NT 2.8 Applications to kinematics

Goal: Better understanding of movements of a particle in a general setting.

/* All physical applications of calculus so far under very limited circumstances. (e.g., movement in 1D space, or 2D movement under a constant acceleration — gravity, etc.) */

Learning objective:

After this lecture, I can decompose the acceleration of a given movement $\vec{r}(t)$ into tangential ($\vec{a}_T$) and normal ($\vec{a}_N$) components.

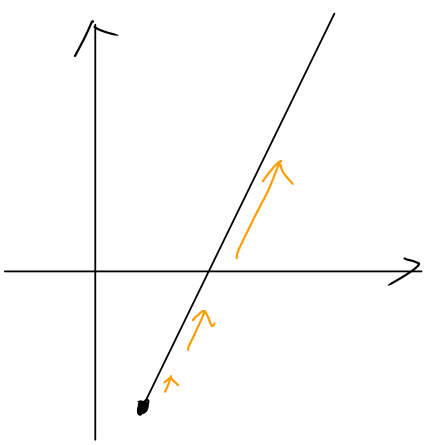
Motivating examples: Two extreme movements (2D for simplicity)

Example 1:

$$\vec{r}_1(t) = (1+t^3, 2t^3-3)$$

Notice that this is actually a line

$$\begin{cases} x = 1+t^3 \\ y = 2t^3-3 \end{cases} \quad t^3 = x-1 \qquad y = 2(x-1)-3 = 2x-5$$



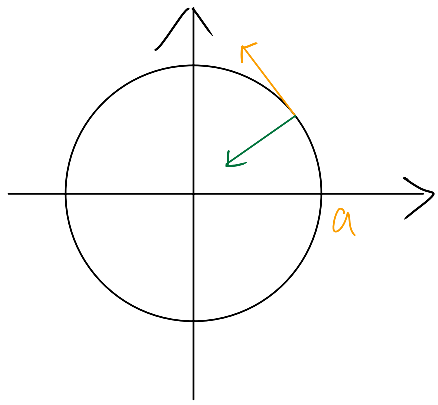
$$\vec{v}_1(t) = \vec{r}_1{}'(t) = (3t^2, 6t^2)$$
$$= 3t^2(1,2)$$
$$\vec{a}_1(t) = \vec{r}_1{}''(t) = (6t, 12t)$$
$$= 6t(1,2)$$

Acceleration acts in the same direction of the velocity. $\longrightarrow$ move straight

Example 2 :

$$\vec{r_2}(t) = (a\cos bt, a\sin bt)$$

$(a, b > 0)$



$$\vec{v_2}(t) = \vec{r_2}'(t)$$
$$= (-ab\sin bt, ab\cos bt)$$
$$\vec{a_2}(t) = \vec{r_2}''(t)$$
$$= (-ab^2\cos bt, -ab^2\sin bt)$$

$$\vec{V_2}(t) \cdot \vec{a_2}(t) = +a^2b^3\sin bt\cos bt - a^2b^3\cos bt\sin bt$$
$$= 0$$

Acceleration $\perp$ velocity all the time

$\Rightarrow \mathrm{proj}_{\vec{v}}\vec{a} = \vec{0}$

$\Rightarrow$ acceleration never changes the speed, but only changes the direction of the movement.

HW5 will be assigned next week.

$\hookrightarrow$ covers next topics.

◎ In general, movements in real world possess features of both kinds

Idea: Isn't it going to be helpful if we can decompose the acceleration into direction of velocity and one perpendicular to it?

$\boxed{\text{Thm}}$ Let $\vec{r}(t)$ be a C^2-smooth path. Then, its acceleration has following decomposition.

$$\boxed{\begin{aligned} \vec{a}(t) &= \vec{a}_T(t) + \vec{a}_N(t), \text{ where} \\ \vec{a}_T(t) &= \frac{d^2 s}{dt^2} \vec{T}(t) \\ \vec{a}_N(t) &= \|\vec{v}(t)\| T'(t) \quad \text{— ①} \\ &= \|\vec{v}(t)\|^2 k(t) N(t) \quad \text{— ②} \end{aligned}}$$

$\vec{T} \perp \vec{N}$

⊙ $\vec{a}_T$ changes speed, while $\vec{a}_N$ changes the direction.

⊙ Usually, ① is easier to compute while ② tells us the relation b/w $\vec{a}_N$ and curvature.

