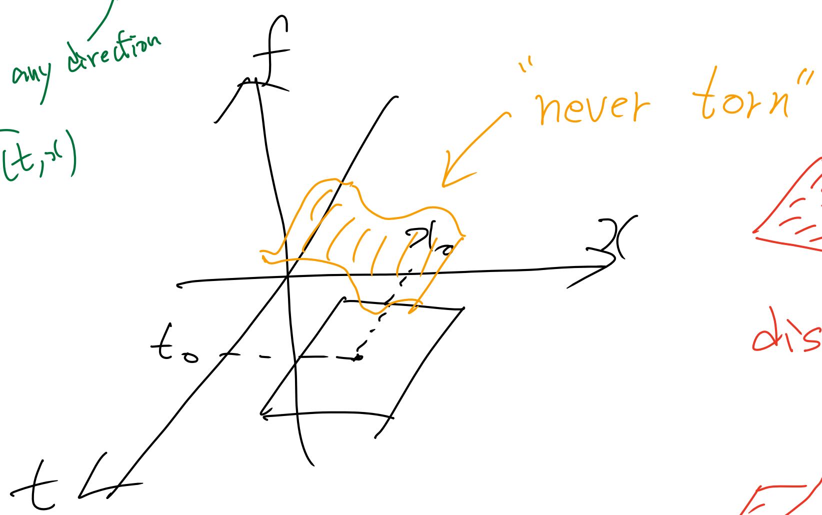
(example) $x'(t) = x^{\frac{2}{3}}$.

This corresponds to $f(t,x) = x^{\frac{2}{3}}$

f is continuous (as a function of two variables).

/* $\displaystyle\lim_{(\tau,z)\to(t,x)} f(\tau,z) = f(t,x)$

any direction



*/

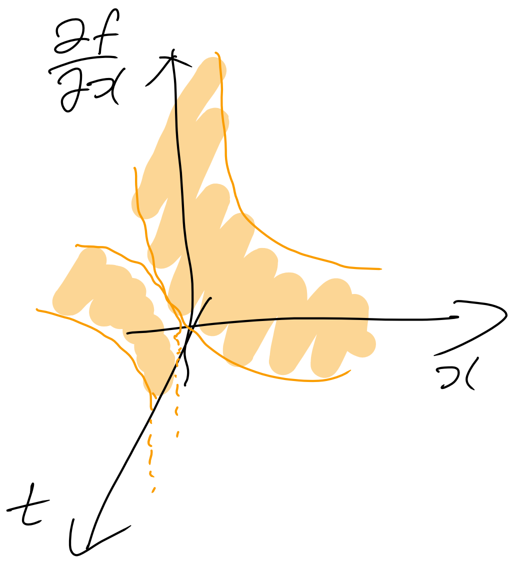
(existence)

Therefore by the theorem, a function $x(t)$ on a small open interval of t $(-\delta,\delta)$ s.t. $x'(t) = x^{\frac{2}{3}}$ and $x(0) = 0$.

For example, $x(t) \equiv 0$: $x'(t) = 0 = 0^{\frac{2}{3}}$

(uniqueness)

However, $\frac{\partial f}{\partial x}(t,x) = \frac{2}{3}x^{-\frac{1}{3}}$ is not continuous on any rectangle around $(t_0, x_0) = (0,0)$.



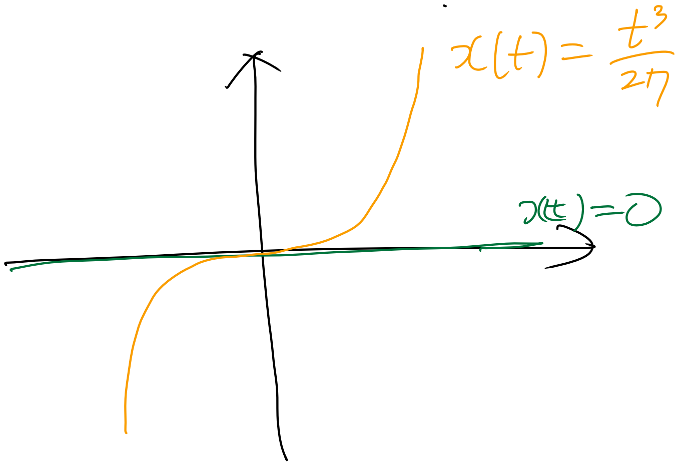
Thus, the theorem does not guarantee uniqueness. In fact, we have another solution.

