Announcement

General Question : Discord $\longrightarrow$ Canvas - [Discussions]

Announcement : Discord $\longrightarrow$ Canvas - [Annoucements]

Recap :

- $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$
- $\text{proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v}$
- $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

len : $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

dir : right hand rule + $\vec{u} \times \vec{v} \perp \vec{u}, \vec{v}$

⊙ Application of dot and cross product.

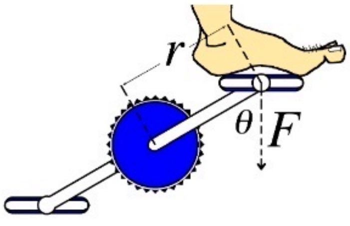
$\vec{a}\cdot\vec{b}$ and $\vec{a}\times\vec{b}$ are motivated more theoretically than intuitively.
(Think of how radians make Calculus easier than degrees do for trig functions.) So, their direct applications are not quite tangible.

- $\int \vec{F}\cdot d\vec{r} \approx$ "$\sum \vec{F}\cdot\Delta\vec{r}$" → work
- "$\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$"

  → allows us to generalize Pythagorian theorem to high dimensions (even ∞-dimension)

  → Flourier series
- Torque ("rotational force")

  $\vec{\tau}=\vec{r}\times\vec{F}$



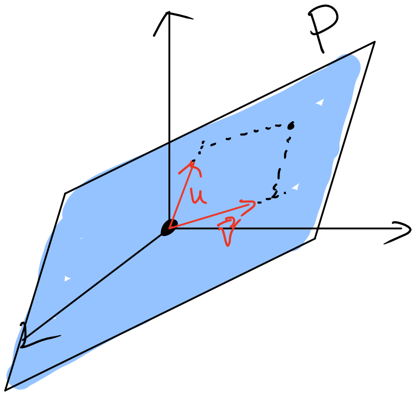
- Maxwell equations

$$\nabla\cdot E=\frac{\rho}{\varepsilon_0},\quad \nabla\cdot B=0$$
$$\nabla\times E=-\frac{\partial B}{\partial t},\quad \nabla\times B=\mu_0\left(J+\varepsilon_0\frac{\partial E}{\partial t}\right)$$

NT 1.14 Equations of planes

(A) parametric form (Ch. 1.2 pp 12-13)

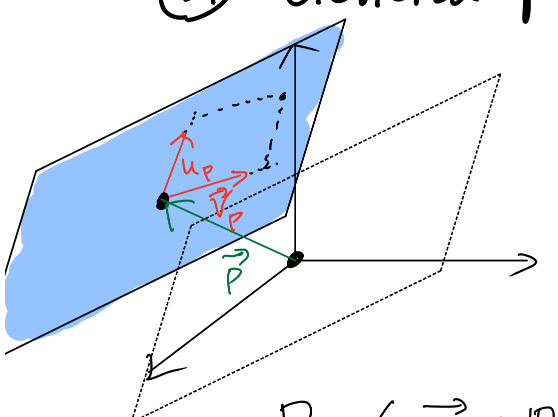
① planes through the origin



Given non-zero and non-parallel 3D vectors $\vec{u}, \vec{v}$ lying in a plane P (i.e., their tails and heads are in P; tail = origin), P can be characterized by collection of all points reachable by some steps in the direction of $\vec{u}$ and $\vec{v}$:

$$P = \{ \underbrace{\vec{a} \in \mathbb{R}^3}_{\text{format}} : \underbrace{\vec{a} = t\vec{u} + s\vec{v},\ t, s \in \mathbb{R}}_{\text{further descriptions (usually on variables)}} \}$$

② General planes



Translation of a plane containing the origin.

$$P = \{ \vec{w} \in \mathbb{R}^3 : \vec{w} = \vec{p} + t\vec{u} + s\vec{v},\ t, s \in \mathbb{R} \}$$

③ We can also write the components

$\boxed{\text{Clicker}}$

Supposing $\vec{u} = (1,2,2)$, $\vec{v} = (0,3,1)$, $\vec{p} = (3,-2,3)$, find a parametric expression of the translation by $\vec{p}$ of the plane spanned by $\vec{u}$ and $\vec{v}$

(A) $\vec{p} + t\vec{u}$ $\qquad t, s, k \in \mathbb{R}$.

