(forward/backward diff. quotient)

Taylor:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2}h^2$$

Rearrange:

$$\underbrace{\frac{f(x+h)-f(x)}{h}}_{e(h)} - f'(x) = \frac{f''(\xi)}{2}h$$

think of this as a function of h that computes an approximation of $f'(x)$.
Then, LHS is the error.

$$f(x-h) = f(x) - f'(x)h + \frac{f''(\hat{\xi})}{2}h^2$$

$$\frac{f(x)-f(x-h)}{h} - f'(x) = -\frac{f''(\hat{\xi})}{2}h$$

(centered difference quotient)

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(\xi)}{6}h^3$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(\tilde{\xi})}{6}h^3$$

subtract and rearrange

$$\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \frac{f'''(\xi)+f'''(\tilde{\xi})}{2}\frac{h^2}{6}$$

$$= \frac{f'''(\xi)}{6}h^2$$

Here, we used a little trick.

lemma: Let g is continuous on $[a,b]$.
Then, for any $x_1, x_2, \cdots, x_n \in [a,b]$ there exists $\xi \in [a,b]$ such that

$$\text{mean} = \frac{g(x_1)+g(x_2)+\cdots+g(x_n)}{n} = g(\xi) \quad \cdots (*)$$

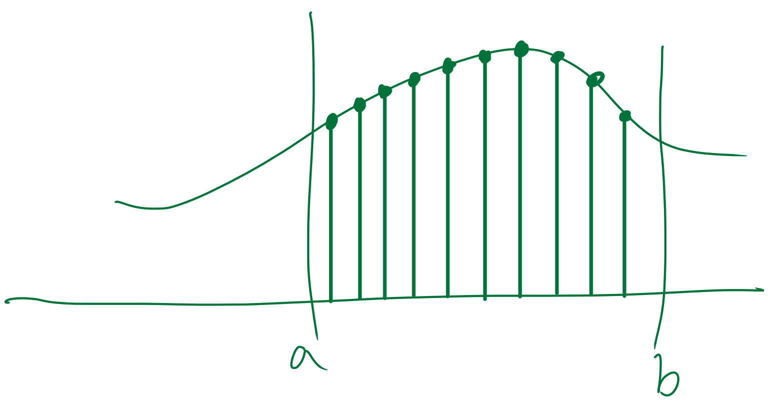
proof) The mean is b/w the maximum and the minimum of g over $[a,b]$. Hence, intermediate value theorem asserts that $\exists \xi$ satisfying $(*)$.

comment: This can be viewed as a discrete version of the mean value theorem of integral: If g is continuous,

$$\frac{1}{b-a}\int_a^b g(x)\,dx = g(\xi)$$

for some $\xi \in [a,b]$

The LHS the mean of g using infinitely many samples.

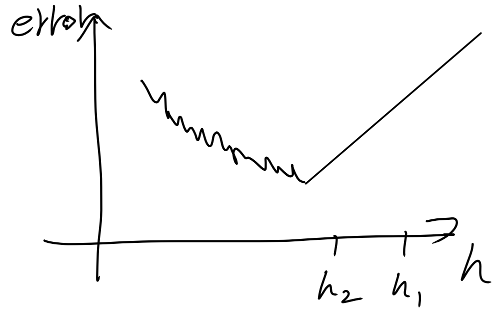


$$\Delta x \approx \frac{b-a}{n} \Rightarrow \frac{1}{n} \approx \frac{\Delta x}{b-a}$$

$$\frac{1}{n}\sum_{i=1}^{n} g(x_i) \approx \frac{1}{b-a}\sum_{i=1}^{n} g(x_i)\Delta x$$

$$\approx \frac{1}{b-a}\int_a^b g(x)\,dx$$

The slope of log-log scale is the rate of convergence.



