Solution to exercises - Modelagem

Exercise 1 Let us denote

(1) x_{ij}: number of units produced by factory U_i for client E_j $i = 1, 2$ and $j = 1, 2, 3$;
(2) c_{ij}: cost of sending 1 unit from factory U_i for client E_j $i = 1, 2$ and $j = 1, 2, 3$ (see the table);

According to the statement of the problem we have the following set of restrictions:

(1) demands of the clients should be fulfilled.
    (a) $x_{11} + x_{21} = 100$;
    (b) $x_{21} + x_{22} = 200$;
    (c) $x_{31} + x_{32} = 300$.
(2) the stokes of the factories must be considered:
    (a) $x_{11} + x_{12} + x_{13} \leq 400$;
    (b) $x_{21} + x_{22} + x_{23} \leq 300$.

The total cost is

$$\sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}x_{ij} = x_{11} + 1.5x_{12} + 3.5x_{13} + 2x_{21} + x_{22} + 2x_{23}.$$

In summary the linear problem to consider is

$$\begin{aligned}
\min \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}x_{ij} &= x_{11} + 1.5x_{12} + 3.5x_{13} + 2x_{21} + x_{22} + 2x_{23}\\
\text{s.t} \quad & x_{i1} + x_{21} = 100\\
& x_{i2} + x_{22} = 200;\\
& x_{i3} + x_{23} = 300;\\
& x_{11} + x_{12} + x_{13} \leq 400;\\
& x_{21} + x_{22} + x_{23} \leq 300;\\
& x_j \in \mathbb{Z}_+ \ i = 1, 2, \ j = 1, 2, 3.
\end{aligned}$$

Exercise 2 Let us denote: p_1 the produced amount of product 1 and p_2 the produced amount of product 2; x_{11} amount of nitrates used for the production of product 1; x_{12} amount of nitrates used for the production of product 2; x_{21} amount of potasio salt used for the production of product 1; x_{22} amount of potasio salt used for the production of product 2; x_{32} amount of phosphates used for the production of product 2 (all non-negative variables that will be measured in kilograms).

According to the statement of the problem we have the following set of restrictions:

(1) the stocks of the fertilizers should be considered
    (a) $x_{11} + x_{12} \leq 8$;
    (b) $x_{21} + x_{22} \leq 19$;
    (c) $x_{32} \leq 4$.
(2) The required combination of fertilizers to obtain 1 unit of each product is predetermined. Hence we can model the situations as follows: let
    (a) $2x_{11} = x_{21} = p_1 \longrightarrow$ to obtain p_1 units of product 1;
    (b) $3x_{12} = x_{22} = 3x_{32} = p_2 \longrightarrow$ to obtain p_2 units of product 2.

Hence, for a produced amount of p_1 kg of product 1 and p_2 kg of product 2, the profit obtained by selling the production is given by

$$7p_1 + 9p_2 = 7x_{11} + 9x_{22}.$$

In summary, the LPP to be considered in the first question is:

$$\begin{aligned}
&\max\ 7x_{21} + 9x_{22} \\
&\text{s.t} \quad \frac{1}{2}x_{21} + \frac{1}{3}x_{22} \leq 8; \\
&\qquad x_{21} + x_{22} \leq 19; \\
&\qquad \frac{1}{3}x_{22} \leq 4; \\
&\qquad x_{21},\ x_{22} \geq 0.
\end{aligned}$$

Clearly the problem is feasible and bounded above, then it has an optimal solution (x^*_{21}, x^*_{22}) The dual of this problem is

$$\begin{aligned}
&\min\ 8\lambda_1 + 19\lambda_2 + 4\lambda_3 \\
&\text{s.t} \quad \frac{1}{2}\lambda_1 + \lambda_2 \geq 7; \\
&\qquad \frac{1}{3}\lambda_1 + \lambda_2 + \frac{1}{3}\lambda_3 \geq 9; \\
&\qquad \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0.
\end{aligned}$$

For the second problem, let us denote by p_1 the selling price for the nitrate, p_2 the selling price for the phosphates and p_3 the selling price for the salt of potassio.

Exercise 3 Let us denote: x_{ij} is the number of engines delivered from the factory in city V_i to the factory in city U_j, and by c_{ij} the cost of transporting 1 engine from the factory in city V_i to the factory in city U_j (see the table), $i = 1, 2$ and $j = 1, 2, 3$;

According to the statement of the problem we have the following set of restrictions:

(1) the stocks of engines in the factories of cities $V_!$, V_2 and V_3 should be considered:
   (a) $\sum_{j=1}^{3} x_{ij} \leq 6\ i = 1, 2$;

(2) the demands of the factories in cities U_1, U_2 and U_3 should be fulfilled:
   (a) $x_{11} + x_{21} \geq 5$;
   (b) $x_{12} + x_{22} \geq 4$;
   (c) $x_{13} + x_{23} \geq 3$;

The total cost of delivering the engines is $\sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij}x_{ij}$

