MA2104 CheatSheet AY23/24 —— @Jin Hang

Vector

Projections: vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$

Scalar Projection: The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ (also called the component of $\boldsymbol{b}$ along $\boldsymbol{a}$) is defined to be the signed magnitude of the vector projection: $\text{comp}_{\mathbf{a}}\mathbf{b} = \|\mathbf{b}\|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$

Relationship:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{a}}\mathbf{b} \times \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\cdot\mathbf{a}$$

Theorem 6: The vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Cross product angle formula: $\|\boldsymbol{a}\times\boldsymbol{b}\| = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\sin\theta$ Note that θ is the angle between vector $\mathbf{a}$ and $\mathbf{b}$

Properties of cross product

- $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$
- $(d\boldsymbol{a})\times\boldsymbol{b} = d(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times(d\boldsymbol{b})$
- $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
- $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
- $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}] = [\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}] = [\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}] = -[\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}] = -[\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}] = -[\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}] = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Distance from a point to a line: Suppose P and R are two points on L. To find the distance from the point Q to the line L

$$\|\overrightarrow{PQ}\|\sin\theta = \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}$$

Relationship between 2 lines In 2-D, two lines are either parallel or intersect. In 3-D, two lines are either

- parallel
- non-parallel and intersect
- non-parallel and non-intersecting (**skew lines**)

Show that the lines are skew

Assume for a contradiction that L_1 and L_2 intersect.
Then there must exist a choice of the parameter t and s such that the values of x; y and z are the same.

Find the line of intersection of two planes

Solving both equations for x and setting them to be equal. Let $y = t$ be the parameter, we obtain a parametric equation for the line of intersection.

简便方法：交线的方向向量为两平面法向量的叉乘

Find the curve of intersection of a surface and a plane

Step 1 For $\begin{cases} a_1x+b_1+c_1z=d_1 \\ a_2x^2+b_2y^2+c_2z^2+ex+fy+gz=d_2 \end{cases}$

let $z = \frac{1}{c_1}(d_1 - a_1x - b_1y)$ then subsitute to the surface equation

Step 2 整理成$m^2(x+p)^2+n^2(y+q)^2=r^2$的形式

Step 3 参数化，令

$$\begin{cases} x(t) = \frac{1}{m}r\cos t - p \\ y(t) = \frac{1}{n}r\sin t - q \\ z(t) = \text{Combin.}(x,y) \end{cases}$$

Step 4 曲线用向量函数表示为$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$

Find the curve of intersection of two surfaces

空间曲线的性质 (1) 设空间曲线 Γ 由参数方程

$$\begin{cases} x = \varphi(t) \\ y = \psi(t), (t\in[\alpha,\beta]) \\ z = \omega(t) \end{cases}$$

给出，其中 $\varphi(t),\psi(t),\omega(t)$ 均可导，$P_0(x_0,y_0,z_0)$是 Γ 上对应 $t=t_0$ 的点，且当 $t=t_0$ 时，$\varphi'(t_0),\psi'(t_0),\omega'(t_0)$ 都不为 0，则曲线 Γ 在点 $P_0(x_0,y_0,z_0)$ 处的

1. 切向量为 $\tau = (\varphi'(t_0),\psi'(t_0),\omega'(t_0))$.
2. 切线方程为 $\frac{x-x_0}{\varphi'(t_0)} = \frac{y-y_0}{\psi'(t_0)} = \frac{z-z_0}{\omega'(t_0)}$.
3. 法平面 (过点 $P_0(x_0,y_0,z_0)$ 且与切线垂直的平面) 方程为
$$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=0$$

(2) 设空间曲线 Γ 由交面式方程 $\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$ 给出，则在以下表达式有意义的条件下，曲线 Γ 在点 $P_0(x_0,y_0,z_0)$ 处的

(1) 切向量为

$$\tau = \left(\begin{vmatrix} F_y' & F_z' \\ G_y' & G_z' \end{vmatrix}_{P_0}, \begin{vmatrix} F_z' & F_x' \\ G_z' & G_x' \end{vmatrix}_{P_0}, \begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix}_{P_0}\right)$$

(2) 切线方程为

$$\frac{x-x_0}{\begin{vmatrix} F_y' & F_z' \\ G_y' & G_z' \end{vmatrix}_{P_0}} = \frac{y-y_0}{\begin{vmatrix} F_z' & F_x' \\ G_z' & G_x' \end{vmatrix}_{P_0}} = \frac{c_x-z_0}{\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix}_{P_0}}.$$

(3)法平面方程为

$$\begin{vmatrix} F_y' & F_z' \\ G_y' & G_z' \end{vmatrix}(x-x_0)+\begin{vmatrix} F_z' & F_x' \\ G_z' & G_x' \end{vmatrix}(y-y_0)+\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix}(z-z_0)=0$$

旋转曲面问题

曲线 $\Gamma: \begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$ 绕直线 $L: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$ 旋转形成一个旋转曲面,旋转曲面方程的求法如下.

