⊙ Midterm 2 → next Thursday
see Canvas frontpage

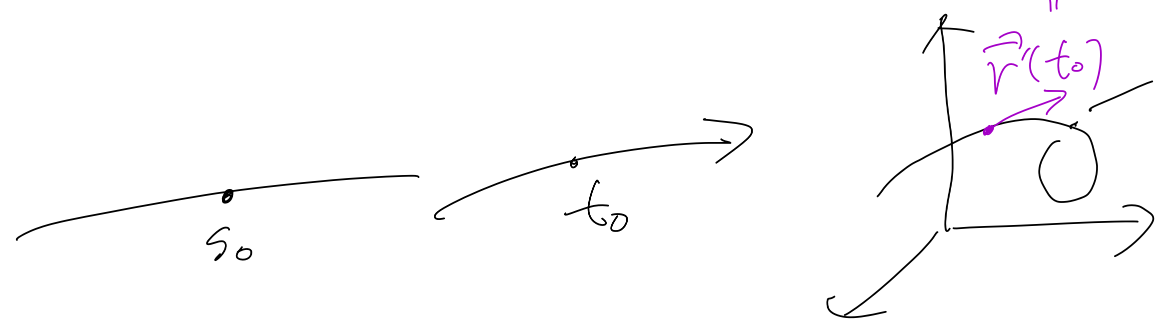
Recap:

- Chain rule

$$\vec{r}(s) = \vec{r}(t(s))$$

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ s & & t & & \vec{r} \end{array}$$

$$\frac{d}{ds}\vec{r}(s_0) = \frac{d}{dt}\vec{r}(t_0)\cdot\frac{dt}{ds}(s_0)$$



- Arc length fn

$$s(t) = \int_a^t \|\vec{r}'(z)\|\,dz \xrightarrow{\text{inverse}} t = t(s)$$

- Arc length reparametrization

$$\vec{r}(t(s)) \longrightarrow \text{unit speed.}$$

/* $[a,b] = \{a \le x \le b\}$
$(a,b] = \{a < x \le b\}$
$(a,b) = \{a < x < b\}$ */

$\boxed{\text{Def}}$ (Unit tangent vector)

Given a smooth path $\vec{r} : [a,b] \to \mathbb{R}^3$,

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$$

is called the **unit tangent vector at t.**

- $\|\vec{T}(t)\| = \left\| \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \right\| = \frac{\|\vec{r}\,'(t)\|}{\|\vec{r}\,'(t)\|} = 1$ (unit)

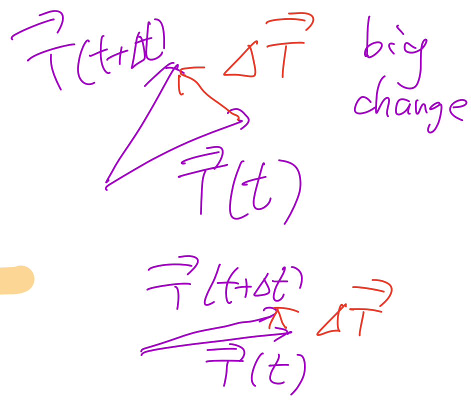
- Since $\vec{r}$ is smooth, $\|\vec{r}\,'(t)\| \neq 0$. (No problem dividing by $\|\vec{r}\,'(t)\|$.)

- If $\vec{r}$ is parametrized by arc length s, $\vec{r}\,'(s) = \underline{\vec{T}(s)}$ b/c $\|\vec{r}\,'(s)\| = 1$.

$\boxed{\text{Def}}$ (Curvature)

Let $\vec{r}(s)$ be a C^2 smooth path, where s is the arc length. Then, **curvature** is defined:

$$\boxed{\kappa(s) = \| \vec{T}'(s) \|}$$



① This exactly reflects our motivation mentioned earlier.

② WARNING

If the curve is not parametrized by arc length, this you cannot use this definition. In that case, you either

(a) reparametrize it and compute $\|\vec{T}'(s)\|$.

