MA2104 CheatSheet AY23/24 —— @Jin Hang

Vector

Projections: vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$

Scalar Projection: The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ (component of $\boldsymbol{b}$ along $\boldsymbol{a}$) $\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\| \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Relationship:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\cdot\mathbf{a}$$

Theorem 6: The vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Cross product angle formula: $\|\boldsymbol{a}\times\boldsymbol{b}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\sin(\widehat{\boldsymbol{a},\boldsymbol{b}})$

Properties of cross product

- $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$
- $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
- $(d\boldsymbol{a})\times\boldsymbol{b} = d(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times(d\boldsymbol{b})$
- $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
- $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b})$
- $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\cdot\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$
- $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}] = [\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}] = [\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}] = -[\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}] = -[\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}] = -[\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}] = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Distance from a point to a line: Suppose P and R are two points on L. Distance from point Q to line L is $\|\overrightarrow{PQ}\|\sin\theta = \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}$

Relationship between 2 lines In 2-D, two lines are either parallel or intersect. In 3-D, two lines are either

- parallel
- non-parallel and intersect
- non-parallel and non-intersecting (**skew lines**)

Show that the lines are skew

Assume for a contradiction that L_1 and L_2 intersect.
Then there must exist a choice of the parameter t and s such that the values of x; y and z are the same.

Find the line of intersection of two planes

Solving both equations for x and setting them to be equal. Let $y = t$ be the parameter, obtain a parametric equation for line of intersection

简便方法: 交线的方向向量为两平面法向量的叉乘

Find the distance of two planes

Find two point on P_1 and P_2, assume they are (a_1, b_2, c_3) and (a_2, b_2, c_2), then let $\boldsymbol{u} = (a_1-a_2, b_1-b_2, c_1-c_2)$. Assume the normal vector to the planes is $\boldsymbol{n}$, then $d = \|\mathbf{u}\|\,|\cos\theta| = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|}\right|$

Properties of Vector

1. $\text{comp}_{\mathbf{b}}(\mathbf{a}+\mathbf{c}) = \text{comp}_{\mathbf{b}}(\mathbf{a}) + \text{comp}_{\mathbf{b}}(\mathbf{c})$
2. **Triangle Inequality**
   - $||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$
   - $||\mathbf{a}|| - ||\mathbf{b}|| \le ||\mathbf{a}-\mathbf{b}||$

Vector-valued function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We say that r(t) is a **parametrization** of C. A curve C can have more than one **parameterizations**.

- $\mathbf{r(t)}$ = the position vector of a particle in space at time t
- $\mathbf{r}'(t)$ = the velocity of the particle at time t
- $\|\mathbf{r}'(t)\|$ = speed of the particle at time t
- $\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a)\rangle$
- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)\cdot\mathbf{s}(t) = \mathbf{r}'(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\mathbf{s}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)\times\mathbf{s}(t)) = \mathbf{r}'(t)\times\mathbf{s}(t) + \mathbf{r}(t)\times\mathbf{s}'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}|r(\mathbf{t})| = \frac{1}{|r(\mathbf{t})|}\cdot r(\mathbf{t})\cdot r'(\mathbf{t})$
- **Chain Rule** $\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{r}(t)) = \bigtriangledown f(\mathbf{r}(t))\cdot\mathbf{r}'(t)$
- **求两向量函数在某一点的夹角** $\Leftrightarrow$ 求导 $\rightarrow$ 方向向量夹角

Arc Length: Let C be the curve given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \le t \le b$$

where f', g', h' are continuous, its length is

$$s = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\mathrm{d}t = \int_a^b \|\mathbf{r}'(t)\|\,\mathrm{d}t$$

Partial Derivatives

Find the range of $f(x,y)$: i.e. for $f(x,y) = x\ln(y^2-x)$
Only when $y^2 - x > 0 \Leftrightarrow y^2 > x \Leftrightarrow$ Domain$=\{(x,y)\in\mathbf{R} : y^2 > x\}$

Level Curve: A level curve of $f(x,y)$ is the two-dimensional graph of the equation $f(x,y) = k$ for some constant k.

Contour Plot: A contour plot of $f(x,y)$ is a graph of numerous level curves $f(x,y) = k$, for representative values of k.

