

(proof of convergence of bisection method; slides
P.6)
Chop the task into pieces.

① $\lim a_n, \lim b_n, \lim c_n$ exist and they all the same.

② Call the limit, ξ , then $f(\xi) = 0$.

③ $|c_n - \xi| < 2^{-(n+1)}(b-a)$

① Observe $a_0 \leq a_1 \leq a_2 \leq \dots \leq b$ by construction. Therefore $\lim a_n$ exists.

/* Math 3B - monotone sequence theorem.

If $\{a_n\}$ is nondecreasing (or nonincreasing) and bounded above (or bounded below), the limit exists. [This is "half-version"]

If $\{a_n\}$ is monotonic (i.e., only nonincreasing or only nondecreasing) and bounded (i.e., bounded from above and below), it converges. [This is "two-sided-ver."] */

Likewise $b_0 \geq b_1 \geq b_2 \geq \dots \geq a$. Therefore b_n also converges.

Let $\lim a_n = \xi_1$, and $\lim b_n = \xi_2$.

We know the length of $[a_n, b_n]$ gets halved from the construction. Therefore, $b_n - a_n \rightarrow 0$ as $n \rightarrow \infty$. Then, we must have

$$0 = \lim(b_n - a_n) = \lim b_n - \lim a_n \\ = \xi_2 - \xi_1,$$

$\Rightarrow \xi_1 = \xi_2$ Call this common limit ξ .

Lastly for ①, we have $a_n \leq c_n \leq b_n$.

Therefore, sandwich theorem says

$$\lim a_n \leq \lim c_n \leq \lim b_n$$

\downarrow $\underbrace{\lim c_n}$ \downarrow
 ξ ξ

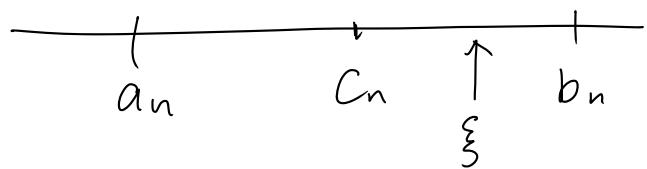
Thus, $\lim c_n = \xi$.

(2) Taking limit of $f(a_n)f(b_n) \leq 0$, we have
 $f(\xi)f(\xi) \leq 0$. The only possibility is
 $f(\xi) = 0$.

* Here, continuity of f is used.

f conti at $a \iff f(a_n) \rightarrow f(a)$ & $\{a_n\}$ st. $a_n \rightarrow a$

(3) Observe $\xi \in [a_n, b_n] \quad \forall n$. Thus,



$$\begin{aligned} |c_n - \xi| &< \frac{1}{2} \cdot \text{length}([a_n, b_n]) \\ &= \frac{1}{2} \cdot \frac{1}{2} \text{length}([a_{n-1}, b_{n-1}]) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{2}\right)^n \text{length}([a_1, b_1]) \\ &= \underbrace{\left(\frac{1}{2}\right)^{n+1} \text{length}([a_0, b_0])}_{= b_0 - a_0 = b - a} \end{aligned}$$

(proof of quadratic conv. of Newton's method.)

Put $e_n = x_n - \xi$.

Subtract ξ from the method and sneak in $f(\xi)$

$$x_{n+1} - \xi = x_n - \xi - \frac{f(x_n) - f(\xi)}{f'(x_n)}$$

$$e_{n+1} = \frac{1}{f'(x_n)} (f(\xi) - f(x_n) - f'(x_n)(\xi - x_n))$$

$$\left. \begin{aligned} f(\xi) &= f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(c_n)}{2!} (\xi - x_n)^2, & c_n \in (\xi, x_n) \\ &\text{or } (x_n, \xi) \end{aligned} \right\}$$

$$= \frac{1}{f'(x_n)} \cdot \frac{f''(c_n)}{2} e_n^2 \quad \text{as } e_n \rightarrow 0$$

Therefore, IF $e_n \rightarrow 0$ as $n \rightarrow \infty$, then

$x_n \rightarrow \xi$, and in turn, $f'(x_n) \rightarrow f'(\xi)$

and $f''(c_n) \rightarrow f''(\xi)$ as $n \rightarrow \infty$.

