(proof of unique interpolation - Newton form)
Let k be the number of nodes.

$(k=1)$ $\quad P_0 = y_0 = c_0$

$(k=2)$ $\quad P_1 = c_0 + c_1(x - x_0)$

From $y_1 = P_1(x_1) = c_0 + c_1(x_1 - x_0)$

$$c_1 = [\,y_1 - c_0\,]/(x_1 - x_0)$$

$(k=3)$ $\quad P_2 = P_1(x) + c_2(x - x_1)(x - x_0)$

This construction allows us to only consider the new condition $P_2(x_2) = y_2$ b/c we already know $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$ while $c_2(x-x_1)(x-x_0)$ vanishes if $x = x_0$ or x_1.

From $y_2 = P_2(x_2) = P_1(x_2) + c_2(x_2 - x_1)(x_2 - x_0)$

$$c_2 = [\,y_2 - P_1(x_2)\,]/(x_2 - x_1)(x_2 - x_0)$$

Repeat this until $k=n$. Since the process is constructive, the resulting $P_n(x)$ must be unique.

Example : Find $p \in \Pi_3$ such that

x	0	1	2	3
y	1	1	2	3

$$p_0 = 1 = c_0$$

$$p_1(x) = c_0 + \frac{y_1 - c_0}{x_1 - x_0} = 1 + \frac{1-1}{1-0}(x-0) = 1$$

$$p_2(x) = p_1(x) + \frac{y_2 - p_1(x_2)}{(x_2-x_1)(x_2-x_0)}(x-x_1)(x-x_0)$$

$$= 1 + \frac{2-1}{2 \cdot 1}(x-1)x = 1 + \frac{x}{2}(x-1)$$

$$p_3(x) = p_2(x) + \frac{y_3 - p_2(x_3)}{(x_3-x_2)(x_3-x_1)(x_3-x_0)}(x-x_2)(x-x_1)(x-x_0)$$

$$= 1 + \frac{x}{2}(x-1) + \frac{3 - 1 - \frac{3}{2} \cdot 2}{1 \cdot 2 \cdot 3}(x-2)(x-1)x$$

$$= \underbrace{\underbrace{\underbrace{\underbrace{1}_{p_0}}_{p_1} + \frac{x}{2}(x-1)}_{p_2} - \frac{x}{6}(x-2)(x-1)}_{p_3}$$

Horner's algorithm

number of multiplications

$$c_0 + c_1 d_1 + c_2 d_1 d_2 + c_3 d_1 d_2 d_3 \qquad 1+2+3$$

$$= c_0 + c_1 d_1 + (c_2 + c_3 d_3) d_1 d_2$$

$$= c_0 + (c_1 + (c_2 + c_3 d_3) d_2) d_1$$

nested $\underbrace{\alpha}_{c\text{'s}} + \underbrace{\beta}_{\text{previous result}} \underbrace{\gamma}_{d\text{'s}}$ form

$$= \underbrace{c_0 + (\underbrace{c_1 + (\underbrace{c_2 + c_3 d_3}_{\alpha_1 + \beta_1 \gamma_1}) d_2}_{\alpha_2 + \beta_2 \cdot \gamma_2}) d_1}_{\alpha_3 + \beta_3 \cdot \gamma_3} \qquad 1+1+1$$

example : Lagrange basis subordinate to nodes $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4$

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{\underbrace{(x_0-x_1)}_{-1}\underbrace{(x_0-x_2)}_{-2}\underbrace{(x_0-x_3)}_{-3}} = -\frac{1}{6}(x-2)(x-3)(x-4)$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{\underbrace{(x_1-x_0)}_{1}\underbrace{(x_1-x_2)}_{-1}\underbrace{(x_1-x_3)}_{-2}} = \frac{1}{2}(x-1)(x-3)(x-4)$$

$$\ell_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{\underbrace{(x_2-x_0)}_{2}\underbrace{(x_2-x_1)}_{1}\underbrace{(x_2-x_3)}_{-1}} = -\frac{1}{2}(x-1)(x-2)(x-4)$$

$$\ell_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{\underbrace{(x_3-x_0)}_{3}\underbrace{(x_3-x_1)}_{2}\underbrace{(x_3-x_2)}_{1}} = \frac{1}{6}(x-1)(x-2)(x-3)$$

(proof of linear independence)