Sketch contour plots Use values of k that are equally spaced

- The traces in $x = k/y = k/z = k$ is ...
- Draw diagram of curve in xy, yz, xz plane and decide outline

Level Surface: A level surface of $f(x,y,z)$ is the 3-D graph of the equation $f(x,y,z) = k$ for some constant k.

Limit: Limit of $f(x,y)$ as (x,y) approaches (a,b) is $L\in\mathbb{R}$, denoted by

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any number $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x,y) - L| < \epsilon$, whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$

How to show limit does not exist

- Consider the limit along the path P_1. We have ...
- Consider the limit along the path P_2. We have ...
- Since this limit along P_2 is different from the one we had previously for P_1, we conclude that the limit does not exist.

How to show limit exists

1. we can deduce it from known/simple functions using properties of limits or continuity
2. we can use the **Squeeze theorem**
   - $|f(x,y) - L| \le g(x,y)\forall(x,y)$ close to (a,b)
   - $\lim_{(x,y)\to(a,b)} g(x,y) = 0$

   Then, $\lim_{(x,y)\to(a,b)} f(x,y) = L$

Limit Theorems

- $\lim_{(x,y)\to(a,b)}(f\pm g) = \lim_{(x,y)\to(a,b)} f \pm \lim_{(x,y)\to(a,b)} g$
- $\lim_{(x,y)\to(a,b)}(f\cdot g) = \left(\lim_{(x,y)\to(a,b)} f\right)\cdot\left(\lim_{(x,y)\to(a,b)} g\right)$
- $\lim_{(x,y)\to(a,b)}\frac{f}{g} = \frac{\lim_{(x,y)\to(a,b)} f}{\lim_{(x,y)\to(a,b)} g}$, where $\lim_{(x,y)\to(a,b)} g(x,y) \ne 0$

极坐标换元法证明极限存在/不存在（通法）

- Let $f(x,y) = 0 \rightarrow f(r\cos\theta, r\sin\theta) = 0$, 即让极径$r$来代表趋近值，让极角 θ 来代表趋近路径
- $\lim_{(x,y)\to(a,b)} f(x,y) \Longrightarrow \lim_{r\to k^+} f(r\cos\theta, r\sin\theta)$
- 最终结果
  1. 如果**只**包含θ变量，则该极限不存在
  2. 如果$\ldots = r^n\sin^a\theta\cos^b\theta$则极限为0
  3. 若$\ldots = \frac{r^n}{\sin^a\theta\pm\cos^b\theta}$ $\Rightarrow$讨论是否存在分母为0的情况判断极限是否存在

Continuity of $f(x,y)$: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Continuity Theorems: If $f(x,y)$ and $g(x,y)$ are continuous at (a,b)

- $f\pm g$ is is continuous at (a,b)
- $f\cdot g$ is is continuous at (a,b)
- $\frac{f}{g}$ is is continuous at (a,b), provided $g(a,b)\ne 0$

Continuity and Composition: Suppose $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at $f(a,b)$. Then

$$h(x,y) = (g\circ f)(x,y) = g(f(x,y))$$

is continuous at (a,b).

Continuity Functions: Following are continuous in its domain.

- Polynomial in x and y;
- Trigonometric and exponential functions in x and y;
- Rational function in x and y.

Partial Derivative (Definition)

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h}$$

Differentiability Let $z = f(x,y)$, f is differentiable at (a,b) if

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy which vanish(i.e. $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$)

Increment: Let $z = f(x,y)$. Suppose Δx and Δy are increments in the independent variable x and y respectively from a fixed point (a,b), then $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

Linear approximation: $\Delta z \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$

Implicit Differentiation: Suppose the equation $F(x,y,z) = 0$, where F is differentiable, defines z implicitly as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

Directional derivative: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of unit vector $\boldsymbol{u} = \langle a,b\rangle$ is

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = \bigtriangledown f\cdot\boldsymbol{u}$$

Geometric: Suppose surface S given by $z = f(x,y)$, where f is differentiable function of x and y at (x_0,y_0) and $\bigtriangledown f(x_0,y_0)\ne\mathbf{0}$.

