⊙ HW6 is open and it covers today's topic.
Deadline is as usual, but start early for your practice.

Midterm 3 Coverage

Tue	Thu
HW5 (5/9)	HW6 (5/11)
HW6 (5/16)	HW6 (5/18)

Example:

$$h(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find $\displaystyle\lim_{(x,y)\to(0,0)} h(x,y)$.

Is this continuous at the point of interest?
Not sure.

Again, aim at DNE. → Find bad paths.

① x-axis : $h(x,0) = \dfrac{x^2 \cdot 0^2}{x^2+0^2} = 0$ (constant)
$y=0$.
$\displaystyle\lim_{x\to 0} h(x,0) = 0$

② y-axis : Similarly, $\displaystyle\lim_{y\to 0} h(0,y) = 0$.
$x=0$

③ $y = mx$ $(m \neq 0)$: $h(x, mx) = \dfrac{\cancel{x^2} \cdot m^2x^2}{\cancel{x^2} + m^2\cancel{x^2}} = \dfrac{m^2x^2}{1+m^2}$

$\longrightarrow 0$ as $x \to 0$.

④ $y = mx^2$: $h(x, mx^2) = \dfrac{\cancel{x^2} \cdot m^2 x^4}{\cancel{x^2} + m^2 x^{4\,2}}$

$= \dfrac{m^2x^4}{1+m^2x^2} \longrightarrow 0$ as $x \to 0$.

Maybe, limit exists. How can we test all possible paths? → Go back to definition.

why? candidate from previous examination or squeeze thm.

$$0 \le |h(x,y) - 0| = \left|\frac{x^2 y^2}{x^2+y^2}\right| \le \frac{(x^2+y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$$

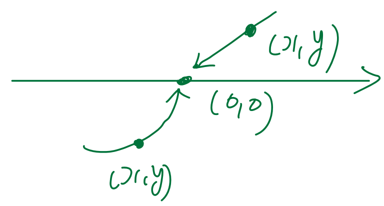
$$= x^2+y^2 \to 0 \quad \text{as } (x,y) \to (0,0)$$

$x^2 \le x^2+y^2$ and $y^2 \le x^2+y^2$

b/c $(x,y)\to(0,0) \iff (x,y)-(0,0) \to (0,0)$,

$\iff \|(x,y)-(0,0)\| \to 0$

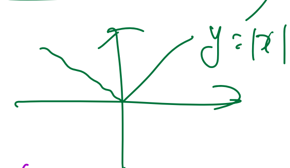
$\iff \sqrt{x^2+y^2} \to 0$



By squeeze/sandwich thm.

$$\lim_{\|(x,y)-(0,0)\|\to 0} |h(x,y)| = 0$$

$$\Rightarrow \lim_{\|(x,y)-(0,0)\|\to 0} h(x,y) = 0$$ (Why? Continuity of absolute value)



$$\left(\text{“}\lim_{\|(x,y)-(0,0)\|\to 0}\text{”} = \text{“}\lim_{(x,y)\to(0,0)}\text{”} \right)$$

Therefore, the limit exists and it is 0.

Example: $g(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

① x-axis: $\underline{y=0}$, $g(x,0) = \dfrac{x^2 \cdot 0}{x^4+0^2} = \underset{\sim}{0}$

$\Longrightarrow \lim_{x \to 0} g(x,0) = 0$ $\qquad$ $\underline{\text{constant}}$

② y-axis: $\underline{x=0}$, $g(0,y) = \dfrac{0^2 \cdot y}{0^4+y^2} = \underset{\sim}{0}$

$\Longrightarrow \lim_{y \to 0} g(0,y) = 0$

③ $y = mx$ $\quad (m \neq 0)$

$(x,y) = (x, mx)$, $g(x, mx) = \dfrac{x^2 \cdot mx}{x^4 + m^2 x^2}$

$= \dfrac{m \cancel{x^3}^{\,1}}{\cancel{x^2}(x^2+m^2)} = \dfrac{mx}{x^2+m^2} \longrightarrow \underline{0}$ as $x \to 0$.