spacing : $h_1 = 2h_2$ (h_1 : big , h_2 : small)

error : $e_1 \leftrightarrow h_1$, $e_2 \leftrightarrow h_2$

$$0 < \text{slope} = r = \frac{\Delta \log(\text{err})}{\Delta \log(\text{spacing})} = \frac{\log e_1 - \log e_2}{\log h_1 - \log h_2}$$

$$= \frac{\log e_1/e_2}{\log h_1/h_2} = \frac{\log e_1/e_2}{\log 2}$$

$$\log e_1/e_2 = r \log 2$$

$$\frac{e_1}{e_2} = e^{r \log 2} = 2^r$$

$$\frac{e_2}{e_1} = 2^{-r}$$

$e_2 = 2^{-r} e_1 \rightarrow$ If $r=1$, $e_2 = \frac{1}{2} e_1$

half spacing $\rightarrow$ half error

If $r=2$, $e_2 = \frac{1}{4} e_1$

half spacing $\rightarrow$ a fourth error

and so on.

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(x)}{24}h^4 + \cdots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + \frac{f^{(4)}(x)}{24}h^4 + \cdots$$

subtract and rearrange:

$$(*) \quad f'(x) = \varphi(h) - \frac{f'''(x)}{6}h^2 - \frac{f^{(5)}}{5!}h^4 - \cdots$$

$$= \underbrace{\varphi(h)}_{(f(x+h)-f(x-h))/2h} - a_2 h^2 - a_4 h^4 - \cdots$$

This hold for any small enough $h>0$.

$h \leftarrow h/2$ yields

$$f'(x) = \varphi(h/2) - \frac{a_2}{4}h^2 - \frac{a_4}{16}h^4 - \cdots$$

$$(**) \quad 4f'(x) = 4\varphi(h/2) - a_2 h^2 - \frac{a_4}{4}h^4 - \cdots$$

$$(**) - (*): \quad 3f'(x) = 4\varphi(h/2) - \varphi(h) + \frac{3}{4}a_4 h^4 + \cdots$$

$$\Rightarrow f'(x) - \underbrace{\frac{1}{3}\left(4\varphi(h/2) - \varphi(h)\right)}_{=: \psi(h)} = \frac{3}{4}a_4 h^4 + \cdots$$

where

$$
\begin{aligned}
\psi(h) &= \frac{1}{3}\left(4\varphi(h/2) - \varphi(h)\right) \\
&= \frac{1}{3}\left(4\,\frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{\cancel{2}\cdot\frac{h}{\cancel{2}}} - \frac{f(x+h) - f(x-h)}{2h}\right) \\
&= \frac{1}{6h}\left(8f(x+\tfrac{h}{2}) - 8f(x-\tfrac{h}{2}) - f(x+h) + f(x-h)\right)
\end{aligned}
$$

Since $f(x) \approx p(x)$ on $[a, b]$

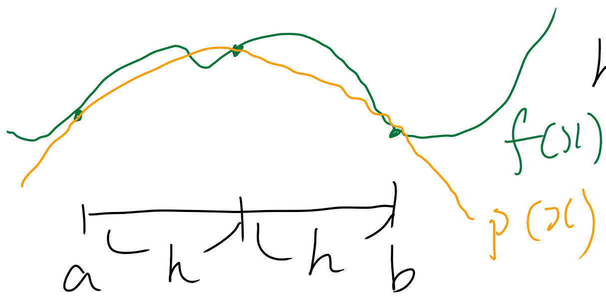
we expect $\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$.

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

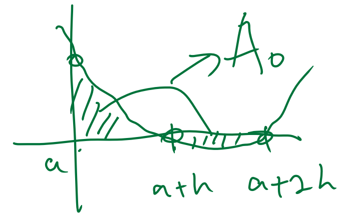
$$\int_a^b p(x)\,dx = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x)\,dx$$

$$= \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)\,dx}_{=:A_i}$$

$$= \sum_{i=0}^{n} f(x_i)\, A_i$$



$$h = \frac{b-a}{2}$$



$$A_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-a-h)(x-a-2h)}{(\cancel{a}-\cancel{a}-h)(\cancel{a}-\cancel{a}-2h)}\,dx$$