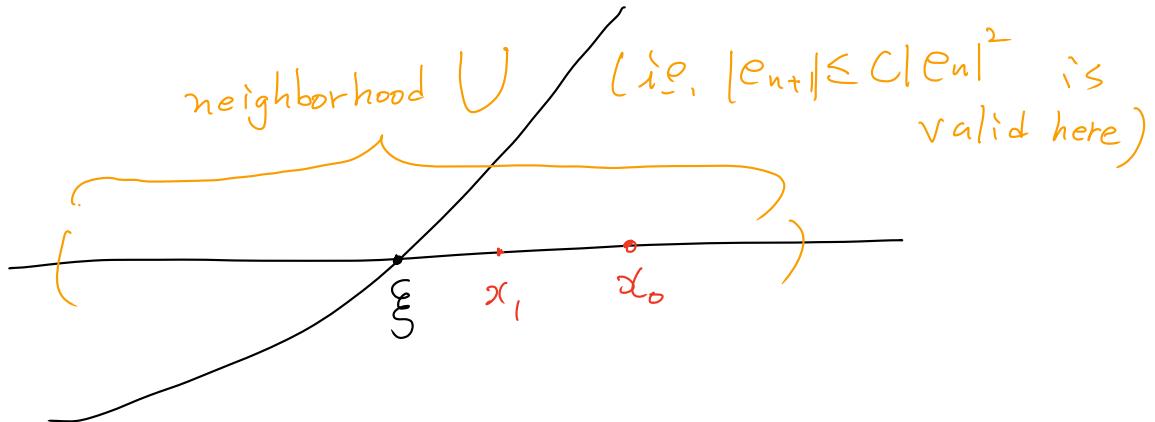
Then, dividing the error e_{n+1} by e_n^2 and taking limit $e_n \rightarrow 0$, we have

$$\frac{e_{n+1}}{e_n^2} \rightarrow \frac{f''(\xi)}{2f'(\xi)} \text{ as } e_n \rightarrow 0. \text{ That is,}$$

$$\left| \frac{e_{n+1}}{e_n^2} \right| \leq C \quad \text{if } e_n \approx 0. \text{ Or roughly } |e_{n+1}| \approx C|e_n|^2$$

just a generic constant as long as it is fixed.

Now, we prove the IF part.



Let $|e_{n+1}| \leq C|e_n|^2$ on U

Choose x_0 so that $|e_0| = |x_0 - \xi| < \frac{1}{2C}$
and $x_0 \in U$.

/* This is where "if the initial guess
is sufficiently close to the zero"
comes in. */

Then, we see

$$|e_1| \leq C|e_0|^2 \leq C \cdot \frac{1}{2C} \cdot |e_0| = \frac{1}{2} |e_0|$$

Therefore, $x_1 \in U$ too.

Repeat this so that

$$\begin{aligned} |e_2| &\leq C|e_1|^2 \leq C \cdot \left(\frac{1}{2}\right)^2 |e_0| \cdot |e_0| \\ &\leq C \cdot \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2C} \cdot |e_0| = \left(\frac{1}{2}\right)^3 |e_0| \end{aligned}$$

$$\dots |e_n| \leq \left(\frac{1}{2}\right)^{2n-1} |e_0|$$

Thus, $e_n \rightarrow 0$ as $n \rightarrow \infty$.



[Thm] (Taylor thm)

If $f \in C^{n+1}$ near a point x (on $(x-\delta, x+\delta)$),

then for any $y \in (x-\delta, x+\delta)$, we have

(Lagrange remainder ver.)

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(y-x)^n \\ + \frac{f^{(n+1)}(\xi_y)}{(n+1)!}(y-x)^{n+1} \quad (\xi_y \in (x, y) \text{ or } (y, x))$$

and

(Integral remainder ver.)

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(y-x)^n \\ + \int_x^y \frac{f^{(n+1)}(t)}{n!} \cdot (y-t)^n dt$$

(proof of Taylor theorem with Lagrange remainder)

If $y=x$, there is nothing to prove. So, $y \neq x$.

$$\text{Set } M = \frac{(f(y) - T_n(y; x))}{(y-x)^{n+1}}$$

$$\text{so that } f(y) - T_n(y; x) - M(y-x)^{n+1} = 0.$$

We want : there is $\xi_y \in (x, y)$ st. $f^{(n+1)}(\xi_y) = (n+1)! M$

Introduce

$$g(t) = f(t) - T_n(t; x) - M(t-x)^{n+1}$$

Note that $f(x) = T_n(x; x)$, $f'(x) = T'_n(x; x)$,
 \dots , $f^{(n)}(x) = T_n^{(n)}(x; x)$. Therefore,

$$g(x) = g'(x) = g''(x) = \dots = g^{(n)}(x) = 0 \quad \text{since}$$

$M(t-x)^{n+1}$ has zero at x of order $n+1$.

