(proof of convergence of bisection method; slides p.6)

Chop the task into pieces.

1. $\lim a_n$, $\lim b_n$, $\lim c_n$ exist and they all the same.
2. Call the limit, ξ, then $f(\xi)=0$.
3. $|c_n - \xi| < 2^{-(n+1)}(b-a)$

---

1. Observe $\underbrace{a_0 \leq a_1 \leq a_2 \leq \cdots}_{\text{nondecreasing}} \underbrace{\leq b}_{\text{bdd above}}$ by construction. Therefore $\lim a_n$ exists.

/* Math 3B – monotone sequence theorem.
If $\{a_n\}$ is nondecreasing (or nonincreasing) and bounded above (or bounded below), the limit exists. [This is "half-version"]
If $\{a_n\}$ is monotonic (i.e., only nonincreasing or only nondecreasing) and bounded (i.e., bounded from above and below), it converges. [This is "two-sided-ver"] */

Likewise $b_0 \geq b_1 \geq b_2 \geq \cdots \geq a$. Therefore b_n also converges.

Let $\lim a_n = \xi_1$ and $\lim b_n = \xi_2$.
We know the length of $[a_n, b_n]$ gets halved from the construction. Therefore, $b_n - a_n \to 0$ as $n \to \infty$. Then, we must have

$$0 = \lim(b_n - a_n) = \lim b_n - \lim a_n$$
$$= \xi_2 - \xi_1$$

$\Longrightarrow \quad \xi_1 = \xi_2$ Call this common limit ξ.

Lastly for ①, we have $a_n \le c_n \le b_n$.
Therefore, sandwich theorem says

$$\underset{\xi}{\underset{\downarrow}{\lim a_n}} \le \underbrace{\lim c_n}_{\substack{\downarrow \\ \xi}} \le \underset{\xi}{\underset{\downarrow}{\lim b_n}}$$

Thus, $\lim c_n = \xi$.