$$= \int_a^{a+2h} \frac{(x-a-h)(x-a-2h)}{2h^2}\,dx$$

$z = x-a-h$

$dz = dx$

$$= \frac{1}{2h^2}\int_{-h}^{h} z(z-h)\,dz$$

$$= \frac{1}{2h^2}\int_{-h}^{h} \underbrace{z^2}_{\text{even}} - \xrightarrow{0}{\cancel{hz}}\,dz \quad = 2\int_0^h z^2\,dz$$

$$= \frac{1}{2h^2}\cdot 2\left[\frac{z^3}{3}\right]_0^h$$

$$= \frac{h}{3}$$

Similarly,

$$A_1 = \int_a^{a+2h} \frac{(x-a)(x-a-2h)}{(\cancel{a}+h-\cancel{a})(\cancel{a}+h-\cancel{a}-2h)}\,dx$$

$$= -\frac{1}{h^2}\int_{-h}^{h} (z+h)(z-h)\,dz$$

$x-a-h = z$

$dx = dz$

$$= -\frac{1}{h^2}\int_{-h}^{h} \underbrace{z^2-h^2}_{\text{even}}\,dz \quad = 2\int_0^h z^2-h^2\,dz$$

$$= -\frac{1}{h^2}\cdot 2\underbrace{\left[\frac{z^3}{3}-h^2 z\right]_0^h}_{=\frac{h^3}{3}-h^3=-\frac{2}{3}h^3}$$

$$= \frac{4}{3}h$$

$$A_2 = \int_a^{a+2h} \frac{(x-a)(x-a-h)}{(\cancel{a}+2h-\cancel{a})(\cancel{a}+2h-\cancel{a}-h)}\,dx$$

$$= \frac{1}{2h^2}\int_a^{a+2h}(x-a)(x-a-h)\,dx \qquad x-a-h=z,\quad dx=dz$$

$$= \frac{1}{2h^2}\int_{-h}^{h}\underbrace{(z+h)z}_{\text{even } z^2 + \cancel{hz} \to 0 \text{ (b/c odd)}}\,dx$$

$$= \frac{1}{2h^2}\cdot 2\left[\frac{z^3}{3}\right]_0^h$$

$$= \frac{h}{3}$$

Thus, $\displaystyle\int_a^b f(x)\,dx \approx \sum_{i=0}^{2} f(x_i)A_i$

$$= f(a)\frac{h}{3} + f(a+h)\frac{4h}{3} + f(a+2h)\frac{h}{3}$$

$$= \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right)$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)$$

(Quadrature formula for a different interval)

① find $\lambda(t)$

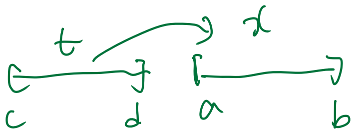
t	c	d
x	a	b

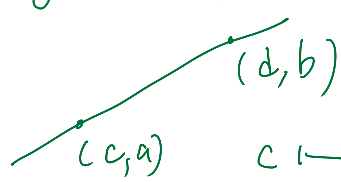
Lagrange : $\lambda(t) = a \cdot \dfrac{t-d}{c-d} + b\,\dfrac{t-c}{d-c}$

$$= \frac{-a}{d-c}t + \frac{ad}{d-c} + \frac{b}{d-c}t - \frac{bc}{d-c}$$

$$= \frac{b-a}{d-c}t + \frac{ad-bc}{d-c}$$

/* Or good old High School:





$y - c = \dfrac{b-a}{d-c}(t-a)$

$c \longmapsto a$

$d \longmapsto b$

*/

② $$\int_a^b f(x)\,dx = \int_c^d f(\lambda(t))\,\underbrace{\lambda'(t)}_{\text{constant}}\,dt$$

