# ST2334 Cheatsheet AY23/24 —— @Jin Hang

## Chapter 1

**Observation:** Recording of information, numerical or categorical
**Statistical Exp:** Procedure that generates a set of observations
**Sample Space:** Set of all possible outcomes of a statistical experiment, represented by the symbol $S$.
**Sample Points:** Every outcome in a sample space.
**Events:** Subset of sample space.
**Simple/Compound Event:** Exactly one/More than one outcome or sample point.
**Sure/null Event:** Sample space/Event with no outcomes.

### Operations on Events

- **Complement**: $A'$ Elements not in $A$
- **Mutually Exclusive:(互斥)** $A \cap B = \emptyset$
- **Independent(独立):** $A \perp B \Rightarrow P(AB) = P(A) \cdot P(B)$
  1. **独立和互斥的关系:** 独立不互斥, 互斥不独立
  2. $S$ and $\emptyset$ are independent of any other event.
  3. If $A \perp B$, then $A \perp B', A' \perp B$, and $A' \perp B'$.
  4. $A \perp B \Rightarrow P(A) = P(A|B)$ & $P(B) = P(B|A)$
- **不相关:** $cov(X, Y) = 0$, 独立必不相关, 不相关未必独立
- **Union:** $A \cup B$ contains $A$ or $B$ or both elements
- **Intersection:** $A \cap B$ contains elements common to both

1. $A \cap A' = \emptyset$
2. $A \cap \emptyset = \emptyset$
3. $A \cup A' = S$
4. $(A')' = A$
5. $(A \cap B)' = A' \cup B'$
6. $(A \cup B)' = A' \cap B'$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup B = A \cup (B \cap A')$
10. $A = (A \cap B) \cup (A \cap B')$

$A \subset B$ if all elements in event $A$ are in event $B$, if $A \subset B$ and $B \subset A$ then $A = B$. We assume **contained** means proper subset.

### Permutations and Combinations

- P: Arrange $r$ objects from $n$ objects where $r \leq n$, ${}_nP_r = \frac{n!}{(n-r)!}$
- P: Number of ways around a circle $= (n-1)!$
- C: No. of ways to select $r$ from $n$ without regard to order: $\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$

### Properties

- $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$
- $\binom{n}{r} = \binom{n}{n-r}$
- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

### Permutations with Sets of Indistinguishable Objects

Suppose a collection consists of $n$ objects of which $n_1$ are of type 1 and are indistinguishable from each other $n_2$ are of type 2 and are indistinguishable from each other $n_k$ are of type $k$ and are indistinguishable from each other and suppose that $n_1 + n_2 + \ldots + n_k = n$. Then the number of distinguishable permutations of the $n$ objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

### 常用方法

| | Order Matters | Order Does Not Matter |
|---|---|---|
| Repetition | $n^k$ | $\binom{k+n-1}{k}$ |
| No Repetition | $P(n,k)$ | $\binom{n}{k}$ |

**方程模型** How many solutions are there to the give equations?
**(a)** $x_1 + x_2 + x_3 = 20$, each $x_i$ is a nonnegative integer.
**Solution.** $n = 3, r = 20$.

$$\binom{r+(n-1)}{r} = \binom{20+2}{20} = \binom{22}{20} = \frac{22 \cdot 21}{2} = 231$$

**(b)** $x_1 + x_2 + x_3 = 20$, each $x_i$ is a positive integer.
**Solution.** $\Leftrightarrow y_1 + y_2 + y_3 = 17$, each $y_i$ is a nonnegative integer

$$\binom{r+(n-1)}{r} = \binom{17+2}{17} = \binom{19}{17} = \frac{19 \cdot 18}{2} = 171$$

### Probability

- **Conditional:** $\mathrm{P}(A|B) = \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$, if $\Pr(A) \neq 0$
- **Multiplicative:** $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A) = \mathrm{P}(B)\mathrm{P}(A|B)$
- **LoTP:** $\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(B \cap A_i) = \sum_{i=1}^{n} \mathrm{P}(A_i)\mathrm{P}(B|A_i)$
- **Bayes:** $\mathrm{P}(A_k|B) = \frac{\mathrm{P}(A_k)\mathrm{P}(B|A_k)}{\sum_{i=1}^{n}\mathrm{P}(A_i)\mathrm{P}(B|A_i)}$

### Probability Axioms

- $P(A \cup B) = P(A) + P(B)$
- $P(A) = P(A \cap B) + P(A \cap B')$
- If $A \subset B$, then $P(A) \leq P(B)$
- If $P(A) \neq 0, A \perp B$ if and only if $P(B \mid A) = P(B)$

### Properties of Independent Events

1. $\mathrm{P}(B|A) = \mathrm{P}(B)$ and $\mathrm{P}(A|B) = \mathrm{P}(A)$
2. $A$ and $B$ cannot be mutually exclusive if they are independent, supposing $\mathrm{P}(A), \mathrm{P}(B) > 0$
3. $A$ and $B$ cannot be independent if they are mutually exclusive
4. Sample space $S$ and empty set $\emptyset$ are independent of any event
5. If $A \subset B$, then $A$ and $B$ are dependent unless $B = S$.

