Announcement

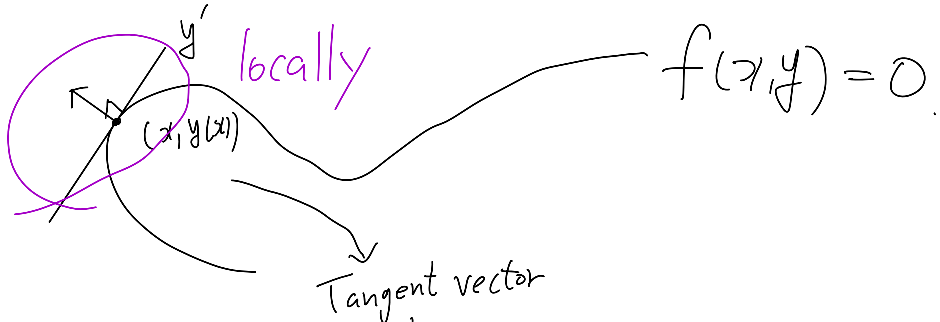
- ESCI (Currently less than 18%)
- Regular office hours end this week.

Exam week $\longrightarrow$ Special office hours (TBA on Thursday)

Application of $\nabla f \perp$ level set

Implicit differentiation using partials.

Want: $y'(x_0)$ at (x_0, y_0) when the function y is given implicitly via $f(x,y)=0$



$$\nabla f(x,y) \cdot (1, y'(x)) = 0.$$

$$(f_x, f_y) \cdot (1, y'(x)) = 0$$

$$f_x + f_y\, y'(x) = 0 \qquad \text{solve for } y'$$

$$y'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$$

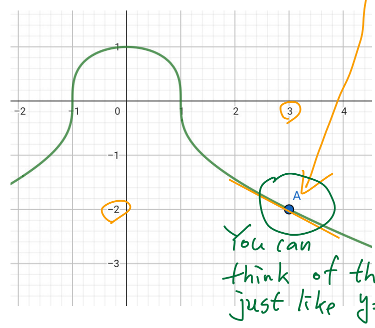
You can also derive this using chain rule.

$$x \rightrightarrows \begin{matrix} x \\ y \end{matrix} \rightrightarrows f(x,y) \qquad \frac{d}{dx}(0) = 0 = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

/* We are recast the problem as a level set situation. (a trick) */

Example: Find $\frac{dy}{dx}$ at $(3,-2) = A$

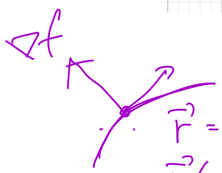
$$\underbrace{x^2+y^3-1}_{f(x,y)} = 0$$



You can think of this just like $y=f(x)$ near A

(Way 1: Partials)

Quick derivation



$\vec{r} = (x, y(x))$

$\vec{r}\,' = (1, y'(x))$

Model the curve locally by $(x, y(x))$ and use orthogonality

$$\nabla f \cdot (1, y'(x)) = 0 \Rightarrow (f_x, f_y)\cdot(1, y'(x)) = 0$$

$$\Rightarrow \boxed{y'(x) = -\frac{f_x}{f_y}}$$

Let $f(x,y) = x^2+y^3-1$

$$f_x(3,-2) = 2x\Big|_{\substack{x=3\\y=-2}} = 2\cdot 3 = 6$$

$$f_y(3,-2) = 3y^2\Big|_{\substack{x=3\\y=-2}} = 3\cdot(-2)^2 = 12$$

$$y'(3) = -\frac{f_x(3,-2)}{f_y(3,-2)} = -\frac{6}{12} = -\frac{1}{2}$$

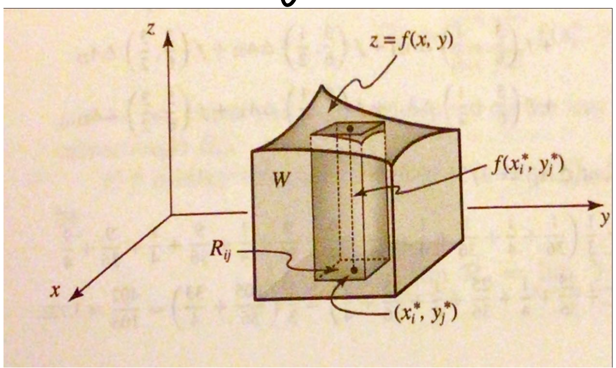
(Way 2: Implicit differentiation – Cal 1)

