Announcement

- Final exam info → Canvas front page.

- FAQ : Accommodation on HW, Quiz
  → You already have it : drops

- ESCI : Please Join! (27 %)

- Kenny's mind map has been updated.

- Special OH :
  Jea-Hyun : Mon, Wed 3-4 pm
  Alex : Tue 11 AM – 1 PM

Survey on mind map

How much do you think are they useful?

(A) I spent less than 30 min on them
and they were useful
considering time spent

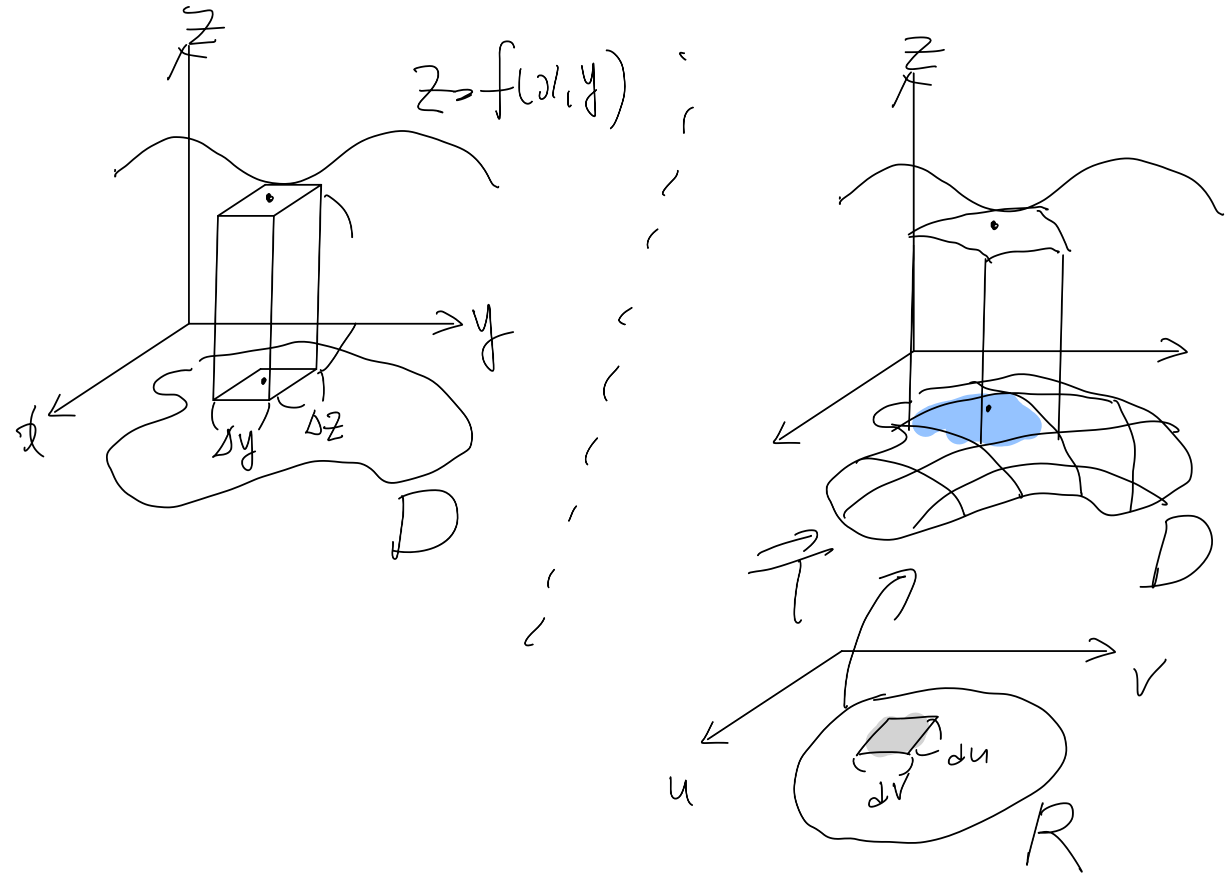
(B) I spent more than 30 min on them
and they were useful
considering time spent

(C) I spent less than 30 min on them
and they were not very useful
considering time spent

(D) I spent more than 30 min on them
and they were not very useful
considering time spent

(E) I never looked at it so I have
no opinion about them.

