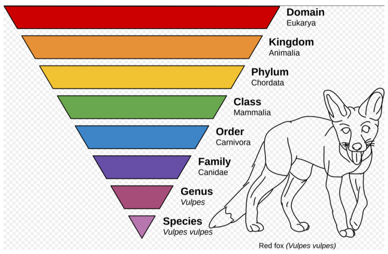
Announcement

- Midterm / Quiz result → within 1 week

NT 1.16 Four species of functions



We use functions to describe quantities that depend on other quantities. We can classify them in many different ways. For the purpose of vector calculus, it is helpful to classify functions based on the dimensions of input and output.

input   output   put   put

1D → 1D : Real-valued univariate functions

1D → n-D : Vector-valued univariate functions

n-D → 1-D : Real-valued multivariate functions

n-D → m-D : Vector-valued multivariate functions

/* 1D−1D fn's are the topics of Math 3A and 3B (Differential and integral Calculus of single variable). */

Examples of the four species.

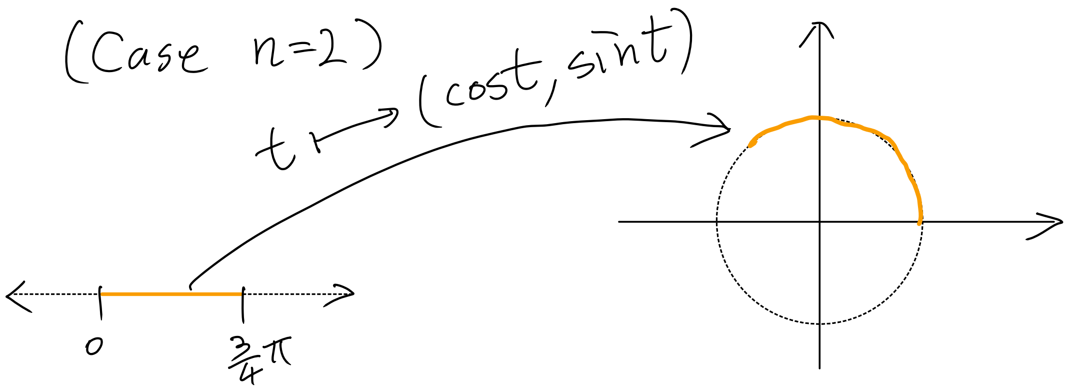
⓪ 1D − 1D : You already know many examples.

$$y = \sin x, \quad y = e^{ax}, \quad y = x^2 - 2x,$$

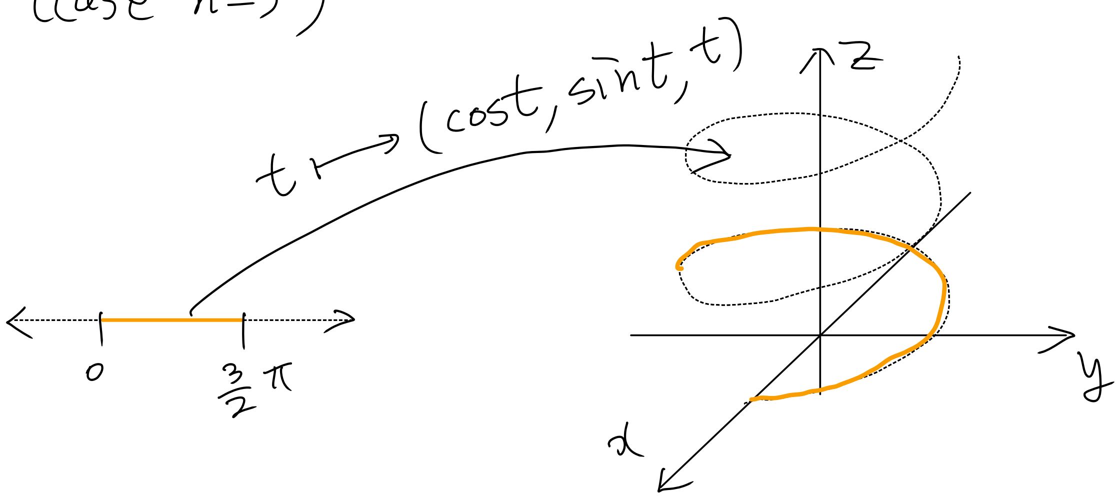
$$y = |x|, \quad y = \begin{cases} 1 & (x \geq 0) \\ -1 & (x < 0) \end{cases}, \quad \ldots$$

