Signals & Systems

Homework 5

ECE 315 - Fall 2020

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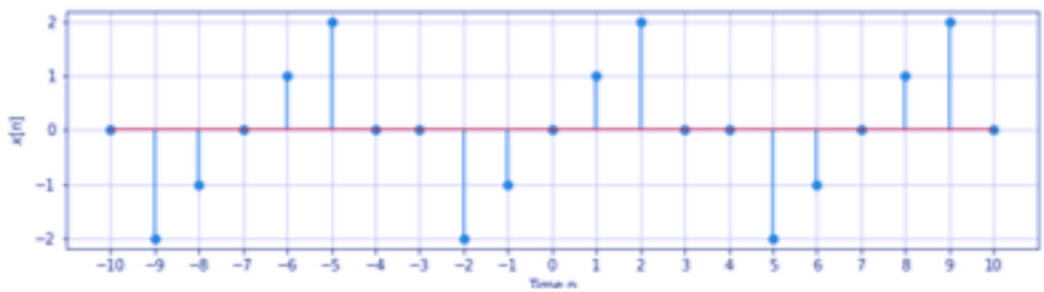
Problem 1.

Consider the periodic discrete-time signal x[n] shown below.



a. Calculate the DTFS coefficients of x[n] by hand. (16 points)

From the graph: $N_{period} = 7, \;\; x_n(n); 0_0, 1_1, 2_2, 0_3, 0_4, -2_5, -1_6$

Angular frequency: $w_0 = \frac{2\pi}{N_0}$

A periodic discrete time signal can be expressed in the following terms

$$C_x[k] = \frac{1}{N_0}\sum_{n=N} x[n]e^{-jkw_0 n} \quad , analysis\ eq. to\ break\ x[n]\ into\ complex\ \sin components$$

$$= \frac{1}{7}\sum_{N=0}^{7-1} x(n)e^{-j\frac{2\pi kn}{7}}$$

$$= \frac{1}{7}\left(0e^{-j\frac{2\pi k0}{7}} + 1e^{-j\frac{2\pi k1}{7}} + 2e^{-j\frac{2\pi k2}{7}} + 0e^{-j\frac{2\pi k3}{7}} + 0e^{-j\frac{2\pi k4}{7}} - 2e^{-j\frac{2\pi k5}{7}} - 1e^{-j\frac{2\pi k6}{7}}\right)$$

$$= \frac{1}{7}(e^{-jwk} + 2e^{-jwk2} - 2e^{-jwk5} - e^{-jwk6}) \ldots, on\ next\ page$$

$$C_k[k] = \frac{1}{7}(e^{-jw_0k} + 2e^{-jw_02k} - 2e^{-jw_05k} - e^{-jw_06k})$$

$$C_k[0] = \frac{1}{7}(e^{-jw_00} + 2e^{-jw_00} - 2e^{-jw_00} - e^{-jw_00})$$

$$= \frac{1}{7}(1 + 2 - 2 - 1)$$

$$= \mathbf{0}$$

$$C_k[1] = \frac{1}{7}(e^{-jw_01} + 2e^{-j2w_01} - 2e^{-j5w_01} - e^{-j6w_01})$$

$$= \frac{1}{7}\big((0.6235 - 0.7818i) + (-0.4450 - 1.9499i) - (-0.4450 - 1.9499i) - (-0.6235 - 0.7818i)\big)$$

$$= \mathbf{0.0000 - 0.7805}\boldsymbol{i}$$

$$C_k[2] = \frac{1}{7}(e^{-jw_02} + 2e^{-j2w_02} - 2e^{-j5w_02} - e^{-j6w_02})$$

$$= (0.6235 - 0.7818i) + (-0.4450 - 1.9499i) - (0.4450 - 1.9499i) - (-0.6235 - 0.7818i)$$

