(forward / backward diff. quotient)

Taylor :

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2}h^2$$

Rearrange :

$$\underbrace{\frac{f(x+h)-f(x)}{h}}_{e(h)} - f'(x) = \frac{f''(\xi)}{2}h$$

$e(h)$ think of this as a function of h that computes an approximation of $f'(x)$.
Then, LHS is the error.

$$f(x-h) = f(x) - f'(x)h + \frac{f''(\hat{\xi})}{2}h^2$$

$$\frac{f(x)-f(x-h)}{h} - f'(x) = -\frac{f''(\hat{\xi})}{2}h$$

(centered difference quotient)

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(\tilde{\xi})}{6}h^3$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(\hat{\xi})}{6}h^3$$

subtract and rearrange

$$\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \frac{f'''(\tilde{\xi})+f'''(\hat{\xi})}{2}\frac{h^2}{6}.$$

$$= \frac{f'''(\xi)}{6}h^2$$

Here, we used a little trick.

lemma : Let g is continuous on $[a,b]$.
Then, for any $x_1, x_2, \cdots, x_n \in [a,b]$ there exists $\xi \in [a,b]$ such that

$$\text{mean} = \frac{g(x_1)+g(x_2)+\cdots+g(x_n)}{n} = g(\xi) \quad \cdots (*)$$

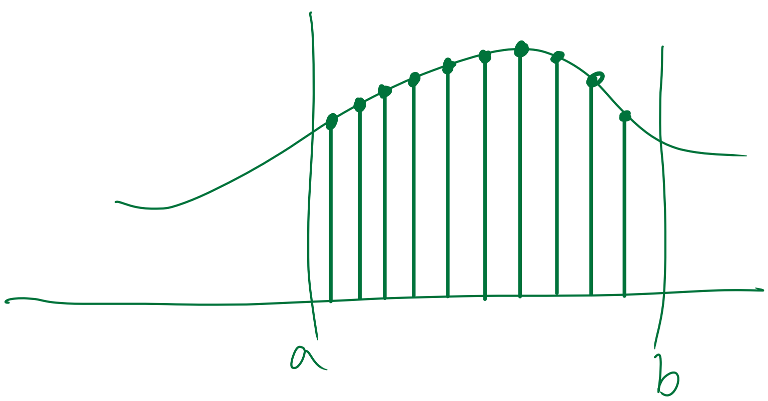
proof) The mean is b/w the maximum and the minimum of g over $[a,b]$. Hence, intermediate value theorem asserts that $\exists\, \xi$ satisfying $(*)$.

comment: This can be viewed as a discrete version of the mean value theorem of integral: If g is continuous,

$$\frac{1}{b-a}\int_a^b g(x)\,dx = g(\xi)$$

for some $\xi \in [a,b]$

The LHS the mean of g using infinitely many samples.

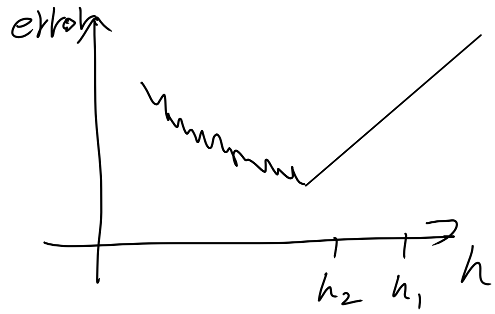


$$\Delta x \approx \frac{b-a}{n} \Rightarrow \frac{1}{n} \approx \frac{\Delta x}{b-a}$$

$$\frac{1}{n}\sum_{i=1}^{n} g(x_i) \approx \frac{1}{b-a}\sum_{i=1}^{n} g(x_i)\Delta x$$

$$\approx \frac{1}{b-a}\int_a^b g(x)\,dx$$

The slope of log-log scale is the rate of convergence.



spacing : $h_1 = 2h_2$ (h_1 : big , h_2 : small)

error : $e_1 \leftrightarrow h_1$, $e_2 \leftrightarrow h_2$

$$0 < \text{slope} = r = \frac{\Delta \log(\text{err})}{\Delta \log(\text{spacing})} = \frac{\log e_1 - \log e_2}{\log h_1 - \log h_2}$$

$$= \frac{\log e_1/e_2}{\log h_1/h_2} = \frac{\log e_1/e_2}{\log 2}$$

$$\log e_1/e_2 = r \log 2$$

$$\frac{e_1}{e_2} = e^{r\log 2} = 2^r$$

$$\frac{e_2}{e_1} = 2^{-r}$$

$e_2 = 2^{-r} e_1 \rightarrow$ If $r=1$, $e_2 = \frac{1}{2} e_1$

half spacing $\rightarrow$ half error

If $r=2$, $e_2 = \frac{1}{4} e_1$

half spacing $\rightarrow$ a fourth error

and so on.

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(x)}{24}h^4 + \cdots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(x)}{24}h^4 + \cdots$$

subtract and rearrange:

$$(*)\quad f'(x) = \varphi(h) - \frac{f'''(x)}{6}h^2 - \frac{f^{(5)}}{5!}h^4 - \cdots$$

$$= \underbrace{\varphi(h)}_{(f(x+h)-f(x-h))/2h} - a_2h^2 - a_4h^4 - \cdots$$

This hold for any small enough $h>0$.