⓪ 1D − n-D : seems fair to call them curves

(Case $n=2$)



(case $n=3$)



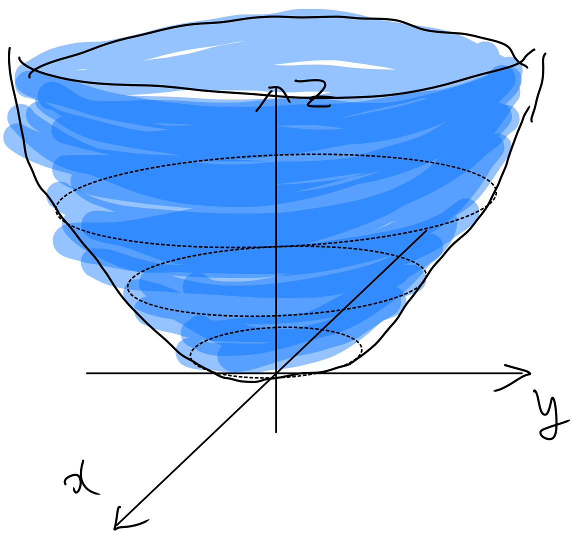
⓪ n-D → 1D : not had an idea to call them

(case n=2)

(x, y) vector

$$\boxed{z = x^2 + y^2}$$

(z) scalar

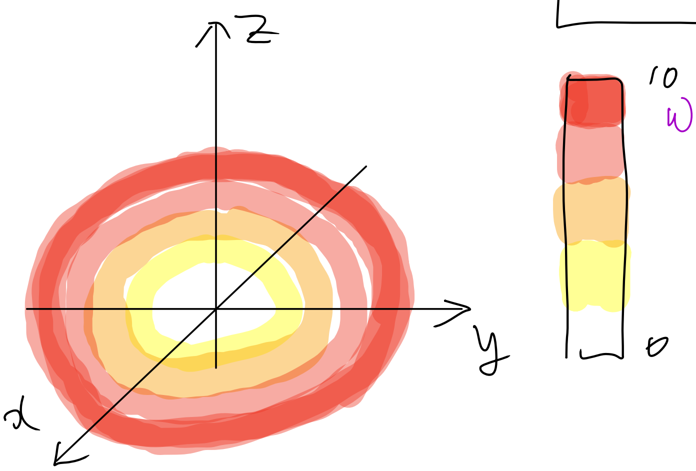


(case n=3)

(x, y, z)

$$\boxed{x^2 + y^2 + z^2 = w}$$

(w)



⑥ n-D → m-D :

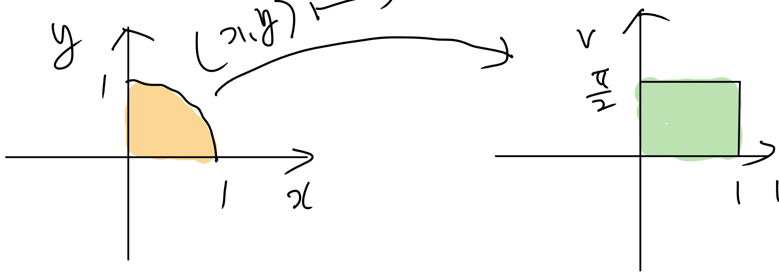
(case $n=2$, $m=2$) : Good name/term ? transformation

$$(x,y) \longmapsto (x^2+y^2,\ \tan^{-1}(y/x))$$

(x,y)

