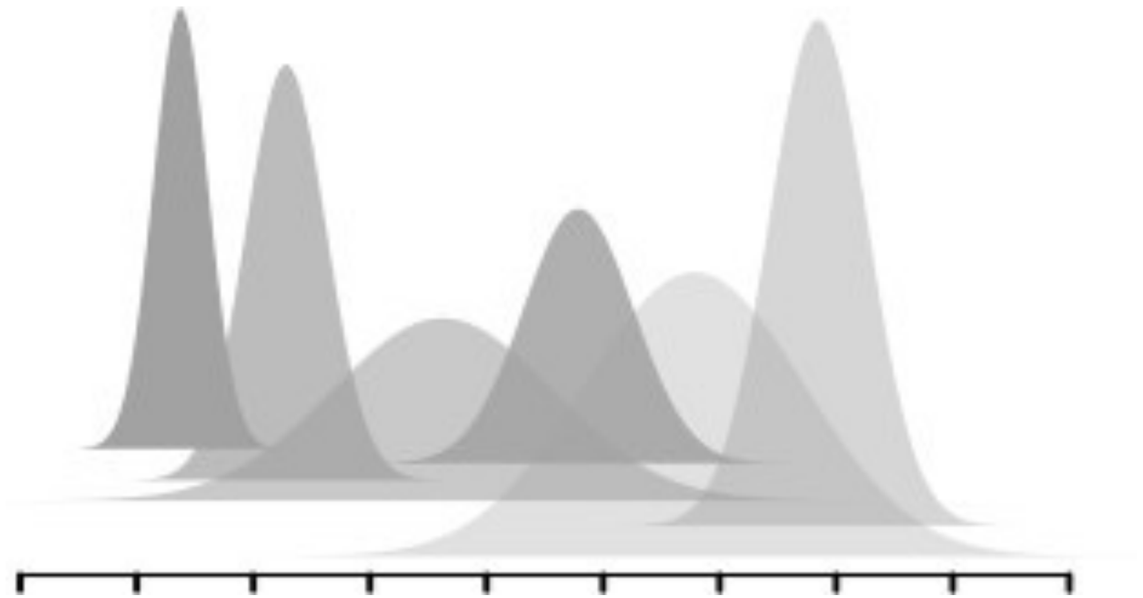
2.1 Population & Community Ecological Models



Jelena H. Pantel

Faculty of Biology

University of Duisburg-Essen

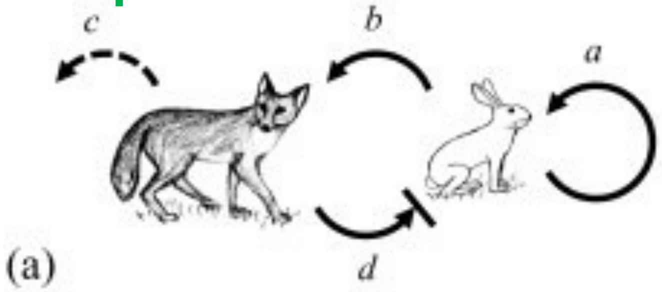
jelena.pantel@uni-due.de

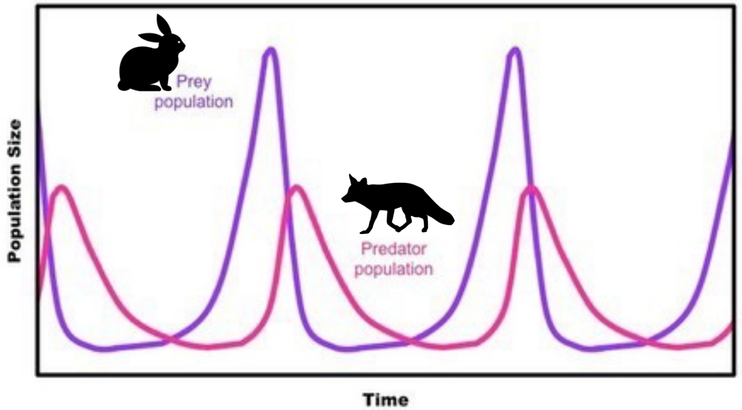
What is a model?

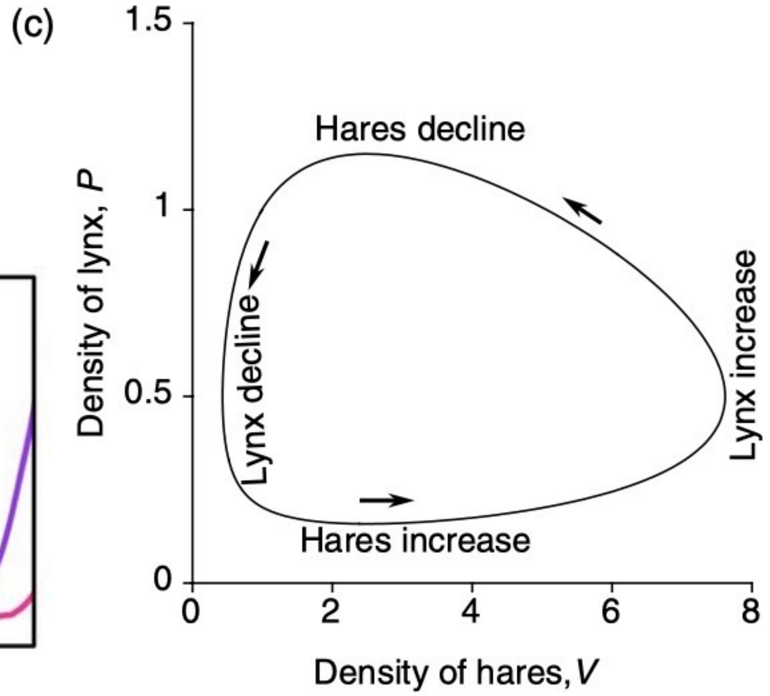
"A *model* is a representation of a particular thing, idea, or condition."

