Announcement

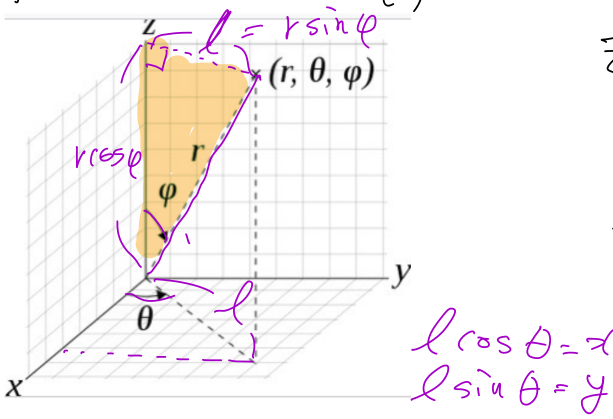
- Classroom tech will be really used as of next week.
  - Discord → Canvas front page
  - iClicker
  - Gradescope app → see Discord — # tech-settin
- → No excuse. Drops will apply.
- Discord will be deleted after final grades are posted
  - → Back up what's important to you for a long term.
- HW1 is due next Thu.
- Communication efficiency

Recap

- Position vector : tail = origin
- $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $k\vec{a}$
- $\mathbb{R}^n$ is a vector space
- norm $\|\vec{a}\| = \sqrt{\sum_{i=1}^{n} a_i^2}$
- polar coordinate (2D), cylindrical (3D), spherical (3D)

⊙ spherical coordinate

Similarly, we can specify 3D location by distance (r), angle from x-axis (θ), and angle from z-axis (φ)



$$z = r\cos\varphi$$

$$y = \underbrace{(r\sin\varphi)}_{=\ell}\sin\theta$$

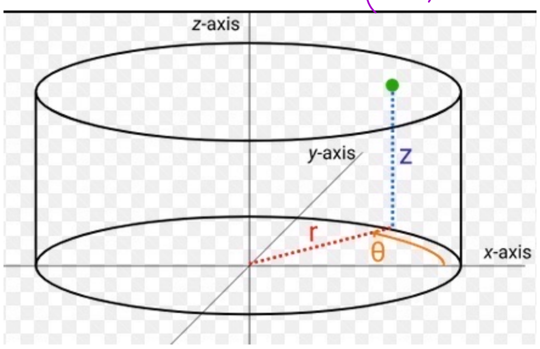
$$x = \underbrace{(r\sin\varphi)}_{=\ell}\cos\theta$$

⊙ We see $r = \sqrt{x^2+y^2+z^2}$

⓪ Cylindrical coordinates

polar coordinate system for xy-plane plus z-axis gives rise to the cylindrical coordinate system.

(r, θ, z)



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

⓪ Observe $r = \sqrt{x^2 + y^2}$

↳ different from spherical coordinates.

⓪ Graphs in 3D

The following intuition continues to hold:
(if there is no "special issue")

dimension = degrees of freedom − # constraints.

constraint	geom. obj. (deg. free)
No constraint (0)	→ whole space $\mathbb{R}^3$ (3)
An equation (1)	→ surface (2D) (2)
Two equations (2)	→ curve (1D) (1)
Three equations (3)	→ point (0D) (0)
Flour equations	→ no point (contradiction)

Exercise: Graph the following in 3D

Use the Geogebra links posted on Discord or fancier ones if you have any.

Cartesian (observe features and sanity-check some pts)

$x = 3$

$y = -2$

$z = 1$

$x - z = 2$

$x + 2y + z = 0$

$x^2 + y^2 + z^2 = 9$

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

$x^2 + y^2 = 4$

$z = x^2 + 3y^2$

$z = xy$

$z = x^2 - 2y^2$

$z = -x^2 - y^2$

$z = \sqrt{x^2 + y^2}$

$z = \sin(x) \cdot \sin(y)$

$z = \sin(xy)$

Cylindrical

If you have Mac, also try:

$r = 3 \longrightarrow r(\theta, z) = "3"$ $\quad \theta = \frac{\pi}{4}$

Spherical

$r = 2 \longrightarrow \rho(\theta, \varphi) = "2"$ $\quad \varphi = \frac{\pi}{4}$

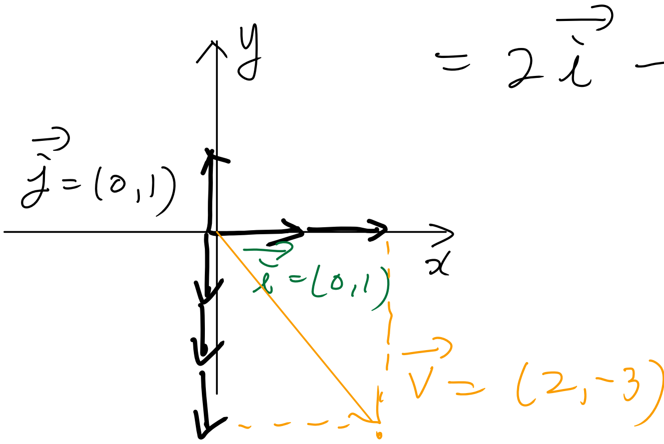
$r = \cos(\varphi) \quad \rho(\theta, \varphi) = "\cos(y)"$

NT 1.6 Basis representation

It is often convenient to write vectors using **canonical basis.**

(2D) $\vec{i} = (1,0)$ $\vec{j} = (0,1)$

$$(2,-3) = 2\cdot(1,0) - 3\cdot(0,1)$$
$$= 2\vec{i} - 3\vec{j}$$



(3D) Similarly $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$

$\boxed{\text{Clicker}}$: What is $3\vec{i} - \vec{j}$ in 3D?

