

(proof of unique interpolation - Newton form)

Let k be the number of nodes.

$$(k=1) \quad P_0 = Y_0 = C_0$$

$$(k=2) \quad P_1 = C_0 + C_1(x - x_0)$$

From $Y_1 = P_1(x_1) = C_0 + C_1(x_1 - x_0)$

$$C_1 = [Y_1 - C_0] / (x_1 - x_0)$$

$$(k=3) \quad P_2 = P_1(x) + C_2(x - x_1)(x - x_0)$$

This construction allows us to only consider the new condition

$$P_2(x_2) = Y_2 \quad \text{b/c we already know}$$

$P_1(x_0) = Y_0$ and $P_1(x_1) = Y_1$, while $C_2(x - x_1)(x - x_0)$ vanishes if $x = x_0$ or x_1 .

From $Y_2 = P_2(x_2) = P_1(x_2) + C_2(x_2 - x_1)(x_2 - x_0)$

$$C_2 = [Y_2 - P_1(x_2)] / (x_2 - x_1)(x_2 - x_0)$$

Repeat this until $k=n$. Since the process is constructive, the resulting $P_n(x)$ must be unique.

Example : Find $p \in \mathbb{P}_3$ such that

x	0	1	2	3
y	1	2	3	

$$p_0 = 1 = C_0$$

$$P_1(x) = C_0 + \frac{y_1 - C_0}{x_1 - x_0} = 1 + \frac{1 - 1}{1 - 0} (x - 0) = 1$$

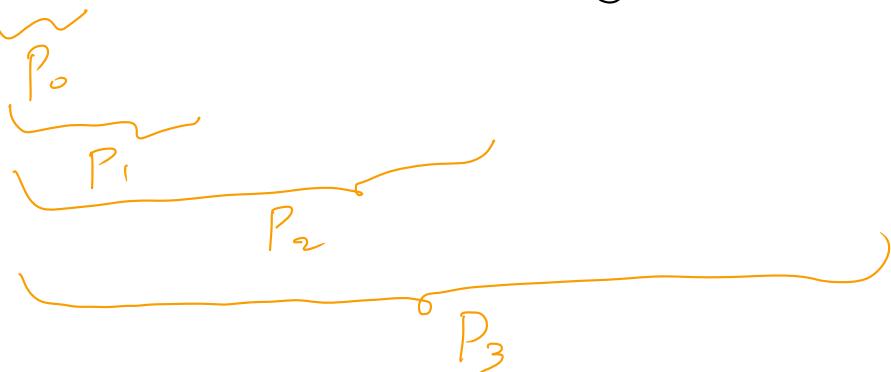
$$P_2(x) = P_1(x) + \frac{y_2 - P_1(x_2)}{(x_2 - x_1)(x_2 - x_0)} (x - x_1)(x - x_0)$$

$$= 1 + \frac{2 - 1}{2 \cdot 1} (x - 1) x = 1 + \frac{x}{2} (x - 1)$$

$$P_3(x) = P_2(x) + \frac{y_3 - P_2(x_3)}{(x_3 - x_2)(x_3 - x_1)(x_3 - x_0)} (x - x_2)(x - x_1)(x - x_0)$$

$$= 1 + \frac{x}{2} (x - 1) + \frac{3 - 1 - \frac{3}{2} \cdot 2}{1 \cdot 2 \cdot 3} (x - 2)(x - 1) x$$

$$= 1 + \frac{x}{2} (x - 1) - \frac{x}{6} (x - 2)(x - 1)$$



Horner's algorithm

number of
multiplications

$$c_0 + c_1 d_1 + c_2 \underline{\underline{d_1 d_2}} + c_3 \underline{\underline{d_1 d_2 d_3}} \quad 1+2+3$$

$$= c_0 + c_1 \underline{\underline{d_1}} + (c_2 + c_3 d_3) \underline{\underline{d_1 d_2}}$$

$$= c_0 + (c_1 + (c_2 + c_3 d_3) d_2) d_1$$

nested $\underbrace{\alpha}_{C's} + \underbrace{\beta r}_{\substack{\text{previous} \\ \text{result}}} \underbrace{\text{ form}}_{d's}$