"The *modeling process* is the series of steps taken to convert an idea first into a conceptual model and then into a quantitative model"

Step 1. Formulate a conceptual model







Step 2. Formulate a quantitative model

$$\frac{\mathrm{d}V}{\mathrm{d}t} = aV - bVP$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -cP + dVP$$

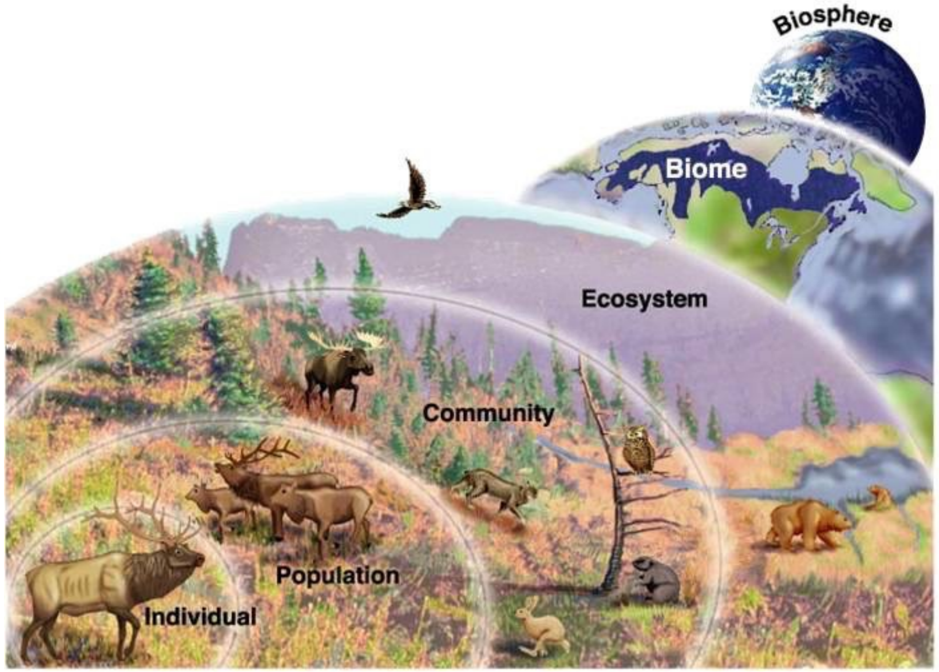
Step 3. Learn about study system through analysis of model behavior

Kinoshita, 2013, Intro to Nonequilibrium Phenomena

What is ecology?

The study of interactions between organisms and their environment, and with one another

The science that investigates the abundance and distribution of organisms



Step 3. Learn about study system through analysis of model behavior

Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

Graphical techniques: develop a feeling for your model

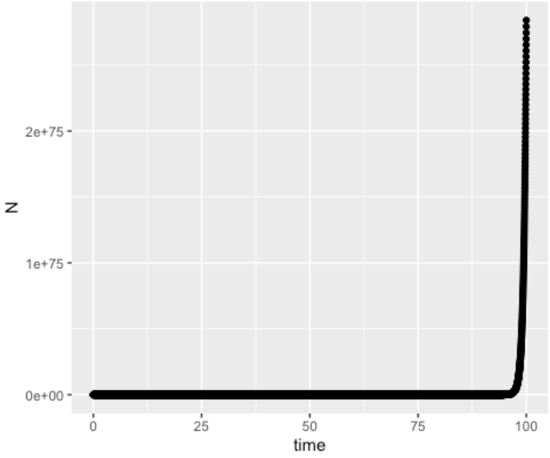
Expected dynamics → depends on r and n_0 (initial population size)

When r > 0?

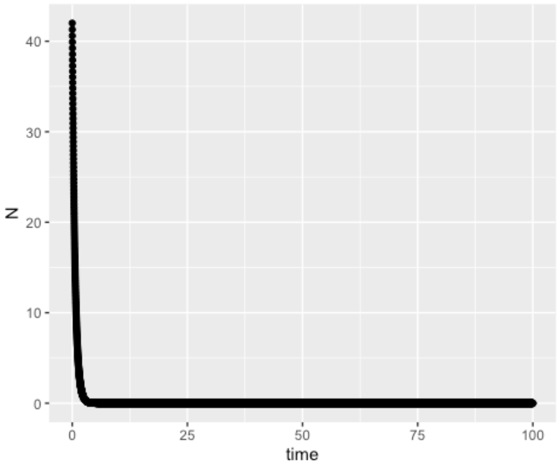
When r < 0?

When r = 0?

