MA1522 Linear Algebra for Computing Final Trick Sheet
Methodology

Find the condition when A is invertible

Let $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the condition when $\boldsymbol{A}$ is invertible.

Let $\boldsymbol{B}=\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Suppose that $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{I}$.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\boldsymbol{AB}=\begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$.

Solve a linear system in x, y, z, w:

- $$\begin{cases} ax \quad\; + bz \quad\;\; = 1 \\ \quad ay \quad\; + bw = 0 \\ cx \quad\; + dz \quad\;\; = 0 \\ \quad cy \quad\; + dw = 1 \end{cases} \Rightarrow \begin{cases} ax+bz=1 \\ cx+dz=0 \end{cases} \quad \begin{cases} ay+bw=0 \\ cy+dw=1 \end{cases}$$
  - They are inconsistent $\Leftrightarrow a:c=b:d \Leftrightarrow ad=bc$.
  - They are consistent $\Leftrightarrow ad\neq bc$.

- If $ad=bc$, then $\boldsymbol{A}$ is singular. Suppose that $ad\neq bc$.
  - $\begin{cases} ax+bz=1 \\ cx+dz=0 \end{cases} \Rightarrow x=\dfrac{d}{ad-bc}, z=\dfrac{-c}{ad-bc}$.
  - $\begin{cases} ay+bw=0 \\ cy+dw=1 \end{cases} \Rightarrow y=\dfrac{-b}{ad-bc}, w=\dfrac{a}{ad-bc}$.

  Let $\boldsymbol{B}=\begin{pmatrix} x & y \\ z & w \end{pmatrix}=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

  - One verifies that $\boldsymbol{AB}=\boldsymbol{I}$ and $\boldsymbol{BA}=\boldsymbol{I}$.

Find Inverse

Example. Let $\boldsymbol{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. Find $\boldsymbol{A}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2+3R_3]{R_1-3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Therefore, $\boldsymbol{A}^{-1}=\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$.

LU Decomposition

Let $\boldsymbol{A}=\begin{pmatrix} 2 & 1 & 3 & 0 & 4 \\ 5 & 2 & 4 & 3 & 1 \\ 2 & 1 & 4 & -1 & 2 \\ 5 & 5 & 2 & 3 & 4 \end{pmatrix}$.

- $$\boldsymbol{A} \xrightarrow[R_4-\frac{5}{2}R_1]{\substack{R_2-\frac{5}{2}R_1 \\ R_3-R_1}} \bullet \xrightarrow{R_4+5R_2} \bullet \xrightarrow{R_4+23R_3} \begin{pmatrix} 2 & 1 & 3 & 0 & 4 \\ 0 & -\frac{1}{2} & -\frac{7}{2} & 3 & -9 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -5 & -97 \end{pmatrix}=\boldsymbol{U}.$$
  - $$\boldsymbol{U} \xrightarrow{R_4-23R_3} \bullet \xrightarrow{R_4-5R_2} \bullet \xrightarrow[R_4+\frac{5}{2}R_1]{\substack{R_2+\frac{5}{2}R_1 \\ R_3+R_1}} \boldsymbol{A}.$$
  - $$\boldsymbol{I} \xrightarrow{R_4-23R_3} \bullet \xrightarrow{R_4-5R_2} \bullet \xrightarrow[R_4+\frac{5}{2}R_1]{\substack{R_2+\frac{5}{2}R_1 \\ R_3+R_1}} \boldsymbol{L}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{5}{2} & -5 & -23 & 1 \end{pmatrix}.$$

Solve *Ax*=*B* using LU Decomposition

Solve $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

- $\boldsymbol{A} = \boldsymbol{LU}$, $\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$, $\boldsymbol{U} = \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$.

1. Let $\boldsymbol{y} = \boldsymbol{Ux}$ and solve $\boldsymbol{Ly} = \boldsymbol{b}$.
   - $(\boldsymbol{L} \mid \boldsymbol{b}) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 & 2 \\ 1 & -3 & 1 & 0 & 3 \\ -3 & 4 & 2 & 1 & 4 \end{array}\right) \Rightarrow \boldsymbol{y} = \begin{pmatrix} 1 \\ 4 \\ 14 \\ -37 \end{pmatrix}$.