(b) or, use the formula below.

Thm (Curvature formula for general curves)

Let $\vec{r}(t)$ be a C^{2} smooth path
(2^{nd} der. is continuous)

(not necessarily parametrized by arc length)

Then, its curvature at t is

$$\boxed{\kappa(t) = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{\|\vec{v}(t)\|^3}}$$

where $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$.

Clicker: What curves would you test this formula for sanity?

Question: Does this formula physically make sense?

(proof of curvature formula)

$$\|\vec{T}'(s)\|^2 = \left\|\frac{d}{ds}\vec{T}(s)\right\|^2$$

$s = s(t)$

$$= \left\|\frac{d}{dt}\vec{T}(t)\cdot\frac{dt}{ds}\right\|^2$$

chain rule.

$dt/ds = \|\vec{r}'(t)\|^{-1}$

product rule to $(\vec{r}' \cdot \|\vec{r}'\|^{-1})$

$$= \frac{1}{\|\vec{r}'(t)\|^2}\left\|\frac{d}{dt}\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\right\|^2$$

$\vec{r}'(t) = \vec{v}(t)$

$\vec{r}''(t) = \vec{a}(t)$

every input is 't' so omit it

$$= \frac{1}{\|\vec{v}\|^2}\left\|\frac{\vec{a}\cdot\|\vec{v}\| - \vec{v}\,\frac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}}{\|\vec{v}\|^2}\right\|^2$$

Exercise: prove

$$\left(\frac{\vec{c}(t)}{\|\vec{c}(t)\|}\right)' = \frac{1}{\|\vec{c}(t)\|^2}\cdot\left(\vec{c}'(t)\|\vec{c}(t)\| - \vec{c}(t)\cdot\frac{(\vec{c}(t)\cdot\vec{c}'(t))}{\|\vec{c}(t)\|}\right)$$

$$= \frac{1}{\|\vec{v}\|^{\cancel{8}\,6}}\underbrace{\left\|\vec{a}\cdot\|\vec{v}\|^2 - \vec{v}\,(\vec{v}\cdot\vec{a})\right\|^2}_{\|\vec{v}\|^{\cancel{4}\,2}\|a\|^2 + (\vec{v}\cdot\vec{a})^2\cancel{\|\vec{v}\|^2} - 2\cancel{\|\vec{v}\|^2}(\vec{v}\cdot\vec{a})^2}$$

Exercise: $\longrightarrow$

$$= \frac{1}{\|\vec{v}\|^6}\left(\|\vec{v}\|^2\|\vec{a}\|^2 - (\vec{v}\cdot\vec{a})^2\right)$$

prove

$\|\vec{v}\|^2\|\vec{u}\|^2 = (\vec{v}\cdot\vec{u})^2 + \|\vec{v}\times\vec{u}\|^2$

$$= \frac{\|\vec{v}\times\vec{a}\|^2}{\|\vec{v}\|^6}$$

The result follows by taking square root. $\blacksquare$

Example: Predict the curvature and compare it with the one found by definition and the formula.

(A) $\vec{r}(t) = \vec{p} + t\vec{d}$, where $\vec{p}, \vec{d}$ are fixed.

① prediction: $\underline{0}$ (b/c it is a $\underline{\text{line}}$)

② Definition:

$$\vec{r}'(t) = \vec{d} \quad \left(r'(t) = \left((p_1 + td_1)', (p_2 + td_2)', (p_3 + td_3)' \right) = (d_1, d_2, d_3) = \vec{d} \right)$$

$$s = \int_0^t \|\vec{r}'(\tau)\| \, d\tau \quad \text{(Let's make starting point } 0\text{.)}$$

$$= \int_0^t \|\vec{d}\| \, d\tau = \|\vec{d}\| t$$

$$t = \frac{s}{\|\vec{d}\|}$$

$$\vec{r}(s) = \vec{p} + \frac{s}{\|\vec{d}\|} \vec{d}$$

$$\vec{r}'(s) = \frac{\vec{d}}{\|\vec{d}\|} = \vec{T}(s)$$

$$k(s) = \|\vec{T}'(s)\| = \|\vec{0}\| = 0. \quad \left(\frac{\vec{d}}{\|\vec{d}\|} \text{ is } \underline{\text{constant}} \right).$$

