

MA2104 CheatSheet AY23/24 — @Jin Hang

Vector

Projections: vector projection of **b** onto **a**, denoted by proj_a**b**

Scalar Projection:The scalar projection of **b** onto **a** (component of **b** along **a**) comp_a**b** = ||**b**|| cos θ = $\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$

Relationship:

proj_a**b** = comp_a**b** × $\frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \cdot \mathbf{a}$

Theorem 6: The vector **a** × **b** is orthogonal to both **a** and **b**

Cross product angle formula: ||**a** × **b**|| = ||**a**|| ||**b**|| sin(α, **b**)

Properties of cross product

- a** × **b** = −**b** × **a**
- (**da**) × **b** = **d** (**a** × **b**) = **a** × (**db**)
- (**a** − **b**) × (**a** + **b**) = 2(**a** × **b**)
- (**a** × **b**) · **c** = [**a**, **b**, **c**] = [**b**, **c**, **a**] = [**c**, **a**, **b**] = −[**b**, **a**, **c**] = −[**c**, **b**, **a**] =
- a** × (**b** + **c**) = **a** × **b** + **a** × **c**
- (**a** + **b**) × **c** = **a** × **c** + **b** × **c**
- a** × (**b** × **c**) = (**a** · **c**) · **b** − (**a** · **b**) · **c**

−[**a**, **c**, **b**] = $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Distance from a point to a line: Suppose *P* and *R* are two points

on *L*. Distance from point *Q* to line *L* is $\|\overrightarrow{PQ}\| \sin \theta = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}$

Relationship between 2 lines In 2-D, two lines are either parallel or intersect. In 3-D, two lines are either

- parallel
- non-parallel and intersect
- non-parallel and non-intersecting (skew lines)

Show that the lines are skew

Assume for a contradiction that *L*₁ and *L*₂ intersect.

Then there must exist a choice of the parameter *t* and *s* such that the values of *x*; *y* and *z* are the same.

Find the line of intersection of two planes

Solving both equations for *x* and setting them to be equal. Let *y* = *t*

be the parameter, obtain a parametric equation for line of intersection
简便方法: 交线的方向向量为两平面向量的叉乘

Find the distance of two planes

Find two point on *P*₁ and *P*₂, assume they are (*a*₁, *b*₂, *c*₃) and (*a*₂, *b*₂, *c*₂), then let **u** = (*a*₁ − *a*₂, *b*₁ − *b*₂, *c*₁ − *c*₂). Assume the normal

vector to the planes is **n**, then *d* = ||**u**|| |cos θ| = $\left|\frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|}\right|$

Properties of Vector

1. comp_b(**a** + **c**) = comp_b(**a**) + comp_b(**c**)

2. Triangle Inequality

- ||**u** + **v**|| ≤ ||**u**|| + ||**v**||
- ||**a**|| − ||**b**|| ≤ ||**a** − **b**||

Vector-valued function:

r(*t*) = ⟨*f*(*t*), *g*(*t*), *h*(*t*)⟩ = *f*(*t*)**i** + *g*(*t*)**j** + *h*(*t*)**k**

We say that **r**(*t*) is a parametrization of *C*. A curve *C* can have more than one parameterizations.

- r**(*t*) = the position vector of a particle in space at time *t*
- r**[′](*t*) = the velocity of the particle at time *t*
- ||**r**[′](*t*)|| = speed of the particle at time *t*
- r**[′](*a*) = ⟨*f*[′](*a*), *g*[′](*a*), *h*[′](*a*)⟩

- $\frac{d}{dt} \mathbf{r}(t) \cdot \mathbf{s}(t) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$
- $\frac{d}{dt} (\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$
- $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \cdot \mathbf{r}(t) \cdot \mathbf{r}'(t)$

- Chain Rule $\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$

- 求两向量函数在某一点的夹角 → 求导 → 方向向量夹角

Arc Length: Let *C* be the curve given by

r(*t*) = ⟨*f*(*t*), *g*(*t*), *h*(*t*)⟩ = *f*(*t*)**i** + *g*(*t*)**j** + *h*(*t*)**k** *a* ≤ *t* ≤ *b*

where *f*[′], *g*[′], *h*[′] are continuous, its length is

$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt$

Partial Derivatives

Find the range of *f*(*x*, *y*): i.e. for *f*(*x*, *y*) = *x* ln(*y*² − *x*)

Only when *y*² − *x* > 0 ⇔ *y*² > *x* ⇔ Domain={(*x*, *y*) ∈ **R**: *y*² > *x*}

Level Curve: A level curve of *f*(*x*, *y*) is the two-dimensional graph of the equation *f*(*x*, *y*) = *k* for some constant *k*.