4.2 Change of variables (Ch. 6.4)



- Want $\iint_D f(x,y)\,dA$

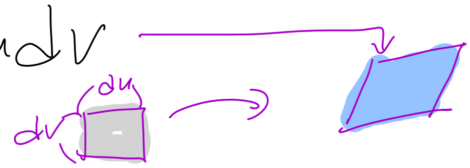
- Introduce smooth one-to-one and onto transformation $\vec{T} : R \to D$ ($R, D \subseteq \mathbb{R}^2$)
  $\vec{T}(u,v) = (x,y) = (x(u,v), y(u,v))$

- But $\iint_R f(\vec{T}(u,v))\,du\,dv$ is not quite the same as $\iint_D f(\vec{T}(u,v))\,du\,dv$

because base area elements are distorted.

* $D\vec{T}(u,v)$ is a linear map, and and the distorted area is

$$|\det(D\vec{T}(u,v))|\, du\,dv$$



* Thus, $\iint_R f(\vec{T}(u,v))\, \underline{|J(\vec{T})|}\, du\,dv$ is the right quantity.

Here, $J(\vec{T}) = \det(D\vec{T}(u,v))$ is called **Jacobian** of $\vec{T}$ at (u,v).

* Jacobian is also denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$ since

$$\det(D\vec{T}) = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right)$$

* Quite often, $|J(\vec{T})|$ is also called Jacobian. As with "directional derivative",

check with people or books that you are communicating with.

⊙ In actual applications, you may need to find an inverse mapping to obtain $\vec{T}$.
(see example)

⊙ More formal and detailed explanation can be found in the textbook (pp. 401–415)

Change of variables (Typical steps)

Given $\iint_D f(x,y)\,dx\,dy$,

① Draw the region D

② Introduce a change of variables
(look at D and $f(x,y)$ for a good one)

ⓐ (direct ver.) $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

or ⓑ (two step ver.) $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ and invert this

③ Write a new integral

ⓐ Find the Jacobian

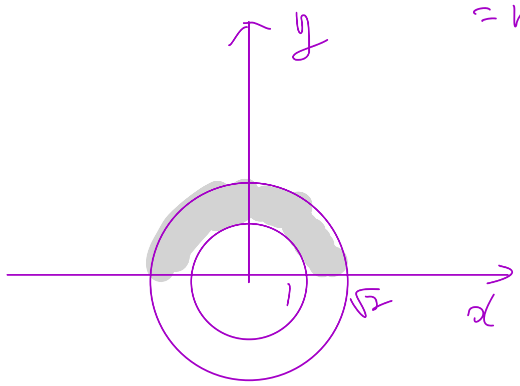
ⓑ Find the new region R.

$$\boxed{\iint_R f(u,v)\,|J(\vec{T})|\,du\,dv}$$

④ Evaluate it.

Example: Evaluate $\iint_D e^{x^2+y^2}\,dA$, where

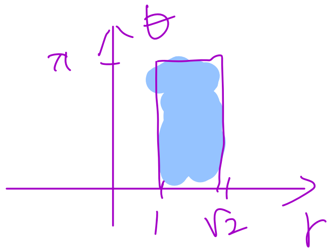
$D=\{(x,y): 1\le \underbrace{x^2+y^2}_{=r^2}\le 2,\ y\ge 0\}$ (p. 413)



Change of variables
(polar coordinate)
/* "x^2+y^2" is a hint on this choice. */

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(or $\vec{T}(r,\theta)=(r\cos\theta, r\sin\theta)$)



