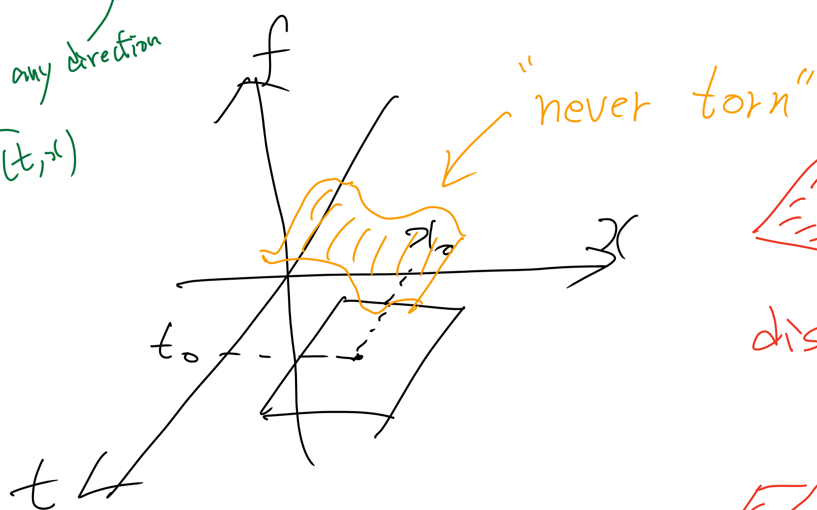
(example) $x'(t) = x^{\frac{2}{3}}$.

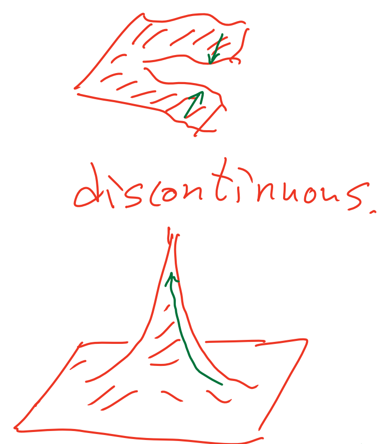
This corresponds to $f(t,x) = x^{\frac{2}{3}}$

f is continuous (as a function of two variables).

/* $\displaystyle\lim_{(\tau,z)\to(t,x)} f(\tau,z) = f(t,x)$

any direction





*/

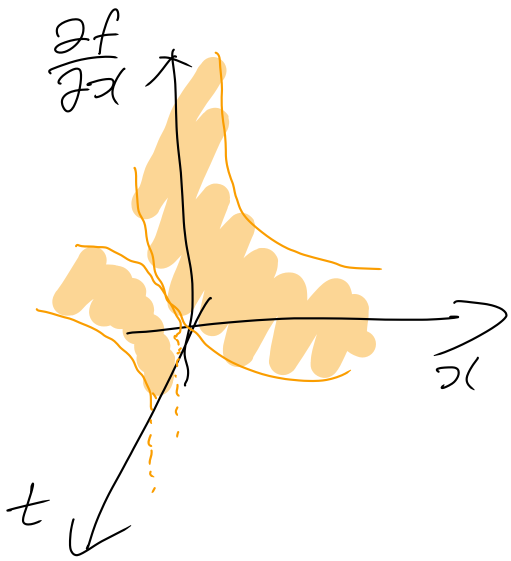
(existence)

Therefore by the theorem, a function $x(t)$ on a small open interval of t $(-\delta,\delta)$ s.t. $x'(t) = x^{\frac{2}{3}}$ and $x(0) = 0$.

For example, $x(t) \equiv 0$: $x'(t) = 0 = 0^{\frac{2}{3}}$

(uniqueness)

However, $\frac{\partial f}{\partial x}(t,x) = \frac{2}{3}x^{-\frac{1}{3}}$ is not continuous on any rectangle around $(t_0, x_0) = (0,0)$.



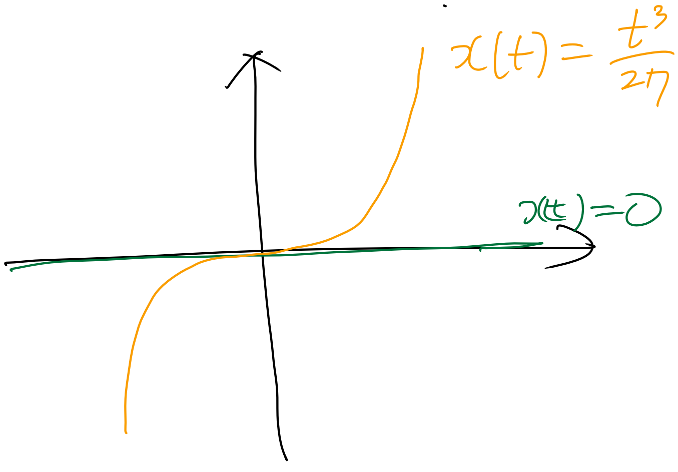
Thus, the theorem does not guarantee uniqueness. In fact, we have another solution.