已知点 $M_0(x_0,y_0,z_0)$, 方向向量 $\boldsymbol{s}=(m,n,p)$. 在母线 Γ 上任取一点 $M_1(x_1,y_1,z_1)$, 则过 M_1 的纬圆上任意一点 $P(x,y,z)$ 满足条件

$$\overrightarrow{M_1P}\perp\boldsymbol{s},\quad \left|\overrightarrow{M_0P}\right| = \left|\overrightarrow{M_0M_1}\right|,$$

$$即\begin{cases} m(x-x_1)+n(y-y_1)+p(z-z_1)=0 \\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = \\ (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2 \end{cases}$$

与方程 $F(x_1,y_1,z_1)=0$ 和 $G(x_1,y_1,z_1)=0$ 联立消去 x_1,y_1,z_1, 便可得到旋转曲面的方程.

Coplanar(共面) $\Rightarrow \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \mathbf{0}$

Find the distance of two planes

Find two point on P_1 and P_2, assume they are (a_1,b_2,c_3) and (a_2,b_2,c_2), then let

$$\boldsymbol{u} = (a_1-a_2, b_1-b_2, c_1-c_2)$$

Assume the normal vector to the planes is $\boldsymbol{n}$, then

$$d = \|\mathbf{u}\|\,|\cos\theta| = \left|\frac{\mathbf{u}\cdot\mathbf{n}}{\|\mathbf{n}\|}\right|$$

Vector-valued function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$$

We say that r(t) is a **parametrization** of C. A curve C can have more than one **parameterizations**.

Sketch the curve traced out by the vector-valued function

Interpretation: $\mathbf{r(t)}$ = the position vector of a particle in space at time t, then $\mathbf{r}'(t)$ = the velocity of the particle at time t. Hence, $\|\mathbf{r}'(t)\|$ = speed of the particle at time t:

Derivative of Vector-valued Function: $\mathbf{r}$ is differentiable at $t=a$ and its derivative is given by $\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a)\rangle$.

- $\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{s}(t) = \mathbf{r}'(t)\cdot\mathbf{s}(t)+\mathbf{r}(t)\cdot\mathbf{s}'(t)$
- $\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{s}(t)) = \mathbf{r}'(t)\times\mathbf{s}(t)+\mathbf{r}(t)\times\mathbf{s}'(t)$

Find the tangent line L to the curve r(t)

- We need to determine t first
- Find vector $\boldsymbol{u} = \mathbf{r}'(t)$
- Then, vector $\boldsymbol{u}$ is the direction vector of the tangent line of $\mathbf{r(t)}$ at point P

求两向量函数在某一点的夹角 $\Leftrightarrow$ 求导 $\rightarrow$ 方向向量夹角

Arc Length: Let C be the curve given by

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}\quad a\le t\le b$$

where f', g', h' are continuous. If C is traversed exactly once as t increases from a to b, then its length is

$$s = \int_a^b\sqrt{f'(t)^2+g'(t)^2+h'(t)^2}\,\mathrm{d}t = \int_a^b\|\mathbf{r}'(t)\|\,\mathrm{d}t$$

Partial Derivatives

Find the range of $f(x,y)$: i.e. for $f(x,y) = x\ln(y^2-x)$
Only when $y^2-x>0 \Leftrightarrow y^2>x \Leftrightarrow$ Domain=$\{(x,y)\in\mathbf{R}: y^2>x\}$

Level Curve: A level curve of $f(x,y)$ is the two-dimensional graph of the equation $f(x,y)=k$ for some constant k.

Contour Plot: A contour plot of $f(x,y)$ is a graph of numerous level curves $f(x,y)=k$, for representative values of k.

Sketch contour plots Use values of k that are equally spaced

- The traces in $x=k$ is ...
- The traces in $y=k$ is ...
- The traces in $z=k$ is ...
- Try to draw the diagram of curve in xy, yz, xz plane and decide the outline. (can refer to the appendix)

Level Surface: A level surface of $f(x,y,z)$ is the 3-D graph of the equation $f(x,y,z)=k$ for some constant k.