$$= \mathbf{-0.0000 + 0.7805}\boldsymbol{i}$$

$$C_k[3] = trivial\ steps$$

$$= \mathbf{-0.0000 + 0.3228}\boldsymbol{i}$$

$$C_k[4] = trivial\ steps$$

$$= \mathbf{-0.0000 + 0.7805}\boldsymbol{i}$$

$$C_k[5] = trivial\ steps$$

$$= \mathbf{-0.0000 + 0.0306}\boldsymbol{i}$$

$$C_k[6] = trivial\ steps$$

$$= \mathbf{-0.0000 + 0.7805}\boldsymbol{i}$$

b. Show that $c_x[-k] = -c_x[k]$ for the coefficients that you calculated. What symmetry do the coefficients have as a result of this relationship? (5 points)

$$c_k = \frac{1}{7}\left(e^{-j\frac{2\pi kn}{7}} + 2e^{-j\frac{2\pi kn}{7}} - 2e^{-j\frac{2\pi n}{7}} - e^{-j\frac{2\pi n}{7}}\right)$$

$$= \frac{1}{7}\left(\frac{-2j}{2j}\left(e^{j\frac{2\pi k}{7}} + 2e^{j\frac{2\pi k}{7}} - 2e^{j\frac{2\pi k}{7}} - e^{j\frac{2\pi k}{7}}\right)\right)$$

$$= \frac{1}{7}\left(-2j\left(\frac{e^{j\frac{2\pi k}{7}} - e^{-j\frac{2\pi k}{7}}}{2j}\right) + 2\left(2j\left(\frac{e^{j\frac{2\pi k}{7}} - e^{-j\frac{2\pi k}{7}}}{2j}\right)\right)\right)$$

$$= -\frac{2j}{7}(sin(kw) + 2\sin(k2w))$$

$$c_{-k} = -\frac{2j}{7}(sin(-kw) + 2\sin(-k2w))$$

$$= \frac{2j}{7}(sin(kw) + 2\sin(k2w))$$

$$\therefore$$

$$c_x[-k] = -c_x[k]$$

Or

Using the negative values in the original function we see.

$$C_x[-k] = C_x[k]$$

$$= \frac{1}{7}(e^{-jw_0k} + 2e^{-jw_02k} - 2e^{-jw_05k} - e^{-jw_06k})$$

$$= \frac{2j}{7}(sin(kw) + 2\sin(k2w))$$

$$C_x[-1] = C_x[1]$$

$$= \frac{1}{7}(e^{-jw_0k} + 2e^{-jw_02k} - 2e^{-jw_05k} - e^{-jw_06k})$$

$$= 0.0000 + 0.7805i$$

$$C_x[-2] = C_x[2] = 0.0000 + 0.7805i$$

$$C_k[-3] = C_x[3] = trivial\ steps$$

$$= \boldsymbol{-0.0000 + 0.3228i}$$

$$C_k[-4] = C_x[4] = trivial\ steps$$

$$= \boldsymbol{-0.0000 + 0.7805i}$$

$$C_k[-5] = C_x[5] = trivial\ steps$$

$$= \boldsymbol{-0.0000 + 0.0306i}$$

$$C_k[-6] = C_x[6] = trivial\ steps$$

$$= \boldsymbol{-0.0000 + 0.7805i}$$

c. Use MATLAB to show that the Fourier series representation of x[n] has the expected values for one period of x[n]. Include your MATLAB code in your answer. (14 points)

We find the process in lecture 15:

Reverse process:

Use coefficients for $x[n] = \sum_{k=0}^{6} c_x[k]e^{jwkn}$

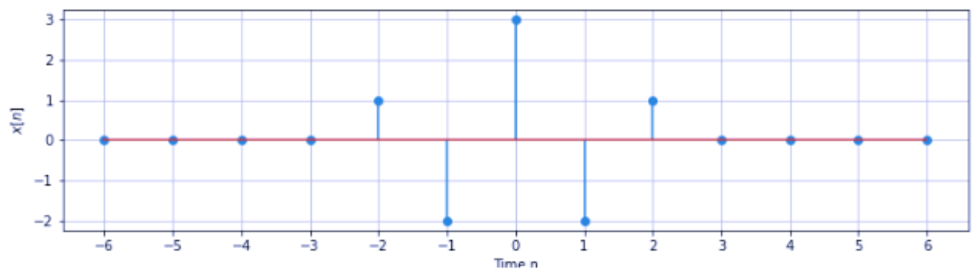
```
clc
clear
close all
c = [0 -.78*1i -.031*1i .32*1i -.32*1i .031*1i .78*1i];
N = 7;
w = (2*pi/N);
n = (0:1:N-1);
x = zeros(1,7);

for i = 0:N-1
    hold = 0;
    for k = 0:N-1
        hold = hold + (c(k+1)*exp(1i*w*k*n(i+1)));
    end
    x(i+1) = hold;
end
```

