Announcement

- Quiz 3 on Wed

- Midterm 3 next Tuesday

Advice on current topics

- Plotting will help you feel more concrete about the material.

- Existence of limits and derivatives
  → try to find a good balance
    - (precision) Be aware that fn's can be "wild"
    - (enjoy) Once "well-behaving" is checked, apply amazing results you learned.

Example: Find the partials of

$f(x,y) = (x-3)^2 e^y$ (exist?) Yes!

↓ Think of y as constant

$$\frac{\partial f}{\partial x}(x,y) = 2(x-3)\, e^y$$

Treat y as const.

$$D_y f(x,y) = (x-3)^2 \, e^y$$

Notation

$\frac{\partial f}{\partial x}(x,y)$

↳ x value of evaluation

(i.e., at what point the derivative is being investigated.)

coordinate direction w.r.t. which the derivative is taken

Confusing? → use x_0, y_0 for evaluation.

Example: $f(x,y,z) = e^{xy}\sin(y^2+z^2)$.

Find the gradient $\nabla f(x,y,z)$.

$$\nabla f(x,y,z) = \left[\, \underline{f_x}(x,y,z) \quad \underline{f_y}(x,y,z) \quad \underline{f_z}(x,y,z) \,\right]$$

"Freeze"

$$\begin{aligned} = \Big[\, & e^{xy}\cdot y\cdot \sin(y^2+z^2) \\ & e^{xy}\cdot x\cdot \sin(y^2+z) + e^{xy}\cos(y^2+z^2)\cdot 2y \\ & e^{xy}\cos(y^2+z^2)\cdot 2z \,\Big] \end{aligned}$$

Derivatives of Vector fields

$\boxed{\text{Def}}$

Let $\vec{F}: \mathbb{R}^m \to \mathbb{R}^n$ be a vector field.
If there exists a linear mapping $L: \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\| \vec{F}(\vec{x}) - \overbrace{\vec{F}(\vec{x}_0) - L(\vec{x} - \vec{x}_0)}^{\text{(lin. appr.)}} \|}{\| \vec{x} - \vec{x}_0 \|} = 0,$$

then we say $\vec{F}$ is **differentiable at $\vec{x}_0$**.
And we call L **derivative of $\vec{F}$ at $\vec{x}_0$**.
Usually, L is denoted by $D\vec{F}$.

⊙ The norms appearing in the limit are different!

$\boxed{\text{Clicker}}$ Give the correct pair of norms in the form (numerator norm, denominator norm)

(A) ($\mathbb{R}^m$-norm, $\mathbb{R}^n$-norm) (B) ($\mathbb{R}^n$-norm, $\mathbb{R}^n$-norm)

(C) ($\mathbb{R}^n$-norm, $\mathbb{R}^m$-norm) D) ($\mathbb{R}^{n+1}$-norm, $\mathbb{R}^{m+1}$-norm)

⊙ Observe the similarity with $\mathbb{R}^n \to \mathbb{R}$ setting in what the definition is trying to capture despite some minute details are different.

$\boxed{\text{Thm}}$ (continuous partials imply continuous derivatives – vector field ver.)

If each component of output of $\vec{F}: \mathbb{R}^m \to \mathbb{R}^n$ has continuous partial derivatives <u>near</u> $\vec{x_0} \in \mathbb{R}^m$, then $\vec{F}$ is differentiable at $\vec{x_0}$.

Furthermore, $D\vec{F}(\vec{x_0})$ is given by the matrix J (n-by-m), where

$$J_{ij} = \frac{\partial y_i}{\partial x_j}(\vec{x_0}) \qquad \begin{array}{l} i = 1, 2, \cdots, n \\ j = 1, 2, \cdots, m \end{array}$$

x_j's are components of input $\vec{x}$, and y_i's are components of output.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{\vec{F}} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x, y, z) \\ v(x, y, z) \end{pmatrix}$$

How to compute $D\vec{F}$

Once m-D → 1D fn's are understood m-D → n-D fn's are easy to grasp. Just think of input and output as column vectors and treat output component-wisely

Example : $\vec{F}(x,y) = (x^3+y, e^{xy}, 2+xy)$

Find $D\vec{F}(1,1)$.

$\vec{F}$ is a 2D → 3D function.