$R=\{(r,\theta): 1\le r\le \sqrt{2},\ 0\le\theta\le\pi\}$

$$J(\vec{T}) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta-(-r\sin^2\theta)=r.$$

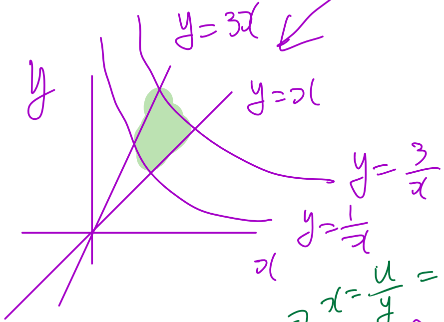
$$\iint_D e^{x^2+y^2}\,dA = \iint_R e^{r^2}\,|J(\vec{T})|\,\underbrace{d\tilde{A}}_{dr\,d\theta}$$

$$= \int_0^\pi\int_1^{\sqrt{2}} e^{r^2}\cdot r\,dr\,d\theta \longleftarrow \left((e^{r^2})' = 2re^{r^2}\right)$$

$$= \int_0^\pi \left[\frac{e^{r^2}}{2}\right]_1^{\sqrt{2}} d\theta = \frac{1}{2}(e^2-e)\cdot\int_0^\pi d\theta$$

$$= \frac{\pi}{2}(e^2-e)$$

Example: Compute $\iint_D xy \, dA$, where D is the first quadrant bounded by $y=x$, $y=3x$, and hyperbolas $xy=1$ and $xy=3$.



Hard to split regions.

Idea: introduce $\underline{u=xy}$

$(v=x$ or $\underline{v=y})$

$\rightarrow x=\frac{u}{y}=\frac{u}{v}$

Let $\begin{cases} xy=u \\ \underline{y=v} \text{ minor role} \end{cases} \underset{x,y}{\overset{\text{solve for}}{\Longrightarrow}} \begin{cases} x=\frac{u}{v} \\ y=v \end{cases}$ (i.e. find $\underline{\text{inverse}}$) $\left(\vec{T}(u,v)=\left(\frac{u}{v},v\right)\right)$

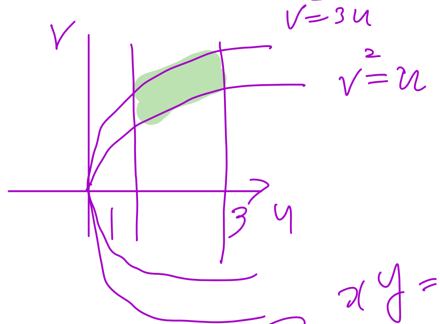
(just choose something simple)

$$J(\vec{T}) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -uv^{-2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$y=mx \qquad v=m\frac{u}{v}$

$\boxed{v^2=mu}$

$D \xleftarrow{\vec{T}} R = \{(u,v) : \underline{1} \le u \le \underline{3}, \sqrt{u} \le v \le \sqrt{3u}\}$ $\quad v^2=3u$, $v^2=u$



$(v=y>0)$

$xy=\frac{u}{v}\cdot v$

$$\iint_D xy \, dA = \iint_R \underline{u} \, \underline{\tfrac{1}{v}} \, d\tilde{A}$$

$\rightarrow$ Jacobian.

$$= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, dv \, du$$

$$= \int_1^3 u \cdot \Big[\ln|v| \Big]_{\sqrt{u}}^{\sqrt{3u}} du$$

$$= \int_1^3 u \, (\underbrace{\ln\sqrt{3u} - \ln\sqrt{u}}_{= \ln\left(\frac{\sqrt{3u}}{\sqrt{u}}\right) = \ln\sqrt{3}}) \, du$$