For example, use the ansatz

$$x(t) = at^b$$

$$\left.\begin{aligned} x'(t) &= abt^{b-1} \\ (x(t))^{\frac{2}{3}} &= a^{\frac{2}{3}} t^{\frac{2}{3}b} \end{aligned}\right] \Rightarrow \begin{aligned} &b-1 = \tfrac{2}{3}b \Rightarrow b = 3 \\ &ab = a^{\frac{2}{3}} \Rightarrow a^3 b^3 = a^2 \\ &\Rightarrow a = \frac{1}{b^3} = \frac{1}{27}. \end{aligned}$$

This also satisfies the IC
(initial condition).

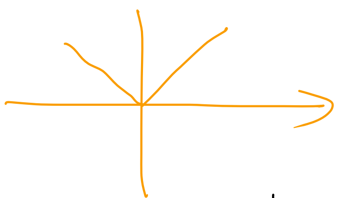


$f(x) = x^2 \in C^1[-1,1]$

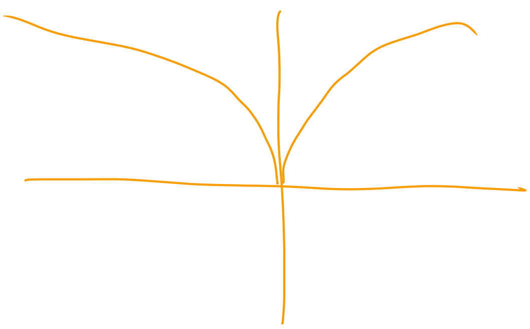
$g(x) = |x| \in Lip[-1,1]$ but $g \notin C^1[-1,1]$

$|g(x)-g(y)| = \big||x|-|y|\big| \le |x-y| \quad L=1.$

inverse triangle ineq.



$h(x) = |x|^{\frac{1}{2}} \in C[-1,1]$ but $h \notin Lip[-1,1]$.



$$|h(x) - h(y)| = \left| |x|^{\frac{1}{2}} - |y|^{\frac{1}{2}} \right|$$

$$= |\delta - 2\delta| = \delta$$

Choose $x = \delta^2$, $h = 4\delta^2$, then $|x-y| = 3\delta^2$

For any $L>0$, if $\delta < \frac{1}{3L}$

$$|h(x)-h(y)| = \delta \frac{3L}{3L} > 3L\delta^2 = L|x-y|.$$

Example : Refresh calculus.

$$\frac{d}{dt} f(x(t), y(t))$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & f(x,y) \\ & y \arrow[ur] & \end{tikzcd}$$

$$= f_x(x(t), y(t))\, x'(t)$$
$$+ f_y(x(t), y(t))\, y'(t)$$

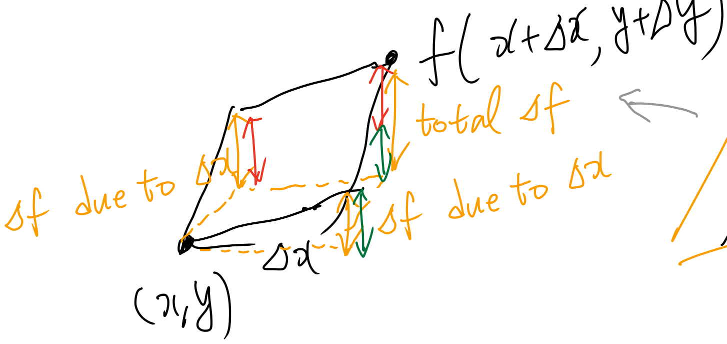
$$\begin{tikzcd} & \Delta x \arrow[dr] & \\ \Delta t \arrow[ur] \arrow[dr] & & \Delta f \\ & \Delta y \arrow[ur] & \end{tikzcd}$$

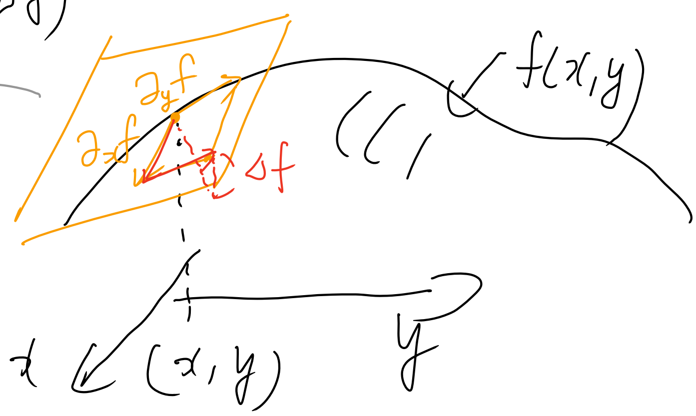
$\Delta x \simeq x' \cdot \Delta t$

$\Delta y \simeq y' \cdot \Delta t$

$\Delta f \simeq \partial_t f \cdot \Delta y + \partial_x f \cdot \Delta x$

$$\Longrightarrow \frac{\Delta f}{\Delta t} \simeq \frac{\partial_y f \cdot \Delta y}{\Delta t} + \frac{\partial_x f \cdot \Delta x}{\Delta t}$$
$$= \partial_y f \cdot y' + \partial_x f \cdot x'$$





