# NT. 1.11 Matrix

**Def** (Matrix)

A matrix is a collection of numbers arranged in a rectangular form. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

is a $\underbrace{2}_{\text{row}} \times \underbrace{3}_{\text{col.}}$ (two-by-three) matrix

and its $(2,3)$ entry is $6$. Its $1^{st}$ row is $[1, 2, 3]$ and its $2^{nd}$ column is $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. We often view rows or columns as a vector.

## Equality of Matrices

We say $A = B$ if they have the same size and each corresponding entries are equal.

/* Def. of Matrix itself does not do much. All the magic you saw from Math 4A comes from how we interpret those numbers and more structures added (e.g. matrix multiplication etc.)

# Basic properties of matrices

(Assuming the sizes of matrices all match) For matrices $A, B, C$ and scalars $\alpha, \beta$

- $A+B = B+A$
- $(A+B)+C = A+(B+C)$
- $\alpha(A+B) = \alpha A + \alpha B$
- $(\alpha+\beta)A = \alpha A + \beta A$
- $AB := \begin{bmatrix} (1^{st} \text{ row } A)\cdot(1^{st} \text{ col } B) & \cdots & (1^{st} \text{ row } A)\cdot(\text{last col } B) \\ \vdots & \ddots & \vdots \\ (\text{last row } A)\cdot(1^{st} \text{ col } B) & & (\text{last row } A)\cdot(\text{last col } B) \end{bmatrix}$

dot product

$(n\times m)\cdot(m\times k) = (n\times k)$

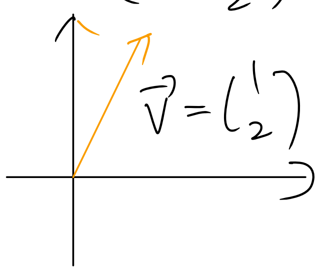
- $A(BC) = (AB)C$
- $(A+B)C = AC+BC$
- $A(B+C) = AB+AC$
- $AB \neq BA$ in general

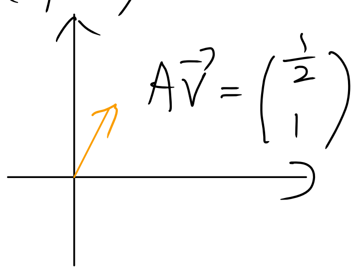
# NT 1.12 Matrix as linear mapping

- Matrix multiplication defines a linear mapping (transformation) acting on vectors.

/* You should interpret vectors as column vectors */

E.g. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$



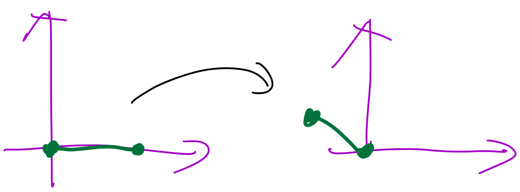



⊙ Any linear mapping can be expressed by a multiplication by a matrix.

"matrix (mult.) = lin. map"

⊙ We can study how such mappings deform shapes (i.e., collection of points):

- point $\longmapsto$ point
- line/segment $\longmapsto$ $\boxed{\text{line/segment (usually)}}$ or a point → "Degenerate"

