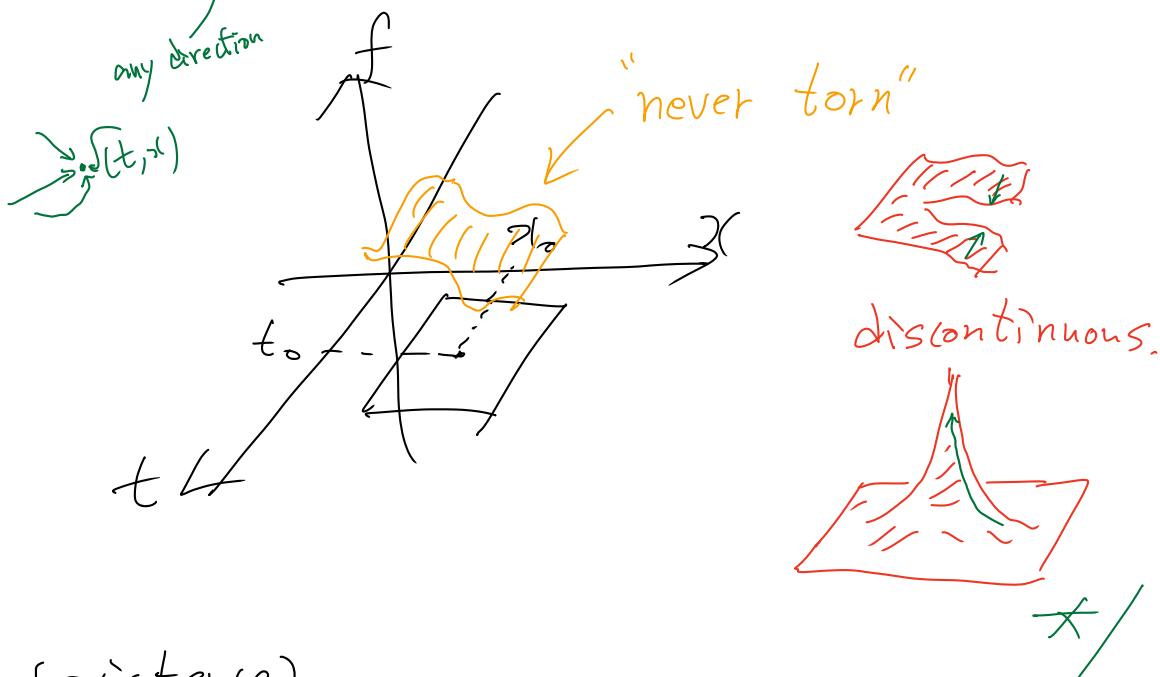


(example) $x'(t) = x^{\frac{2}{3}}$.

This corresponds to $f(t, x) = x^{\frac{2}{3}}$

f is continuous (as a function of two variables).

/* $\lim_{(z, z) \rightarrow (t, x)} f(z, z) = f(t, x)$



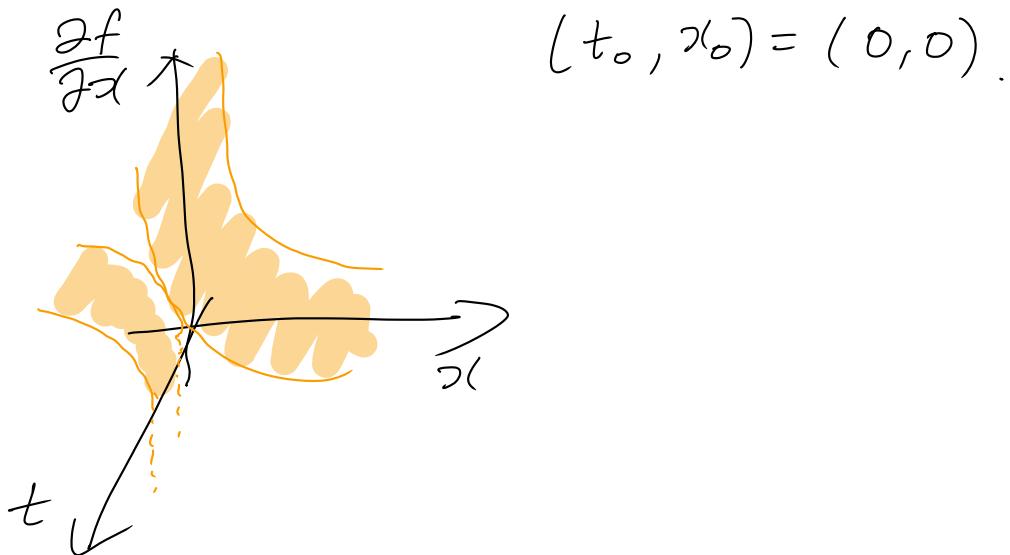
(existence)

Therefore by the theorem, a function $x(t)$ on a small open interval of t $(-\delta, \delta)$ s.t. $x'(t) = x^{\frac{2}{3}}$ and $x(0) = 0$.