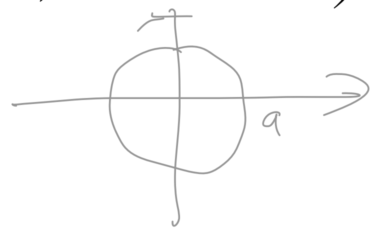
③ formula: $\vec{v}(t) = \vec{r}'(t) = \vec{d}$, $\vec{a}(t) = \vec{v}'(t) = \vec{0}$

$$k(t) = \frac{\|\vec{a}(t) \times \vec{v}(t)\|}{\|\vec{v}(t)\|^3} = 0.$$

(Example continued) $\nearrow$ circle with radius a

(B) $\vec{r}(t) = (a\cos t, a\sin t)$ $(a>0)$

at $t = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.



① prediction: constant.

② formula: $\vec{v}(t) = (-a\sin t, a\cos t)$

$\vec{a}(t) = \vec{v}'(t) = (-a\cos t, -a\sin t)$

$$\|\vec{a}(t)\times\vec{v}(t)\| = \operatorname{abs}\left(\begin{vmatrix} -a\cos t & -a\sin t \\ -a\sin t & a\cos t \end{vmatrix}\right)$$

(see below)

$$= \operatorname{abs}\left(-a\begin{vmatrix} \cos t & \sin t \\ -a\sin t & a\cos t \end{vmatrix}\right)$$

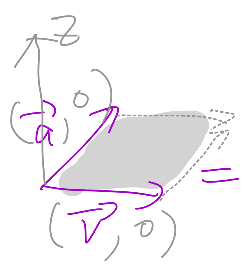
linearity of determinant in each row.

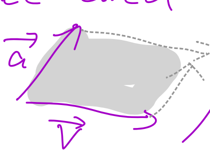
$$= \operatorname{abs}\left(-a^2 \underbrace{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}}_{=1}\right)$$

$$= a^2.$$

$$\|\vec{v}(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a$$

$$k(t) = \frac{a^2}{(a)^3} = \frac{1}{a}$$

$\|\vec{a}\times\vec{v}\|$ = area of 

= abs (signed area of )

$$= \operatorname{abs}\left(\begin{vmatrix} \vec{a} \\ \vec{v} \end{vmatrix}\right)$$

NT. 2.7 Principle unit normal vector and osculating circle

Motivation

When we say "instantaneous velocity is 55 mi/hr now," we are implicitly comparing with a constant velocity:
"we are driving as fast as a vehicle at a constant rate that travels 55 miles in an hour."
Likewise, when we study curvatures of a curve, it is very natural to compare it with another shape that has a constant curvature.

$\boxed{\text{Clicker}}$ What is that shape?
Discuss and type in (open).

$\boxed{\text{circle.}}$

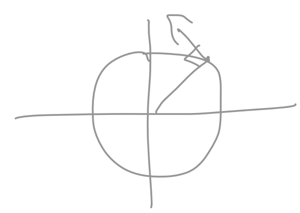
Thm Any smooth paths of constant length is orthogonal to its deriative. That is, if

$$\boxed{\|\vec{r}(t)\| = \underbrace{\text{const}}_{=C} \implies \vec{r}(t)\cdot\vec{r}\,'(t) = 0.}$$

(proof) /* $\vec{r}(t)$ must live in a sphere */

$C^2 = \|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t)$. Take $\frac{d}{dt}$

Then, $0 = \cancel{2}\,\vec{r}\,'(t)\cdot\vec{r}(t)$



Cor $\boxed{\vec{T}(t)\cdot\vec{T}\,'(t) = 0.}$

Def (**principal unit normal**)