By construction of M , we have $g(y) = 0$

Apply MVT (mean value theorem) to g . Then,

there is $\xi_1 \in (x, y)$ st. $g'(\xi_1) = 0$. Apply

MVT to g' on $[x, \xi_1]$. Then, there is

$\xi_2 \in (x, \xi_1)$ st. $g''(\xi_2) = 0$. Repeat this

so that there is $\xi_n \in (x, \xi_{n-1})$ st. $g^{(n)}(\xi_n) = 0$.

Repeat once more to have $\xi_{n+1} = \xi_y \in (x, \xi_n) \subset (x, y)$

st. $g^{(n+1)}(\xi_y) = 0$. That is $f^{(n+1)}(\xi_y) = M(n+1)!$ \square

(proof of global conv. of Newton's method
for convex fn's)

Since f is increasing and has a zero by the assumption, zero is unique.

Since $f \in C^2$ and convex, $f''(x) > 0 \forall x \in \mathbb{R}$. Also, $f'(x) > 0 \forall x \in \mathbb{R}$ since f is increasing.

Now, recall the error equation (*)

$$e_{n+1} = \frac{1}{f'(x_n)} (f(\xi) - f(x_n) - f'(x_n)(\xi - x_n)) \\ = \underbrace{\frac{1}{f'(x_n)}}_{positive} \cdot \underbrace{\frac{f''(c_n)}{2} e_n^2}_{positive}$$

Deduce $e_{n+1} = x_{n+1} - \xi \geq 0$ no matter

$e_n > 0$ or not. $\Rightarrow e_1, e_2, e_3, \dots \geq 0$.

This, in turn, yield $f(x_n) \geq f(\xi) = 0 (n=1,2,\dots)$
since f increasing: $x_n \geq \xi \Rightarrow f(x_n) \geq f(\xi)$

From $\underbrace{x_{n+1} - \xi}_{e_{n+1}} = \underbrace{x_n - \xi}_{e_n} - \underbrace{\frac{f(x_n)}{f'(x_n)}}_{positive}$,

We have $e_{n+1} \leq e_n$ for $n=1,2,\dots$

Again, monotone sequence theorem says $\{e_n\}$ converges. But we don't know the limit. Call it e .

$e_{n+1} \leq e_n$ also implies

$x_n > x_{n+1} > \dots \geq \xi$. The same theorem applies so that $\lim x_n$ exists.

Call it z . Taking $n \rightarrow \infty$ in the following

$$\underbrace{x_{n+1} - \xi}_{e_{n+1}} = \underbrace{x_n - \xi}_{e_n} - \frac{f(x_n)}{\underbrace{f'(x_n)}_{\text{positive}}} \downarrow$$

$$\cancel{e} = e - \frac{f(z)}{f'(z)} \Rightarrow f(z) = 0$$

That is $z = \xi$, the unique zero.

(Newton's method for \sqrt{a})

Given $a > 0$, \sqrt{a} can be defined by a positive root of

$$f(x) = x^2 - a.$$

Thus,

$$f'(x) = 2x.$$

Therefore the Newton's method reads

$$\begin{aligned}x_{n+1} &= x_n - f(x_n)/f'(x_n) \\&= x_n - (x_n^2 - a)/2x_n \\&= x_n - \frac{x_n}{2} + \frac{a}{2x_n} \\&= \frac{x_n}{2} + \frac{a}{2x_n}\end{aligned}$$

proof of superlinear conv. of secant method.