For example, $x(t) \equiv 0$: $x'(t) = 0 = 0^{\frac{2}{3}}$

(uniqueness)

However, $\frac{\partial f}{\partial x}(t, x) = \frac{2}{3}x^{-\frac{1}{3}}$ is not continuous on any rectangle around



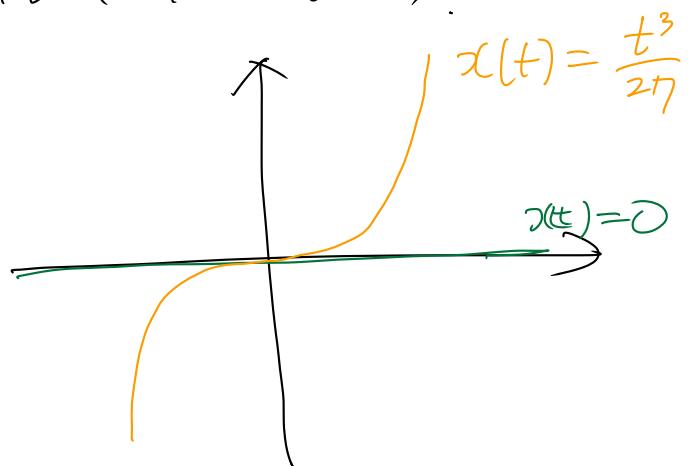
Thus, the theorem does not guarantee uniqueness. In fact, we have another solution.

For example, use the ansatz

$$x(t) = at^b$$

$$\begin{aligned} x'(t) &= abt^{b-1} & b-1 = \frac{2}{3}b \Rightarrow b = 3 \\ (x(t))^{\frac{2}{3}} &= a^{\frac{2}{3}} t^{\frac{2}{3}b} \end{aligned} \quad \Rightarrow ab = a^{\frac{2}{3}} \Rightarrow a^{\frac{1}{3}}b^3 = a^2$$
$$\Rightarrow a = \frac{1}{b^3} = \frac{1}{27}.$$

This also satisfies the IC
(initial condition).

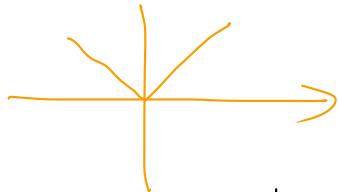


$$f(x) = x^2 \in C[-1, 1]$$

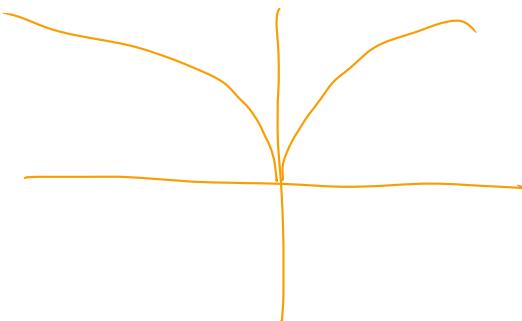
$$g(x) = |x| \in \text{Lip}[-1, 1] \text{ but } g \notin C^1[-1, 1]$$

$$|g(x) - g(y)| = ||x| - |y|| \leq |x - y| \quad L=1.$$

↑ inverse triangle ineq.



$$h(x) = |x|^{\frac{1}{2}} \in C[-1, 1] \text{ but } h \notin \text{Lip}[0, 1].$$



$$\begin{aligned}|h(x) - h(y)| &= \left| |x|^{\frac{1}{2}} - |y|^{\frac{1}{2}} \right| \\ &= |\delta - 2\delta| = \delta\end{aligned}$$

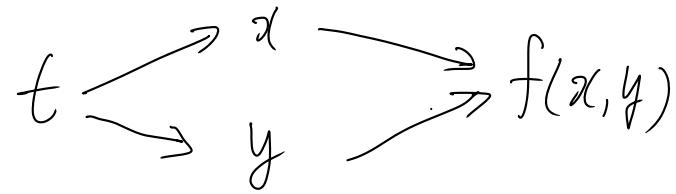
Choose $x = \delta^2$, $y = 4\delta^2$, then $|x - y| = 3\delta^2$