Assume $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ equality as fn's

$$a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x) \overset{\text{equality as fn's}}{=} 0$$

plug in $x = x_0$, then

$$0 = a_0 \underbrace{f_0(x_0)}_{\delta_{00}\ \ 1} + a_1 \underbrace{f_1(x_0)}_{\delta_{10}\ \ 0} + \cdots + a_n \underbrace{f_n(x_0)}_{\delta_{n0}\ \ 0}$$

$$= a_0 .$$

plug in $x = x_1$, then

$$0 = a_0 \underbrace{f_0(x_1)}_{\delta_{01}\ \ 0} + a_1 \underbrace{f_1(x_1)}_{\delta_{11}\ \ 1} + \cdots + a_n \underbrace{f_n(x_1)}_{\delta_{n1}\ \ 0}$$

$$= a_1 .$$

Repeat this to conclude $a_0 = a_1 = \cdots = a_n = 0$ $\blacksquare$

(proof of interpolation error)

① If x is one of the nodes we have

$0 = 0$. ✓

② Assume $x \neq x_i$ $(i = 0,1,2,\cdots,n)$.

/* The trick is (due to Cauchy) to think of x as a new node. */

Put $w(t) = \prod_{i=0}^{n} (t - x_i)$, then $w(x) \neq 0$.

Let $\lambda = (f(x) - p(x))/w(x)$ and introduce

$$\varphi(t) = f(t) - p(t) - \lambda w(t) \in C^{n+1}[a,b]$$

Observe that $\varphi(x_i) = 0$ $(i = 0,1,2,\cdots,n)$

and $\varphi(x) = 0$ (by the construction of λ).

Thus, use Rolle's theorem $(n+1)$ times to argue $\exists\, \xi_j$ $(j = 0,1,\cdots,n)$ s.t. $\varphi'(\xi_j) = 0$.

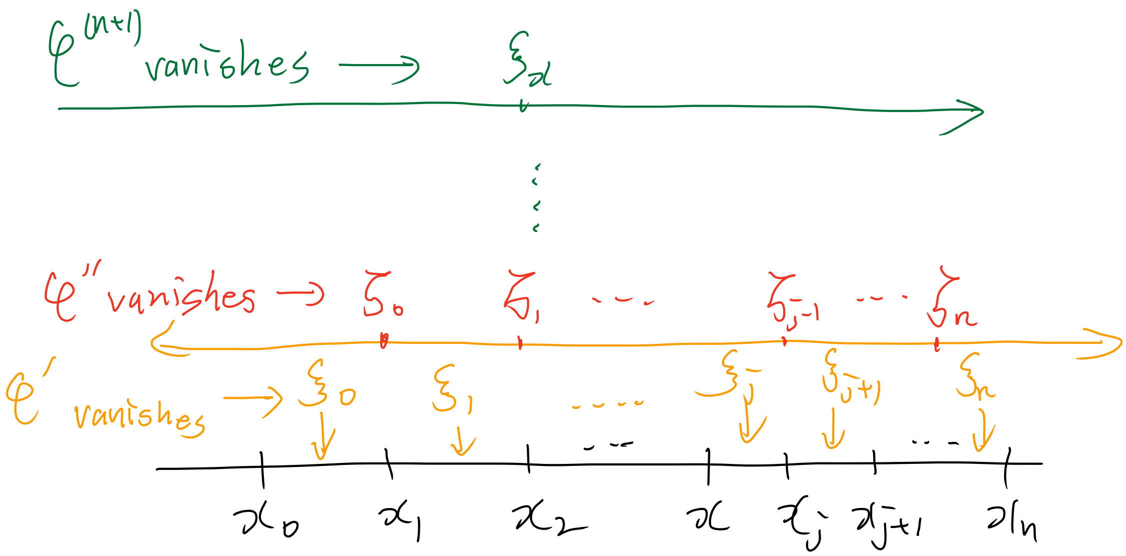
Next, do the similar to argue $\exists\, \zeta_k$ $(k = 0,1,\cdots,n-1)$ s.t. $\varphi''(\zeta_k) = 0$. Repeat this to show that $\exists\, \xi_x$ s.t. $\varphi^{(n+1)}(\xi_x) = 0$. (see picture below). But

$$0 = \varphi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - \lambda (n+1)! \quad \text{(why?)}$$

$$= f^{(n+1)}(\xi_x) - \frac{(f(x)-p(x))}{w(x)} \cdot (n+1)!$$

Rearranging, we obtain

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$



example of error bound on sine interpolation.