Contour Plot: A contour plot of *f*(*x*, *y*) is a graph of numerous level curves *f*(*x*, *y*) = *k*, for representative values of *k*.

Sketch contour plots Use values of *k* that are equally spaced

- The traces in *x* = *k*/*y* = *k*/*z* = *k* is ...
- Draw diagram of curve in *xy*, *yz*, *xz* plane and decide outline

Level Surface: A level surface of *f*(*x*, *y*, *z*) is the 3-D graph of the equation *f*(*x*, *y*, *z*) = *k* for some constant *k*.

Limit: Limit of *f*(*x*, *y*) as (*x*, *y*) approaches (*a*, *b*) is *L* ∈ **R**, denoted by

$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

if for any number ε > 0 there exists a number δ > 0 such that

|*f*(*x*, *y*) − *L*| < ε, whenever 0 < √(*x* − *a*)² + (*y* − *b*)² < δ

How to show limit does not exist

- Consider the limit along the path *P*₁. We have ...
- Consider the limit along the path *P*₂. We have ...
- Since this limit along *P*₂ is different from the one we had previously for *P*₁, we conclude that the limit does not exist.

How to show limit exists

- we can deduce it from known/simple functions using properties of limits or continuity
- we can use the Squeeze theorem

- |*f*(*x*, *y*) − *L*| ≤ *g*(*x*, *y*) ∀(*x*, *y*) close to (*a*, *b*)
- $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = 0$

Then, $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

Limit Theorems

- $\lim_{(x,y) \rightarrow (a,b)} (f \pm g) = \lim_{(x,y) \rightarrow (a,b)} f \pm \lim_{(x,y) \rightarrow (a,b)} g$
- $\lim_{(x,y) \rightarrow (a,b)} (f \cdot g) = (\lim_{(x,y) \rightarrow (a,b)} f) \cdot (\lim_{(x,y) \rightarrow (a,b)} g)$

- $\lim_{(x,y) \rightarrow (a,b)} \frac{f}{g} = \frac{\lim_{(x,y) \rightarrow (a,b)} f}{\lim_{(x,y) \rightarrow (a,b)} g}$, where $\lim_{(x,y) \rightarrow (a,b)} g(x,y) \neq 0$

极坐标换元法证明极限存在/不存在 (通法)

- Let *f*(*x*, *y*) = 0 → *f*(*r* cos θ, *r* sin θ) = 0, 即让极径*r*来代表趋近值, 让极角 θ 来代表趋近路径
- $\lim_{(x,y) \rightarrow (a,b)} f(x,y) \implies \lim_{r \rightarrow k^+} f(r \cos \theta, r \sin \theta)$
- 最终结果
 - 如果只包含θ变量, 则该极限不存在
 - 如果... = *r*^{*n*} sin^{*n*} α cos^{*b*} θ 则极限为0
 - 若... = $\frac{\sin \alpha \pm \cos \theta}{r^n} \implies$ 讨论是否存在分母为0的情况判断极限是否存在

Continuity of *f*(*x*, *y*): $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$

Continuity Theorems: If *f*(*x*, *y*) and *g*(*x*, *y*) are continuous at (*a*, *b*)

- f* ± *g* is is continuous at (*a*, *b*)
- f* · *g* is is continuous at (*a*, *b*)
- $\frac{f}{g}$ is is continuous at (*a*, *b*), provided *g*(*a*, *b*) ≠ 0

Continuity and Composition: Suppose *f*(*x*, *y*) is continuous at (*a*, *b*) and *g*(*x*) is continuous at *f*(*a*, *b*). Then

h(*x*, *y*) = (*g* ∘ *f*)(*x*, *y*) = *g*(*f*(*x*, *y*))

is continuous at (*a*, *b*).

Continuity Functions: Following are continuous in its domain.

- Polynomial in *x* and *y*;
- Trigonometric and exponential functions in *x* and *y*;
- Rational function in *x* and *y*.

