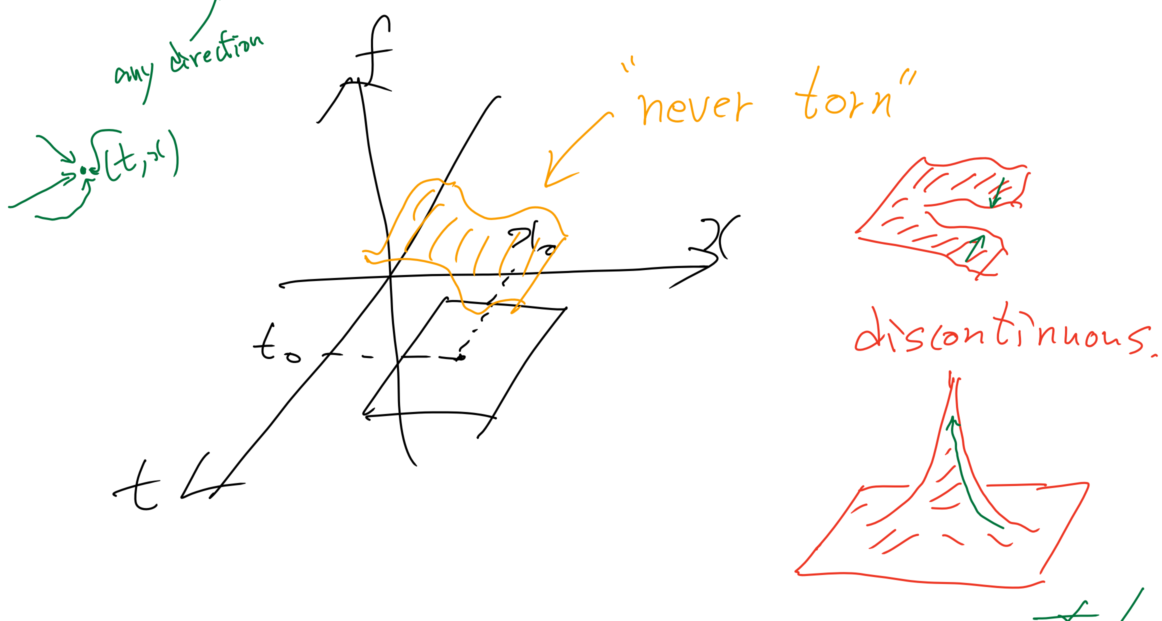
(example) $x'(t) = x^{\frac{2}{3}}$.

This corresponds to $f(t,x) = x^{\frac{2}{3}}$

f is continuous (as a function of two variables).

/* $\lim\limits_{(\tau,z)\to(t,x)} f(\tau,z) = f(t,x)$



*/

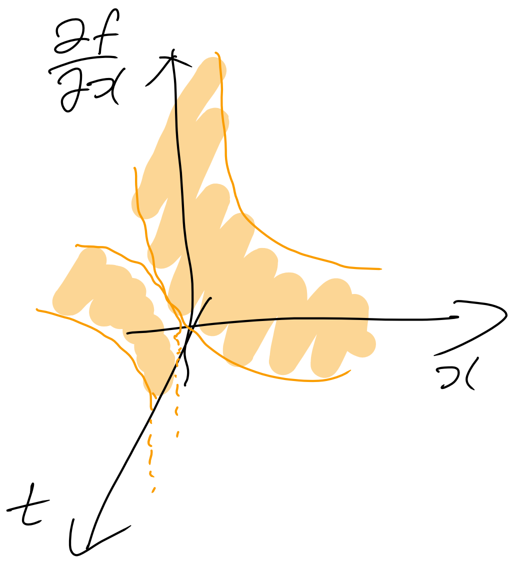
(existence)

Therefore by the theorem, a function $x(t)$ on a small open interval of t $(-\delta,\delta)$ s.t. $x'(t) = x^{\frac{2}{3}}$ and $x(0)=0$.

For example, $x(t)\equiv 0$: $x'(t) = 0 = 0^{\frac{2}{3}}$

(uniqueness)

However, $\frac{\partial f}{\partial x}(t,x) = \frac{2}{3}x^{-\frac{1}{3}}$ is not continuous on any rectangle around $(t_0, x_0) = (0,0)$.