- $\bigtriangledown f(x_0,y_0)$ is normal to level curve $f(x,y) = k$ containing (x_0,y_0)
- **Normal vector** to S at (a,b): $\langle f_x(a,b), f_y(a,b), -1\rangle$
- **Tangent Plane** $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
- **Tangent Plane to Level Surface**

$$\bigtriangledown F(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$$

Saddle Point: Let $f(x,y): D\to\mathbb{R}$. Point (a,b) is saddle point if

- $f_x(a,b) = f_y(a,b) = 0$; AND
- every neighborhood at (a,b) contains points $(x,y)\in 2D$ for which $f(x,y) < f(a,b)$ and points $(x,y)\in 2D$ for which $f(x,y) > f(a,b)$

Closed Set in $\mathbb{R}^2$ Contains all its boundary points.

Boundary Point Point (a,b) s.t. every disk with center (a,b) contains points in R and also points in $\mathbb{R}^2\setminus R$)

Bounded Set in $\mathbb{R}^2$ A set $R\subseteq\mathbb{R}^2$ bounded if it is contained within some disk. In other words, it is finite in extent

The Closed Interval Method / Find the extreme value

- Find the values of f at its **critical points** in D.
- Find the **extreme values** of f on the **boundary** of D. Let $x = n$, represent $f(x,y)$ only in y, same for let $y = n$ ...
- The largest of the values from Step 1 and Step 2 is the **absolute** max. value; the smallest of these values is the **absolute** min. value.

Lagrange Multiplier: Suppose $f(x,y)$ and $g(x,y)$ are differentiable s.t. $\bigtriangledown g(x,y)\ne\mathbf{0}$ on constraint curve $g(x,y) = k$. If (x_0,y_0) is a local minimum/maximum of $f(x,y)$ constrained by $g(x,y) = k$. Then $\bigtriangledown f(x_0,y_0) = \lambda\bigtriangledown g(x_0,y_0)$ for some constant λ (Lagrange Multiplier).

- Solve the following system of equations:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$$

- Find the point(s) (x,y) satisfying the above equations
- Evaluate f at all the points obtained. The largest of these values is the maximum value of f; The smallest is the minimum value of f.

Double Integral

$$\iint_R f(x,y)\mathrm{d}A = \lim_{n\to\infty}\sum_{i=1}^n\sum_{j=1}^n f\left(a+\frac{b-a}{n}i, c+\frac{d-c}{n}j\right)\cdot\frac{b-a}{n}\cdot\frac{d-c}{n}$$

- **Volume** $V = \iint_D f(x,y)\mathrm{d}A$
- **Area** $A(D) = \iint_D 1\mathrm{d}A$

Fubini's Theorem If f is continuous on $R = [a,b]\times[c,d]$, then $\iint_R f(x,y)\mathrm{d}A = \int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x = \int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y$

Polar Coordinates in Double Integral If f is continuous on polar rectangle $R = \{(r,\theta): 0\le a\le r\le b, \alpha\le\theta\le\beta\}$, $0\le\beta-\alpha\le 2\pi$, then $\iint_R f(x,y)\mathrm{d}A = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta$

Tripe Integral over Type 1/2/3 Regions $\iiint_E f(x,y,z)dV =$

1. $\iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\mathrm{d}z\right]\mathrm{d}A$
2. $\iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\mathrm{d}x\right]\mathrm{d}A$
3. $\iint_D\left[\int_{u_1(z,x)}^{u_2(z,x)} f(x,y,z)\mathrm{d}y\right]\mathrm{d}A$

Spherical Coordinates: $\iiint_E f(x,y,z)dV =$
$\int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$

重积分性质

- **可加性** $\iint_{D_1\cup D_2} f(x,y)\mathrm{d}A = \iint_{D_1} f(x,y)\mathrm{d}A + \iint_{D_2} f(x,y)\mathrm{d}A$
- **保号性** $f(x,y)\le g(x,y)\Leftrightarrow\iint_D f(x,y)\mathrm{d}A\le\iint_D g(x,y)\mathrm{d}A$
- **轮换对称性** x, y对调,积分域D不变,则 $\iint_D f(x,y)\mathrm{d}A = \iint_D f(y,x)\mathrm{d}A$

Jacobian Transformation: $x = x(u,v), y = y(u,v)$

- $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$ $\left(\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2\sin\phi\right)$
- $\iint_R f(x,y)dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$

常见Jacobian行列式

- **Spherical Coordinates:** $\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\rho^2\sin\phi$
- **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Set $au = x, bv = y, cw = z$
  $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc \Rightarrow V = \frac{4}{3}\pi abc$

Line Integral of scalar field $\int_C f(x,y)ds = \sum\int_{C_i} f(x,y)ds =$
$\int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int_a^b f(x(t),y(t))\,\|\mathbf{r}'(t)\|\,dt$

- represents the area of the **'fence'**
- independent of the orientation. of $\mathbf{r}(t)$.

