- Practice exam updated
  → a few more problems.

- Recommend walking around mind maps.
  → There are many connections.

NT. 3.4 Properties of der's of multivariate functions. (Ch. 2.6)

/* Even in Cal I, definition of der. is not very efficient and combinations of differentiation rules help a lot. In high dimensions, that is more true. */

These resemble Cal I.

(a) Assume that the functions $\mathbf{F}, \mathbf{G}: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ are differentiable at $\mathbf{a} \in U$. Then the sum $\mathbf{F}+\mathbf{G}$ and the difference $\mathbf{F}-\mathbf{G}$ are differentiable at $\mathbf{a}$ and

$$D(\mathbf{F} \pm \mathbf{G})(\mathbf{a}) = D\mathbf{F}(\mathbf{a}) \pm D\mathbf{G}(\mathbf{a}).$$

(b) If the function $\mathbf{F}: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{a} \in U$ and $c \in \mathbb{R}$ is a constant, then the product $c\mathbf{F}$ is differentiable at $\mathbf{a}$ and

$$D(c\mathbf{F})(\mathbf{a}) = c\,D\mathbf{F}(\mathbf{a}).$$

(c) If the real-valued functions $f, g: U \subseteq \mathbb{R}^m \to \mathbb{R}$ are differentiable at $\mathbf{a} \in U$, then their product fg is differentiable at $\mathbf{a}$ and

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$

1D: $f'g + fg'$

(d) If the real-valued functions $f, g: U \subseteq \mathbb{R}^m \to \mathbb{R}$ are differentiable at $\mathbf{a} \in U$, and $g(\mathbf{a}) \neq 0$, then their quotient f/g is differentiable at $\mathbf{a}$ and

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}.$$

$\frac{f}{g} = fg^{-1}$

(e) If the vector-valued functions $\mathbf{v}, \mathbf{w}: U \subseteq \mathbb{R} \to \mathbb{R}^n$ are differentiable at $a \in U$, then their dot (scalar) product $\mathbf{v} \cdot \mathbf{w}$ is differentiable at a and

$$(\mathbf{v} \cdot \mathbf{w})'(a) = \mathbf{v}'(a) \cdot \mathbf{w}(a) + \mathbf{v}(a) \cdot \mathbf{w}'(a).$$

(f) If the vector-valued functions $\mathbf{v}, \mathbf{w}: U \subseteq \mathbb{R} \to \mathbb{R}^3$ are differentiable at $a \in U$, their cross (vector) product $\mathbf{v} \times \mathbf{w}$ is differentiable at a and

$$(\mathbf{v} \times \mathbf{w})'(a) = \mathbf{v}'(a) \times \mathbf{w}(a) + \mathbf{v}(a) \times \mathbf{w}'(a).$$

/* Pay attention to dimensions */

Example: Compute $\nabla(f/g)(x,y,z)$ if $f(x,y,z) = -x^2y^2$ and $g(x,y,z) = 2yz$.

(p. 125)

(Quotient rule)

$$\frac{\nabla f \cdot g - f \cdot \nabla g}{g^2} =$$

$$\frac{(-2xy^2, -2x^2y, 0)\cdot 2yz - (-x^2y^2)(0, 2z, 2y)}{4y^2z^2}$$

$$= \frac{(-4xy^3z, \overbrace{-4x^2y^2z + 2x^2y^2z}^{-2x^2y^2z}, 2x^2y^3)}{4y^2z^2}$$

$$= \left(-\frac{xy}{z}, -\frac{x^2}{2z}, \frac{x^2y}{2z^2}\right)$$

Way 2: Brute force (simpler here, but quotient rule is the only way when we don't know f and g)

$$\nabla\left(\frac{f}{g}\right) = \nabla\left(-\frac{x^2y}{2z}\right) = \left(-\frac{\cancel{2}xy}{\cancel{2}z}, -\frac{x^2}{2z}, \frac{x^2y}{2z^2}\right)$$

[Thm] (Chain rule)

