- Quizz → Tomorrow

Recap:

- path (or vector-valued fn) 1D → n-D
- multivariate fn m-D → 1D
- vector field m-D → n-D

- Limits, Der., Int. → component-wise

- $\text{len} = \int_a^b \|\vec{r}\,'(t)\|\,dt$

① $\vec{r}(a)=\vec{r}(b)$ is allowed. In this case the path is called closed.

(c)

② A smooth path cannot pass through itself, be tangent to itself, or "stop" (b) (when it stops, it may make a sharp turn. You will see how zero velocity may mess up theory)

Def (Smooth curve) image path function

A curve is called smooth if it has a C^1-parametrization that is smooth, that is, it is the image of a smooth path.

The length of curve (or **arc length**) is the length of that smooth path.

Example: Find the length of hilex

$\vec{r}(t) = (\overset{\text{radius.}}{1}\cos t, 1\sin t, t) \quad (0 \le t \le 2\pi)$

Recall $\ell(\vec{r})$ (arc length of $\vec{r}$)

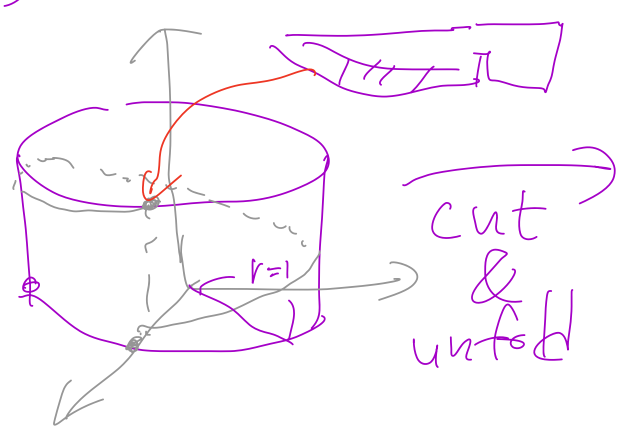
$$\ell = \int_a^b \|\vec{r}\,'(t)\| \, dt$$

$$\vec{r}\,'(t) = (-\sin t, \cos t, 1)$$

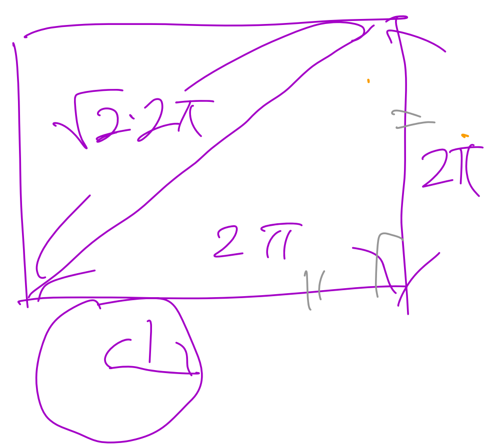
$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2}$$

$$\int_0^{2\pi} \|\vec{r}\,'(t)\| \, dt = \int_0^{2\pi} \sqrt{2} \, dt = 2\sqrt{2}\pi$$

Any idea to check this?



cut & unfold



⑥ Application : length of 2D graph

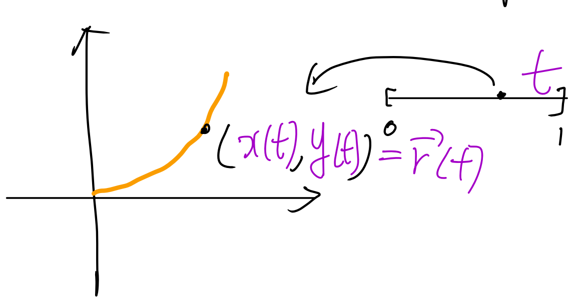
The length of graph $y=f(x)$ $(a \le x \le b)$ is

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

proof) This follows directly by setting $\vec{r}(x) = (x, f(x))$ (See the example below) ▨

Example : Find the length of the parabola $y = x^2/2$ $(0 \le x \le 1)$

(Step 1) Recast the problem using <u>parametrization</u>



$$\boxed{\begin{aligned} x &= x(t) = t \\ y &= y(t) = t^2/2 \end{aligned}} \rightarrow \vec{r}(t) = (t, t^2/2)$$

