(proof of convergence of bisection method; slides p.6)

Chop the task into pieces.

① $\lim a_n$, $\lim b_n$, $\lim c_n$ exist and they all the same.

② Call the limit, ξ, then $f(\xi)=0$.

③ $|c_n - \xi| < 2^{-(n+1)}(b-a)$

---

① Observe $\underbrace{a_0 \le a_1 \le a_2 \le}_{\text{nondecreasing}} \cdots \underbrace{\le b}_{\text{bdd above}}$ by construction. Therefore $\lim a_n$ exists.

/* Math 3B — monotone sequence theorem.

If $\{a_n\}$ is nondecreasing (or nonincreasing) and bounded above (or bounded below), the limit exists. [This is "half-version"]

If $\{a_n\}$ is monotonic (i.e., only nonincreasing or only nondecreasing) and bounded (i.e., bounded from above and below), it converges. [This is "two-sided-ver"] */

Likewise $b_0 \ge b_1 \ge b_2 \ge \cdots \ge a$. Therefore b_n also converges.

Let $\lim a_n = \xi_1$ and $\lim b_n = \xi_2$.
We know the length of $[a_n, b_n]$ gets halved from the construction. Therefore, $b_n - a_n \to 0$ as $n \to \infty$. Then, we must have

$$0 = \lim(b_n - a_n) = \lim b_n - \lim a_n$$
$$= \xi_2 - \xi_1$$

$\Longrightarrow \xi_1 = \xi_2$ Call this common limit ξ.
Lastly for ①, we have $a_n \leq c_n \leq b_n$.
Therefore, sandwich theorem says

$$\underset{\downarrow \atop \xi}{\lim a_n} \leq \underbrace{\lim c_n}_{\downarrow \atop \xi} \leq \underset{\downarrow \atop \xi}{\lim b_n}$$

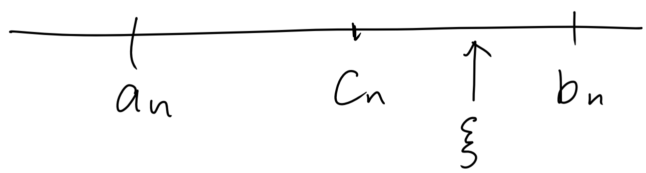
Thus, $\lim c_n = \xi$.

(2) Taking limit of $f(a_n)f(b_n) \lesssim 0$, we have $f(\xi)f(\xi) \leq 0$. The only possibility is $f(\xi) = 0$.

/* Here, continuity of f is used.

f conti at $a \iff f(a_n) \to f(a)$ $\forall \{a_n\}$ s.t. $a_n \to a$

(3) Observe $\xi \in [a_n, b_n]$ $\forall n$. Thus,



$$
\begin{aligned}
|c_n - \xi| &< \frac{1}{2} \cdot \text{length}([a_n, b_n]) \\
&= \frac{1}{2} \cdot \frac{1}{2} \text{length}([a_{n-1}, b_{n-1}]) \\
&\vdots \\
&= \left(\frac{1}{2}\right)^n \text{length}([a_1, b_1]) \\
&= \left(\frac{1}{2}\right)^{n+1} \underbrace{\text{length}([a_0, b_0])}_{= b_0 - a_0 = b - a}
\end{aligned}
$$

(proof of quadratic conv. of Newton's method.)

Put $e_n = x_n - \xi$.

Subtract ξ from the method and sneak in $f(\xi)$

$$x_{n+1} - \xi = x_n - \xi - \frac{f(x_n) - f(\xi)}{f'(x_n)}$$

$$(*) \begin{cases} e_{n+1} = \dfrac{1}{f'(x_n)}\left(f(\xi) - f(x_n) - f'(x_n)(\xi - x_n) \right) \\ \left(f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \dfrac{f''(c_n)}{2!}(\xi - x_n)^2, \quad c_n \in (\xi, x_n) \text{ or } (x_n, \xi) \right) \\ = \dfrac{1}{f'(x_n)} \cdot \dfrac{f''(c_n)}{2} e_n^2 \qquad \text{as } e_n \to 0 \end{cases}$$

Therefore, IF $e_n \to 0$ as $n \to \infty$, then $x_n \to \xi$, and in turn, $f'(x_n) \to f'(\xi)$ and $f''(c_n) \to f''(\xi)$ as $n \to \infty$.