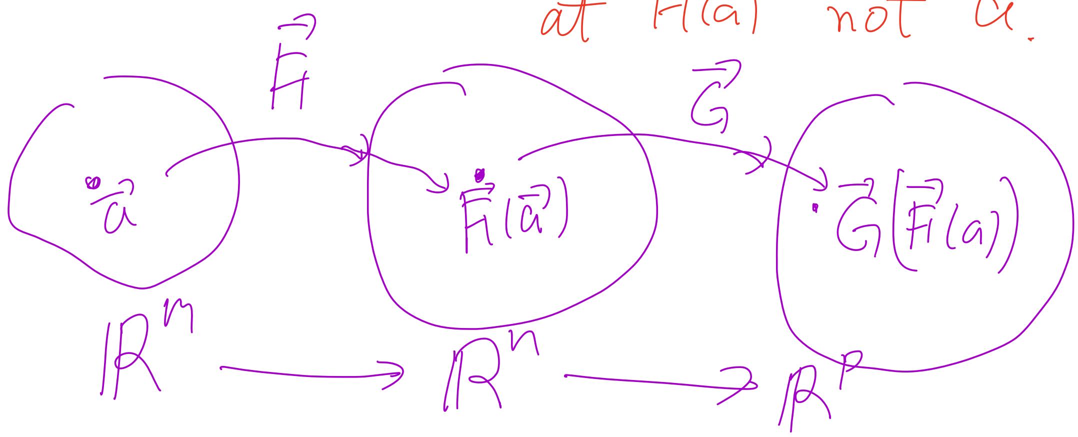
Suppose that $\mathbf{F}: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{a} \in U$, U is open in $\mathbb{R}^m$, $\mathbf{G}: V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $\mathbf{F}(\mathbf{a}) \in V$, V is open in $\mathbb{R}^n$, and $\mathbf{F}(U) \subseteq V$ (so that the composition $\mathbf{G} \circ \mathbf{F}$ is defined). Then $\mathbf{G} \circ \mathbf{F}$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{G} \circ \mathbf{F})(\mathbf{a}) = D\mathbf{G}(\mathbf{F}(\mathbf{a})) \cdot D\mathbf{F}(\mathbf{a}),$$

where $\cdot$ denotes matrix multiplication.

1D : $(f \circ g)'(x)$
$= f'(g(x))\, g'(x)$

evaluate outside fn at $F(a)$ not a.



$\underline{DG} \cdot \underline{DF}$ (suppressing evaluation)

$(p \times n)(n \times m) = p \times m$

$\vec{F} \circ \vec{G} : \mathbb{R}^m \to \mathbb{R}^p$

$D(\vec{F} \circ \vec{G}) : p \times m$

Example: Given $\vec{F}(x,y) = (x^3+y, e^{xy}, 2+xy)$ and $\vec{G}(u,v,w) = (u^2+v, uv+w^3)$, compute $D(\vec{G}\circ\vec{F})(0,1)$ (p. 127)

(Setting)

$$\mathbb{R}^2 \xrightarrow{\vec{F}} \mathbb{R}^3 \xrightarrow{\vec{G}} \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \vec{F}(x,y) \longrightarrow \vec{G}\circ\vec{F}(x,y)$$

(Evaluation points)

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left.\begin{pmatrix} x^3+y \\ e^{xy} \\ 2+xy \end{pmatrix}\right|_{(x,y)=(0,1)} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$D\vec{F}(0,1) = \begin{bmatrix} 3x^2 & 1 \\ e^{xy}\cdot y & e^{xy}\cdot x \\ y & x \end{bmatrix}_{(x,y)=(0,1)} \begin{array}{l} \to \text{grad of } 1^{st} \text{ component} \\ \to \text{grad of } 2^{nd} \text{ component} \\ \to \text{grad of } 3^{rd} \text{ component} \end{array}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$D\vec{G}(\vec{F}(0,1)) = D\vec{G}(1,1,2)$$

$$= \begin{bmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{bmatrix}\Bigg|_{(u,v,w)=(1,1,2)}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 12 \end{bmatrix}$$

chain rule

$$D(\vec{G}\circ\vec{F})(0,1) = D\vec{G}(1,1,2)\cdot D\vec{F}(0,1)$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 12 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 13 & 1 \end{bmatrix}$$

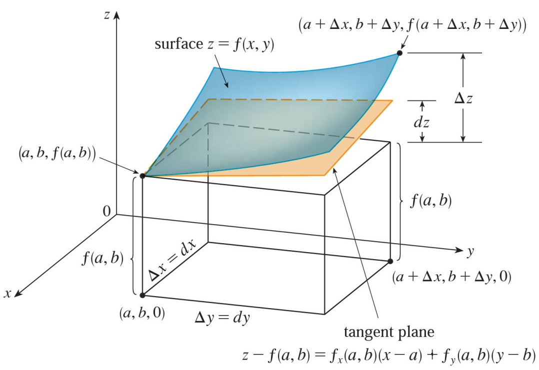
Clicker Does this has the right size?

