(proof of unique interpolation - Newton form)
Let k be the number of nodes.

$(K=1)$ $P_0 = y_0 = C_0$

$(k=2)$ $P_1 = C_0 + C_1(x - x_0)$

From $y_1 = P_1(x_1) = C_0 + C_1(x_1 - x_0)$

$C_1 = [y_1 - C_0]/(x_1 - x_0)$

$(k=3)$ $P_2 = P_1(x) + C_2(x - x_1)(x - x_0)$

This construction allows us to only consider the new condition $P_2(x_2) = y_2$ b/c we already know $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$ while $C_2(x - x_1)(x - x_0)$ vanishes if $x = x_0$ or x_1.

From $y_2 = P_2(x_2) = P_1(x_2) + C_2(x_2 - x_1)(x_2 - x_0)$

$C_2 = [y_2 - P_1(x_2)]/(x_2 - x_1)(x_2 - x_0)$

Repeat this until $k = n$. Since the process is constructive, the resulting $P_n(x)$ must be unique.

Example: Find $p \in \Pi_3$ such that

x	0	1	2	3
y	1	1	2	3

$$p_0 = 1 = c_0$$

$$p_1(x) = c_0 + \frac{y_1 - c_0}{x_1 - x_0} = 1 + \frac{1-1}{1-0}(x-0) = 1$$

$$p_2(x) = p_1(x) + \frac{y_2 - p_1(x_2)}{(x_2-x_1)(x_2-x_0)}(x-x_1)(x-x_0)$$

$$= 1 + \frac{2-1}{2 \cdot 1}(x-1)x = 1 + \frac{x}{2}(x-1)$$

$$p_3(x) = p_2(x) + \frac{y_3 - p_2(x_3)}{(x_3-x_2)(x_3-x_1)(x_3-x_0)}(x-x_2)(x-x_1)(x-x_0)$$

$$= 1 + \frac{x}{2}(x-1) + \frac{3 - 1 - \frac{3}{2} \cdot 2}{1 \cdot 2 \cdot 3}(x-2)(x-1)x$$

$$= \underbrace{\underbrace{\underbrace{\underbrace{1}_{p_0}}_{p_1} + \frac{x}{2}(x-1)}_{p_2} - \frac{x}{6}(x-2)(x-1)}_{p_3}$$

Horner's algorithm

number of multiplications

$$c_0 + c_1 d_1 + c_2 d_1 d_2 + c_3 d_1 d_2 d_3 \qquad 1+2+3$$

$$= c_0 + c_1 d_1 + (c_2 + c_3 d_3) d_1 d_2$$

$$= c_0 + (c_1 + (c_2 + c_3 d_3) d_2) d_1$$

nested $\underbrace{\alpha}_{c's} + \underbrace{\beta}_{\text{previous result}} \underbrace{\gamma}_{d's}$ form

$$= c_0 + \underbrace{(c_1 + \underbrace{(\underbrace{c_2 + c_3 d_3}_{\alpha_1 + \beta_1 \gamma_1}) d_2}_{\alpha_2 + \beta_2 \cdot \gamma_2}) d_1}_{\alpha_3 + \beta_3 \cdot \gamma_3} \qquad 1+1+1$$

example : Lagrange basis subordinate to nodes $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{\underbrace{(x_0-x_1)}_{-1}\underbrace{(x_0-x_2)}_{-2}\underbrace{(x_0-x_3)}_{-3}} = -\frac{1}{6}(x-2)(x-3)(x-4)$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{\underbrace{(x_1-x_0)}_{1}\underbrace{(x_1-x_2)}_{-1}\underbrace{(x_1-x_3)}_{-2}} = \frac{1}{2}(x-1)(x-3)(x-4)$$

$$\ell_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{\underbrace{(x_2-x_0)}_{2}\underbrace{(x_2-x_1)}_{1}\underbrace{(x_2-x_3)}_{-1}} = -\frac{1}{2}(x-1)(x-2)(x-4)$$

$$\ell_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{\underbrace{(x_3-x_0)}_{3}\underbrace{(x_3-x_1)}_{2}\underbrace{(x_3-x_2)}_{1}} = \frac{1}{6}(x-1)(x-2)(x-3)$$

(proof of linear independence)

equality as fn's

Assume

$$a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x) = 0$$

plug in $x = x_0$, then

$$0 = a_0 \underbrace{f_0(x_0)}_{\delta_{00} \quad 1} + a_1 \underbrace{f_1(x_0)}_{\delta_{10} \quad 0} + \cdots + a_n \underbrace{f_n(x_0)}_{\delta_{n0} \quad 0}$$

$$= a_0 .$$

plug in $x = x_1$, then

$$0 = a_0 \underbrace{f_0(x_1)}_{\delta_{01} \quad 0} + a_1 \underbrace{f_1(x_1)}_{\delta_{11} \quad 1} + \cdots + a_n \underbrace{f_n(x_1)}_{\delta_{n1} \quad 0}$$

$$= a_1 .$$

Repeat this to conclude $a_0 = a_1 = \cdots = a_n = 0$ $\blacksquare$

(proof of interpolation error)

① If x is one of the nodes we have $0 = 0$. ✓

② Assume $x \neq x_i$ $(i=0,1,2,\cdots,n)$.