$|f^{(n)}(x)| \le 1$ for all $x \in [0,1]$.

(an answer) Let us use the theorem we just proved.

$$|f(x) - p(x)| = \left| \frac{f^{(11)}(x)}{11!} \prod_{i=1}^{10} (x - x_i) \right|$$

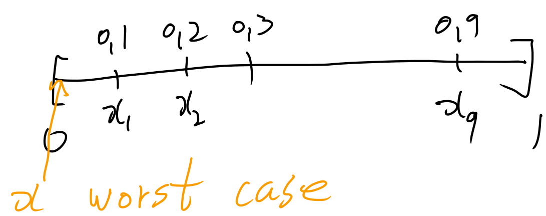
$$\le \frac{1}{11!} \prod_{i=0}^{10} \underbrace{|x - x_i|}_{\le 1} \longrightarrow (*)$$

$$= \frac{1}{11!} \simeq 2.5 \times 10^{-8}$$

If x_i's are equally spaced, (*) continues

$$\le \underbrace{\frac{1}{11!} \prod_{i=1}^{10} \frac{i}{10}} = \frac{1}{10^{10} \cdot 11} \simeq 9.1 \times 10^{-12}$$

very pessimistic bound $|x - x_0| \le 1$ is used.



/*
If you don't like this the pessimistic bound $|x-x_0| \leq 1$ (even though you that must be very small), you can do the following. For the first two nodes, you can model $|x-x_0||x-x_1|$ as

$$|x(x-0.1)| \quad \text{on} \quad [0, 0.1]$$

But we know

$$|x(x-0.1)| \leq 0.5^2 = 0.0025$$

Then, the error bound reads

$$\leq \frac{1}{11!} 0.0025 \prod_{i=2}^{10} \frac{i}{10}$$

$$= \frac{0.0025}{10^9 \cdot 11} \simeq 2.27 \times 10^{-13}$$

But when we bound errors, too precise calculations are not what we are after. Find a good balance. */

(proof of charaterization of Chebyshev poly's)

We argue by an induction.

① $T_0(x) = \cos(0 \cdot \cos^{-1} x) = \cos 0 = 1$

② $T_1(x) = \cos(\cos^{-1} x) = x$

③ Suppose $T_k(x) = \cos(k \cos^{-1} x)$ for $k = 0, 1, \cdots, n$.

/* $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\Rightarrow \begin{array}{rl} & \cos((n+1)\theta) = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta \\ + & \cos((n-1)\theta) = \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta \\ \hline \end{array}$$

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos(n\theta)\cos\theta$$

Take $\theta = \cos^{-1} x$ for $x \in [-1, 1]$:

$$\begin{aligned} \cos((n+1)\cos^{-1} x) &= 2\cos(n\cos^{-1} x)\overbrace{\cos(\cos^{-1} x)}^{x} \\ &\quad - \cos((n-1)\cos^{-1} x) \\ &= 2x T_n(x) - T_{n-1}(x) \\ &= T_{n+1}(x) \end{aligned}$$

by recursive def. of Chebyshev

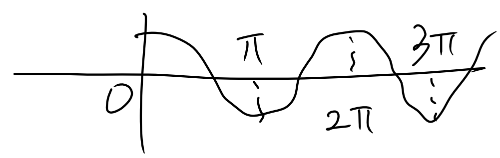
(proof of basic properties of Chebyshev poly's.)

① $|T_n(x)| = |\cos(n\cos^{-1}x)| \le 1$

② $T_n\left(\cos\frac{j\pi}{n}\right) = \cos\left(\cancel{n}\,\cancel{\cos^{-1}}\left(\cancel{\cos}\frac{j\pi}{\cancel{n}}\right)\right)$

$= \cos(j\pi)$

$= (-1)^j$

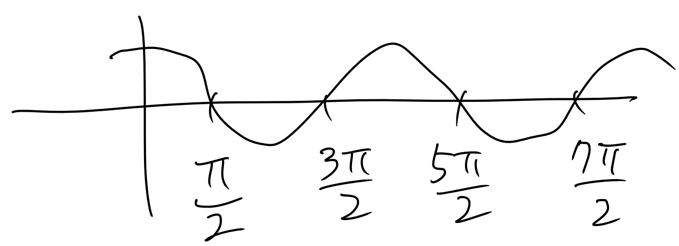


③ $T_n\left(\cos\frac{2j-1}{2n}\pi\right)$

$= \cos\left(\cancel{n}\,\cancel{\cos^{-1}}\left(\cancel{\cos}\frac{2j-1}{2\cancel{n}}\pi\right)\right)$

$= \cos\left(j\pi - \frac{\pi}{2}\right)$

$= 0$



(proof of maximum bound of monic poly.)