Parametrization of a line segment: $\mathbf{r}(t)=\mathbf{r}_0+t\left(\mathbf{r}_1-\mathbf{r}_0\right)$

Vector Field: Let $D \subseteq \mathbb{R}^3$. A vector field on D is a function $\mathbf{F}$ that assigns to each point $(x, y, z) \in D$ a 3 -D vector

$$\mathbf{F}(x, y, z)=P(x, y, z) \mathbf{i}+Q(x, y, z) \mathbf{j}+R(x, y, z) \mathbf{k}$$

Line Integral of Vector Field Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C, $\mathbf{r}(t)=\langle x(t), y(t), z(t)\rangle, a \leq t \leq b$. Then, $\int_C \mathbf{F} \cdot d\mathbf{r}=\int_C \mathbf{F} \cdot \mathbf{T} d s=\int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}^{\prime}(t) d t$

Property: $\int_{-C} \mathbf{F} \cdot d\mathbf{r}=-\int_C \mathbf{F} \cdot d\mathbf{r}$

Component Form: If $\mathbf{F}(x, y, z)=\langle P, Q, R\rangle$ then we can write $\int_C \mathbf{F} \cdot d\mathbf{r}=\int_C P d x+Q d y+R d z=\int_a^b \mathbf{F}(x(t), y(t), z(t)) \mathbf{r} \prime(t) d t$

Conservative Vector A vector field $\mathbf{F}$ is **conservative vector:** field on D if we can write $\mathbf{F}=\nabla f$ for some **potential function** f on D.

Test for Conservative: $\mathbf{F}$ is conservative on D if

- P, Q and R have **continuous** first-order partial **derivatives** on D
- $\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$

Fundamental Theorem: (**Independent** of paths) From A to B

$$\int_C \mathbf{F} \cdot d\mathbf{r}=\int_C \nabla f \cdot d\mathbf{r}=f(B)-f(A)=f\left(x_2, y_2\right)-f\left(x_1, y_1\right)$$

Theorem P, Q在单连通域D上有一阶连续偏导数, 则以下四条等价

- 线积分 $\int_L P \mathrm{~d} x+Q \mathrm{~d} y$ 与路径无关
- $\oint_L P \mathrm{~d} x+Q \mathrm{~d} y=0$, 其中 L 为 D 中任一分段光滑闭曲线
- $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}, \forall(x, y) \in D$
- $P(x, y) \mathrm{d} x+Q(x, y) \mathrm{d} y=\mathrm{d} F(x, y)$

Green's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r}=\int_C P d x+Q d y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) d A=\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

- C is positively oriented(逆时针), piecewisesmooth, simple **closed**
- D is region bounded by C.
- P, Q have continuous partial derivatives on open region contains D
- If $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, but P, Q not continuous at (a, b)
  - All integral, whose region contains (a, b) have the same value Introduce a **unit** circle T centered at (a, b)
  $$\iint_S \mathbf{F} \cdot d\mathbf{S}-\iint_T \mathbf{F} \cdot d\mathbf{T}=\iiint_{E^{\prime}} \operatorname{div}(\mathbf{F}) d V=0$$
  $$\therefore \iint_S \mathbf{F} \cdot d\mathbf{S}=\iint_T \mathbf{F} \cdot d\mathbf{T}$$
  - All integral, whose region **do not** contains (a, b) are 0

Theorem 2: Area $=\int_C x d y=-\int_C y d x=\frac{1}{2}\left(\int_C x d y-y d x\right)$

Surface Integral

Parametric Surface: $\mathbf{r}(u, v)=\langle x(u, v), y(u, v), z(u, v)\rangle$

Smooth Surface: For $\mathbf{r}(u, v)$, $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(u, v) \in D$.