(B) $\vec{p} + t\vec{u} + s\vec{v}$

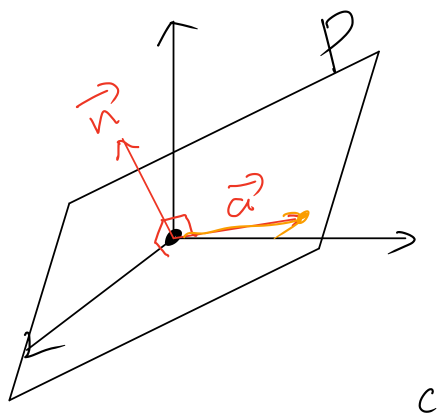
(C) $t\vec{p} + s\vec{u} + k\vec{v}$

$= (3,-2,3) + (t, 2t, 2t) + (0, 3s, s)$

$= (3+t, -2+2t+3s, 3+2t+s)$

(B) Vector equation (ch. 1.3 ; pp. 27–28)

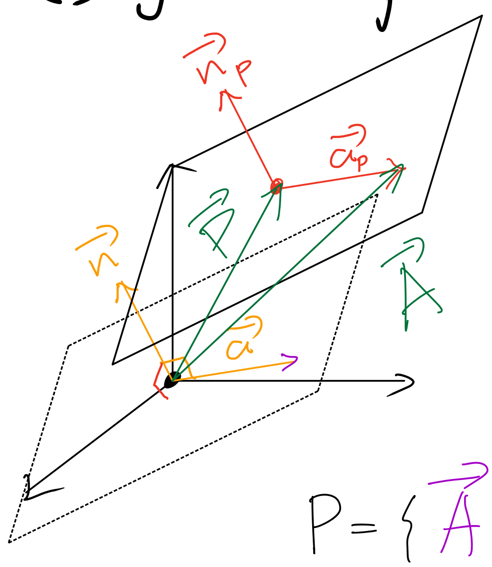
① planes through the origin



Any point in a plane P passing through the origin can be characterized by a collection of points whose corresponding position vectors are orthogonal to a fixed vector $\vec{n}$, called **normal vector**.

$$P = \{ \vec{a} \in \mathbb{R}^3 : \vec{a} \cdot \vec{n} = 0 \}$$

② general planes



Do the same but consider translation: relative position in ref. to $\vec{p}$ is perpendicular to $\vec{n}$

$$P = \{ \vec{A} \in \mathbb{R}^3 : (\vec{A} - \vec{p}) \cdot \vec{n} = 0 \}$$

/* $(\vec{A}-\vec{p})\cdot\vec{n}=0$ is reminiscent of "Translation of $f(x,y)=0$ by a in x direction and b in y direction is $f(x-a,\,y-b)=0$ */

(C) **Equation of a plane** : writing out the orthogonality condition using coordinates, one ends up with: $\vec{p}=(x_0,y_0,z_0)$, $\vec{n}=(a,b,c)$

$$(x-x_0,\ y-y_0,\ z-z_0)\cdot(a,b,c)=0$$

$\vec{A}=(x,y,z)$

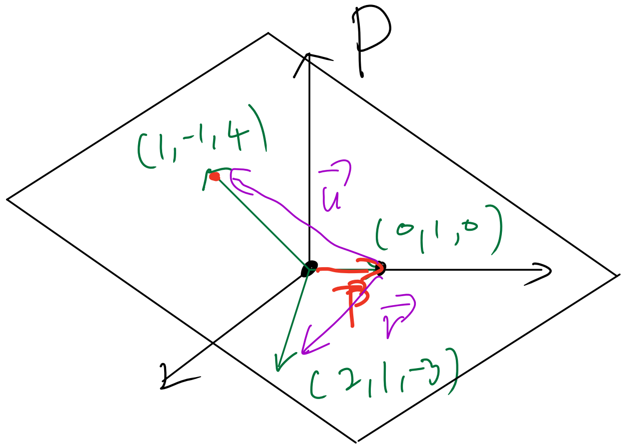
$$ax-\underline{ax_0}+by-\underline{by_0}+cz-\underline{cz_0}=0$$

$$\boxed{ax+by+cz=d}$$

eqn of a plane

$(d=ax_0+by_0+cz_0)$

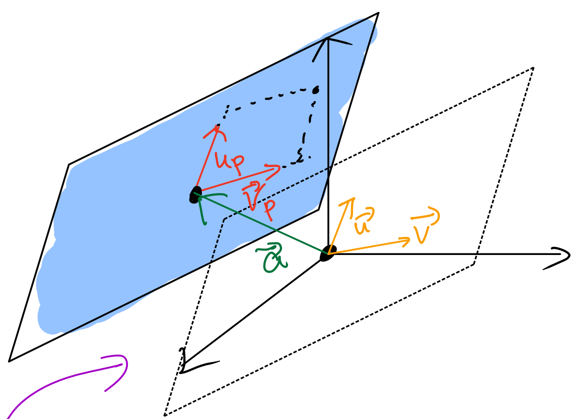
Example: Find the plane that contains $(0,1,0)$, $(2,1,-3)$, $(1,-1,4)$ in parametric form.