$$\begin{aligned} &= c_0 + (c_1 + (c_2 + c_3 d_3) d_2) d_1 \quad 1+1+1 \\ &\quad \quad \quad \underbrace{\alpha_1 + \beta_1 r_1}_{\alpha_1 + \beta_1 r_1} \\ &\quad \quad \quad \underbrace{\alpha_2 + \beta_2 \cdot r_2}_{\alpha_2 + \beta_2 \cdot r_2} \\ &\quad \quad \quad \underbrace{\alpha_3 + \beta_3 \cdot r_3}_{\alpha_3 + \beta_3 \cdot r_3} \end{aligned}$$

example : Lagrange basis subordinate to
nodes $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{1}{6} (x-2)(x-3)(x-4)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{1}{2} (x-1)(x-3)(x-4)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{2} (x-1)(x-2)(x-4)$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{1}{6} (x-1)(x-2)(x-3)$$

(proof of linear independence)

Assume

equality as fn's.

$$a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x) = 0$$

Plug in $x = x_0$, then

$$0 = a_0 \underbrace{f_0(x_0)}_{\delta_{00} \neq 0} + a_1 \underbrace{f_1(x_0)}_{\delta_{10} = 0} + \cdots + a_n \underbrace{f_n(x_0)}_{\delta_{n0} = 0}$$

$$= a_0.$$

Plug in $x = x_1$, then

$$0 = a_0 \underbrace{f_0(x_1)}_{\delta_{01} = 0} + a_1 \underbrace{f_1(x_1)}_{\delta_{11} \neq 0} + \cdots + a_n \underbrace{f_n(x_1)}_{\delta_{n1} = 0}$$

$$= a_1.$$

Repeat this to conclude $a_0 = a_1 = \cdots = a_n = 0$ 

(proof of interpolation error)

① If x is one of the nodes we have

$$D = 0 \quad \checkmark$$

② Assume $x \neq x_i$ ($i=0, 1, 2, \dots, n$).

/* The trick is (due to Cauchy) to think of x as a new node. */

Put $w(t) = \prod_{i=0}^n (t - x_i)$, then $w(x) \neq 0$.

Let $\lambda = (f(x) - p(x)) / w(x)$ and introduce

$$\varphi(t) = f(t) - p(t) \rightarrow w(t) \in C^{n+1}[a, b]$$

Observe that $\varphi(x_i) = 0$ ($i=0, 1, 2, \dots, n$)

and $\varphi(x) = 0$ (by the construction of λ).

Thus, use Rolle's theorem $(n+1)$ times to

argue $\exists \xi_j$ ($j=0, 1, \dots, n$) s.t. $\varphi'(\xi_j) = 0$.

Next, do the similar to argue $\exists \zeta_k$ ($k=0, 1, \dots, n-1$)
s.t. $\varphi''(\zeta_k) = 0$. Repeat this to show that

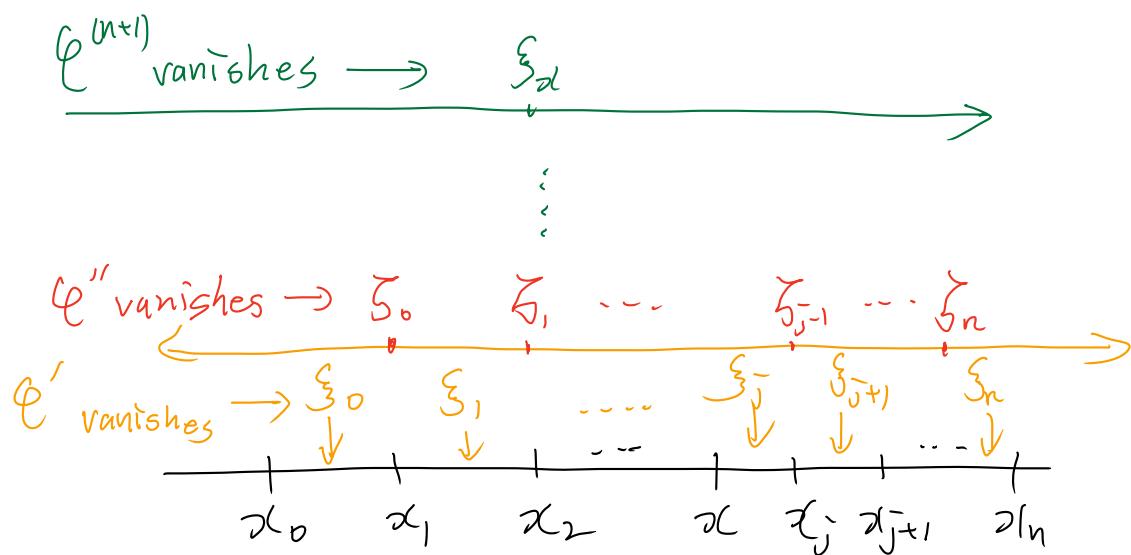
$\exists \xi_x$ s.t. $\varphi^{(n+1)}(\xi_x) = 0$. (see picture
below). But