④ $y = kx^2$ $\quad (k \neq 0)$

$(x,y) = (x, kx^2)$, $g(x, kx^2) = \dfrac{x^2 \cdot kx^2}{x^4 + k^2 x^4}$

$= \dfrac{k \cancel{x^4}}{(1+k^2)\cancel{x^4}} \longrightarrow \underline{\dfrac{k}{1+k^2}}$ as $x \to 0$.

"Limit" depends on paths. $\longrightarrow$ $\underline{\text{DNE}}$

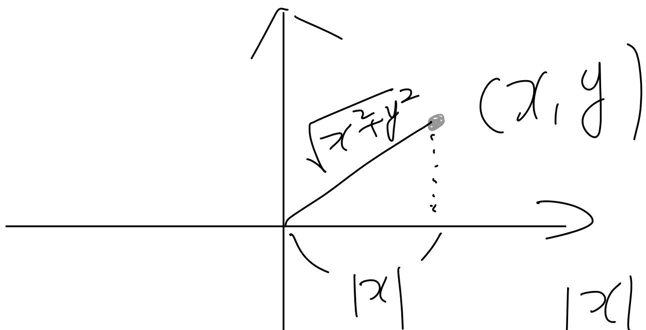
⊙ The moral of the previous examples:
2D → 1D functions bear more complexity than 1D → 1D.
We need to pay more attention to details, sometimes even when they seem minor.

⊙ How to show limits does not exist
Typically, find "bad paths"

⊙ How to show limits exist or find them
Easy case → use continuity (Limit-Conti thm)
Otherwise → Use squeez theorem on $|f(x,y) - L|$

$\boxed{\text{Clicker}}$ Choose true statement

(A) For any $x, y \in \mathbb{R}$, it is the case $|x| \le \sqrt{x^2 + y^2}$.
(B) For some $x, y \in \mathbb{R}$, the inequality may not be true.



$$|x| \leq \sqrt{x^2+y^2}$$

$$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} \quad (\text{b/c } y^2 \geq 0)$$

NT 3.3 Derivatives (Ch. 2.4)

Recall that the essenle of derivatives is a linear approximation of a function.

Question: Given $f: \mathbb{R}^2 \to \mathbb{R}^3$, say with $f(x,y) = x^2+2y^2$, guess what a derivative of f at $(1,2)$ would look like?

Hint 1 - essenle of der.

Hint 2 - building block

Hint 3 - review of linear algebra

Hint 4 - dimension of input (column vector)

$\boxed{\text{Def}}$ (Derivative) A linear mapping $L: \mathbb{R}^m \to \mathbb{R}$ is called the **derivative** of $f: \mathbb{R}^m \to \mathbb{R}$ at $\vec{a}$ if

$$\lim_{\vec{x} \to \vec{a}} \frac{|f(\vec{x}) - \overbrace{f(\vec{a}) - L(\vec{x} - \vec{a})}^{\text{— (linear approximation)}}|}{\|\vec{x} - \vec{a}\|} = 0$$

Such L may not exist. If it actually exists, we say f is **differentiable** at $\vec{x} = \vec{a}$. ("diff." for short)

/* Textbook uses a slightly different definition. The official definition in this course is the one above (in fact, it is more standard). */

/* Since derivatives stem from limits, all thier complications carry over to derivatives. In particular, it is possible that the derivative may not exist while the rate of change in a particular direction still makes sense and exists, hence the following definition */

$\boxed{\text{Def}}$ (Directional derivatives)

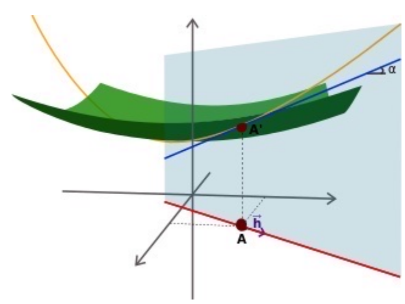
Let $\vec{d} \in \mathbb{R}^m$ be a unit vector.
Given a function $f: \mathbb{R}^m \to \mathbb{R}$ and $\vec{a} \in \mathbb{R}^m$, if $l \in \mathbb{R}$ exists such that

$$\lim_{t \to 0} \frac{f(\vec{a}+t\vec{d})-f(\vec{a})}{t} = l$$

If exists, l is called **directional derivative of f at $\vec{a}$ in the direction $\vec{d}$.**