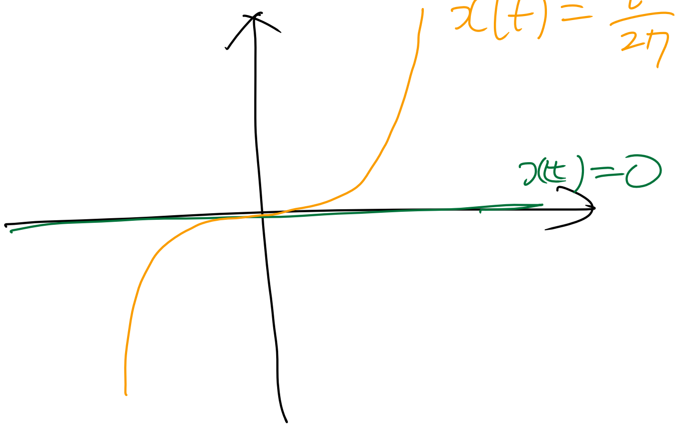
Thus, the theorem does not guarantee uniqueness. In fact, we have another solution.

For example, use the ansatz

$$x(t) = at^b$$

$$\left.\begin{aligned} x'(t) &= abt^{b-1} \\ (x(t))^{\frac{2}{3}} &= a^{\frac{2}{3}} t^{\frac{2}{3}b} \end{aligned}\right] \Rightarrow \begin{aligned} &b-1 = \tfrac{2}{3}b \Rightarrow b = 3 \\ &ab = a^{\frac{2}{3}} \Rightarrow a^3 b^3 = a^2 \\ &\Rightarrow a = \frac{1}{b^3} = \frac{1}{27}. \end{aligned}$$

This also satisfies the IC (initial condition).

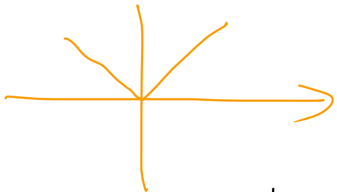


$f(x) = x^2 \in C^1[-1,1]$

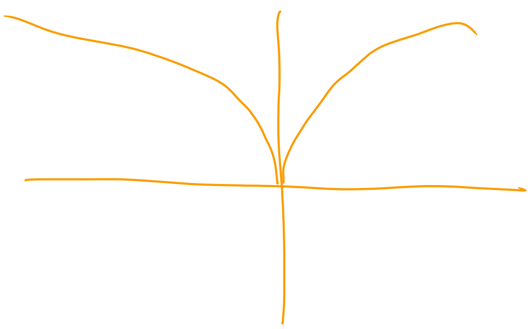
$g(x) = |x| \in Lip[-1,1]$ but $g \notin C^1[-1,1]$

$|g(x)-g(y)| = \big||x|-|y|\big| \le |x-y|$ $L=1$.

inverse triangle ineq.



$h(x) = |x|^{\frac{1}{2}} \in C[-1,1]$ but $h \notin Lip[-1,1]$.



$$|h(x)-h(y)| = \big|\,|x|^{\frac{1}{2}} - |y|^{\frac{1}{2}}\big|$$

$$= |\delta - 2\delta| = \delta$$

Choose $x = \delta^2$, $h = 4\delta^2$, then $|x-y| = 3\delta^2$

For any $L>0$, if $\delta < \frac{1}{3L}$

$$|h(x)-h(y)| = \delta\,\frac{3L}{3L} > 3L\delta^2 = L|x-y|.$$

Example : Refresh calculus.