(A) Yes, it should be 2×2.

(B) No, we did something wrong.

Partial der. of a composite function

/* This is just another way of organizing Chain rule */



For concreteness consider the case $\mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \to f(x,y) =: z$$

How can we compute $\frac{\partial z}{\partial u}$?

/* We are assuming only u changes (v fixed) */

$$dz = f_x \underbrace{dx}_{\substack{\text{change in } x \\ (\text{due to } u)}} + f_y \underbrace{dy}_{\substack{\text{change in } y \\ (\text{due to } u)}}$$

$$= f_x \cdot \frac{\partial x}{\partial u} du + f_y \frac{\partial y}{\partial u} du$$

$$\boxed{\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}}$$

$$\begin{tikzcd} & x \arrow[dr, "\frac{\partial f}{\partial x}"] & \\ u \arrow[ur, "\frac{\partial x}{\partial u}"] \arrow[dr, "\frac{\partial y}{\partial u}"'] & & f \\ & y \arrow[ur, "\frac{\partial f}{\partial y}"'] & \end{tikzcd}$$

$\shortparallel$ (under f): z

$\longrightarrow$ "Dependency diagram"

Example: Let $(f\circ\vec{c})(t)$ be a composition of $f(x,y)=x^2+2y^2$ and $\vec{c}(t)=(e^t, te^t)$.

Compute $(f\circ\vec{c})'(0)$. (p. 127)

$$\begin{array}{ccccc} \mathbb{R}^1 & & \mathbb{R}^2 & & \mathbb{R}^1 \\ t & \longrightarrow & \begin{pmatrix}x\\y\end{pmatrix}=\vec{c}(t) & \longrightarrow & f(x,y) \end{array}$$

$$0 \longmapsto \begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x(0)\\y(0)\end{pmatrix},$$ that is, $\underset{t=0}{\downarrow}\ (x,y)=(1,0)$

$$\begin{tikzcd} & x \arrow[dr, "\frac{\partial f}{\partial x}"] & \\ t \arrow[ur, "\frac{dx}{dt}"] \arrow[dr, "\frac{dy}{dt}"'] & & f \\ & y \arrow[ur, "\frac{\partial f}{\partial y}"'] & \end{tikzcd}$$

$$\Longrightarrow \frac{df}{dt} = \underline{\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$$

$$=(2x)\cdot e^t+(4y)\cdot(e^t+te^t)$$

$$=2\cdot 1+0\cdot 1=2$$

vector version

$$(f\circ\vec{c})'(0)=\nabla f(1,0)\cdot(\vec{c}\,'(0))$$

$$=\begin{bmatrix}2x & 4y\end{bmatrix}\Big|_{(x,y)=(1,0)}\cdot(e^t, e^t+te^t)_{t=0}$$

$$=\begin{bmatrix}2 & 0\end{bmatrix}\cdot(1,1)=2$$

Directional derivative via chain rule.

Given $f: \mathbb{R}^m \to \mathbb{R}$, $\vec{a} \in \mathbb{R}^n$, and a unit vector $\vec{d} \in \mathbb{R}^m$, directional der. is defined by

$$D_{\vec{d}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{d}) - f(\vec{a})}{t}$$

If we view $f(\vec{a} + t\vec{d})$ as $(f \circ g)(t)$, where $\vec{g}(t) = \vec{a} + t\vec{d}$, this is nothing but $\frac{d}{dt}(f \circ g)(0)$.

/*
$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{g} & \mathbb{R}^m & \xrightarrow{f} & \mathbb{R} \\ t & & \vec{a} + t\vec{d} & & f(\vec{a} + t\vec{d}) \end{array}$$
*/

Using chain rule,

$$\frac{d}{dt}(f \circ g)(0) = \nabla f(g(0)) \cdot g'(0).$$

Thus,

$$\boxed{D_{\vec{d}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{d}}$$

directional derivative = Gradient • direction

Example: (Directional der. revisited)

Given $f(x,y) = \sqrt{10 - x^2 - 2y^2}$, find the directional der. $D_{\vec{u}} f(2,1)$, where $\vec{u} = \left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$.