$\vec{u} = (1,-1,4)-(0,1,0)$
$= (1,-2,4)$

$\vec{v} = (2,1,-3)-(0,1,0)$
$= (2,0,-3)$

$\vec{p} = (0,1,0)$



Plan: ① Find a parallel one passing $\vec{0}$ (think of $\vec{p}$ as $\vec{0}$)

② Translatate it by $\vec{p}$

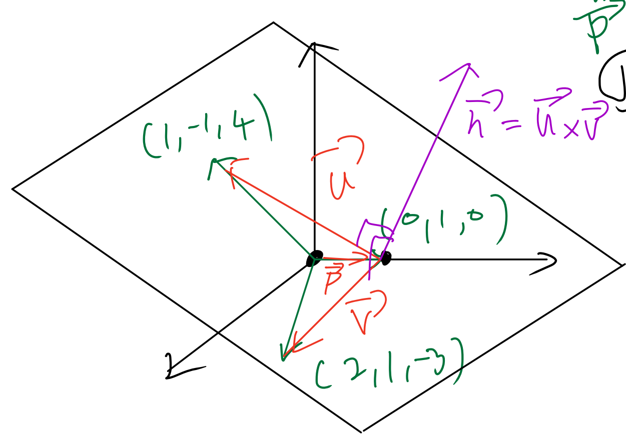
$\vec{p} + t\vec{u} + s\vec{v}$

$= (0,1,0) + t(1,-2,4)$
$s(2,0,-3)$

$= (t+2s, 1-2t, 4t-3s)$

(This is copy from the formula, not the plane of the question)

Example: Find the equation of the plane that contains $\underbrace{(0,1,0)}_{\vec{p}}$, $(2,1,-3)$, $(1,-1,4)$



① $\vec{u} = (1,-1,4) - (0,1,0)$
$= (1,-2,4)$

$\vec{v} = (2,1,-3) - (0,1,0)$
$= (2,0,-3)$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 2 & 0 & -3 \end{vmatrix}$$

$$= \vec{i}(6) + \vec{j}(+11) + \vec{k}(4)$$

$$= (6,11,4)$$

Plan: ① Find the normal vector (cross product)
② write out orthogonality (use relative position!)

② Letting (x,y,z) be the position vector of an (arbitrary) point on the plane,

$$0 = \left((x,y,z) - (0,1,0)\right) \cdot (6,11,4) = (x, y-1, z) \cdot (6,11,4)$$

$$= 6x + 11(y-1) + 4z$$

Rearranging,

$$6x + 11y + 4z = 11$$

(Another way) — Carry out details yourself

Use the fact that a plane can be express as an equation of the form:

$$ax+by+cz=d$$

Plug in $(0,1,0)$, $(2,1,-3)$, $(1,-1,4)$ (why?)

$$\begin{cases} \boxed{b \quad = d} \\ 2a+b-3c=d \\ a-b+4c=d \end{cases} \Rightarrow \begin{matrix} 2a+\cancel{d}-3c=\cancel{d} \\ a-d+4c=d \end{matrix} \Rightarrow \begin{matrix} 2a-3c=0 \quad \boxed{2a=3c} \\ (a+4c=2d)\times 2 \end{matrix}$$

$$2a+8c=4d \overset{2a=3c}{\Longrightarrow} 3c+8c=4d \Rightarrow 11c=4d$$

$$\Rightarrow \boxed{c=\frac{4}{11}d} \Rightarrow a=\frac{3}{2}c=\frac{3}{2}\cdot\frac{4}{11}d = \frac{6}{11}d$$