(equality) Let $q(x) = 2^{1-n} T_n(x)$. This is monic and of degree n:

$$T_2(x) = \underbrace{2x}_{\times 2} \underbrace{T_1(x)}_{\text{1 higher deg}} - T_0(x) = 2x^2 - 1 \in \Pi_2$$

$$T_3(x) = \underbrace{2x}_{\times 2} \underbrace{T_2(x)}_{\text{1 higher deg}} - T_1(x) = 2^2 x^3 - 3x \in \Pi_3$$

$\vdots$

and so on

$$T_n(x) = 2x\, T_{n-1}(x) - T_{n-2}(x) = 2^{1-n} x^n + \cdots \in \Pi_n$$

Also, $|T_n(x)| \leq 1$ on $[-1, 1]$ by previous corollary.

Thus, for all $x \in [-1, 1]$, we have

$$\left|2^{1-n} \cdot T_n(x)\right| = 2^{1-n} \left|T_n(x)\right| \leq 2^{1-n}$$

(Inequality) Suppose $\exists p \in \Pi_n$, monic such that

$$\max_{|x| \leq 1} |P(x)| < 2^{1-n}$$

Let $x_j = \cos\left(\frac{j\pi}{n}\right)$. Then, observe

(*) $(-1)^j \cdot p(x_j) \le |p(x_j)| < 2^{1-n} = (-1)^j q(x_j)$

The last equality holds b/c

$T_n(x_j) = (-1)^j$ (previous corollary)

$\Rightarrow \quad q(x_j) = 2^{1-n} \cdot T_n(x_j) = 2^{1-n}(-1)^j$

$\Rightarrow \quad (-1)^j q(x_j) = (-1)^j \cdot (-1)^j \cdot 2^{1-n} = 2^{1-n}$.

Rearranging (*), we have

(**) $(-1)^j \left(q(x_j) - p(x_j) \right) > 0 \quad (i = 0, 1, \cdots, n)$

$\Rightarrow$ $q(x) - p(x)$ changes sign n times $\Rightarrow$ n roots

But $\deg\left(q(x) - p(x)\right) = n-1$ since x^n terms have been cancelled out. Besides,

$q(x) - p(x)$ is not zero function

since, e.g., $q(x_0) - p(x_0) > 0$ from (**)

Thus, we have contradiction. $\blacksquare$

(proof of error bound with Chebyshev nodes)

$$|f(x)-p(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)\right|$$

$$\leq \frac{1}{(n+1)!}\max_{|t|\leq 1}\left|f^{(n+1)}(t)\right| \cdot \underbrace{\max_{|x|\leq 1}\left|\overbrace{\prod_{i=0}^{n}(x-x_i)}^{\in \Pi_{n+1}}\right|}$$

minimized by $2^{-n}T_{n+1}$, which is achieved by choosing x_i's to be the zeros of T_{n+1}.

$$= \frac{1}{2^{n}(n+1)!}\max_{|t|\leq 1}\left|f^{(n+1)}(t)\right|$$

(proof recursive formula of divided differences)

Idea: Use HW3 #4 and look at x^n term.

[[If g interplates f at $x_0, x_1, \cdots x_{n-1}$, and h does at $x_1, x_2, \cdots, x_n$, then

$$\phi(x) = g(x) + \frac{x - x_0}{x_n - x_0}(h(x) - g(x))$$

interpolates f at $x_0, x_1, \cdots, x_n$.]]

