Announcement

- Quiz 4 tomorrow.
- ESCI → Please join.

  Particularly welcome are specific comments

  (Notes, organization, clicker, mind map, communication, etc.)
- HW8 → unusual deadline

  Due on Jun 11th (Sun)

  late sub. until Jun 15th (Thu)

  (b/c of exam/grades schedule)

- optional HW9 → no credit (for practice)
- extra credit HW6.5 → (2pt / 30pt) × 20%)

  will be announced for directional der prob.
- Curve → if overall grades suffer

  → Roughly 70% of class B^- or better.

/* How to apply and not to apply a theorem.

"$\boxed{\text{Thm}}$ If A is true, then B holds."

(A) Thm applies

① If A is checked, B must be true

② If B is not true, A cannot be true.

→ If A were true, B would've been true.

(B) Thm does not applies.

(→ Look for other ways.)

③ If A is not the case, the theorem does not say anything about B.

④ If B is true, A may or may not be ture.

*/

⊙ As theorems of form "If A, then B", these theorems do not tell us the all possible stories, there is no simple summary. This is just because functions are so diverse.

⊙ A good way is to use only what's clearly known and to see many different examples focusing on what condition is violated when theorems do not apply.

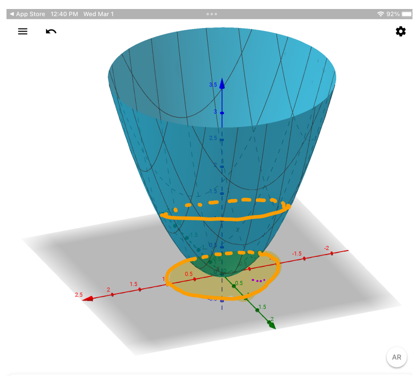
Examples: Determine whether the following functions have absolute max/min.

(A) $f(x,y) = x^2 + y^2$ on $D = B_1(0,0)$ → open ball of radius 1 centered at $(0,0)$.

$\{(x,y) : x^2+y^2<1\}$

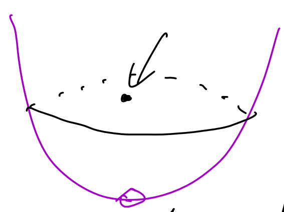
f has a global min, but not global max.

Domain not closed



(B) $f(x,y) = \begin{cases} x^2+y^2 & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$ on $\overline{B_1(0,0)}$

not continuous



closed, bounded

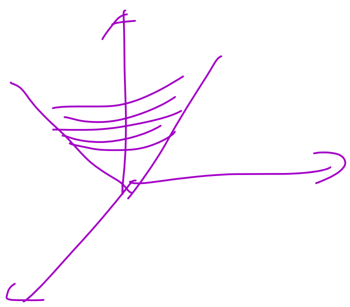
f has maximum 1 at $(0,0)$, but no min.

$\boxed{\text{Clicker}}$ Do these make extreme value thm false?

(A) Of course! We don't have maximum promised.

(B) No. The theorem does not apply.

Example: Show that $z = \sqrt{x^2+y^2}$ has a global max and min on $D = \overline{B_1(0,0)}$. Discuss previous two theorems for this function.



① $f(x,y) = \sqrt{x^2+y^2}$ is continuous as a composition of other continuous functions:

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\text{square}} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \xrightarrow{\text{sum}} x^2+y^2 \xrightarrow{\text{sq. root}} \sqrt{x^2+y^2}$$

② $\overline{B_1(0,0)}$ is bounded and closed.

Thus, extreme value thm applies to guarantee that there are global max and min.

Fermat does not apply since it is not continuous at $(0,0)$.