⊙ Notation warning: Some authors and our HW4-#6 us a_T and a_N to denote the scalar part of $\vec{a}_T$ and $\vec{a}_N$, resp. (ans to HW4-#6 is a number, not vector) proof) To make use of $\vec{T} \perp \vec{T}'$ and $\vec{T}' \parallel \vec{N}$, express velocity as

$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$

$$\vec{v}(t) = \underline{\|\vec{v}(t)\|}\ \vec{T}(t). \text{ Then,}$$

$$\vec{a}(t) = \vec{v}'(t) \underset{\uparrow}{=} \underbrace{\|\vec{v}(t)\|' \vec{T}(t)}_{ⓐ\ \vec{a}_T} + \underbrace{\|\vec{v}(t)\| \vec{T}'(t)}_{ⓑ\ \vec{a}_N}$$

of $\underline{P}$ rule

$\underline{s}\cdot\underline{v}$

ⓐ is immediate by realizing

$$\|\vec{v}(t)\|' = \left(\frac{ds}{dt}\right)' = \frac{d^2 s}{dt^2} \quad \text{(hence ① follows)}$$

② : First, note that

$$k(t_0) = \left\| \frac{d}{ds} \vec{T}(t(s_0)) \right\| \quad (t_0 = t(s_0))$$

$$= \left\| \vec{T}'(t_0) \cdot \frac{dt}{ds}(s_0) \right\| = \|\vec{T}'(t_0)\| \cdot \underbrace{\left|\frac{dt}{ds}(s_0)\right|}$$

$$= \frac{\|\vec{T}'(t_0)\|}{\|\vec{v}(t_0)\|} \qquad = \left|\frac{ds}{dt}\right|^{-1} = \|\vec{v}(t_0)\|^{-1}$$

Now, drop subscript from t_0, and conclude:

$$\|\vec{v}(t)\| \vec{T}'(t) = \|\vec{v}(t)\|^2 \underbrace{\frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|}}_{k(t)} \underbrace{\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}}_{\vec{N}(t)}$$

⓪ For explicit calculations, an intermediate step in the proof can be more practical.

$$\boxed{\vec{a} = (\vec{v})' = (\|\vec{v}\|\vec{T})' = \underbrace{\|\vec{v}\|'\cdot\vec{T}}_{=\vec{a}_T} + \underbrace{\|\vec{v}\|\vec{T}'}_{=\vec{a}_N}}$$

product rule

This is a good example of how proofs can be useful.

$\boxed{\text{Clicker}}$ What should you do if you obtained $\vec{a} = \vec{a}_T + \vec{a}_N$, but you want only scalar part of $\vec{a}_T$ and $\vec{a}_N$ (short answer)

scalar direction (unit vector)

$$\vec{u} = \|\vec{u}\| \frac{\vec{u}}{\|\vec{u}\|} = \|\vec{u}\| \frac{\vec{u}}{\|\vec{u}\|}$$

But be careful of sign (i.e. same dir.? or opposite dir.?)

$\vec{a}_T = \|\vec{a}_T\| \frac{\vec{a}_T}{\|\vec{a}_T\|} \rightarrow$ check if 1st component has the same sign as $\vec{v}$ or $\vec{T}$.

same sign $\longrightarrow$ $\|\vec{a}_T\|$ is what we are looking for

opposite sign $\longrightarrow$ $= \underbrace{-\|\vec{a}_T\|}_{\text{This is what we want.}} \left(-\dfrac{\vec{a}_T}{\|\vec{a}_T\|}\right)$

For $\vec{a}_N$, no worries since we know the scalar part is positive.

$$\vec{a}_N = \|\vec{a}_N\| \underbrace{\frac{\vec{a}_N}{\|\vec{a}_N\|}}_{\vec{N}}.$$

Example: Suppose a particle's position at time t is described by
$\vec{r}(t) = (\cos t + t\sin t, -\sin t + t\cos t, t^2/2)$
$(t \geq 0)$. Find the acceleration and decompose it into tangential and normal components $\vec{a}_T$ and $\vec{a}_N$

$$\vec{v}(t) = \vec{r}\,'(t) = (\cancel{-\sin t} + \cancel{\sin t} + t\cos t,$$
$$\cancel{-\cos t} + \cancel{\cos t} - t\sin t,\ t)$$
$$= t\,(\cos t, -\sin t, 1).$$
$$\|\vec{v}(t)\| = \|t(\cos t, -\sin t, 1)\|$$
$$= t\,\|(\cos t, -\sin t, 1)\|$$
$$= t\sqrt{\cos^2 t + (-\sin t)^2 + 1^2}$$
$$= \sqrt{2}\,t.$$

same sign $(t \geq 0)$

$$\vec{v}(t) = \|\vec{v}(t)\| \cdot \vec{T}(t) = \underbrace{\sqrt{2}\,t}_{\|\vec{v}(t)\|} \cdot \underbrace{\left(\frac{1}{\sqrt{2}}(\cos t, -\sin t, 1)\right)}_{\vec{T}(t)}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \underbrace{\sqrt{2} \cdot \left(\frac{1}{\sqrt{2}}(\cos t, -\sin t, 1)\right)}_{= \vec{a}_T}$$

product rule

scalar part of $\vec{a}_T$

$$+ \underbrace{\sqrt{2}\,t\left(\frac{1}{\sqrt{2}}(-\sin t, -\cos t, 0)\right)}_{= \vec{a}_N}$$

