⊙ In general, movements in real world possess features of both kinds

Idea: Isn't it going to be helpful if we can decompose the acceleration into direction of velocity and one perpendicular to it?

$\boxed{\text{Thm}}$ Let $\vec{r}(t)$ be a C^2-smooth path. Then, its acceleration has following decomposition.

$$\boxed{\begin{aligned} \vec{a}(t) &= \vec{a}_T(t) + \vec{a}_N(t), \quad \text{where} \\ \vec{a}_T(t) &= \frac{d^2 s}{dt^2}\vec{T}(t) \\ \vec{a}_N(t) &= \|\vec{v}(t)\|\, T'(t) \quad \text{— ①} \\ &= \|\vec{v}(t)\|^2 \kappa(t) N(t) \quad \text{— ②} \end{aligned}}$$

$\vec{T} \perp \vec{N}$

⊙ $\vec{a}_T$ changes <u>speed</u>, while $\vec{a}_N$ changes the <u>direction</u>.

⊙ Usually, ① is easier to compute while ② tells us the relation b/w $\vec{a}_N$ and curvature.

⊙ Notation warning: Some authors and our HW4-#6 us a_T and a_N to denote the scalar part of $\vec{a}_T$ and $\vec{a}_N$, resp. (ans to HW4-#6 is a number, not vector) proof) To make use of $\vec{T} \perp \vec{T}'$ and $\vec{T}' \parallel \vec{N}$, express velocity as

$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$

$$\vec{v}(t) = \underline{\|\vec{v}(t)\|}\ \vec{T}(t).$$

Then,

$$\vec{a}(t) = \vec{v}'(t) \underset{\uparrow}{=} \underbrace{\|\vec{v}(t)\|'\, \vec{T}(t)}_{\text{ⓐ } \vec{a}_T} + \underbrace{\|\vec{v}(t)\|\, \vec{T}'(t)}_{\text{ⓑ } \vec{a}_N}$$

P___ rule of s · v

ⓐ is immediate by realizing

$$\|\vec{v}(t)\|' = \left(\frac{ds}{dt}\right)' = \frac{d^2 s}{dt^2} \quad (\text{hence ① follows})$$

② : First, note that

$$k(t_0) = \left\| \frac{d}{ds} \vec{T}(t(s_0)) \right\| \quad (t_0 = t(s_0))$$

$$= \left\| \vec{T}'(t_0) \cdot \frac{dt}{ds}(s_0) \right\| = \|\vec{T}'(t_0)\| \cdot \underbrace{\left|\frac{dt}{ds}(s_0)\right|}_{= \left|\frac{ds}{dt}\right|^{-1} = \|\vec{v}(t_0)\|^{-1}}$$

$$= \frac{\|\vec{T}'(t_0)\|}{\|\vec{v}(t_0)\|}$$

Now, drop subscript from t_0, and conclude:

$$\|\vec{v}(t)\|\, \vec{T}'(t) = \|\vec{v}(t)\|^2 \underbrace{\frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|}}_{k(t)} \underbrace{\frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}}_{\vec{N}(t)}$$

⓪ For explicit calculations, an intermediate step in the proof can be more practical.

$$\boxed{\vec{a} = (\vec{v})' = \left(\|\vec{v}\|\,\vec{T}\right)' = \underbrace{\|\vec{v}\|'\cdot\vec{T}}_{=\vec{a}_T} + \underbrace{\|\vec{v}\|\,\vec{T}'}_{=\vec{a}_N}}$$

product rule

This is a good example of how proofs can be useful.

$\boxed{\text{Clicker}}$ What should you do if you obtained $\vec{a} = \vec{a}_T + \vec{a}_N$, but you want only scalar part of $\vec{a}_T$ and $\vec{a}_N$ (short answer)

scalar　direction (unit vector)

$$\vec{u} = \frac{\|\vec{u}\|}{\|\vec{u}\|}\vec{u} = \|\vec{u}\|\,\frac{\vec{u}}{\|\vec{u}\|}$$

But be careful of sign (i.e. same dir.? or opposite dir.?)

$$\vec{a}_T = \|\vec{a}_T\|\left(\frac{\vec{a}_T}{\|\vec{a}_T\|}\right) \rightarrow$$ check if 1^{st} component has the same sign as $\vec{v}$ or $\vec{T}$.

same sign $\longrightarrow$ $\|\vec{a}_T\|$ is what we are looking for

opposite sign $\longrightarrow$ $= \underbrace{-\|\vec{a}_T\|}\left(-\dfrac{\vec{a}_T}{\|\vec{a}_T\|}\right)$

This is what we want.