If the $\vec{d}$ coincides with one of the standard basis vectors (e.g., $(1,0)$ or $(0,1)$ in 2D, or $\vec{i}, \vec{j}, \vec{k}$ in 3D), the derectional derivative is called **partial derivative or partial(s)**

$\boxed{\text{Clicker}}$ what is the dimension of l? (A) 1D (B) (m-1) D
(C) m D (D) (m+1) D



The above limit can be viewed as "$\underline{\text{1D}}$ derivative of f restricted to a $\underline{\text{section}}$ (along the direction $\vec{d}$)

$$\lim_{t\to 0} \frac{\left| \overbrace{f(\vec{a}+t\vec{d})-f(\vec{a})}^{\text{1D fn in } t} - t\overset{\nearrow \text{rate of change in direction } \vec{d}}{\ell} \right|}{\|t\vec{d}\|} = 0$$

$$\Longleftrightarrow \lim_{t\to 0} \left| \frac{f(\vec{a}+t\vec{d})-f(\vec{a})-t\ell}{t} \right| = 0$$

$$\Longleftrightarrow \lim_{t\to 0} \frac{f(\vec{a}+t\vec{d})-f(\vec{a})-t\ell}{t} = 0$$

$$\Longleftrightarrow \lim_{t\to 0} \frac{f(\vec{a}+t\vec{d})-f(\vec{a})}{t} - \ell = 0$$

$$\Longleftrightarrow \underbrace{\lim_{t\to 0} \frac{f(\vec{a}+t\vec{d})-f(\vec{a})}{t} = \ell}_{\text{looks more like Cal1}}$$

Recap:

- Behaviors of multivariate functions are way more diverse than in 1D setting

  → Limits/continuity can be harder to predict.

- Derivative → linear mapping that behave similarly near a point (input)

  - Der. of $f : \mathbb{R}^m \to \mathbb{R}$ is $1 \times m$ matrix

$$\lim_{\vec{x} \to \vec{a}} \frac{|f(\vec{x}) - f(\vec{a}) - L(\vec{x} - \vec{a})|}{\|\vec{x} - \vec{a}\|} = 0.$$

- Directional der. : rate of change when input changes in a certain <u>direction</u>. must be unit vector

  Otherwise, normalize it.

$$\lim_{t \to 0} \frac{f(\vec{a} + t\vec{d}) - f(\vec{a})}{t} = l \quad \text{(number)}$$

- Partial der. : directional der. in the coordinate direction.

/* A remarkable fact is if a function is "nice", derivatives in several directions determine the whole derivative. */

$\boxed{\text{Thm}}$ (Continuous partials imply differentiability)

Let $f:\mathbb{R}^m \to \mathbb{R}$ and $\vec{a} \in \mathbb{R}^m$.

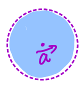
Suppose partial derivatives of f exist and continuous near $\vec{a}$, and let $L_1, L_2, \cdots, L_m$ be partial der.'s in each coordinate direction.

Then, f is diff. at $\vec{a}$. Furthermore, the linear mapping L appearing in the definition of derivative can be represented as a multiplication by $(1 \times m)$ matrix

$$[L_1\ L_2\ \cdots\ L_m]\cdot[\ \cdot\ ] : \mathbb{R}^m \to \mathbb{R}$$

where we treat input as column vectors.

/* "near $\vec{a}$" means at all points ~~of~~ an open disk centered at $\vec{a}$.



↳ no boundary */

/* Such functions are called C^1 functions. I.e.,

each partial der.'s of an m-D → 1D fn exists and continuous. */

/* (Technical comment – feel free to ignore)

C^1-fn's derivative is, in fact, "continuous". But this involves continuity in a more general setting. Thus, we don't discuss it here. */