Command window

```
>> x

x =

   0.0000 + 0.0000i   1.0024 - 0.0000i   1.9944 - 0.0000i   0.0044 + 0.0000i…

  -0.0044 + 0.0000i  -1.9944 + 0.0000i  -1.0024 + 0.0000i

>>
```

Problem 2

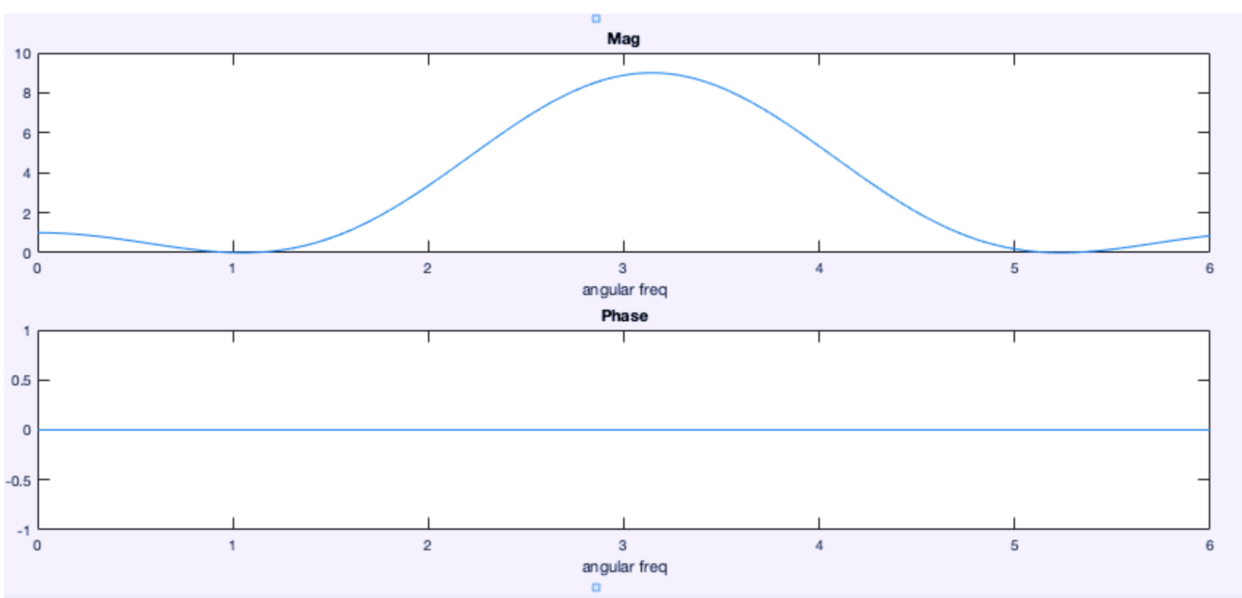
Consider the discrete-time signal x[n] shown below, where x[n] = 0 for n ≤ −3 and for n ≥ 3.



