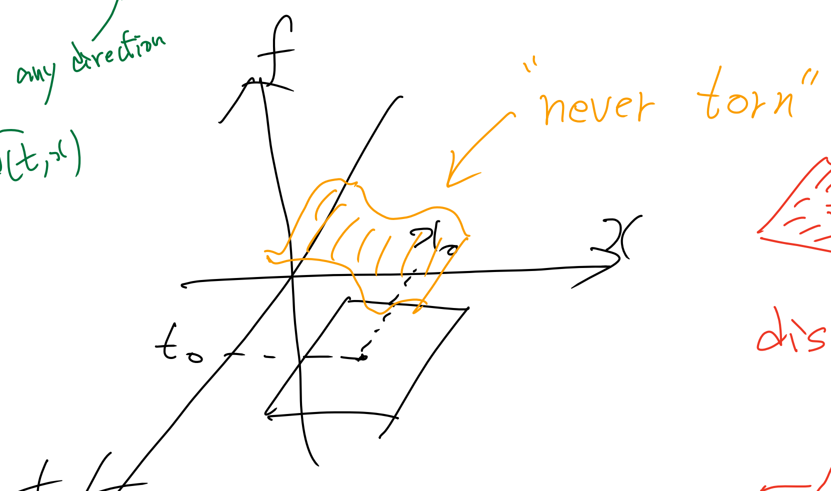
(example) $x'(t) = x^{\frac{2}{3}}$.

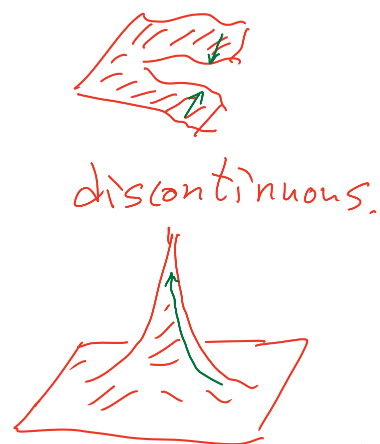
This corresponds to $f(t,x) = x^{\frac{2}{3}}$

f is continuous (as a function of two variables).

/* $\lim_{(\tau,z)\to(t,x)} f(\tau,z) = f(t,x)$

any direction





*/

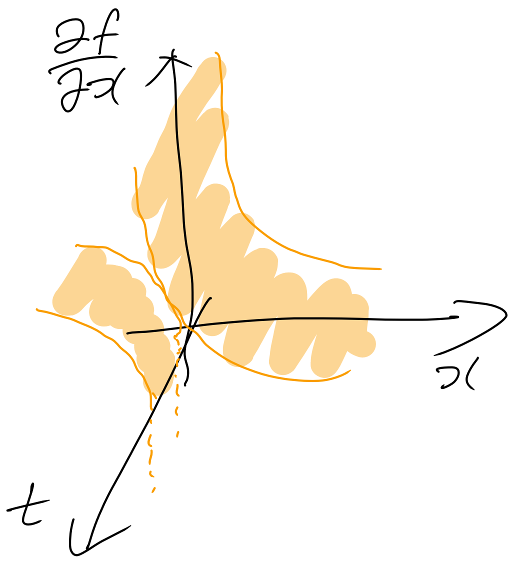
(existence)

Therefore by the theorem, a function $x(t)$ on a small open interval of t $(-\delta,\delta)$ s.t. $x'(t) = x^{\frac{2}{3}}$ and $x(0) = 0$.

For example, $x(t) \equiv 0$: $x'(t) = 0 = 0^{\frac{2}{3}}$

(uniqueness)

However, $\frac{\partial f}{\partial x}(t,x) = \frac{2}{3}x^{-\frac{1}{3}}$ is not continuous on any rectangle around $(t_0, x_0) = (0,0)$.



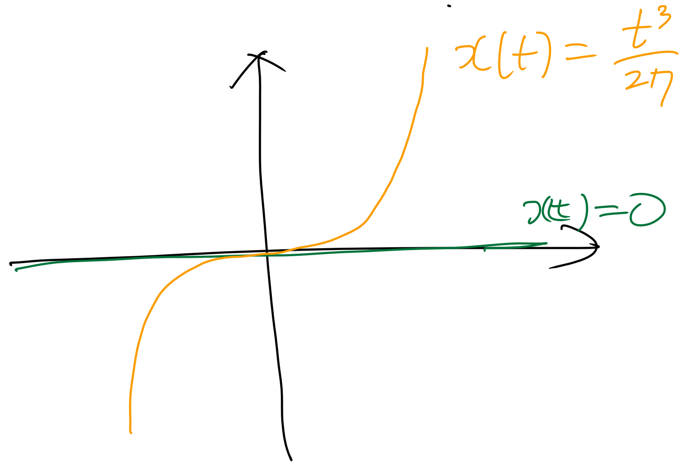
Thus, the theorem does not guarantee uniqueness. In fact, we have another solution.

