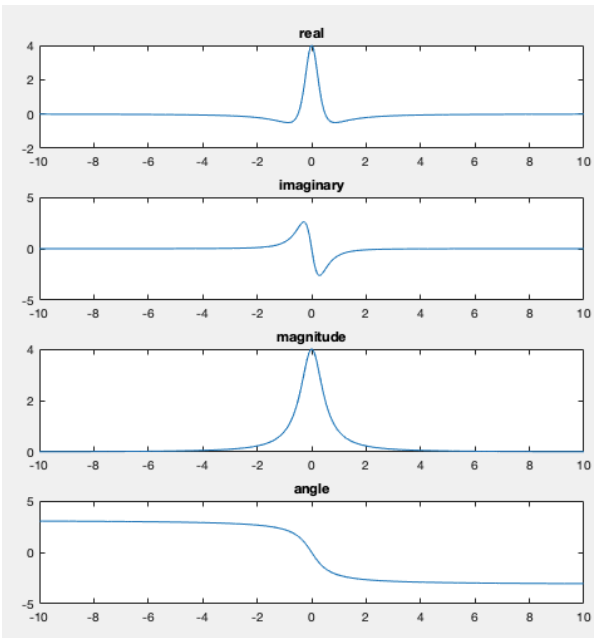
$$= \underbrace{\frac{\frac{1}{4} - \omega^2}{\left(\frac{1}{4} - \omega^2\right)^2 + \frac{1}{2}\omega^2}}_{real} - \underbrace{\frac{j\omega}{\left(\frac{1}{4} - \omega^2\right)^2 + \frac{1}{2}\omega^2}}_{imaginary}$$

Phase: $X(j\omega) = -\tan^{-1}\left(\frac{\omega}{\frac{1}{4} - \omega^2}\right)$,

$$magnitude: |X(j\omega)| = \frac{1}{\sqrt{\left(\frac{1}{2} + j\omega\right)^2}}$$

Part (c)

Use the MATLAB function fft, which calculates the discrete Fourier transform, to approximate the CTFT of x(t). Create the approximation using N = 32768 samples of x(t) over the time interval $0 \le t < 50$. Use MATLAB to plot the real and imaginary parts and the magnitude and phase of the approximation as a function of angular frequency ω for $-10 \le \omega \le 10$. Include your MATLAB code in your solution.




```
w = (-10:.01:10);
X = (1./2+1i.*w).^-2;
r = real(X);
i = imag(X);
mag = abs(X);
angle = angle(X);

subplot(4,1,1);
plot(w,r);
title('real');
subplot(4,1,2);
plot(w,i);
title('imaginary');
subplot(4,1,3);
plot(w,mag);
title('magnitude');
subplot(4,1,4);
plot(w,angle);
title('angle');
```

Problem 4

Consider the system:

$$y(\mathrm{t})+2\mathrm{y}'(\mathrm{t})+\mathrm{y}(\mathrm{t})=\mathrm{x}(t)+2x(t)$$

Part (a)

Find the transfer function for the system.

$$H(s)=\frac{C_x(s)}{C_y(s)}$$

$$=\frac{s^2+2}{s^2+2s+1}$$

Part(b)

Use a partial fraction expansion and a table of CTFT pairs from the textbook to determine the response y(t) of the system when the input is

$$x(t)=u(t)e^{-t/2}$$

$$\boldsymbol{x(t)=u(t)e^{-\frac{t}{2}}\Rightarrow x(s)=\frac{1}{s+\frac{1}{2}}}$$

$$\boldsymbol{Y(s)=H(s)X(s)}$$

$$=\frac{s^2+2}{s^2+2s+1}$$

$$=\frac{A}{\boldsymbol{s+1}}+\frac{B}{\boldsymbol{s+1}}+\frac{C}{\boldsymbol{s+\frac{1}{2}}}$$

$$=-\frac{-8}{\boldsymbol{s+1}}+\frac{-6}{\boldsymbol{s+1}}+\frac{9}{\boldsymbol{s+\frac{1}{2}}}$$

$$=\left(-8e^{-t}-6te^{-t}+9e^{-1/2t}\right)u(t)$$