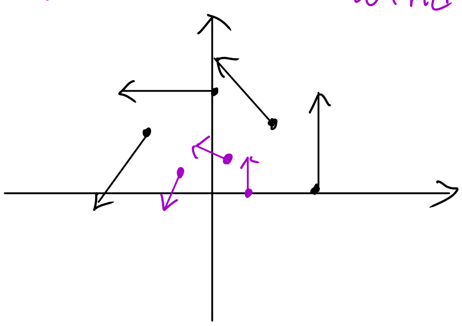
$$\boxed{\begin{aligned} u &= x^2+y^2 \\ v &= \tan^{-1}(y/x) \end{aligned}}$$

(u,v)



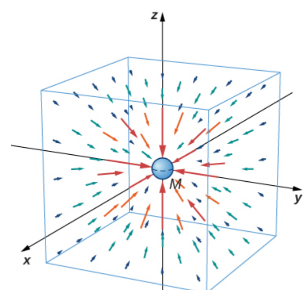
($n=2$, $m=2$)

$$\underbrace{(x,y)}_{\text{location}} \longmapsto \underbrace{(-y,x)}_{\text{wind velocity}}$$

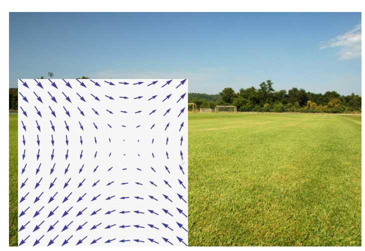


(case $n=3$, $m=3$)

gravitation



When 2^{nd} interpretation is helpful, we call them vector field.



/* You can imagine $n=2$ $m=3$ case, $n=3$, $m=2$ case, etc. However, more important is to study their behaviors species by species. (BTW, 'species' is not a standard terminology in mathematics. It is just a metaphor) */

- 1D → n-D functions are our target for the next 2 weeks.
- n-D → 1D functions are our topic of the rest of 6A after studying 1D → n-D.
- n-D → m-D functions will be covered in Math 6B, with beautiful and far-reaching results. But we need some elementary facts about them in this course.

$\boxed{\text{Clicker}}$ Classify $f(x,y) = (x+y, xy)$
$g(x) = (3x-2, x+3)$, $h(x,y) = e^x \cdot \sin y$

	(A)	(B)	(C) ✓	(D)
Vector valued fn	f	g, f	g	h
multivariate fn	g	h	h	g
vector field	h		f	f

NT 2.1 Paths and Curves (Ch. 3.2)

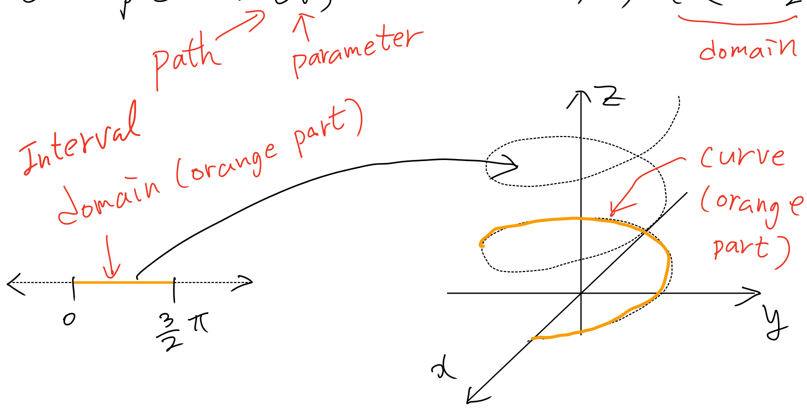
Our 2nd chunk of this quarter is devoted to studying 1D → n-D fn's. We call them **vector-valued (univariate) functions** or **paths**.

$\boxed{\text{Def}}$ (Path and curve)

A **path** is a function from an interval (i.e. $a<t<b$) to $\mathbb{R}^n$ (mostly interested in $n=2,3$) (i.e. 1D → n-D function.)

The variable for the domain (i.e., the set of inputs) is often called parameter. And the range / image (i.e., the set of outputs) is called **curve**.

Example: $\vec{r}(t) = (\cos t, \sin t, t)$ $\underbrace{(0 \le t \le \tfrac{3}{2}\pi)}_{\text{domain}}$



/* 'Curve' is often used when it really means 'path'. But when there are possible confusions, they must be clearly distinguished. */

/* **Notation**

It is often useful to specify only domain and the codomain (i.e., the space where the range lives) b/c this gives an idea what the inputs and outputs look like.