Treat y as a fn of x: $y = y(x)$

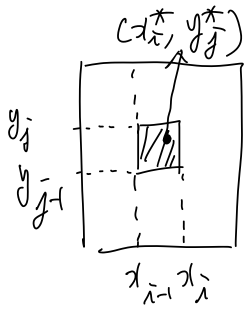
$$x^2+y^3-1=0 \xrightarrow{d/dx} 2x + 3y^2\cdot y' = 0$$

$$\Rightarrow y' = -\frac{2x}{3y^2}\Big|_{\substack{x=3\\y=-2}} = -\frac{2\cdot 3}{3\cdot(-2)^2} = -\frac{1}{2}$$

4.1 Double integrals over rectangle (Ch. 6.1)



Let $f : D \to \mathbb{R}$, where $D = \{(x,y) : a \le x \le b,\ c \le y \le d\}$.
If $\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A_{ij}$ converges to a number as $n \to \infty$, we say f is **integrable** and call the limit **(double) integral**, and denote it $\iint_D f(x,y)\,dA$, where $\Delta A_{ij} = \Delta x_i \Delta y_j$



$\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \cdots, n)$,
$\Delta y_j = y_j - y_{j-1} \quad (j = 1, 2, \cdots, n)$,
$a = x_0 < x_1 < x_2 < \cdots < x_n = b$,
$c = y_0 < y_1 < y_2 < \cdots < y_n = d$,
$x_i^* \in [x_{i-1}, x_i]$ and $y_j^* \in [y_{j-1}, y_j]$ are arbitrary.

$\boxed{\text{Thm}}$ If $f : D \to \mathbb{R}$ (rectangle) is bounded and its discontinuity can be covered by curves of finite lengths, f is integrable.

/* Don't worry too much about integrability */

Basic properties of double integrals.

⊙ $\iint_D f \pm g \, dA$

$$= \iint_D f \, dA \pm \iint_D g \, dA$$

⊙ $\iint_D cf \, dA = c \iint_D f \, dA$ (c: constant)

⊙ If $f(x,y) \le g(x,y)$ on D,

$$\iint_D f \, dA \le \iint_D g \, dA$$

⊙ $\left| \iint_D f \, dA \right| \le \iint_D |f| \, dA$

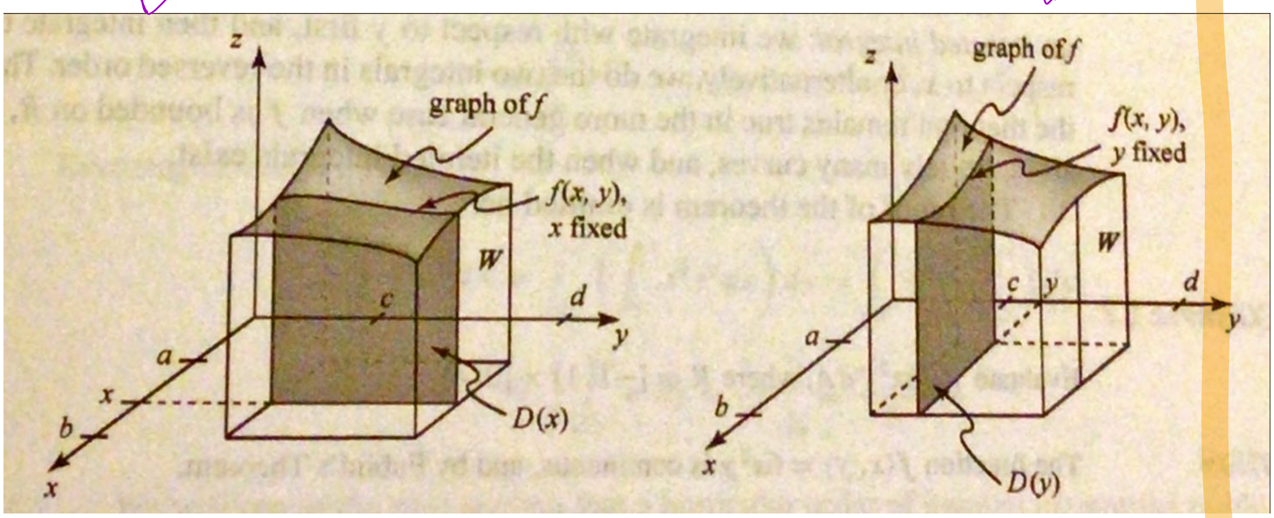
Fubini theorem

(Double integral = Iterated integral)