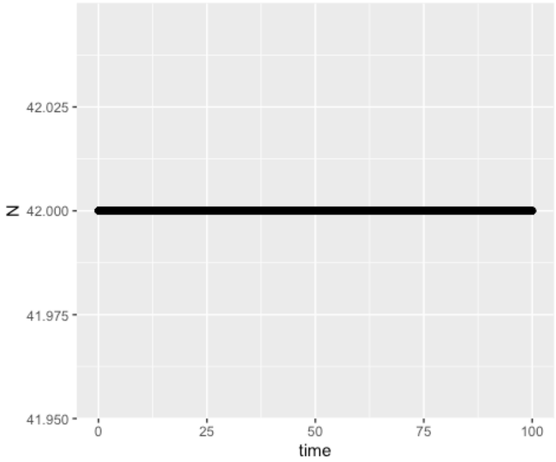
r = 1.7



r = -1.7



r = 0



parameter	
r	population growth rate

Step 3. Learn about study system through analysis of model behavior

Population ecology

Exponential growth

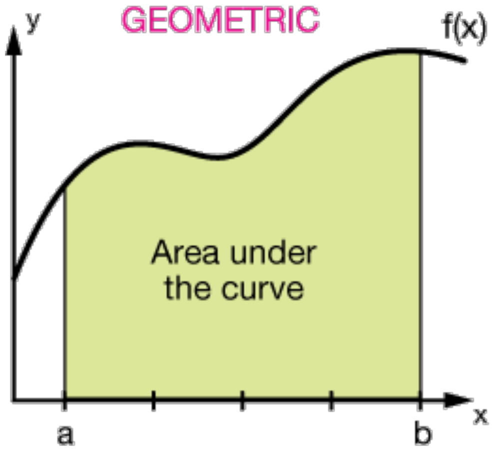
$$\frac{dn}{dt} = rn$$

Graphical techniques: develop a feeling for your model

How did I develop a graph of population size over time for this equation?

- I solved for values of n(t) at different points in time.
- Let's solve the equation to get a formula for this: we'll need to integrate the formula

$$\frac{dn}{dt} = rn \rightarrow \frac{dn}{n} = r dt \rightarrow \int_{n_0}^{n(t)} \frac{dn}{n} = r\int_0^t dt \rightarrow ln\frac{n(t)}{n_0} = rt \rightarrow n(t) = n_0 e^{rt}$$



ANALYTIC

$$A = \int_a^b f(x)\,dx$$

The definite integral of f(x) between x=a & x=b

Table of Integrals

BASIC FORMS

(1) $\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1}}$

(2) $\boxed{\int \frac{1}{x} dx = \ln x}$

(3) $\int u dv = uv - \int v du$

(4) $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

RATIONAL FUNCTIONS

(5) $\int \frac{1}{ax+b} dx = \frac{1}{a}\ln(ax+b)$

(6) $\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$

INTEGRALS WITH ROOTS

(18) $\int \sqrt{x-a}\,dx = \frac{2}{3}(x-a)^{3/2}$

(19) $\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a}$

(20) $\int \frac{1}{\sqrt{a-x}} dx = 2\sqrt{a-x}$

(21) $\int x\sqrt{x-a}\,dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$

(22) $\int \sqrt{ax+b}\,dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{b+ax}$

(23) $\int (ax+b)^{3/2} dx = \sqrt{b+ax}\left(\frac{2b^2}{5a} + \frac{4bx}{5} + \frac{2ax^2}{5}\right)$

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Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

