NT. 3.9 Constrained optimization and Lagrange multiplier. (Ch. 4.4)

Goal: We want to solve something like "what is max/min of $f(x,y) = 3x + y - 4$ when (x,y) satisfies $x^2 + 2y^2 = 38$."

Namely, x and y are not totally free but only certain pairs are allowed

⊙ Terminology

The quantity to optimize ($f(x,y)$ above) is called **objective (function)** and the relations that inputs must satisfy are called **constraints**.

We usually summarize constrained optimization in the following way:

$$\text{maximize } f(x,y) = 3x + y - 4,$$
$$\text{subject to } x^2 + 2y^2 = 38$$

/* You could think of elementary strategies like maximiz $(f(x,y))^2$ instead, and plug in $x^2 = 38 - 2y^2$ and so on.

But we are going to listen to a devilishly smart person and learn his idea, which extends to high dimensions. */

Lagrange multiplier for

maximize (or minimize) $f(x,y)$

subject to $g(x,y) = c$ (constant).

① Find $\nabla f(x,y)$ and $\nabla g(x,y)$

② Solve the system

(*) $\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = c \end{cases}$ (λ: scalar)

③ Plug the solutions (x,y) into $f(x,y)$ and summarize max/min.

λ is called **Lagrange multiplier**.

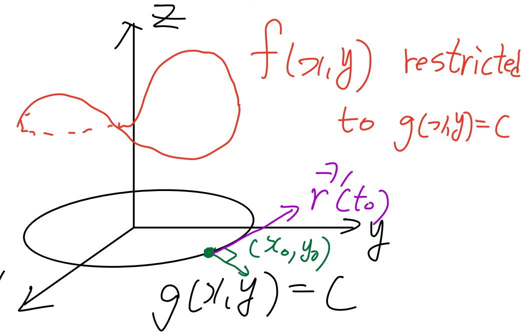
Q: What does (*) mean?

(proof for a "nice" case) $x^2+y^2=1$

/* Assume everything is smooth and $g(x,y)=C$ is a closed loop for simplicity. */

Model the curve of constraint, $g(x,y)=C$, by $\vec{r}:[a,b]\longrightarrow \mathbb{R}^2$

so that $\vec{r}(t)=(x(t),y(t))$



$g\circ\vec{r}(t)=g(x(t),y(t))=C.$

Observe that $f\circ\vec{r}$ reduces to

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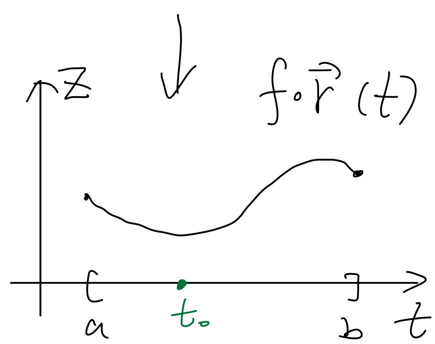
$$\mathbb{R}^1\longrightarrow\mathbb{R}^2\longrightarrow\mathbb{R}^1$$

$$t\longmapsto\vec{r}(t)\longmapsto f(\vec{r}(t))$$



By Fermat, if $f\circ\vec{r}$ has a local max/min at t_0 with $(x_0,y_0)=(x(t_0),y(t_0))$, then we have

$$0=\frac{d}{dt}f\circ\vec{r}(t_0)\underset{\text{chain rule}}{=}\underbrace{\nabla f(x_0,y_0)}_{\text{gradient of } f}\cdot\underbrace{\vec{r}\,'(t_0)}_{\text{tangent to } g(x,y)=C}\quad\text{—}(*)$$

On the other hand, from the theorem about gradient (applied to $g(x,y)$) and level sets, we know

$\nabla g(x(t_0), y(t_0)) \perp g(x,y)=c$ — (**)

at $(x(t_0), y(t_0))$ $\underbrace{}_{\text{namely}}$ tangent direction

From (*) and (**), we conclude

$\nabla f \parallel \nabla g$ at $(x(t_0), y(t_0))$, that is

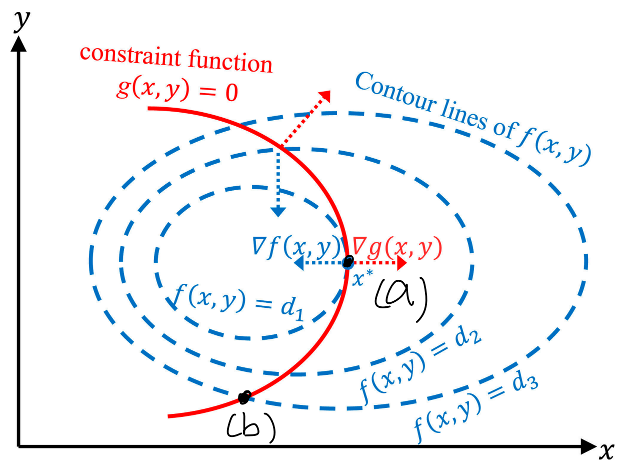
$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$. — (***)