$$\left(= \frac{1}{2} \ln 3\right)$$

$$= \int_1^3 u \ln\sqrt{3} \, du$$

$$= \left[\frac{u^2}{2} \cdot \ln\sqrt{3} \right]_1^3$$

$$= \frac{\ln 3}{4} \cdot \left(3^2 - 1^2 \right)$$

$$= 2 \ln 3$$

NT 4.3 Triple integrals

/* The same idea as in double integrals */

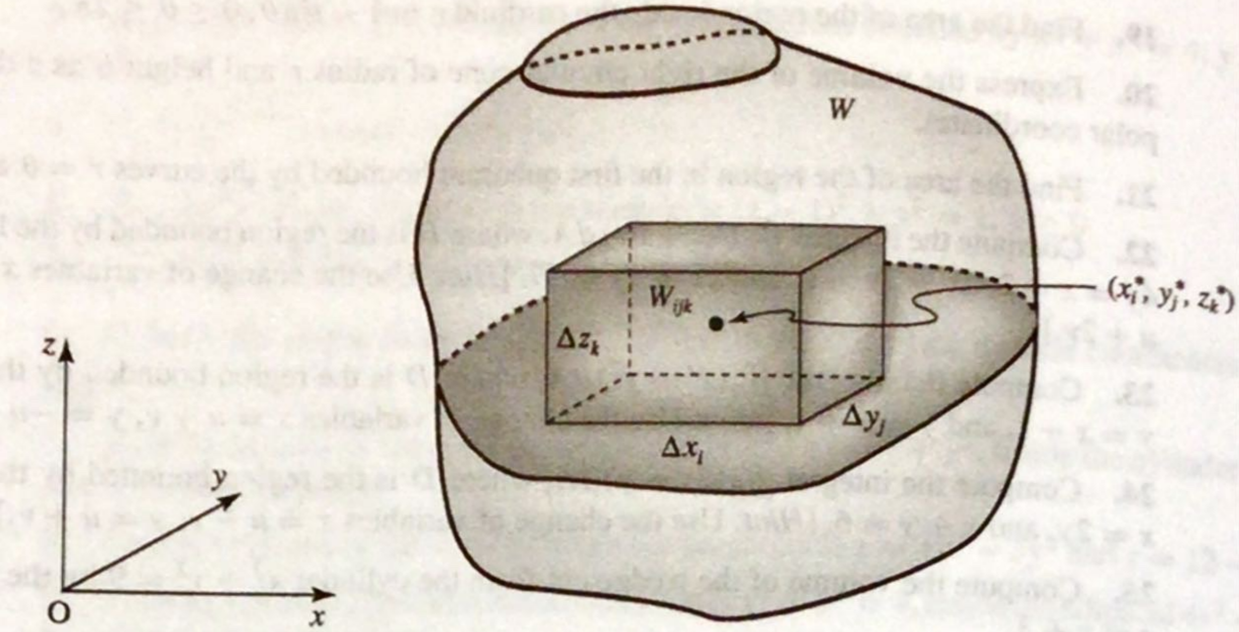


Figure 6.51 A subbox W_{ijk} from the definition of the Riemann sum of f.

coordinate planes; see Figure 6.51. Form the *(triple) Riemann sum*

$$\mathcal{R}_n = \sum_i \sum_j \sum_k f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j \Delta z_k,$$

where (x_i^*, y_j^*, z_k^*) is any point in W_{ijk}, the product $\Delta x_i \Delta y_j \Delta z_k$ is the volume
the sums run over those W_{ijk} that have a non-empty intersection with W.

DEFINITION 6.8 Triple Integral

The *triple integral* $\iiint_W f\,dV$ *of* f *over* W is defined by

$$\iiint_W f\,dV = \lim_{n\to\infty} \mathcal{R}_n,$$

whenever the limit on the right side exists.

◎ For a more concrete idea, think of f as density. Then, $\iiint_W f\,dV$ is mass.

◎ All the properties that are true for double integrals also hold for triple integrals

THEOREM 6.8 Fubini's Theorem

Assume that $f = f(x, y, z)$ is a continuous function defined on a rectangular box $W = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $\mathbb{R}^3$. Then

$$\iiint_W f\,dV = \int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y, z)dx \right) dy \right) dz$$

$$= \int_{a_1}^{b_1} \left(\int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} f(x, y, z)dy \right) dz \right) dx$$

$$= \int_{a_3}^{b_3} \left(\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y, z)dy \right) dx \right) dz,$$

etc. There are six iterated integrals altogether.

