Midterm 2 keys are found

[Canvas] - [Files] - [keys]

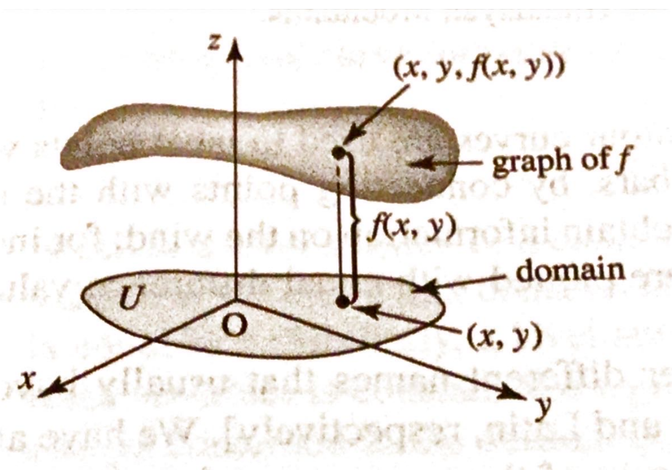
→ may be posted the next day after grades are published.

NT. 3.1 Graphical representations of multivariate functions. (Ch. 2.2)

/* see the classification of functions */

For simplicity, focus on 2D $\to$ 1D.

Surface representation



$$G(f) = \{ (x,y,z) \in \mathbb{R}^3 : z = f(x,y),\ (x,y) \in \underbrace{U}_{\text{Domain}} \}$$

Graph : collection of tuples $(\underbrace{\text{input}}_{\text{2D}}, \text{output})$

For function $z = f(x,y)$, think of

$((x,y), f(x,y))$ as $(x, y, f(x,y))$

E.g. If $(1,2) \longmapsto 5$, then think of $((1,2),5)$ as $(1,2,5)$

Level $\underbrace{\text{set}}_{\text{general}}$ / $\underbrace{\text{curve}}_{\text{2D}}$ / $\underbrace{\text{surface}}_{\text{3D}}$ (or contour)

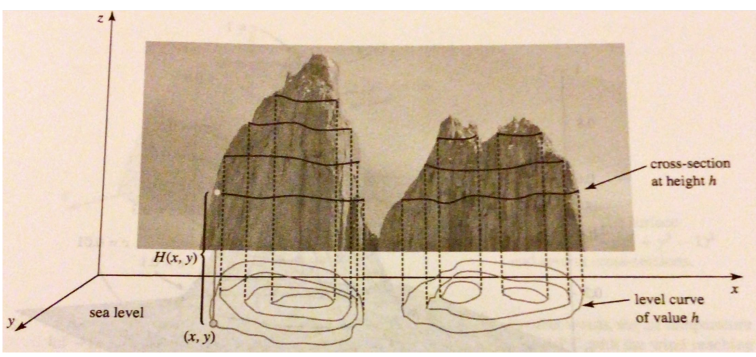
Given $f: U \subset \mathbb{R}^n \to \mathbb{R}$, **level set of value $c \in \mathbb{R}$** is defined by

$$L_c = \{ \vec{x} \in \mathbb{R}^n : f(\vec{x}) = c \}$$

Collection of all inputs that outputs c.

If 2D $(n=2)$, it is normally called **level curve** (of value c) or **contour**.

If 3D $(n=3)$, it is usually called **level surface** (of value c).



Example: Draw several level curves of

$z = 9 - x^2 - y^2$.

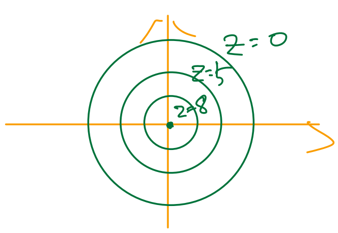
$z = 0 \Rightarrow 0 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 = 9$

$z = 5 \Rightarrow 5 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$

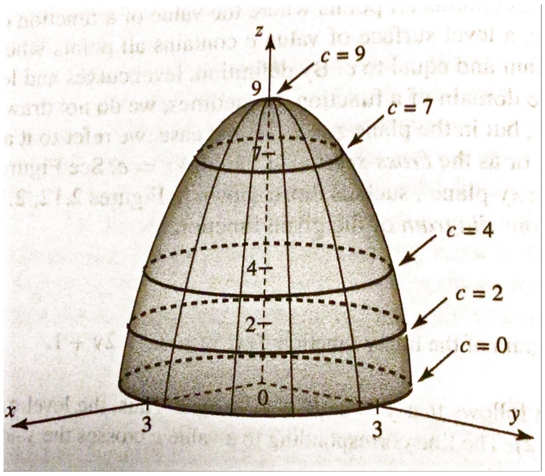
$z = 8 \Rightarrow 8 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$