$$\frac{d}{dt} f(x(t), y(t))$$

$$= f_x(x(t), y(t))\, x'(t)$$

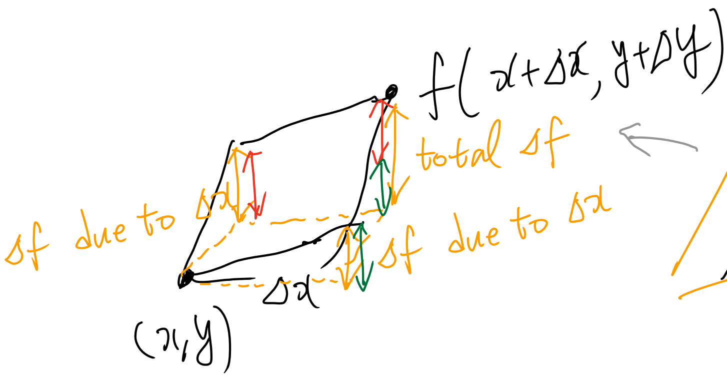
$$+ f_y(x(t), y(t))\, y'(t)$$

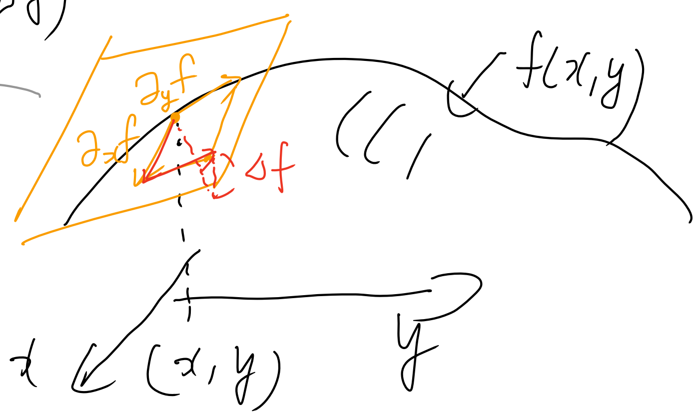
$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & f(x,y) \\ & y \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & \Delta x \simeq x' \cdot \Delta t \arrow[dr] & \\ \Delta t \arrow[ur] \arrow[dr] & & \Delta f \simeq \partial_t f \cdot \Delta y + \partial_x f \cdot \Delta x \\ & \Delta y \simeq y' \cdot \Delta t \arrow[ur] & \end{tikzcd}$$

$$\Longrightarrow \frac{\Delta f}{\Delta t} \simeq \frac{\partial_y f \cdot \Delta y}{\Delta t} + \frac{\partial_x f \cdot \Delta x}{\Delta t}$$

$$= \partial_y f \cdot y' + \partial_x f \cdot x'$$





$$x'' = \big(f(t,x)\big)' = \underbrace{f_t(t,x) + f_x(t,x)\, x'}$$

$$x''' = \big(f_t(t,x) + f_x(t,x)\, x'\big)'$$

$$= f_{tt}(t,x) + f_{tx}(t,x)\, x'$$

$$+ \big(f_{xt}(t,x) + f_{xx}(t,x)x'\big)x' + f_x(t,x)\, x''$$

$$x''(t) = -\sin t + 2t - \cos x \cdot x'$$

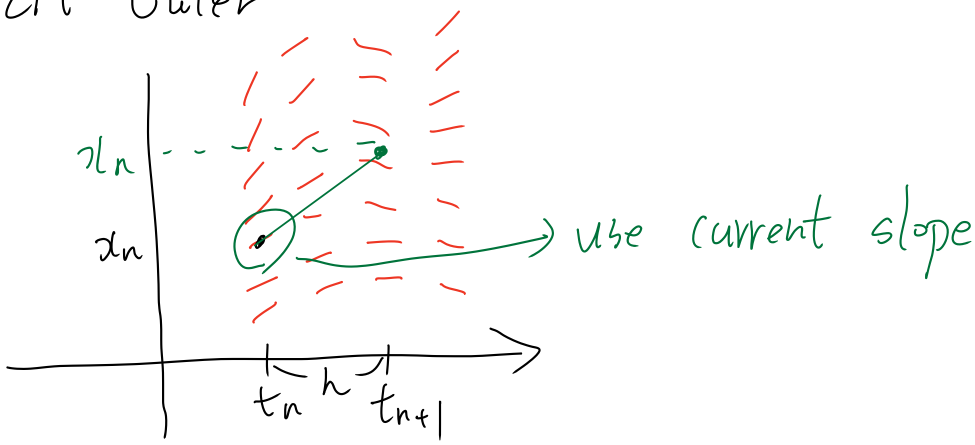
$$x'''(t) = -\cos t + 2 - \cos x \cdot x'' + (\sin x \cdot x') \cdot x'$$