$$f(x,y) = (10 - x^2 - 2y^2)^{\frac{1}{2}}$$

$$\partial_x f(x,y) = \frac{1}{\cancel{2}} (10 - x^2 - 2y^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)$$

$$\partial_y f(x,y) = \frac{1}{\cancel{2}} (10 - x^2 - 2y^2)^{-\frac{1}{2}} \cdot (-\overset{2}{\cancel{4}}y)$$

$$\underline{(10 - \underbrace{2^2}_{4} - \underbrace{2 \cdot 1^2}_{2})^{-\frac{1}{2}}} = (4)^{-\frac{1}{2}} = \frac{1}{2}$$

Therefore,

$$\nabla f(2,1) = \left(-\frac{1}{2} \cdot 2, -\frac{1}{2} 2 \cdot 1\right) = (-1,-1)$$

$$D_{\vec{u}} f(2,1) = \underline{\nabla f(2,1) \cdot \vec{u}}$$

$$= (-1,-1) \cdot \left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$$

$$= -\frac{1}{3} - \frac{2\sqrt{2}}{3} = \frac{-1-2\sqrt{3}}{3}$$

Leave it but aware of it.

$\boxed{\text{Clicker}}$

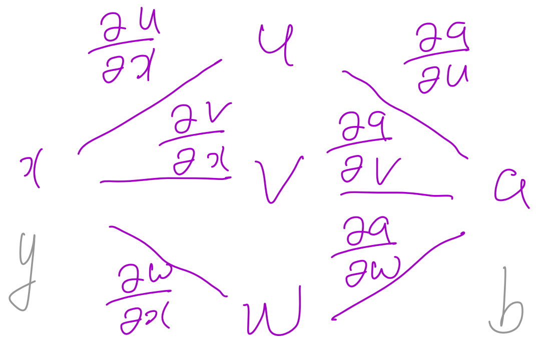
What if $\vec{u}$ is not a unit vector?

/* Directional der. depends on the length of $\vec{d}$, not only on the direction */

Example: Given $\vec{F}(x,y) = (x^3+y,\ e^{xy},\ 2+xy)$
$\vec{G}(u,v,w) = (u^2+v,\ uv+w^3)$, find $\frac{\partial a}{\partial x}(0,1)$, where a is the 1^{st} component of $\vec{G}\circ\vec{F}$.

Setting:

$$\mathbb{R}^2 \xrightarrow{\vec{F}} \mathbb{R}^3 \xrightarrow{\vec{G}} \mathbb{R}^2$$



$$\frac{\partial a}{\partial x} = \frac{\partial a}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial a}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial a}{\partial w}\frac{\partial w}{\partial x}$$

$$= (2u)\cdot(3x^2) + (1)\cdot(e^{xy}\cdot y) + (0)\cdot y$$

$$= (2\cdot 1)(3\cdot 0^2) + 1\cdot e^{0\cdot 1}\cdot 1 + 0\cdot 1 = 1.$$

In fact, if you carefully keep track of the calculation, these are nothing but part of the (full) derivative

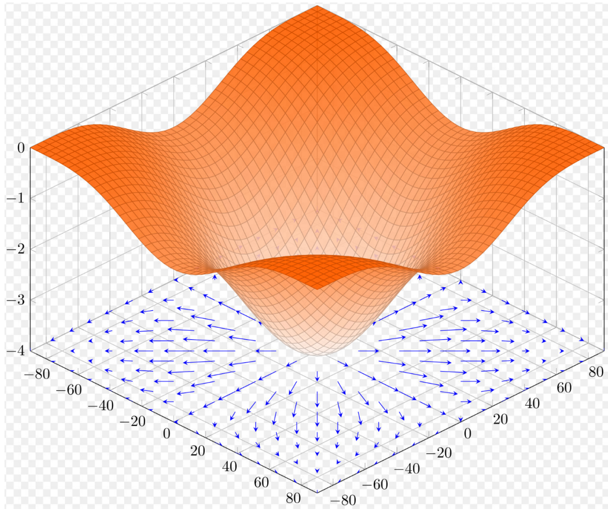
example (the matrix version).

- matrix version is easier for full der.
- Dependency diagram (current version) is easier for a particular partial derivative.

(A) Taday's pace is too fast
(B) It is alright.
(C) You can do faster.

NT. 3.5 Gradient and directional der. (Ch. 2.7)

Gradient Field



Given $f : \mathbb{R}^m \to \mathbb{R}$, $\nabla f(\vec{x})$ is an $\mathbb{R}^m$ vector for each $\vec{x}$: vector field.