/* This statement itself holds even if g and h are not polynomials. But in our main proof, they will be polynomials. */

Now, let $g(x)$ and $h(x)$ to be the unique polynomials that interpolate $x_0, x_1, \cdots, x_{n-1}$ and $x_1, x_2, \cdots, x_n$ respectively. That is

$$g(x) = \sum_{i=0}^{n-1} f[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1} (x - x_j),$$

$$h(x) = \sum_{i=1}^{n} f[x_1, x_2, \cdots, x_i] \prod_{j=1}^{i-1} (x - x_j),$$

$$P(x) = \sum_{i=0}^{n} f[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

(Coeff. of x^n)

$LHS = f[x_0, x_1, \cdots, x_n]$

$$RHS = \frac{1}{x_n - x_0} \cdot \left(f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}] \right)$$

$$
\begin{array}{lllll}
5 & 4 & 3 & 2 & 1 \\
x_0 & f(x_0) & f[x_0,x_1] & f[x_0,x_1,x_2] & f[x_0,x_1,x_2,x_3] \\
x_1 & f(x_1) & f[x_1,x_2] & f[x_1,x_2,x_3] & \\
x_2 & f(x_2) & f[x_2,x_3] & & \\
x_3 & f(x_3) & & & \\
 & x_0,x_1,x_2,x_3 & x_0,x_1,x_2,x_3 & x_0,x_3 & \\
 & 4 & 3 & 2 &
\end{array}
$$

1 – 5 logical order to get interpolation

(e.g., to get 1, we need 2. For that, we need 3 and so on)

The construction goes in the opposite order.

$$
\begin{array}{lllll}
x_0 & f(x_0) & f[x_0,x_1] & f[x_0,x_1,x_2] & f[x_0,x_1,x_2,x_3] \\
x_1 & f(x_1) & f[x_1,x_2] & f[x_1,x_2,x_3] & \\
x_2 & f(x_2) & f[x_2,x_3] & & \\
x_3 & f(x_3) & & & \\
 & 1 & 2 & 3 & 4
\end{array}
$$

5 $p(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$
$\qquad + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$

(example)

0	1	$\frac{1-1}{1-0}=0$	$\frac{1-0}{2-0}=\frac{1}{2}$	$\frac{0-\frac{1}{2}}{3-0}=-\frac{1}{6}$
1	1	$\frac{2-1}{2-1}=1$	$\frac{1-1}{3-1}=0$	3
2	2	$\frac{3-2}{3-2}=1$	2	
3	3	1		

$$p(x) = 1 + \frac{1}{2}(x)(x-1) - \frac{1}{6}(x)(x-1)(x-2)$$

(proof of permutation invariance of divided differences)

$f[z_0, z_1, \cdots, z_n]$ is the coeff. of x^n of the polynomial that interpolates $z_0, z_1, \cdots, z_n$ while $f[x_0, x_1, \cdots, x_n]$ is the coeff. of x^n of the polynomial that interpolates $x_0, x_1, \cdots, x_n$. But they are the same set of nodes (in different order). Since such a polynomial is unique the coeff.'s of x^n must equal.

(proof of the error eqn in (divided differences) Newton form.)

CASE I) If $t = x_i$ ($\exists i = 0, 1, \cdots, n$), the equality hold: $0 = 0$.

CASE II) Assume $t \neq x_i$ ($\forall i = 0, 1, \cdots, n$).

Think of $q(x)$ that interpolates $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n)), (t, f(t))$.

We have learned that

$$q(x) = \underbrace{p(x)}_{} + f[x_0, x_1, \cdots, x_n, t] \prod_{j=0}^{n} (x - x_j)$$

Without $(t, f(t))$, this would be $\prod_{j=0}^{n-1} (x - x_j)$

Evaluate this at $x = t$, and realize that $q(t) = f(t)$,

$$q(t) = f(t) = p(t) + f[x_0, x_1, \cdots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Rearrange this to obtain the desired result.

(proof of "$\underbrace{\text{Discrete derivatives}}_{\text{not an official name}}$")

Consider that $p(x)$ interpolates $f(x)$ at $x_0, x_1, \cdots, x_{n-1}$ and the error at the node x_n. Then, from the interpolation error eqn, we know that there is $\xi \in [a,b]$ s.t.

$$(*) \quad f(x_n) - p(x_n) = \frac{1}{n!} f^{(n)}(\xi_{x_n}) \prod_{i=0}^{n-1} (x_n - x_i).$$

On the other hand, the previous thm asserts

$$(**) \quad f(x_n) - p(x_n) = f[x_0, x_1, \cdots, x_n] \prod_{i=0}^{n-1} (x_n - x_i).$$

Now, solve $(*)$ for $\frac{1}{n!} f^{(n)}(\xi_{x_n})$ and $(**)$ for $f[x_0, x_1, \cdots, x_n]$, the we see that they are equal as $f(x_n) - p(x_n) \Big/ \prod_{i=0}^{n-1} (x_n - x_i)$.