Def (Critical point in 2D)

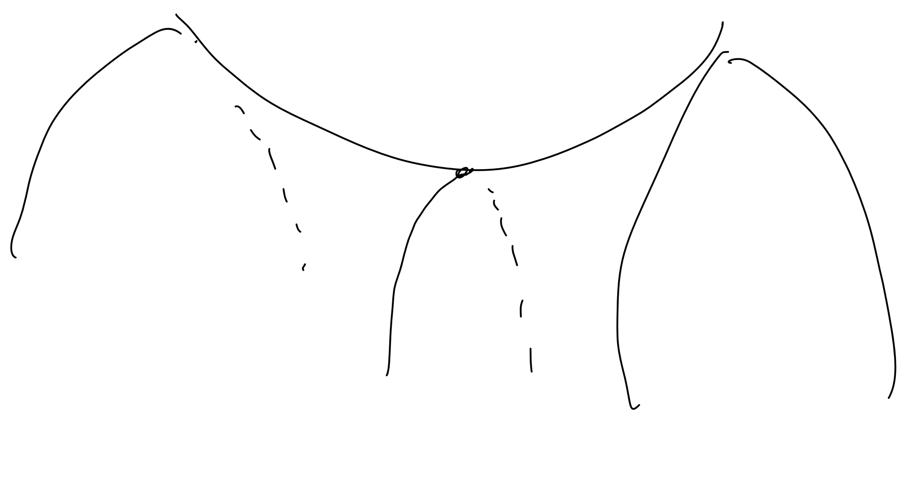
$\vec{a} \in D$ is called a **critical point** of $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$ if

ⓐ $\nabla f(\vec{a}) = \vec{0}$, or

ⓑ $f_x(\vec{a})$ or $f_y(\vec{a})$ does not exist.

Def (saddle point)

A critial point that is neither a local maximizer nor local minimizer is called a **saddle point**.



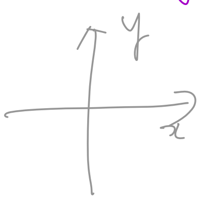
Example: Find all critical point of

$f(x,y) = 2x^2 - y^2 + 3$ on $D = \{(x,y) : -1 < x, y < 1\}$

And determine whether f has any extreme value there. (p. 248)

Discuss extreme value thm and Fermat thm.

$$\left.\begin{aligned} f_x &= 4x \\ f_y &= -2y \end{aligned}\right] \nabla f(x,y) = [4x, -2y] = \underline{(0,0)}$$



$\rightarrow$ $4x = 0$ and $-2y = 0$

$\rightarrow$ $(x,y) = (0,0)$

Along x-axis

$f(x,0) = 2x^2 + 3 > 3 = f(0,0)$ if $x > 0$

$\rightarrow$ (always sth <u>larger</u> nearby)

review def.

$\rightarrow$ can never be a local max.

Along y-axis

$f(0,y) = -y^2 + 3 \leq 3 = f(0,0)$

for any $y > 0$

$\rightarrow$ (always sth <u>smaller</u> nearby)

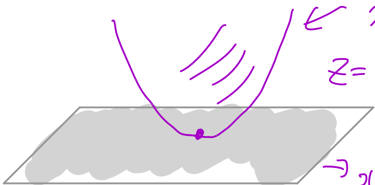
$\rightarrow$ can never be a local min.

Def (positive/negative definite)

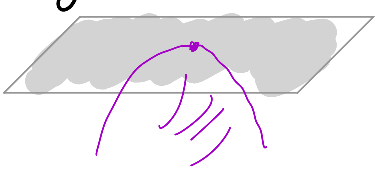
/* This generalizes 'concave up/down' */

Given $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is called

positive definite if $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq (0,0)$, and



negative definite if $\vec{x}^T A \vec{x} < 0$ for all $\vec{x} \neq (0,0)$.



Fact (Sylvester criterion)

Given $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \det(A)$

A is positive definite iff

$a>0$ and $D>0$.

Fact

A is negative definite if

$a<0$ and $D>0$.

THEOREM 4.10 Second Derivatives Test

Assume that $z = f(x, y)$ is a C^2 function defined on an open set containing (x_0, y_0), and let (x_0, y_0) be a critical point of f [i.e., $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$].

(a) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.

(b) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.

(c) If $D(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a neither a local maximum nor a local minimum (i.e., it is a saddle point).

If $f_{xx}(x_0, y_0) = 0$ ~~and $f_{yy}(x_0, y_0) \neq 0$, then, in (a) and (b), we use $f_{yy}(x_0, y_0)$ instead of $f_{xx}(x_0, y_0)$. In the case $f_{xx}(x_0, y_0) = f_{yy}(x_0, y_0)$~~ $= 0$ and $f_{xy}(x_0, y_0) \neq 0$, the discriminant $D(x_0, y_0)$ is negative, so case (c) applies. If $D(x_0, y_0) = 0$, then the Second Derivatives Test provides no answer: f could have a local extreme at (x_0, y_0), or (x_0, y_0) can be a saddle point. See Examples 4.27 and 4.28 below.