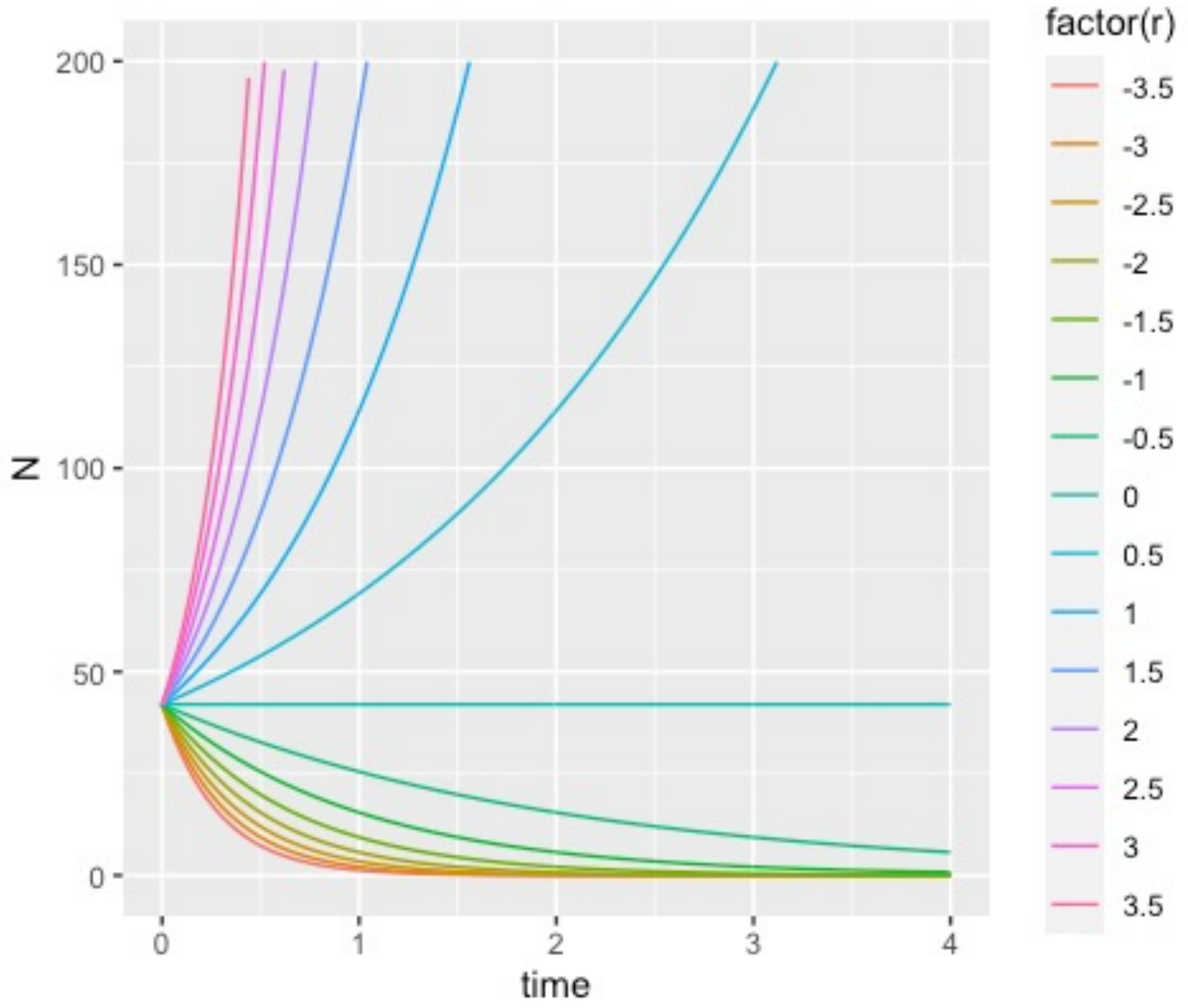
Graphical techniques: develop a feeling for your model

Expected dynamics

When r > 0?

When r < 0?

When r = 0?



parameter	
r	population growth rate

Step 3. Learn about study system through analysis of model behavior

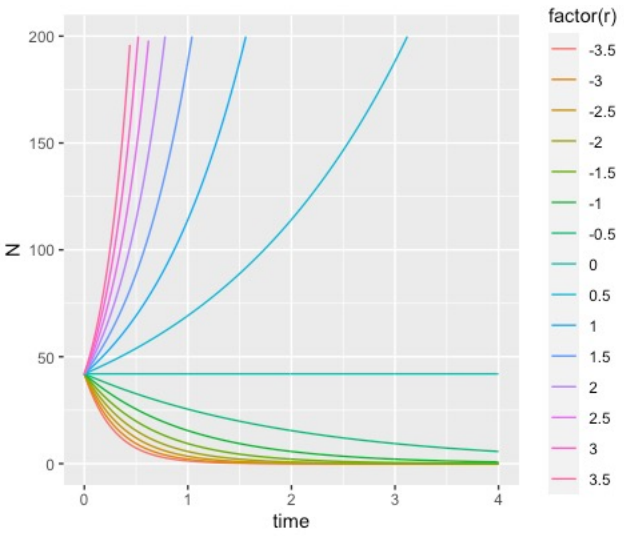
Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

Graphical techniques: develop a feeling for your model

Expected dynamics



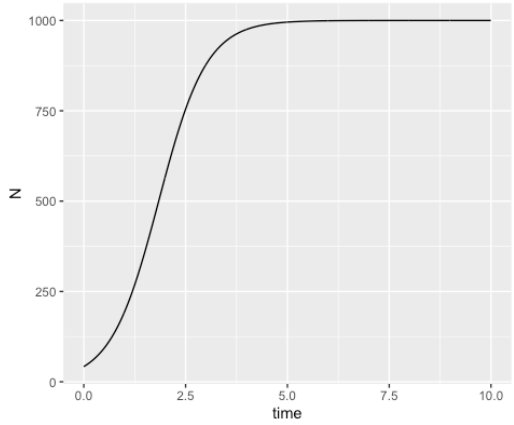
parameter	
r	population growth rate

Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Expected dynamics

r = 1.7



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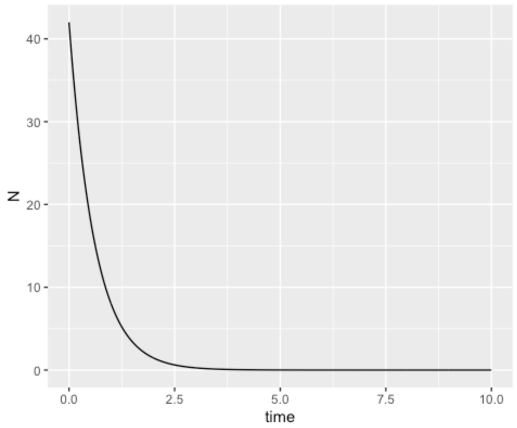
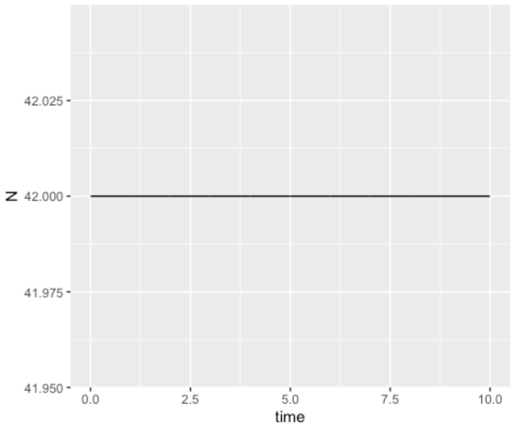