Partial Derivative (Definition)

$f_x(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$

$f_y(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$

Differentiability Let *z* = *f*(*x*, *y*), *f* is differentiable at (*a*, *b*) if

Δ*z* = *f_x*(*a*, *b*)Δ*x* + *f_y*(*a*, *b*)Δ*y* + ε₁Δ*x* + ε₂Δ*y*

where ε₁ and ε₂ are functions of Δ*x* and Δ*y* which vanish(i.e.

ε₁, ε₂ → 0 as Δ*x*, Δ*y* → 0)

Increment: Let *z* = *f*(*x*, *y*). Suppose Δ*x* and Δ*y* are increments in the independent variable *x* and *y* respectively from a fixed point (*a*, *b*), then Δ*z* = *f*(*a* + Δ*x*, *b* + Δ*y*) − *f*(*a*, *b*)

Linear approximation: Δ*z* ≈ *f_x*(*a*, *b*)Δ*x* + *f_y*(*a*, *b*)Δ*y*

Implicit Differentiation: Suppose the equation *F*(*x*, *y*, *z*) = 0, where *F* is differentiable, defines *z* implicitly as a differentiable function of *x* and *y*. Then,

$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$

Directional derivative: The directional derivative of *f*(*x*, *y*) at (*x*₀, *y*₀) in the direction of unit vector **u** = ⟨*a*, *b*⟩ is

$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h} = \nabla f \cdot \mathbf{u}$

Geometric: Suppose surface *S* given by *z* = *f*(*x*, *y*), where *f* is differentiable function of *x* and *y* at (*x*₀, *y*₀) and ∇*f*(*x*₀, *y*₀) ≠ 0.

- ∇*f*(*x*₀, *y*₀) is normal to level curve *f*(*x*, *y*) = *k* containing (*x*₀, *y*₀)

- Normal vector to *S* at (*a*, *b*): ⟨*f_x*(*a*, *b*), *f_y*(*a*, *b*), −1⟩
- Tangent Plane *z* = *f*(*a*, *b*) + *f_x*(*a*, *b*)(*x* − *a*) + *f_y*(*a*, *b*)(*y* − *b*)
- Tangent Plane to Level Surface

∇*F*(*x*₀, *y*₀, *z*₀) · ⟨*x* − *x*₀, *y* − *y*₀, *z* − *z*₀⟩ = 0

Saddle Point: Let *f*(*x*, *y*) : *D* → **R**. Point (*a*, *b*) is saddle point if

- f_x*(*a*, *b*) = *f_y*(*a*, *b*) = 0; AND
- every neighborhood at (*a*, *b*) contains points (*x*, *y*) ∈ 2*D* for which *f*(*x*, *y*) < *f*(*a*, *b*) and points (*x*, *y*) ∈ 2*D* for which *f*(*x*, *y*) > *f*(*a*, *b*)

Closed Set in **R**² Contains all its boundary points.

Boundary Point Point (*a*, *b*) s.t. every disk with center (*a*, *b*)

contains points in *R* and also points in **R**² \ *R*)

Bounded Set in **R**² A set *R* ⊆ **R**² bounded if it is contained within some disk. In other words, it is finite in extent

The Closed Interval Method / Find the extreme value

- Find the values of *f* at its critical points in *D*.
- Find the extreme values of *f* on the boundary of *D*. Let *x* = *n*, represent *f*(*x*, *y*) only in *y*, same for let *y* = *n* ...
- The largest of the values from Step 1 and Step 2 is the absolute max. value; the smallest of these values is the absolute min. value.

Lagrange Multiplier: Suppose *f*(*x*, *y*) and *g*(*x*, *y*) are differentiable s.t. ∇*g*(*x*, *y*) ≠ 0 on constraint curve *g*(*x*, *y*) = *k*. If (*x*₀, *y*₀) is a local minimum/maximum of *f*(*x*, *y*) constrained by *g*(*x*, *y*) = *k*. Then ∇*f*(*x*₀, *y*₀) = λ ∇*g*(*x*₀, *y*₀) for some constant λ (Lagrange Multiplier).

- Solve the following system of equations:

$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$

- Find the point(s) (*x*, *y*) satisfying the above equations
- Evaluate *f* at all the points obtained. The largest of these values is the maximum value of *f*; The smallest is the minimum value of *f*.