For any $L > 0$, if $\delta < \frac{1}{3L}$

$$|h(x) - h(y)| = \delta \frac{3L}{3L} > 3L \delta^2 = L|x - y|.$$

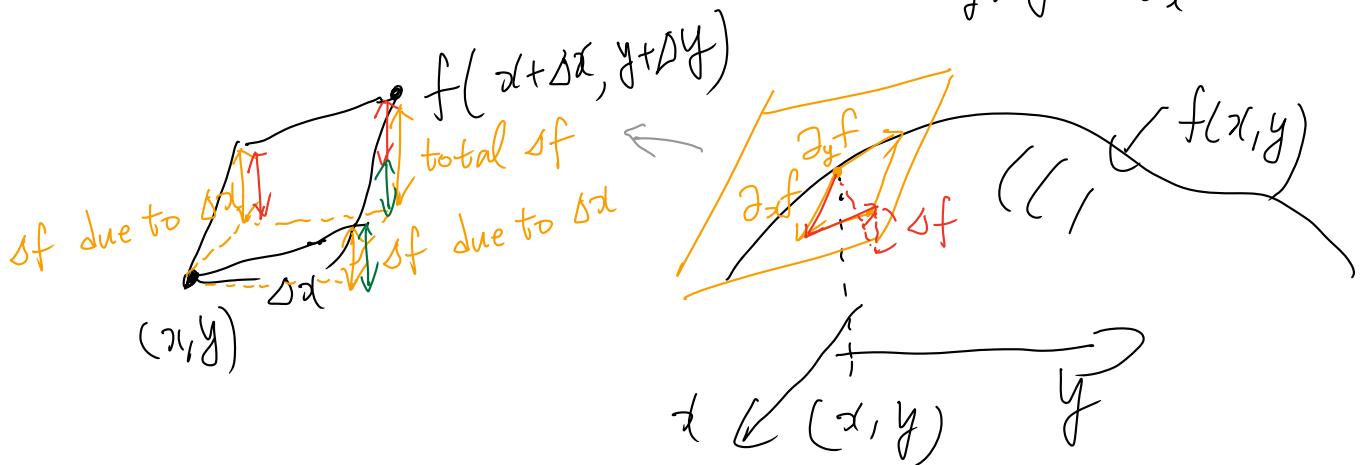
Example : Refresh calculus.

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$



$$\Delta t \rightarrow \begin{array}{l} \Delta x \\ \approx x' \cdot \Delta t \\ \Delta y \\ \approx y' \cdot \Delta t \end{array} \rightarrow \begin{array}{l} \Delta f \\ \approx \partial_y f \cdot \Delta y + \partial_x f \cdot \Delta x \end{array}$$

$$\Rightarrow \frac{df}{dt} \approx \frac{\partial_y f \cdot \Delta y}{\Delta t} + \frac{\partial_x f \cdot \Delta x}{\Delta t} = \partial_y f \cdot y' + \partial_x f \cdot x'$$



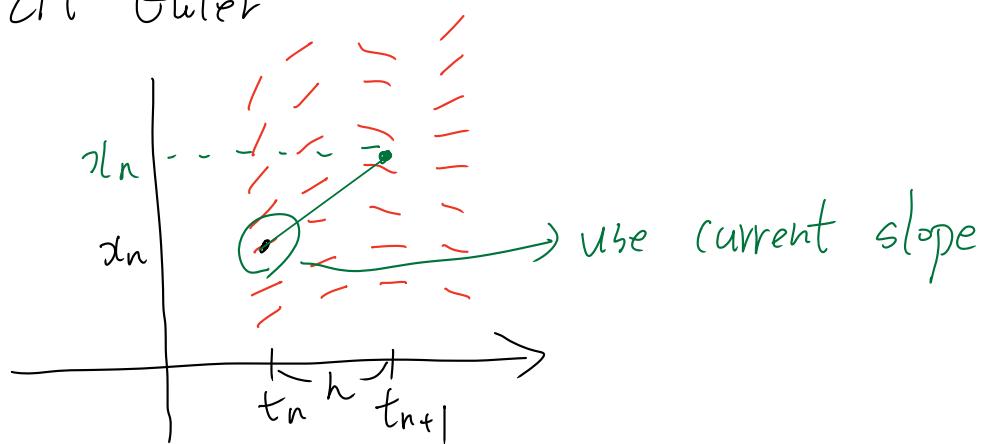
$$x'' = (f(t, x))' = \underbrace{f_t(t, x)}_{f_{tt}(t, x)} + \underbrace{f_{xx}(t, x)x'}_{f_{tx}(t, x)x'}$$

$$\begin{aligned} x''' &= (f_t(t, x) + f_{xx}(t, x)x')' \\ &= f_{tt}(t, x) + f_{tx}(t, x)x' \\ &\quad + f_{xt}(t, x) + f_{xx}(t, x)x' x' + f_{xx}(t, x)x'' \end{aligned}$$