For $\vec{a}_N$, no worries since we know the scalar part is positive.

$$\vec{a}_N = \|\vec{a}_N\| \underbrace{\frac{\vec{a}_N}{\|\vec{a}_N\|}}_{\vec{N}}.$$

Example: Suppose a particle's position at time t is described by
$$\vec{r}(t) = (\cos t + t\sin t, -\sin t + t\cos t, t^2/2)$$
$(t \geq 0)$. Find the acceleration and decompose it into tangential and normal components $\vec{a}_T$ and $\vec{a}_N$

$$\vec{v}(t) = \vec{r}\,'(t) = (\cancel{-\sin t} + \cancel{\sin t} + t\cos t, \; \cancel{-\cos t} + \cancel{\cos t} - t\sin t, \; t)$$
$$= t(\cos t, -\sin t, 1).$$

same sign $(t>0)$

$$\|\vec{v}(t)\| = \|t(\cos t, -\sin t, 1)\|$$
$$= t\,\|(\cos t, -\sin t, 1)\|$$
$$= t\sqrt{\cos^2 t + (-\sin t)^2 + 1^2}$$
$$= \sqrt{2}\,t.$$

$$\vec{v}(t) = \|\vec{v}(t)\| \cdot \vec{T}(t) = \overbrace{\sqrt{2}\,t}^{\|\vec{v}(t)\|} \cdot \overbrace{\left(\frac{1}{\sqrt{2}}(\cos t, -\sin t, 1)\right)}^{\vec{T}(t)}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \underbrace{\sqrt{2} \cdot \left(\frac{1}{\sqrt{2}}(\cos t, -\sin t, 1)\right)}_{= \vec{a}_T} + \underbrace{\sqrt{2}\,t\left(\frac{1}{\sqrt{2}}(-\sin t, -\cos t, 0)\right)}_{= \vec{a}_N}$$

product rule

scalar part of $\vec{a}_T$

NT 2.8 Local geometry of 3D curves.

(Frenet frame / TNB system)

Motivation: We studied <u>local</u> behaviors of a curve:
near a point

- direction of curve $\rightarrow \vec{T}(s)$
- how it is bent $\rightarrow k(s)\vec{N}(s)$

But these are not enough to give a complete descriptions of curves.
It turns out studying <u>how the osculating plane changes</u> gives the whole picture of curves.

$\boxed{\text{Clicker}}$ What should we study for this mission?

(A) $(\vec{T})'$ (B) $(\vec{N})'$ (C) $(\vec{T}\times\vec{N})'$

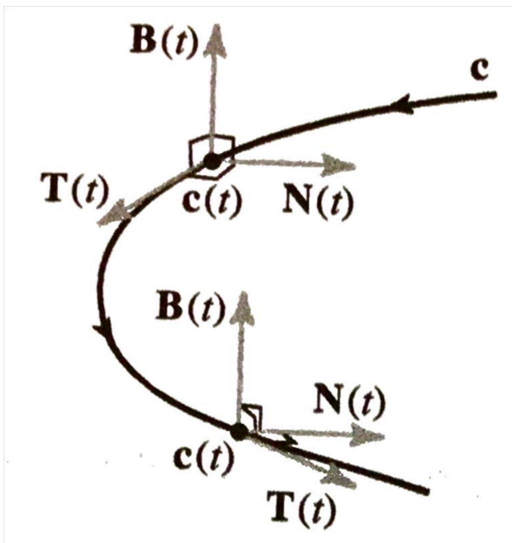
Learning objective:

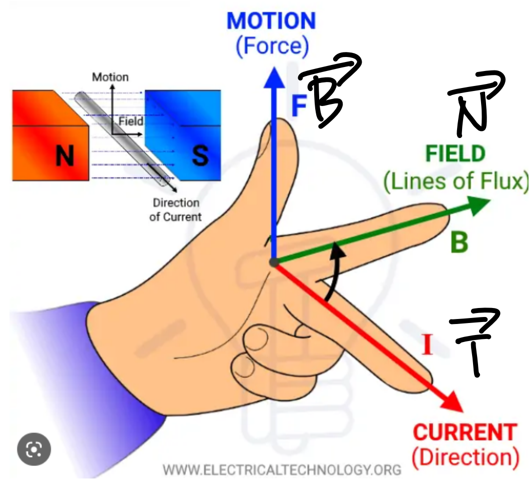
- I can find TNB frame and torsion.
- I can explain behaviors of a curve based on k, τ, T, N, B

Def (binormal vector)

Given a C^2-smooth curve $\vec{r}$

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called **binormal vector**, where $\vec{T}(t)$ and $\vec{N}(t)$ are unit tangent and principal normal vector of $\vec{r}$.





Left hand rule.

- $\vec{T}(t), \vec{N}(t), \vec{B}(t)$ are called **Frenet frame** or **TNB frame**.
- $\frac{d}{ds}\vec{B}(s)$ let us know how fast the osculating plane changes $\rightarrow$ "twist".

$\boxed{\text{Thm}}$

Given a C^2-smooth curve parametrized by arc length $\vec{r}(s)$, its TNB frame, the curvature $k(s)$, and the **torsion** $\tau(s)$ satisfy

$$\boxed{\begin{aligned} \vec{T}'(s) &= \phantom{-k(s)\vec{T}} \quad k(s)\vec{N} \\ \vec{N}'(s) &= -k(s)\vec{T} \qquad\qquad + \tau(s)\vec{B} \\ \vec{B}'(s) &= \phantom{-k(s)\vec{T}} \quad -\tau(s)\vec{N} \end{aligned}}$$

where, $\boxed{\tau(s) := -\vec{N}(s) \cdot \vec{B}'(s)}$

- Everything is defined for arc length parametrization.
- There is a formula for $\tau(t)$, but it is not considered an acceptable tool in this course. (learning purpose)
- Note that the curvature is always positive (or zero) while the torsion can be both positive and negative.
- In this course, the formulas in the 1st box is not the focus.