If $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable (don't worry)
(D is a rectangle), then

$$\iint_D f\, dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\, dy\right)}_{\text{a function of } x} dx$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy$$

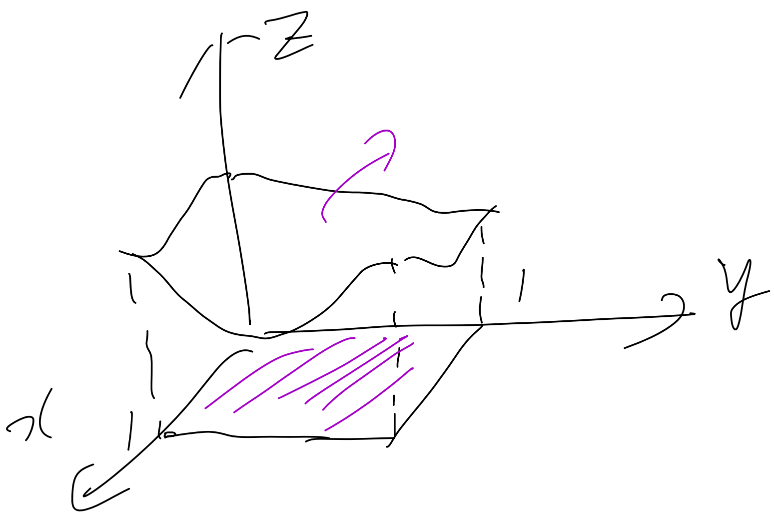


$$\int_a^b \underbrace{\int_c^d f(x,y)\, dy}_{\text{section areas at each } x=k}\, dx$$

$$\int_c^d \underbrace{\int_a^b f(x,y)\, dx}_{\text{section areas at each } y=k}\, dy$$

Example: (p. 380)

Find the volume of solid bounded by the surface $z = 5 - 2x - y^2$, the three coordinate planes and the planes $x = 1$ and $y = 1$.



$$\iint_D (5 - 2x - y^2)\, dA$$ ← double int.

iterated int.

$$= \int_0^1 \underbrace{\int_0^1 5 - 2x - y^2 \, dx}\, dy$$

y is a constant

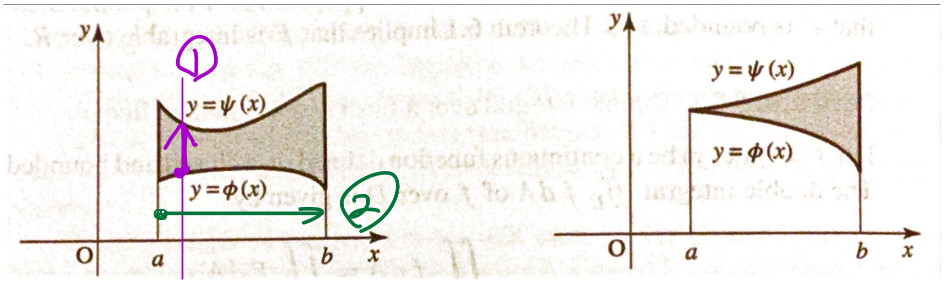
$$= \int_0^1 \left[5x - x^2 - y^2 x\right]_0^1 dy$$