r = 0



parameter	
r	population growth rate
K	Carrying capacity

Step 3. Learn about study system through analysis of model behavior

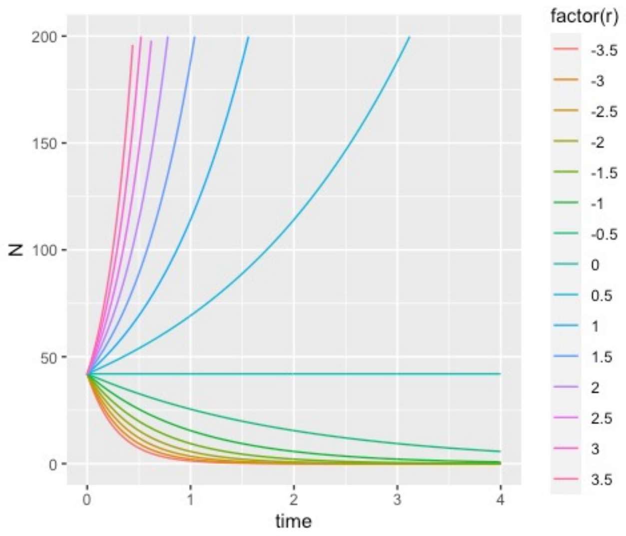
Population ecology

Exponential growth

$$\frac{dn}{dt} = rn \longrightarrow n(t) = n_0 e^{rt}$$

Graphical techniques: develop a feeling for your model

Expected dynamics

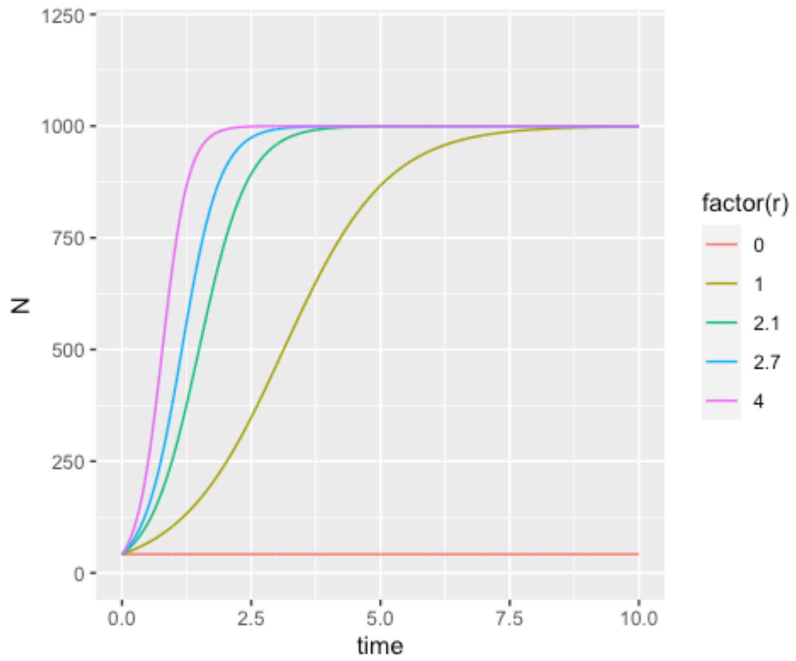


parameter	
r	population growth rate

Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) \longrightarrow n(t) = \frac{K}{1 + n_0 e^{-rt}}$$

Expected dynamics



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Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Graphical techniques: develop a feeling for your model

Equilibria and stability analyses

Equilibrium

- a system at *equilibrium* does not change over time
- A particular value of a variable is called an *equilibrium value* if, when the variable is *started* at this value, the system never changes
- At equilibrium in a continuous-time model, dn/dt must equal 0 for each variable

How to solve for equilibrium values

- Replace desired value with «the equilibrium value»

$$\frac{dn}{dt} = rn \longrightarrow \frac{dn}{dt} = rn^*$$

- We would like to know n*, «the value of n at which the population size is no longer changing»

$$\frac{dn}{dt} = 0 \longrightarrow 0 = rn^*$$

- Solve for the equilibrium value of n (n*)