Double Integral

$\iint_R f(x,y) dA = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n f(a + \frac{b-a}{n}i, c + \frac{d-c}{n}j) \cdot \frac{b-a}{n} \cdot \frac{d-c}{n}$

- Volume *V* = ∫∫_{*D*} *f*(*x*, *y*) *dA*
- Area *A*(*D*) = ∫∫_{*D*} 1 *dA*

Fubini's Theorem If *f* is continuous on *R* = [*a*, *b*] × [*c*, *d*], then

∫∫_{*R*} *f*(*x*, *y*) *dA* = ∫_{*a*}^{*b*} ∫_{*c*}^{*d*} *f*(*x*, *y*) *dydx* = ∫_{*c*}^{*d*} ∫_{*a*}^{*b*} *f*(*x*, *y*) *dx dy*

Polar Coordinates in Double Integral If *f* is continuous on polar

rectangle *R* = {(*r*, θ) : 0 ≤ *a* ≤ *r* ≤ *b*, α ≤ θ ≤ β}, 0 ≤ β − α ≤ 2π,

then ∫∫_{*R*} *f*(*x*, *y*) *dA* = ∫_α^β ∫_{*a*}^{*b*} *f*(*r* cos θ, *r* sin θ) *rdrdθ*

Tripe Integral over Type 1/2/3 Regions ∫∫∫_{*E*} *f*(*x*, *y*, *z*) *dV* =

- ∫∫_{*D*} $\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) dz$ *dA*
- ∫∫_{*D*} $\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z) dx$ *dA*
- ∫∫_{*D*} $\int_{u_1(z,x)}^{u_2(z,x)} f(x,y,z) dy$ *dA*

Spherical Coordinates: ∫∫∫_{*E*} *f*(*x*, *y*, *z*) *dV* =

∫_{*c*}^{*d*} ∫_α^β ∫_{*a*}^{*b*} *f*(ρ sin φ cos θ, ρ sin φ sin θ, ρ cos θ) ρ² sin φ *dρdθdφ*

重积分性质

- 可加性 ∫∫_{*D*₁ ∪ *D*₂} *f*(*x*, *y*) *dA* = ∫∫_{*D*₁} *f*(*x*, *y*) *dA* + ∫∫_{*D*₂} *f*(*x*, *y*) *dA*
- 保号性 *f*(*x*, *y*) ≤ *g*(*x*, *y*) ⇔ ∫∫_{*D*} *f*(*x*, *y*) *dA* ≤ ∫∫_{*D*} *g*(*x*, *y*) *dA*
- 轮换对称性 *x*, *y*对调,积分域*D*不变则 ∫∫_{*D*} *f*(*x*, *y*) *dA* = ∫∫_{*D*} *f*(*y*, *x*) *dA*

Jacobian Transformation: *x* = *x*(*u*, *v*), *y* = *y*(*u*, *v*)

- $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2 \sin \phi$

- ∫∫_{*R*} *f*(*x*, *y*) *dA* = ∫∫_{*S*} *f*(*x*(*u*, *v*), *y*(*u*, *v*)) $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$

常见Jacobian行列式

- Spherical Coordinates: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\rho^2 \sin \phi$

- Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Set *au* = *x*, *bv* = *y*, *cw* = *z*

Parametrization of a line segment: $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$
Vector Field: Let $D \subseteq \mathbb{R}^3$. A vector field on D is a function \mathbf{F} that assigns to each point $(x, y, z) \in D$ a 3-D vector

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

Line Integral of Vector Field Let \mathbf{F} be a continuous vector field defined on a smooth curve C , $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. Then, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt$

Property: $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$

Component Form: If $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ then we can write $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz = \int_a^b \mathbf{F}(x(t), y(t), z(t))\mathbf{r}'(t) dt$
Conservative Vector A vector field \mathbf{F} is **conservative vector**: field on D if we can write $\mathbf{F} = \nabla f$ for some **potential function** f on D .
Test for Conservative: \mathbf{F} is conservative on D if

- P, Q and R have **continuous** first-order partial **derivatives** on D
- $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Fundamental Theorem: (Independent of paths) From A to B

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = f(x_2, y_2) - f(x_1, y_1)$$

Theorem P, Q in单连通域 D 上有一阶连续偏导数, 则以下四条等价

- 线积分 $\int_L P \, dx + Q \, dy$ 与路径无关
- $\oint_L P \, dx + Q \, dy = 0$, 其中 L 为 D 中任一段光滑闭曲线
- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \forall (x, y) \in D$
- $P(x, y)dx + Q(x, y)dy = dF(x, y)$