Tangent Plane of Smooth Surface $\boldsymbol{n}=\mathbf{r}_u(a, b) \times \mathbf{r}_v(a, b)$

Surface Integral: $\iint_S f(x, y, z) d S=\sum \iint_{S_i} f(x, y, z) d S$

$=\iint_D f(x(u, v), y(u, v), z(u, v))\left\|\mathbf{r}_u \times \mathbf{r}_v\right\| d A$

$=\iint_D f(x, y, g(x, y))\left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\right) d A$

- For surface $z=g(x, y)$, $\mathbf{r}(x, y)=\langle x, y, g(x, y)\rangle$
  $\Rightarrow \mathbf{r}_x \times \mathbf{r}_y=\left\langle-g_x,-g_y, 1\right\rangle$

Surface Area: $\iint_S d S=\iint_D\left\|\mathbf{r}_u \times \mathbf{r}_v\right\| d A$

Oriented Surface: A surface S is **orientable** (or two-sided) if it is possible to define a unit normal vector $\mathbf{n}$ at each point (x, y, z) not on the boundary of surface s.t. $\mathbf{n}$ is a **continuous** function of (x, y, z).

Positive Orientation: For a **closed surface** that is the boundary of a solid region E, the convention is that: The **positive orientation** is the one for normal vectors point **outward** from E. **Inward**-pointing normals give the **negative orientation**.

Flux: If $\mathbf{F}$ is **continuous** on oriented surface S with unit normal vector $\mathbf{n}$, then: $\iint_S \mathbf{F} \cdot d\mathbf{S}=\iint_S \mathbf{F} \cdot \mathbf{n} d S=\iint_D \mathbf{F} \cdot\left(\mathbf{r}_u \times \mathbf{r}_v\right) d A$

- Upward orientation: $\iint_S \mathbf{F} \cdot d\mathbf{S}=\iint_D\left(-P \frac{\partial g}{\partial x}-Q \frac{\partial g}{\partial y}+R\right) d A$
- Downward orientation: $\iint_S \mathbf{F} \cdot d\mathbf{S}=\iint_D\left(P \frac{\partial g}{\partial x}+Q \frac{\partial g}{\partial y}-R\right) d A$

Divergence: Let $\mathbf{F}(x, y, z)=P(x, y, z) \mathbf{i}+Q(x, y, z) \mathbf{j}+R(x, y, z) \mathbf{k}$, where P, Q and R have first order derivatives in some region D.

$\operatorname{div} \mathbf{F}=\nabla \cdot \mathbf{F}=\left\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot\langle P, Q, R\rangle=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$, where $\nabla=\frac{\partial}{\partial x} \mathbf{i}+\frac{\partial}{\partial y} \mathbf{j}+\frac{\partial}{\partial z} \mathbf{k}$.

- Divergence is positive $\Longrightarrow$ There is a net outflow.
- Divergence is negative $\Longrightarrow$ There is a net inflow.

Gauss Theorem: $\iint_S \mathbf{F} \cdot d\mathbf{S}=\iiint_E \operatorname{div} \mathbf{F} d V$, when

- E is solid region where boundary surface S is piecewise smooth
- Positive (outward) orientation
- Component functions of $\mathbf{F}(x, y, z)$ have continuous partial derivatives on an open region that contains E

Curl: $\operatorname{curl} \mathbf{F}=\nabla \times \mathbf{F}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right) \mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right) \mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) \mathbf{k}$.

- $\operatorname{div}(\operatorname{curl}(\mathbf{F}))=0 \Rightarrow$ 判断是否是curl时看div是否为0
- $\nabla \cdot \nabla=\nabla^2=\triangle=\frac{\partial^2}{\partial x^2} \boldsymbol{i}+\frac{\partial^2}{\partial y^2} \boldsymbol{j}+\frac{\partial^2}{\partial z^2} \boldsymbol{k}$
- curl($\mathbf{F}$)=0 for conservative vector field $\mathbf{F}$
- For Sphere with ρ
  $\mathbf{r}_\phi \times \mathbf{r}_\theta=\left\langle\rho^2 \sin ^2 \phi \cos \theta, \rho^2 \sin ^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi\right\rangle$

Stoke's Theorem $\int_C \mathbf{F} \cdot d\boldsymbol{r}=\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, when

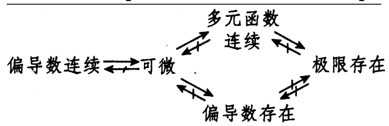
- C is boundary curve (simple closed) of surface S with u.n.v. $\mathbf{n}$
- Positively oriented with respect to $\mathbf{n}$
- Components have continuous partial derivatives on an open region