Similarly to Newton's method, subtract the zero ξ from the method

$$x_{n+1} - \xi = x_n - \xi - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad (1)$$

Use MVT: $f(x_n) - f(x_{n-1}) = f'(z_n)(x_n - x_{n-1})$

Plug this in, we obtain

$$\begin{aligned} e_{n+1} &= e_n - \underbrace{f(x_n)/f(z_n)} \\ &= \underbrace{f(\xi) + f'(\xi)(\xi - z_n)}_{=0} + \underbrace{f''(z_n)/2}_{\cdot (\xi - z_n)^2} \\ &= e_n \left(1 - \underbrace{\frac{f'(\xi)}{f'(z_n)}}_{\rightarrow 1} + \underbrace{\frac{f''(z_n)}{2} (\xi - z_n)}_{\rightarrow 0} \right) \\ &\quad \rightarrow 0 \end{aligned}$$

We can still extract something tending to 0!

This time we continue (*)

$$\begin{aligned}
 e_{n+1} &= \frac{(f(x_n) - f(x_{n-1}))e_n - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\
 &= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})} \approx "f'(x_n) - f'(x_{n-1})" \\
 &\quad \text{(see below)} \\
 &= e_n e_{n-1} \frac{\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{f(x_n) - f(x_{n-1})} \\
 &= e_n e_{n-1} \frac{(x_n - x_{n-1}) \left(\frac{f''(\xi)}{2} + O(e_n^2) + O(e_{n-1}^2) \right)}{f(x_n) - f(x_{n-1})}
 \end{aligned}$$

MVT

$$f(x_n) - f(x_{n-1}) = f'(\theta_n)(x_n - x_{n-1}) \rightarrow C \text{ as } e_n \rightarrow 0$$

$$= e_n e_{n-1} \frac{1}{f'(\theta_n)} \left(\frac{f''(\xi)}{2} + O(e_n^2) + O(e_{n-1}^2) \right)$$

$$f(x_n) = f(\xi) + f'(\xi)(\xi - x_n) + \frac{f''(\xi)}{2} (\xi - x_n)^2 + \dots$$

$$\Rightarrow f(x_n)/e_n = -f'(\xi) - \frac{f''(\xi)}{2} (\xi - x_n) + O(e_n^2)$$

$$f(x_{n-1}) = f(\xi) + f'(\xi)(\xi - x_{n-1}) + \frac{f''(\xi)}{2} (\xi - x_{n-1})^2 + O(e_{n-1}^2)$$

$$\Rightarrow f(x_{n-1})/e_{n-1} = -f'(\xi) - \frac{f''(\xi)}{2} (\xi - x_{n-1}) + \dots$$

$$\Rightarrow f(x_n)/e_n - f(x_{n-1})/e_{n-1} = \frac{f''(\xi)}{2} (x_n - x_{n-1}) + O(e_n^2) + O(e_{n-1}^2)$$

In short,

$$|e_{n+1}| \approx C |e_n| e_{n-1} \quad (\star\star)$$

Let's assume that a fixed rate of convergence is available and call it φ :

$$|e_{n+1}| \approx A |e_n|^{\varphi} \quad \text{when } |e_n| \text{ is small}$$

Plug this into $(\star\star)$.

$$\begin{aligned} |e_{n+1}| &\approx C |e_n| |e_{n-1}| \\ A |e_n|^{\varphi} &\quad C A |e_{n-1}|^{\varphi+1} \\ A (A |e_{n-1}|^{\varphi})^{\varphi} & \\ A^{1+\varphi} |e_{n-1}|^{\varphi^2} & \\ \Rightarrow |e_{n-1}|^{\varphi^2 - \varphi - 1} &\approx C A^{-\varphi} \quad (\text{constant}) \\ \text{tending to } 0 \text{ or } \infty \text{ unless } \varphi^2 - \varphi - 1 = 0 & \end{aligned}$$

$$\Rightarrow \varphi = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow \varphi = \frac{1 + \sqrt{5}}{2} \quad (\text{take positive one})$$

A common trick using big oh.

Def If there is a neighborhood I of a and $C > 0$ such that

$$|f(x)| \leq C(g(x)) \quad \forall x \in I \setminus \{a\}$$

we denote $f(x) = O(g(x))$ as $x \rightarrow a$ and say " $f(x)$ is big oh of $g(x)$ " or " $f(x)$ is of order of $g(x)$ ".

Intuition: $f(x)$ is almost constant multiple of $g(x)$ near $x=a$.

Advantage:

give minimal attention to what does not matter much.

- sometimes, big o term does not matter (our current situation)
- other times, everything other than big o term does not matter.

Example : part of the proof above

Compare the two scenarios.