$$= \int_0^1 \underbrace{5 - 1 - y^2}_{4 - y^2}\, dy$$

$$= \left[4y - \frac{y^3}{3}\right]_0^1 = 4 - \frac{1}{3} - 0 = \frac{11}{3}.$$

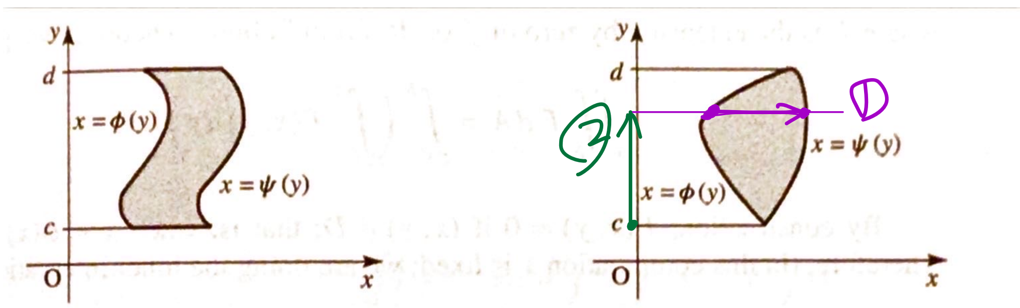
Iterated integrals for general domains

(Ch. 6.2) Choose the order (domain type matter)



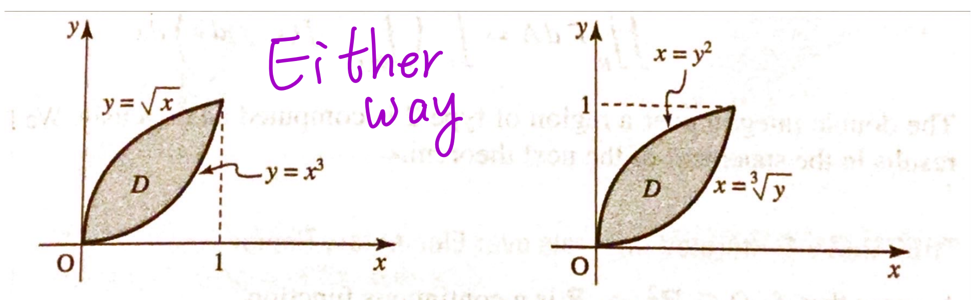
Type 1

$$\int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right)dx$$



Type 2

$$\int_c^d \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$



Either way

(a) As a type-1 region (b) As a type-2 region

Figure 6.24 The region $D = \{(x, y) \mid x^3 \le y \le \sqrt{x}, 0 \le x \le 1\}$.

/* Often, one way is easier than the other */

$\boxed{\text{◎ All basic properties for rectangular domains hold for ("nice") general domain.}}$

This can be seen using cut-off functions.