$$= -\cos t + 2 - \cos x \cdot x'' + \sin x \cdot (x')^2$$

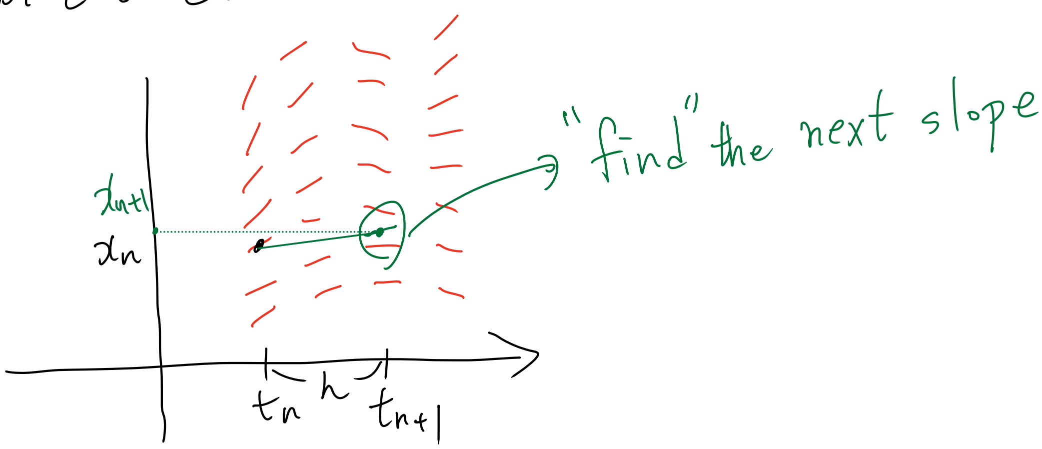
$$x''''(t) = \sin t + \sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot x' \cdot (x')^2 + \sin x \cdot 2(x') x''$$

$$= \sin t + 3 \sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot (x')^3$$

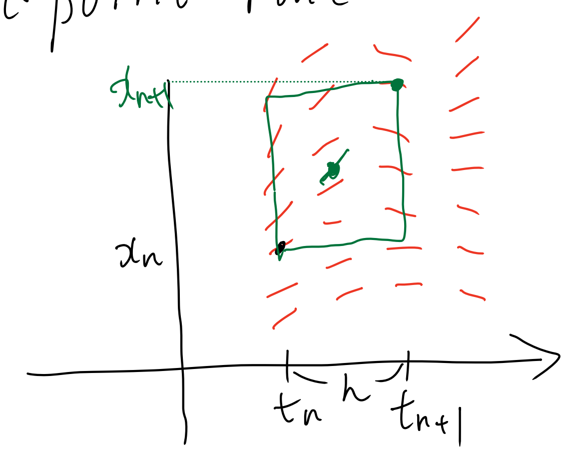
Explicit Euler



Implicit Euler

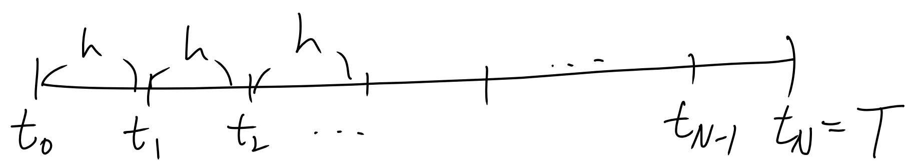


midpoint rule



Global truncation error.

Suppose local truncation error is $O(h^{n+1})$.
Then, global error accumulates over the time steps.



$$h = \frac{T-t_0}{N} \implies N = \frac{T-t_0}{h}$$

$$\sum_{i=1}^{N} O(h^{n+1}) = N \cdot O(h^{n+1}) = \frac{T-t_0}{h} O(h^{n+1})$$

$$= \underbrace{(T-t_0)}_{\text{fixed.}} O(h^n)$$

$$= O(h^n)$$

Derivation of 2^{nd} order RK method

1. Truncate 3^{rd} or higher order terms.
2. $x'(t) = f(t, x(t))$ no problem.

$$x''(t) = f_t(t, x(t)) + f_x(t, x(t))\, x'(t)$$

$$= f_t(t, x(t)) + f_x(t, x(t))\, f(t, x(t))$$

Thus,

$$x(t+h) \approx x + hf + \frac{h^2}{2}\left(f_t + f_x \cdot f\right)$$

3. $f(t+h, \underbrace{x + hf}_{\text{explicit Euler prediction of } 2^{nd} \text{ slot}})$

$$= f + f_t h + f_x \cdot hf$$

4. $x(t+h) \approx x + hf + \frac{h}{2}\left(f(t+h, x+hf) - f\right)$

$$= x + \frac{1}{2}\underbrace{hf}_{F_1} + \underbrace{\frac{1}{2}hf(t+h, x+\underbrace{hf}_{F_1})}_{F_2}$$

$$= x + \frac{1}{2}(F_1 + F_2)$$

where $F_1 = hf$

$F_2 = hf(t+h, x+F_1)$

/* Taylor theorem in two variables

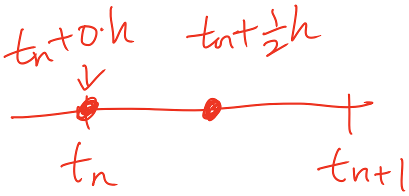
$$f(x+\Delta x,\, y+\Delta y) = f(x,y)$$ — 0-th order appr.

