CS1231S Trick Summary

Basic Theorem and Reference

Theorem 2.1.1

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \textbf{true} \equiv p$	$p \vee \textbf{false} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \textbf{true}$	$p \wedge {\sim}p \equiv \textbf{false}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \textbf{true} \equiv \textbf{true}$	$p \wedge \textbf{false} \equiv \textbf{false}$
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \quad \equiv p$	$p \wedge (p \vee q) \quad \equiv p$
11	Negation of **true** and **false**	${\sim}\textbf{true} \equiv \textbf{false}$	${\sim}\textbf{false} \equiv \textbf{true}$

Implication law, contrapositive, converse and inverse

Inplication law: $p \rightarrow q \equiv \sim p \vee q$

Contrapositive: $p \rightarrow q \equiv \sim q \rightarrow \sim p$

Converse $\Leftrightarrow$ Inverse: $q \rightarrow p \equiv \sim p \rightarrow \sim q$

Rule of inference		
Modus Ponens	$p \rightarrow q$ p • q	
Modus Tollens	$p \rightarrow q$ ${\sim}q$ • ${\sim}p$	
Generalization	p • $p \vee q$	q • $p \vee q$
Specialization	$p \wedge q$ • p	$p \wedge q$ • q
Conjunction	p q • $p \wedge q$	

Rule of inference		
Elimination	$p \vee q$ ${\sim}q$ • p	$p \vee q$ ${\sim}p$ • q
Transitivity	$p \rightarrow q$ $q \rightarrow r$ • $p \rightarrow r$	
Proof by Division Into Cases	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ • r	
Contradiction Rule	${\sim}p \rightarrow \textbf{false}$ • p	

Theorem 6.2.2 Set Identities

1	Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
2	Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
3	Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4	Identity laws	$A \cup \varnothing = A$	$A \cap U = A$
5	Complement laws	$A \cup \bar{A} = U$	$A \cap \bar{A} = \varnothing$
6	Double Complement Law	$\bar{\bar{A}} = A$	
7	Idempotent laws	$A \cup A = A$	$A \cap A = A$
8	Universal bound laws	$A \cup U = U$	$A \cap \varnothing = \varnothing$
9	De Morgan's laws	$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$
10	Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
11	Complements of U and $\varnothing$	$\bar{U} = \varnothing$	$\bar{\varnothing} = U$
12	Set Difference Law	$A \setminus B = A \cap \bar{B}$	

Advanced Concept (Only use them in MCQ)

变元易名后的分配律	常见等值公式
$(\forall x)(\forall y)(P(x) \vee Q(y)) = (\forall x)P(x) \vee (\forall x)Q(x)$ $(\exists x)(\exists y)(P(x) \wedge Q(y)) = (\exists x)P(x) \wedge (\exists x)Q(x)$	$P \to Q = \neg P \vee Q$ $P \to Q = \neg Q \to \neg P$ $P \to (Q \to R) = (P \wedge Q) \to R$ $P \leftrightarrow Q = (P \wedge Q) \vee (\neg P \wedge \neg Q)$ $P \leftrightarrow Q = (P \vee \neg Q) \wedge (\neg P \vee Q)$ $P \leftrightarrow Q = (P \to Q) \wedge (Q \to P)$ $P \to (Q \to R) = Q \to (P \to R)$ $(P \to R) \wedge (Q \to R) = (P \vee Q) \to R$ $P \to (Q \to R) = (P \wedge Q) \to R$
量词对合取、析取的分配律 2 $(\forall x)(P(x) \wedge Q(x)) = (\forall x)P(x) \wedge (\forall x)Q(x)$ $(\exists x)(P(x) \vee Q(x)) = (\exists x)P(x) \vee (\exists x)Q(x)$	

基本推理公式	
$P \wedge Q \Rightarrow P$	$\neg Q \wedge (P \to Q) \Rightarrow \neg P$
$\neg(P \to Q) \Rightarrow P$	$(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$
$\neg(P \to Q) \Rightarrow \neg Q$	$(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow P \leftrightarrow R$
$P \Rightarrow P \vee Q$	$(P \to R) \wedge (Q \to R) \wedge (P \vee Q) \Rightarrow R$
$\neg P \Rightarrow P \to Q$	$(P \to Q) \wedge (R \to S) \wedge (P \vee R) \Rightarrow Q \vee S$
$Q \Rightarrow P \to Q$	$(P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S) \Rightarrow \neg P \vee \neg R$
$\neg P \wedge (P \vee Q) \Rightarrow Q$	$(Q \to R) \Rightarrow ((P \vee Q) \to (P \vee R))$
$P \wedge (P \to Q) \Rightarrow Q$	$(Q \to R) \Rightarrow ((P \to Q) \to (P \to R))$

Prove a Statement is valid / sound

Given an argument: $p1$ $p2$ … pk $\therefore q$ where $p1$, $p2$, … , pk are the k premises and q the conclusion	we can say that **the argument is valid** if and only if **$(p1 \wedge p2 \wedge \cdots \wedge pk) \longrightarrow q$ is a tautology** This serves as an alternative way to check whether an argument is valid, besides the critical row method shown in lecture.

