

(proof of unique interpolation - Newton form)

Let k be the number of nodes.

$$(k=1) \quad P_0 = y_0 = C_0$$

$$(k=2) \quad P_1 = C_0 + C_1(x-x_0)$$

$$\text{From } y_1 = P_1(x_1) = C_0 + C_1(x_1 - x_0)$$

$$C_1 = [y_1 - C_0] / (x_1 - x_0)$$

$$(k=3) \quad P_2 = P_1(x) + C_2(x-x_1)(x-x_0)$$

This construction allows us to only consider the new condition

$$P_2(x_2) = y_2 \quad \text{b/c we already know}$$

$$P_1(x_0) = y_0 \quad \text{and} \quad P_1(x_1) = y_1 \quad \text{while}$$

$C_2(x-x_1)(x-x_0)$ vanishes if $x=x_0$ or x_1 .

$$\text{From } y_2 = P_2(x_2) = P_1(x_2) + C_2(x_2-x_1)(x_2-x_0)$$

$$C_2 = [y_2 - P_1(x_2)] / (x_2-x_1)(x_2-x_0)$$

Repeat this until $k=n$. Since the process is constructive, the resulting $P_n(x)$ must be unique.

Example: Find $p \in \Pi_3$ such that

x	0	1	2	3
y	1	1	2	3

$$p_0 = 1 = C_0$$

$$p_1(x) = C_0 + \frac{y_1 - C_0}{x_1 - x_0} = 1 + \frac{1 - 1}{1 - 0} (x - 0) = 1$$

$$p_2(x) = p_1(x) + \frac{y_2 - p_1(x_2)}{(x_2 - x_1)(x_2 - x_0)} (x - x_1)(x - x_0)$$

$$= 1 + \frac{2 - 1}{2 \cdot 1} (x - 1)x = 1 + \frac{x}{2} (x - 1)$$

$$p_3(x) = p_2(x) + \frac{y_3 - p_2(x_3)}{(x_3 - x_2)(x_3 - x_1)(x_3 - x_0)} (x - x_2)(x - x_1)(x - x_0)$$

$$= 1 + \frac{x}{2} (x - 1) + \frac{3 - 1 - \frac{3}{2} \cdot 2}{1 \cdot 2 \cdot 3} (x - 2)(x - 1)x$$

$$= 1 + \frac{x}{2} (x - 1) - \frac{x}{6} (x - 2)(x - 1)$$

p_0

p_1

p_2

p_3

Horner's algorithm

number of
multiplications

$$c_0 + c_1 d_1 + c_2 \underline{d_1 d_2} + c_3 \underline{d_1 d_2 d_3} \quad 1+2+3$$

$$= c_0 + c_1 \underline{d_1} + (c_2 + c_3 d_3) \underline{d_1 d_2}$$

$$= c_0 + (c_1 + (c_2 + c_3 d_3) d_2) d_1$$

nested $\frac{\alpha}{c's} + \underbrace{\beta r}_{\substack{\text{previous} \\ \text{result}}} \text{ form}$
previous result

$$= c_0 + (c_1 + \underbrace{(c_2 + c_3 d_3)}_{\alpha_1 + \beta_1 r_1}) d_2 \quad 1+1+1$$

$$\underbrace{\alpha_2 + \beta_2 \cdot r_2}$$

$$\underbrace{\alpha_3 + \beta_3 \cdot r_3}$$

example : Lagrange basis subordinate to
nodes $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{1}{6} (x-2)(x-3)(x-4)$$

-1 -2 -3

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{1}{2} (x-1)(x-3)(x-4)$$

1 -1 -2

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{2} (x-1)(x-2)(x-4)$$

2 1 -1

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{1}{6} (x-1)(x-2)(x-3)$$

3 2 1

(proof of linear independence)

Assume

equality as fn's.

$$a_0 f_0(x) + a_1 f_1(x) + \dots + a_n f_n(x) = 0$$

plug in $x = x_0$, then

$$\begin{aligned} 0 &= a_0 \underbrace{f_0(x_0)}_{\delta_{00} \quad 1} + a_1 \underbrace{f_1(x_0)}_{\delta_{10} \quad 0} + \dots + a_n \underbrace{f_n(x_0)}_{\delta_{n0} \quad 0} \\ &= a_0. \end{aligned}$$

plug in $x = x_1$, then

$$\begin{aligned} 0 &= a_0 \underbrace{f_0(x_1)}_{\delta_{01} \quad 0} + a_1 \underbrace{f_1(x_1)}_{\delta_{11} \quad 1} + \dots + a_n \underbrace{f_n(x_1)}_{\delta_{n1} \quad 0} \\ &= a_1. \end{aligned}$$

Repeat this to conclude $a_0 = a_1 = \dots = a_n = 0$ 