$$0 = rn^* \longrightarrow \frac{0}{r} = \frac{n^*}{r} \longrightarrow 0 = n^*$$

- The system is at equilibrium (n is not changing, dn/dt = 0) when n* = 0

Step 3. Learn about study system through analysis of model behavior

Population ecology

Exponential growth

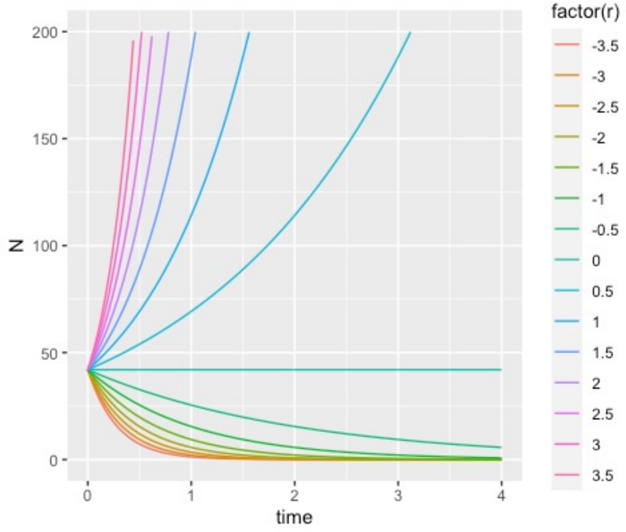
$$\frac{dn}{dt} = rn$$

Graphical techniques: develop a feeling for your model

Equilibria and stability analyses

Equilibrium

$$\frac{dn}{dt} = 0 \longrightarrow n_t^* = 0$$



Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

$$0 = rn\left(1 - \frac{n}{K}\right) \longrightarrow n^* = 0$$

$$\left(1 - \frac{n^*}{K}\right) = 0 \longrightarrow 1 = \frac{n^*}{K} \longrightarrow K = n^*$$

Step 3. Learn about study system through analysis of model behavior

Population ecology

Exponential growth

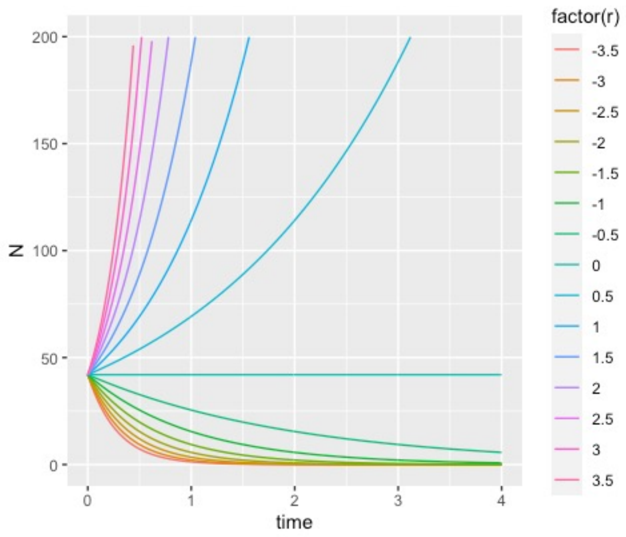
$$\frac{dn}{dt} = rn$$

Graphical techniques: develop a feeling for your model

Equilibria and stability analyses

Equilibrium

$$\frac{dn}{dt} = 0 \longrightarrow n_t^* = 0$$



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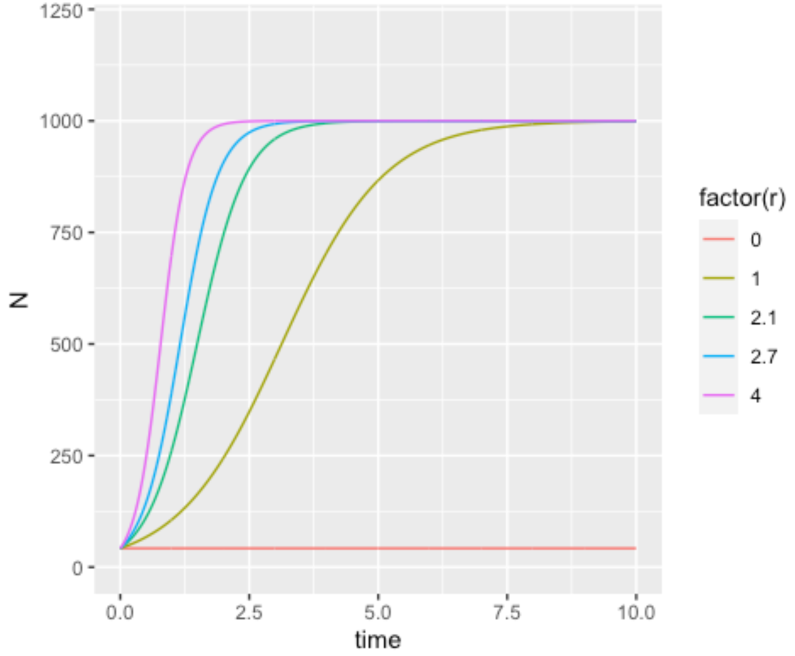
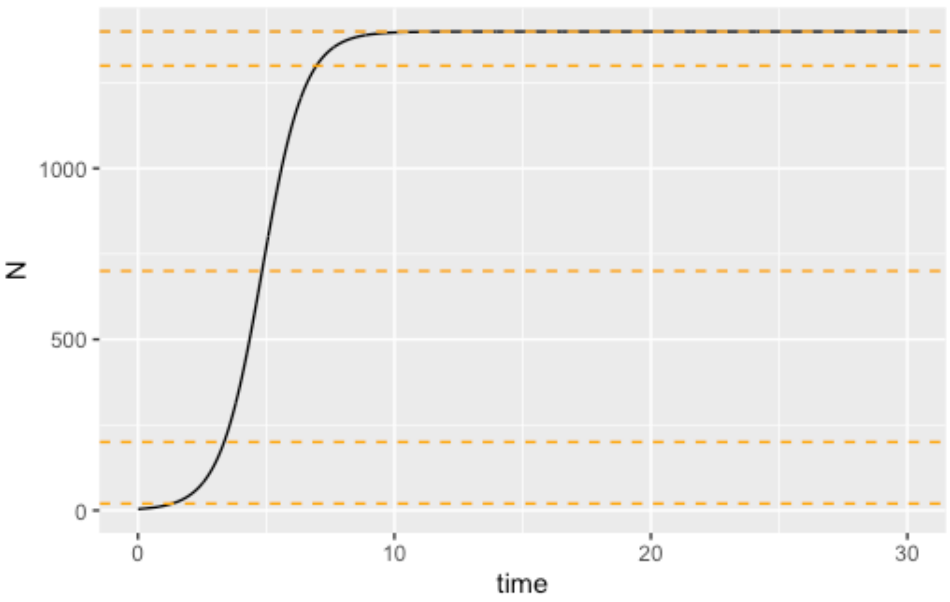