We do this by fn : domain $\longrightarrow$ codomain.

E.g., $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^3$. Suppose $\vec{r}(t) = (1, t, 3t-1)$.

" $\vec{r}$ " refers to the function as a whole (domain + assignment)

" $\vec{r}(t)$ " refers to the output when input is t. It can be arbitrary, but still an output. */

/* (Codomain vs range/image)

Codomain : set of all possible output

range/image : set of all actual output

E.g., in the example of helix, the whole $\mathbb{R}^3$ space is the codomain while the image orange part of the helix is the image. */

NT 2.2 Basic Calculus of paths

Limits of vector-valued functions

Given $\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^n$, where (say $n=3$)

$$\vec{r}(t) = (x(t), y(t), z(t)),$$

the **limit of $\vec{r}(t)$ as $t \to a$** is defined by the component-wis limit:

$$\boxed{\lim_{t \to a} \vec{r}(t) = \left(\lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)}$$

Derivative of vector-valued fn

Given a path $\vec{r}(t)$, the derivative with respect to t is defined

time der.

$$\boxed{\vec{r}'(t)} = \frac{d}{dt}(\vec{r}(t)) = \dot{\vec{r}}(t)$$

input of evaluation

$$= \lim_{h \to 0} \frac{1}{h}(\vec{r}(t+h) - \vec{r}(t))$$

$$= \lim_{h \to 0} \left(\frac{1}{h} x(t+h) - x(t), \frac{1}{h} y(t+h) - y(t), \frac{1}{h} z(t+h) - z(t) \right)$$

$$= \boxed{(x'(t), y'(t), z'(t))}$$

In short, it is component-wise derivative. Notice $\vec{r}'(t)$ has the same dimension as $r(t)$.

/* For derivative to be defined, each component must be differentiable. But we don't worry much about non-differentiable cases. */

(plain) integral of vector-valued fn

Given a path $\vec{r}(t)$ $= (x(t), y(t), z(t))$, the (definite) integral from a to b is defined

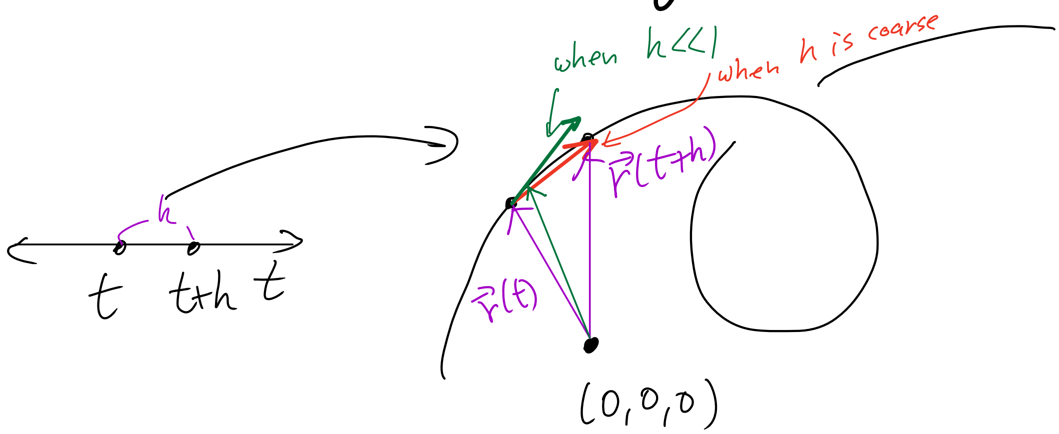
$$\underbrace{\int_a^b \vec{r}(t)\,dt}_{\text{what is this? } \boxed{\text{Vector}} \text{ or scalar?}} = \left(\int_a^b x(t)\,dt,\ \int_a^b y(t)\,dt,\ \int_a^b z(t)\,dt\right)$$

If $\vec{p}'(t) = \vec{r}(t)$, $\vec{p}(t)$ is called indefinite integral of $\vec{r}(t)$, and we have

$$\vec{p}(t) = \int_a^t \vec{r}(s)\,ds + \underbrace{\vec{c}}_{\text{constant vector}}$$