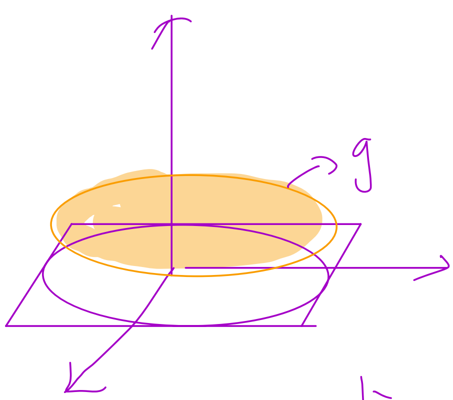
e.g.) If $f(x,y) = xe^{y^2} + y$, and

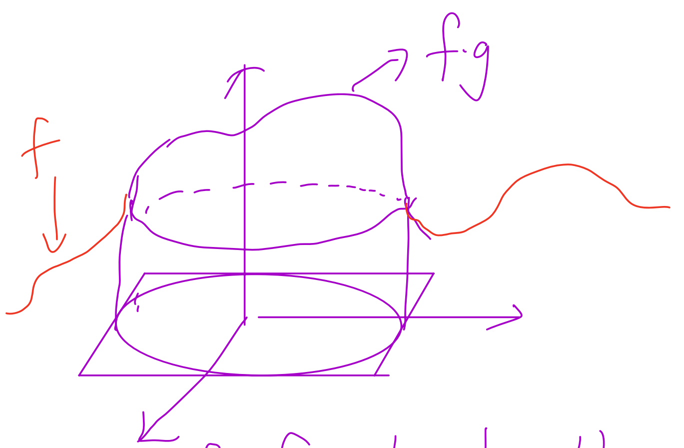
$$D = \{(x,y) : x^2+y^2 \le 1\}$$

$$R = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$$

$$g(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

$$\iint_D f \, dA = \iint_R f \cdot g \, dA$$

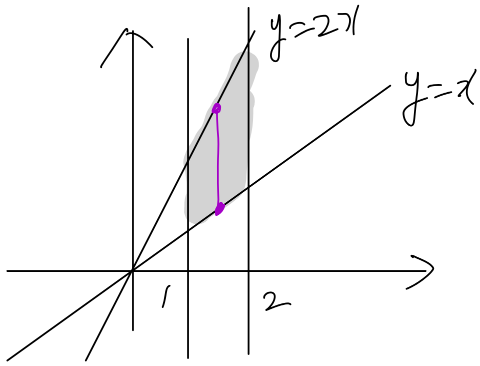




discontinuity of finite length

Example: Evaluate $\iint_D e^{2x+y}\, dA$, where D is the region bounded by $y=2x$, $y=x$, $x=1$, and $x=2$. (p. 385)

① Draw the region of integral



② Decide the order of iterated integral

scratch work

$$\left[e^{2x+y}\right]_{x}^{2x} = e^{4x} - e^{3x}$$

$$= \int_1^2 \int_x^{2x} e^{2x+y}\, dy\, dx$$

$$= \int_1^2 e^{4x} - e^{3x}\, dx$$

$$= \frac{1}{4}(e^8 - e^4) - \frac{1}{3}(e^6 - e^3)$$

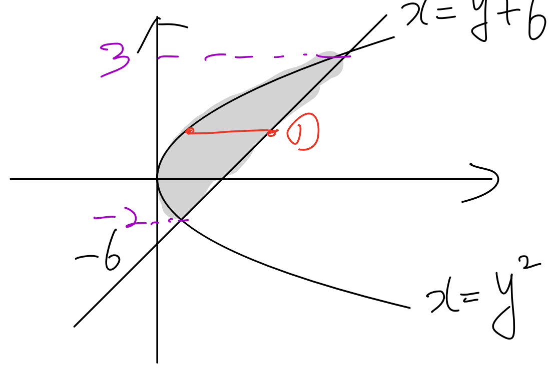
$\boxed{\text{Clicker}}$ Choose the correct one(s) about dimension.

(A) A double integral is a number.

(B) A double integral is a 2D vector

Example: Evaluate $\iint_D 2y\,dA$, where D is a region bounded by $y = x-6$ and $y^2 = x$.

① Draw the region of double integral



② Decide iterated integral

(order and upper/lower limits)

point of intersection

$$y^2 = y+6 \Rightarrow y^2 - y - 6 = 0$$
$$\Rightarrow (y-3)(y+2) = 0$$
$$\Rightarrow y = 3, -2$$

$$= \int_{-2}^{3}\int_{y^2}^{y+6} 2y\,dx\,dy$$

scratch work

$$\left[2xy\right]_{y^2}^{y+6} = 2(y+6)y - 2y^3 = -2y^3+2y^2+12y$$

$$= \left[-\cancel{2}\cdot\frac{y^4}{\cancel{4}\,2} + 2\cdot\frac{y^3}{3} + \cancel{12}\,6\cdot\frac{y^2}{\cancel{2}}\right]_{-2}^{3}$$

$$= -\frac{1}{2}(\overbrace{81-16}^{65}) + \frac{2}{3}(\overbrace{27-(-8)}^{35}) + 6(\overbrace{9-4}^{5})$$

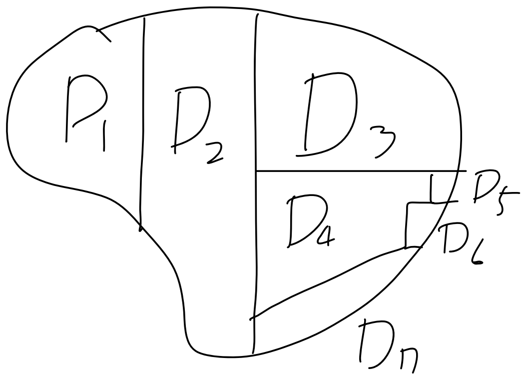
$$= -\frac{65}{2} + \frac{70}{3} + 30 = \frac{140-195+180}{6}$$

$$= \frac{125}{6}$$

Some properties of double integrals

(0) If D_i's partition the region D, then $\iint_D f\,dA = \sum_{i=1}^{n} \iint_{D_i} f\,dA$

(split)