NT 2.8 Local geometry of 3D curves.
(Frenet frame / TNB system)

Motivation: We studied local behaviors of a curve:
near a point

- direction of curve → $\vec{T}(s)$
- how it is bent → $\kappa(s)\vec{N}(s)$

But these are not enough to give a complete descriptions of curves.
It turns out studying how the osculating plane changes gives the whole picture of curves.

$\boxed{\text{Clicker}}$ What should we study for this mission?

(A) $(\vec{T})'$ (B) $(\vec{N})'$ (C) $(\vec{T}\times\vec{N})'$

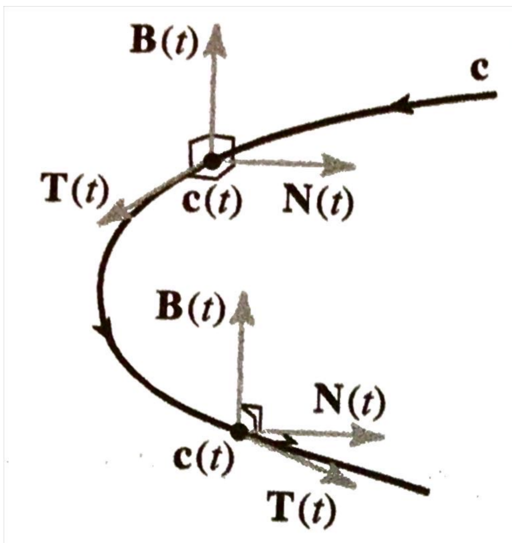
Learning objective:

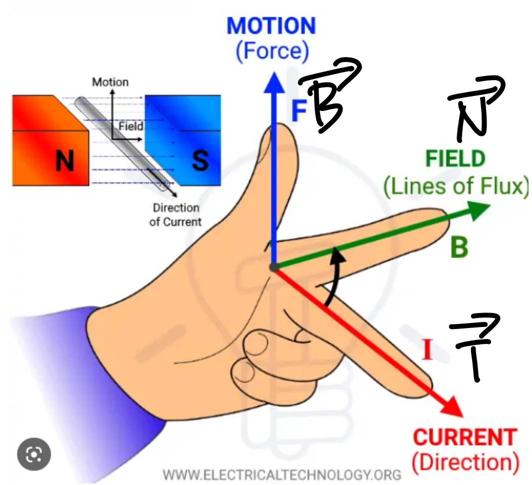
- I can find TNB frame and torsion.
- I can explain behaviors of a curve based on κ, τ, T, N, B

Def (binormal vector)

Given a C^2-smooth curve $\vec{r}$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called **binormal vector**, where $\vec{T}(t)$ and $\vec{N}(t)$ are unit tangent and principal normal vector of $\vec{r}$.





Left hand rule.

◎ $\vec{T}(t), \vec{N}(t), \vec{B}(t)$ are called **Frenet frame** or **TNB frame**.

◎ $\frac{d}{ds}\vec{B}(s)$ let us know how fast the osculating plane changes $\rightarrow$ "twist".

Thm

Given a C^2-smooth curve parametrized by arc length $\vec{r}(s)$, its TNB frame, the curvature $k(s)$, and the **torsion** $\tau(s)$ satisfy

$$\boxed{\begin{aligned}\vec{T}'(s) &= \phantom{-k(s)\vec{T}} \quad k(s)\vec{N} \\ \vec{N}'(s) &= -k(s)\vec{T} \qquad\qquad + \tau(s)\vec{B} \\ \vec{B}'(s) &= \qquad\qquad -\tau(s)\vec{N}\end{aligned}}$$

where, $\boxed{\tau(s) := -\vec{N}(s)\cdot\vec{B}'(s)}$

- Everything is defined for arc length parametrization.
- There is a formula for $\tau(t)$, but it is not considered an acceptable tool in this course. (learning purpose)
- Note that the curvature is always positive (or zero) while the torsion can be both positive and negative.
- In this course, the formulas in the 1^{st} box is not the focus.

(proof) For ease of notation, omit input s, and all derivatives are $\frac{d}{ds}$.