1) Using big o (See above)

2) Full details without big o.

$$\begin{aligned}
 & \frac{f(x_n) - f(x_{n-1})}{e_n e_{n-1}} = \frac{\overbrace{e_n}^{\text{MVT}} \underbrace{\frac{f'(x_n) - f'(x_{n-1})}{e_n}}_{\text{see below}}}{e_{n-1}} \\
 &= e_n e_{n-1} \frac{(x_n - x_{n-1}) \left(\frac{f''(\xi)}{2} + \frac{f'''(a_n)}{6} e_n^2 - \frac{f'''(a_{n-1})}{6} e_{n-1}^2 \right)}{f(x_n) - f(x_{n-1})} \\
 &= e_n e_{n-1} \frac{1}{f'(x_n)} \left(\frac{f''(\xi)}{2} + \frac{f'''(a_n)}{6} e_n^2 - \frac{f'''(a_{n-1})}{6} e_{n-1}^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 f(x_n) &= f(\xi) + f'(\xi)(\xi - x_n) + \frac{f''(\xi)}{2} (\xi - x_n)^2 + \frac{f'''(a_n)}{6} (\xi - x_n)^3 \quad a_n \in (x_n, \xi) \\
 \Rightarrow f(x_n)/e_n &= -f'(\xi) - \frac{f''(\xi)}{2} (\xi - x_n) + \frac{f'''(a_n)}{6} e_n^2 \quad a_n \in (x_n, \xi) \\
 f(x_{n-1}) &= f(\xi) + f'(\xi)(\xi - x_{n-1}) + \frac{f''(\xi)}{2} (\xi - x_{n-1})^2 + \frac{f'''(a_{n-1})}{6} (\xi - x_n)^3 \\
 \Rightarrow f(x_{n-1})/e_{n-1} &= -f'(\xi) - \frac{f''(\xi)}{2} (\xi - x_{n-1}) + \frac{f'''(a_{n-1})}{6} e_{n-1}^2 \\
 \Rightarrow f(x_n)/e_n - f(x_{n-1})/e_{n-1} &= \frac{f''(\xi)(x_n - x_{n-1})}{2} \\
 &\quad + \frac{f'''(a_n)}{6} e_n^2 - \frac{f'''(a_{n-1})}{6} e_{n-1}^2
 \end{aligned}$$

(proof of contraction mapping thm)

$$\begin{aligned}
 |x_{n+1} - x_n| &= |F(x_n) - F(x_{n-1})| \\
 &\leq \lambda |x_n - x_{n-1}| \\
 &\leq \lambda^2 |x_{n-1} - x_{n-2}| \\
 &\quad \vdots \\
 &\leq \lambda^n |x_1 - x_0|
 \end{aligned}$$

Observe $x_n = S_n - S_{n-1}$ ($n \geq 1$)

$$\begin{aligned}
 &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots \\
 &\quad + (x_1 - x_0) + x_0,
 \end{aligned}$$

where $S_n = \sum_{j=0}^n x_j$.

$$\begin{aligned}
 \sum_{n=0}^{\infty} |x_{n+1} - x_n| &\leq \sum_{n=0}^{\infty} \lambda^n |x_1 - x_0| \\
 &= |x_1 - x_0| \frac{1}{1-\lambda} \text{ (convergent.)}
 \end{aligned}$$

$\Rightarrow \sum_{n=0}^{\infty} (x_{n+1} - x_n)$ converges.

(absolute conv. implies conv.)

$$\Rightarrow \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sum_{n=0}^{\infty} (x_{n+1} - x_n) + x_0 \right) = x$$

Then, take $n \rightarrow \infty$ in $x_{n+1} = F(x_n)$:

$$x = F(x).$$

* Contraction is continuous

If $|x-y| < \epsilon/\lambda$, then

$$\begin{aligned}|F(x) - F(y)| &\leq \lambda |x-y| \\&< \lambda \cdot \frac{\epsilon}{\lambda} = \epsilon.\end{aligned}$$

Therefore, the limit x is the fixed point of F .

- $x \in D$ b/c D is closed.
- Fixed point is unique since if y is another one,

$$\begin{aligned}|x-y| &= |F(x) - F(y)| \leq \lambda |x-y| \\&\Rightarrow (\lambda - 1) |x-y| \leq 0 \Rightarrow x=y.\end{aligned}$$

example of contraction mapping
diverging $\lambda = 1$.