$z = 10 \Rightarrow 10 = 9 - x^2 - y^2 \Rightarrow \underline{x^2 + y^2 = -1}$ empty set

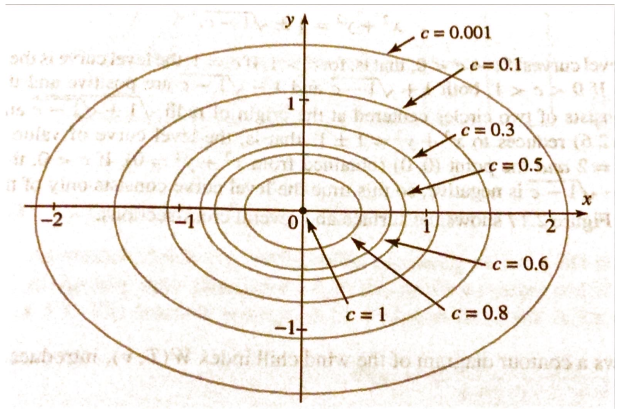
$z = 9 \Rightarrow x^2 + y^2 = 0 \Leftrightarrow (x, y) = (0, 0)$



More terminology



cross-sections at (heigts) 0, 2, 4, 7, 9 or level curves lifted to the surface



contour diagram

⊙ dim(graph) = dim(domain) + 1

dim(level set) = dim(domain)

Overview

We will adopt the same approach as we did in Cal I:

"Smooth changes are linear in close up."

Visual demonstration.

- Thus, we will develop tools needed to study tangent planes, or more generally linear approximations.
  - But things are more complicated than in univariate case by nature, which requires quite a bit of preparation.

- We will also study other things.
  - Taylor theorem for $2D \to 1D$ fn.
  - maximum/minimum of $2D \to 1D$ fn.
  - Lagrange multiplier, etc.
  - Multiple integrals

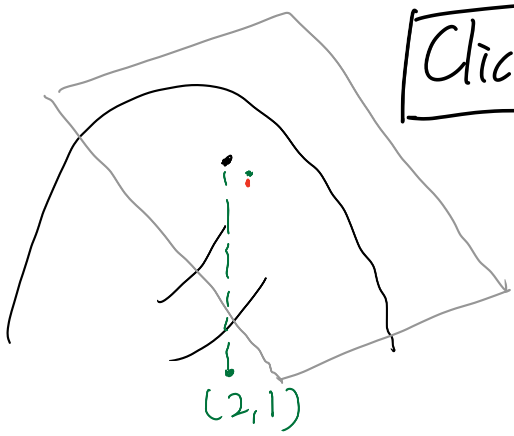
Example: Suppose you somehow know that the tangent plane to a surface $f(x,y) = \sqrt{10 - x^2 - 2y^2}$ at $(2,1)$ is

$$z = -(x-2) - (y-1) + 2$$

(1) What is $f(2.1, 1.1) = \sqrt{10 - (2.1)^2 - 2\cdot(1.1)^2}$

(2) What is linear approximation of $f(2.1, 1.1)$?

$\boxed{\text{Clicker}}$



Plug into the plane.

$$-(2.1-2)-(1.1-1)+2$$

$$= -(0.1) - (0.1) + 2$$

$$= 1.8$$

⊙ (2) is way faster for computations.

⊙ In many applications, (written in diff. eqn.) the ground truth (corresponding to (1)) is mostly unknown while its linear appr. is known. Therefore, it is very natural to study linear appr.

NT 3.2 Limits and continuity. (Ch. 2.3)

$\boxed{\text{Def}}$ (continuity)

Let $f: \mathbb{R}^2 \to \mathbb{R}$. Then, it is said to be **continuous at** $\vec{x} = \vec{x_0}$ if the following holds:

For any $\varepsilon > 0$, there is $\delta > 0$ such that
$\|\vec{x} - \vec{x_0}\| < \delta$ implies $|f(\vec{x}) - f(\vec{x_0})| < \varepsilon$.

① In words, "if input is close enough, then the output is also close."

Why do we care continuity?

small changes in input
$\longrightarrow$ small changes in output
(desired in all science.)

② This can also be expressed in terms of limits. (see below)

$\boxed{\textbf{Def}}$ (Limits)

Let $f : \mathbb{R}^2 \to \mathbb{R}$. The fn f is said to converge to $L \in \mathbb{R}$ as $\vec{x}$ approaches $\vec{x}_0$ or the limit of f exists as $\vec{x}$ approaches $\vec{x}_0$ and the limit is L if the following holds:

For any $\varepsilon > 0$, there is $\delta > 0$ such that $0 < \|\vec{x} - \vec{x}_0\| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$.

When this is the case, we denote this fact by

In 1D.