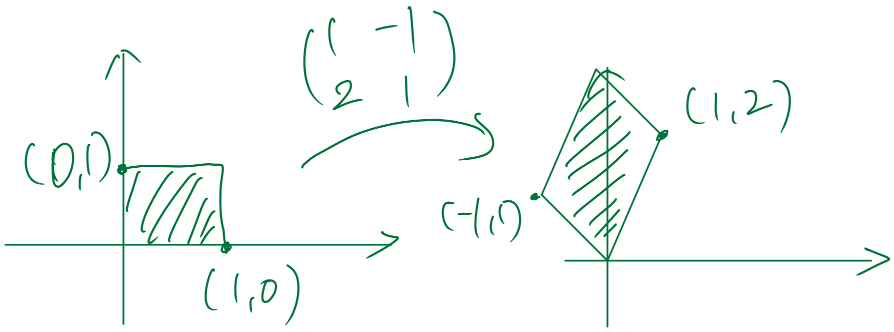


e.g. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

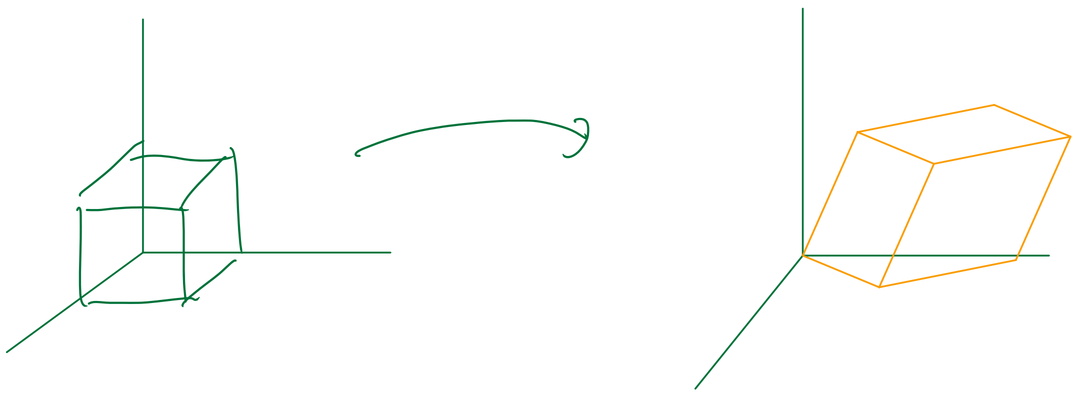
$(x,y)$ from a segment

- rectangle $\longmapsto$ $\boxed{\text{parallelogram (usually)}}$ or a segment or a point — degenerate

e.g.



rectangular prism $\longmapsto$ parallelopiped



① In 2D, the ratio of deformed output area and the original area input remains fixed no matter what shape input it is. And the ratio coincides with the (absolute value) of the determinant of the matrix.

② In 3D, the ratio of deformed volume and the original volume remains fixed no matter what shape it is. And the ratio coincides with the (absolute value) of the determinant of the matrix.

# NT 1.12 Determinant

(2x2) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

simple

$= ad - bc$

(3x3) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$A_{11} = \begin{matrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{matrix}$$

$$A_{12} = \begin{matrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{matrix}$$

$$A_{13} = \begin{matrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{matrix}$$

/* You can choose any other row or column for "cover up expansion". However, be careful of signs. You have to add up

$$(-1)^{i+j} \cdot a_{ij} |A_{ij}|.$$

For example, if you choose the second column and expand around it.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12}|A_{12}| + a_{22}|A_{22}| - a_{32}|A_{32}|$$

b/c $(-1)^{1+2} = -1$ $\quad (-1)^{2+2} = 1$ $\quad (-1)^{3+2} = -1$

**Example**: determinant of $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= (-1) \cdot (\underbrace{2 \cdot 1 - 1 \cdot 0}_{=2})$$

$$= -2$$

## ⊕ Properties of determinant

- Two of the rows/columns are switched $\longrightarrow$ sign flipped

e.g.
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Determinant is linear in each column/row (with all others fixed)

That is, "$f(ax) = a f(x)$"

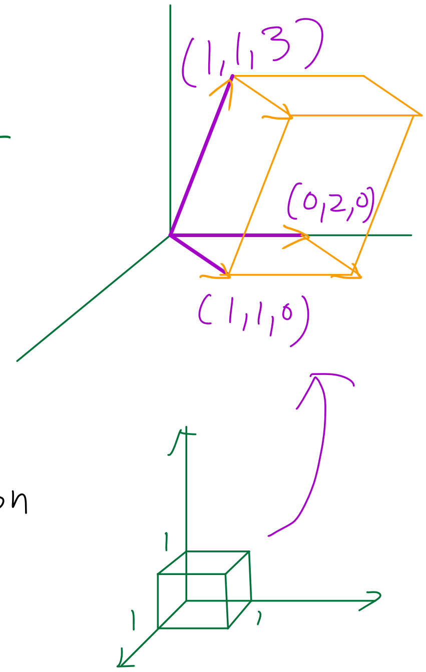
$$\begin{vmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \alpha \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

"$f(x+y) = f(x)+f(y)$"

$$\begin{vmatrix} a_1+d_1 & a_2+d_2 & a_3+d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Determinant = signed volume of the parallelopiped determined by the row vectors (or column vectors).

e.g. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{vmatrix} = \pm$




Another way to view this is det(A) is volume distorsion of a unit cube.

- Two of the rows/columns are equal → determinant = 0.

e.g. $\begin{vmatrix} 3 & 100 & 7 \\ 3 & 100 & 7 \\ 1 & 4 & 5 \end{vmatrix} = 0$

- Determinant of triangular matrix is the product of diagonal entries

e.g. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24.$