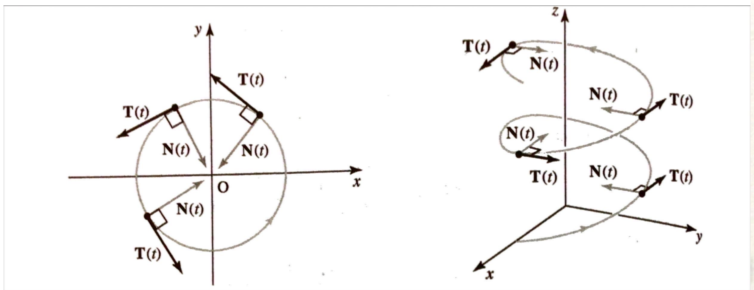
Given a smooth curve with a parametrization $\vec{r}(t)$, the (**pricipal**) **unit normal** (**vector**) is defined

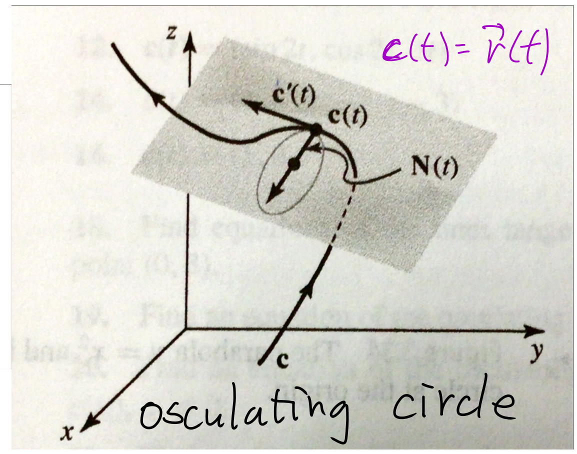
$$\boxed{\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}},$$

where $\vec{T}(t)$ is the unit tangent vector.

⊙ $\vec{N}$ tells us in what direction the curve is bent "in the current plane."
(Curves "instantaneously live" in a plane)

⊙ For circles, $\vec{N}(t)$ points to its center.

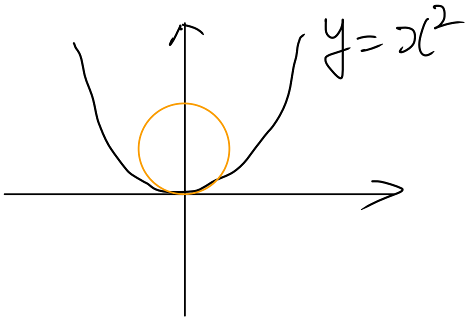




$\boxed{\text{Def}}$ (**Osculating plane and circle**)

Let $\vec{r}$ be a C^2 smooth path with $\vec{r}''(t) \neq \vec{0}$. The plane spanned by $\vec{T}(t)$ and $\vec{N}(t)$ passing through $\vec{r}(t)$ is called **osculating plane**. And the circle on that plane with raduis $1/\kappa(t)$ that is tangent to the curve at $\vec{r}(t)$ and its center is on the side of $\vec{N}(t)$ is called **osculating circle**.

Example: Find the largest radius of a wheel that can roll smoothly inside the parabola $y=x^2$ in 2D world.



The curvature of the wheel must be larger than anywhere on the parabola.

Find the largest curvature. x y

Vector form: $\vec{r}(t) = (t, t^2)$

$\vec{v}(t) = \vec{r}\,'(t) = (1, 2t)$

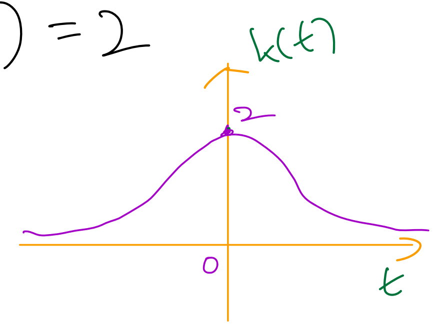
$\vec{a}(t) = \vec{v}\,'(t) = (0, 2)$

$\|\vec{v}(t)\| = \sqrt{1+(2t)^2} = \sqrt{1+4t^2}$

$\|\vec{a}(t) \times \vec{v}(t)\| = \text{abs}\left(\begin{vmatrix} 0 & 2 \\ 1 & 2t \end{vmatrix} \right)$

$= \text{abs}(-2) = 2$

$k(t) = \dfrac{2}{(\sqrt{1+4t^2})^3} = \dfrac{2}{(1+4t^2)^{3/2}}$



(based on "Example circle")

↳ Thus, the radius of wheel must be less than $\frac{1}{2}$.