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} f(x,y) \\ g(x,y) \\ h(x,y) \end{pmatrix} = \begin{pmatrix} x^3+y \\ e^{xy} \\ 2+xy \end{pmatrix}$$

$D\vec{F}(x,y)$ must be a linear mapping $\mathbb{R}^2 \to \mathbb{R}^3$: (3×2) matrix

1st row → derivative of 1st component of output (setting $\mathbb{R}^2 \to \mathbb{R}^1$)

→ $\nabla f(x,y)$ (1×2)

2nd row → $\nabla g(x,y)$ (simiarly)

3rd row → $\nabla h(x,y)$

Find $D\vec{F}(1,1)$.

$$\begin{pmatrix} x^3+y \\ e^{xy} \\ 2+xy \end{pmatrix} =: \begin{pmatrix} f \\ g \\ h \end{pmatrix} \longrightarrow D\vec{F} = \begin{bmatrix} Df \\ Dg \\ Dh \end{bmatrix} \quad \begin{bmatrix} f_x & f_y \\ g_x & g_y \\ \underline{h_x} & h_y \end{bmatrix}$$

reads "h sub x"

$$D\vec{F}(x,y) = \begin{bmatrix} 3x^2 & 1 \\ e^{xy}\cdot y & e^{xy}\cdot x \\ y & x \end{bmatrix}$$

$(x,y)=(1,1)$

$$D\vec{F}(1,1) = \begin{bmatrix} 3\cdot 1^2 & 1 \\ e^{1\cdot 1}\cdot 1 & e^{1\cdot 1}\cdot 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ e & e \\ 1 & 1 \end{bmatrix}$$

Talk with peers: what does this mean?

Def (Linear approximation)

Given a C^1-function $f:\mathbb{R}^m \to \mathbb{R}$ and $\vec{a} \in \mathbb{R}^m$,

$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

is called **linear approximation** of f at $\vec{a}$.

◎ Notice $\vec{a}$ is fixed and $\vec{x}$ is the variable.

◎ If $m=2$, $L(\vec{x})$ gives rise to the tangent plane:

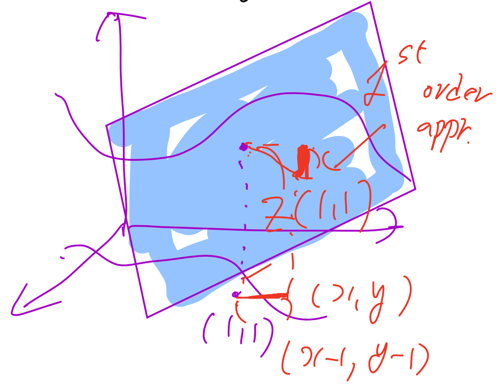
$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

set $\vec{a} = (x_0, y_0)$ $\vec{x} = (x, y)$

$$z = L(\vec{x})$$
$$= f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$
$$= f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)\ \frac{\partial f}{\partial y}(x_0, y_0)\right) \cdot (x - x_0, y - y_0)$$

Example: Find the equation of the tangent plane to $z = \tan^{-1}(xy)$ at $(x,y) = (1,1)$.

(p. 104)



Draw a rough picture inside.

appr. by const. fn.

$$L(x,y) = z(1,1) + \nabla z(1,1) \cdot ((x,y)-(1,1))$$

$$= \tan^{-1}(1\cdot 1) + \left(\frac{\partial}{\partial x}\tan^{-1}(xy), \frac{\partial}{\partial y}\tan^{-1}(xy)\right)\Bigg|_{(x,y)=(1,1)} \cdot (x-1, y-1)$$

(Cal1) Recall $(\tan^{-1}x)' = \dfrac{1}{1+x^2}$

$$= \frac{\pi}{4} + \left(\frac{1}{1+x^2y^2}\cdot y, \frac{1}{1+x^2y^2}\cdot x\right)\Bigg|_{(x,y)=(1,1)} \cdot (x-1, y-1)$$

$$= \frac{\pi}{4} + \left(\frac{1}{2}, \frac{1}{2}\right)\cdot(x-1, y-1)$$

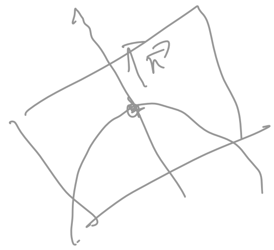
$$z = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1).$$

What if you want the normal line?

$\vec{n} = \left(\frac{1}{2}, \frac{1}{2}, -1\right) \longrightarrow$ normal vector of tangent plane

$\vec{p} = (1, 1, z(1,1)) \longrightarrow$ point on the surface

$= \left(1, 1, \frac{\pi}{4}\right)$



Thus, the normal line is

$t\vec{n} + \vec{p} =$ exercise.

