Announcement

- Midterm Exam 1

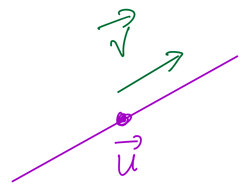
  Apr 18 (Tue), lecture time, lecture room

  Quiz 1 Tomorrow (Section) → Course cal.

- Discord → Course communication

  learning, help, general questions

Recap:



lines : $\underset{\text{point}}{\vec{u}} + t\,\underset{\text{direction}}{\vec{V}}$

(not an equation)

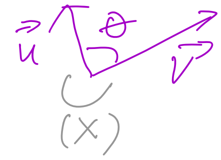
→ may look simple, but require a lot of "speaking" and "writing" in the language of vector and geometry.

Dot prod : $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Also basic properties.

NT 1.9 Dot product and angles

$\boxed{\text{Thm}}$ Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ (mostly interested in $n=2,3$). Then,



$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta,$$

where θ is the angle b/w $\vec{u}$ and $\vec{v}$ so that $\underline{0 \leq \theta \leq \pi}$

proof) If $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$, $\vec{u}\cdot\vec{v} = 0$ and $\underbrace{\|\vec{u}\|\,\|\vec{v}\| \cos\theta = 0}_{\text{RHS}}$

LHS (Left Hand side)

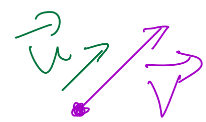
Now, assume $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$

If $\theta = 0$, then $\vec{v} = k\vec{u}$ for some $k>0$.

LHS $\vec{u}\cdot\vec{v} = \vec{u}\cdot(k\vec{u}) = k\|\vec{u}\|^2$

RHS $\|\vec{u}\|\,\|\vec{v}\| \cos\theta = \|\vec{u}\|\,\|k\vec{u}\| \underbrace{\cos 0}_{=1}$

$= |k|\,\|\vec{u}\|^2 = k\|\vec{u}\|^2$



$k>0$

If $\theta = \pi$, then $\vec{v} = k\vec{u}$ for some $k<0$.

LHS $\vec{u}\cdot\vec{v} = \vec{u}\cdot(k\vec{u}) = k\|\vec{u}\|^2$

RHS $\|\vec{u}\|\,\|\vec{v}\| \cos\theta = \|\vec{u}\|\,\|k\vec{u}\| \underbrace{\cos(\pi)}_{=-1}$

$= -|k|\,\|\vec{u}\|^2$

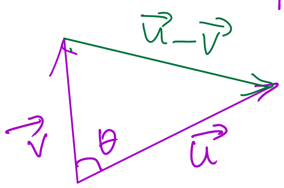
$= k\|\vec{u}\|^2$ ($|k| = -k$ since $k<0$)

For all other cases, we have a triangle.

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}$$



On the other hand, cosine law asserts

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Comparing the two equalities, we have

$$-\cancel{2}\,\vec{u}\cdot\vec{v} = \cancel{-2}\|\vec{u}\|\|\vec{v}\|\cos\theta,$$ that is

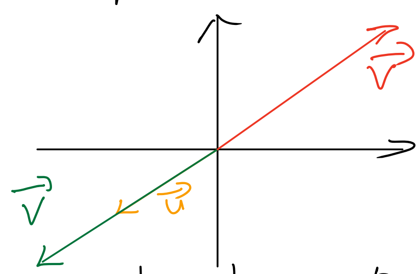
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$\blacksquare$

Cor /* short for "corollary" – immediate result */

- $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$ if $\vec{u}\neq\vec{0}, \vec{v}\neq\vec{0}$

- $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|$ precisely when $\theta=0$, that is $\vec{u}$ and $\vec{v}$ point to the same direction.



- $\vec{u}\cdot\vec{v} = -\|\vec{u}\|\|\vec{v}\|$ precisely when $\theta=\pi$, that is $\vec{u}$ and $\vec{v}$ point to the opposite direction.

- $\vec{u}\cdot\vec{v} = 0$ precisely when $\vec{u}\perp\vec{v}$, $\vec{v}=\vec{0}$, or $\vec{u}=\vec{0}$.

Def (orthogonality)

Two vectors $\vec{u}$ and $\vec{v}$ are said to be **orthogonal** ("$\vec{u}$ is orthogonal to $\vec{v}$ or vice versa) if $\vec{u}\cdot\vec{v}=0$.

◎ We also say $\vec{u}$ is **perpendicular** to $\vec{v}$.

◎ This includes the case $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$.