Some Concepts

sound	if, and only if, it is ***valid*** and ***all its premises are true***
fallacies	The argument is **valid** by modus ponens, but its **major premise is false**
Vacuously true	A conditional statement that is **true** by virtue of the fact that its **hypothesis is false**. e.g. F→T is true; If 1=0, then 1=2(F→F); P(x)→Q(x) and P(x) is false for every x in D

Prove a argument is valid **by contradiction**.

- List the premises given by the qns
- Use the (negative conclusion)[Start of proof by contradiction]
- Basic Operation **remember to use generalization and specialization.**
- When it come to a false, end of the proof

Write down the statement

1. be careful with “and” and “imply”
2. when encounter “exist”, think “what if the premise have no element”

3. when encounter “all” be careful with “and” between p and q

Basic Concepts

(tut1 Q4) $(p \to q) \to r$ is **NOT** logically equivalent to $p \to (q \to r)$

(tut1 Q10) The product of any two odd integers is an odd integer

(tut1 Q11) Let n be an integer. Then n^2 is odd if and only if n is odd.

comma does not represent conjunction, disjunction, or any logical connective

(tut2 Q11) Prove that if n is a product of two positive integers a and b, then $a \le n^{1/2}$ or $b \le n^{1/2}$

(Lec3#38) $\forall x \in A,\ P(x)$ is vacuously true if $A = \varnothing$

Sets

1. Suppose $|A| = n$, $|B| = k$.

$|A \times B| = nk$ (cardinality of Cartesian product)

$|\mathcal{P}(B)| = 2^k$

$|\mathcal{P}(A \times B)| = 2^{nk}$

2. For all sets A, B, C

$A \cap (B \setminus C) = (A \cap B) \setminus C$

$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Advance Concept (Only use them in MCQ)

Subset $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$ $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$ $(A \subseteq B) \wedge (C \subseteq D) \Rightarrow (A \cup C) \subseteq (B \cup D)$ $(A \subseteq B) \wedge (C \subseteq D) \Rightarrow (A \cap C) \subseteq (B \cap D)$ $(A \subseteq B) \wedge (C \subseteq D) \Rightarrow (A \setminus D) \subseteq (B \setminus C)$ $C \subseteq D \Rightarrow (A \setminus D) \subseteq (A \setminus C)$	**Power Set** $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ $A = B \Leftrightarrow P(A) = P(B)$ $P(A) \in P(B) \Rightarrow A \in B$ $P(A) \cap P(B) = P(A \cap B)$ $P(A) \cup P(B) \subseteq P(A \cup B)$ $P(A - B) \subseteq (P(A) - P(B)) \cup \{\varnothing\}$
Set difference $A \setminus B = A \setminus (A \cap B)$ $A \setminus B = A \cap (A \setminus B)$ $A \cup (B \setminus A) = A \cup B$ $A \cap (B \setminus C) = (A \cap B) \setminus C$	**Unions and Intersections of an Indexed Collection of Sets** $A \subseteq B \Rightarrow \cup A \subseteq \cup B$ $A \subseteq B \Rightarrow \cap B \subseteq \cap A$(其中$A$, B非空) $\cup(A \cup B) = (\cup A) \cup (\cup B)$ $\cap(A \cup B) = (\cap A) \cap (\cap B)$(其中$A$, B非空) $\cup(P(A)) = A$
Cartesian Products $A \times \varnothing = \varnothing \times B = \varnothing$ 若$A \neq \varnothing$, $B \neq \varnothing$且$A \neq B$, 则$A \times B \neq B \times A$ $A \times (B \times C) \neq (A \times B) \times C$ $A \times (B \cup C) = (A \times B) \cup (A \times C)$ $A \times (B \cap C) = (A \times B) \cap (A \times C)$ $(B \cup C) \times A = (B \times A) \cup (C \times A)$ $(B \cap C) \times A = (B \times A) \cap (C \times A)$ (若$C \neq \varnothing$, 则$(A \subseteq B) \Leftrightarrow (A \times C \subseteq B \times C) \Leftrightarrow (C \times A \subseteq C \times B)$) $(A \times B \subseteq C \times D) \Leftrightarrow (A \subseteq C \wedge B \subseteq D)$	**Other** $A \cup B = A \cup C, A \cap B = A \cap C \Rightarrow B = C$

Relation

Consider the relation $S = \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$, then $S^{-1} = S \circ S = S \circ S^{-1} = S$

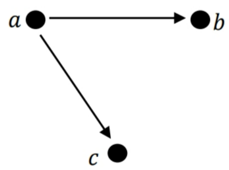
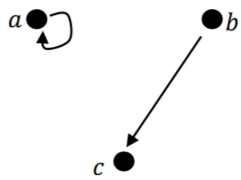
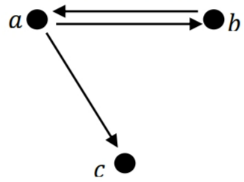
$T \circ (S \circ R) = (T \circ S) \circ R$ (composition of relations is associative)