For general domains, iterated integrals read

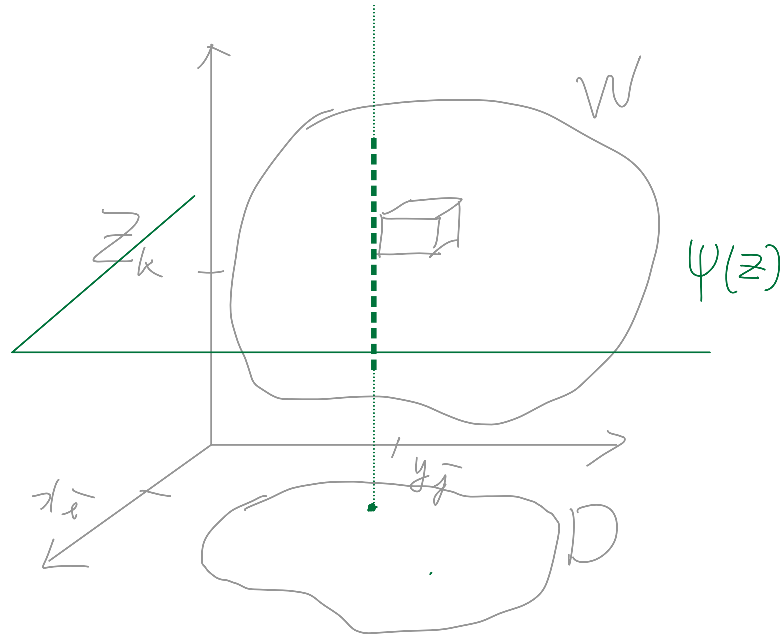
$$\int_{a_1}^{b_1} \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\,dy\,dx$$

or in some other order in dz, dy, and dx.

Clicker What is the order of iterated integral above?

(A) x, y, then z (B) z, y, then x (C) z, x, then y

(*) While reducing triple integral to "stacking slices" makes more sense physically, reducing it to "collecting fibers" makes it easier mathematically.



First, "accumulate the density along a line in z-direction at a fixed (x_i, y_j)"

$$\ell(x,y) := \int_{a_3}^{b_3} f(x, y, z)\, dz$$

$$\approx \sum_k f(x, y, z_k) \cdot \Delta z_k$$

If we further integrate these "fibers" across the region D (xy-projection of W), we will get what we want.

$$\iint_D \ell(x,y)\,dA \approx \sum_{i,j}\left(\sum_k f(x_i, y_j, z_k)\cdot\Delta z_k\right)\Delta x_i \Delta y_k$$

$$= \sum_{i,j,k} f(x_i, y_j, z_k)\,\Delta V_{ijk}$$

$$\approx \iiint_W f(x,y,z)\,dV$$

If we further apply 2D Fubini, then we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \ell(x,y)\,dy\,dx$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(\int_{a_3}^{b_3} f(x, y, z)\,dz\right)dy\,dx$$

Example: Evaluate $\iiint_W f\,dV$, where $f(x,y,z) = ze^{x+y}$ and $W = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$.

$$\int_0^1 \int_0^2 \int_0^3 z \underbrace{(e^x e^y)}_{\text{constant in } z} \, dz\,dy\,dx = \int_0^1 \int_0^2 e^x \underbrace{e^y \int_0^3 z\,dz}_{\text{constant in } y} \, dy\,dx$$