$$x'' = \big(f(t,x)\big)' = \underbrace{f_t(t,x) + f_x(t,x)\, x'}$$

$$x''' = \big(f_t(t,x) + f_x(t,x)\, x'\big)'$$
$$= f_{tt}(t,x) + f_{tx}(t,x)\, x'$$
$$+ \big(f_{xt}(t,x) + f_{xx}(t,x)\, x'\big)\, x' + f_x(t,x)\, x''$$

$$x''(t) = -\sin t + 2t - \cos x \cdot x'$$

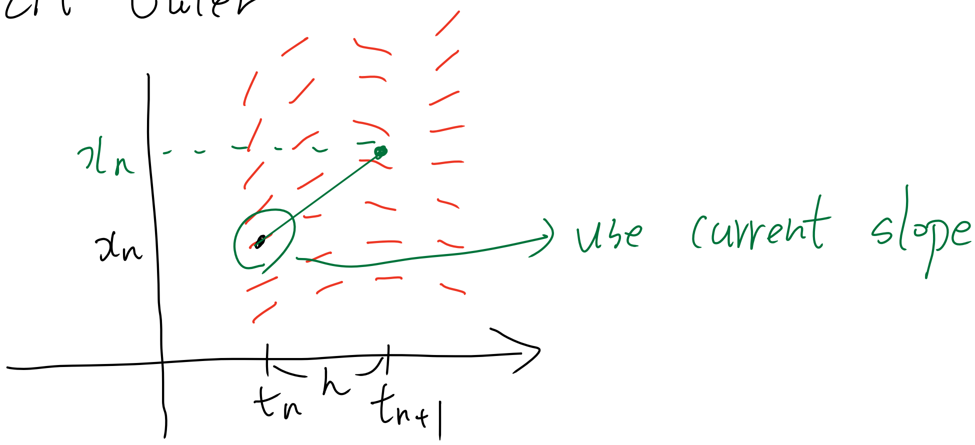
$$x'''(t) = -\cos t + 2 - \cos x \cdot x'' + (\sin x \cdot x') \cdot x'$$

$$= -\cos t + 2 - \cos x \cdot x'' + \sin x \cdot (x')^2$$

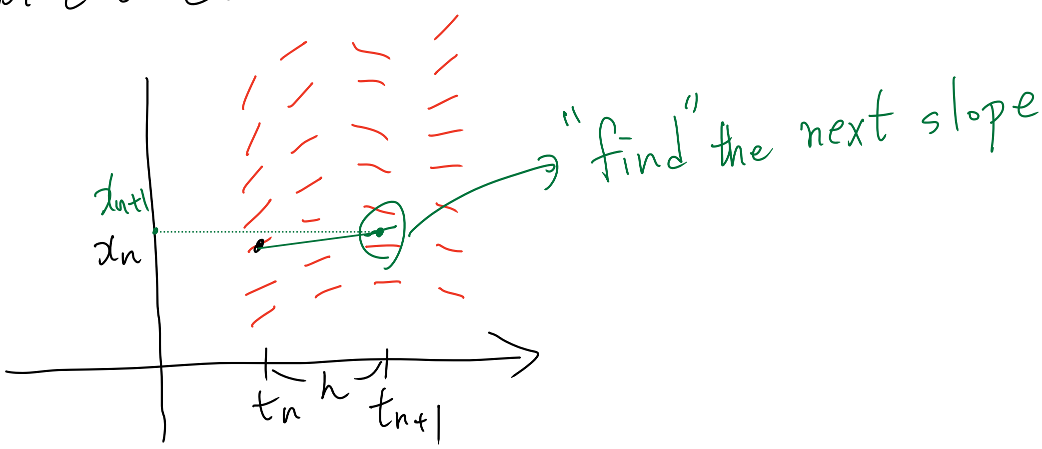
$$x''''(t) = \sin t + \sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot x' \cdot (x')^2 + \sin x \cdot 2(x')x''$$

$$= \sin t + 3\sin x \cdot x' \cdot x'' - \cos x \cdot x''' + \cos x \cdot (x')^3$$