Common Relation

Relation	Definition	
reflexive	$\forall x \in A\ (x, x) \in R$	Check all elements
irreflexive	$\forall x \in A\ (x, x) \notin R$	Check specific
symmetric	$\forall x, y \in A\ ((x, y) \in R \Rightarrow (y, x) \in R)$	
antisymmetric	$\forall x, y \in A\ ((x, y) \in R \land (y, x) \in R \Rightarrow x = y)$	
asymmetric	$\forall x, y \in A\ ((x, y) \in R \Rightarrow (y, x) \notin R)$	
transitive	$\forall x, y, z \in A\ ((x, y) \in R \land (y, z) \in R \Rightarrow (x, z) \in R)$	Try to draw Hasse diagram st

Note:

1. Asymmetric and Antisymmetric
   a) Every asymmetric relation is antisymmetric
   b) Let $A = \{a, b, c\}$

	antisymmetric	Not antisymmetric
asymmetric		**Every asymmetric relation is antisymmetric**
Not asymmetric		

2. Suppose a Relation R on Set A, consider the follow situation

<table>
<tr><td></td><td>$A = \varnothing$</td><td>$A \neq \varnothing$</td></tr>
<tr><td>$R = \varnothing$</td><td rowspan="2"><table><tr><td>ref</td><td>irr</td><td>sym</td><td>anti</td><td>assy</td><td>trans</td></tr><tr><td>T</td><td>T</td><td>T</td><td>T</td><td>T</td><td>T</td></tr></table>Relation on Empty Set is Equivalence</td><td><table><tr><td>ref</td><td>irr</td><td>sym</td><td>anti</td><td>assy</td><td>trans</td></tr><tr><td>F</td><td>T</td><td>T</td><td>T</td><td>T</td><td>T</td></tr></table></td></tr>
<tr><td>$R \neq \varnothing$</td><td>It depends</td></tr>
</table>

Maximal, minimal, largest, smallest

Suppose c is a element of A

	Definition	
maximal	either $x \leqslant c$, or x and c are **not comparable:** $\forall x \in A\ (c \leqslant x \Rightarrow c = x.)$	
largest	$\forall x \in A\ (x \leqslant c)$	
minimal	either $c \leqslant x$, or x and c are **not comparable:** $\forall x \in A\ (x \leqslant c \Rightarrow c = x)$	
smallest	$\forall x \in A\ (c \leqslant x)$	

1. In a partially ordered set, if there is **a smallest element**, then there must be **exactly one minimal element**. But, when there is **exactly one minimal element**, we **CANNOT** say there is **a smallest element**
   Example: Suppose an infinite partial order. There is exactly one element a_0 and an infinite chain $a_1 > a_2 > \cdots > a_n > \cdots$, there is a minimal element a_0 but there is no smallest element as there will always be an element a_{n+1} smaller than a_n.

2.

Comparable and Compatible

comparable	$a \leqslant b$ or $b \leqslant a$	
compatible	there exists $c \in A$ such that $a \leqslant c$ and $b \leqslant c$	

In all partially ordered sets

any two comparable elements are compatible.	$comparable \underset{\not\Leftarrow}{\overset{\Longrightarrow}{}} compatible$
any two compatible elements **may NOT** be comparable.	

Partial Order Relation, Total Order Relation and Linearization

R	Definition	rk
partial order	R is reflexive, antisymmetric and transitive	1
total order	all the elements of a partial order relation R are **comparable**: $\forall x, y \in A\ x R y \vee y R x$	2
Linearization	linearization of partial order $\leqslant$ is a **total order** $\leqslant^*$ on A s.t. $\forall x, y \in A\ x \leqslant y \Rightarrow x \leqslant^* y$.	3
Well-ordered	Let $\leqslant$ be a total order on set A, A is well-ordered iff every **non-empty** subset of A contains a **smallest element**.	

Note:

1. Hasse diagram for a **total order** relation can be drawn as a **single vertical "chain."**

   i.e. "divides" relation is not a total order relation unless the elements are all powers of a single integer.

2. any **subset** of a partially ordered set is partially ordered.
3. (Extra Definition) A subset B of A is called a **chain** iff. the elements in each pair of elements in B are **comparable**.
4.

Constructing a Topological Sorting

Let $\preceq$ be a partial order relation on a nonempty finite set A. To construct a topological sorting:

1. Pick any minimal element x in A. *[Such an element exists since A is nonempty.]*
2. Set $A' := A - \{x\}$.
3. Repeat steps a–c while $A' \neq \emptyset$.
   a. Pick any minimal element y in A'.
   b. Define $x \preceq' y$.
   c. Set $A' := A' - \{y\}$ and $x := y$.

   [Completion of steps 1–3 of this algorithm gives enough information to construct the Hasse diagram for the total ordering $\preceq$. We have already shown how to use the Hasse diagram to obtain a complete directed graph for a relation.]

Special Concepts in tutorial Question Sheet

对称差： $A \oplus B = (A \backslash B) \cup (B \backslash A)$

交换律 $A \oplus B = B \oplus A$

结合律 $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

分配率 $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

同一率 $A \oplus \varnothing = A$

零率 $A \oplus A = \varnothing$

$A \oplus (A \oplus B) = B$