For example, use the ansatz

$$x(t) = at^b$$

$$\left.\begin{array}{l} x'(t) = abt^{b-1} \\ (x(t))^{\frac{2}{3}} = a^{\frac{2}{3}} t^{\frac{2}{3}b} \end{array}\right] \Rightarrow \begin{array}{l} b-1 = \frac{2}{3}b \Rightarrow b = 3 \\ ab = a^{\frac{2}{3}} \Rightarrow a^3 b^3 = a^2 \\ \Rightarrow a = \frac{1}{b^3} = \frac{1}{27}. \end{array}$$

This also satisfies the IC (initial condition).

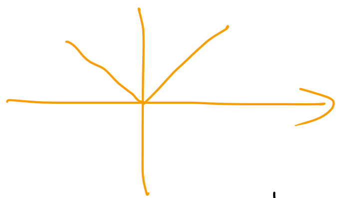


$f(x) = x^2 \in C^1[-1,1]$

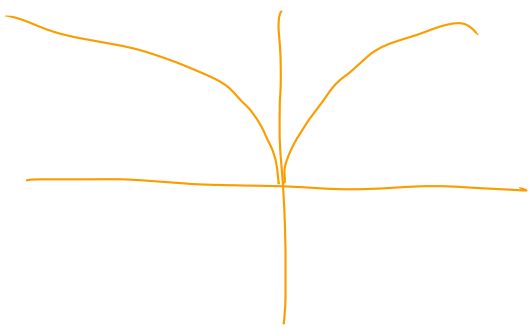
$g(x) = |x| \in Lip[-1,1]$ but $g \notin C^1[-1,1]$

$|g(x)-g(y)| = \big||x|-|y|\big| \le |x-y|$ $\quad L=1$.

inverse triangle ineq.



$h(x) = |x|^{\frac{1}{2}} \in C[-1,1]$ but $h \notin Lip[-1,1]$.



$$|h(x)-h(y)| = \left| |x|^{\frac{1}{2}} - |y|^{\frac{1}{2}} \right|$$

$$= |\delta - 2\delta| = \delta$$

Choose $x=\delta^2$, $h=4\delta^2$, then $|x-y| = 3\delta^2$

For any $L>0$, if $\delta < \frac{1}{3L}$

$$|h(x)-h(y)| = \delta \frac{3L}{3L} > 3L\delta^2 = L|x-y|.$$

Example : Refresh calculus.

$$\frac{d}{dt} f(x(t), y(t))$$

$$= f_x(x(t), y(t))\, x'(t)$$
$$+ f_y(x(t), y(t))\, y'(t)$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & f(x,y) \\ & y \arrow[ur] & \end{tikzcd}$$

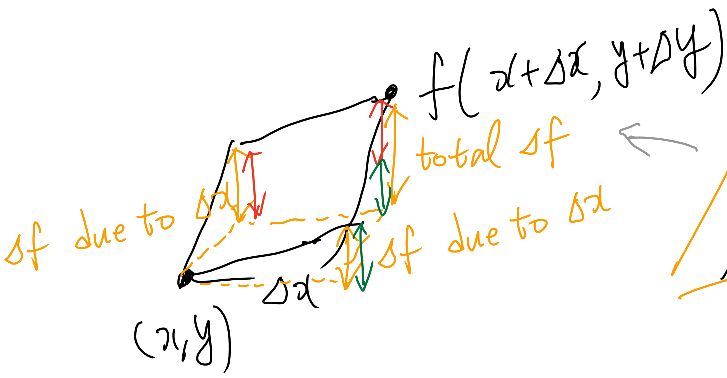
$$\begin{tikzcd} & \Delta x \arrow[dr] & \\ \Delta t \arrow[ur] \arrow[dr] & & \Delta f \\ & \Delta y \arrow[ur] & \end{tikzcd}$$