$$+\; f_x(x,y)\cdot\Delta x + f_y(x,y)\,\Delta y$$ — linear appr.

$$+\; \frac{1}{2}\Big(f_{xx}(\xi_x,\xi_y)\,\Delta x^2 + 2 f_{xy}(\xi_x,\xi_y)\,\Delta x\,\Delta y + f_{yy}(\xi_x,\xi_y)\,\Delta y^2\Big),$$ quadratic appr.

where (ξ_x, ξ_y) is a point on the segment connecting (x,y) and $(x+\Delta x,\, x+\Delta y)$.

Recover modified Euler from tableau



nodes of stages

$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & \underbrace{0 \quad\quad 1} &
\end{array}
$$

← weights for stage 1

← weights for stage 2.

main ← weight for advance

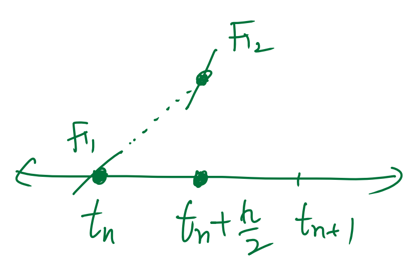
$x_{n+1} = x_n + h\left(0 \cdot F_1 + 1 F_1\right)$

Slope from stage 1

$$F_1 = f\left(t_n + 0 \cdot h,\ x_n + h\left(0 \cdot F_1 + 0 \cdot F_1\right)\right)$$

Slope from stage 2

$$F_2 = f\left(t_n + \tfrac{1}{2} \cdot h,\ x_n + h\left(\tfrac{1}{2} F_1 + 0 \cdot F_2\right)\right)$$



$$x_{n+1} = x_n + h \cdot f\left(t_n + \tfrac{h}{2},\ x_n + \tfrac{h}{2} f(t_n, x_n)\right)$$

Butcher's tableau for RK4.

stages : t, $t+\frac{1}{2}h$, $t+\frac{1}{2}h$, $t+h$

(can be repeated) $\rightarrow 0, \frac{1}{2}, \frac{1}{2}, 1$

weights for advance : $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$

inner weights for stage1 : $0, 0, 0, 0$

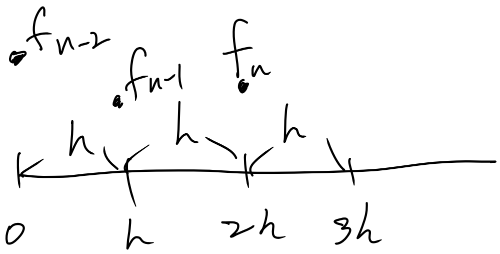
inner weights for stage2 : $\frac{1}{2}, 0, 0, 0$

inner weights for stage3 : $0, \frac{1}{2}, 0, 0$

inner weights for stage4 : $0, 0, 1, 0$

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

(Verifying AB3)



$$b_0 = \int_{2h}^{3h} \frac{(t-h)(t-2h)}{(0-h)(0-2h)}\,dt$$

$$s = t-2h \qquad dt = ds$$

$$= \frac{1}{2h^2}\int_0^h \underbrace{(s+h)s}_{s^2+hs}\,ds$$

$$= \frac{1}{2h^2}\left[\frac{s^3}{3} + h\frac{s^2}{2}\right]_0^h$$

$$= \frac{1}{2h^2}\left[\frac{h^3}{3} + \frac{h^3}{2}\right]$$

$$= h\cdot\frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}h$$

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t))\, dt$$

$$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t)\, dt$$ $p(t)$ interpolates

(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}) (t_n, f_n) 3 nodes ($m+1$) nodes with $m=2$

Since we assume x_{n-2}, x_{n-1}, x_n are exact,

e.g., $f_{n-2} = f(t_{n-2}, x(t_{n-2})) = \underbrace{f(t_{n-2}, x_{n-2})}$ computable.