Therefore, $\vec{0}$ is orthogonal to all vector.

perp. normalized = unit length.

$\boxed{\text{Def}}$ (**orthonormal set**)

If $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_\ell\}$ consists of unit vectors and any two of them are orthogonal to each other, it is called an **orthonormal set**.

Example: $\{\vec{i}, \vec{j}, \vec{k}\}$

$\boxed{\text{Clicker}}$

Find the angle b/w $\vec{v} = \underbrace{4\vec{i} - \vec{j} + \vec{k}}_{= (4,-1,1)}$

and $\vec{w} = \underbrace{\vec{j} + 3\vec{k}}_{= (0,1,3)}$ in $\mathbb{R}^3$.

(A) $\pi/4$ (B) 120° ~~(C)~~ $\cos^{-1} \frac{1}{3\sqrt{5}}$

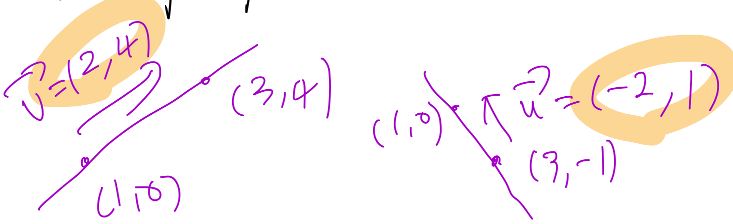
$\vec{v} \circ \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$

$$\cos\theta = \frac{\vec{v} \circ \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$= \frac{-1+3}{\sqrt{16+1+1}\sqrt{1+9}} = \frac{2}{\sqrt{18}\sqrt{10}} = \frac{2}{3\sqrt{2}\sqrt{5}\sqrt{2}}$$

$$= \frac{1}{3\sqrt{5}}$$

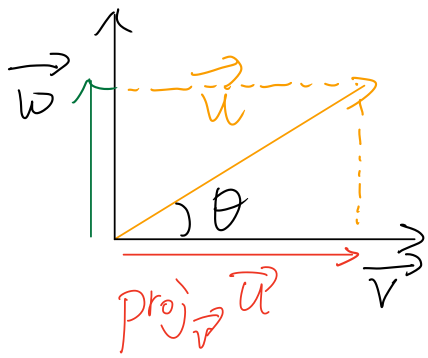
(b) Show ℓ_1 (a line passing through $(1,0)$ and $(3,4)$) and ℓ_2 (a line passing through $(3,-1)$ and $(1,0)$) are perpendicular.



$\vec{v} \circ \vec{u} = 2 \cdot (-2) + 4 \cdot 1$
$= 0$

NT 1.10 projection onto a vector/line

Motivation: How much of the force actually is exerted in a certain direction?



Def (projection onto a vector/line)

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ (mostly $n=2,3$) and ℓ be a line whose direction vector is $\vec{v}$.

Then, the **projection of $\vec{u}$ onto $\vec{v}$** or **projection of $\vec{u}$ onto ℓ** is defined

$$\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\ell}\vec{u} = \underbrace{\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}}_{①}\ \underbrace{\vec{v}}_{②}$$

- ① is a scalar, ② is a vector
- The projection is in direction $\vec{v}$ (not $\vec{u}$)
- The projection is linear in $\vec{u}$: "$\approx f(x)=ax$"

$$\text{proj}_{\vec{v}}(k\vec{u}) = k\,\text{proj}_{\vec{v}}(\vec{u}) \quad (k\in\mathbb{R})$$

$$\text{proj}_{\vec{v}}(\vec{u}+\vec{w}) = \text{proj}_{\vec{v}}(\vec{u}) + \text{proj}_{\vec{v}}(\vec{w})$$

- The projection is independent of the length of $\vec{v}$, but depends only on the direction of $\vec{v}$.

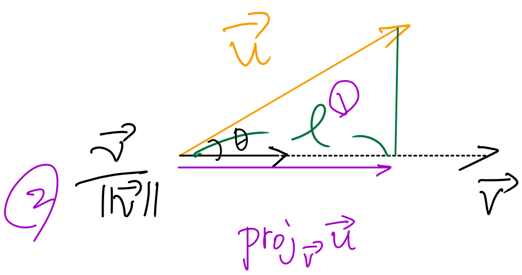
Exercise:
- Explain the first two observations.
- Prove the last two statements.