Furthermore, if $\vec{r}$ is a closed loop, it is closed and bounded.

Therefore, by the extreme value thm, there are absolute max and min. And these must be among local max or min since there is no critical point of rough kind.

To sum up, if we can find (x_0, y_0) satisfying (***) and $g(x_0, y_0) = c$, they are the only possible max/min.

(Intuition)



(a) Walk along the curve $g(x,y) = c$ (g <u>fixed</u>)
When you reach the highest or lowest point (locally) in f, ∇f and ∇g are parallel: level curve of f must be parallel to $g(x,y)=c$ (in terms of tangent direction)

→ no increase or decrease for a moment

→ likely to have local min/max

(b) If this ∇f is not parallel to ∇g, hence level curves of f are not parallel to $g(x,y)=c$, f increases or decreases as you move. ▨

Example: (p. 264)

Find the maximum and minimum of $f(x,y) = 3x + y - 4$ subject to $x^2+2y^2 = 38$.

Use Lagrange multiplier.

Let $g(x,y) = x^2+2y^2$ (x^2+2y^2-38 is also fine)

$\nabla f(x,y) = (3,1)$. $\nabla g(x,y) = (2x, 4y)$

$$\begin{cases} \lambda(3,1) = (2x,4y) & \text{——— ⓐ} \\ x^2+2y^2 = 38. & \text{——— ⓑ} \end{cases}$$

ⓐ: $3\lambda = 2x$, $\lambda = 4y \longrightarrow x = \frac{3}{2}\lambda$, $y = \frac{\lambda}{4}$ ⓒ

Plug into ⓑ

$$\left(\frac{3}{2}\lambda\right)^2 + 2\left(\frac{\lambda}{4}\right)^2 = 38$$

$$\left(\frac{9\lambda^2}{4} + \frac{\lambda^2}{8} = 38\right) \times 8$$

$$\left(19\lambda^2 = 18\lambda^2 + \lambda^2 = 38 \cdot 8\right) \times \frac{1}{19}$$

$$\lambda^2 = 16 \quad \boxed{\lambda = \pm 4} \text{ (a)}$$

(by ⓒ)

(a)-1 If $\lambda = 4$, then $x = \frac{3}{2} \cdot 4 = 6$, $y = \frac{1}{4} \cdot 4 = 1$

$f(6,1) = 3 \cdot 6 + 1 - 4 = \boxed{15}$

(by ⓒ)

(a)-2 If $\lambda = -4$, then $x = \frac{3}{2} \cdot (-4) = -6$, $y = \frac{1}{4}(-4) = 1$

$f(-6,1) = 3 \cdot (-6) + (-1) - 4 = \boxed{-23.}$

Thus, max 15 occurs at (6,1), min -23 at (-6,1)

Lagrange multiplier for

maximize (or minimize) $f(x,y,z)$

subject to $g(x,y,z) = c$

① Find $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$

② Solve the system

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = c \end{cases}$$

③ Plug the solutions (x,y,z) into $f(x,y,z)$ and find max/min.

① Idea is similar to 2D case. But you need to imagine walking around the surface $g(x,y,z) = c$ and "feel" level surfaces of f.