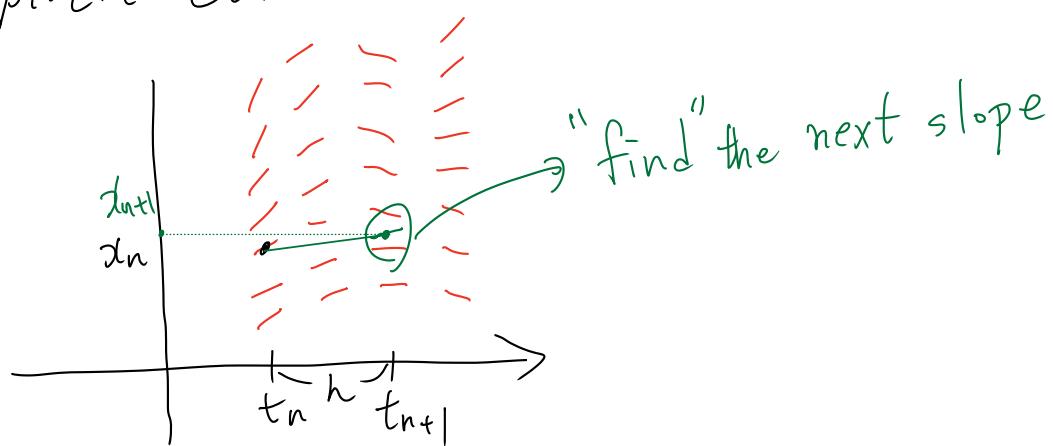
$$\begin{aligned}
 x''(t) &= -\sin t + 2t - \cos x \cdot x' \\
 x'''(t) &= -\cos t + 2 - \cos x \cdot x'' \\
 &\quad + (\sin x \cdot x') \cdot x' \\
 &= -\cos t + 2 - \cos x \cdot x'' + \sin x \cdot (x')^2
 \end{aligned}$$

$$\begin{aligned}
 x''''(t) &= \sin t + \sin x \cdot x' \cdot x'' - \cos x \cdot x''' \\
 &\quad + \cos x \cdot x' \cdot (x')^2 + \sin x \cdot 2(x') \cdot x'' \\
 &= \sin t + 3 \sin x \cdot x' \cdot x'' - \cos x \cdot x''' \\
 &\quad + \cos x \cdot (x')^3
 \end{aligned}$$

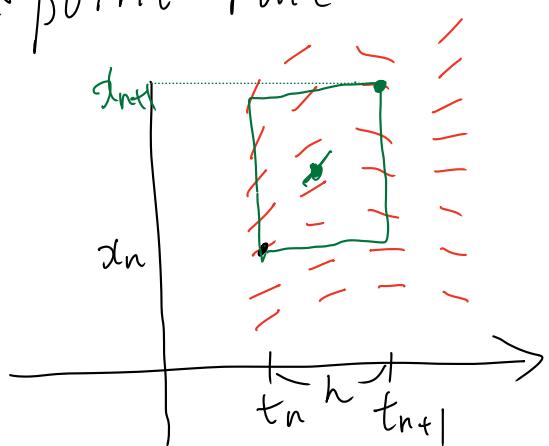
Explicit Euler



Implicit Euler



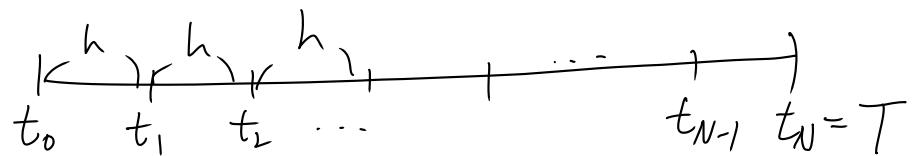
mid point rule



Global truncation error.

Suppose local truncation error is $O(h^{n+1})$.

Then, global error accumulates over the time steps.



$$h = \frac{T-t_0}{N} \quad \Rightarrow \quad N = \frac{T-t_0}{h}$$

$$\begin{aligned} \sum_{i=1}^N O(h^{n+1}) &= N \cdot O(h^{n+1}) = \frac{T-t_0}{h} O(h^{n+1}) \\ &= \underbrace{(T-t_0)}_{\text{fixed.}} O(h^n) \\ &= O(h^n) \end{aligned}$$

Derivation of 2nd order RK method

1. Truncate 3rd or higher order terms.

2. $x'(t) = f(t, x(t))$ no problem.

$$x''(t) = f_t(t, x(t)) + f_{x_1}(t, x(t)) x'(t)$$

$$= f_t(t, x(t)) + f_{x_1}(t, x(t)) f(t, x(t))$$

Thus,

$$x(t+h) \approx x + hf + \frac{h^2}{2} (f_t + f_{x_1} \cdot f)$$

3. $f(t+h, \underline{x+hf})$

explicit Euler prediction of 2nd slot

$$= f + f_t h + f_x \cdot hf$$

$$4. x(t+h) \approx x + hf + \frac{h}{2} (f(t+h, \underline{x+hf}) - f)$$

$$= x + \frac{1}{2} \underbrace{hf}_{F_1} + \frac{1}{2} hf(t+h, \underbrace{x+hf}_{F_1})$$