$\to \quad \nabla f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$

$\quad\quad \vec{x} \longmapsto \nabla f(\vec{x})$

vector $\quad\quad$ vector

Thm Let $f:\mathbb{R}^m \to \mathbb{R}$ be diff. Then,

(Case I) **(flat)** if $\nabla f(\vec{a}) = \vec{0}$, then $D_{\vec{u}} f(\vec{a}) = \underline{0}$ for $\underline{\text{all}}$ direction $\vec{u}$ ($\vec{u} \neq \vec{0}$).

(Case II) if $\nabla f(\vec{a}) \neq \vec{0}$, then

(steepest ascent) f "increases the most rapidly" along the direction $\underline{\nabla f(\vec{a})}$

(steepest descent) f "decreases the most rapidly" along the direction $\underline{-\nabla f(\vec{a})}$

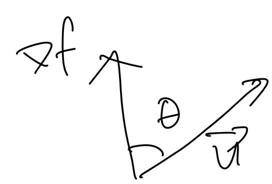
(orthogonal) $\nabla f(\vec{a})$ is $\underline{\text{orthogonal}}$ to the $\underline{\text{level set}}$ of $\underline{\text{value}}$ $f(\vec{a})$.

(proof) (Case I) is obvious. $D_{\vec{u}} f(\vec{a}) = \underbrace{\nabla f}_{=\vec{0}} \cdot \vec{u}$

Assume $\nabla f(\vec{a}) \neq \vec{0}$. Then,

$$D_{\vec{u}} f(\vec{a}) = \underline{\nabla f(\vec{a}) \cdot \vec{u}}$$

$$= \underbrace{\|\nabla f(\vec{a})\|}_{\text{fixed}} \cdot 1 \cdot \cos\theta$$

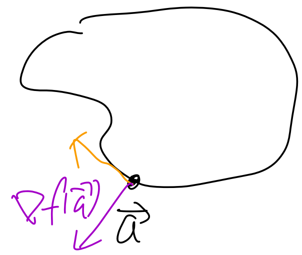


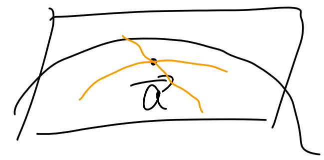
→ maximum when $\underline{\cos\theta = 1}$ ($\theta = \underline{0}$)
$\nabla f(\vec{a})$ and $\vec{u}$ head to the same direction.

→ minimum when $\underline{\cos\theta = -1}$ ($\theta = \underline{\pi}$)
$\nabla f(\vec{a})$ and $\vec{u}$ head to the opposite direction.

Next, think of a level set

$f(\vec{x}) = c$





$$(*) \quad f(\vec{x}) = c = f(\vec{a})$$

For any curve on the level set passing through $\vec{a}$

$$\vec{r}(t) = (r_1(t), r_2(t), \cdots, r_n(t))$$

$$\vec{r}(0) = \vec{a}$$

chain rule

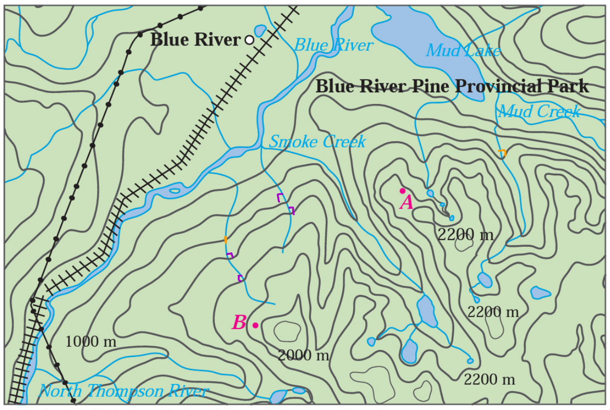
$$\frac{d}{dt} f(\vec{r}(0)) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) \qquad \text{(LHS of } (*))$$

$$= \frac{d}{dt}(c) = \underline{0}. \qquad \text{(RHS of } (*))$$

$$\Longrightarrow \underbrace{\nabla f(\vec{a})}_{\text{gradient at } \vec{a}} \perp \underbrace{\vec{r}\,'(0)}_{\text{tangent vector}}$$

of any curve passing $\vec{a}$ on the level set $f(\vec{x}) = c \; (= f(\vec{a}))$

Example



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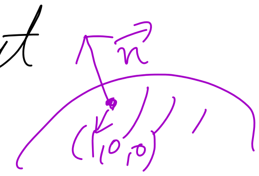
Q: Any observation in relation to the theorem we just proved?