Step 3. Learn about study system through analysis of model behavior

Population ecology

Logistic growth

$$\frac{dn}{dt} = rn_t\left(1 - \frac{n_t}{K}\right)$$

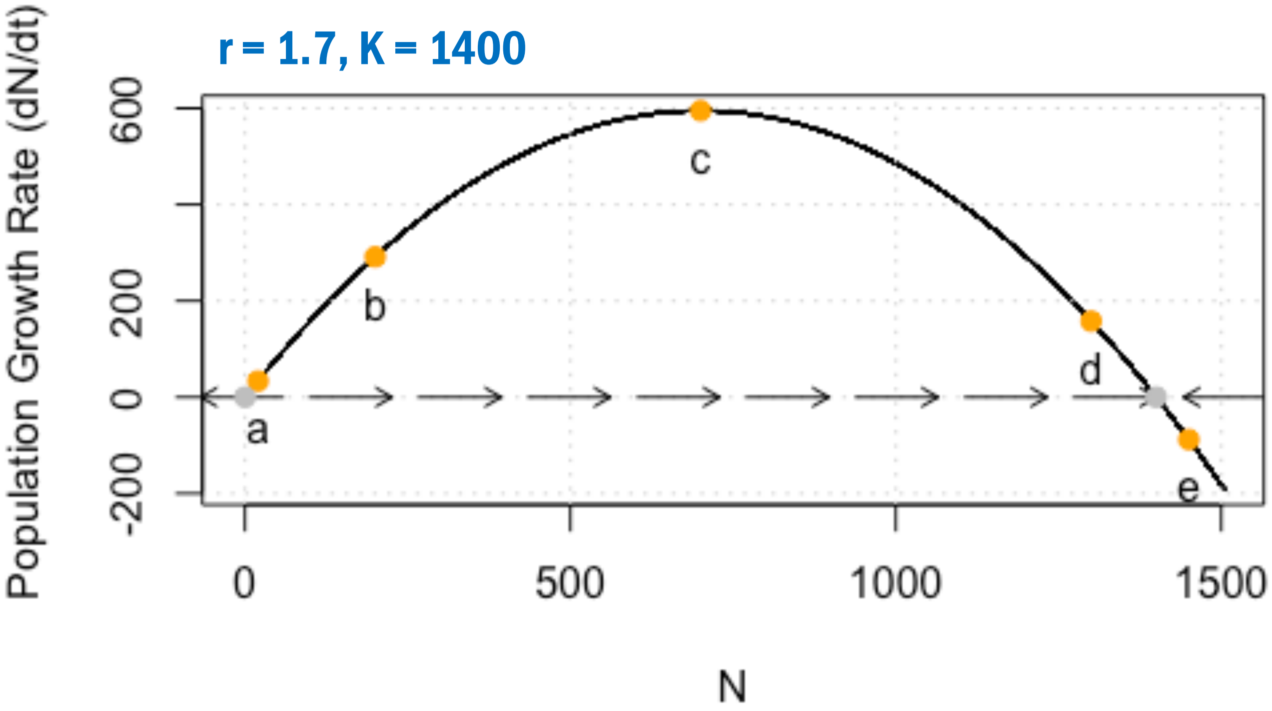


Graphical techniques: develop a feeling for your model
Equilibria, Stability, Phase plane diagram

How does the growth rate of the system (dn/dt) change for different values of *n*?