Special Case: Reduce to Green's Theorem where

1. S is flat.
2. S lies in the xy-plane with upward orientation $\mathbf{n}=\mathbf{k}$.

Appendix

Relationship between basic concept



标准积分表

- $\int \frac{1}{a x+b} d x=\frac{1}{a} \ln |a x+b|+C$
- $\int \tan (a x+b) d x=\frac{1}{a} \ln |\sec (a x+b)|+C$
- $\int \sec (a x+b) d x=\frac{1}{a} \ln |\sec (a x+b)+\tan (a x+b)|+C$
- $\int \csc (a x+b) d x=-\frac{1}{a} \ln |\csc (a x+b)+\cot (a x+b)|+c$
- $\int \cot (a x+b) d x=-\frac{1}{a} \ln |\csc (a x+b)|+C$
- $\int \sec ^2(a x+b) d x=\frac{1}{a} \tan (a x+b)+C$
- $\int \csc ^2(a x+b) d x=-\frac{1}{a} \cot (a x+b)+C$
- $\int \sec (a x+b) \tan (a x+b) d x=\frac{1}{a} \sec (a x+b)+C$
- $\int \csc (a x+b) \cot (a x+b) d x=-\frac{1}{a} \csc (a x+b)+C$
- $\int \frac{1}{a^2+(x+b)^2} d x=\frac{1}{a} \tan ^{-1}\left(\frac{x+b}{a}\right)+C$
- $\int \frac{1}{\sqrt{a^2-(x+b)^2}} d x=\sin ^{-1}\left(\frac{x+b}{a}\right)+C$
- $\int \frac{-1}{\sqrt{a^2-(x+b)^2}} d x=\cos ^{-1}\left(\frac{x+b}{a}\right)+C$
- $\int \frac{1}{a^2-(x+b)^2} d x=\frac{1}{2 a} \ln \left|\frac{x+b+a}{x+b-a}\right|+C$
- $\int \frac{1}{(x+b)^2-a^2} d x=\frac{1}{2 a} \ln \left|\frac{x+b-a}{x+b+a}\right|+C$
- $\int \frac{1}{\sqrt{(x+b)^2+a^2}} d x=\ln \left|(x+b)+\sqrt{(x+b)^2+a^2}\right|+C$
- $\int \frac{1}{\sqrt{(x+b)^2-a^2}} d x=\ln \left|(x+b)+\sqrt{(x+b)^2-a^2}\right|+C$
- $\int \sqrt{a^2-x^2} d x=\frac{x}{2} \sqrt{a^2-x^2}+\frac{a^2}{2} \sin ^{-1} \frac{x}{a}+C$
- $\int \sqrt{x^2-a^2} d x=\frac{x}{2} \sqrt{x^2-a^2}-\frac{a^2}{2} \ln \left|x+\sqrt{x^2-a^2}\right|+C$

因式分解

- $(a+b)^3=a^3+3 a^2 b+3 a b^2+b^3$
- $(a-b)^3=a^3-3 a^2 b+3 a b^2-b^3$
- $a^3-b^3=(a-b)\left(a^2+a b+b^2\right)$
- $a^3+b^3=(a+b)\left(a^2-a b+b^2\right)$
- $a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2} b+\cdots+a b^{n-2}+b^{n-1}\right)$

求和 $\sum_{k=1}^n k^2=\frac{n(n+1)(2 n+1)}{6}$

三角恒等变换 半角公式

- $\cos ^2 \alpha=\frac{1+\cos 2 \alpha}{2}$
- $\sin ^2 \alpha=\frac{1-\cos 2 \alpha}{2}$
- $\tan 2 \alpha=\frac{2 \tan \alpha}{1-\tan ^2 \alpha}$
- $\sin \frac{\alpha}{2}=\pm \sqrt{\frac{1-\cos \alpha}{2}}$
- $\cos \frac{\alpha}{2}=\pm \sqrt{\frac{1+\cos \alpha}{2}}$
- $\tan \frac{\alpha}{2}=\frac{\sin \alpha}{1+\cos \alpha}=\frac{1-\cos \alpha}{\sin \alpha}$

和差化积公式

- $\sin \alpha+\sin \beta=2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$
- $\sin \alpha-\sin \beta=2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$
- $\cos \alpha+\cos \beta=2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$
- $\cos \alpha-\cos \beta=-2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}$