/*

Exercises are not for credit. But they are very good for a better understanding. You are welcome to talk about them during office hours. */

⑤ (Derivation of projection)

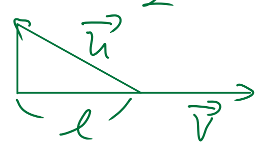
Projection formula can be better understood using unit (direction) vector:



$$\Rightarrow \text{proj}_{\vec{v}}\vec{u} = \underbrace{\ell}_{\text{length ①}} \cdot \underbrace{\frac{\vec{v}}{\|\vec{v}\|}}_{\text{direction (unit vector) ②}}$$

① $$\ell = \|\vec{u}\|\cos\theta = \|\vec{u}\| \overbrace{\left\|\frac{\vec{v}}{\|\vec{v}\|}\right\|}^{=1} \cos\theta = \vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|}$$

(It can be $\ell<0$ if $\vec{u}$ is more in the opposite direction than the same direction: $\theta > \frac{\pi}{2}$

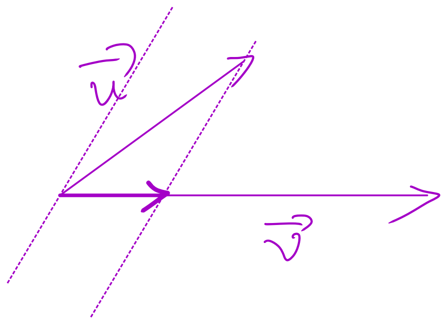


What we want:

"a vector with length ℓ in the direction of $\vec{v}$":

$$\ell\cdot\frac{\vec{v}}{\|\vec{v}\|} = \left(\vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\cdot\vec{v}$$

⑥ Precise name of the above is **orthogonal projection** to distinguish from other "slanted projections":



But in this course, we are not interested in those general cases. Thus, we often omit "**orthogonal**."

$\boxed{\text{Clicker}}$: Draw an appropriate picture and and find the projection of $\vec{a} = (2,1,4)$ onto $\vec{v} = (0,2,3)$

(Type the answer without parentheses)

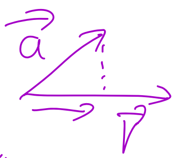
e.g. "1,4,9"

(A) $\frac{14}{13}(2,1,4)$

(B) $\frac{14}{13}(0,2,3)$

(C) $(0,2,3)$

$$\frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}$$



$$= \frac{\overbrace{2\cdot 0 + 1\cdot 2 + 4\cdot 3}^{14}}{\underbrace{0^2+2^2+3^2}_{=13}}(0,2,3)$$

$$= \frac{14}{13}(0,2,3)$$

NT 1.13 Cross product (Defined only on $\mathbb{R}^3$)

$\boxed{\text{Def}}$ For $\vec{a}, \vec{b} \in \mathbb{R}^3$, their cross product $\vec{a} \times \vec{b} \in \mathbb{R}^3$ is defined by

$$\vec{c} = (a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1)$$

- ◎ $\vec{a} \times \vec{b}$ is a $\mathbb{R}^3$-vector.
- ◎ Usually, it is computed by a formal trick.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$\boxed{\text{Clicker}}$ Choose $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$, where $\vec{a} = (1, 0, -2)$, $\vec{b} = (-2, 3, -4)$.

(A) $(6, 8, 3)$ ✓ (B) $(6, -8, 3)$

(C) $(-6, -8, -3)$ (D) $(-6, 8, -3)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ -2 & 3 & -4 \end{vmatrix} = 6\vec{i} + (\cancel{-}8)\vec{j} + 3\vec{k}$$

◎ Basic properties of cross product

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ anticommutative
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ distributive
- $\alpha(\vec{a} \times \vec{b}) = (\alpha \vec{a}) \times (\vec{b}) = \vec{a} \times (\alpha \vec{b})$
- $\vec{a} \times \vec{a} = \vec{0}$ associative

prove only 1^{st} and last items.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\vec{b} \times \vec{a}$$

$$\vec{a} \times \vec{a} \underset{\text{by } 1^{st} \text{ property}}{=} -\vec{a} \times \vec{a} \Rightarrow 2(\vec{a} \times \vec{a}) = \vec{0} \Rightarrow \vec{a} \times \vec{a} = \vec{0}.$$

⑤ Scalar triple product

$$\underbrace{\vec{a}\cdot(\underbrace{\vec{b}\times\vec{c}}_{\text{vector}})}_{\text{scalar}} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

proof)