/* This is direct consequence of Cal2 : Do Cal2 business component-wisely */

0 meaning derivative: tangent, velocity



$$\underbrace{\underbrace{\frac{1}{h}}_{\substack{\text{very large}\\ \text{iff } h\ll 1}}\left(\underbrace{\vec r(t+h)-\vec r(t)}\right)}_{\text{stay fixed in size as } h\to 0}$$

"chord vector"

(very small in size if $h \ll 1$)

$\ll$ means very small

$\boxed{\text{Direction: tangent}}$ to the curve at t.

$\boxed{\text{Length of } \vec r'(t)\text{: How fast}}$ $\vec r(t)$ changes

$\boxed{\text{speed}}$

Def (velocity and acceleration) (Ch. 3.2)

Given a path $\vec{r}(t): \mathbb{R} \to \mathbb{R}^n$ $(n=2,3)$

$$\boxed{\begin{aligned}\vec{v}(t) &:= \vec{r}\,'(t) = (x'(t), y'(t), z'(t)) \text{ and} \\ \vec{a}(t) &:= \vec{v}\,'(t) = \vec{r}\,''(t) = (x''(t), y''(t), z''(t))\end{aligned}}$$

are called **velocity** and **acceleration** resp. (short for respectively)

/* For velocity and acceleration to be defined, the path must once and twice differentiable resp. But we don't worry much about non-differentiable cases. */

⊙ Basic differentiation rules

(a) $(\vec{r}(t)+\vec{p}(t))' = \vec{r}'(t)+\vec{p}'(t)$

(b) $(\alpha\vec{r}(t))' = \alpha\vec{r}'(t)$ (where $\alpha\in\mathbb{R}$)

(c) $(\vec{r}(t)\cdot\vec{p}(t))' = \vec{r}'(t)\cdot\vec{p}(t)+\vec{r}(t)\cdot\vec{p}'(t)$

(d) $(\|\vec{r}(t)\|^2)' = 2\,\vec{r}(t)\cdot\vec{r}'(t)$

(e) $(\alpha(t)\vec{r}(t))' = \alpha'(t)\vec{r}(t)+\alpha(t)\vec{r}'(t)$

(f) $(\vec{r}(t)\times\vec{p}(t))' = \vec{r}'(t)\times\vec{p}(t)+\vec{r}(t)\times\vec{p}'(t)$ (only for $\mathbb{R}^3$)

(proof) $(\vec{r}(t)\cdot\vec{p}(t))'$

$$= \big(r_1(t)p_1(t)+r_2(t)p_2(t)+r_3(t)p_3(t)\big)'$$

$$\begin{aligned}= {} & r_1'(t)p_1(t)+r_1(t)p_1'(t)\\ & + r_2'(t)p_2(t)+r_2(t)p_2'(t)\\ & + r_3'(t)p_3(t)+r_3(t)p_3'(t)\end{aligned}$$

$$= \vec{r}'(t)\cdot\vec{p}(t)+\vec{r}(t)\cdot\vec{p}'(t)$$

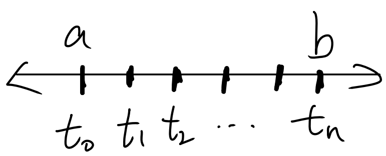
⊙ (a) and (b) are linearity of differentiation.

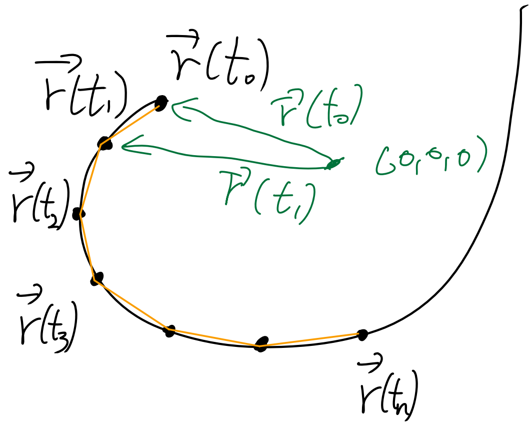
⊙ (c), (e), (f) remind of <u>product</u> rule.