Differential is an estimate of (small) change using linear approximation.

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$\Rightarrow \quad f(\vec{x}) - f(\vec{a}) \approx \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$\Rightarrow \quad \Delta f \approx \nabla f \cdot \Delta \vec{x}.$$

If $m = 2$,

$$f(x,y) - f(x_0,y_0) \approx \frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

$$\underbrace{\Delta f}_{\text{real change}} \approx \underbrace{f_x(x_0,y_0) \cdot \Delta x + f_y(x_0,y_0) \Delta y}_{\text{differential}}$$

When $\Delta x, \Delta y$ are considered small we usually write

$$\boxed{df = f_x(x_0,y_0)\, dx + f_y(x_0,y_0)\, dy}$$

- This is not new. We just focus on increment instead of full output.
- But sometimes this is used to indicate infinitesimal change. In this case, it is just a symbol rather than real estimates.

Example : Find the differential df of $f(x,y,z) = xy^2e^z$ at $(2,1,0)$.
What is approximate change in f when the change of input is
$\vec{dx} = (dx, dy, dz) = (0.1, -0.1, 0.1)$?

$$df = \nabla f \cdot \vec{dx} \qquad \text{(vector)}$$
$$= f_x \cdot dx + f_y\, dy + f_z\, dz \qquad \text{(expanded)}$$
$$= y^2e^z dx + x\cdot(2y)e^z dy + xy^2e^z \cdot dz$$

at $(2,1,0)$

$$= 1^2 \cdot e^0 dx + 2\cdot(2\cdot 1)\cdot e^0 dy + 2\cdot 1^2 e^0 dz$$
$$= dx + 4dy + 2dz.$$

If $(dx, dy, dz) = (0.1, -0.1, 0.1)$

$$= 0.1 + 4\cdot(-0.1) + 2(0.1)$$
$$= -0.1$$

skip.

$\boxed{\text{Clicker}}$ What is $\left(\lim_{x\to 0^+} \frac{x}{|x|}, \lim_{x\to 0^-} \frac{x}{|x|} \right)$

$= \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

$= \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^-} -1 = -1$

(A) $(1, 1)$ (B) $(1, -1)$
(C) $(-1, -1)$ (D) (DNE, DNE)

Example: Determine whether the cone

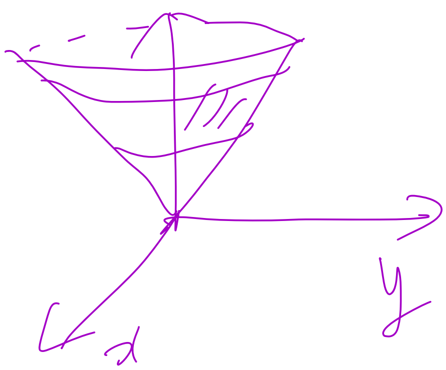
$z = \sqrt{x^2+y^2}$ is differentiable at $(0,0)$.

(p.105)

Not me  or Not brushing teeth, which she has never skipped.

/* Indirect proof:

"Suppose what you said is true, then we have this absurdity." The absurdity may not be directly related to what we want to prove, but anything that does not make sense works. */



Let us suppose that a derivative exists and call it

$L : \mathbb{R}^2 \to \mathbb{R}$

$L = [a \quad b]$. Then, we must have

$$\lim_{(x,y)\to(0,0)} \frac{\left| z(x,y) - z(0,0) - L\cdot\big((x,y)-(0,0)\big) \right|}{\|(x,y)-(0,0)\|} = \underline{0}.$$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2} - ax - by}{\sqrt{x^2+y^2}} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{ax+by}{\sqrt{x^2+y^2}} = 1$$

Interpret this as limit problem

$\sqrt{x^2} =$ ______

x-axis: $(x,0) \longrightarrow \lim_{x\to 0} \frac{ax}{|x|} = 1$

left limit

$$\lim_{x\to 0^-} \frac{ax}{|x|} = \lim_{x\to 0^-} \frac{ax}{-x} = -a$$

$$= 1 \Rightarrow a = -1$$

Right limit

$$\lim_{x\to 0^+} \frac{ax}{|x|} = \lim_{x\to 0^-} \frac{ax}{+x} = +a$$

$$= 1 \Rightarrow a = +1$$

$a = 1$ and $a = -1$ (Impossible)

Therefore, at least, "derivative exists" is not true

$\Rightarrow$ not differentiable.