$a=1$ ↙ From integral table

$\vec{r}'(t) = (1, t)$ $\quad \int \sqrt{x^2 \pm a^2}\;dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

$$\int_0^1 \|\vec{r}'(t)\|dt = \int_0^1 \sqrt{1+t^2}\,dt \qquad t \to x, \; 1 \to a$$

$$= \left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|\right]_0^1$$

$$= \frac{1}{2}\cdot 1 \cdot \sqrt{2} + \frac{1}{2}\ln|1+\sqrt{2}| - 0 - \frac{1}{2}\underbrace{\ln|0+\sqrt{1}|}_{=0}$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})$$

NT. 2.4 Chain rule for paths

Settings:

$$s \qquad t = t(s) \qquad \vec{r}(t) = \vec{r}(t(s))$$

$$\mathbb{R} \xrightarrow[1D \to 1D]{} \mathbb{R} \xrightarrow[1D \to n\text{-}D]{} \mathbb{R}^n$$

$$s \longrightarrow \vec{r}(t(s)) = (\vec{r} \circ t)(s)$$

Suppose $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ and $t : \mathbb{R} \to \mathbb{R}$ so that $\vec{r} \circ t : \mathbb{R} \to \mathbb{R}^n$ is defined $\vec{r} \circ t(s) = \vec{r}(t(s))$. Also, $\vec{r}$ and t are both C^1-fn's. Then, $\vec{r} \circ t$ is also C^1 and

$$\boxed{\frac{d}{ds}(\vec{r} \circ t)(s) = \frac{d}{dt}\vec{r}\,(t(s))\,\frac{d}{ds}t(s)}$$

Component-wise perspective

$$s \longmapsto t \longmapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x(t(s)) \\ y(t(s)) \\ z(t(s)) \end{pmatrix}$$

Thus, $\frac{d}{ds}(\vec{r} \circ t)(s) = \begin{pmatrix} \frac{d}{ds} x(t(s)) \\ \frac{d}{ds} y(t(s)) \\ \frac{d}{ds} z(t(s)) \end{pmatrix}$

$$= \begin{pmatrix} \frac{dx}{dt} \cdot \frac{dt}{ds} \\ \frac{dy}{dt} \cdot \frac{dt}{ds} \\ \frac{dz}{dt} \cdot \frac{dt}{ds} \end{pmatrix} \leftarrow \text{Cal 1}$$

common scalar

$$= \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} \frac{dt}{ds} = \frac{d}{dt} \vec{r}(t) \cdot \frac{dt}{ds}$$

$$= \vec{r}\,'(t) \cdot t'(s)$$

$\frac{d}{dt}$ $\frac{d}{ds}$

/* **Notation** : our textbook uses the convention

$\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t)$, $t'(s) = \frac{d}{ds} t(s)$

$\cancel{\vec{r}\,'(t(s))}$ confusing ($\frac{d}{dt}\vec{r}(t(s))$ or $\frac{d}{ds}\vec{r}(t(s))$?) */

Example : Find $\vec{r}'(s)$, where

$$\vec{r}(s) = (\cos(3s+\tfrac{\pi}{4}), \sin(3s+\tfrac{\pi}{4}), e^{3s+\frac{\pi}{4}})$$

Put $t(s) = 3s + \frac{\pi}{4}$. $t'(s) = \frac{d}{ds} t(s)$

$= 3$ evaluation.

$\vec{r}(t) = (\cos t, \sin t, e^{t})$

$\vec{r}'(t) = (-\sin \underline{t}, \cos \underline{t}, e^{\underline{t}})$ d/dt

Clicker	(A) d/ds
(B) d/dt	(C) $d/d\vec{r}$

By <u>chain</u> rule,

$$\frac{d}{ds}\vec{r}(s) = \frac{d}{ds}\vec{r}(\underline{t(s)}) = \underline{\frac{d}{dt}\vec{r}(t)} \cdot \underline{\frac{d}{ds}t(s)}$$

$$= (-\sin t, \cos t, e^{t}) \cdot 3$$

$$= 3(-\sin(3s+\tfrac{\pi}{4}), \cos(3s+\tfrac{\pi}{4}), e^{3s+\frac{\pi}{4}})$$

/* (brute-force)

$$\frac{d}{ds}\vec{r}(s) = (-\sin(3s+\tfrac{\pi}{4})\cdot 3, \cos(3s+\tfrac{\pi}{4})\cdot 3, e^{3s+\frac{\pi}{4}}\cdot 3)$$

$$= 3 (-\sin(3s+\tfrac{\pi}{4}), \cos(3s+\tfrac{\pi}{4}), e^{3s+\frac{\pi}{4}})$$

*/

Use chain rule whenever possible. When settings are more involved, brute force does not work.