(A) $(3,-1)$ (B) $(-1,3)$ (C) $(3,-1,0)$ ✓

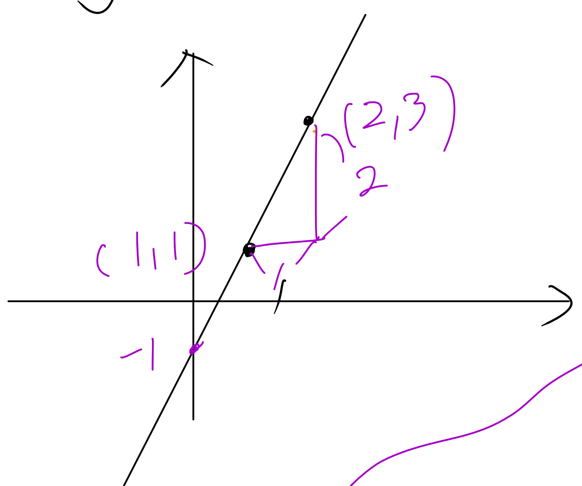
$$\underline{3\vec{i} - \vec{j}} = 3\cdot(1,0,0) - (0,1,0)$$
$$= (3,0,0) + (0,-1,0)$$
$$= (3,-1,0)$$
$$\Rightarrow (3,-1,0)$$

NT 1.7 Application - eqn of lines
(Ch. 1.2)

Motivation : Recall high school math

Consider the following problem.

Find the equation of line passing through $(1,1)$, $(2,3)$



slope $= \dfrac{3-1}{2-1}$

$= \dfrac{2}{1} = 2$

a point passed $= (1,1)$

Slope - point formula

$y - 1 = 2(x-1)$

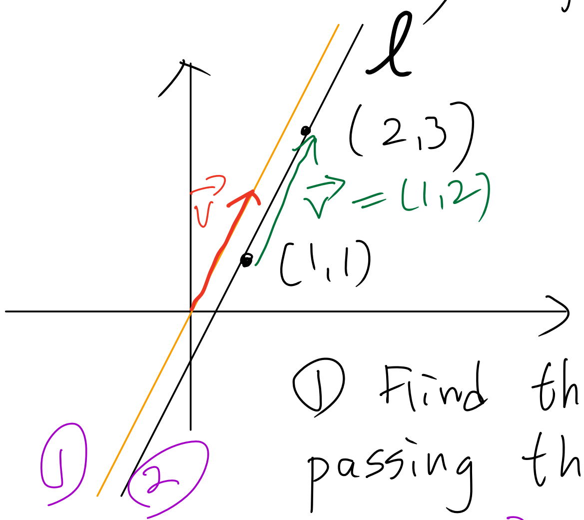
$y = 2x - 2 + 1 = 2x - 1$

Thus, the line can be expressed as

$\{ \underbrace{(x,y)}_{\text{what you are collecting}} : \underbrace{y = 2x - 1}_{\text{further description}} \}$

In the language of vectors

(identify points with vectors)



direction vector

(end - start)

$\vec{v} = (2,3)-(1,1) = (1,2)$

① Find the line in the direction $\vec{v}$ passing through the origin.

$t\vec{v}$, where $t \in \mathbb{R}$

② translate all such vectors (points) by $\vec{u}=(1,1)$:

$$\vec{u}+t\vec{v}$$
$$= (1,1)+t(1,2)$$
$$= (1+t,\ 1+2t)$$

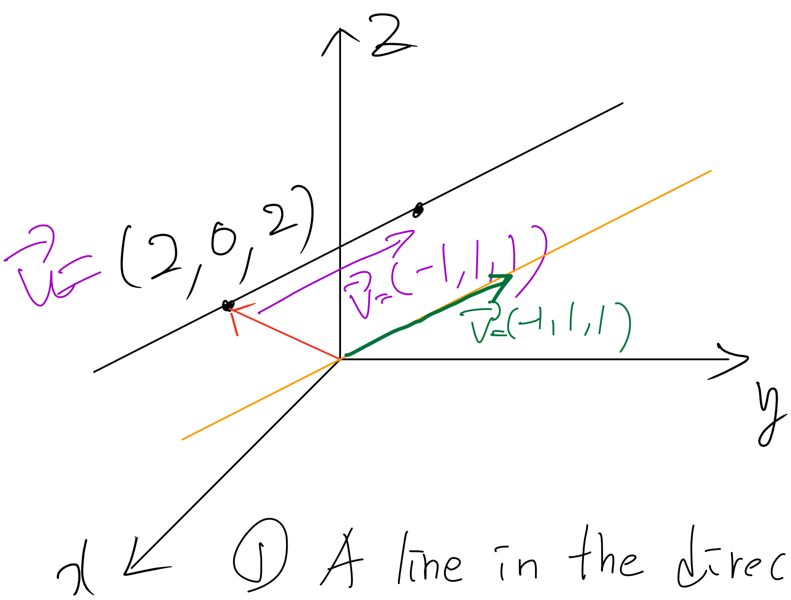
Check: $t=0 \Rightarrow (1,1)$ $t=1 \Rightarrow (2,3)$
$t=2 \Rightarrow (3,5)$