Here, $D(x_0, y_0) = \det\left(H_f(x_0, y_0) \right)$

(a) corresponds to negative definite H_f

(b) corresponds to positive definite H_f

(c) corresponds to saddle point H_f

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad \det(H_f) = f_{xx} f_{yy} - (f_{xy})^2$$

If $D(x_0, y_0) < 0$, then f has an increasing direction and a decreasing direction at (x_0, y_0) $\longrightarrow$ saddle point.

(Sketch of proof)

At a critical point $\vec{a}$, we have $\underbrace{\nabla f(\vec{a}) = \vec{0}}_{\text{no 1st order term}}$

Thus, by Taylor theorem

$$f(\vec{x}) = \underbrace{f(\vec{a}) + \frac{1}{2}\vec{h}^T H_f \vec{h}}_{T_2(\vec{x})} + \underbrace{R_2(\vec{a}, \vec{x})}_{\text{very } \underline{\text{small}} \text{ near } \vec{a}.}$$

where, $\vec{h} = \vec{x} - \vec{a}$. Behavior of f is very similar to $T_2(\vec{x})$ near $\vec{x} = \vec{a}$.

If H_f is negative definite,

$\frac{1}{2}\vec{h}^T H_f(\vec{a})\vec{h} < 0$ for any $\vec{h} \neq \vec{0}$

Thus, $f(\vec{x}) < f(\vec{a})$ near $\vec{a}$.

$\rightarrow f(\vec{a})$ is local $\underline{\text{max}}$.

If H_f is positive definite,

$\frac{1}{2}\vec{h}^T H_f(\vec{a})\vec{h} > 0$ for any $\vec{h} \neq \vec{0}$

Thus, $f(\vec{x}) > f(\vec{a})$ near $\vec{a}$.

$\rightarrow f(\vec{a})$ is local $\underline{\text{min}}$.

Example: Find and classify critical points of $f(x,y) = x^3 - 3x^2 - 3y^2 + 3xy^2$.

- Critical point : $\nabla f = 0$ or ~~partial DNE~~

$f_x(x,y) = 3x^2 - 6x + 3y^2 = 0$ —— $(*)$

$f_y(x,y) = -6y + 6xy = 6y(x-1) = 0$

$\iff \underline{y=0}_{(a)}$ or $\underline{x=1}_{(b)}$

by $(*)$

(a) If $y = 0$, then $f_x = 3x^2 - 6x = 3x(x-2) = 0$

$\iff x = 0$ or $x = 2$

$\boxed{(0,0),\ (2,0)}$ ① ②

(b) If $x = 1$, $f_x \overset{\text{by }(*)}{=} 3 - 6 + 3y^2 = 3(y^2 - 1)$

$= 3(y+1)(y-1) = 0 \iff y = 1$ or $y = -1$.

$\boxed{(1,-1),\ (1,1)}$ ③ ④

$f_{xx}(x,y) = 6x - 6$

$f_{xy}(x,y) = 6y$

$f_{yy}(x,y) = -6 + 6x$

① $(0,0)$: local max

$f_{xx}(0,0) = -6 < 0$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-6)\cdot(-6) - 0$

$= 36 > 0$

(negative definite)

② $(2,0)$: local min

$f_{xx}(2,0) = 6 \cdot 2 - 6 = 6 > 0$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 6 \cdot 6 - 0 = 36 > 0$

(positive definite Hessian)

③ $(1,-1)$

$f_{xx}(1,-1) = 6 - 6 = 0$

$f_{xx} f_{yy} - (f_{xy})^2 = 0 - (-6)^2 = -36 < 0$

saddle point

④ $(1,1)$

$f_{xx}(1,1) = 6-6 = 0$

$f_{xx}f_{yy} - (f_{xy})^2 = 0-(6)^2 = -36 < 0$

saddle point

Exercise: graph this fn

check what we did.

(Absolute max/min)

Typical steps to find global max/min

① find the values of f at critical points in the interior

② find max/min on the boundry.

③ Choose the smallest / largest value from ① and ②.