Thus, $\frac{6}{11}dx+dy+\frac{4}{11}dz=d$

But $d\neq 0$, b/c otherwise, we have

$$0\cdot x+0\cdot y+0\cdot z=0$$

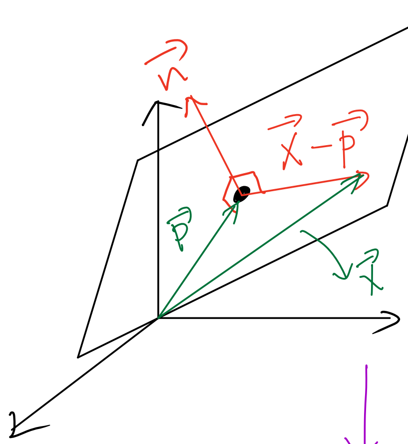
$\Rightarrow$ all $(x,y,z)\in\mathbb{R}^3$ can be in the plane.

Multiplying by $11/d$

$$6x+11y+4z=11$$

Interpret a plane back from an eqn.

$\vec{x} = (x, y, z)$, $\vec{p} = (x_0, y_0, z_0)$, $\vec{n} = (a, b, c)$



$$\downarrow$$

$$(\vec{x} - \vec{p}) \cdot \vec{n} = 0$$

$$\downarrow$$

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

$$\downarrow$$

$$ax + by + cz = d$$

$\boxed{\text{Clicker}}$ Find the normal vector to the plane $3x - 2y = 4$

(A) $(3, -2, 4)$ (B) $(3, -2, 0)$

(C) $(3, -2, -4)$ (D) $(3, 2, 4)$

Parametric form $\longleftrightarrow$ equation of planes

(This involves only elementary calculations, but not so much intuition. So, we briefly mention outline and leave details to you. If you need help, ask in person: OH, Math Lab, CLAS)

1) Parametric form $\longrightarrow$ equation

Having known $(x, y, z) = (\boxed{t+2s, 1-2t}, \boxed{4t-3s})$ (a), (b)

"eliminating t and s" $\rightarrow$ reveals how x, y, z are related

(a) $\begin{cases} t+2s = x \\ 1-2t = y \end{cases} \Longrightarrow$ $t = \frac{1}{2}(1-y) = \frac{1}{2} - \frac{1}{2}y$

$s = (x-t)\cdot\frac{1}{2}$

$= \frac{1}{2}x - \frac{1}{2}t$

$= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}y\right)$

$= \frac{1}{2}x + \frac{1}{4}y - \frac{1}{4}$

(b) $z = 4t - 3s = \cancel{4}^{2}\cdot\cancel{\tfrac{1}{2}}(1-y) - 3\left(\frac{1}{2}x + \frac{1}{4}y - \frac{1}{4}\right)$

Multiply by 4 and rearrange:

$4z = 8 - 8y - 6x - 3y + 3$

$6x + 11y + 4z = 11$

② Equation $\longrightarrow$ parametric

$$6x + 11y + 4z = 11$$

$$z = \frac{1}{4}(11 - 6x - 11y)$$

Introduce the parameters so that

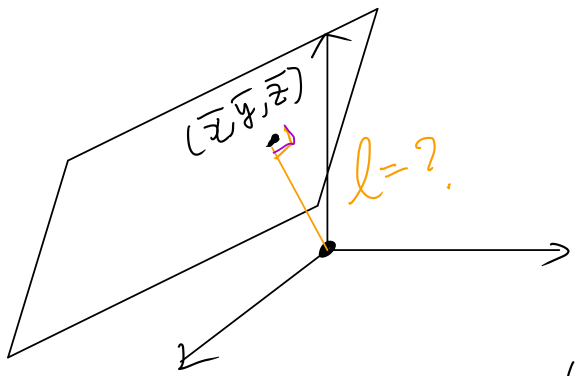
$$x = t$$
$$y = s$$
$$z = \frac{1}{4}(11 - 6t - 11s)$$

$$P: \left(t, s, \frac{11}{4} - \frac{3}{2}t - \frac{11}{4}s\right)$$

/* This looks different from the previous result $(t+2s, 1-2t, 4t-3s)$.
However, this is a matter of different labellings of parameters.

NT 1.15 Distance from a point to a plane.