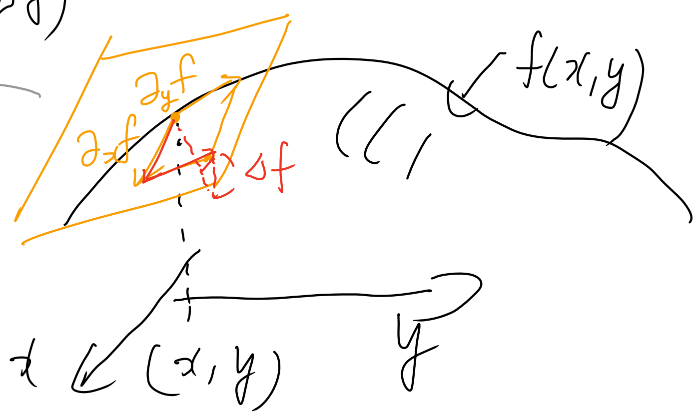
$\Delta x \simeq x' \cdot \Delta t$

$\Delta y \simeq y' \cdot \Delta t$

$\Delta f \simeq \partial_y f \cdot \Delta y + \partial_x f \cdot \Delta x$

$$\Longrightarrow \frac{\Delta f}{\Delta t} \simeq \frac{\partial_y f \cdot \Delta y}{\Delta t} + \frac{\partial_x f \cdot \Delta x}{\Delta t}$$
$$= \partial_y f \cdot y' + \partial_x f \cdot x'$$





$$x'' = \big(f(t,x)\big)' = \underbrace{f_t(t,x) + f_x(t,x)\, x'}$$

$$x''' = \big(f_t(t,x) + f_x(t,x)\, x'\big)'$$
$$= f_{tt}(t,x) + f_{tx}(t,x)\, x'$$
$$+ \big(f_{xt}(t,x) + f_{xx}(t,x)\, x'\big)\, x' + f_x(t,x)\, x''$$

$$x''(t) = -\sin t + 2t - \cos x \cdot x'$$

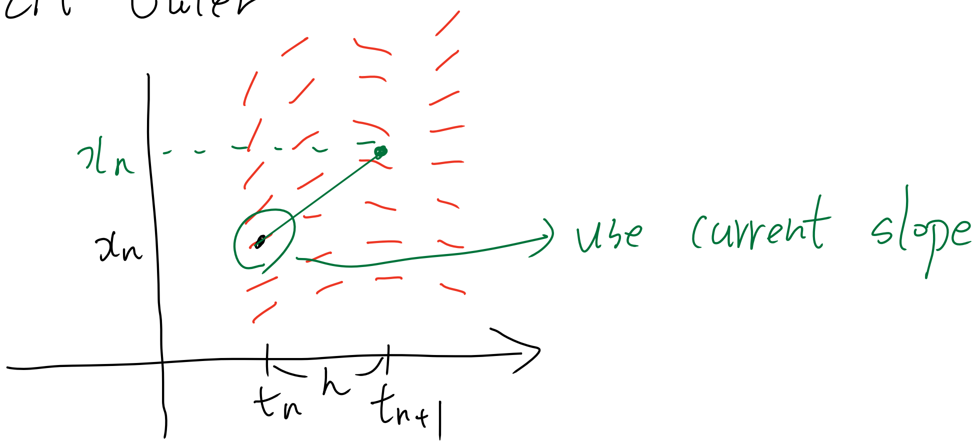
$$x'''(t) = -\cos t + 2 - \cos x \cdot x'' + (\sin x \cdot x') \cdot x'$$

$$= -\cos t + 2 - \cos x \cdot x'' + \sin x \cdot (x')^2$$

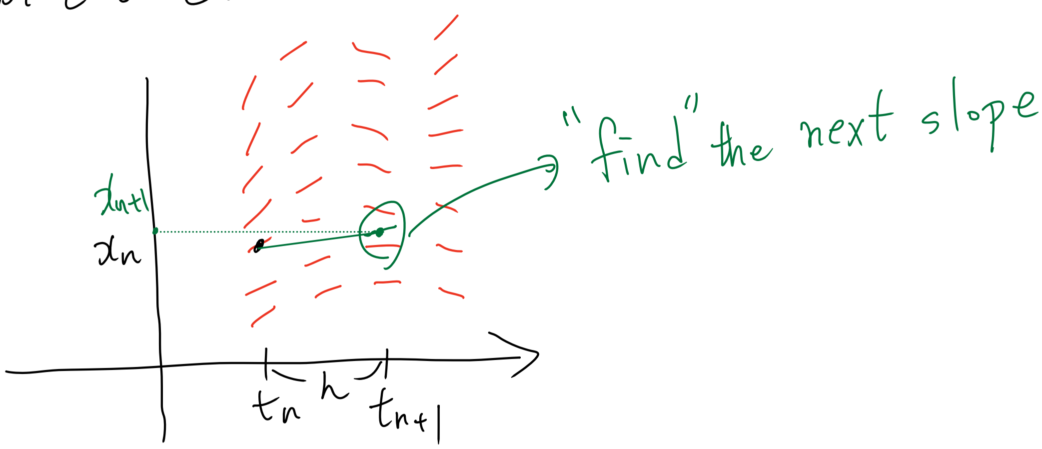
$$x''''(t) = \sin t + \sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot x' \cdot (x')^2 + \sin x \cdot 2(x')x''$$