Limit: Let f be a function of two variables whose domain D contains points arbitrarily close to (a,b). We say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is $L\in\mathbb{R}$, denoted by

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any number $\epsilon>0$ there exists a number $\delta>0$ such that $|f(x,y)-L|<\epsilon$, whenever $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$

How to show limit does not exist

- Consider the limit along the path P_1. We have ...
- Consider the limit along the path P_2. We have ...
- Since this limit along P_2 is different from the one we had previously for P_1, we conclude that the limit does not exist.

How to show limit exists

- we can deduce it from known/simple functions using properties of limits or continuity
- we can use the **Squeeze theorem**

Limit Theorems

- $\lim_{(x,y)\to(a,b)}(f(x,y)\pm g(x,y)) = \lim_{(x,y)\to(a,b)}f(x,y)\pm\lim_{(x,y)\to(a,b)}g(x,y)$
- $\lim_{(x,y)\to(a,b)}(f(x,y)\cdot g(x,y)) = (\lim_{(x,y)\to(a,b)}f(x,y))\cdot(\lim_{(x,y)\to(a,b)}g(x,y))$
- $\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)}f(x,y)}{\lim_{(x,y)\to(a,b)}g(x,y)}$, where $\lim_{(x,y)\to(a,b)}g(x,y)\ne 0$

Squeeze Theorem Suppose

- $|f(x,y)-L|\le g(x,y)\forall(x,y)$ close to (a,b)
- $\lim_{(x,y)\to(a,b)}g(x,y) = 0$

Then, $\lim_{(x,y)\to(a,b)}f(x,y) = L$

极坐标换元法证明极限存在/不存在（通法）

- Let $f(x,y)=0 \to f(r\cos\theta, r\sin\theta)=0$, 即让极径$r$来代表趋近值，让极角 θ 来代表趋近路径
- $\lim_{(x,y)\to(a,b)} f(x,y) \implies \lim_{r\to k^+} f(r\cos\theta, r\sin\theta)$
- 最终结果
  1. 如果只包含θ变量，则该极限不存在
  2. 如果$\ldots = r^n \sin^a\theta\cos^b\theta$则极限为0
  3. 若$\ldots = \frac{r^n}{\sin^a\theta \pm \cos^b\theta}$则需要讨论是否存在分母为0的情况判断极限是否存在

Continuity of $f(x,y)$: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Continuity Theorems: If $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then

- $f \pm g$ is is continuous at (a,b)
- $f \cdot g$ is is continuous at (a,b)
- $\frac{f}{g}$ is is continuous at (a,b), provided $g(a,b)\neq 0$

Continuity and Composition: Suppose $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at $f(a,b)$. Then

$$h(x,y) = (g\circ f)(x,y) = g(f(x,y))$$

is continuous at (a,b).

Continuity Functions: Subsequently, the following classes of functions are continuous in its domain.

- Polynomial in x and y;
- Trigonometric and exponential functions in x and y;
- Rational function in x and y.

Partial Derivative: If f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Equation of Tangent Plane: Consider the surface S given by $z=f(x,y)$. A normal vector to the tangent plane to S at (a,b) is $\langle f_x(a,b), f_y(a,b), -1\rangle$. Hence, tangent plane

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Increment: Let $z=f(x,y)$. Suppose Δx and Δy are increments in the independent variable x and y respectively from a fixed point (a,b), then $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

Differentiability - Two Variable Let $z=f(x,y)$. We say that f is differentiable at (a,b) if we can write

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1\Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy which vanish(i.e. $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$)

Linear approximation: $\Delta z \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$

Implicit Differentiation: Suppose the equation $F(x,y,z)=0$, where F is differentiable, defines z implicitly as a differentiable function of x and y. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

Directional derivative: The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of unit vector $\boldsymbol{u} = \langle a,b\rangle$ is

$$D_{\mathbf{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h} = \nabla f \cdot \boldsymbol{u}$$

Level Curve vs ∇f: Suppose $f(x,y)$ is differentiable function of x and y at (x_0,y_0). Suppose $\nabla f(x_0,y_0) \neq \mathbf{0}$. Then $\nabla f(x_0,y_0)$ is normal to the level curve $f(x,y)=k$ that contains the point (x_0,y_0). Similar in x-y-z for $\nabla F(x_0,y_0,z_0)$