NT 2.5 Parametrization by arc length.

Motivation: We observed that there can be many different paths for the same curve. For example, for a circle

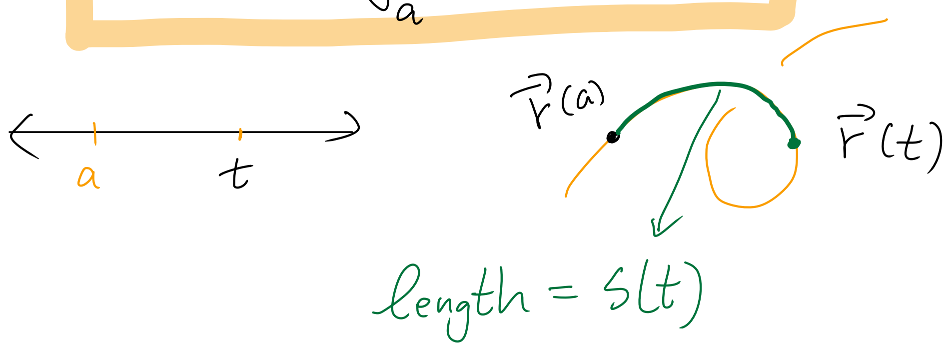
$\vec{r}_1(t) = (\cos t, \sin t)$, $\vec{r}_2(t) = (\cos 3t, \sin 3t)$

Is there some "good normalization"?

$\boxed{\text{Def}}$ (Arc length fn)

Given a smooth path $\vec{r}: [a,b] \to \mathbb{R}^n$ the **arc length** function is defined by

$$\boxed{s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\, d\tau}$$



$\boxed{\text{Cor}}$ Let $s(t)$ and $\vec{r}$ be as above.

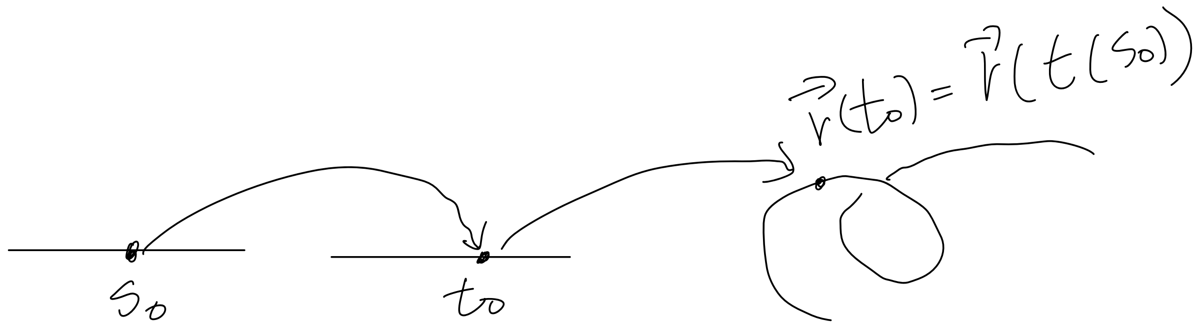
Then, (by FTC)

$$\boxed{s'(t_0) = \frac{d}{dt}s(t_0) = \|\vec{r}\,'(t_0)\|.}$$

Furthermore, $s(t)$ is strictly increasing.

Consequently, it is one-to-one and has an inverse function, and for $s_0 = s(t_0)$ (or $t_0 = t(s_0)$)

$$\boxed{t'(s_0) = \frac{d}{ds}t(s_0) = \frac{1}{\|\vec{r}\,'(t_0)\|} = \frac{1}{\|\frac{d}{dt}\vec{r}(t_0)\|}}$$

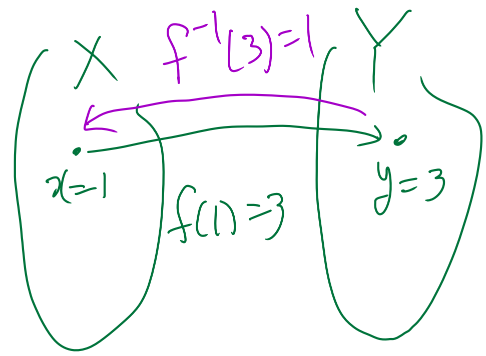


/* In words, (length)$'$ $\frac{d}{dt}$ = speed */

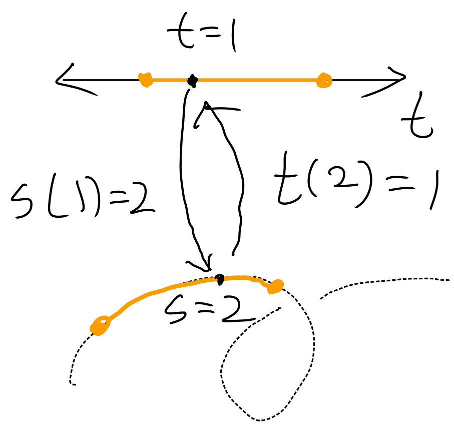
/* **Notation**

Usual notation for inverse of f is f^{-1}.