Then, dividing the error eqn $(*)$ by e_n^2 and taking limit $e_n \to 0$, we have

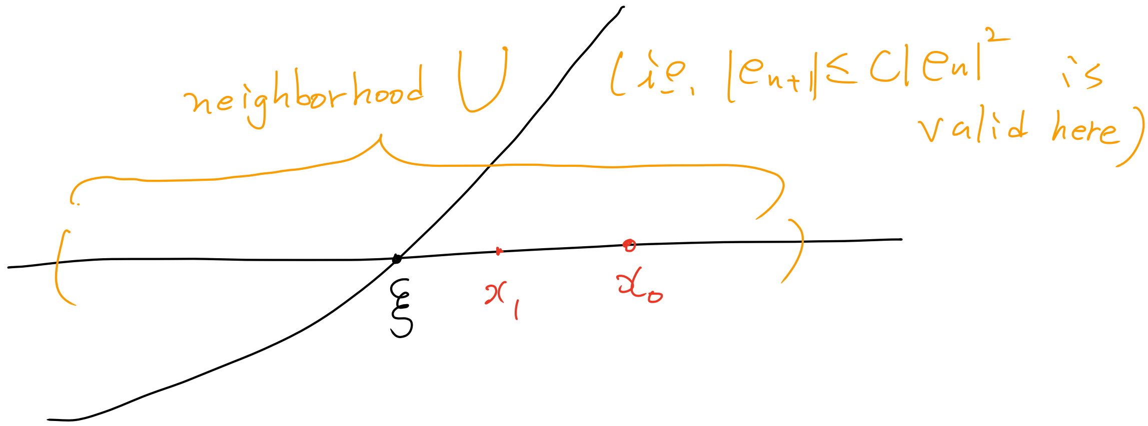
$$\frac{e_{n+1}}{e_n^2} \longrightarrow \frac{f''(\xi)}{2f'(\xi)} \quad \text{as } e_n \to 0.$$

That is,

$$\left| \frac{e_{n+1}}{e_n^2} \right| \leq C \quad \text{if } e_n \simeq 0. \text{ Or roughly } "|e_{n+1}| \simeq C|e_n|^2"$$

↳ just a generic constant as long as it is fixed.

Now, we prove the IFF part.



Let $|e_{n+1}| \le C|e_n|^2$ on U

Choose x_0 so that $|e_0| = |x_0 - \xi| < \frac{1}{2C}$ and $x_0 \in U$.

/* This is where "if the initial guess is sufficiently close to the zero" comes in. */

Then, we see

$$|e_1| \le C|e_0|^2 \le C \cdot \frac{1}{2C} \cdot |e_0| = \frac{1}{2}|e_0|$$

Therefore, $x_1 \in U$ too.

Repeat this so that

$$|e_2| \le C|e_1|^2 \le C \cdot \left(\frac{1}{2}\right)^2 |e_0| \cdot |e_0|$$

$$\le C \cdot \left(\frac{1}{2}\right)^2 \frac{1}{2C} \cdot |e_0| = \left(\frac{1}{2}\right)^3 |e_0|$$

$$\cdots \quad |e_n| \le \left(\frac{1}{2}\right)^{2n-1} |e_0|$$

Thus, $e_n \to 0$ as $n \to \infty$. $\blacksquare$

$\boxed{\text{Thm}}$ (Taylor thm)

If $f \in C^{n+1}$ near a point x (i.e. on $(x-\delta, x+\delta)$),
then for any $y \in (x-\delta, x+\delta)$, we have

(Lagrange remainder ver.)

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(y-x)^n + \frac{f^{(n+1)}(\xi_y)}{(n+1)!}(y-x)^{n+1} \quad (\xi_y \in (x,y) \text{ or } (y,x))$$

and

(Integral remainder ver.)