2. Solve $\boldsymbol{Ux} = \boldsymbol{y}$.
   - $(\boldsymbol{U} \mid \boldsymbol{y}) = \left(\begin{array}{ccccc|c} 2 & 4 & -1 & 5 & -2 & 1 \\ 0 & 3 & 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 0 & 0 & 0 & 5 & -37 \end{array}\right) \Rightarrow \boldsymbol{x} = \begin{pmatrix} \frac{7}{6}t - \frac{29}{4} \\ -\frac{1}{3}t - \frac{66}{5} \\ t \\ \frac{107}{10} \\ -\frac{37}{5} \end{pmatrix}$.

Find Determinants using Elementary Matrix

Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

$$\boldsymbol{A} \xrightarrow[\boldsymbol{E}_1]{R_2+\frac{4}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_2]{R_3-6R_2} \bullet \xrightarrow[\boldsymbol{E}_3]{-\frac{1}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_4]{3R_2} \bullet$$
$$\xrightarrow[\boldsymbol{E}_5]{-\frac{1}{34}R_3} \bullet \xrightarrow[\boldsymbol{E}_6]{R_1+\frac{4}{3}R_3} \bullet \xrightarrow[\boldsymbol{E}_7]{R_2-19R_3} \bullet \xrightarrow[\boldsymbol{E}_8]{R_1-\frac{2}{3}R_2} \boldsymbol{I}.$$

- $\det(\boldsymbol{E}_i^{-1}) = 1$ for $i = 1, 2, 4, 6, 7, 8$.
- $\det(\boldsymbol{E}_3^{-1}) = -3$, $\det(\boldsymbol{E}_4^{-1}) = \frac{1}{3}$, $\det(\boldsymbol{E}_5^{-1}) = -34$.

$\det(\boldsymbol{A}) = \det(\boldsymbol{E}_1^{-1}) \cdots \det(\boldsymbol{E}_8^{-1}) = (-3) \cdot \frac{1}{3} \cdot (-34) = 34$.

Find Determinants using Elementary Row Operation

Find $\det(\boldsymbol{A})$, where $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_1]{R_2-R_1} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$
$$\xrightarrow[\boldsymbol{E}_2]{R_2 \leftrightarrow R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_3]{R_4-2R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \boldsymbol{B}.$$

- $\det(\boldsymbol{B}) = 3 \cdot 2 \cdot 1 \cdot (-1) = -6.$
  - $\det(\boldsymbol{A}) = (-1)^1 \det(\boldsymbol{B}) = 6$.

Determinant of 2×2 Matrix

Consider the linear system $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

- Suppose that $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible.
  - $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{pmatrix}$$

  - $x_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$ and $x_2 = \dfrac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$.

Cramer's Rule

$\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. Verify that $\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix} = 60$.

- $x = \dfrac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{60} = \dfrac{132}{60} = 2.2$
- $y = \dfrac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{60} = \dfrac{-24}{60} = -0.4$
- $z = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{60} = \dfrac{-36}{60} = -0.6$

Find Implicit Forms

Example. $\begin{cases} x + y + z = 0, \\ x - y + 2z = 1. \end{cases}$

- An implicit form of the solution set is
  - $\{(x, y, z) \mid x + y + z = 0 \text{ and } x - y + 2z = 1\}$.

  The solution set is the intersection of two non-parallel plane in $\mathbb{R}^3$.
  - It is a straight line in $\mathbb{R}^3$.

Find Explicit Forms

Example. $\begin{cases} x + y + z = 0, \\ x - y + 2z = 1. \end{cases}$

- $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right)$.
  - $x = \frac{1}{2} - \frac{3}{2}t$, $y = -\frac{1}{2} + \frac{1}{2}t$, $z = t$, where $t \in \mathbb{R}$.
- An explicit form of the solution set is
  - $\{(\frac{1}{2} - \frac{3}{2}t, -\frac{1}{2} + \frac{1}{2}t, t) \mid t \in \mathbb{R}\}$.

Lines in $\mathbf{R}^2$

Example. Find an implicit form of the following line:

- $\{(2 + 5t, 3 - 2t) \mid t \in \mathbb{R}\}$.

Solution. $x = 2 + 5t$ and $y = 3 - 2t$.

- $t = \dfrac{x-2}{5}$ and $t = \dfrac{3-y}{2}$.
- $\dfrac{x-2}{5} = \dfrac{3-y}{2} \Rightarrow \{(x, y) \mid 2x + 5y = 19\}$.

Planes in $\mathbb{R}^3$

Example. A plane is given by

- $\{(1+s-t, 2+s-2t, 4-s-3t) \mid s,t \in \mathbb{R}\}$.

Let $x = 1+s-t$, $y = 2+s-2t$, $z = 4-s-3t$.