This is convenient when we interpret inputs just as numbers. (input of f^{-1} is in fact y values while input of f is x value).



However, when the input and output have concrete meanings, it is often good to name the inverse function after the input.



t: time

s: arc length

$S(\overset{1}{t_0}) = \overset{2}{s_0}$

$\iff t(s_0) = t_0$

$\Leftarrow t(s(t_0)) = t_0$

proof) Since $\vec{r}(t)$ is a smooth path, $\vec{r}'(t) \neq \vec{0}$, hence $\|\vec{r}'(t)\| \neq 0$.

Therefore

FTC : $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$s(t) = \int_a^t \|\vec{r}'(\tau)\|\,d\tau$$ is strictly

increasing. $\Rightarrow$ 1-1 $\Rightarrow$ invertible

And from Cal 1, derivatives of inverse fn's are inverse to each other at $t = t(s)$ and $s = s(t)$. To be more specific,

$t \circ s$ = identify on t domain, that is,

$$t(s(\tau)) = \tau$$

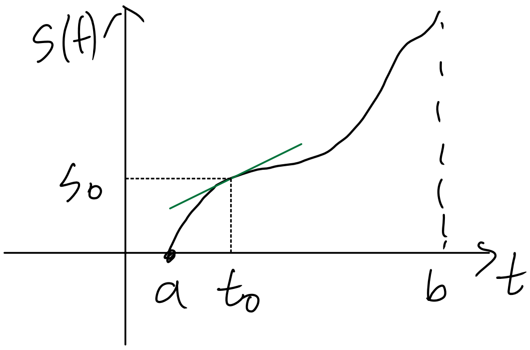
Differentiate this w.r.t. τ (short for "with respect to") and evaluate at $\tau = t_0$ and put $s_0 = s(t_0)$ (or $t_0 = t(s_0)$)

$$\frac{d}{ds} t(s_0) \cdot \frac{d}{d\tau} s(t_0) = 1$$

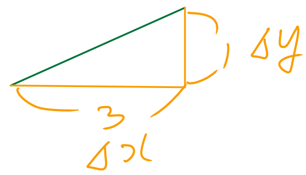
$$\frac{d}{ds} t(s_0) \cdot \|\vec{r}\,'(t_0)\| = 1$$

d/dt

$$\Rightarrow \frac{d}{ds} t(s_0) = t'(s_0) = \frac{1}{\|\vec{r}'(t_0)\|} = \frac{1}{\|\frac{d}{dt}\vec{r}(t(s_0))\|}$$



e.g.



$$\frac{\Delta y}{\Delta x} = \frac{1}{3} \Rightarrow \frac{\Delta x}{\Delta y} = 3.$$

Example: Reparametrize $\vec{r}(t) = (3\cos t, 3\sin t)$ by its arc length. (Start from $t=0$)

$$\vec{r}\,'(t) = (-3\sin t, 3\cos t) = 3(-\sin t, \cos t)$$

$$\int_0^t \|\vec{r}\,'(\tau)\|\,d\tau = \int_0^t \sqrt{\overbrace{(-3\sin\tau)^2 + (3\cos\tau)^2}^{=3}}\,d\tau$$

$$= \int_0^t 3\,d\tau = 3t$$

$$s = s(t) = 3t$$

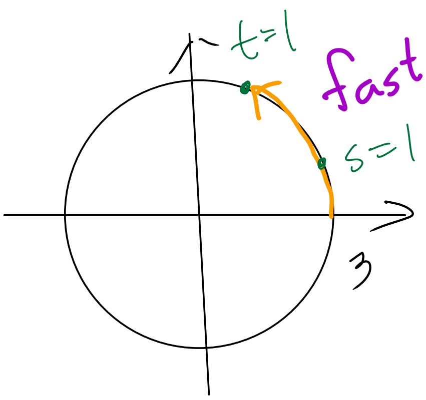
$$t = t(s) = \frac{s}{3}$$

(here we have found the inverse)

$$\vec{r}(s) = \vec{r}(t(s))$$

$$= (3\cos t, 3\sin t)$$

$$= \left(3\cos\frac{s}{3}, 3\sin\frac{s}{3}\right)$$



fast b/c of radius.