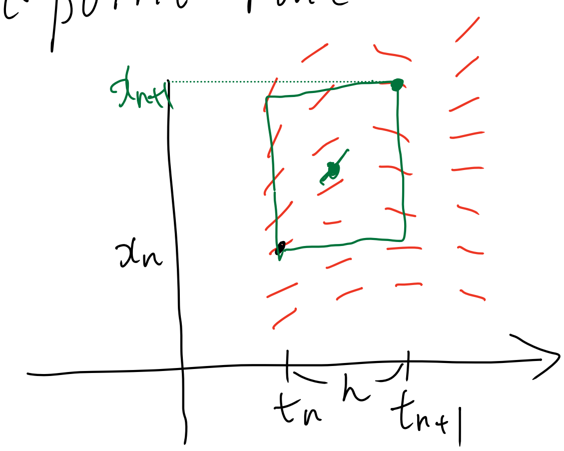
Explicit Euler



Implicit Euler

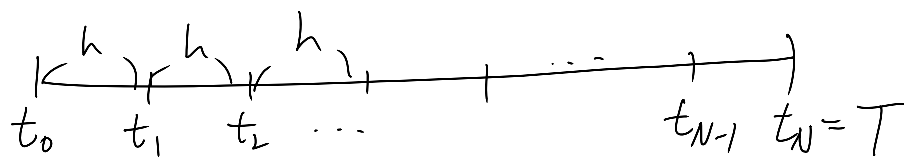


midpoint rule



Global truncation error.

Suppose local truncation error is $O(h^{n+1})$.
Then, global error accumulates over the time steps.



$$h = \frac{T - t_0}{N} \Rightarrow N = \frac{T - t_0}{h}$$

$$\sum_{i=1}^{N} O(h^{n+1}) = N \cdot O(h^{n+1}) = \frac{T - t_0}{h} O(h^{n+1})$$

$$= \underbrace{(T - t_0)}_{\text{fixed.}} O(h^{n})$$

$$= O(h^{n})$$

Derivation of 2^{nd} order RK method

1. Truncate 3^{rd} or higher order terms.
2. $x'(t) = f(t, x(t))$ no problem.

$$x''(t) = f_t(t, x(t)) + f_x(t, x(t))\, x'(t)$$

$$= f_t(t, x(t)) + f_x(t, x(t))\, f(t, x(t))$$

Thus,

$$x(t+h) \approx x + hf + \frac{h^2}{2}\left(\underline{f_t + f_x \cdot f}\right)$$

3. $f(t+h, \underbrace{x+hf}_{\text{explicit Euler prediction of } 2^{nd} \text{ slot}})$

$$= \underline{f + f_t h + f_x \cdot hf}$$

4. $x(t+h) \approx x + hf + \frac{h}{2}\left(f(t+h, x+hf) - f\right)$

$$= x + \frac{1}{2}\underbrace{hf}_{F_1} + \underbrace{\frac{1}{2}hf(t+h, x+\underbrace{hf}_{F_1})}_{F_2}$$

$$= x + \frac{1}{2}(F_1 + F_2)$$

where $F_1 = hf$

$F_2 = hf(t+h, x+F_1)$

/* Taylor theorem in two variables

$$f(x+\Delta x, y+\Delta y) = f(x,y) \quad \text{— 0-th order appr.}$$

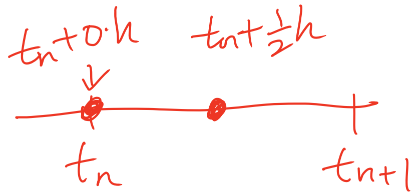
$$+ f_x(x,y)\cdot\Delta x + f_y(x,y)\Delta y \quad \text{— linear appr.}$$

$$\left.\begin{aligned} &+ \frac{1}{2}\Big(f_{xx}(\xi_x,\xi_y)\Delta x^2 + 2 f_{xy}(\xi_x,\xi_y)\Delta x\Delta y \\ &\quad + f_{yy}(\xi_x,\xi_y)\Delta y^2\Big), \end{aligned}\right\} \text{quadratic appr.}$$

where (ξ_x, ξ_y) is a point on the segment connecting (x,y) and $(x+\Delta x, x+\Delta y)$.

Recover modified Euler from tableau

nodes of stages



$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & 0 & 1
\end{array}
$$

← weights for stage 1

← weights for stage 2.