$$= x + \frac{1}{2} (F_1 + F_2)$$

where $F_1 = hf$

$$F_2 = hf(t+h, x+F_1)$$

/* Taylor theorem in two variables

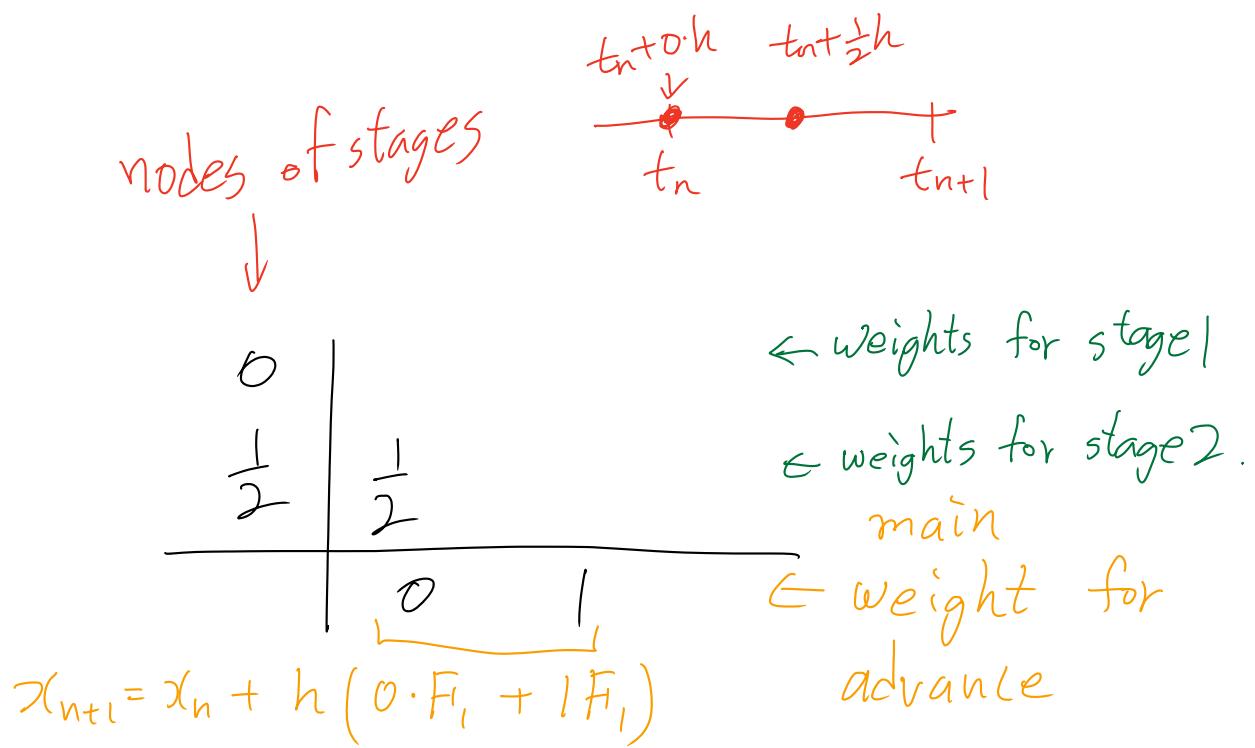
$$f(x+\Delta x, y+\Delta y) = f(x, y) - \text{0-th order appr.}$$

$$+ f_x(x, y) \cdot \Delta x + f_y(x, y) \Delta y - \text{linear appr.}$$

$$+ \frac{1}{2} \left(f_{xx}(x, y) \Delta x^2 + 2 f_{xy}(x, y) \Delta x \Delta y + f_{yy}(x, y) \Delta y^2 \right), \quad \text{quadratic appr.}$$

where (ξ_x, ξ_y) is a point on the segment connecting (x, y) and $(x+\Delta x, y+\Delta y)$.

Recover modified Euler from tableau

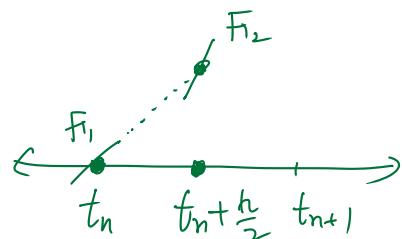


Slope from stage 1

$$F_1 = f(t_n + 0 \cdot h, x_n + h(0 \cdot F_1 + 0 \cdot F_2))$$

Slope from stage 2

$$\bar{F}_2 = f(t_n + \frac{1}{2} \cdot h, x_n + h(\frac{1}{2} F_1 + 0 \cdot F_2))$$



$$x_{n+1} = x_n + h \cdot f(t_n + \frac{h}{2}, x_n + \frac{h}{2} f(t_n, x_n))$$

Butcher's tableau for RK4.