$$0 = \varphi^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - \lambda(n+1)! \quad (\text{why?})$$

$$= f^{(n+1)}(\xi_x) - \frac{(f(x) - p(x))}{w(x)} \cdot (n+1)!$$

Rearranging, we obtain

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$



example of error bound on sine interpolation.

$$|f^{(n)}(x)| \leq 1 \quad \text{for all } x \in [0, 1].$$

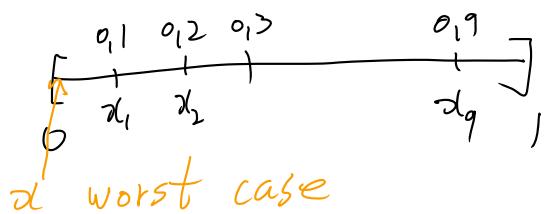
(an answer) Let us use the theorem we just proved.

$$\begin{aligned} |f(x) - p(x)| &= \left| \frac{f^{(11)}(x)}{11!} \prod_{i=1}^{10} (x - x_i) \right| \\ &\leq \frac{1}{11!} \prod_{i=0}^{10} |x - x_i| \xrightarrow{\substack{|x-x_0| \leq 1 \\ \dots}} (\times) \\ &= \frac{1}{11!} \approx 2.5 \times 10^{-8} \end{aligned}$$

If x_i 's are equally spaced, (*) continues

$$\leq \frac{1}{11!} \prod_{i=1}^{10} \frac{1}{10} = \frac{1}{10^{10} \cdot 11!} \approx 9.1 \times 10^{-12}$$

very pessimistic bound $|x - x_0| \leq 1$ is used.



/* If you don't like this the pessimistic bound $|x - x_0| \leq 1$ (even though you that must be very small), you can do the following. For the first two nodes, you can model $|x - x_0| |x - x_1|$ as

$$|x(x-0.1)| \quad \text{on } [0, 0.1]$$

But we know

$$|x(x-0.1)| \leq 0.5^2 = 0.0025$$

Then, the error bound reads

$$\begin{aligned} &\leq \frac{1}{11!} 0.0025 \prod_{i=2}^{10} \frac{i}{10} \\ &= \frac{0.0025}{10^9 \cdot 11} \approx 2.27 \times 10^{-13} \end{aligned}$$

But when we bound errors, too precise calculations are not what we are after. Find a good balance. */

(Proof of characterization of Chebyshev poly's)
 We argue by an induction.

$$\textcircled{1} \quad T_0(x) = \cos(0 \cdot \cos^{-1}x) = \cos 0 = 1$$

$$\textcircled{2} \quad T_1(x) = \cos(\cos^{-1}x) = x$$

$$\textcircled{3} \quad \text{Suppose } T_k(x) = \cos(k \cos^{-1}x) \text{ for } k=0, 1, \dots, n$$

$$/* \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\Rightarrow \boxed{\cos((n+1)\theta) = \cos(n\theta) \cos \theta - \sin(n\theta) \sin \theta}$$

$$+ \boxed{\cos((n-1)\theta) = \cos(n\theta) \cos \theta + \sin(n\theta) \sin \theta}$$

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2 \cos(n\theta) \cos \theta$$