NT 2.8 Applications to kinematics

Goal : Better understanding of movements of a particle in a general setting.

/* All physical applications of calculus so far under very limited circumstances. (e.g., movement in 1D space, or 2D movement under a constant acceleration — gravity, etc.) */

Learning objective:

After this lecture, I can decompose the acceleration of a given movement $\vec{r}(t)$ into tangential ($\vec{a}_T$) and normal ($\vec{a}_N$) components.

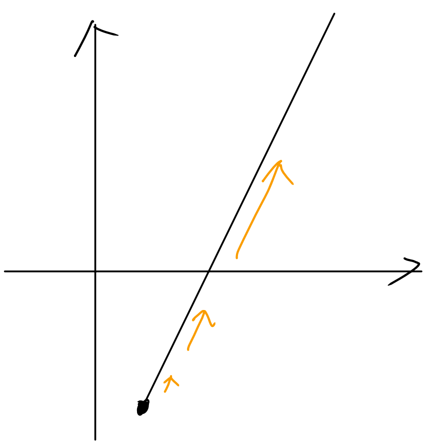
Motivating examples: Two extreme movements (2D for simplicity)

Example 1:

$$\vec{r}_1(t) = (1+t^3, 2t^3-3)$$

Notice that this is actually a line

$$\begin{cases} x = 1+t^3 \\ y = 2t^3-3 \end{cases} \qquad t^3 = x-1 \qquad y = 2(x-1)-3 = 2x-5$$



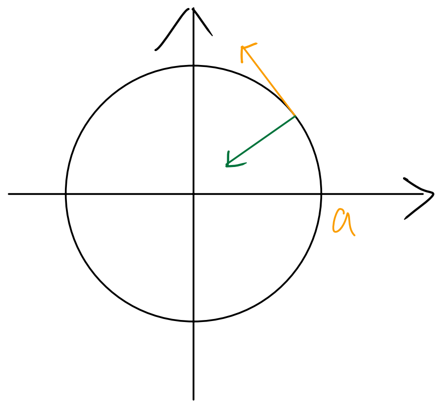
$$\vec{v}_1(t) = \vec{r}_1{}'(t) = (3t^2, 6t^2)$$
$$= 3t^2(1,2)$$
$$\vec{a}_1(t) = \vec{r}_1{}''(t) = (6t, 12t)$$
$$= 6t(1,2)$$

Acceleration acts in the same direction of the velocity. $\longrightarrow$ move straight

Example 2 :

$$\vec{r_2}(t) = (a\cos bt,\ a\sin bt)$$

$(a, b > 0)$



$$\vec{v_2}(t) = \vec{r_2}'(t)$$
$$= (-ab\sin bt,\ ab\cos bt)$$
$$\vec{a_2}(t) = \vec{r_2}''(t)$$
$$= (-ab^2\cos bt,\ -ab^2\sin bt)$$

$$\vec{V_2}(t) \cdot \vec{a_2}(t) = +a^2b^3\sin bt\cos bt - a^2b^3\cos bt\sin bt$$
$$= 0$$

Acceleration $\perp$ velocity all the time

$\Rightarrow \mathrm{proj}_{\vec{v}}\vec{a} = \vec{0}$

$\Rightarrow$ acceleration never changes the speed, but only changes the direction of the movement.

⊙ In general, movements in real world possess features of both kinds.

Idea: Isn't it going to be helpful if we can decompose the acceleration into direction of velocity and one perpendicular to it?