Stages : $t, t + \frac{1}{2}h, t + \frac{1}{2}h, t + h$
(can be repeated) $\rightarrow 0, \frac{1}{2}, \frac{1}{2}, 1$

weights for advance : $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$

inner weights for stage 1 : 0, 0, 0, 0

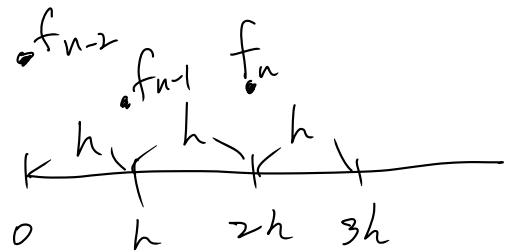
inner weights for stage 2 : $\frac{1}{2}, 0, 0, 0$

inner weights for stage 3 : $0, \frac{1}{2}, 0, 0$

inner weights for stage 4 : $0, 0, 1, 0$

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

(Verifying AB3)



$$b_0 = \int_{2h}^{3h} \frac{(t-h)(t-2h)}{(0-h)(0-2h)} dt$$

$$s = t - 2h \quad dt = ds$$

$$= \frac{1}{2h^2} \int_0^h (s+h)s ds$$

$\underbrace{s^2 + hs}$

$$= \frac{1}{2h^2} \left[\frac{s^3}{3} + h \frac{s^2}{2} \right]_0^h$$

$$= \frac{1}{2h^2} \left[\frac{h^3}{3} + \frac{h^3}{2} \right]$$

$$= h \cdot \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12} h$$

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t)) dt$$

$$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t) dt \quad p(t) \text{ interpolates}$$

$(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n)$ 3 nodes $(m+1)$ nodes with $m=2$

Since we assume x_{n-2}, x_{n-1}, x_n are exact,

Thus, e.g., $f_{n-2} = f(t_{n-2}, x(t_{n-2})) = \underbrace{f(t_{n-2}, x_{n-2})}_{\text{computable}}$.

$$\begin{aligned} |x(t_{n+1}) - x_{n+1}| &\leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)| dt \\ &\leq \int_{t_n}^{t_{n+1}} \frac{1}{3!} \left| \frac{d^3}{dt^3} f(t, x(t)) \right| \prod_{i=0}^2 |t - t_{n-i}| dt \\ &\leq \frac{3^3}{3!} M \cdot h^3 \int_{t_n}^{t_{n+1}} dt \end{aligned}$$

$$\left(\text{where } M = \max_{t_0 \leq t \leq T} \left| \frac{d^3}{dt^3} f(t, x(t)) \right| \right)$$

$$= Ch^4$$

(where $C = \frac{3^3 M}{3!}$. Important thing is this is independent of h)

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t)) dt$$

$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t) dt$ $p(t)$ interpolates
 $(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})$
4 nodes $(m+1)$ nodes with $m=3$

Thus,

$$\begin{aligned} |x(t_{n+1}) - x_{n+1}| &\leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)| dt \\ &\leq \int_{t_n}^{t_{n+1}} \frac{1}{4!} \left| \frac{d^4}{dt^4} f(t, x(t)) \right|_{t=t_n}^3 |t - t_{n+1}| dt \\ &\leq \frac{4^4}{4!} M \cdot h^4 \int_{t_n}^{t_{n+1}} dt \end{aligned}$$

(where $M = \max_{t_0 \leq t \leq T} \left| \frac{d^4}{dt^4} f(t, x(t)) \right|$)

$$= Ch^5$$

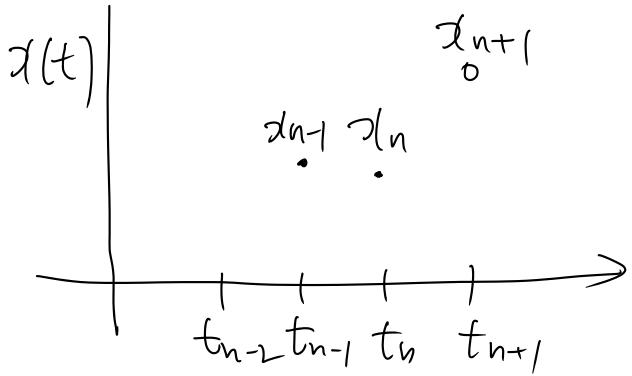
(where $C = \frac{4^4 M}{4!}$ is independent of h)

/* This proof is not rigorous because we don't have $x_{n+1} = x(t_{n+1})$ in practice.

Therefore our error formula from the interpolation theory is not quite true.

Nonetheless, there are other approaches that rigorously provide the same conclusion. $\star /$.