(proof) For ease of notation, omit input s, and all derivatives are $\frac{d}{ds}$.

First, we already know

$$\vec{T}' = k\vec{N}$$

Also note $\vec{N} \cdot \vec{N}' = 0$ since $\vec{N}$ has constant length. Therefore, when we decompose $\vec{N}'$ using TNB frame, it does not have $\vec{N}$ component. That is,

$$\vec{N}' = a\vec{T} + b\vec{B} \qquad \text{———————} \quad (*)$$

for some scalar function $a(s)$ and $b(s)$.

Now, differentiating $\vec{B} = \vec{T} \times \vec{N}$ and using the product rule for cross product

$$\vec{B}' = \underbrace{\vec{T}' \times \vec{N}}_{=0 \text{ b/c } \vec{T}' /\!/ \vec{N}} + \vec{T} \times \vec{N}' = \vec{T} \times \vec{N}'$$

$$= \vec{T} \times (a\vec{T} + b\vec{B}) \qquad \vec{N}' = a\vec{T} + b\vec{B}$$

$$= b\,\vec{T} \times \vec{B} \qquad \vec{T} \times \vec{B} = -\vec{N}$$

$$= -b\vec{N}$$

Here $b(s)$ tells us how fast $\vec{B}$ changes.
Call it **torsion** and label $\tau(s)$.
Plug this back into $(*)$. Then, we are left to show $a(s) = -k(s)$. To show this, differentiate $\vec{N} \cdot \vec{T} = 0$ to get

$$\vec{N}' \cdot \vec{T} + \vec{T}' \cdot \vec{N} = 0 \implies$$

$$\vec{N}' \cdot \vec{T} = -\underbrace{\vec{T}'}_{= k\vec{N}} \cdot \vec{N} = -(k\vec{N}) \cdot \vec{N} = -k$$

But taking dot product with $\vec{T}$ on both sides of $(*)$, we have

$$-k = \vec{N}' \cdot \vec{T} = a \underbrace{\vec{T} \cdot \vec{T}}_{=1} + b \underbrace{\vec{B} \cdot \vec{T}}_{=0} = a$$

Thus, we showed all relation b/w $\vec{T}, \vec{N}, \vec{B}$ and their derivatives.
Lastly, it follows

$$\tau(s) = -\vec{N}(s) \cdot \vec{B}'(s)$$

by taking dot product with $\vec{N}$ on $\vec{B}' = -\tau \vec{N}$. $\blacksquare$

Example : Find the torsion of the helix

$\vec{r}(t) = (a\cos t, a\sin t, bt)$ $t \geq 0$, $a, b > 0$.

(You will end finding every thing: κ, τ, T, N, B)

Step 1: reparametrize it.

$$\vec{r}'(t) = (-a\sin t, a\cos t, b)$$

$$s = \int_0^t \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2 + b^2} = \int_0^t \sqrt{a^2+b^2}\, d\theta$$

$$= t\sqrt{a^2+b^2} \implies t = \frac{s}{\sqrt{a^2+b^2}}$$

Put $C = \sqrt{a^2+b^2}$

Step 2: carry out calculations.

$$\vec{r}(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c} \right)$$

$$\vec{T}(s) = \vec{r}'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c} \right)$$

$$\vec{T}'(s) = \left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0 \right)$$

$$= \underbrace{\frac{a}{c^2}}_{\kappa(s)} \underbrace{\left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0 \right)}_{\text{already unit vector} \to \vec{N}(s)}$$

↳ if not $\vec{N}$ = normalized ver.

Thus,

$$\kappa(s) = \frac{a}{c^2} = \frac{a}{a^2+b^2}$$

$$\vec{N}(s) = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$$

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) \qquad (\underline{\text{order}}\ \text{matters})$$

$$= \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right), & \frac{a}{c}\cos\left(\frac{s}{c}\right), & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right), & -\sin\left(\frac{s}{c}\right), & 0 \end{matrix} \right|$$

$$= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)$$

$$\tau(s) = -\vec{B}'(s) \cdot \vec{N}(s)$$

$$= -\left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), +\frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right)$$

$$\cdot \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$$

$$= +\frac{b}{c^2} = \frac{b}{a^2+b^2}$$

$\boxed{\text{Thm}}$ (Congruence of 3D curves)

Two 3D curves with non-zero curvature are congruent if and only if their arc length parametrizations have the same <u>curvature</u> and <u>torsion</u> at each $0 \leq s \leq l$, where l is the length of the two curves.

(proof omitted)

$\boxed{\text{Thm}}$ 3D curve is a line or part of a line if and only if the <u>curvature</u> is zero everywhere.

(proof omitted)

$\boxed{\text{Thm}}$ 3D curve is contained in a plane if and only if the <u>torsion</u> is zero everywhere.

(proof omitted)