## Chapter 2

**Range Space:** The range space of $X$ is the set of real numbers $R_X = \{x | x = X(s), s \in S\}$. The set $X = x = s \in S : X(s) = x$ is a subset of $S$.

### Probability Mass Function

**Discrete Random Variable:**

1. $f(x) = \begin{cases} P(X = x), & for \quad x \in R_X \\ 0, & for \quad x \notin R_X \end{cases}$
2. $f(x_i) \geq 0$ for all $x_i$
3. $\sum_{i=1}^{\infty} f(x_i) = 1$
4. For $B \subset \mathbb{R}, P(X \in B) = \sum_{x_i \in B \cap R_X} f(x_i)$

**Continuous Random Variable:** The **probability density function** (p.d.f.) $f(x)$ of a continuous random variable must satisfy the following conditions

- $f(x) \geq 0$ for all $x \in R_X$ and $f(x) = 0$ for $x \notin R_X$ i.e. $\mathrm{P}(A) = 0$ does not imply $A = \emptyset$
- $\int_{R_X} f(x)dx = 1 \Rightarrow \int_{-\infty}^{+\infty} f(x)dx = 1$
- For any $(c, d) \subset R_X, c < d$, $\mathrm{P}(c \leq X \leq d) = \int_c^d f(x)dx$
- Specially $\mathrm{P}(X = x_0) = \int_{x_0}^{x_0} f(x)dx = 0$
- $P(a < X < b) = P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b) = \int_a^b f(x)dx$

### Cumulative Distribution Function c.d.f

$F(x)$ as cumulative distribution function (c.d.f.) of the random variable $X$ where $F(x) = \mathrm{P}(X \leq x)$. For any $a \leq b$, $\mathrm{P}(a \leq B \leq b) = \mathrm{P}(X \leq b) - \mathrm{P}(X < a) = F(b) - F(a^-)$ where $a^-$ is the largest possible value of $X$ that is strictly less than $a$. Actually, $F(a^-) = \lim_{x \to a^-} F(x)$

- $F(x) = \int_{-\infty}^{x} f(t)dt$
- $f(x) = \frac{dF(x)}{dx}$(连续点处)

**Properties:**

1. $F(x)$ is **non-decreasing**.
2. **one-to-one** correspondence
3. $F(x)$右连续, $F(a+0) = \lim_{x \to a^+} F(x) = F(a)$

**Range:** The ranges of $F(x)$ and $f(x)$ satisfy

- $0 \leq F(x) \leq 1$;
- for discrete distributions, $0 \leq f(x) \leq 1$;
- for continuous distributions, $f(x) \geq 0$, but **not necessary** that $f(x) \leq 1$.

### 利用CDF求概率问题

- $P\{X = x\} = F(x) - F(x-0)$
- $P\{X < x\} = F(x-0)$
- $P\{X > x\} = 1 - F(x)$
- $P\{X \geq x\} = 1 - F(x-0)$
- $P\{a < X \leq b\} = F(b) - F(a)$
- $P\{a < X < b\} = F(b-0) - F(a)$
- $P\{a \leq X < b\} = F(b-0) - F(a-0)$
- $P\{a \leq X \leq b\} = F(b) - F(a-0)$

### Expectation / Mean

- **Discrete:** $\mu_X = E(X) = \sum_i x_i f_X(x_i) = \sum_x x f_X(x)$
- **Cont.:** $\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$; $E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$

**Properties**

- $E(aX + b) = aE(X) + b$, where $a$ and $b$ are constants
- $E(X + Y) = E(X) + E(Y)$

**Variance** $g(x) = (x - \mu_X)^2$, leads us to the definition of variance.

$$\sigma_X^2 = V(X) = E[g(x)] = E[(X - \mu_X)^2] = \begin{cases} \sum_x (x - \mu_X)^2 f_X(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)dx, & \text{if } X \text{ is continuous.} \end{cases}$$

**Properties**

- $V(X) \geq 0$ for any $X$. Equality holds iff. $P(X = E(X)) = 1$ ($X$ is a constant).
- $\sigma_X = \sqrt{V(X)}$ (**Standard deviation**)
- $V(X) = E(X^2) - [E(X)]^2$
- $V(aX + b) = a^2V(X)$
- $V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCov(X, Y)$, if $X, Y$ independent, $Cov(X, Y) = 0$

## Chapter 3

$(X, Y)$ is a two-dimensional random variable, where $X, Y$ are functions assigning a real number to each $s \in S$.
**Range Space:** $R_{X,Y} = \{(x, y) | x = X(s), y = Y(s), s \in S\}$

### Joint Probability Function — Discrete Random Variables

$f_{X,Y}(x_i, y_j) = \mathrm{P}(X = x_i, Y = y_j)$

1. $f_{X,Y}(x_i, y_j) \geq 0$ for all $(x_i, y_j) \in R_{X,Y}$.
2. $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{X,Y}(x_i, y_j) = 1$.