← main weight for advance

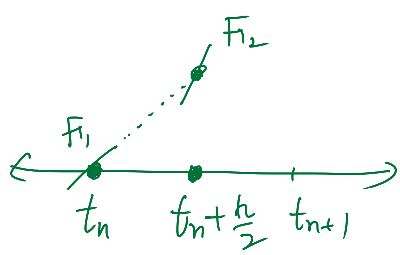
$$x_{n+1} = x_n + h\,(\underbrace{0 \cdot F_1 + 1 F_1}_{})$$

Slope from stage 1

$$F_1 = f(t_n + 0 \cdot h,\; x_n + h(0 \cdot F_1 + 0 \cdot F_2))$$

Slope from stage 2

$$F_2 = f(t_n + \tfrac{1}{2} \cdot h,\; x_n + h(\tfrac{1}{2} F_1 + 0 \cdot F_2))$$



$$x_{n+1} = x_n + h \cdot f(t_n + \tfrac{h}{2},\; x_n + \tfrac{h}{2} f(t_n, x_n))$$

Butcher's tableau for RK4.

stages : t, $t+\frac{1}{2}h$, $t+\frac{1}{2}h$, $t+h$

(can be repeated) $\rightarrow 0, \frac{1}{2}, \frac{1}{2}, 1$

weights for advance : $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$

inner weights for stage1 : $0, 0, 0, 0$

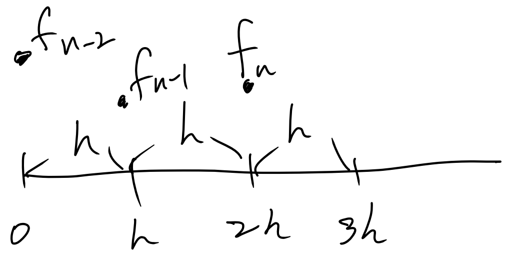
inner weights for stage2 : $\frac{1}{2}, 0, 0, 0$

inner weights for stage3 : $0, \frac{1}{2}, 0, 0$

inner weights for stage4 : $0, 0, 1, 0$

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

(Verifying AB3)



$$b_0 = \int_{2h}^{3h} \frac{(t-h)(t-2h)}{(0-h)(0-2h)}\,dt$$

$$s = t-2h \qquad dt = ds$$

$$= \frac{1}{2h^2}\int_0^h \underbrace{(s+h)s}_{s^2+hs}\,ds$$

$$= \frac{1}{2h^2}\left[\frac{s^3}{3} + h\frac{s^2}{2}\right]_0^h$$

$$= \frac{1}{2h^2}\left[\frac{h^3}{3} + \frac{h^3}{2}\right]$$

$$= h\cdot\frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}h$$

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t))\, dt$$

$$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t)\, dt$$ $p(t)$ interpolates

$(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}) (t_n, f_n)$ 3 nodes ($m+1$) nodes with $m=2$

Since we assume x_{n-2}, x_{n-1}, x_n are exact,

e.g., $f_{n-2} = f(t_{n-2}, x(t_{n-2})) = \underbrace{f(t_{n-2}, x_{n-2})}$ computable.

Thus,

$$|x(t_{n+1}) - x_{n+1}| \leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)|\, dt$$

$$\leq \int_{t_n}^{t_{n+1}} \frac{1}{3!} \left| \frac{d^3}{dt^3} f(t, x(t)) \right| \prod_{i=0}^{2} \underbrace{|t - t_{n-i}|}_{\leq 3h} dt$$

$$\leq \frac{3^3}{3!} M \cdot h^3 \int_{t_n}^{t_n+h} dt$$

(where $M = \max_{t_0 \leq t \leq T} \left| \frac{d^3}{dt^3} f(t, x(t)) \right|$)

$$= C h^4$$

(where $C = \frac{3^3 M}{3!}$. Important thing is this is independent of h

order of 3-step AB method.

$$x(t_{n+1}) = \int_{t_n}^{t_{n+1}} f(t, x(t))\, dt$$

$$x_{n+1} = \int_{t_n}^{t_{n+1}} p(t)\, dt$$ $p(t)$ interpolates

$(t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n), (t_{n+1}, f_{n+1})$

4 nodes $(m+1)$ nodes with $m=3$

Thus,

$$|x(t_{n+1}) - x_{n+1}| \leq \int_{t_n}^{t_{n+1}} |f(t, x(t)) - p(t)|\, dt$$

$$\leq \int_{t_n}^{t_{n+1}} \frac{1}{4!} \left| \frac{d^4}{dt^4} f(t, x(t)) \right| \prod_{i=0}^{3} \underbrace{|t - t_{n-i+1}|}_{\leq 3h} dt$$

$$\leq \frac{4^4}{4!} M \cdot h^4 \int_{t_n}^{t_n+h} dt$$

(where $M = \max_{t_0 \leq t \leq T} \left| \frac{d^4}{dt^4} f(t, x(t)) \right|$)