Thm (differentiability implies continuity)
If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then it is continuous at $\vec{a}$.

proof) Since

$$\lim_{\vec{x} \to \vec{a}} \frac{1}{\|\vec{x}-\vec{a}\|} \left| f(\vec{x}) - f(\vec{a}) - L(\vec{x}-\vec{a}) \right| = 0$$

$$\Rightarrow \quad \lim_{\vec{x} \to \vec{a}} \left| f(\vec{x}) - f(\vec{a}) - L(\vec{x}-\vec{a}) \right| = 0$$

$$\Rightarrow \quad \lim_{\vec{x} \to \vec{a}} \left(f(\vec{x}) - f(\vec{a}) - \underbrace{L(\vec{x}-\vec{a})}_{\to 0} \right) = 0$$

$$\Rightarrow \quad \lim_{\vec{x} \to \vec{a}} f(\vec{x}) - f(\vec{a}) = 0$$

◎ However, unlike (full) differentiability, existence of partials does not guarantee continuity. See the following examples.

Example : Given

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ((x,y) \neq (0,0)) \\ 0 & ((x,y) = (0,0)), \end{cases}$$

show that f_x, f_y exist and f is not continuous at $(0,0)$. (p. 106)

According to def. of partial derivative,

$$f_x(0,0) = \lim_{t \to 0} \frac{\overset{=0}{f(t,0)} - \overset{=0}{f(0,0)}}{t}$$

$$= \lim_{t \to 0} \frac{0}{t} = 0.$$

Similarly,

$$f_y(0,0) = \lim_{t \to 0} \frac{\overset{=0}{f(0,t)} - \overset{=0}{f(0,0)}}{t} = 0$$

Therefore, f_x and f_y exist at $(0,0)$ and $f_x(0,0) = f_y(0,0) = 0$.

But it is not continuous.
To this end, we show its limit does not exist.

/* Logic: continuous iff (a) limit exists, (b) output defined, and (c) they are equal

"if and only if" = equivalent to

One of (a), (b), (c) is not true $\longrightarrow$ discontinuous */

Along $y = mx$, we have

$$f(x, mx) = \frac{mx^{\cancel{2}}}{\cancel{x^2} + m^2 \cancel{x^2}} = \frac{m}{1+m^2}$$

Hence, $\lim\limits_{x \to 0} f(x, mx) = \lim\limits_{x \to 0} \dfrac{m}{1+m^2}$

$$= \frac{m}{1+m^2} \rightarrow \text{depends on } m.$$

Therefore, the limit $\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does not exist.

Conclusion: f is not continous at $(0,0)$

$\boxed{\text{Clicker}}$ We could have computed partials:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$f_x(x,y) = \frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

Why did Prof. Park show a hard way?

(A) He overcomplicated stuff.

(B) ✓ The above approach is wrong, and the previous one is correct.

Okay good effort. What is $f_x(0,0)$?

→ Recall "Shortcuts are shortcuts."

When existence of something is in question, go back to definition.

(2) We can show that it is not diff. using definition.

If it were diff. at $(0,0)$, then $L=[a\ b]$ exists s.t.

$$\frac{f(x,y)-f(0,0)-L(x,y)}{\sqrt{(x-0)^2+(y-0)^2}} \to 0 \quad \text{as} \quad (x,y)\to(0,0)$$

That is,

$$\frac{\frac{xy}{x^2+y^2}-ax-by}{\sqrt{x^2+y^2}} = \underbrace{\frac{xy-(ax+by)\cdot(x^2+y^2)}{\left(\sqrt{x^2+y^2}\right)^3} \to 0}_{(*)}$$

as $(x,y)\to(0,0)$.

(x-axis): $\dfrac{-ax^3}{|x|^3} \to a$ or $-a$ depending on $x\to 0^-$, $x\to 0^+$

(Left and right limits as before).

They must be equal: $a=-a \Rightarrow a=0$.

Similarly, we get $b=0$ by examining y-axis.

Plug $a=0$ and $b=0$ into (*), then we must have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}^{\,3}} = 0.$$

However, along $y=x$

$$\frac{x\cdot x}{\sqrt{x^2+x^2}^{\,3}} = \frac{x^2}{(2x^2)^{3/2}} = \frac{1}{2^{3/2}\cdot|x|}$$

$$\to \pm\infty \quad \text{as } x\to 0$$

This is a contradiction.

Therefore, L does not exist.