This reparametrization slows down to speed 1.

$\boxed{\text{Thm}}$ (Arc length and unit speed)

Given a smooth path $\vec{r}(t)$, let $s(t)$ be its arc length fn. Then, its reparametrization by arc length has unit speed. That is,

$$\| \vec{r}\,'(s) \| = \| \frac{d}{ds} \vec{r}(s) \| = 1$$

proof)

For clarity, let us use s_0 for evaluation.

$$\left\| \frac{d}{ds} \vec{r}(t(s_0)) \right\| = \left\| \frac{d}{dt} \vec{r}(t(s_0)) \frac{dt}{ds}(s_0) \right\| \quad (\text{chain rule})$$

$$= \left\| \frac{d}{dt} \vec{r}(t(s_0)) \right\| \left| \frac{dt}{ds}(s_0) \right| \quad (\|\alpha \vec{v}\| = |\alpha| \|\vec{v}\|)$$

$$= \left\| \frac{d}{dt} \vec{r}(t(s_0)) \right\| \frac{1}{\left\| \frac{d}{dt} \vec{r}(t(s_0)) \right\|} \quad \text{Previous thm}$$

$$= 1.$$

Since s_0 is arbitrary, replace s_0 with s.

$\blacksquare$

Summary of important facts

Suppose

(smooth path) $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$ $\qquad t_0 \longmapsto \vec{r}(t_0)$

(arc length) $s : [a,b] \longrightarrow \mathbb{R}$ $\qquad t_0 \longmapsto s(t_0) = \int_a^{t_0} \|\vec{r}\,'(\tau)\| \, d\tau$.

Then,

- s has inverse t: $s(t(s_0)) = s_0$, $t(s(t_0)) = t_0$.
- $s(t_0)$ = arc length from (time) $\underline{a}$ to t_0.
  — starting time
- $t(s_0)$ = time when arc length is s_0.
- The arc length reparametrization $\vec{r}(s) = \vec{r}(t(s))$
  has unit speed : $\left\| \frac{d}{ds} \vec{r}(s_0) \right\| = 1$

$\boxed{\text{Clicker}}$ $\quad s'(t_0) = (\underbrace{\|\vec{r}\,'(t_0)\|}^{\text{speed}} \quad)$

(A) $\dfrac{1}{\|\vec{r}\,'(t_0)\|}$ $\qquad$ ($\cancel{B}$) $\|\vec{r}\,'(t_0)\|$

(C) $\dfrac{1}{\|\vec{r}\,'(s_0)\|}$ $\qquad$ (D) $\|\vec{r}\,'(s_0)\|$

NT 2.6 Curvature

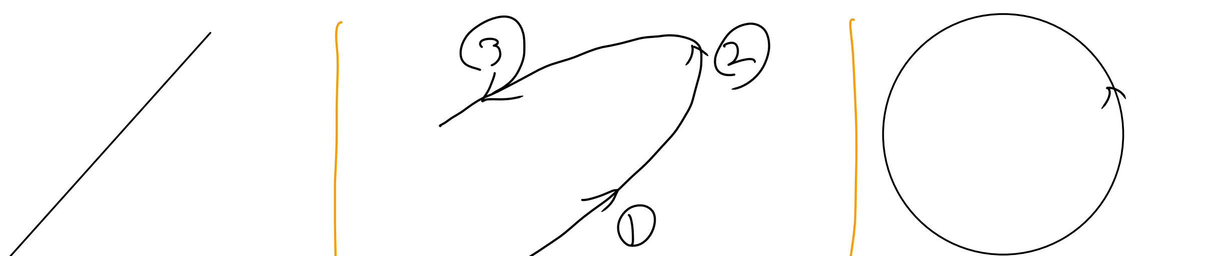
Motivation

We have seen that an identical curve may have different parametrization (i.e., movements along the same curve with different speeds).

However, from geometric point of view the curve (i.e., the image of a path) is fixed. Moreover, studying characteristic features of such shapes is a natural question.

In particular, how can we measure how much a curve is bent?

Imagine a vehicle moving at a fixed speed



never bent | ① little ② a lot ③ little | equally bent

Idea: How fast the direction of movement changes (b) when moving at a constant speed? (a)

(a) can be achieved by reparametrization by arc length.

(b) is exactly what derivatives are about.