Tangent Plane to Level Surface:

$$\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$$

Saddle Point: Let $f(x,y): D \to \mathbb{R}$. Then A point (a,b) is called a saddle point of f if

- $f_x(a,b) = f_y(a,b) = 0$; AND
- every neighborhood at (a,b) contains points $(x,y)\in 2D$ for which $f(x,y) < f(a,b)$ and points $(x,y)\in 2D$ for which $f(x,y) > f(a,b)$

Closed Set in $\mathbb{R}^2$ A set $R \subseteq \mathbb{R}^2$ is closed if it contains all its boundary points. (A boundary point of R is a point (a,b) such that every disk with center (a,b) contains points in R and also points in $\mathbb{R}^2 \setminus R$)

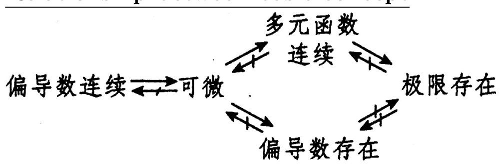
Bounded Set in $\mathbb{R}^2$ A set $R\subseteq\mathbb{R}^2$ bounded if it is contained within some disk. In other words, it is finite in extent

The Closed Interval Method / Find the extreme value

- Find the values of f at its critical points in D.
- Find the extreme values of f on the boundary of D. Let $x=n$, represent $f(x,y)$ only in y, same for let $y=n$ ...
- The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Appendix

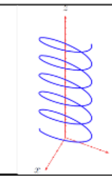
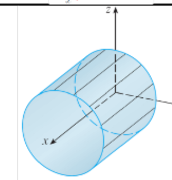
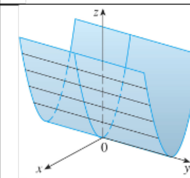
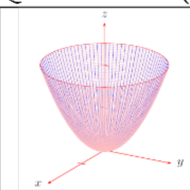
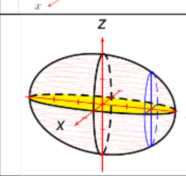
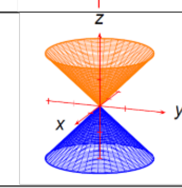
Relationship between basic concept

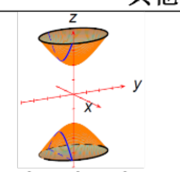
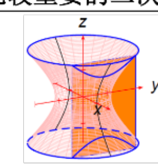
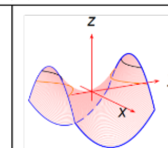


标准积分表

- $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$
- $\int \tan(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b)| + C$
- $\int \sec(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)| + C$
- $\int \csc(ax+b)dx = -\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)| + c$
- $\int \cot(ax+b)dx = -\frac{1}{a}\ln|\csc(ax+b)| + C$
- $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$
- $\int \csc^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$
- $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$
- $\int \csc(ax+b)\cot(ax+b)dx = -\frac{1}{a}\csc(ax+b) + C$
- $\int \frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx = \sin^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{-1}{\sqrt{a^2-(x+b)^2}}dx = \cos^{-1}\left(\frac{x+b}{a}\right) + C$
- $\int \frac{1}{a^2-(x+b)^2}dx = \frac{1}{2a}\ln\left|\frac{x+b+a}{x+b-a}\right| + C$
- $\int \frac{1}{(x+b)^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x+b-a}{x+b+a}\right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2+a^2}}dx = \ln\left|(x+b)+\sqrt{(x+b)^2+a^2}\right| + C$
- $\int \frac{1}{\sqrt{(x+b)^2-a^2}}dx = \ln\left|(x+b)+\sqrt{(x+b)^2-a^2}\right| + C$
- $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
- $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

3-D diagram

名称	图像	方程
elliptical helix		$x^2+\left(\frac{y}{3}\right)^2=1$
Cylinder (柱面) [1] 圆柱面		$y^2+z^2=1$
Cylinder (柱面) [2] 抛物柱面		$z=x^2$
Quadric Surface (二次曲面)		
Elliptic paraboloid 椭圆抛物面		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$
Ellipsoid 椭球面		$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 特别的，$a=b=c$ 时为球面
Double Cone 椭圆锥面		$\frac{z^2}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

其他比较重要的二次曲面		
 $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Hyperboloid of 2 sheet	 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of 1 sheet	 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ Hyperbolic Paraboloid
双叶双曲面	单叶双曲面	双曲抛物面（马鞍面）