积化和差公式

- $\sin \alpha \cos \beta=\frac{1}{2}[\sin (\alpha+\beta)+\sin (\alpha-\beta)]$
- $\cos \alpha \sin \beta=\frac{1}{2}[\sin (\alpha+\beta)-\sin (\alpha-\beta)]$
- $\cos \alpha \cos \beta=\frac{1}{2}[\cos (\alpha+\beta)+\cos (\alpha-\beta)]$
- $\sin \alpha \sin \beta=-\frac{1}{2}[\cos (\alpha+\beta)-\cos (\alpha-\beta)]$

万能公式

- $\sin \alpha=\frac{2 \tan \frac{\alpha}{2}}{1+\tan ^2 \frac{\alpha}{2}}$
- $\cos \alpha=\frac{1-\tan ^2 \frac{\alpha}{2}}{1+\tan ^2 \frac{\alpha}{2}}$
- $\tan \alpha=\frac{2 \tan \frac{\alpha}{2}}{1-\tan ^2 \frac{\alpha}{2}}$
- $\tan (\alpha+\beta)=\frac{\tan \alpha+\tan \beta}{1-\tan \alpha \tan \beta}$

三角函数积分公式

$$\int_0^{\frac{\pi}{2}} \sin ^n x \mathrm{~d} x=\int_0^{\frac{\pi}{2}} \cos ^n x \mathrm{~d} x=\begin{cases}\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, \text{正奇数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \text{正偶数}\end{cases}$$

$$\int_0^\pi \sin ^n x \mathrm{~d} x=\begin{cases}2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, \text{正奇数} \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \text{正偶数}\end{cases}$$

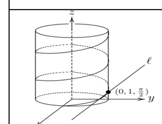
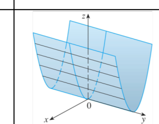
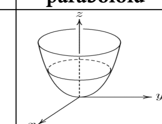
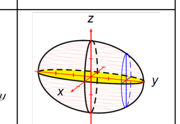
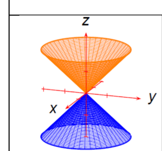
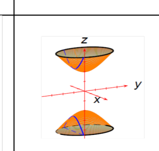
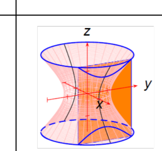
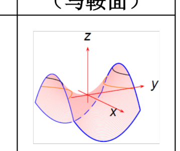
$$\int_0^\pi \cos ^n x \mathrm{~d} x=\begin{cases}0 & \text{, 正奇数} \\ 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{, 正偶数}\end{cases}$$

$$\int_0^{2 \pi} \cos ^n x \mathrm{~d} x=\int_0^{2 \pi} \sin ^n x \mathrm{~d} x=\begin{cases}0 & \text{, 正奇数} \\ 4 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{, 正偶数}\end{cases}$$

积分的对称性

- 若E关于xOy对称，且$f(x, y, z)=f(x, y,-z)$
$$\iiint_E f(x, y, z) \mathrm{d} V=2 \iiint_{E_{z \geq 0}} f(x, y, z) \mathrm{d} V$$
- 若积分曲线C关于y轴对称，且$f(x, y)=f(-x, y)$
$$\int_C f(x, y, z) \mathrm{d} r=2 \int_{C_{x \geq 0}} f(x, y) \mathrm{d} r$$
- 若曲面S关于xOy对称，且$f(x, y, z)=f(x, y,-z)$
$$\iint_S f(x, y, z) \mathrm{d} S=2 \iint_{S_{z \geq 0}} f(x, y, z) \mathrm{d} S$$

3-D diagram

Elliptical Helix	Cylinder	Elliptic paraboloid	Ellipsoid
			
$x^2+\frac{y^2}{3}=1$	$z=x^2$	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Double Cone	**双叶双曲面**	**单叶双曲面**	**单叶双曲面（马鞍面）**
			
$\frac{z^2}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

参数方程与极坐标

Cycloid(摆线): $\begin{cases}x=a(t-\sin t) \\ y=a(1-\cos t)\end{cases}(0 \leq x \leq 2 \pi)$

Cardioid(心脏线): $r(\theta)=2 a(1-\cos \theta)$

提醒

- 求方向导数时要用单位向量
- 曲线积分中需要用到法向量时应该用单位法向量！