- $\left(\begin{array}{cc|c} 1 & -1 & x-1 \\ 1 & -2 & y-2 \\ -1 & -3 & z-4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & -1 & x-1 \\ 0 & -1 & -x+y-1 \\ 0 & 0 & 5x-4y+z-1 \end{array}\right)$.

The system is consistent. So $5x-4y+z-1=0$.

- Implicit form:
  - $\{(x,y,z) \mid 5x-4y+z=1\}$.

Lines in $\mathbb{R}^3$ (Explicit Form)

Example. Suppose a line is the intersection of

- $x+2y+3z=4$ and $2x+3y+4z=5$.

Solve the system to have

- $x = t-2$, $y=-2t+3$ and $z=t$, where t is an arbitrary parameter.

An explicit form of the line:

- $\{(t-2, -2t+3, t) \mid t \in \mathbb{R}\}$.

Note that $(t-2, -2t+3, t) = (-2,3,0) + t(1,-2,1)$.

Lines in $\mathbb{R}^3$ (Implicit Form)

Example. A line is given explicitly by $\{(t-2, -2t+3, t+1) \mid t \in \mathbb{R}\}$.

- Express t in terms of x,y,z: $\begin{cases} t = x+2 \\ -2t = y-3 \\ t = z-1 \end{cases}$
- Augmented matrix: $\left(\begin{array}{c|c} 1 & x+2 \\ -2 & y-3 \\ 1 & z-1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2+2R_1} \left(\begin{array}{c|c} 1 & x+2 \\ 0 & 2x+y+1 \\ 0 & -x+z-3 \end{array}\right)$.

Implicit form: $\{(x,y,z) \mid 2x+y+1=0 \text{ and } -x+z-3=0\}$.

Determine whether v is a linear combination

Let $\boldsymbol{v}_1 = (2,1,3)$, $\boldsymbol{v}_2 = (1,-1,2)$ and $\boldsymbol{v}_3 = (3,0,5)$.

- Is $\boldsymbol{v} = (3,3,4)$ a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$?

Suppose that $\boldsymbol{v} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$, i.e.,

$$\begin{aligned}(3,3,4) &= a(2,1,3) + b(1,-1,2) + c(3,0,5) \\ &= (2a+b+3c, a-b, 3a+2b+5c).\end{aligned}$$

Solve the linear system $\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4. \end{cases}$

- $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$

The system is consistent.

- Therefore, $(3,3,4)$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Determine whether v is NOT a linear combination

- $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \end{array}\right)$

The system is inconsistent.

- Therefore, $(1,2,4)$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Prove span{V} = R[n]

Prove that $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

- It is clear: $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} \subseteq \mathbb{R}^3$.

Is $\mathbb{R}^3 \subseteq \mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\}$?

Let $(x,y,z) \in \mathbb{R}^3$. We shall show that there exist $a,b,c \in \mathbb{R}$ such that

- $(x,y,z) = a(1,0,1) + b(1,1,0) + c(0,1,1)$.
- Equivalently, $\begin{cases} a+b \quad\;\; = x \\ \quad\; b+c = y \\ a \quad\; +c = z \end{cases}$
  - $\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z-x+y \end{array}\right)$

  The system is always consistent for any $(x,y,z) \in \mathbb{R}^3$.

Therefore, $\mathrm{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

Determine the relation between spans

Let $\boldsymbol{u}_1 = (1,0,1)$, $\boldsymbol{u}_2 = (1,1,2)$, $\boldsymbol{u}_3 = (-1,2,1)$, and

- $\boldsymbol{v}_1 = (1,2,3)$, $\boldsymbol{v}_2 = (2,-1,1)$.

Determine the relation between $\mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$ and $\mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$.

Solution. Step 1: Is $\mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\} \subseteq \mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$?

- Check whether $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3 \in \mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$. Let $\boldsymbol{A} = (\boldsymbol{v}_1 \quad \boldsymbol{v}_2)$.

$$\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -5 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- The systems $\boldsymbol{Ax} = \boldsymbol{u}_j$, $j = 1,2,3$, are all consistent.
  - Then $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3 \in \mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$.

  Therefore, $\mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\} \subseteq \mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$.

Solution. Step 2: Is $\mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\} \subseteq \mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$?

- Check whether $\boldsymbol{v}_1,\boldsymbol{v}_2 \in \mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$. Let $\boldsymbol{B} = (\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3)$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- The systems $\boldsymbol{Bx} = \boldsymbol{v}_j$, $j = 1,2$, are all consistent.
  - Then $\boldsymbol{v}_1,\boldsymbol{v}_2 \in \mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$.