Example: Find the line that passes through $(1,0,0)=\vec{a}$ on the surface $z=x^2+y^2-1$ and normal to this surface.



Make it a level set (level surface in this case) of another fn.

Let $f(x,y,z)=x^2+y^2-z-1=0$

Then, the surface given is a level set of f of value 0.

Therefore, by the previous theorem

$\nabla f(1,0,0)$ is orthogonal to the surface.

$$= (2x, 2y, -1)\Big|_{(x,y,z)=(1,0,0)} = (2,0,-1) = \vec{n}$$

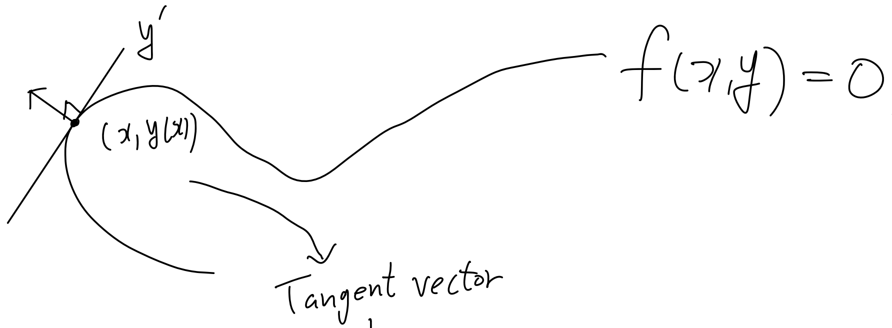
Therefore, the line we are looking for is: $\vec{n} \to (2,0,-1)$, $\vec{a} \to (1,0,0)$

$$\vec{a}+t\vec{n} = (1,0,0)+t(2,0,-1)$$

/* Tangent plane is now easier to find b/c ∇f gives the normal vector */

Application of $\nabla f \perp$ level set (Skip for now)

Implicit differentiation using partials.



$$\nabla f(x,y) \cdot (\quad , \quad) = 0.$$

$$(f_x, f_y) \cdot (\quad , \quad) = 0$$

$$f_x + f_y\, y'(x) = 0 \qquad \text{solve for } y'$$

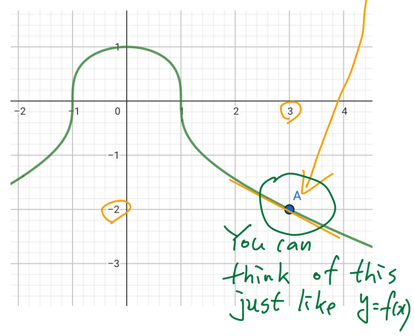
$$y'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$$

We are recast the problem as a level set situation. (a trick)

Typically $y = f(x) \to f'(x)$
but here $f(x,y)=0$

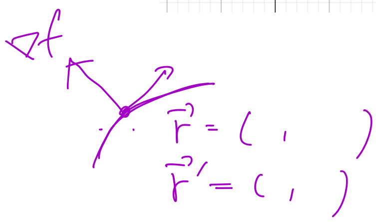
Example: Find $\frac{dy}{dx}$ at $(3,-2) = A$

$$x^2 + y^3 - 1 = 0$$



(Way 1: Partials)

Quick derivation



Let $f(x,y) =$ ______

$$f_x(3,-2) = 2x \Big|_{\substack{x=3 \\ y=-2}} = 2 \cdot 3 = 6$$

$$f_y(3,-2) = 3y^2 \Big|_{\substack{x=3 \\ y=-2}} = 3\cdot(-2)^2 = 12$$

$$y'(3) = -\frac{f_x(3,-2)}{f_y(3,-2)} = -\frac{6}{12} = -\frac{1}{2}$$

(Way 2: Implicit differentiation – Cal 1)

Treat y as a fn of x: $y = y(x)$

$$x^2 + y^3 - 1 = 0 \xrightarrow{d/dx} 2x + 3y^2 \cdot y' = 0$$

$$\Rightarrow y' = -\frac{2x}{3y^2}\Big|_{\substack{x=3 \\ y=-2}} = -\frac{2\cdot 3}{3\cdot(-2)^2} = -\frac{1}{2}$$

NT. 3.6 Potential function and conservative field

Thm (Mixed partials are the same)
Given $f: \mathbb{R}^m \to \mathbb{R}$, if all 1^{st} and 2^{nd} order partials are continuous on its domain, $\partial_{ij} f = \partial_{ji} f$ for all $i, j = 1, 2, \cdots, m$.