$$= \sin t + 3\sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot (x')^3$$

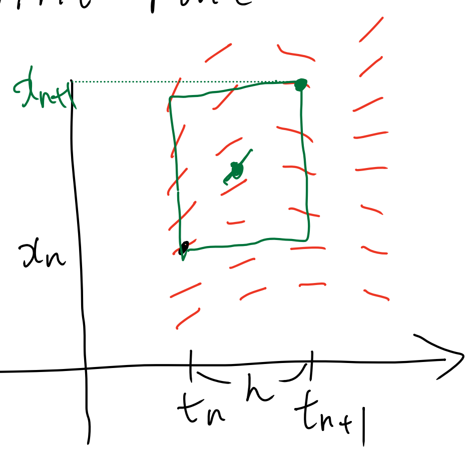
Explicit Euler



Implicit Euler

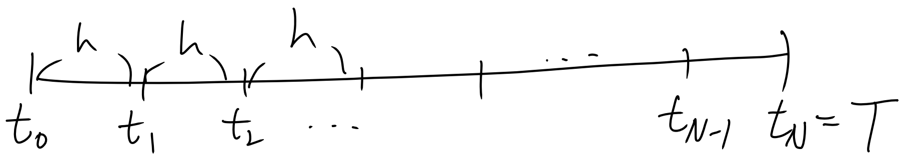


midpoint rule



Global truncation error.

Suppose local truncation error is $O(h^{n+1})$.
Then, global error accumulates over the time steps.



$$h = \frac{T - t_0}{N} \implies N = \frac{T - t_0}{h}$$

$$\sum_{i=1}^{N} O(h^{n+1}) = N \cdot O(h^{n+1}) = \frac{T - t_0}{h} O(h^{n+1})$$

$$= \underbrace{(T - t_0)}_{\text{fixed.}} O(h^{n})$$

$$= O(h^{n})$$

Derivation of 2^{nd} order RK method

1. Truncate 3^{rd} or higher order terms.
2. $x'(t) = f(t, x(t))$ no problem.

$$x''(t) = f_t(t, x(t)) + f_x(t, x(t))\, x'(t)$$

$$= f_t(t, x(t)) + f_x(t, x(t))\, f(t, x(t))$$

Thus,

$$x(t+h) \approx x + hf + \frac{h^2}{2}\left(\underline{f_t + f_x \cdot f}\right)$$

3. $f(t+h, \underbrace{x+hf}_{\text{explicit Euler prediction of } 2^{nd} \text{ slot}})$

$$= f + \underline{f_t\, h + f_x \cdot hf}$$

4. $x(t+h) \approx x + hf + \frac{h}{2}\left(f(t+h, x+hf) - f\right)$

$$= x + \frac{1}{2}\underbrace{hf}_{F_1} + \underbrace{\frac{1}{2}hf(t+h, x+\underbrace{hf}_{F_1})}_{F_2}$$

$$= x + \frac{1}{2}(F_1 + F_2)$$

where $F_1 = hf$

$F_2 = hf(t+h, x+F_1)$

/* Taylor theorem in two variables

$$f(x+\Delta x, y+\Delta y) = f(x,y) \quad \text{— 0-th order appr.}$$

$$+ f_x(x,y)\cdot\Delta x + f_y(x,y)\Delta y \quad \text{— linear appr.}$$

$$\left.\begin{aligned} &+ \frac{1}{2}\Big(f_{xx}(\xi_x, \xi_y)\Delta x^2 + 2 f_{xy}(\xi_x, \xi_y)\Delta x \Delta y \\ &\quad + f_{yy}(\xi_x, \xi_y)\Delta y^2\Big), \end{aligned}\right\} \text{quadratic appr.}$$

where (ξ_x, ξ_y) is a point on the segment connecting (x,y) and $(x+\Delta x, x+\Delta y)$.