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2!}(y-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(y-x)^n + \int_x^y \frac{f^{(n+1)}(t)}{n!} \cdot (y-t)^n \, dt$$

(proof of Taylor theorem with Lagrange remainder)

If $y=x$, there is nothing to prove. So, $y \neq x$.

Set $M = \left(f(y) - T_n(y;x)\right) / (y-x)^{n+1}$

so that $f(y) - T_n(y;x) - M(y-x)^{n+1} = 0$.

We want: there is $\xi_y \in (x,y)$ s.t. $f^{(n+1)}(\xi_y) = (n+1)!M$

Introduce

$$g(t) = f(t) - T_n(t;x) - M(t-x)^{n+1}$$

Note that $f(x) = T_n(x;x)$, $f'(x) = T_n'(x;x)$, $\ldots$, $f^{(n)}(x) = T_n^{(n)}(x;x)$. Therefore,

$g(x) = g'(x) = g''(x) = \cdots g^{(n)}(x) = 0$ since $M(t-x)^{n+1}$ has zero at x of order $n+1$.

By construction of M, we have $g(y)=0$

Apply MVT (mean value theorem) to g. Then, there is $\xi_1 \in (x,y)$ s.t. $g'(\xi_1)=0$. Apply MVT to g' on $[x,\xi_1]$. Then, there is $\xi_2 \in (x,\xi_1)$ s.t. $g''(\xi_2)=0$. Repeat this so that there is $\xi_n \in (x,\xi_{n-1})$ s.t. $g^{(n)}(\xi_n)=0$.

Repeat once more to have $\xi_{n+1} = \xi_y \in (x,\xi_n) \subset (x,y)$ s.t. $g^{(n+1)}(\xi_y)=0$. That is $f^{(n+1)}(\xi_y) = M(n+1)!$ ∎

(proof of global conv. of Newton's method for convex fn's)

Since f is increasing and has a zero by the assumption, zero is unique.

Since $f \in C^2$ and convex, $f''(x) \geq 0$ $\forall x \in \mathbb{R}$. Also, $f'(x) > 0$ $\forall x \in \mathbb{R}$ sinle f is increasing.

Now, recall the error equation (*)

$$e_{n+1} = \frac{1}{f'(x_n)}\left(f(\xi) - f(x_n) - f'(x_n)(\xi - x_n) \right)$$

$$= \underbrace{\frac{1}{f'(x_n)} \cdot \frac{f''(c_n)}{2}}_{\text{positive}} e_n^2$$

Deduce $e_{n+1} = x_{n+1} - \xi \geq 0$ no matter $e_n \geq 0$ or not. $\Longrightarrow$ $e_1, e_2, e_3, \cdots \geq 0$.

This, in turn, yield $f(x_n) \geq f(\xi) = 0$ $(n = 1, 2, \cdots)$

since f increasing: $x_n \geq \xi \Rightarrow f(x_n) \geq f(\xi)$

From $\underbrace{x_{n+1} - \xi}_{e_{n+1}} = \underbrace{x_n - \xi}_{e_n} - \underbrace{\frac{f(x_n)}{f'(x_n)}}_{\text{positive}}$,

We have $e_{n+1} \leq e_n$ for $n = 1, 2, \cdots$

Again, monotone sequence theorem says $\{e_n\}$ converges. But we don't know the limit. Call it e.

$e_{n+1} \leq e_n$ also implies

$x_n \geq x_{n+1} \geq \cdots \geq \xi$. The same theorem applies so that $\lim x_n$ exists. Call it z. Taking $n \to \infty$ in the following

$$\underbrace{x_{n+1} - \xi}_{e_{n+1}} = \underbrace{x_n - \xi}_{e_n} - \underbrace{\frac{f(x_n)}{f'(x_n)}}_{\text{positive}}$$

$$\downarrow$$

$$\cancel{e} = \cancel{e} - \frac{f(z)}{f'(z)} \Rightarrow f(z) = 0$$

That is $z = \xi$, the unique zero.