Example: (p. 268)

Find the point on the surface $z = 2xy + 4$ that are closest to the origin. (Assume that point exists.)

minimize $f(x,y,z) = x^2+y^2+z^2$

subject to $g(x,y,z) = 2xy - z + 4 = 0$

/* $f(x,y,z) = \sqrt{x^2+y^2+z}$ leads to an equivalent problem. But it will get messy */

$\underline{\nabla f(x,y,z)} = (2x, 2y, 2z)$, $\underline{\nabla g(x,y,z)} = (2y, 2x, -1)$

System $\begin{cases} (2x, 2y, 2z) = \lambda(2y, 2x, -1) & \text{—— } (*) \\ 2xy - z + 4 = 0. & \text{—— } (**) \end{cases}$

From $(*)$,

$2x = 2\lambda y$, $2y = 2\lambda x$, $2z = -\lambda$

$x = \lambda y$, $\underbrace{y = \lambda x}_{(d)}$

$x = \lambda(\lambda x) = \lambda^2 x \longrightarrow 0 = x - \lambda^2 x = x(1-\lambda^2)$

$(\underbrace{x = 0}_{(a)}$ or $\underbrace{\lambda = \pm 1}_{(b)})$ and $(\underbrace{z = -\frac{\lambda}{2}}_{(c)})$

(a) If $x = 0$, from $(**)$, we have $z = 4$.

Then, $\lambda = -8$ from (c). And $y = 0$ (from (d))

In this case, $(x,y,z) = \boxed{(0,0,4)}$ gives

$f(0,0,4) = \boxed{16}$.

(b)-1 If $\lambda = 1$, $x = y$ (d) and $z = -\frac{1}{2}$ (c). Plug these into (**) to obtain

$$2x^2 - \left(-\frac{1}{2}\right) + 4 = 0 \Rightarrow 2x^2 = -\frac{9}{2}$$

no real number x satisfies this.

(b)-2 If $\lambda = -1$, $x = -y$ (d) and $z = \frac{1}{2}$ (c). Plug these into (**) to obtain

$$-2x^2 - \frac{1}{2} + 4 = 0 \Rightarrow 2x^2 = \frac{7}{2} \Rightarrow x^2 = \frac{7}{4}$$

$$x = \pm\frac{\sqrt{7}}{2} \Rightarrow y = \mp\frac{\sqrt{7}}{2}.$$

In this case, $\boxed{\left(\frac{\sqrt{7}}{2}, -\frac{\sqrt{7}}{2}, \frac{1}{2}\right)}$ and $\boxed{\left(-\frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}, \frac{1}{2}\right)}$ are possible and give

$$f\left(\frac{\sqrt{7}}{2}, -\frac{\sqrt{7}}{2}, \frac{1}{2}\right) = f\left(-\frac{\sqrt{7}}{2}, +\frac{\sqrt{7}}{2}, \frac{1}{2}\right)$$

$$= \left(\frac{\sqrt{7}}{2}\right)^2 + \left(-\frac{\sqrt{7}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{7+7+1}{4} = \boxed{\frac{15}{4}}$$

Since this is smaller than 16, $\left(\frac{\sqrt{7}}{2}, -\frac{\sqrt{7}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{7}}{2}, \frac{\sqrt{7}}{2}, \frac{1}{2}\right)$ are the closest points from the origin.

/* The shortest distance is $\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$. */

Lagrange multiplier for

maximize (or minimize) $f(x,y,z)$

subject to (a) $g_1(x,y,z) = C_1$ and ——— (a)

(b) $g_2(x,y,z) = C_2$. ——— (b)

① find ∇f, ∇g_1, and ∇g_2

② solve the system

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1(x,y,z) = C_1 \\ g_2(x,y,z) = C_2 \end{cases}$$

(everything evaluated at (x,y,z))

③ plug the solutions (x,y,z) into f to determine max/min.

(c) Similar idea to the previous ones:

- Each of (a) and (b) confines (x, y, z) to a surface

  $\to$ "(a) and (b) simultaneously" confine (x,y,z) to a

$\boxed{\text{Clicker}}$ (A) surface (B) curve (C) point

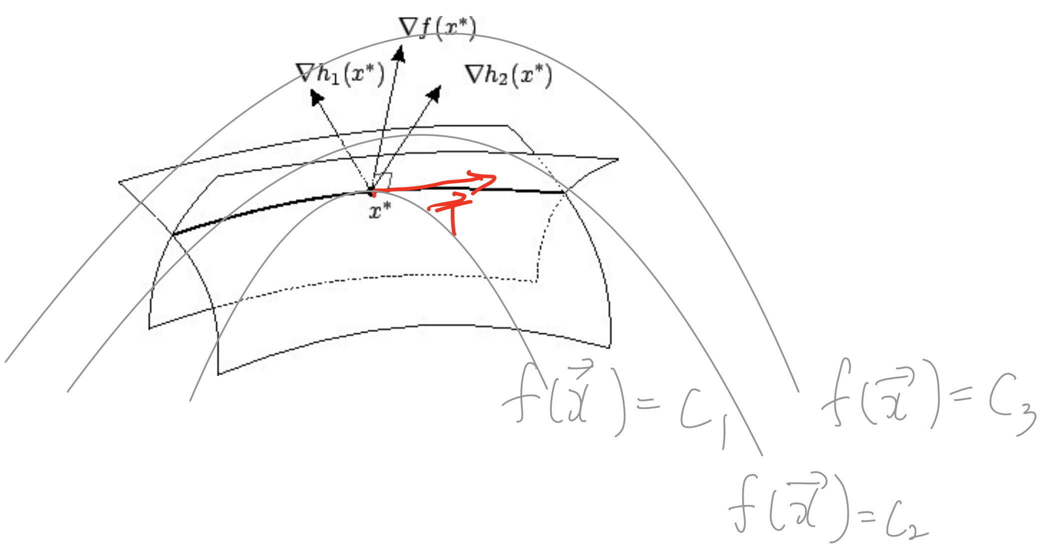
- Max/min occurs when this object is tangent to a level surface of f:

  $$\nabla f \perp \vec{T}$$

- But $\vec{T} \perp \nabla g_1$ and $\vec{T} \perp \nabla g_2$

  $\to$ $\nabla f, \nabla g_1, \nabla g_2$ all live in a plane that is orthogonal to $\vec{T}$

  $\to$ $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$



Example: Find the maximum and minimum values of $f(x,y,z) = 2x+y-z$ subject to

$$2x+z = 2/\sqrt{5} \quad \text{and} \quad y^2+z^2 = 1$$

$$\begin{cases} g_1(x,y,z) = 2x+z = 2/\sqrt{5} & \text{———} \ (*) \\ g_2(x,y,z) = y^2+z^2 = 1 & \text{———} \ (**) \\ \left(\nabla f = (2,1,-1),\ \nabla g_1 = (2,0,1),\ \nabla g_2 = (0,2y,2z)\right) & \\ (2,1,-1) = \lambda(2,0,1) + \mu(0,2y,2z) & \end{cases}$$

$\Rightarrow \quad 2 = 2\lambda, \quad 1 = 2\mu y, \quad -1 = \overset{1}{\cancel{\lambda}} + 2\mu z$

$\Rightarrow \boxed{\lambda = 1}, \ \boxed{y = \frac{1}{2\mu}}, \ \boxed{z = -\frac{1}{\mu}}$ ——— $(***)$

Plug these into $(**)$: ($(*)$ is not very useful: why?)

$$\left(\frac{1}{2\mu}\right)^2 + \left(-\frac{1}{\mu}\right)^2 = 1 \Rightarrow 1+4 = 4\mu^2$$

$$\Rightarrow \boxed{\mu = \pm \frac{\sqrt{5}}{2}} \quad (a)$$

(a)-1 If $\mu = \frac{\sqrt{5}}{2}$, then from $(***)$,

$\boxed{y = \frac{1}{\sqrt{5}}}, \ \boxed{z = -\frac{2}{\sqrt{5}}}$, and from $(*)$,

$$2x+z = 2x - \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \Rightarrow \boxed{x = \frac{2}{\sqrt{5}}}.$$

$$\Rightarrow (x,y,z) = \boxed{\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)}$$

In this case, $f\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = 2\cdot\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} - \left(-\frac{2}{\sqrt{5}}\right)$

$$= \boxed{\frac{7}{\sqrt{5}}}.$$

(a)-2 If $\mu = -\frac{\sqrt{5}}{2}$, then from (***),

$\boxed{y = -\frac{1}{\sqrt{5}}}$, $\boxed{z = \frac{2}{\sqrt{5}}}$, and from (*),

$$2x + z = 2x + \left(\frac{2}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} \Rightarrow \boxed{x = 0}.$$

$$\Rightarrow (x, y, z) = \boxed{\left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)}$$

In this case, $f\left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = 2 \cdot 0 - \frac{1}{\sqrt{5}} - \left(\frac{2}{\sqrt{5}}\right)$

$$= \boxed{-\frac{3}{\sqrt{5}}}$$

Thus the maximum is $\frac{7}{\sqrt{5}}$, which occurs at $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

and the minimum is $-\frac{3}{\sqrt{5}}$, which is attained at $\left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

Exercise: Try elementary strategies (e.g, algebra, etc.) for a few of these examples.