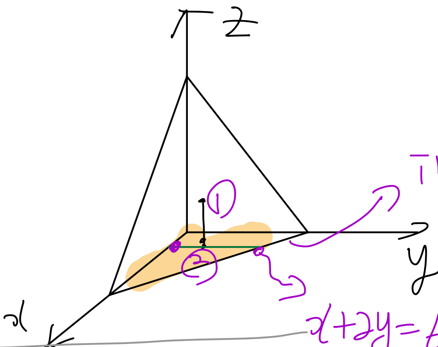
$$= \int_0^1 e^x\,dx \int_0^2 e^y\,dy \int_0^3 z\,dz \qquad \Vert \qquad \int_0^3 z\,dz \int_0^1 e^x \underbrace{\int_0^2 e^y\,dy}_{\text{const. in } x}\,dx$$

$$= \left[e^x\right]_0^1 \left[e^y\right]_0^2 \left[\frac{z^2}{2}\right]_0^3$$

$$= \frac{9}{3}(e^2-1)(e-1)$$

/* As in 2D, we have: (product of functions of x, y, z separately)
x, y, z are independent of each other and the region is a box,
$\Longrightarrow$ product of 1D integrals. */

Example: Evaluate $\iiint_W 2y \, dV$, where W is the solid in the first octant bounded by the plane $x+2y+z=6$. (p.421)



intersection of $x+2y+z=6$ and xy-plane $(z=0)$

$x+2y=6$ on xy-plane

① Determine the order of iterated integral and lower and upper limits $dz\,dy\,dx$:

solve for z

(for each (x,y))

lower surface

upper surface

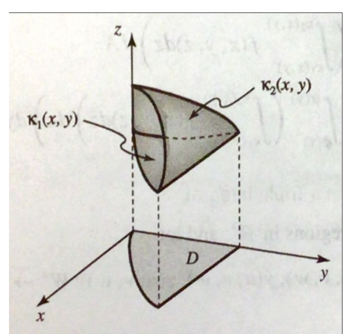
z from $\underline{0}$ to $\underline{6-x-2y}$

(for each x on D)

y from $\underline{0}$ to $\underline{3-\frac{x}{2}}$

x from $\underline{0}$ to $\underline{6}$

x-intercept.



D: projection of W

$$\int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{6-x-2y} 2y \, dz \, dy \, dx$$

$$= \int_0^6 \int_0^{3-\frac{1}{2}x} \Big[2yz \Big]_0^{6-x-2y} dy \, dx$$

$$= \int_0^6 \int_0^{3-\frac{1}{2}x} 12y - 2xy - 4y^2 \, dy \, dx$$

$$= \int_0^6 \left[6y^2 - xy^2 - \frac{4}{3}y^3 \right]_0^{3-\frac{1}{2}x} dx$$

$$= \int_0^6 \underbrace{6\left(3-\frac{1}{2}x\right)^2 - x\left(3-\frac{1}{2}x\right)^2}_{2\left(3-\frac{1}{2}x\right)^3} - \frac{4}{3}\left(3-\frac{1}{2}x\right)^3 dx$$