This means, every time you plug in different real number t, you get a (position) vector pointing to one of the points of the line ℓ. Therefore, if you collect all such vectors, that is one way to represent a line.

$$\ell = \{ \vec{u} + t\vec{v} ;\ t \in \mathbb{R},\ \vec{u} = (1,1),\ \vec{v} = (1,2) \}$$

Upshot: very easy to generalize! to 3D.

Example: Find the vector form of a line passing through $\vec{u} = (2,0,2)$ and in the direction of $\vec{v} = (-1,1,1)$



① A line in the direction $\vec{v}$ passing
$t\vec{v}$

② Translate it by $\vec{u} = (2,0,2)$:

$$\vec{u} + t\vec{v} = (2,0,2) + t(-1,1,1)$$
$$= (2-t,\ t,\ 2+t)$$

The interpretation is the same as before.

⊙ Here, t is called **parameter**.

⊙ For example in the 3D example,

$$\ell(t) = \vec{u} + t\vec{v} \qquad \textbf{vector form}$$

$$\ell(t) = (\underset{x}{\underset{\|}{2-t}},\ \underset{y}{\underset{\|}{t}},\ \underset{z}{\underset{\|}{2+t}}) \qquad \textbf{component form}$$

$$\begin{cases} x = 2-t \\ y = t \\ z = 2+t \end{cases} \qquad \textbf{parametric form}$$

⊙ Observe the way a line is represented. In particular, it involves a parameter (1D object) and a simple equation of x, y, z cannot do the job.

NT 1.8 Dot product

Def (Dot product)

For $\vec{u}, \vec{v} \in \mathbb{R}^n$ (mostly interested $n=2,3$), the **dot product** of them are

$$\vec{u}\cdot\vec{v} = (u_1, u_2, \cdots, u_n)\cdot(v_1, v_2, \cdots, v_n) := u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

↳ equal by definition.

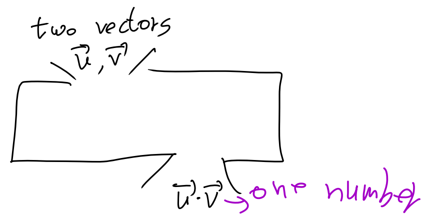
In particular,

$$(u_1, u_2)\cdot(v_1, v_2) = (u_1\vec{i} + u_2\vec{j})\cdot(v_1\vec{i} + v_2\vec{j}) = u_1v_1 + u_2v_2$$

$$(u_1, u_2, u_3)\cdot(v_1, v_2, v_3) = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k})\cdot(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$
$$= u_1v_1 + u_2v_2 + u_3v_3$$

◎ Dot product is always a scalar (i.e. $\vec{a}\cdot\vec{b} \in \mathbb{R}$)

◎ It is also called **scalar product** or **inner product**.



◎ Other notations are $(\vec{u}, \vec{v})$, $\langle \vec{u}, \vec{v} \rangle$, etc.

$\vec{i}\cdot\vec{j} = (1,0,0)\cdot(0,1,0) = 0$

$\vec{i}\cdot\vec{i} = (1,0,0)\cdot(1,0,0) = 1$

$\vec{j}\cdot\vec{k} = (0,1,0)\cdot(0,0,1) = 0$

Clicker What are $\vec{i}\cdot\vec{j}$, $\vec{i}\cdot\vec{i}$, $\vec{j}\cdot\vec{k}$?

(A) 0, 0, 0 (B) 0, 1, 0

(C) 1, 1, 1 (D) 1, 0, 1

⊙ Basic properties of dot product

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ (commutative)
- $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ (distributive)
- $(\alpha\vec{u})\cdot\vec{v} = \alpha(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(\alpha\vec{v})$, (associative)

  where $\alpha \in \mathbb{R}$.
- $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$

All elementary. Just write out.

E.g. $\vec{u}\cdot\vec{u} = (u_1, u_2, \cdots, u_n)\cdot(u_1, u_2, \cdots, u_n)$

$$= u_1u_1 + u_2u_2 + \cdots + u_nu_n$$

$$= u_1^2 + u_2^2 + \cdots + u_n^2$$

$$= \|\vec{u}\|^2$$