Green’s Theorem:

- $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$
- C is positively oriented(逆时针), piecewissmooth, simple **closed**
 - D is region bounded by C .
 - P, Q have continuous partial derivatives on open region contains D
 - If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, but P, Q not continuous at (a, b)
 - All integral, whose region contains (a, b) have the same value

$$\iint_S \mathbf{F} \cdot d\mathbf{S} - \iint_T \mathbf{F} \cdot d\mathbf{T} = \iiint_{E'} \operatorname{div}(\mathbf{F}) \, dV = 0$$

$$\therefore \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F} \cdot d\mathbf{T}$$

- All integral, whose region **do not** contains (a, b) are 0

Theorem 2: Area $= \int_C x dy = - \int_C y dx = \frac{1}{2} \left(\int_C x dy - y dx \right)$

Surface Integral

Parametric Surface: $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$

Smooth Surface: For $\mathbf{r}(u, v)$, \mathbf{r}_u and \mathbf{r}_v are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(u, v) \in D$.

Tangent Plane of Smooth Surface $\mathbf{n} = \mathbf{r}_u(a, b) \times \mathbf{r}_v(a, b)$

Surface Integral: $\iint_S f(x, y, z) dS = \sum \iint_{S_i} f(x, y, z) dS$

$$= \iint_D f(x(u, v), y(u, v), z(u, v)) \parallel \mathbf{r}_u \times \mathbf{r}_v \parallel dA$$

$$= \iint_D f(x, y, g(x, y)) \left(\sqrt{\left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 + 1} \right) dA$$

- For surface $z = g(x, y)$, $\mathbf{r}(x, y) = \langle x, y, g(x, y) \rangle$
 $\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1 \rangle$

Surface Area: $\iint_S dS = \iint_D \parallel \mathbf{r}_u \times \mathbf{r}_v \parallel dA$

Oriented Surface: A surface S is **orientable** (or two-sided) if it is possible to define a unit normal vector \mathbf{n} at each point (x, y, z) not on the boundary of surface s.t. \mathbf{n} is a **continuous** function of (x, y, z) .

Positive Orientation: For a **closed surface** that is the boundary of a solid region E , the convention is that: The **positive orientation** is the one for normal vectors point **outward** from E . **Inward**-pointing normals give the **negative orientation**.

Flux: If \mathbf{F} is **continuous** on oriented surface S with unit normal vector \mathbf{n} , then: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$

- Upward orientation: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$

- Downward orientation: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P \frac{\partial g}{\partial x} + Q \frac{\partial g}{\partial y} - R \right) dA$

Divergence: Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, where P, Q and R have first order derivatives in some region D .

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \text{ where } \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}.$$

- Divergence is positive \implies There is a net outflow.
- Divergence is negative \implies There is a net inflow.
- Gauss Theorem:** $\iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV$, when
- E is solid region where boundary surface S is piecewise smooth
- Positive (outward) orientation
- Component functions of $\mathbf{F}(x, y, z)$ have continuous partial derivatives on an open region that contains E

Curl: $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$.

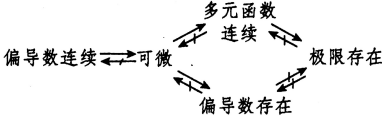
- $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0 \Rightarrow$ 判断是否是curl时看div是否为0
- $\nabla \cdot \nabla = \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} \mathbf{i} + \frac{\partial^2}{\partial y^2} \mathbf{j} + \frac{\partial^2}{\partial z^2} \mathbf{k}$
- $\operatorname{curl}(\mathbf{F})=0$ for conservative vector field \mathbf{F}
- For Sphere with ρ
 $\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \rho^2 \sin^2 \phi \cos \theta, \rho^2 \sin^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi \rangle$
- Stoke’s Theorem** $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, when
- C is boundary curve (simple closed) of surface S with u.n.v. \mathbf{n}
- Positively oriented with respect to \mathbf{n}
- Components have continuous partial derivatives on an open region

Special Case: Reduce to Green’s Theorem where

- S is flat.
- S lies in the xy -plane with upward orientation $\mathbf{n} = \mathbf{k}$.