1 Derivatives

1.1 Notation of derivatives

(Full) Derivative	$Df(\vec{a})$, $\nabla f(\vec{a})$	Derivative of f at $\vec{a}$
	$Df(\vec{a})\begin{bmatrix}2\\3\end{bmatrix}$	linear mapping $Df(\vec{a})$ applied to input $\begin{bmatrix}2\\3\end{bmatrix}$
Directional Derivative	$D_{\vec{d}}f(\vec{a})$, $\nabla_{\vec{d}}f(\vec{a})$	Directional derivative of f at $\vec{a}$ in the direction of $\vec{d}$
Partial derivative	$\frac{\partial f}{\partial x_i}(\vec{a})$, $\partial_i f(\vec{a})$, $D_i f(\vec{a})$, $\nabla_i f(\vec{a})$	Partial derivative of f at $\vec{a}$ with respect to i–th coordinate
	$\frac{\partial f}{\partial y}(x_0, y_0)$	Partial derivative of f at (x_0, y_0) with respect to y (2nd coordinate)
	$\partial_z f(1, 0, -2)$	Partial derivative of f at $(1, 0, -2)$ with respect to z (3rd coordinate)
	$D_x f(x, y, z)$	Partial derivative of f at (x, y, z) with respect to x (1st coordinate)
	$\nabla_w f(z, w)$	Partial derivative of f at (z, w) with respect to w (2nd coordinate)
	$\frac{\partial^2 f}{\partial x^2}(\vec{a})$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(\vec{a})$, $f_{xx}(\vec{a})$	2nd order partial derivative of f at $\vec{a}$ with respect to x
	$\frac{\partial^2 f}{\partial x \partial y}(\vec{a})$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(\vec{a})$, $f_{yx}(\vec{a})$	2nd order partial derivative of f at $\vec{a}$ with respect to y then x The closer to f, the earlier it applies.

Remark 1.1 (column and row vector)**.** One needs to be flexible dealing with input vectors when working in multi-dimensional settings.

- Treating inputs as column vectors leads to a unified way to deal with derivatives and linear approximations. In particular, when dealing with vector fields $f : \mathbb{R}^m \to \mathbb{R}^n$, this is considered standard since there is no other clean notation.

- When the main function is multivariate, but real-valued function, i.e. $f : \mathbb{R}^m \to \mathbb{R}$, inputs are usually treated as row vectors. In this case, the linear map $\nabla f(\vec{a})$ applied to an input $\vec{w}$ is interpreted as a dot product:

$$\underbrace{\nabla f(\vec{a})}_{1\times m}\underbrace{\vec{w}}_{m\times 1} = \nabla f(\vec{a}) \cdot \vec{w} = \partial_1 f(\vec{a})w_1 + \partial_2 f(\vec{a})w_2 \cdots + \partial_m f(\vec{a})w_m$$

  because

$$\nabla f(\vec{a}) = \begin{bmatrix}\partial_1 f(\vec{a}) & \partial_2 f(\vec{a}) & \cdots & \partial_m f(\vec{a})\end{bmatrix}, \quad \vec{w} = \begin{bmatrix}w_1\\ w_2\\ \cdots\\ w_m\end{bmatrix} \text{ (column)}, \quad \vec{w} = \begin{bmatrix}w_1 & w_2 & \cdots & w_m\end{bmatrix} \text{ (row)}.$$

$\boxed{\text{Notation}}$

- $Df(\vec{a})$: derivative of f at $\vec{a}$.
  $\nabla f(\vec{a})$ is ~~often~~ used in place of $Df(\vec{a})$ and it is called **gradient**

- $D_{\vec{d}}f(\vec{a})$: directional derivative of f (or $\nabla_{\vec{d}}f(\vec{a})$) at $\vec{a}$ in the direction $\vec{d}$

- $\frac{\partial f}{\partial x_i}(\vec{a})$, $\partial_i f(\vec{a})$, $D_i f(\vec{a})$:

  partial derivative of f at $\vec{a}$
  in the direction of i-th coordinate
  (or with respect to, w.r.t.)

- In 2D or 3D, the following are common

$f_x(x_0,y_0) = D_x f(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0, z_0)$, $\partial_z f(x_0, y_0, z_0)$

- We are going to focus on 2D, 3D and use

$$\nabla f(\vec{a}) = \left[\frac{\partial f}{\partial x}(\vec{a}), \frac{\partial f}{\partial y}(\vec{a}) \right] \quad (2D)$$

$$\nabla f(\vec{a}) = \left[\frac{\partial f}{\partial x}(\vec{a}), \frac{\partial f}{\partial y}(\vec{a}), \frac{\partial f}{\partial z}(\vec{a}) \right] \quad (3D)$$

◎ If we treat inputs as column vectors, things generalize to m-D $\to$ n-D setting (vector field).

