Math 104A - Intro to Numerical Analysis

Numerical Solution of ODE

Jea-Hyun Park

University of California Santa Barbara

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Intro

Problem of interest

Given $\vec{f} : \mathbb{R}^{1+d} \to \mathbb{R}^d$, and $\vec{x}_0 \in \mathbb{R}^d$, find $\vec{x} : I \to \mathbb{R}^d$, where $t_0 \in I \subset \mathbb{R}$ (often $I = [0, T]$) satisfying

$$\dot{\vec{x}}(t) = \vec{f}(t, \vec{x}(t)) \quad (t \in I), \qquad \vec{x}(t_0) = \vec{x}_0$$

Example: (Lorenz equation; $d = 3$)

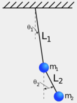
$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \text{ and } f(t, x, y, z) = \begin{bmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{bmatrix}$$

If we set $\sigma = 1, \rho = \frac{1}{9}, \beta = 2.$

$$\begin{cases} x_t = y - x, \\ y_t = -xz + \frac{1}{9}x - y, \\ z_t = xy - 2z, \end{cases} \qquad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

- $\dot{(\ )}$ denotes time derivative $\frac{\mathrm{d}}{\mathrm{dt}}(\ )$.
- $\vec{f}$ is called the **slope function**.
- The first piece is called ordinary differential equation (**ODE**) while the second **initial condition**, and altogether an initial value problem (**IVP**).
- f is independent of t in this example, but may depend on time in general.

Problem of interest



Plan

- We mainly focus on one dimensional case ($d = 1$). However, most of the important concepts and intuition are readily extended to higher dimensions (assuming proficiency in vector calculus).

Problem of interest (IVP)

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

- ODE (more or less synonymous to dynamical system) is a rather general model for physics, biology, etc, anything that depends on time smoothly.
- Since the solution is a function of t (time), it is often called a **trajectory**.

Existence and uniqueness of exact solution

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{IVP}$$

Theorem (Existence and uniqueness 1)

If f is continuous on a rectangle centered at (t_0, x_0), $D = \{(t,x) \ : \ |t - t_0| \le \alpha, |x - x_0| \le \beta\}$, then (IVP) has a solution on $(t_0 - r, t_0 + r)$, where $r = \min(\alpha, \beta/M)$ and $M = \max_{(t,x)\in D} |f(t,x)|$. If, in addition, $\partial f/\partial x$ is continuous on D, then the solution is unique.

Example

Verify that an IVP $x'(t) = x^{2/3}$ subject to $x(0) = 0$ has a solution around $t = 0$, but it is not unique.

- Are you trying to find something that exists?
- If so, does it stay the same every time you find it?
- We don't prove existence theorem
- Don't get overwhelmed by the theorem, in particular, by its details. Focus on the big picture to begin with.
- In words, "if slope function is nice, the system evolves deterministically at least for a short time."

Theorem (Existence and uniqueness 2)

If f is continuous on $[a,b]\times\mathbb{R}$ satisfies the Lipschitz condition in the second variable, x, i.e.,

$$|f(t,x)-f(t,y)|\le L|x-y|$$

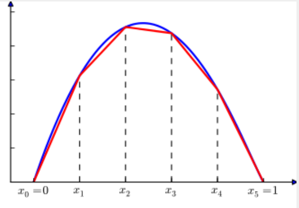
the (IVP) has a unique solution on $[a,b]$.

Remark (Continuous, Lipschitz continuous, continosly differentiable functions of one variable)

Note that the following inclusions, where UC (nonstandard notation) means uniformly continuous functions,

$$C^1[a,b]\subset \mathrm{Lip}[a,b]\subset UC[a,b]=C[a,b].$$

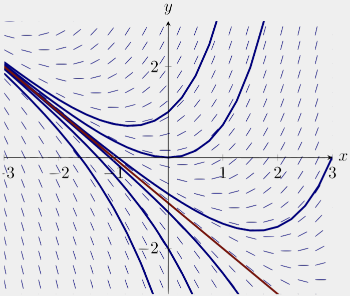
- To make the statement true, we end up needing to classify functions.
- **Subjective question**: Lipschitz functions are very important class. Would you come up with a more intuitive, informal description?