  Therefore, $\mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\} \subseteq \mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$.

We can conclude that $\mathrm{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\} = \mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$.

Find Solution Space

Example. $\begin{cases} x - 2y + 3z = 0 \\ 2x - 4y + 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases}$

- $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
- $(x, y, z) = (2s - 3t, s, t) = s(2, 1, 0) + t(-3, 0, 1)$.

The solution space is $\text{span}\{(2, 1, 0), (-3, 0, 1)\}$.

Vector Spaces and Subspace Relation

- Let $U = \text{span}\{(1, 1, 1)\}$, $V = \text{span}\{(1, 1, -1)\}$ and $W = \text{span}\{(1, 0, 0), (0, 1, 1)\}$.

  Then U, V, W are vector spaces (subspace of $\mathbb{R}^3$).
  - $(1, 1, 1) = (1, 0, 0) + (0, 1, 1) \in W$.
    - Then $U \subseteq W$; so U is a subspace of W.
  - $(1, 1, -1) \notin \text{span}\{(1, 0, 0), (0, 1, 1)\}$.
    - Then $V \nsubseteq W$; so V is NOT a subspace of W.

Prove S is a basis of T

Example. Show that $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ is a basis for $\mathbb{R}^3$.

1. Show that S is linearly independent.
   - Let $c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4) = \mathbf{0}$.
   - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}$.
     - All columns are pivot.
     - The system has only the trivial solution.

   Therefore, S is linearly independent.

2. Show that $\text{span}(S) = \mathbb{R}^3$.
   - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}$.
     - The row-echelon form has no zero row.

   Therefore, $\text{span}(S) = \mathbb{R}^3$.

We can conclude that S is a basis for $\mathbb{R}^3$.

Example 2

Let $V = \text{span}\{(1,1,1,1),(1,-1,-1,1),(1,0,0,1)\}$.

- $S = \{(1,1,1,1),(1,-1,-1,1)\}$. Is S a basis for V?

1. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{Gaussian elimination}} \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.
   - All columns are pivot.
   - $c_1(1,1,1,1) + c_2(1,-1,-1,1) = \mathbf{0}$ has only the trivial solution.

   So S is linearly independent.
2. $(1,0,0,1) = \frac{1}{2}(1,1,1,1) + \frac{1}{2}(1,-1,-1,1)$.
   - Then $(1,0,0,1) \in \text{span}(S)$.

   So $\text{span}(S) = V$.

Therefore, S is a basis for V.

Disprove

Let $S = \{(1,1,1,1),(0,0,1,2),(-1,0,0,1)\}$.

- Let $|S|$ be the number of vectors in S. Then $|S| = 3$.
- So $\text{span}(S) \neq \mathbb{R}^4$; thus S is NOT a basis for $\mathbb{R}^4$.

Let $V = \text{span}(S)$, $S = \{(1,1,1),(0,0,1),(1,1,0)\}$.

- $(1,1,1) = (0,0,1) + (1,1,0)$.
- So S is linearly dependent; thus S is not a basis for V.

Find Coordinate Vector

Let $S = \{(1,2,1),(2,9,0),(3,3,4)\}$.

- One can check that S is a basis for $\mathbb{R}^3$. (Exercise!)

Let $\boldsymbol{v} = (5,-1,9)$. Solve

- $\boldsymbol{v} = a(1,2,1) + b(2,9,0) + c(3,3,4)$.
  - $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$.

  $(\boldsymbol{v})_S = (a,b,c) = (1,-1,2)$.

Dimension of Solution Space

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0 \end{cases}$$

- $\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
  - The 2nd and 5th columns of the coefficient matrix are non-pivot.
  - The solution space has dimension 2.

Transition Matrix

Example. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

- $\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$,
- $\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

One verifies that they are bases for $\mathbb{R}^3$. View all vectors as column vectors.

- $\left(\begin{array}{ccc|c|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array} \right)$

$$\xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array} \right) = (\boldsymbol{I} \mid \boldsymbol{P}).$$

Then $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$ is the transition matrix from S to T.

- Suppose $\boldsymbol{w} \in \mathbb{R}^3$ such that $(\boldsymbol{w})_S = (2, -1, 2)$.
  - $[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$; or $(\boldsymbol{w})_T = (2, -1, -3)$.