$\boxed{\text{Clicker}}$ Give your opinion about this steps.

(A) We need to classify types of critical points to determine whether they give max or min.

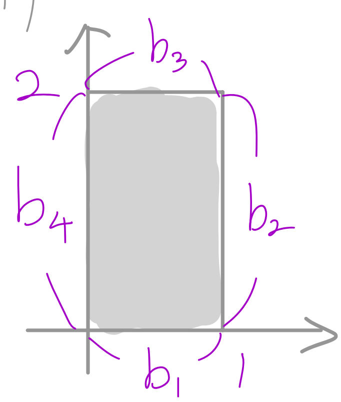
(B) No worries. Just chill out.

Example: Find absolute maximum and minimum of $f(x,y) = x^3 - 3xy + 3y^2$ on the rectangle $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$.
(p. 257)

(Interior)

$f_x = 3x^2 - 3y = 0 \Longrightarrow y = x^2$

$f_y = -3x + 6y = 0 \Longrightarrow \underline{x = 2y}$ (#)



$\Longrightarrow x = 2x^2$

$\Longrightarrow 2x^2 - x = 0$

$\Longrightarrow x(2x-1) = 0$

(a) $\boxed{x = 0 \text{ or } \frac{1}{2}}$

(a)-1 If $x = \frac{1}{2} \rightarrow y = \frac{1}{4}$ (#) $\boxed{\left(\frac{1}{2}, \frac{1}{4}\right)}$

$f\left(\frac{1}{2}, \frac{1}{4}\right) = \left(\frac{1}{2}\right)^3 - 3 \cdot \frac{1}{2} \cdot \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 = \boxed{-\frac{1}{16}}$

(a)-2 If $x = 0 \rightarrow y = 0$

$\longrightarrow$ boundary point. so, ignore this.

(examined later anyway)

(Boundary - vertices) $f(x,y) = x^3 - 3xy + 3y^2$

↳ "boundary of boundary"

$(0,0) \to f(0,0) = \boxed{0}$

$(1,0) \to f(1,0) = 1$

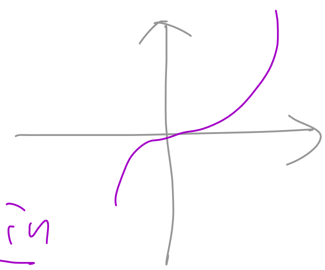
$(0,2) \to f(0,2) = \boxed{12}$

$(1,2) \to f(1,2) = 7$

(Boundary - open interval)

(b_1) $y=0$ and $0<x<1$

$f(x,0) = x^3$ <u>no</u> <u>max/min</u>

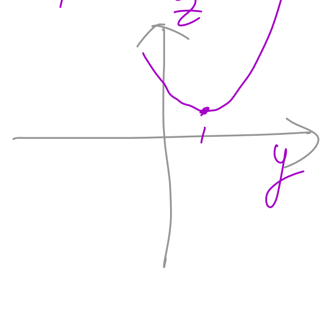


(b_2) $x=1$, $0<y<2$

$f(1,y) = 1-3y+3y^2$ $x^3-3xy+3y^2$

$= 3(y-\frac{1}{2})^2 - \frac{3}{4} + 1 = 3(y-\frac{1}{2})^2 + \frac{1}{4}$

at $(1,\frac{1}{2}) \to f(x,y) = \boxed{\frac{1}{4}}$



(b_3) $y=2$, $0<x<1$

$f(x,2) = x^3 - 6x + 12$

$\xrightarrow{d/dx} 3x^2 - 6 = 0$

$\Rightarrow x^2 = 2$

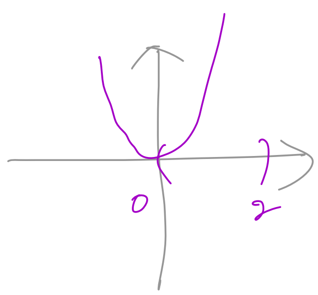
$\Rightarrow x = \pm\sqrt{2}$

↳ doesn't lie in the domain

(b_4) $x=0$, $0<y<2$

$f(0,y) = 3y^2$

Again, no max/min



min: $-\frac{1}{16}$ at $(\frac{1}{2}, \frac{1}{4})$ max: 12 at $(0,2)$