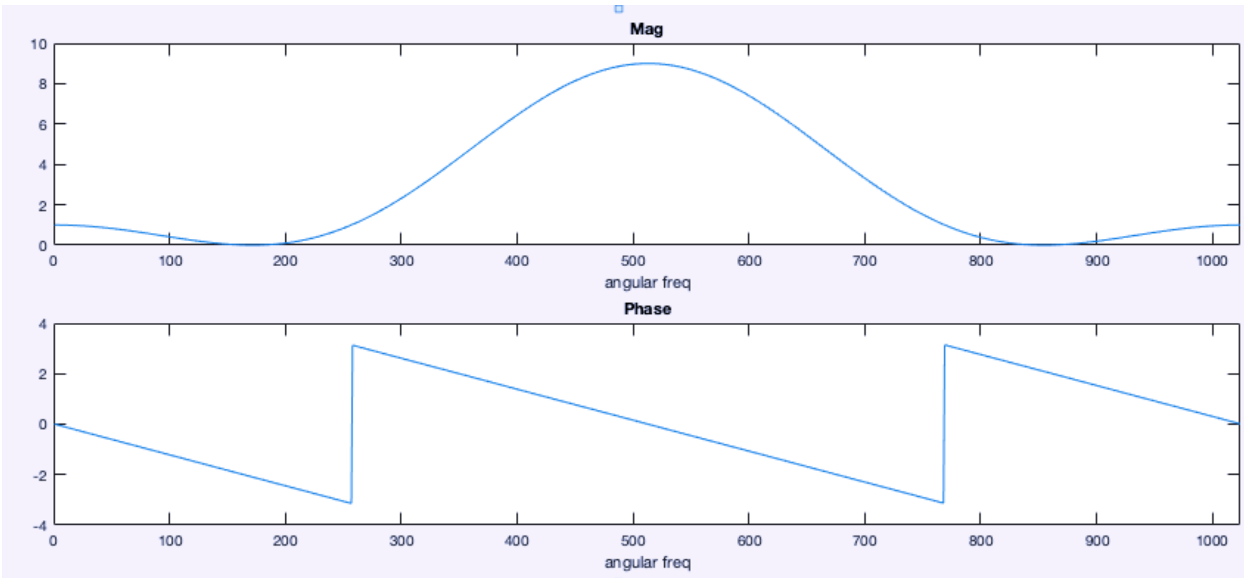
a. Find the DTFT for the signal x[n] by hand. (8 points)

$$X(F) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$$
$$= e^{-j2\pi(-2)f} - 2e^{-j2\pi(-1)f} + 3e^{-j2\pi(0)f} - 2e^{-j2\pi(1)f} + e^{-j2\pi(2)f}$$
$$= e^{j4\pi f} - 2e^{j2\pi f} + 3 - 2e^{-j2\pi f} + e^{-j4\pi f}$$

b. In this part, you'll be comparing the DTFT that you calculated and the DFT calculated using MATLAB.
   i. Use MATLAB to plot the magnitude and phase of the DTFT of x[n] that you calculated in Part (a) as functions of angular frequency Ω. (6 points)



ii. Next, create a MATLAB vector of length 1024 that starts with the nonzero values of x[n] and is otherwise 0. (This is called zero-padding and is often done when working with short signals. In this case, you're just adding values of x[n] at times n when x[n] = 0.) Use the MATLAB function fft to calculate the DFT of this signal and plot the magnitude and phase of the DFT as functions of angular frequency Ω. (10 points)



iii. How do the magnitudes and phases of the DTFT and the DFT compare to each other? If they are different, use the properties of the DTFT or the properties of the DFT to explain why. (5 points)

Their magnitudes are the **same**.

c. Show that the inverse DTFT produces the expected values of x[n] for all integers n. This calculation should be done by hand. (16 points)

$$X(e^{j\Omega}) = e^{j2\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}$$

Inversely:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})\, e^{j\Omega n} d\Omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$x[-2] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n+2)} d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\Omega$$

$$= \frac{2\pi}{2\pi}$$

$$= 1$$

$$n \neq -2, \qquad = \frac{2}{n+2}\sin\left(\pi(n+2)\right) = 0$$

$$x[-1] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(-1+2)} d\Omega$$

$$= -2$$

$$= \lim_{n \to -1} n \neq -1\,, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n+2)} d\Omega = 0$$

Same steps:

$$x[0] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$x[0] = \begin{cases} 3, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$x[1] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$x[1] = \begin{cases} -2, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

$$x[2] = \frac{1}{2\pi}\int_{2\pi} \left(e^{2j\Omega} - 2e^{j\Omega} + 3 - 2e^{-j\Omega} + e^{-j2\Omega}\right) e^{j\Omega n} d\Omega$$

$$x[2] = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}$$