$$= \frac{2}{3} \int_0^6 \left(3-\frac{1}{2}x\right)^3 dx$$

$3-\frac{1}{2}x = t$

$-\frac{1}{2}dx = dt$

$$= \frac{2}{3} \int_3^0 t^3 \cdot (-2\,dt)$$

$$= \frac{4}{3} \int_0^3 t^3 \, dt$$

$$= \frac{4}{3} \left[\frac{t^4}{4} \right]_0^3$$

$$= 27.$$

THEOREM 6.9 Change of Variables in a Triple Integral

Let W and W^* be elementary (3D) regions in $\mathbb{R}^3$, and let

$$T = T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)): W^* \to W$$

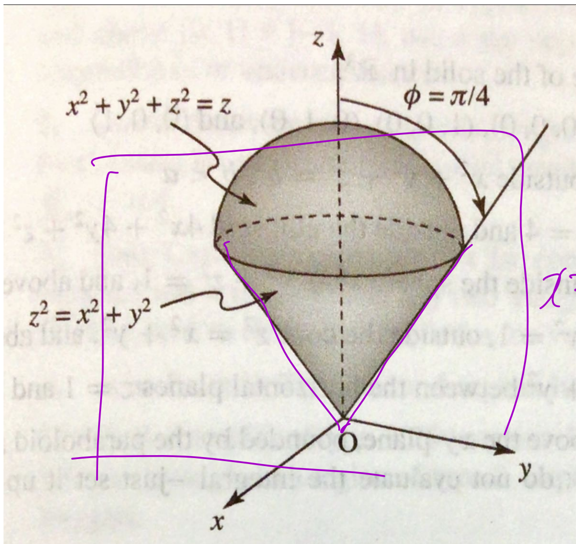
be a C^1, one-to-one function such that $T(W^*) = W$. For an integrable function $f: W \to \mathbb{R}$,

$$\iiint_W f\, dV = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV^*.$$

The integrand on the right side is the composition $f \circ T$ (i.e., f expressed in terms of "new" variables u, v, and w) multiplied by the absolute value of the Jacobian

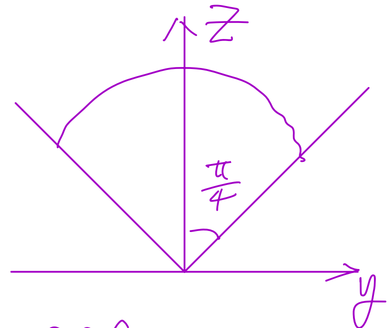
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = det(DT) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}.$$

Example: Find the volume of the solid that lies inside $x^2+y^2+z^2=z$ and above the cone $z^2=x^2+y^2$, $z\geq 0$.



$x=0 \Rightarrow z^2=y^2$

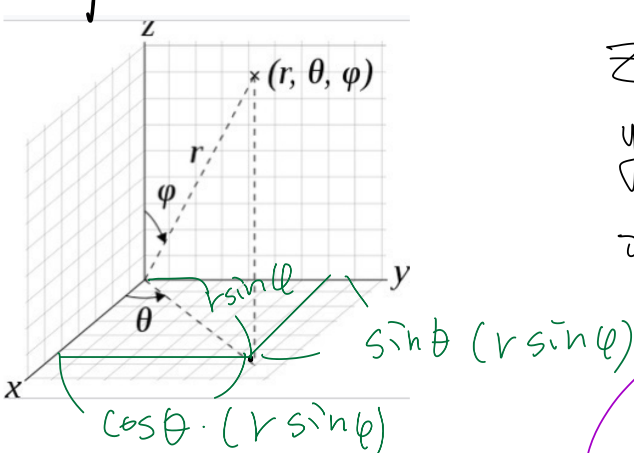
$z=y \ (z\geq 0)$



$\iiint_W 1\, dV$ (volume of W)

$x^2+y^2+(z-\frac{1}{2})^2=(\frac{1}{2})^2$

⊙ spherical coordinate



$z = r\cos(\varphi)$

$y = r\sin\theta\sin\varphi$

$x = r\cos\theta\sin\varphi$

$\vec{T}(r,\theta,\varphi)=(\quad\quad,\quad\quad,\quad\quad)$

① New region

$\underline{0} \leq \varphi \leq \underline{\frac{\pi}{4}}$, $\underline{0} \leq \theta \leq \underline{2\pi}$,

From $x^2+y^2+z^2=z$

$$r^2 = r\cos\varphi \Rightarrow r=\cos\varphi$$

$\underline{0} \leq r \leq \underline{\cos\varphi}$

$$\begin{vmatrix} x_r & x_\theta & x_\varphi \\ y_r & y_\theta & y_\varphi \\ z_r & z_\theta & z_\varphi \end{vmatrix}$$