(6) If f is continuous on D, then

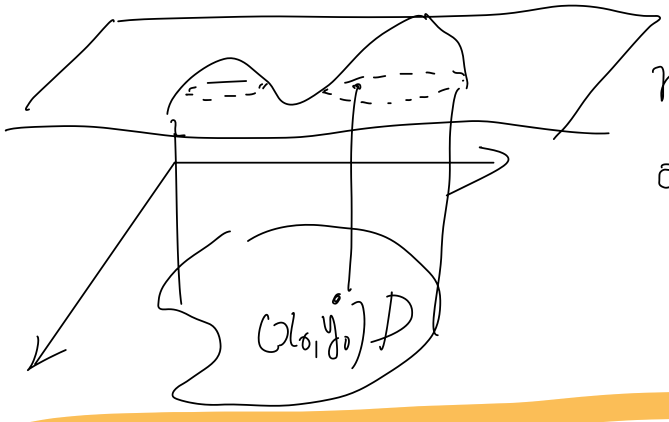
$$mA \le \iint_D f\,dA \le MA,$$

where A is the area of D, and m and M are minimum and maximum of f on D.

(6) If f is continuous on D, then

$$\iint_D f\,dA = f(x_0, y_0)A \quad \text{fo some } (x_0, y_0) \in D.$$

(mean value thm)



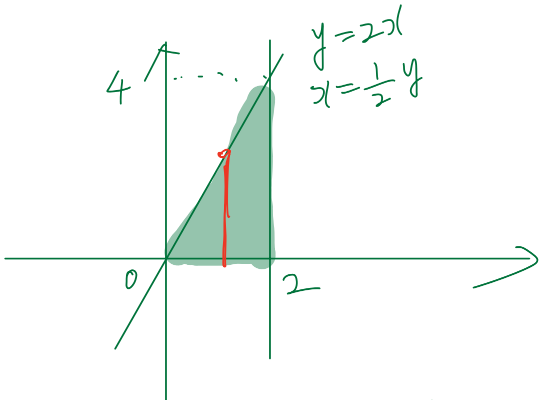
mean height over D.

/* $\boxed{\text{Sometimes, you need to switch the order of iterated integral (Ch. 6.3)}}$ */

Example: Evaluate $\int_0^4 \int_{\frac{1}{2}y}^2 e^{x^2}\, dx\, dy$

① Go back to double integral

(This amounts to finding <u>region</u>)



lower $x=\frac{1}{2}y \Rightarrow y=2x$

upper $x=2$

② Try the other order of iterated integral

(Be careful: upper/lower limits may look significantly different)

scratch work

$\left[y e^{x^2}\right]_0^{2x} = 2x e^{x^2}$

$= \int_0^2 \int_0^{2x} e^{x^2}\, dy\, dx$

$= \int_0^2 2x e^{x^2}\, dx$

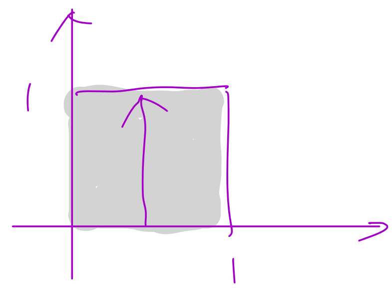
$(e^{x^2})' = 2x e^{x^2}$

$= \left[e^{x^2}\right]_0^2 = e^4 - 1$

/* Sometimes, you can reduce double integral to a product of (single) integrals */

Example: (p.398) Compute $\iint_D e^{x+y}\, dA$, where $D = [0,1]\times[0,1]$.

$$= \int_0^1 \int_0^1 e^x e^y \, dx\, dy$$



$$= \int_0^1 e^y \left(\int_0^1 e^x dx\right) dy$$

$$= \int_0^1 e^x dx \int_0^1 e^y dy \qquad \int_0^1 e^x dx$$

dummy var.

(bares no meaning in the end → may be relabelled by other symbol)

$$= \left(\int_0^1 e^x dx\right)^2$$

$$= \left(\left[e^x\right]_0^1\right)^2$$

$$= (e-1)^2$$

integrand

in the form of $g(x)\cdot h(y) = f(x,y)$

/* x and y independent of each other both in $f(x,y)$ and in D (rectangle) ⟹ product of 1D integrals */