- In 2D, this reads $f_{xy} = f_{yx}$
- By applying this theorem repeatedly, we can say "the order doesn't matter in high order partials (3^{rd}, 4^{th} order, $\cdots$)"

Example: Compute f_{xzzy} if

$f(x,y,z) = x^2 e^{y^2-x^2} + x^2 y z^2 + \cos(x^2+y^2)$

(p. 221)

Observe that the 1st and 3rd terms do not involve 'z'.

recall the order: the closer to the fn, the earlier they are applied.

$$
\begin{aligned}
f_{xzzy}(x,y,z) &= f_{zzyx} \\
&= \frac{\partial^3}{\partial x \partial y \partial z}\left(x^2 y (2z)\right) \\
&= \frac{\partial^2}{\partial x \partial y}\left(2x^2 y\right) \\
&= \frac{\partial}{\partial x}\left(2x^2\right) \\
&= 4x
\end{aligned}
$$

If we didn't switch the order, the calculation would have been messy. (Try it!)

Example: Find all second order partials of $f(x,y) = e^{x^2/2+y^2}$ (exercise)

$$f_x(x,y) = e^{x^2/2+y^2} \cdot \left(\frac{\cancel{2}x}{\cancel{2}}\right) = x e^{x^2/2+y^2}$$

$$f_y(x,y) = e^{x^2/2+y^2} \cdot (2y)$$

$$f_{xx}(x,y) = e^{x^2/2+y^2} + x e^{x^2/2+y^2} \cdot \left(\frac{\cancel{2}x}{\cancel{2}}\right) = (1+x^2) e^{x^2/2+y^2}$$

$$f_{yy}(x,y) = e^{x^2/2+y^2} \cdot (2y)^2 + e^{x^2/2+y^2} \cdot 2 = 2(1+2y^2) e^{x^2/2+y^2}$$

$$f_{xy}(x,y) = x e^{x^2/2+y^2} \cdot 2y.$$

$\boxed{\text{Clicker}}$ What about f_{yx}? (skip)

(A) $e^{x^2/2+y^2}$

(B) $x e^{x^2/2+y^2} \cdot 2y.$

(C) $e^{x^2/2+y^2} \cdot (2y)$

⑥ Potential

If a vector field $\vec{F}: \mathbb{R}^m \to \mathbb{R}^m$ happens to be a gradient field of a scalar field $V: \mathbb{R}^m \to \mathbb{R}$, that is

$$\vec{F}(\vec{x}) = \nabla V(\vec{x}) \quad (\text{or } -\nabla V(\vec{x}))$$

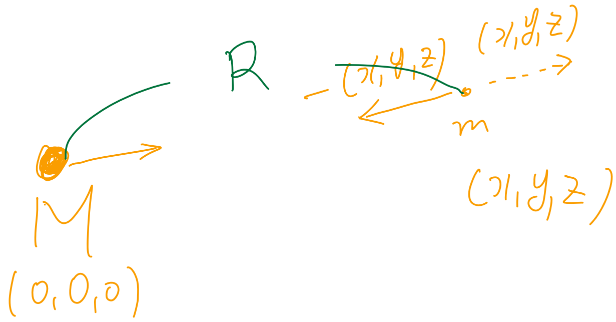
we have some nice properties:

V can be interpreted as a potential energy.

and $\vec{F}$ as force caused by V.

If this is the case, we call V **pontential function** and $\vec{F}$ a **conservative field**.

Example: Gravitational field (exercise)



Potential: $V(\vec{x}) = G\dfrac{Mm}{\|\vec{x}\|} = GMm\,(x^2+y^2+z^2)^{-\frac{1}{2}}$

$$V_x(x,y,z) = GMm\,(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot \left(-\tfrac{1}{\cancel{2}}\right)\cdot \cancel{2}x$$

$$= \frac{GMm\,(-x)}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = -\frac{GMm}{\|\vec{x}\|^3}\cdot y$$

Similarly

$V_y(x,y,z) = -\dfrac{GMm\cdot y}{\|\vec{x}\|^3}$ and $V_z(x,y,z) = -\dfrac{GMm}{\|\vec{x}\|^3}\,z$

$$\vec{F}(\vec{x}) = \nabla V(\vec{x}) = \frac{GMm}{\|\vec{x}\|^3}(-x, -y, -z)$$

$\odot$ $$\|\vec{F}(\vec{x})\| = \frac{GMm}{\|\vec{x}\|^3}\cdot\|(-\vec{x})\| = \frac{GMm}{\|\vec{x}\|^2}$$

$\dfrac{GMm}{R^2}$ R: distance b/w M and m.