Appendix

Relationship between basic concept



标准积分表

- $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax + b| + C$
- $\int \tan(ax + b) dx = \frac{1}{a} \ln | \sec(ax + b) | + C$
- $\int \sec(ax + b) dx = \frac{1}{a} \ln | \sec(ax + b) + \tan(ax + b) | + C$
- $\int \csc(ax + b) dx = -\frac{1}{a} \ln | \csc(ax + b) + \cot(ax + b) | + C$
- $\int \cot(ax + b) dx = -\frac{1}{a} \ln | \csc(ax + b) | + C$
- $\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + C$
- $\int \csc^2(ax + b) dx = -\frac{1}{a} \cot(ax + b) + C$
- $\int \sec(ax + b) \tan(ax + b) dx = \frac{1}{a} \sec(ax + b) + C$
- $\int \csc(ax + b) \cot(ax + b) dx = -\frac{1}{a} \csc(ax + b) + C$
- $\int \frac{1}{a^2 + (x+b)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x+b}{a} \right) + C$
- $\int \frac{1}{\sqrt{a^2 - (x+b)^2}} dx = \sin^{-1} \left(\frac{x+b}{a} \right) + C$
- $\int \frac{-1}{\sqrt{a^2 - (x+b)^2}} dx = \cos^{-1} \left(\frac{x+b}{a} \right) + C$
- $\int \frac{1}{a^2 - (x+b)^2} dx = \frac{1}{2a} \ln \left| \frac{x+b+a}{x+b-a} \right| + C$
- $\int \frac{1}{(x+b)^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x+b-a}{x+b+a} \right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2 + a^2}} dx = \ln \left| (x + b) + \sqrt{(x+b)^2 + a^2} \right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2 - a^2}} dx = \ln \left| (x + b) + \sqrt{(x+b)^2 - a^2} \right| + C$
- $\int \sqrt{a^2 - x^2} dx = \frac{\pi}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$
- $\int \sqrt{x^2 - a^2} dx = \frac{\pi}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + C$

因式分解

- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $a^3 - b^3 = (a - b) (a^2 + ab + b^2)$
- $a^3 + b^3 = (a + b) (a^2 - ab + b^2)$
- $a^n - b^n = (a - b) (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

求和 $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$

三角恒等变换 半角公式

- $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$
- $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$
- $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

和差化积公式

- $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$
- $\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$

- $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$
- $\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$

积化和差公式

- $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$
- $\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$
- $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$
- $\sin \alpha \sin \beta = -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$

万能公式

- $\sin \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$
- $\cos \alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$
- $\tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}$
- $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

三角函数积分公式

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, & \text{正奇数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{正偶数} \end{cases}$$

$$\int_0^{\pi} \sin^n x \, dx = \begin{cases} 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, & \text{正奇数} \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{正偶数} \end{cases}$$

$$\int_0^{\pi} \cos^n x \, dx = \begin{cases} 0, & \text{正奇数} \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{正偶数} \end{cases}$$

$$\int_0^{2\pi} \cos^n x \, dx = \int_0^{2\pi} \sin^n x \, dx = \begin{cases} 0, & \text{正奇数} \\ 4 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{正偶数} \end{cases}$$

积分的对称性

- 若 E 关于 xOy 对称, 且 $f(x, y, z) = f(x, y, -z)$

$$\iiint_E f(x, y, z) dV = 2 \iiint_{E_{z \geq 0}} f(x, y, z) dV$$

- 若积分曲线 C 关于 y 轴对称, 且 $f(x, y) = f(-x, y)$

$$\int_C f(x, y, z) dr = 2 \int_{C_{x \geq 0}} f(x, y) dr$$

- 若曲面 S 关于 xOy 对称, 且 $f(x, y, z) = f(x, y, -z)$

$$\iint_S f(x, y, z) dS = 2 \iint_{S_{z \geq 0}} f(x, y, z) dS$$

3-D diagram

Elliptical Helix	Cylinder	Elliptic paraboloid	Ellipsoid
$x^2 + \frac{y^2}{3} = 1$	$z = x^2$	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Double Cone	双叶双曲面	单叶双曲面	单叶双曲面 (马鞍面)
$\frac{z^2}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

参数方程与极坐标

Cycloid(摆线): $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad (0 \leq x \leq 2\pi)$

Cardioid(心脏线): $r(\theta) = 2a(1 - \cos \theta)$

提醒

- 求方向导数时要用单位向量
- 曲线积分中需要用到法向量时应该用单位法向量!