◎ On the other hand, in m-D $\to$ 1D setting, people usually treat inputs just as a row vector. In this case, the derivative is interpreted as a dot product with $\nabla f(\vec{a})$.
But this is only a cosmetic matter

$$\underbrace{\nabla f(\vec{a})}_{(1\times m)} \underbrace{\left[\vec{x}-\vec{a}\right]}_{\substack{(m\times 1)\\ \text{column}}} = \underbrace{\nabla f(\vec{a})}_{\mathbb{R}^m} \cdot \underbrace{(\vec{x}-\vec{a})}_{\mathbb{R}^m}$$

$$= \partial_1 f(\vec{a}) \cdot (x_1 - a_1) + \partial_2 f(\vec{a})(x_2 - a_2) + \cdots + \partial_m f(\vec{a})(x_m - a_m)$$

◎ We will use higher order partial derivatives a lot.

- $$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y,z)\right)$$
  $$= \frac{\partial^2}{\partial x^2} f(x,y,z) = f_{xx}(x,y,z)$$

- Differentiate f w.r.t. x 1^{st} and then w.r.t. y.
  $$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y,z)\right)$$
  $$= \frac{\partial^2}{\partial y \partial x} f(x,y,z) = f_{xy}(x,y,z)$$

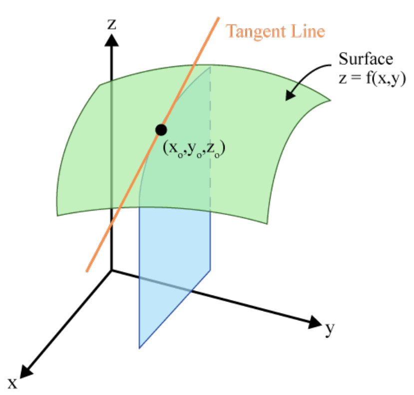
watch the order.

The closer to the function name is, the earlier we differentiate w.r.t.

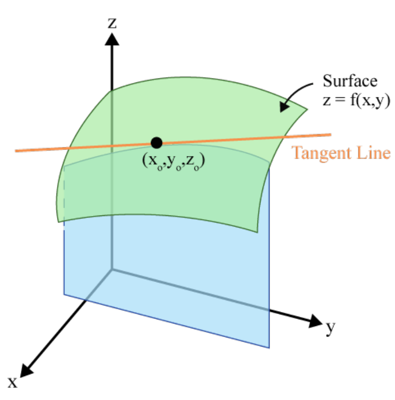
How to compute partial derivatives

To find $\partial_x f(x,y,z)$, treat y and z as constant, and find 1D derivative w.r.t. x.

① $\lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}$ → definition suggest that $\frac{\partial f}{\partial x}(x,y)$ is nothing but fix $y = y_0$ and view $f(x, y_0)$ as a fn of x. Then, its derivative (as univariate) is the partial w.r.t. x.



Slope of the surface in the x-direction



Slope of the surface in the y-direction

Calcworkshop.com

⊙ The above short cut holds only when the partials exist.

⊙ A common mistake is to

① compute partials first without checking that they exist, then

② claim that the full derivative exists b/c we have partials.

/* Advice: Shortcuts are shortcuts. If things are delicate, you need to investigate more closely, usually by going back to definition or something absolutely true. */

Example: Find the partials of

$f(x,y) = (x-3)^2 e^y$ (exist?) Yes!

Think of y as constant

$$\frac{\partial f}{\partial x}(x,y) = 2(x-3)\, e^y$$

Treat y as const.

$$D_y f(x,y) = (x-3)^2\, e^y$$

⊙ Notation

$\frac{\partial f}{\partial x}(x,y)$

↳ x v______ e______

(i.e., at what point the derivative is being investigated.)

c______ d______ w.r.t. which the derivative is taken

Confusing? → use x_0, y_0 for evaluation.