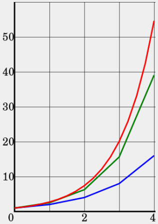
Concrete pictures of what we will do

What does a numerical solution look like?

t_0	t_1	t_2	t_3	$\cdots$
x_0	x_1	x_2	x_3	$\cdots$



(A) Slope field



(B) Solutions of $x' = x$, $x(0) = x_0$. Euler (blue, bottom), Midpoint (green, middle), True (red, top)

- A numerical solution is a list of point values.
- (A) Each curve is a solution to IVP with a different initial value.
- (B) For each IVP, you have different numerical solutions depending on the method used.

Numerical solution of ODE

Taylor-series method

Taylor-series method

How is the next step computed? $\to$ Taylor series

To compute $x(t+h)$, take a few terms from

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \frac{h^4}{4!}x^{(4)}(t) + \cdots$$

Example:

$$\begin{cases} x' = f(t,x) = \cos t - \sin x + t^2 \\ x(-1) = 3 \end{cases}$$

Computation results

Program example desired.

Pros

- Conceptually easy.
- High order methods are obtained easily (just add more terms).

Cons

- Require a high regularity on the slope function.
- Preliminary analytic work must be done.
- During this stage, human-made error can be a disaster.

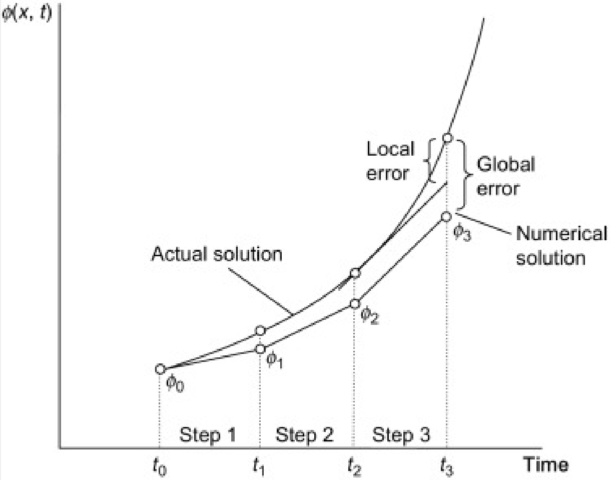
- If we take the first two terms, it is called **(explicit) Euler method**.
- I will show you only once the detailed picture of what is happening when you numerically solve an IVP. We will then focus on methods and analysis.

Errors in a numerical solution to an IVP

1. **Local truncation error (LTE)** : errors caused by including only finite number of calculations out of an exact procedure assuming the current data is exact.
2. **Local roundoff error**: errors caused by limited precision of computers.
3. **Global truncation error**: accumulation of all LTE. Usually, global error is of one lower order than that of LTE: since $n = \frac{T - t_0}{h}$, $\sum_{i=1}^{n} \mathcal{O}\left(h^{k+1}\right) \approx n\mathcal{O}\left(h^{k+1}\right) = \frac{T-t_0}{h}\mathcal{O}\left(h^{k+1}\right) = \mathcal{O}\left(h^{k}\right)$
4. **Global roundoff error**: accumulated roundoff errors.
5. **Total error**: sum of the global truncation errors and global roundoff errors.

Exercise: What is the order of LTE (also called *order of accuracy*) in the previous example of Taylor's method?

- 'global error' usually means global truncation error. But people normally say the full name for 'local truncation error.'
- Truncation errors are inherent in the method chosen, and quite independent of the roundoff errors.
- Roundoff errors depend on the computer environment.



$\phi(x, t)$
Local error
Global error
Numerical solution
Actual solution
ϕ_0
ϕ_1
ϕ_2
ϕ_3
Step 1
Step 2
Step 3
t_0
t_1
t_2
t_3
Time

LTE of Taylor method

For example, if the method include up to 3rd order term, the LTE is of 4th order.

$$\underbrace{x(t+h)}_{\text{target}} - \underbrace{x(t) + hx'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t)}_{\text{approximation}} = \frac{h^4}{4!}x^{(4)}(\xi)$$