In summary, the LPP to be considered is:

$$
\begin{aligned}
\min \ & \sum_{i=}^{2}\sum_{j=1}^{3} c_{ij}x_{ij};\\
\text{s.t} \ & \sum_{j=1}^{3} x_{1j} \leq 6;\\
& \sum_{j=1}^{3} x_{2j} \leq 6;\\
& x_{11} + x_{21} \geq 5;\\
& x_{12} + x_{22} \geq 4;\\
& x_{13} + x_{23} \geq 3;\\
& x_{ij} \in \mathbb{Z}_+ \ i = 1, 2, \ j = 1, 2, 3.
\end{aligned}
$$

Exercise 4 Let us denote: s the number of shoes and c the number of belts produced in one day.

According to the statement of the problem we have the following set of restrictions:

(1) the upper bound on the number of hours dedicated to work should be considered:
   (a) $s/6 + c/5 \leq 10$

(2) the upper bound on the number of units of leather used in making the products (shoes and belts) should be considered:
   (a) $2s + c \leq 78$

The total income due to selling the products is $5s + 4c$

In summary, the LPP to be considered is:

$$
\begin{aligned}
\max \ & 5s + 4c.\\
\text{s.t} \ & s/6 + c/5 \leq 10\\
& 2s + c \leq 78\\
& s, c \in \mathbb{Z},
\end{aligned}
$$

Exercise 5 Let us denote the diesels of type A and B by diesel 1 and 2 and by g_1 and g_2 the amounts of gasoline used in diesels 1 and 2, respectively; let x_{ij} be the amount of gasoline i used to produce diesel j $i = 1, \ldots, 3$, $j = 1, 2$.

According to the statement of the problem we have the following set of restrictions:

(1) the stocks of the three types of gasoline should be considered:
   (a) $x_{11} + x_{12} \leq 500$;
   (b) $x_{21} + x_{22} \leq 200$;
   (c) $x_{31} + x_{32} \leq 200$;

(2) the amounts of each gasoline on each diesel should be considered:
   (a) $x_{11} = .25g_1$;
   (b) $x_{21} = .25g_1$;
   (c) $x_{31} = 0.5g_1$;
   (d) $x_{22} = 0.5g_2$;
   (e) $x_{32} = 0.5g_2$;

In summary, the LPP to be considered is:

$$\begin{aligned} \max\ & 20g_1 + 30g_2; \\ \text{s.t}\quad & .25g_1 \le 500; \\ & .25g_1 + 0.5g_2 \le 2000; \\ & 0.5g_1 + 0.5g_2 \le 4000; \\ & g_1, g_2 \ge 0. \end{aligned}$$

Exercise 6 Let us denote diesels of type A, B and C by diesel 1,2 and 3, respectively; x_{ij} the amount of oil i used to produce diesel j $i = 1, \ldots, 4$, $j = 1, \ldots, 3$; and d_i the amount of oil j produced $j = 1, \ldots, 3$.

According to the statement of the problem we have the following set of restrictions:

(1) the stocks of the four type of oils should be considered:
   (a) $x_{21} + x_{22} \le 2000$;
   (b) $x_{31} + x_{32} \le 4000$;
   (c) $x_{41} + x_{42} \le 1000$;

(2) the limits on the amount of each oil on each diesel should be considered:
   (a) $x_{11} \le 0.3d_1$;
   (b) $x_{31} \le 0.5d_1$;
   (c) $x_{21} \ge 0.4d_2$;
   (d) $x_{22} \ge 0.1d_2$;
   (e) $x_{12} \le 0.5d_2$;
   (f) $x_{13} \le 0.7d_3$;

Other facts: the cost of the production process is $3(x_{11} + x_{12}) + 6(x_{21} + x_{22}) + 4(x_{31} + x_{32}) + 5(x_{41} + x_{42})$, and the total income due to the sale of the diesels is $5.5d_1 + 4.5d_2$

In summary, the LPP to be considered is:

$$\begin{aligned} \max\ & 5.5d_1 + 4.5d_2 - 3(x_{11} + x_{12}) - 6(x_{21} + x_{22}) - 4(x_{31} + x_{32}) - 5(x_{41} + x_{42}) \\ \text{s.t}\quad & x_{13} + x_{11} + x_{12} \le 3000; \\ & x_{21} + x_{22} \le 2000; \\ & x_{31} + x_{32} \le 4000; \\ & x_{41} + x_{42} \le 1000; \\ & x_{11} \le 0.3d_1; \\ & x_{31} \le 0.5d_1; \\ & x_{21} \le 0.4d_2; \\ & x_{31} \le 0.7d_3; \\ & x_{22} \ge 0.1d_2; \\ & x_{12} \ge 0.5d_2; \\ & x_{ij} \ge 0,\ d_1, d_2 \ge 0. \end{aligned}$$

Clearly we can eliminate variables x_{41} and x_{42} in the above formulation