$\dfrac{b-a}{d-c}$ constant

$x = \lambda(t)$

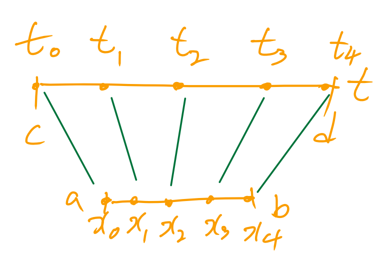
$dx = \lambda'(t)\,dt$

$= \dfrac{b-a}{d-c}\,dt$

$$\approx \frac{b-a}{d-c}\sum_{i=0}^{n} f(\lambda(t_i)) \cdot \int_c^d l_i(t)\,dt$$

$$= \frac{b-a}{d-c}\sum_{i=0}^{n} f(\underbrace{\lambda(t_i)}_{x_i})\,A_i$$

where, $t_i = \lambda^{-1}(x_i)$



(Error of quadrature)

① Single interval.

$$\left|\int_a^{a+h} f(x) - p(x)\,dx\right| \le \int_a^{a+h} |f(x) - p(x)|\,dx$$

$$= \int_a^{a+h} \frac{1}{(n+1)!}\left|f^{(n+1)}(\xi_x)\prod_{i=0}^{n}(x-x_i)\right| dx$$

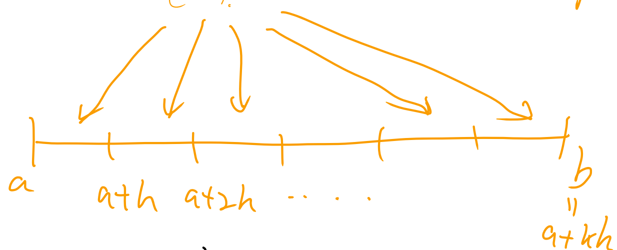
$$\le \frac{1}{(n+1)!}\max_{a\le x\le a+h}\left|f^{(n+1)}(x)\right| \prod_{i=0}^{n} \underbrace{|x-x_i|}_{\le h}\,dx$$

$$= \frac{\hat{M}_{n+1}}{(n+1)!}h^{n+1}\int_a^{a+h} dx \qquad \hat{M}_{n+1} := \max_{a\le x\le a+h}\left|f^{(n+1)}(x)\right|$$

$$= \frac{\hat{M}_{n+1}}{(n+1)!}h^{n+2}$$

$\frac{\hat{M}_{n+1}}{(n+1)!}h^{n+2}$ errors add up.

② Composite rule



$$\left|\int_a^b f(x)\,dx - \sum_{j=1}^{k}\sum_{i=0}^{n} f(x_i+(j-1)h)A_i\right|$$

$$= \left|\sum_{j=1}^{k}\int_{a+(j-1)h}^{a+jh} f(x)\,dx - \sum_{j=1}^{k}\sum_{i=0}^{n} f(x_i+(j-1)h)A_i\right|$$

$$\leq \sum_{j=1}^{k} \underbrace{\int_{a+(j-1)h}^{a+jh} \left| f(x) - \sum_{i=0}^{n} f(x_i+(j-1)h)A_i \right|}_{\leq \frac{\hat{M}_{n+1}}{(n+1)!} h^{n+2}}$$

$$\leq \sum_{j=1}^{k} \frac{M_{n+1}}{(n+1)!} h^{n+2} \qquad M_{n+1} := \max_{a \leq x \leq b} \left| f^{(n+1)}(x) \right|$$

$$= \frac{M_{n+1}}{(n+1)!} h^{n+2} \cdot k \qquad \left(kh = b-a \quad b/c \quad h = \frac{b-a}{k} \right)$$

$$= \frac{M_{n+1}}{(n+1)!} h^{n+1} \cdot (b-a)$$

(Example) Error of trapezoidal rule.

$$\left| \int_a^b f(x)\,dx - \frac{f(b)+f(a)}{2} \cdot (b-a) \right| \leq \frac{\hat{M}_2}{2!} (b-a)^3$$

Composite trapezoidal rule.

$$\left| \int_a^b f(x)\,dx - \frac{h}{2}\left(f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(a+(k-1)h) + f(b) \right) \right|$$

$$\leq \frac{M_2}{2!} h^2 (b-a)$$