② Jacobian

$$J(\vec{T}) = \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$$= \cos\varphi \begin{vmatrix} -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \end{vmatrix}$$

$$-r\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta \end{vmatrix}$$

$$= \cos\varphi \cdot r^2\cos\varphi\cdot\sin\varphi \underbrace{\begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}}_{=-1}$$

$$-r^2\sin\varphi\sin^2\varphi \underbrace{\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}}_{=1}$$

$$= -r^2\sin\varphi$$

$$|J(\vec{T})| = r^2\sin\varphi \qquad (0 \le \varphi \le \tfrac{\pi}{4})$$

$\sin\varphi \geq 0.$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} 1 \cdot r^2\sin\varphi \, dr\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left[\sin\varphi \, \frac{r^3}{3}\right]_0^{\cos\varphi} d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\varphi\cos^3\varphi \, d\varphi\, d\theta$$

$$\left((\cos\varphi)^4\right)' = 4\cos^3\varphi \cdot (-\sin\varphi)$$

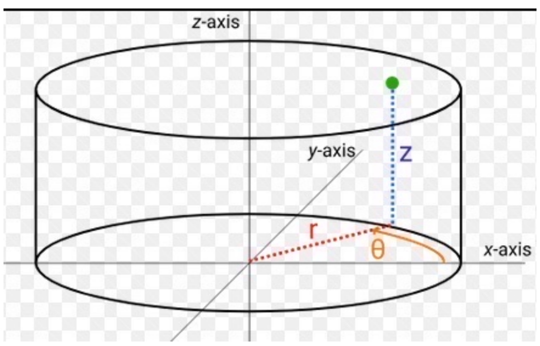
$$= \int_0^{2\pi} -\frac{1}{3}\cdot\frac{1}{4}\left[\cos^4\varphi\right]_0^{\frac{\pi}{4}} d\theta$$

$$= -\frac{1}{12}\underbrace{\left[\underbrace{\left(\frac{1}{\sqrt{2}}\right)^4}_{\frac{1}{4}} - 1^4\right]}_{-\frac{3}{4}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= \frac{\pi}{8}.$$

Example: Consider the solid that lies within the cylinder $x^2+y^2=1$, below $z=4$ and above the paraboloid $z=1-x^2-y^2$. The density of this solid is proportional to the distance from z-axis. Find the mass. (Let $\alpha>0$ be the constant of proportion)

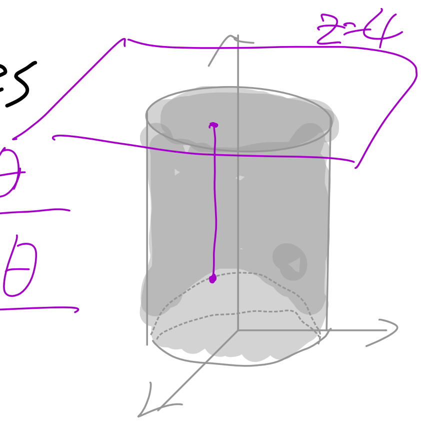
Cylindrical coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



① Jacobian

$$J(\vec{T}) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

① region

$1-x^2-y^2$

$\underline{0} \le r \le \underline{1}$, $\underline{0} \le \theta \le \underline{2\pi}$, $\underline{1-r^2} \le z \le 4$

② Integral : Let density $= \alpha r$. $(\alpha > 0)$

$$\int_{\underline{0}}^{\underline{2\pi}} \int_{\underline{0}}^{\underline{1}} \int_{\underline{1-r^2}}^{\underline{4}} \underline{\alpha r} \cdot \underline{r} \, dz \, dr \, d\theta$$

$$= \alpha \int_0^{2\pi} \int_0^1 \left(4-(1-r^2)\right) r^2 \, dr \, d\theta$$

$$= \alpha \int_0^{2\pi} \int_0^1 3r^2 + r^4 \, dr \, d\theta$$

$$= \alpha \int_0^{2\pi} \left[r^3 + \frac{r^5}{5} \right]_0^1 d\theta$$

$$= \alpha \cdot 2\pi \cdot \left(1 + \frac{1}{5}\right)$$

$$= \frac{12\pi}{5} \alpha$$