How to find potential fn.

(3D or 2D) $\rightarrow$ easier by doing example

① Check whether it is possible

(3D) Let $\vec{F}(x,y,z) = (\overset{V_x}{f(x,y,z)}, \overset{V_y}{g(x,y,z)}, \overset{V_z}{h(x,y,z)})$.

Then, there is a potential if

$f_y = g_x$, $g_z = h_y$, and $h_x = f_z$

/* This is nothing but $V_{xy} = V_{yx}$, $V_{yz} = V_{zy}$, $V_{zx} = V_{xz}$ */

(2D) $\vec{F}(x,y) = (f(x,y), g(x,y))$.

Check only $f_y = g_x$

② How to find potential : "Partial integration"

(3D) ⓐ $\int f(x,y,z)\,dx = V = (\text{antider. in } x) + \underbrace{(\text{fn of } y,z)}_{\hat{V}(y,z)}$ constant in x

ⓑ Determine $\hat{V}(y,z)$ through

$\hat{V}_y = g$ and $\hat{V}_z = h$ (one more "partial integration" is needed)

(2D) ⓐ $\int f(x,y)\,dx = V = (\text{antider in } x) + \underbrace{(\text{fn of } y)}_{\hat{V}(y)}$

ⓑ Determine $\hat{V}(y)$ via

$\hat{V}_y = g$

Example: $\vec{F}(x,y) = \left(e^{xy}(1+xy),\ x^2 e^{xy}\right)$

Find potential of $\vec{F}$ if possible.

↱ Assume V exists.

<u>Check</u>: $f = \underset{= V_x}{e^{xy}(1+xy)}$, $g = \underset{= V_y}{x^2 e^{xy}}$

$$f_y = (V_{xy} =)\ x e^{xy}(1+xy) + x e^{xy} = 2x e^{xy} + x^2 y e^{xy}$$

$$g_x = (V_{yx} =)\ 2x e^{xy} + x^2 y e^{xy}$$

/* $\int f\,dx = \int e^{xy}(1+xy)\,dx$ complicated */

$$V = \int \overset{V_y}{g}\,dy = \int x^2 e^{xy}\,dy = x^2 \cdot \frac{1}{x} e^{xy} + \underline{\hat{V}(x)}$$

("+c" in 1D)

$$= x e^{xy} + \underline{\hat{V}(x)}$$

$$\hookrightarrow V_x = e^{xy} + x e^{xy} \cdot y + \underline{\hat{V}'(x)}$$

$$f = e^{xy}(1+xy) = e^{xy} + xy e^{xy}$$

$$\Rightarrow \underline{\hat{V}'(x) = 0}$$

$$\Rightarrow \underline{\hat{V}(x) = C}$$

(constant)

$$V(x,y) = x e^{xy} + C..$$

Example: $\vec{F}(x,y,z) = (yz,\ zx-1,\ xy)$

Find the potential function if possible:

$$\overset{V_x}{f} = yz,\quad \overset{V_y}{g} = zx-1,\quad \overset{V_z}{h} = xy$$

① <u>Check</u>

$f_y = z,\ g_x = z$ ✓

$g_z = x,\ h_y = x$ ✓

$h_x = y,\ f_z = y$ ✓

② $V = \int f\,dx = \int yz\,dx = xyz + \underbrace{\hat{V}(y,z)}_{\text{constant in } x}$

$V_y = \underline{xz} + \hat{V}_y(y,z) = zx - 1 = g$ ↙from $\vec{F}$ (meaning comparing w/ g)

$\Longrightarrow \hat{V}_y(y,z) = -1 \longrightarrow$ "subproblem"

$\hat{V} = \int \hat{V}_y(y,z)\,dy = \int -1\,dy = -y + \underbrace{\tilde{V}(z)}_{\text{const. in } y}$

$V_z = xy + \hat{V}_z = xy + \tilde{V}'(z) = xy = h$ ↙from $\vec{F}$ (comparing w/ h)

$\Longrightarrow \tilde{V}_z(z) = 0 \Longrightarrow \underline{\tilde{V}(z) \equiv C}$ (constant)

$\hat{V}(y,z) = -y + C$

$V(x,y,z) = xyz - y + C$.