Derivation of BDF1 2 (instead of BDF3)



$$p(t) = x_{n-1} \cdot l_0(t) + x_n \cdot l_1(t) + x_{n+1} \cdot l_2(t)$$

$$l_0(t) = \frac{(t - t_{n-1})(t - t_{n+1})}{(t_{n-1} - t_n)(t_{n-1} - t_{n+1})} = \frac{1}{2h^2}(t - t_n)(t - t_{n+1})$$

$$l_1(t) = \frac{(t - t_{n-1})(t - t_{n+1})}{(t_n - t_{n-1})(t_n - t_{n+1})} = -\frac{1}{h^2}(t - t_{n-1})(t - t_{n+1})$$

$$l_2(t) = \frac{(t - t_{n-1})(t - t_n)}{(t_{n+1} - t_{n-1})(t_{n+1} - t_n)} = \frac{1}{2h^2}(t - t_{n-1})(t - t_n)$$

$$l'_0(t) = \frac{1}{2h^2}(t - t_n + t - t_{n+1})$$

$$l'_1(t) = -\frac{1}{h^2}(t - t_{n-1} + t - t_{n+1})$$

$$l'_2(t) = \frac{1}{2h^2}(t - t_{n-1} + t - t_n)$$

$$\begin{aligned}
 p'(t_{n+1}) &= x_{n-1} \ell_0'(t_{n+1}) + x_n \ell_1'(t_{n+1}) + x_{n+1} \ell_2'(t_{n+1}) \\
 &= x_{n-1} \cdot \frac{1}{2h} + x_n \left(-\frac{2}{h} \right) + x_{n+1} \cdot \frac{3}{2h} \\
 &\approx f_{n+1}
 \end{aligned}$$

Divide through by $\frac{3}{2h}$ (to solve for x_{n+1}), then

$$x_{n+1} - \frac{4}{3}x_n + \frac{1}{3}x_{n-1} = \frac{2}{3}hf_{n+1}$$

(order condition of LMM)

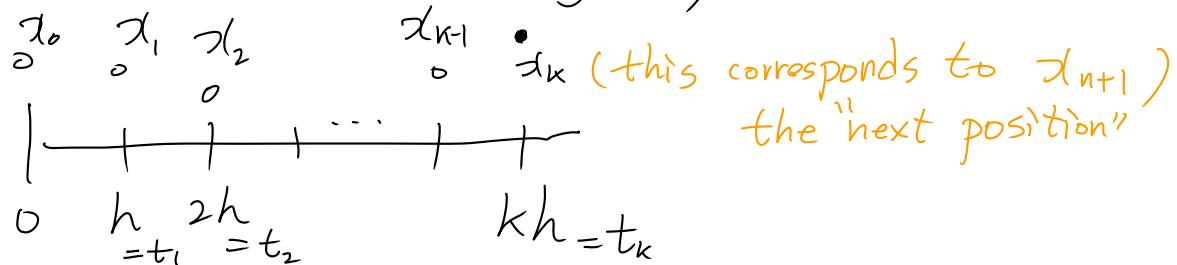
Suppose the LTE is of order $m+1$.

That is, if all the history is exact,

$$|\varphi_{n+1} - \varphi(t_{n+1})| = O(h^{m+1})$$

For ease of notation, assume $t_0=0$.

This does not bring any limitation.



Have (exact history) $x_0 = \varphi(0), x_1 = \varphi(h), \dots, x_{k-1} = \varphi((k-1)h)$

and (order m) $|\varphi_k - \varphi(kh)| = O(h^{m+1})$

Want: $d_0 = d_1 = \dots = d_m = 0$

Assuming enough smoothness of the true solution, for each $i=0, 1, \dots, k$,

Taylor expansion of $\varphi(t)$ and $\varphi'(t)$ at $t=0$ read:

$$\varphi(ih) = \varphi(0) + (ih) \cdot \varphi'(0) + \dots + \frac{(ih)^j}{j!} \varphi^{(j)}(0) + \dots$$

$$\begin{aligned}\partial L'(\tilde{x}h) &= \partial L'(0) + (\tilde{x}h) \cdot \partial L''(0) + \dots + \frac{(\tilde{x}h)^{\tilde{j}}}{\tilde{j}!} x^{(\tilde{j}+1)}(0) + \dots \\ &= f(t_{\tilde{x}}, x_{\tilde{x}})\end{aligned}$$

Then, the error operator reads

$$\begin{aligned}L[x] &= \sum_{i=0}^k (a_i x(t_i) - h b_i x'(t_i)) \\ &= \sum_{i=0}^k (a_i x(0)) \\ &\quad + \sum_{i=0}^k (ia_i - b_i) x'(0) h \\ &\quad + \sum_{i=0}^k \left(\frac{i^2}{2!} a_i - ib_i \right) x''(0) h^2 \\ &\quad + \sum_{i=0}^k \left(\frac{i^3}{3!} a_i - \frac{i^2}{2!} b_i \right) x'''(0) h^3 \\ &\quad + \dots + \sum_{i=0}^k \left(\frac{i^{\tilde{j}}}{\tilde{j}!} a_i - \frac{i^{(\tilde{j}-1)}}{(\tilde{j}-1)!} b_i \right) x^{(\tilde{j})}(0) h^{\tilde{j}} \\ &\quad + \dots\end{aligned}$$