/* The trick is (due to Cauchy) to think of x as a new node. */

Put $w(t) = \prod_{i=1}^{n} (t - x_i)$, then $w(x) \neq 0$.

Let $\lambda = (f(x) - p(x))/w(x)$ and introduce

$$\varphi(t) = f(t) - p(t) - \lambda w(t) \in C^{n+1}[a,b]$$

Observe that $\varphi(x_i) = 0$ $(i=0,1,2,\cdots,n)$ and $\varphi(x) = 0$ (by the construction of λ).

Thus, use Rolle's theorem $(n+1)$ times to argue $\exists \xi_j$ $(j=0,1,\cdots,n)$ s.t. $\varphi'(\xi_j) = 0$.

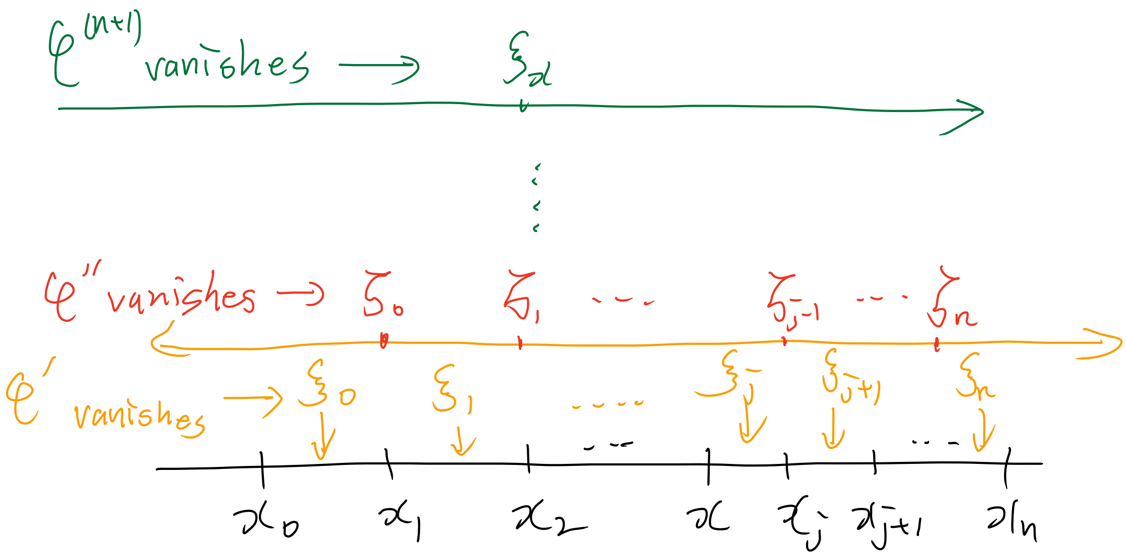
Next, do the similar to argue $\exists \zeta_k$ $(k=0,1,\cdots,n-1)$ s.t. $\varphi''(\zeta_k) = 0$. Repeat this to show that $\exists \xi_x$ s.t. $\varphi^{(n+1)}(\xi_x) = 0$. (see picture below). But

$$0 = \varphi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - \lambda (n+1)! \quad \text{(why?)}$$

$$= f^{(n+1)}(\xi_x) - \frac{(f(x) - p(x))}{w(x)} \cdot (n+1)!$$

Rearranging, we obtain

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$



example of error bound on sine interpolation.

$$|f^{(n)}(x)| \le 1 \quad \text{for all } x \in [0,1].$$

(an answer) Let us use the theorem we just proved.

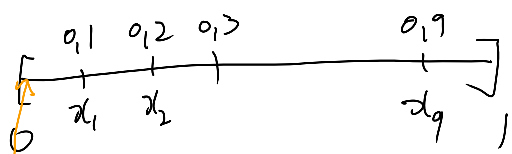
$$|f(x) - p(x)| = \left| \frac{f^{(11)}(x)}{11!} \prod_{i=1}^{10} (x - x_i) \right|$$

$$\le \frac{1}{11!} \prod_{i=0}^{10} \underbrace{|x - x_i|}_{\le 1} \longrightarrow (*)$$

$$= \frac{1}{11!} \simeq 2.5 \times 10^{-8}$$

If x_i's are equally spaced, $(*)$ continues

$$\le \underbrace{\frac{1}{11!} \prod_{i=1}^{10} \frac{i}{10}}_{\text{very pessimistic bound } |x - x_0| \le 1 \text{ is used.}} = \frac{1}{10^{10} \cdot 11} \simeq 9.1 \times 10^{-12}$$



x worst case

/*
If you don't like this the pessimistic bound $|x-x_0| \leq 1$ (even though you that must be very small), you can do the following. For the first two nodes, you can model $|x-x_0||x-x_1|$ as

$$|x(x-0.1)| \quad \text{on} \quad [0, 0.1]$$

But we know

$$|x(x-0.1)| \leq 0.5^2 = 0.0025$$

Then, the error bound reads

$$\leq \frac{1}{11!} 0.0025 \prod_{i=2}^{10} \frac{i}{10}$$

$$= \frac{0.0025}{10^9 \cdot 11} \simeq 2.27 \times 10^{-13}$$

But when we bound errors too precise calculations are not what we are after. Find a good balance. */