Thus,

$$|x(t_{n+1}) - x_{n+1}| \leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)|\, dt$$

$$\leq \int_{t_n}^{t_{n+1}} \frac{1}{3!} \left| \frac{d^3}{dt^3} f(t, x(t)) \right| \prod_{i=0}^{2} \underbrace{|t - t_{n-i}|}_{\leq 3h}\, dt$$

$$\leq \frac{3^3}{3!} M \cdot h^3 \int_{t_n}^{t_n+h} dt$$

(where $M = \max_{t_0 \leq t \leq T} \left| \frac{d^3}{dt^3} f(t, x(t)) \right|$)

$$= C h^4$$

(where $C = \frac{3^3 M}{3!}$. Important thing is this is independent of h

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t))\,dt$$

$$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t)\,dt$$ $p(t)$ interpolates

$(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})$

4 nodes $(m+1)$ nodes with $m=3$

Thus,

$$|x(t_{n+1}) - x_{n+1}| \leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)|\,dt$$

$$\leq \int_{t_n}^{t_{n+1}} \frac{1}{4!} \left|\frac{d^4}{dt^4} f(t, x(t))\right| \prod_{i=0}^{3} \underbrace{|t - t_{n-i+1}|}_{\leq 3h}\,dt$$

$$\leq \frac{4^4}{4!} M \cdot h^4 \int_{t_n}^{t_n+h} dt$$

(where $M = \max_{t_0 \leq t \leq T} \left|\frac{d^4}{dt^4} f(t, x(t))\right|$)

$$= C h^5$$

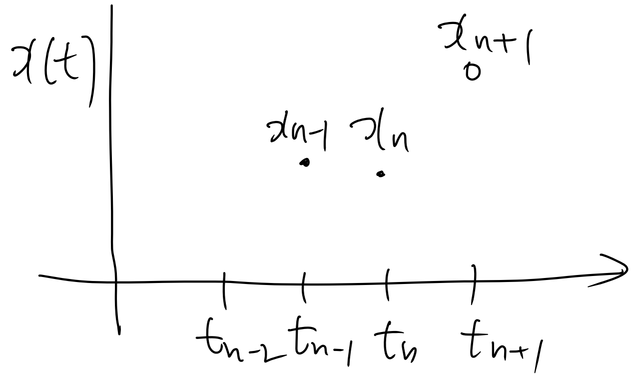
(where $C = \frac{4^4 M}{4!}$ is independent of h

/* This proof is not rigorous because we don't have $x_{n+1} = x(t_{n+1})$ in practice.

Therefore our error formula from the interpolation theory is not quite true. Nonetheless, there are other approaches that rigorously provide the same conclusion.

*/.

Derivation of BDF2 (instead of BDF3)



$$p(t) = x_{n-1} \cdot l_0(t) + x_n \cdot l_1(t) + x_{n+1} \cdot l_2(t)$$

$$l_0(t) = \frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{1}{2h^2}(t-t_n)(t-t_{n+1})$$

$$l_1(t) = \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = -\frac{1}{h^2}(t-t_{n-1})(t-t_{n+1})$$

$$l_2(t) = \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} = \frac{1}{2h^2}(t-t_{n-1})(t-t_n)$$

$$l_0'(t) = \frac{1}{2h^2}(t-t_n+t-t_{n+1})$$

$$l_1'(t) = -\frac{1}{h^2}(t-t_{n-1}+t-t_{n+1})$$

$$l_2'(t) = \frac{1}{2h^2}(t-t_{n-1}+t-t_n)$$

$$p'(t_{n+1}) = x_{n-1}\, l_0'(t_{n+1}) + x_n\, l_1'(t_{n+1}) + x_{n+1}\, l_2'(t_{n+1})$$
$$= x_{n-1} \cdot \frac{1}{2h} + x_n \cdot \left(-\frac{2}{h}\right) + x_{n+1} \cdot \frac{3}{2h}$$
$$\approx f_{n+1}$$

Divide through by $\frac{3}{2h}$ (to solve for x_{n+1}), then

$$x_{n+1} - \frac{4}{3} x_n + \frac{1}{3} x_{n-1} = \frac{2}{3} h f_{n+1}$$