First, we already know

$$\vec{T}' = k\vec{N}$$

Also note $\vec{N}\cdot\vec{N}'=0$ sinle $\vec{N}$ has constant length. Therefore, when we decompose $\vec{N}'$ using TNB frame, it does not have $\vec{N}$ component. That is,

$$\vec{N}' = a\vec{T} + b\vec{B} \qquad (*)$$

for some scalar function $a(s)$ and $b(s)$.

Now, differentiating $\vec{B} = \vec{T}\times\vec{N}$ and using the product rule for cross product

$$\vec{B}' = \underbrace{\vec{T}'\times\vec{N}}_{=0 \text{ b/c } \vec{T}' /\!/ \vec{N}} + \vec{T}\times\vec{N}' = \vec{T}\times\vec{N}'$$

$$= \vec{T}\times(a\vec{T} + b\vec{B}) \qquad \vec{N}' = a\vec{T} + b\vec{B}$$

$$= b\,\vec{T}\times\vec{B} \qquad \vec{T}\times\vec{B} = -\vec{N}$$

$$= -b\vec{N}$$

Here $b(s)$ tells us how fast $\vec{B}$ changes. Call it **torsion** and label $\tau(s)$.

Plug this back into $(*)$. Then, we are left to show $a(s) = -k(s)$. To show this, differentiate $\vec{N} \cdot \vec{T} = 0$ to get

$$\vec{N}' \cdot \vec{T} + \vec{T}' \cdot \vec{N} = 0 \implies$$

$$\vec{N}' \cdot \vec{T} = -\underbrace{\vec{T}'}_{= k\vec{N}} \cdot \vec{N} = -(k\vec{N}) \cdot \vec{N} = -k$$

But taking dot product with $\vec{T}$ on both sides of $(*)$, we have

$$-k = \vec{N}' \cdot \vec{T} = a \underbrace{\vec{T} \cdot \vec{T}}_{=1} + b \underbrace{\vec{B} \cdot \vec{T}}_{=0} = a$$

Thus, we showed all relation b/w $\vec{T}, \vec{N}, \vec{B}$ and their derivatives.

Lastly, it follows

$$\tau(s) = -\vec{N}(s) \cdot \vec{B}'(s)$$

by taking dot product with $\vec{N}$ on

$$\vec{B}' = -\tau \vec{N}.$$

$\boxtimes$

Example: Find the torsion of the helix
$\vec{r}(t) = (a\cos t, a\sin t, bt)$ $t \geq 0$, $a, b > 0$.
(You will end finding everything: κ, τ, T, N, B)

Step 1: reparametrize it.

$$\vec{r}'(t) = (-a\sin t, a\cos t, b)$$

$$s = \int_0^t \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2 + b^2} = \int_0^t \sqrt{a^2+b^2}\, d\theta$$

$$= t\sqrt{a^2+b^2} \implies t = \frac{s}{\sqrt{a^2+b^2}}$$

Put $C = \sqrt{a^2+b^2}$

Step 2: carry out calculations.

$$\vec{r}(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right)$$

$$\vec{T}(s) = \vec{r}'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$$

$$\vec{T}'(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$$

$$= \underbrace{\frac{a}{c^2}}_{\kappa(s)} \underbrace{\left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)}_{\text{already unit vector} \to \vec{N}(s)}$$

↳ if not $\vec{N}$ = normalized ver.

Thus,

$$\kappa(s) = \frac{a}{c^2} = \frac{a}{a^2+b^2}$$

$$\vec{N}(s) = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$$

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) \qquad (\underline{\text{order}} \text{ matters})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right), & \frac{a}{c}\cos\left(\frac{s}{c}\right), & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right), & -\sin\left(\frac{s}{c}\right), & 0 \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)$$

$$\tau(s) = -\vec{B}'(s) \cdot \vec{N}(s)$$

$$= -\left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), +\frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$$

$$\cdot \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$$

$$= +\frac{b}{c^2} = \frac{b}{a^2+b^2}$$

$\boxed{\text{Thm}}$ (Congruence of 3D curves)

Two 3D curves with non-zero curvature are congruent if and only if their arc length parametrizations have the same <u>curvature</u> and <u>torsion</u> at each $0 \leq s \leq \ell$, where ℓ is the length of the two curves.

(proof omitted)

$\boxed{\text{Thm}}$ 3D curve is a line or part of a line if and only if the <u>curvature</u> is zero everywhere.

(proof omitted)

$\boxed{\text{Thm}}$ 3D curve is contained in a plane if and only if the <u>torsion</u> is zero everywhere.

(proof omitted)