For example, use the ansatz

$$x(t) = at^b$$

$$\left.\begin{aligned} x'(t) &= abt^{b-1} \\ (x(t))^{\frac{2}{3}} &= a^{\frac{2}{3}} t^{\frac{2}{3}b} \end{aligned}\right] \Rightarrow \begin{aligned} &b-1 = \tfrac{2}{3}b \Rightarrow b = 3 \\ &ab = a^{\frac{2}{3}} \Rightarrow a^3 b^3 = a^2 \\ &\Rightarrow a = \frac{1}{b^3} = \frac{1}{27}. \end{aligned}$$

This also satisfies the IC (initial condition)

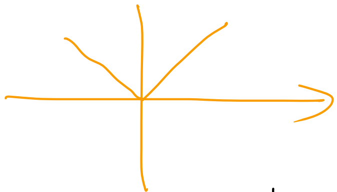


$f(x) = x^2 \in C^1[-1,1]$

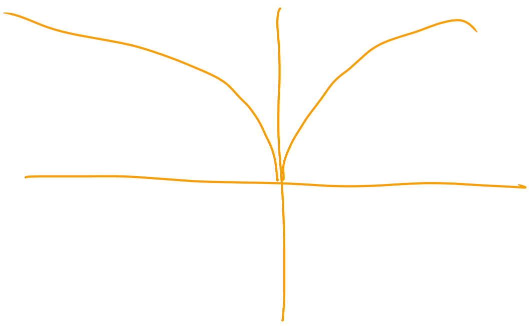
$g(x) = |x| \in Lip[-1,1]$ but $g \notin C^1[-1,1]$

$|g(x)-g(y)| = \big||x|-|y|\big| \le |x-y| \quad L=1.$

inverse triangle ineq.



$h(x) = |x|^{\frac{1}{2}} \in C[-1,1]$ but $h \notin Lip[-1,1]$.



$$|h(x)-h(y)| = \left| |x|^{\frac{1}{2}} - |y|^{\frac{1}{2}} \right|$$
$$= |\delta - 2\delta| = \delta$$

Choose $x = \delta^2$, $h = 4\delta^2$, then $|x-y| = 3\delta^2$

For any $L>0$, if $\delta < \frac{1}{3L}$

$$|h(x)-h(y)| = \delta \frac{3L}{3L} > 3L\delta^2 = L|x-y|.$$

Example : Refresh calculus.

$$\frac{d}{dt} f(x(t), y(t))$$

$$= f_x(x(t), y(t))\, x'(t)$$

$$+ f_y(x(t), y(t))\, y'(t)$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & f(x,y) \\ & y \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & \Delta x \arrow[dr] & \\ \Delta t \arrow[ur] \arrow[dr] & & \Delta f \\ & \Delta y \arrow[ur] & \end{tikzcd}$$

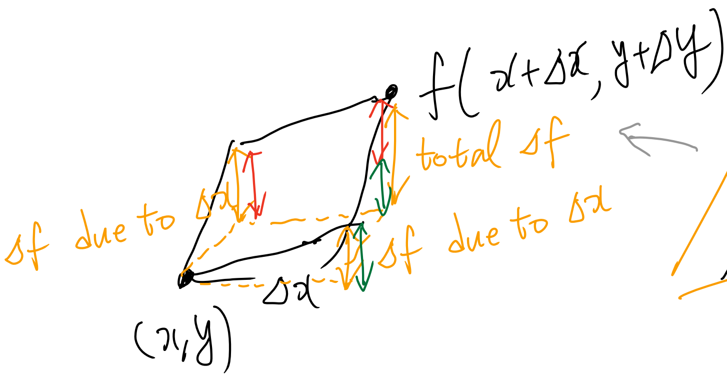
$\Delta x \simeq x' \cdot \Delta t$

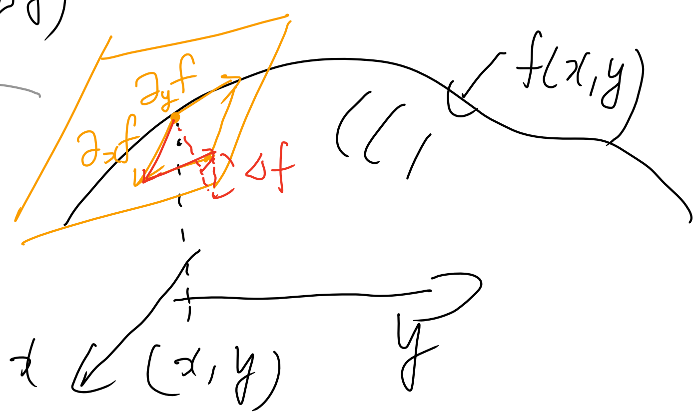
$\Delta y \simeq y' \cdot \Delta t$

$\Delta f \simeq \partial_t f \cdot \Delta y + \partial_x f \cdot \Delta x$

$$\Longrightarrow \frac{\Delta f}{\Delta t} \simeq \frac{\partial_y f \cdot \Delta y}{\Delta t} + \frac{\partial_x f \cdot \Delta x}{\Delta t}$$

$$= \partial_y f \cdot y' + \partial_x f \cdot x'$$





$$x'' = \big(f(t,x)\big)' = \underbrace{f_t(t,x) + f_x(t,x)\, x'}$$

$$x''' = \big(f_t(t,x) + f_x(t,x)\, x'\big)'$$

$$= f_{tt}(t,x) + f_{tx}(t,x)\, x'$$

$$+ \big(f_{xt}(t,x) + f_{xx}(t,x)\, x'\big)\, x' + f_x(t,x)\, x''$$