Take $\theta = \cos^{-1}x$ for $x \in [-1, 1]$:

$$\begin{aligned} \cos((n+1)\cos^{-1}x) &= 2 \cos(n\cos^{-1}x) \cos(\cos^{-1}x) \\ &\quad - \cos((n-1)\cos^{-1}x) \end{aligned}$$

$$= 2x T_n(x) - T_{n-1}(x)$$

$$= T_{n+1}(x) \quad \text{by recursive}$$

def. of Chebyshev

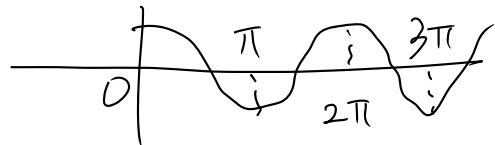
(proof of basic properties of)
 Chebyshov poly's.

$$\textcircled{1} \quad |T_n(x)| = |\cos(n \cos^{-1}x)| \leq 1$$

$$\textcircled{2} \quad T_n(\cos \frac{j\pi}{n}) = \cos(n \cos^{-1}(\cos \frac{j\pi}{n}))$$

$$= \cos(j\pi)$$

$$= (-1)^j$$

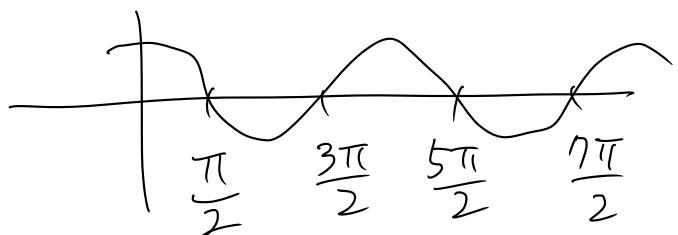


$$\textcircled{3} \quad T_n(\cos \frac{2j-1}{2n}\pi)$$

$$= \cos(n \cos^{-1}(\cos \frac{2j-1}{2n}\pi))$$

$$= \cos(j\pi - \frac{\pi}{2})$$

$$= 0$$



(proof of maximum bound of monic poly.)

(equality) Let $g(x) = 2^{1-n} T_n(x)$. This is monic and of degree n :

$$T_2(x) = 2x T_1(x) - T_0(x) = \underline{2x^2} - 1 \in \mathbb{T}_2$$

$$T_3(x) = \underline{\cancel{2x^2}} T_2(x) - T_1(x) = \underline{2^2 x^3} - 3x \in \mathbb{T}_3$$

\vdots $\underline{\cancel{2x^2}}$ higher deg

and so on

$$T_n(x) = 2x(T_{n-1}(x)) - T_{n-2}(x) = 2^{1-n} x^n + \dots \in \mathbb{T}_n$$

Also, $|T_n(x)| \leq 1$ on $[-1, 1]$ by previous corollary.

Thus, for all $x \in [-1, 1]$, we have

$$|2^{1-n} \cdot T_n(x)| = 2^{1-n} |T_n(x)| \leq 2^{1-n}$$

(Inequality) Suppose $\exists P \in \mathbb{T}_n$, monic such that

$$\max_{|x| \leq 1} |P(x)| < 2^{1-n}$$

Let $x_j = \cos\left(\frac{j\pi}{n}\right)$. Then, observe

$$(*) (-1)^{\bar{j}} \cdot P(x_{\bar{j}}) \leq |P(x_{\bar{j}})| < 2^{1-n} = (-1)^{\bar{j}} f(x_{\bar{j}})$$

The last equality holds b/c

$$T_n(x_{\bar{j}}) = (-1)^{\bar{j}} \quad (\text{previous corollary})$$

$$\Rightarrow f(x_{\bar{j}}) = 2^{\lceil n \rceil} \cdot T_n(x_{\bar{j}}) = 2^{\lceil n \rceil} (-1)^{\bar{j}}$$

$$\Rightarrow (-1)^{\bar{j}} f(x_{\bar{j}}) = (-1)^{\bar{j}} \cdot (-1)^{\bar{j}} \cdot 2^{\lceil n \rceil} = 2^{\lceil n \rceil}$$

Rearranging (*), we have

$$(**) (-1)^{\bar{j}} (f(x_{\bar{j}}) - p(x_{\bar{j}})) > 0 \quad (\bar{j}=0, 1, \dots, n)$$