$$(*) \quad = d_0 + d_1 h + d_2 h^2 + \dots$$

On the other hand, we have, by definition of the method,

$$(\#) \quad \sum_{i=0}^k (a_i x_i - h b_i f_i) = 0.$$

Subtract $(\#)-(\#)$ and use the exactness for $i=0, 1, \dots, k-1$, then we arrive at

$$a_k(x_k - x(t_k)) - h b_k (\underbrace{f_k - x'(t_k)}_{= f(t_k, x_k)}) = f(t_k, x(t_k))$$

$$(\#) = d_0 + d_1 h + d_2 h^2 + \dots$$

Using MVT,

$$\begin{aligned} f_k - x'(t_k) &= f(t_k, x_k) - f(t_k, x(t_k)) \\ &= f_x(\xi) \cdot (x_k - x(t_k)) \end{aligned}$$

Plug this back into $(\#)$ to get

$$(\#) (x_k - x(t_k)) \cdot (a_k - h b_k f_x(\xi)) = d_0 + d_1 h + \dots$$

$$\Rightarrow x_k - x(t_k) = \frac{1}{a_k - h b_k f_x(\xi)} (d_0 + d_1 h + d_2 h^2 + \dots)$$

$\approx a_k$ since we are

considering $h \rightarrow 0$. In particular,
 the whole fraction $\asymp a_k^{-1}$
 since $a_k \neq 0$

$$\asymp a_k^{-1} (d_0 + d_1 h + d_2 h^2 + \dots)$$

Therefore, if LTE is of order $m+1$

$$d_0 = d_1 = \dots = d_m = 0.$$

Conversely, if $d_0 = d_1 = \dots = d_m = 0$,
 the error of the next step is of
 order $m+1$:

$$\begin{aligned} \chi_{k+1} - \chi(t_k) &\asymp a_k^{-1} d_{m+1} h^{m+1} + \dots \\ (\cancel{\text{---}}) &= O(h^{m+1}) . \end{aligned}$$

(Equivalence of 1 and 3)

If 3 is true, we can apply it to the true solution $x(t)$ and repeat the argument to ~~(1, 2, 3)~~ have 1.

Assume 1 is true, that is, LTE of the method is of order $m+1$ for all IVP with smooth enough f . For any smooth function y , it is the solution to

$$\begin{cases} x'(t) = f(t, x(t)) = y'(t) \\ x(0) = y(0) \end{cases}$$

That is, $x(t) = y(t)$, $t \geq 0$.

Thus, by repeating the argument from (*) to (#), we have

$$\begin{aligned} L[y] &= \sum_{i=0}^k (a_i y(t_i) - h b_i y'(t_i)) \\ &= \sum_{i=0}^k (a_i x(t_i) - h b_i x'(t_i)) \\ &= (x_k - x(t_k)) \cdot (a_k - h b_k \underbrace{f_x(\xi)}_{=0}) \\ &\quad \text{since } f(t, x) = y'(t) \text{ depends only on } t. \\ &= O(h^{m+1}) \text{ (by assumed order of method)} \end{aligned}$$

Name Consistency.

Easier to see the reason for the name in Adams family. Suppose LTE of

$$(*) \quad \varphi_{n+1} = \varphi_n + h(Af_{n+1} + Bf_n + Cf_{n-1})$$

is of order $m+1$ with $m \geq 1$, then

$$(**) - \frac{\varphi_{n+1} - \varphi_n}{h} = Af_{n+1} + Bf_n + Cf_{n-1}$$

If we plug in the true solution

$$\begin{aligned} \underbrace{\frac{\varphi(t_{n+1}) - \varphi(t_n)}{h}}_① &= Af(t_{n+1}, \varphi(t_{n+1})) + Bf(t_n, \varphi(t_n)) \\ &\quad + Cf(t_{n-1}, \varphi(t_{n-1})) + O(h^m) \\ &= A\varphi'(t_{n+1}) + B\varphi'(t_n) + C\varphi'(t_{n-1}) \\ &\quad + O(h^m) \end{aligned}$$

As $h \rightarrow 0$, ① $\rightarrow \varphi'(t)$ and ③ $\rightarrow 0$.

Therefore, we must have ② $\rightarrow \varphi'(t) = f(t, \varphi(t))$

This means (**) is a proper discretization of $\varphi' = f(t, \varphi(t))$, hence (*) too, since those calculations approach the ODE that it tries to solve as $h \rightarrow 0$.

consistency $\Leftrightarrow p(1) = 0$ and $p'(1) = g(1)$

$$0 = p(1) = \sum_{i=0}^k a_i = d.$$

$$p'(x) - g(x) = \sum_{i=0}^k (i a_i x^{i-1} - b_i x^i)$$

Therefore, if $p'(1) = g(1)$, then

$$0 = p'(1) - g(1) = \sum_{i=0}^k (i a_i - b_i) = d,$$

Therefore, by the order condition,
the method is at least of order 1.