(Step 1) Distance from the origin to a plane



$P: ax+by+cz=d$

Idea: $(\bar{x},\bar{y},\bar{z})$ can be viewed in two different ways

① a point on the plane:

$$a\bar{x}+b\bar{y}+c\bar{z}=d$$

② normal vector.

$$\underbrace{\frac{(a,b,c)}{\|(a,b,c)\|}}_{\text{(unit) normal vector}} \cdot \underbrace{l}_{\text{length}} = \pm(\bar{x},\bar{y},\bar{z})$$

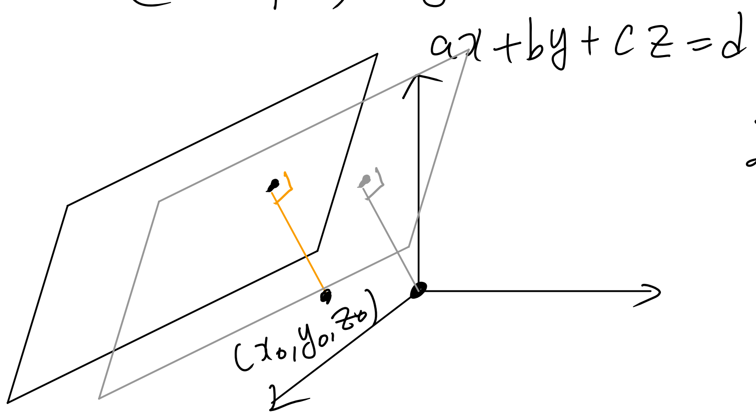
Inner product both sides with $(\bar{x},\bar{y},\bar{z})$ and take absolute value (not to worry about sign)

$$\frac{|(a,b,c)\cdot(\bar{x},\bar{y},\bar{z})|}{\|(a,b,c)\|}\cancel{l} = l^{\cancel{2}} = \|(\bar{x},\bar{y},\bar{z})\|^2$$

$$a\bar{x}+b\bar{y}+c\bar{z}=d$$

$$l=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

(Step 2) general case



Idea: Reduce to (step 1) by translation.

/* Simiarly to 2D equation of translation of $f(x,y,z)=0$ by (x_1, y_1, z_1) is

$$f(x-x_1, y-y_1, z-z_1)=0$$ */

Here, we want to translate P by $(-x_0, -y_0, -z_0)$. Thus, the equation of the dummy plane is

$$a(x+x_0)+b(y+y_0)+c(z+z_0)=d$$

$$ax+by+cz=-(ax_0+by_0+cz_0)+d$$

Using the result of step 1, we have

$$l=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

⊙ This is "easy" to remember

numerator : abs. of eqn of P
with the point plugged in,
from which distance is measured
($\boxed{ax+by+cz-d}=0$)

denominator : norm of normal vector

Clicker Find the shortest distance from $(1,1,1)$ to a plane $x+2y+3z=9$

(A) $\dfrac{3}{\sqrt{14}}$ (B) $\dfrac{6}{\sqrt{81}}$ (C) $\dfrac{-3}{14}$

NT 1.16 Miscellaneous (intersections)

(This involves only elementary calculations, but not so much intuition. So, we briefly mention outline and leave details to you. If you need help, ask in person: OH, Math Lab, CLAS)

⓪ line of intersection of two planes.

Example: Find the line where the following two planes meet:

$P: \ x+2y+3z=4$

$Q: \ 2x \quad +z=2$

(As in point of intersection of two lines, you are looking for points that satisfy equations $\Rightarrow$ solve)

$2P - Q \Rightarrow \quad 4y+5z=6$

$$\boxed{y=\frac{3}{2}-\frac{5}{4}z}$$

$Q \Rightarrow \quad \boxed{x=1-\frac{1}{2}z}$

introduce a parameter $t = z$. Then,

/* Observe that knowing what a line looks like makes this step very natural. */

$$\begin{cases} x = 1 - \frac{1}{2}t \\ y = \frac{3}{2} - \frac{5}{4}t \\ z = t \end{cases}$$

or (vector form with parameter)

$$(1, \tfrac{3}{2}, 0) + t(-\tfrac{1}{2}, -\tfrac{5}{4}, 1)$$

⑥ Point of intersection of a line and a plane.

Example: Find the point of intersection where $l: (1+t, -t, 2-t)$ meets $P: x - y + 3z = 4$

/* For the same reason as above
$\Longrightarrow$ solve system */

$x = 1+t$, $y = -t$, $z = 2-t$ (points on the line)

plug into plane:

$$1+t - (-t) + 3\underbrace{(2-t)}_{6-3t} = 4$$

$$-t = 4 - 7 = -3 \Rightarrow t = 3.$$

when $t=3$, the point $(4,-3,1)$ is on ℓ. This also satisfies

$4-(-3)+3\cdot 1=4$

check