- Note the equilibria N^* values (N=0,K) – where dn/dt = 0
- Note the changes in the growth rate of the system (dn/dt)
  - a (N=20), b (n=200), c (n=700), d (n = 1350)
- Note the direction of the arrows along the axis of N (the state variable) – equilibrium analysis asks – what equilibria does the system move towards when N is moved *away* from one of these
  - This is *perturbation analysis*
  - If N > 0, N=K is the stable equilibrium

r = 1.7, K = 1400



Step 3. Learn about study system through analysis of model behavior

Population ecology

Logistic growth

$$\frac{dn}{dt} = rn_t\left(1 - \frac{n_t}{K}\right)$$

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- Note the direction of the arrows along the axis of N (the state variable) – equilibrium analysis asks – what equilibria does the system move towards when N is moved *away* from one of these
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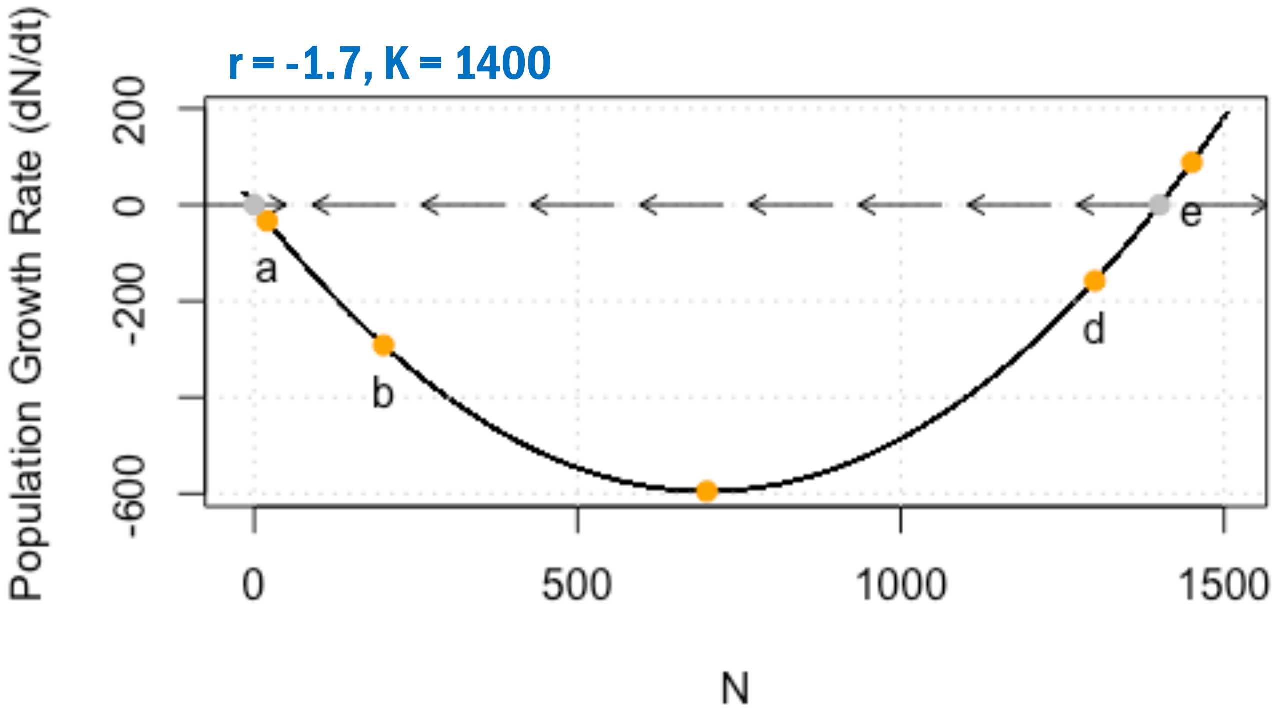
    - If N < K, N=0 is the stable equilibrium



Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Community ecology

Lotka Volterra competition model

$$\frac{dn_i}{dt} = rn_i\left(1 - \frac{n_i + \alpha_{ij}n_j}{K_i}\right) \qquad \frac{dn_j}{dt} = rn_j\left(1 - \frac{n_j + \alpha_{ji}n_i}{K_j}\right)$$

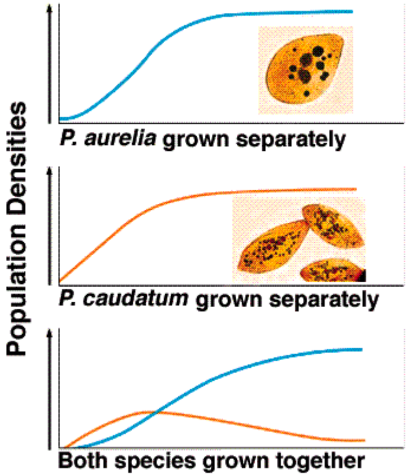
Alfred J. Lotka (1880-1949)
Chemist, ecologist, mathematician
Ukrainian immigrant to the USA

Vito Volterra (1860-1940)
Mathematical Physicist
Italian, refugee of fascist Italy

Testing the consequences of species interactions: Georgii Frantsevich Gause (b. 1910)

Paramecium caudatum

Paramecium aurelia



Population ecology

Exponential growth

$$\frac{dn}{dt} = rn$$

Logistic growth

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right)$$

Community ecology

Lotka Volterra competition model

$$\frac{dn_i}{dt} = rn_i\left(1 - \frac{n_i + \alpha_{ij}n_j}{K_i}\right) \qquad \frac{dn_j}{dt} = rn_j\left(1 - \frac{n_j + \alpha_{ji}n_i}{K_j}\right)$$

Lotka Volterra predator-prey model

$$\frac{dn_i}{dt} = rn_i - acn_in_j \qquad \frac{dn_j}{dt} = \varepsilon acn_in_k - \delta n_j$$

Testing the consequences of species interactions: Georgii Frantsevich Gause (b. 1910)