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = |A_{11}|\vec{i} - |A_{12}|\vec{j} + |A_{13}|\vec{k}$$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$A_{11} = \begin{array}{ccc} \cancel{\vec{i}} & \cancel{\vec{j}} & \cancel{\vec{k}} \\ \cancel{b_1} & b_2 & b_3 \\ \cancel{c_1} & c_2 & c_3 \end{array}$$

Thus

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1|A_{11}| - a_2|A_{12}| + a_3|A_{13}|$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

⊙ $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

= signed volume of parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$

$\boxed{\text{Thm}}$ Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. Then,

1. $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$

2. $\|\vec{a} \times \vec{b}\| = \overbrace{\|\vec{a}\| \|\vec{b}\| \sin\theta}^{\text{area of parallelogram}}$, where 

$0 \le \theta \le \pi$ is the angle b/w $\vec{a}$ and $\vec{b}$.

proof) 1. $\vec{a} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

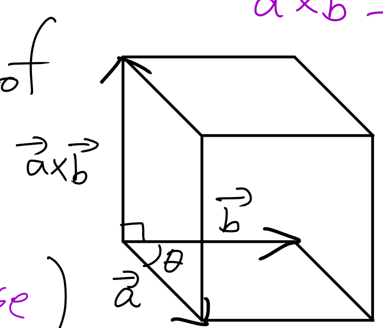
Similarly for $\vec{b} \perp \vec{a} \times \vec{b}$.

2. Assume $\vec{a} \times \vec{b} \neq \vec{0}$.

$$\|\vec{a} \times \vec{b}\|^{\cancel{2}} = (\overset{①}{\vec{a} \times \vec{b}}) \cdot (\overset{②}{\vec{a}} \times \overset{③}{\vec{b}})$$

$\vec{a} \times \vec{b} \perp \vec{a}$
$\vec{a} \times \vec{b} \perp \vec{b}$

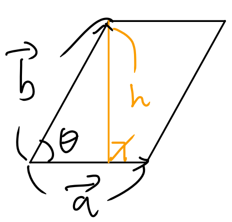
$$= \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{vmatrix} \vec{a} \times \vec{b} \\ \vec{a} \\ \vec{b} \end{vmatrix} = \text{volume of}$$



$$= \underbrace{\cancel{\|\vec{a} \times \vec{b}\|}}_{\text{height}} \cdot (\text{area of } \underbrace{\text{base}}_{\text{parallelogram}})$$

Cancel $\|\vec{a} \times \vec{b}\|$ out

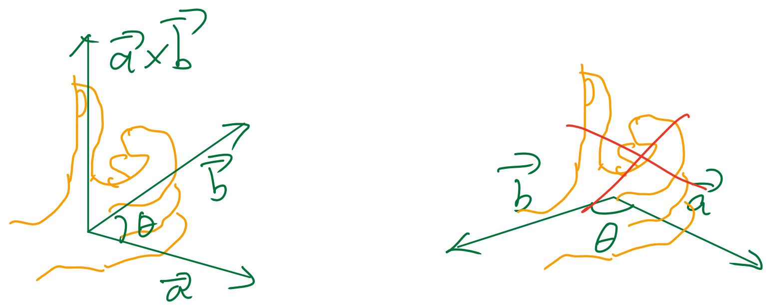
$$\|\vec{a} \times \vec{b}\| = \text{area of base} = \|\vec{a}\| \|\vec{b}\| \sin\theta \text{ (see below)}$$



$h = \|\vec{b}\| \sin\theta \Rightarrow \text{area} = \|\vec{a}\| \|\vec{b}\| \sin\theta$

/* $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, or $\sin\theta = 0$ case is not interesting or inspiring. */

① 1 of thm does not determine the direction completely. It can be by by **right hand rule.**

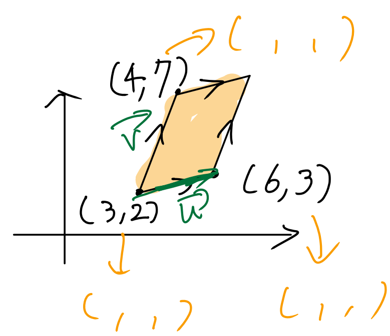


Your four fingers should cover $\vec{a}$, $\underbrace{\theta}_{0\le\theta\le\pi}$, then $\vec{b}$ in order.

① direction of $\vec{a}\times\vec{b}$: orthogonal to $\vec{a}$ and $\vec{b}$ following right hand rule.

length of $\vec{a}\times\vec{b}$: area of parallelogram determined by $\vec{a}$ and $\vec{b}$

Example: Find the area



(A) 34 (B) 8 (C) 14