⊙ (d) is similar to $(y^2)' = 2y\cdot y'$

But I recommend deriving it using (c).

NT. 2.3 Length of path (Ch. 3.3)





$$\ell \approx \|\vec{r}(t_1)-\vec{r}(t_0)\| + \|\vec{r}(t_2)-\vec{r}(t_1)\| + \cdots + \|\vec{r}(t_n)-\vec{r}(t_{n-1})\|$$

$$= \sum_{i=1}^{n} \|\vec{r}(t_i)-\vec{r}(t_{i-1})\| \qquad \vec{r}\,'(t_i) \approx \frac{\vec{r}(t_i)-\vec{r}(t_{i-1})}{\underbrace{t_i - t_{i-1}}_{\Delta t}}$$

$$\approx \sum_{i=1}^{n} \|\vec{r}\,'(t_{i-1})\| \Delta t$$

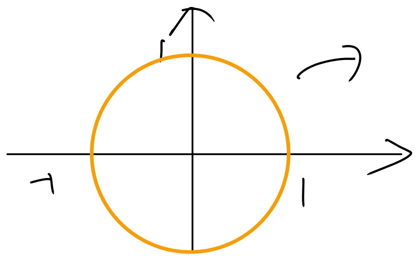
$$\approx \int_a^b \|\vec{r}\,'(t)\|\, dt \rightarrow$$ What is this? vector? Scalar?

$\boxed{\text{Thm}}$ (length of path)

$$\boxed{\ell = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Example: Find the length of the path:

$\vec{r}(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$



Clicker What is the answer from geometry?

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π

How to start? We want length $\int_a^b \|\vec{r}\,'(t)\| dt$

(2) $\vec{r}\,'(t) = (-\sin t, \cos t)$

(3) $\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$

(4)(1) $\int_0^{2\pi} \|\vec{r}\,'(t)\| dt = \int_0^{2\pi} 1 \cdot dt = 2\pi$

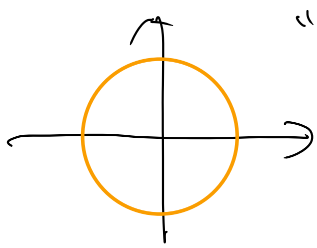
(0) Similar example → $\boxed{\text{Guess the answer}}$

$\vec{p}(t) = (\cos 3t, \sin 3t) \quad (0 \le t \le 2\pi)$

$\vec{p}\,'(t) = (-3\sin 3t, 3\cos 3t)$

$\|\vec{p}\,'(t)\| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2} = 3$

$\int_0^{2\pi} \|\vec{p}\,'(t)\| dt = \int_0^{2\pi} 3\, dt = 3 \cdot 2\pi = 6\pi$



"But picture is the same"

/* $\vec{p}$ travels the circle 3 times. But as curves (image) $\vec{r}$ and $\vec{p}$ are not distinguishable. In cases like this, one must clearly distinguish path and curve */

The following definitions are to exclude cases like the second example when necessary.

$\boxed{\textbf{Def}}$ (**Smooth/regular path**)

A path $\vec{r}(t) : [a,b] \longrightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$) is called **smooth** or **regular** if

(a) it is C^1 (1^{st} derivative of each component is continuous)

(b) $\vec{r}\,'(t) \neq \vec{0}$ for $a < t < b$

(c) $\vec{r}(t) \neq \vec{r}(s)$ for $a < t < s < b$.

/* C^1 means 1^{st} derivative is continuous. */

① $\vec{r}(a)=\vec{r}(b)$ is allowed. In this case the path is called closed.

② A smooth path cannot pass through itself, (c) be tangent to itself, or "stop." (b) (when it stops, it may make a sharp turn. You will see how zero velocity may mess up theory)

$\boxed{\text{Def}}$ (Smooth curve) path

A curve is called smooth if ______________ ______________ that is smooth, that is, it is the ______________ ______________.

The length of curve (or **arc length**) is the length of that smooth path.