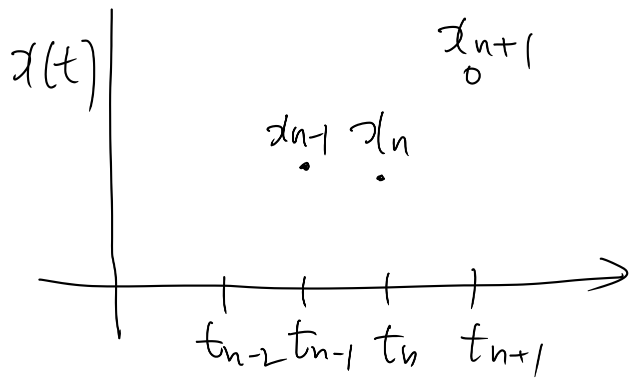
$$= C h^5$$

(where $C = \frac{4^4 M}{4!}$ is independent of h

/* This proof is not rigorous because we don't have $x_{n+1} = x(t_{n+1})$ in practice.

Therefore our error formula from the interpolation theory is not quite true. Nonetheless, there are other approaches that rigorously provide the same conclusion. */

Derivation of BDF2 (instead of BDF3)



$$p(t) = x_{n-1} \cdot l_0(t) + x_n \cdot l_1(t) + x_{n+1} \cdot l_2(t)$$

$$l_0(t) = \frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} = \frac{1}{2h^2}(t-t_n)(t-t_{n+1})$$

$$l_1(t) = \frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} = -\frac{1}{h^2}(t-t_{n-1})(t-t_{n+1})$$

$$l_2(t) = \frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} = \frac{1}{2h^2}(t-t_{n-1})(t-t_n)$$

$$l_0'(t) = \frac{1}{2h^2}(t-t_n+t-t_{n+1})$$

$$l_1'(t) = -\frac{1}{h^2}(t-t_{n-1}+t-t_{n+1})$$

$$l_2'(t) = \frac{1}{2h^2}(t-t_{n-1}+t-t_n)$$

$$p'(t_{n+1}) = x_{n-1}\, l_0'(t_{n+1}) + x_n\, l_1'(t_{n+1}) + x_{n+1}\, l_2'(t_{n+1})$$

$$= x_{n-1} \cdot \frac{1}{2h} + x_n \cdot \left(-\frac{2}{h}\right) + x_{n+1} \cdot \frac{3}{2h}$$

$$\approx f_{n+1}$$

Divide through by $\frac{3}{2h}$ (to solve for x_{n+1}), then

$$x_{n+1} - \frac{4}{3} x_n + \frac{1}{3} x_{n-1} = \frac{2}{3} h f_{n+1}$$

(order condition of LMM)

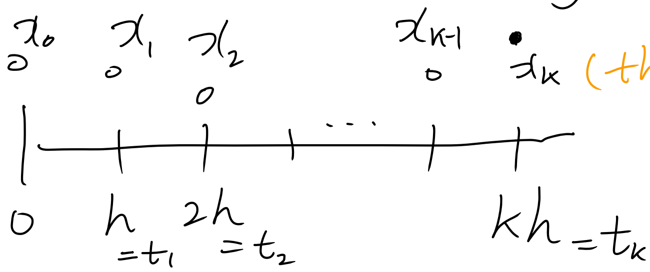
Suppose the LTE is of order $m+1$

That is, if all the history is exact,

$$|x_{n+1} - x(t_{n+1})| = O(h^{m+1}).$$

For ease of notation, assume $t_0 = 0$.

This does not bring any limitation.



(this corresponds to x_{n+1} the "next position")

Have (exact history) $x_0 = x(0)$, $x_1 = x(h)$, $\cdots$, $x_{k-1} = x((k-1)h)$

and (order m) $|x_k - x(kh)| = O(h^{m+1})$

Want : $d_0 = d_1 = \cdots = d_m = 0$

Assuming enough smoothness of the true solution, for each $i = 0, 1, \cdots, k$, Taylor expansion of $x(t)$ and $x'(t)$ at $t=0$ read :

$$x(ih) = x(0) + (ih) \cdot x'(0) + \cdots + \frac{(ih)^j}{j!} x^{(j)}(0) + \cdots$$

$$x'(ih) = x'(0) + (ih)\cdot x''(0) + \cdots + \frac{(ih)^j}{j!} x^{(j+1)}(0) + \cdots$$
$$= f(t_i, x_i)$$