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \lim_{(x,y) \to (x_0, y_0)} f(x,y) = L, \text{ or}$$

$$f(\vec{x}) \to L \text{ as } \vec{x} \to \vec{x}_0$$

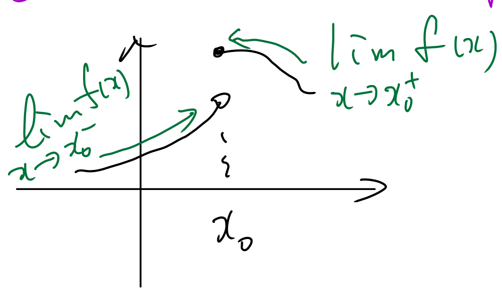
where $\vec{x} = (x, y)$ and $\vec{x}_0 = (x_0, y_0)$.

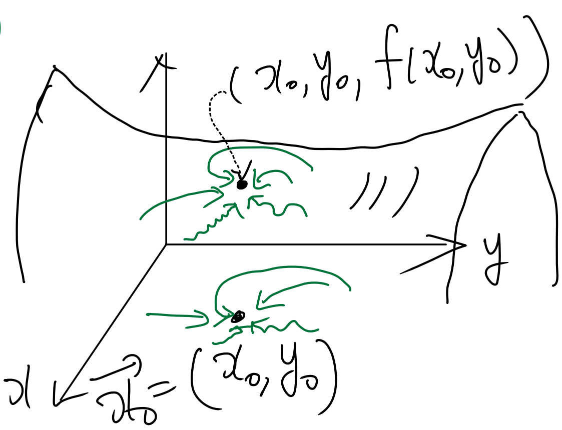
- The existence of limits is "almost continuity."
- In limits, we care how f behaves only near a point $\vec{x}_0$, but not at that point itself.

① Limits involve infinite pieces of information.

① Intuitively, the existence of limits says that "as $\vec{x}$ gets closer to $\vec{x}_0$, the output $f(\vec{x})$ gets closer to a fixed value L."

① There are infinitely many ways that $\vec{x}$ approaches $\vec{x}_0$ in m-D $(m \geq 2)$ while there are only two ways in 1D. (This is why we need to test only left and right limits in 1D.)





But those infinitely many ways can be captured by the condition

$0 < \|\vec{x} - \vec{x}_0\| < \delta$

$\boxed{\text{Thm}}$ (p. 85)

Let $\mathbf{F}, \mathbf{G}: \mathbb{R}^m \to \mathbb{R}^n$, $f, g: \mathbb{R}^m \to \mathbb{R}$ and assume that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x})$, $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x})$, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ exist. Then

(a) $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x}))$ and $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) - \mathbf{G}(\mathbf{x}))$ exist and

$$\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{F}(\mathbf{x}) \pm \mathbf{G}(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) \pm \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x}).$$

(b) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} c\mathbf{F}(\mathbf{x})$ (for any constant c) exist, and

$$\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x})g(\mathbf{x})) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right) \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} (c\mathbf{F}(\mathbf{x})) = c \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}).$$

(c) If $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) \neq 0$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/g(\mathbf{x})$ exists, and

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})}.$$

(d) For any $\mathbf{a} \in \mathbb{R}^m$ and any constant $\mathbf{c} \in \mathbb{R}^n$,

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{x} = \mathbf{a} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{c} = \mathbf{c}.$$

In part (d) the symbol $\mathbf{c}$ denotes the function $\mathbf{F}: \mathbb{R}^m \to \mathbb{R}^n$ given by $\mathbf{F}(\mathbf{x}) = \mathbf{c}$ for all $\mathbf{x} \in \mathbb{R}^m$.

You have all good properties.

Thm (Limit — continuity)
Let $f: \mathbb{R}^2 \to \mathbb{R}$. The fn f is continuous at $\vec{x}_0$ if and only if

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0).$$

That is, (a) the limit as $\vec{x}$ approaches $\vec{x}_0$ exists, (b) $f(\vec{x}_0)$ is defined, and (c) they are equal.

Fact A sum, difference, product, quotient, and composition of continuous fn's are all continuous (as long as the denominator fn of the quotient is not 0 at the input of interest and the dimensions match for the composition)

(More precise statements are found on pp. 88–89.)

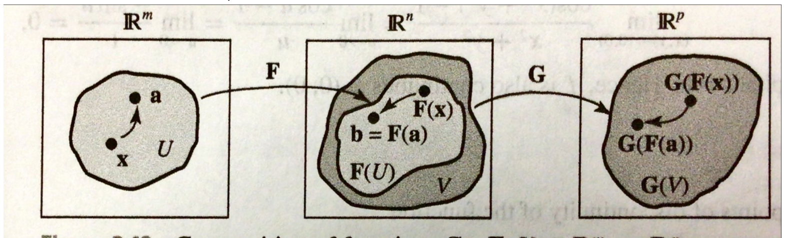


Figure 2.40 Composition of functions $\mathbf{G} \circ \mathbf{F}: U \subseteq \mathbb{R}^m \to \mathbb{R}^p$.