$h \leftarrow h/2$ yields

$$f'(x) = \varphi(h/2) - \frac{a_2}{4}h^2 - \frac{a_4}{16}h^4 - \cdots$$

$$(**)\quad 4f'(x) = 4\varphi(h/2) - a_2h^2 - \frac{a_4}{4}h^4 - \cdots$$

$$(**)-(*):\quad 3f'(x) = 4\varphi(h/2) - \varphi(h) + \frac{3}{4}a_4h^4 + \cdots$$

$$\Rightarrow f'(x) - \underbrace{\frac{1}{3}\left(4\varphi(h/2) - \varphi(h)\right)}_{=:\,\psi(h)} = \frac{3}{4}a_4h^4 + \cdots$$

where

$$\begin{aligned}
\psi(h) &= \frac{1}{3}\left(4\varphi(h/2) - \varphi(h)\right) \\
&= \frac{1}{3}\left(4\,\frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{\cancel{2}\cdot\frac{h}{\cancel{2}}} - \frac{f(x+h)-f(x-h)}{2h}\right) \\
&= \frac{1}{6h}\left(8f(x+\tfrac{h}{2}) - 8f(x-\tfrac{h}{2}) - f(x+h) + f(x-h)\right)
\end{aligned}$$

Since $f(x) \approx p(x)$ on $[a, b]$

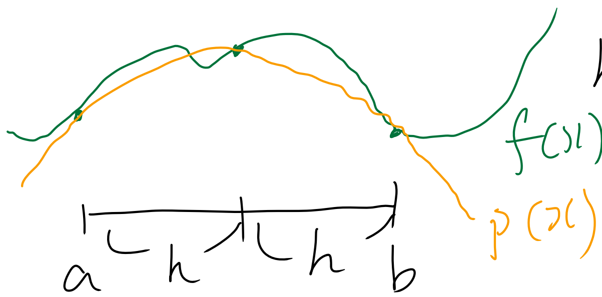
we expect $\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$.

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

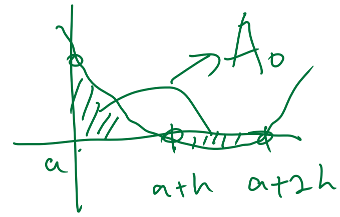
$$\int_a^b p(x)\,dx = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x)\,dx$$

$$= \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)\,dx}_{=:A_i}$$

$$= \sum_{i=0}^{n} f(x_i) A_i$$



$$h = \frac{b-a}{2}$$



$$A_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-a-h)(x-a-2h)}{(\cancel{a}-\cancel{a}-h)(\cancel{a}-\cancel{a}-2h)}\,dx$$

$$= \int_a^{a+2h} \frac{(x-a-h)(x-a-2h)}{2h^2}\,dx$$

$z = x-a-h$

$dz = dx$

$$= \frac{1}{2h^2}\int_{-h}^{h} z(z-h)\,dz$$

$$= \frac{1}{2h^2}\int_{-h}^{h} \underbrace{z^2}_{\text{even}} - \underbrace{\cancel{hz}}_{\to 0}\,dz \quad = 2\int_0^h z^2\,dz$$

$$= \frac{1}{2h^2}\cdot 2\left[\frac{z^3}{3}\right]_0^h$$

$$= \frac{h}{3}$$

Similarly,

$$A_1 = \int_a^{a+2h} \frac{(x-a)(x-a-2h)}{(\cancel{a}+h-\cancel{a})(\cancel{a}+h-\cancel{a}-2h)}\,dx$$

$$= -\frac{1}{h^2}\int_{-h}^{h} (z+h)(z-h)\,dz$$

$x-a-h = z$

$dx = dz$

$$= -\frac{1}{h^2}\int_{-h}^{h} \underbrace{z^2-h^2}_{\text{even}}\, dz \quad = 2\int_0^h z^2-h^2\,dz$$

$$= -\frac{1}{h^2}\cdot 2\left[\underbrace{\frac{z^3}{3}-h^2 z}\right]_0^h \quad = \frac{h^3}{3}-h^3 = -\frac{2}{3}h^3$$

$$= \frac{4}{3}h$$

$$A_2 = \int_a^{a+2h} \frac{(x-a)(x-a-h)}{(\cancel{a}+2h-\cancel{a})(\cancel{a}+2h-\cancel{a}-h)}\, dx$$

$$= \frac{1}{2h^2}\int_a^{a+2h} (x-a)(x-a-h)\, dx \qquad x-a-h=z,\quad dx=dz$$

$$= \frac{1}{2h^2}\int_{-h}^{h} \underbrace{(z+h)z}_{\text{even }z^2+\cancel{hz} \to 0 \text{ (b/c odd)}}\, dx$$

$$= \frac{1}{2h^2}\cdot 2\left[\frac{z^3}{3}\right]_0^h$$

$$= \frac{h}{3}$$

Thus, $\int_a^b f(x)\,dx \approx \sum_{i=0}^{2} f(x_i)A_i$

$$= f(a)\frac{h}{3} + f(a+h)\frac{4h}{3} + f(a+2h)\frac{h}{3}$$

$$= \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right)$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)$$