Then, the error operator reads

$$L[x] = \sum_{i=0}^{k} \left(a_i x(t_i) - h b_i x'(t_i)\right)$$
$$= \sum_{i=0}^{k} \left(a_i x(0)\right)$$
$$+ \sum_{i=0}^{k} (i a_i - b_i) x'(0) h$$
$$+ \sum_{i=0}^{k} \left(\frac{i^2}{2!} a_i - i b_i\right) x''(0) h^2$$
$$+ \sum_{i=0}^{k} \left(\frac{i^3}{3!} a_i - \frac{i^2}{2!} b_i\right) x'''(0) h^3$$
$$+ \cdots + \sum_{i=0}^{k} \left(\frac{i^j}{j!} a_i - \frac{i^{(j-1)}}{(j-1)!} b_i\right) x^{(j)}(0) h^j$$
$$+ \cdots$$

(*) —— $= d_0 + d_1 h + d_2 h^2 + \cdots$

On the other hand, we have, by definition of the method,

$(**)$ $\sum_{i=0}^{k} (a_i x_i - h b_i f_i) = 0$.

Subtract $(**) - (*)$ and use the exactness for $i = 0, 1, \cdots, k-1$, then we arrive at

$$a_k (x_k - x(t_k)) - h b_i (\underbrace{f_k}_{= f(t_k, x_k)} - \underbrace{x'(t_k)}_{= f(t_k, x(t_k))})$$

$(***)$ $= d_0 + d_1 h + d_2 h^2 + \cdots$

Using MVT,

$$f_k - x'(t_k) = f(t_k, x_k) - f(t_k, x(t_k))$$
$$= f_x(\xi) \cdot (x_k - x(t_k))$$

Plug this back into $(***)$ to get

$(\#)$ $(x_k - x(t_k)) \cdot (a_k - h b_i f_x(\xi)) = d_0 + d_1 h + \cdots$

$$\Rightarrow x_k - x(t_k) = \frac{1}{\underbrace{a_k - h b_k f_x(\xi)}_{\simeq a_k \text{ since we are}}} (d_0 + d_1 h + d_2 h^2 + \cdots)$$

considering $h \to 0$. In particular, the whole fraction $\simeq a_k^{-1}$ since $a_k \neq 0$

$$\simeq a_k^{-1}(d_0 + d_1 h + d_2 h^2 + \cdots)$$

Therefore, if LTE is of order $m+1$

$$d_0 = d_1 = \cdots = d_m = 0.$$

Conversely, if $d_0 = d_1 = \cdots = d_m = 0$, the error ~~of~~ the next step is of order $m+1$:

$$x_k - x(t_k) \simeq a_k^{-1} d_{m+1} h^{m+1} + \cdots$$

(***)

$$= O(h^{m+1}).$$

(Equivalence of 1 and 3)

If 3 is true, we can apply (#) to the true solution $x(t)$ and repeat the argument to (***) have 1.

Assume 1 is true, that is, LTE of the the method is of order $m+1$ for all IVP with smooth enough f. For any smooth function y, it is the solution to

$$\begin{cases} x'(t) = f(t, x(t)) = y'(t) \\ x(0) = y(0) \end{cases}$$

That is, $x(t) = y(t)$, $t \geq 0$.

Thus, by repeating the argument from (*) to (#), we have

$$L[y] = \sum_{i=0}^{k} \left(a_i y(t_i) - h b_i y'(t_i)\right)$$

$$= \sum_{i=0}^{k} \left(a_i x(t_i) - h b_i x'(t_i)\right)$$

$$= \left(x_k - x(t_k)\right) \cdot \left(a_k - h b_i \underbrace{f_x(\xi)}_{=0} \right)$$

since $f(t, x) = y'(t)$ depends only on t.

$$= O(h^{m+1})$$ (by assumed order of method)

Name Consistency.

Easier to see the reason for the name in Adams family. Suppose LTE of

(*) $$x_{n+1} = x_n + h\left(Af_{n+1} + Bf_n + Cf_{n-1}\right)$$

is of order $m+1$ with $m \geq 1$, then

(**) — $$\frac{x_{n+1} - x_n}{h} = Af_{n+1} + Bf_n + Cf_{n-1}$$

If we plug in the true solution

$$\underbrace{\frac{x(t_{n+1}) - x(t_n)}{h}}_{①} = Af(t_{n+1}, x(t_{n+1})) + Bf(t_n, x(t_n)) + Cf(t_{n-1}, x(t_{n-1})) + O(h^m)$$

$$= \underbrace{Ax'(t_{n+1}) + Bx'(t_n) + Cx'(t_{n-1})}_{②} + \underbrace{O(h^m)}_{③}$$

As $h \to 0$, ① $\to x'(t)$ and ③ $\to 0$.