### Joint Probability Function — Continuous Random Variables

1. $P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)dydx$

2. $f_{X,Y}(x,y) \le 0$ for all $(x,y) \in R_{X,Y}$.
3. $\iint_{(x,y)\in R_{X,Y}} f_{X,Y}(x,y)dxdy = \int_{\infty}^{\infty}\int_{\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$.

**Marginal Probabilty Densities**

- **Discrete:** $f_X(x) = \sum_y f_{X,Y}(x,y)$ and $f_Y(y) = \sum_x f_{X,Y}(x,y)$
- **Cont:** $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

**Conditional Probabilty Densities**

The conditional distribution of $Y$ given that $X = x$ is given by $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$, iff. $f_X(x) > 0$, for each $x$ within the range of $X$. Flip the variables for $X$ given $Y = y$.

1. Consider $y$ as the variable (and $x$ as a fixed value), $f_{Y|X}(y|x)$ is a **probability function**.
2. $f_{Y|X}(y|x)$ is NOT a probability function for $x \Rightarrow$ No need for $\int_{-\infty}^{+\infty} f_{Y|X}(y|x)dx = 1$ or $\sum_x f_{Y|X}(y|x) = 1$
3. if $f(x) > 0 \Rightarrow f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x)$
4. For Cont. $P(Y \le y|X = x) = \int_{-\infty}^{y} f_{Y|X}(y|x)dy$
   $E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$

**Independent Random Variables**

Random variables $X$ and $Y$ are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x, y$, extendable to $n$ variables.
$\Rightarrow$ Implying $R_{X,Y} = \{(x,y)|x \in R_X; y \in R_Y\}$
$\Rightarrow$ If $f_X(x) > 0$ then $f_{Y|X}(y|x) = f_Y(y)$
To Check independence, we have $f_{X,Y}(x,y) = C \times g_1(x) \times g_2(y)$

**Expectation**

1. Discrete, $E(g(X,Y)) = \sum_x \sum_y g(x,y) f_{X,Y}(x,y)$
2. Continous $E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dydx$
3. $E(XY) = \int\int xy(f(x,y))dydx$
4. $X \perp Y \Rightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \Rightarrow E(XY) = E(X)E(Y)$

**Covariance**

1. $cov(X,Y) = \sum_x \sum_y (x - \mu_X)(y - \mu_Y) f_{X,Y}(x,y)$
2. $cov(X,Y) = E[(X - E(X)(Y - E(Y)] = E(XY) - \mu_X \mu_Y$.
3. $X \perp Y \Rightarrow cov(X,Y) = 0$.
   $cov(X,Y) = 0 \not\Rightarrow X \perp Y$
4. $cov(X,Y) = cov(Y,X)$
5. $cov(X + b, Y) = cov(X,Y)$
6. $cov(aX, Y) = acov(X,Y)$
7. $cov(aX + b, cY + d) = accov(X,Y)$.
8. $V(X + Y) = V(X) + V(Y) + 2cov(X,Y)$
9. $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab \cdot cov(X,Y)$.
10. If $X \perp Y$, $V(aX - bY) = a^2V(X) + b^2V(Y)$ $(cov = 0)$
11. $V(X - Y) = V(X) + V(Y)$

## Appendix

**标准积分表**

- $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$
- $\int \tan(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b)| + C$
- $\int \sec(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C$
- $\int \csc(ax+b)dx = -\frac{1}{a}\ln|\csc(ax+b) + \cot(ax+b)| + c$
- $\int \cot(ax+b)dx = -\frac{1}{a}\ln|\csc(ax+b)| + C$
- $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$
- $\int \csc^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$
- $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$
- $\int \csc(ax+b)\cot(ax+b)dx = -\frac{1}{a}\csc(ax+b) + C$
- $\int \frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx = \sin^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{-1}{\sqrt{a^2-(x+b)^2}}dx = \cos^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{1}{a^2-(x+b)^2}dx = \frac{1}{2a}\ln\left|\frac{x+b+a}{x+b-a}\right| + C$
- $\int \frac{1}{(x+b)^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x+b-a}{x+b+a}\right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2+a^2}}dx = \ln\left|(x+b) + \sqrt{(x+b)^2+a^2}\right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2-a^2}}dx = \ln\left|(x+b) + \sqrt{(x+b)^2-a^2}\right| + C$
- $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
- $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$

**三角恒等变换**

半角公式

- $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$
- $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$
- $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$
- $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$
- $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$
- $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$

和差化积公式

- $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
- $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
- $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
- $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

积化和差公式

- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
- $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$
- $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$
- $\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

万能公式

- $\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$
- $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$
- $\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$
- $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$