Example: Let $f(x,y) = \dfrac{\sin(\pi(x^2+y^2))}{xy}$

Find $\displaystyle\lim_{(x,y)\to(\frac{1}{2},\frac{1}{2})} f(x,y)$.

/* **Convention (in this course)**

When you are asked to "find" something but if that does not exist, you are expected to answer "that does not exist" or "DNE." If that exists, go ahead and give the answer. In other words, "find" is short for "find if that exist, and otherwise answer DNE." */

The fn f can be seen as a composition of operations and functions.

$$\begin{tikzcd}
\mathbb{R}^2 & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\
& x^2 \arrow[r] & \pi(x^2+y^2) \arrow[r] & \sin(\pi(x^2+y^2)) \arrow[dr] & \\
\begin{pmatrix} x \\ y \end{pmatrix} \arrow[ur] \arrow[r] \arrow[dr] & y^2 \arrow[ur] & & & \dfrac{\sin(\pi(x^2+y^2))}{xy} \\
& xy \arrow[r] & xy \arrow[r] & xy \arrow[ur] &
\end{tikzcd}$$

Or if we use m-D → n-D fn's

$$\underset{\mathbb{R}^2}{\begin{pmatrix} x \\ y \end{pmatrix}} \longrightarrow \underset{\mathbb{R}^3}{\begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}} \longrightarrow \underset{\mathbb{R}^2}{\begin{pmatrix} \sin(\pi(x^2+y^2)) \\ xy \end{pmatrix}} \longrightarrow \underset{\mathbb{R}}{\frac{\sin(\pi(x^2+y^2))}{xy}}$$

Since all compositions in between are continuous, the whole composition

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \frac{\sin(\pi(x^2+y^2))}{xy} = f(x,y)$$

is continuous. Thus, use (c) of "Limit-continuity thm".

$$\lim_{(x,y)\to(\frac{1}{2},\frac{1}{2})} f(x,y) \overset{①}{=} f\left(\tfrac{1}{2},\tfrac{1}{2}\right) \overset{②}{=} \frac{\sin(\pi\underbrace{((\frac{1}{2})^2+(\frac{1}{2})^2)}_{\frac{1}{4}+\frac{1}{4}=\frac{1}{2}})}{\frac{1}{2}\cdot\frac{1}{2}}$$

$$\overset{③}{=} \frac{(\sin\frac{\pi}{2})\times 4}{(\frac{1}{4})\times 4} \overset{④}{=} 4.$$

$\boxed{\text{Clicker}}$ Where did we use the thm?

(A) ① (B) ② (C) ③ (D) ④

- Observe that we need a lot of reasoning for what your answer would be a one-line calculation.

- Things are not always as simple as this example. In particular, when it is not easy to see if the fn is <u>continuous</u>, we need use the <u>definition</u> or some tool. See the following examples.

Example : ("Limit" depends on paths → DNE)

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{xy} & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

Convention: "find" means "find if exists, answer DNE if it doesn't"

Is f continuous at the point involved? Not sure

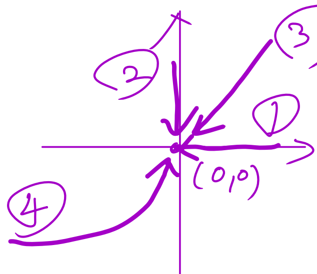
Recall the definition and comments.

→ Aim at DNE → find "bad paths"

Let us test different paths.

① x-axis

What is limit of constant?



$y = 0$

$f(x,y) = f(x,0) = 0$ (constant)

Therefore, $\lim\limits_{x\to 0} f(x,0) = 0$ (limit of constant is that constant)

② y-axis : Similarly, $\lim\limits_{y\to 0} f(0,y) = 0$.

$x = 0$

$f(x,y) = f(0,y) = 0$ (constant) (So far, no issues)

③ along $y = mx$ for some $m \in \mathbb{R}$.

$(x, y) = (x, mx)$, $f(x,y) = \dfrac{\sin(x^2 + m^2x^2)}{x \cdot m \cdot x}$

$$= \frac{\sin(x^2(1+m^2))}{x^2 m}$$

$$= \frac{\sin(x^2(1+m^2))}{x^2(1+m^2)} \cdot \frac{\cancel{x^2}(1+m^2)}{\cancel{x^2}\, m}$$

$\to 1$

From Cal1, $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

$(\theta = x^2(1+m^2))$

$\to \dfrac{1+m^2}{m}$ as $x \to 0$.

E.g., $m = 1 \to 2$

$m = 2 \to \dfrac{5}{2}$

$m = -1 \to -2$

There is no ~~Flixed~~ value that the expression approaches.

$\to$ DNE.