Therefore, we must have ② $\to x'(t) = f(t, x(t))$

This means (**) is a proper discretization of $x' = f(t, x(t))$, hence (*) too, since those calculations approach the ODE that it tries to solve as $h \to 0$.

consistency $\Longleftrightarrow$ $p(1)=0$ and $p'(1)=q(1)$

$$0 = p(1) = \sum_{i=0}^{k} a_i = d_0$$

$$p'(x) - q(x) = \sum_{i=0}^{k} \left(i a_i x^{i-1} - b_i x^i\right)$$

Therefore, if $p'(1)=q(1)$, then

$$0 = p'(1) - q(1) = \sum_{i=0}^{k} (i a_i - b_i) = d_1$$

Therefore, by the order condition, the method is at least of order 1.

BDF2 is stable.

$$p(x) = \frac{1}{3}(3x^2 - 4x + 1)$$

$$= \frac{1}{3}(3x-1)(x-1)$$

$$\Rightarrow x = 1, \frac{1}{3} \qquad \text{root condition.}$$

BDF3 is stable.

$$p(x) = \frac{1}{11}(11x^3 - 18x^2 + 9x - 2)$$

Sinle a reasonable method is consistent, we can guess $p(1) = 0$.

$$
\begin{array}{r|l}
 & 11x^2 - 7x + 2 \\
\hline
x-1 & 11x^3 - 18x^2 + 9x - 2 \\
 & 11x^3 - 11x^2 \\
\hline
 & \quad -7x^2 + 9x \\
 & \quad -7x^2 + 7x \\
\hline
 & \qquad 2x - 2 \\
 & \qquad 2x - 2 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Thus $p(x) = \frac{1}{11}(x-1)\underbrace{(11x^2-7x+2)}$

determinant

$49-88<0$

complex root.

Hence the two roots are complex conjugate.

$x = 1, z, \bar{z}$

But

$|z| = |\bar{z}|$

$= (z\bar{z})^{\frac{1}{2}} = \left(\frac{2}{11}\right)^{\frac{1}{2}} < 1.$

Thus, satisfies root condition.

$$p(z) = \sum_{i=0}^{k} a_i z^i$$
$$q(z) = \sum_{i=0}^{k} b_i z^i$$

$$d_0 = \sum_{i=0}^{k} a_i = p(1)$$

=0 when $i=0$ b/c this comes from a_k.

$$p'(1) - q(1) = \sum_{i=0}^{k} \left(i a_i z^{i-1} - b_i z^i \right)_{z=1}$$
$$= \sum_{i=0}^{k} \left(i a_i - b_i \right)$$
$$= d_1$$

Therefore, by the order condition,

$d_0 = d_1 = 0 \implies$ LTE of the method is $O(h^m)$ $m \geq 2$

$\underset{p(1)}{\underset{\|}{d_0}} = \underset{p'(1)-q(1)}{\underset{\|}{d_1}}$

$\implies$ method is $O(h^m)$ $m \geq 1$.

Convergence of explicit Euler method: $x_{n+1} = x_n + hf(t_n, x_n)$

$$x_{n+1} - x_n = h(f_n)$$

1-step method with

$$a_1 = 1, \quad a_0 = -1, \quad b_1 = 0, \quad b_0 = 1$$

$$p(z) = z - 1, \quad q(z) = 1$$

(consistency)

$$p(1) = 1 - 1 = 0,$$

$$p'(1) = 1 = q(1) = 1$$

(stability)

$$p(z) = 0 \iff z = 1$$ root cond.

By Dahlquist equivalence theorem, the method converges.

AB3 is convergent.

(stability)

$$x_{n+1} - x_n = \frac{h}{12}(23 f_n - 16 f_{n-1} + 5 f_{n-2})$$

$$p(x) = x^3 - x^2 = x^2(x-1) \Rightarrow x=1,\ x=0.$$

satisfies root condition.

(consistency)

$$p(1) = 1^3 - 1^2 = 0$$

$$q(x) = (23x^2 - 16x + 5)/12$$

$$p'(1) = 3 \cdot 1^2 - 2 \cdot 1 = 1$$

$$q(1) = \frac{1}{12}(\underbrace{23-16+5}_{12}) = 1.$$

Therefore, by the equivalence thm,

AB3 is convergent.