$\Rightarrow g(x) - p(x)$ changes sign n times $\Rightarrow n$ roots

But $\deg(g(x) - p(x)) = n-1$ since x^n terms have been cancelled out. Besides,

$g(x) - p(x)$ is not zero function

since, e.g., $g(x_0) - p(x_0) > 0$ from (**)

Thus, we have contradiction. \square

(proof of error bound with Chebyshev nodes)

$$|f(x) - p(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) \right|$$

$$\leq \frac{1}{(n+1)!} \max_{|t| \leq 1} |f^{(n+1)}(t)|$$

$\in T_{n+1}$

$$\cdot \max_{|x| \leq 1} \left| \prod_{i=0}^n (x - x_i) \right|$$

minimized by $2^{-n} T_{n+1}$

, which is achieved by
choosing x_i 's to be the
zeros of T_{n+1} .

$$= \frac{1}{2^n (n+1)!} \max_{|t| \leq 1} |f^{(n+1)}(t)|$$

(Proof recursive formula of divided differences)

Idea: Use HW3 #4 and look at x^n term.
If g interpolates f at x_0, x_1, \dots, x_{n-1} , and
 h does at x_1, x_2, \dots, x_n , then

$$p(x) = g(x) + \frac{x_0 - x}{x_n - x_0} (g(x) - h(x))$$

Interpolates f at $x_0, x_1, \dots, x_n.$]

/* This statement itself hold even
if g and h are not polynomials.

But in our main proof, they will be
polynomial. */

Now, let $g(x)$ and $h(x)$ to be the
unique polynomials that interpolate
 x_0, x_1, \dots, x_{n-1} and x_1, x_2, \dots, x_n
respectively. That is

$$g(x) = \sum_{i=0}^{n-1} f[x_0, x_1, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j),$$

$$h(x) = \sum_{i=1}^n f[x_1, x_2, \dots, x_i] \prod_{j=1}^{i-1} (x - x_j),$$

$$P(x) = \sum_{i=0}^n f[x_0, x_1, \dots; x_i] \prod_{j=0}^{i-1} (x - x_j)$$

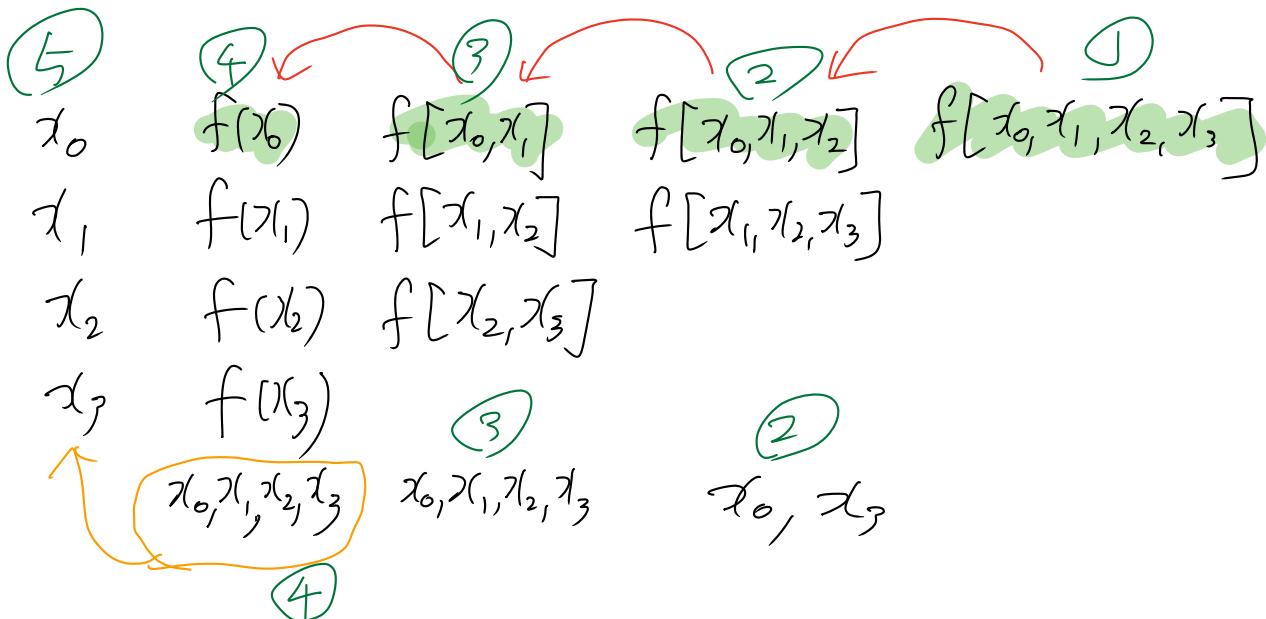
(Coeff. of x^n)

$$\text{LHS} = f[x_0, x_1, \dots; x_n]$$

$$\text{RHS} = \frac{-1}{x_n - x_0} \cdot (f[x_0, x_1, \dots, x_{n-1}]$$

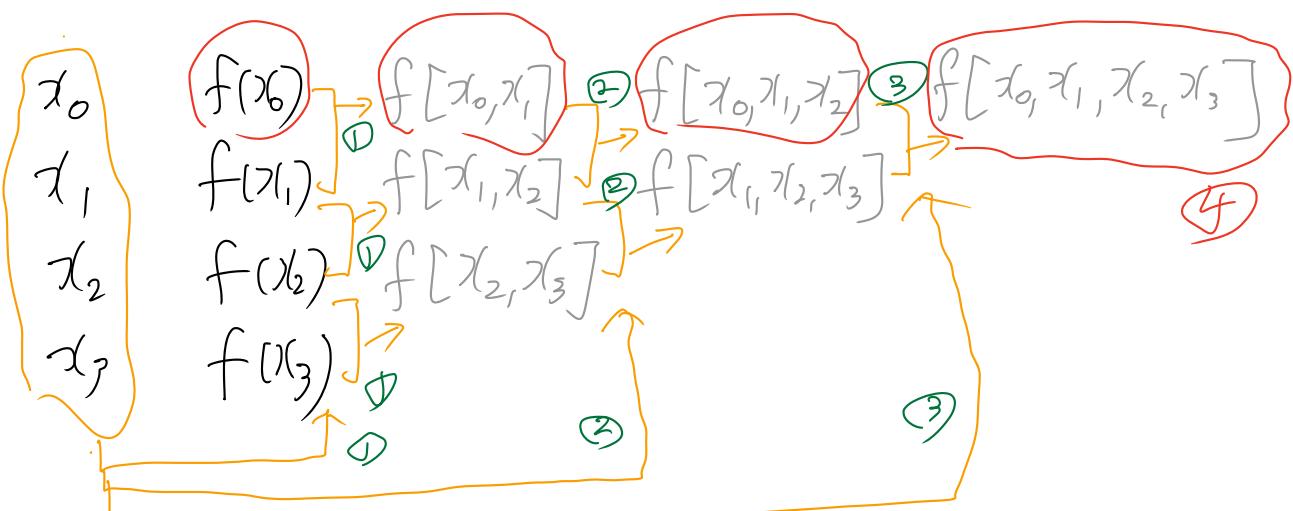
$$-f[x_1, x_2, \dots, x_n])$$

$$= \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$



① - ⑤ logical order to get interpolation
 (e.g., to get ①, we need ②. For that,
 we need ③ and so on)

The construction goes in the opposite order.



$$\begin{aligned}
 ⑤ \quad p(x) = & f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
 & + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)
 \end{aligned}$$

(example)

$$\begin{array}{ccccc} 0 & 1 & \frac{1-1}{1-0} = 0 & \frac{1-0}{2-0} = \frac{1}{2} & \frac{0-\frac{1}{2}}{3-0} = -\frac{1}{6} \\ 1 & 1 & \frac{2-1}{2-1} = 1 & \frac{1-1}{3-1} = 0 & \textcircled{3} \\ 2 & 2 & \frac{3-2}{3-2} = 1 & & \\ 3 & 3 & \frac{3-2}{3-2} = 1 & \textcircled{2} & \end{array}$$

$$P(x) = 1 + \frac{1}{2}(x)(x-1) - \frac{1}{6}(x)(x-1)(x-2)$$