Example 2

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where

$\boldsymbol{u}_1 = (1, 1)$, $\boldsymbol{u}_2 = (1, -1)$, $\boldsymbol{v}_1 = (1, 0)$, $\boldsymbol{v}_2 = (1, 1)$.

Since S and T are linearly independent, they are bases for $\mathbb{R}^2$.

- $\left(\begin{array}{cc|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{array} \right) = \left(\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{array} \right)$
  - Transition matrix from S to T: $\boldsymbol{P} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.
- $\left(\begin{array}{cc|c|c} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{array} \right) = \left(\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array} \right)$
  - Transition matrix from T to S: $\boldsymbol{Q} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$.
- One checks immediately that $\boldsymbol{PQ} = \boldsymbol{QP} = \boldsymbol{I}$.

Example 3

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

- $S = \{(1, 0, -1), (0, -1, 0), (1, 0, 2)\}$;
- $T = \{(1, 1, 1), (1, 1, 0), (-1, 0, 0)\}$.

We have computed the transition matrix from S to T:

- $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$.

Then the transition matrix from T to S is

- $\boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

For any $\boldsymbol{w} \in \mathbb{R}^3 = \mathrm{span}(S) = \mathrm{span}(T)$,

- $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ and $\boldsymbol{P}^{-1}[\boldsymbol{w}]_T = [\boldsymbol{w}]_S$.

Basis of Column Space

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

- $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \boldsymbol{B} \xrightarrow{2R_1} \boldsymbol{C} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{D}$.
- Then $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ have the same row space.
  - $\operatorname{span}\{(0,0,1), (0,2,4), (\frac{1}{2}, 1, 2)\}$
    $= \operatorname{span}\{(1,0,0), (0,2,4), (0,0,1)\}$.

  Note that $\boldsymbol{D}$ is in row-echelon form.
  - $\{(1,0,0), (0,2,4), (0,0,1)\}$ is a basis for the row space of $\boldsymbol{D}$.
    - It is also a basis for the row space of $\boldsymbol{A}$ (and $\boldsymbol{B}, \boldsymbol{C}$).
  - The row space of $\boldsymbol{A}$ has dimension 3.

Basis of Row Space

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

- A row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Then $\boldsymbol{A}$ and $\boldsymbol{R}$ has the same row space.

- $\boldsymbol{R}$ has 3 nonzero rows.
- Basis of the row space of $\boldsymbol{A}$:
  - $\{(2,2,-1,0,1), (0,0,\frac{3}{2},-3,\frac{3}{2}), (0,0,0,3,0)\}$.
- Dimension of the row space of $\boldsymbol{A}$ is 3.

Pivot Column and Basis of Column Space

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

- Row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

In $\boldsymbol{R}$, the 1st, 3rd and 4th columns are pivot columns.

- So they form a basis for the column space of $\boldsymbol{R}$.

Then the 1st, 3rd and 4th columns of $\boldsymbol{A}$ form

a basis for the column space of $\boldsymbol{A}$:

- $\{(2,-1,0,1), (-1,2,1,-2), (0,-3,1,0)\}$.

How to find a basis for a vector space V

Solution 1

Let $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$.

- $\boldsymbol{v}_1 = (1,2,2,1)$, $\boldsymbol{v}_2 = (3,6,6,3)$, $\boldsymbol{v}_3 = (4,9,9,5)$,
- $\boldsymbol{v}_4 = (-2,-1,-1,1)$, $\boldsymbol{v}_5 = (5,8,9,4)$, $\boldsymbol{v}_6 = (4,2,7,3)$.

View each $\boldsymbol{v}_i$ as a **row vector** and form a matrix.

- $$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \\ \boldsymbol{v}_5 \\ \boldsymbol{v}_6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
  - V has a basis $\{(1,2,2,1), (0,1,1,1), (0,0,1,1)\}$.
  - $\dim(V) = 3$.

Solution 2

View each $\boldsymbol{v}_j$ as a **column vector** and form a matrix.

- $$\begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
  - The 1st, 3rd, 5th columns of row-echelon form are pivot columns.
    - They form a basis for the column space of the row-echelon form.
  - V has a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_5\}$.
  - $\dim(V) = 3$.

Extend S to a basis for $\mathbb{R}^n$

Let $S = \{(1,4,-2,5,1), (2,9,-1,8,2), (2,9,-1,9,3)\}$.

- $$\begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \boldsymbol{R}.$$
  - S is linearly independent.
  - The 1st, 2nd, 4th columns of $\boldsymbol{R}$ are pivot.
    - Add rows to $\boldsymbol{R}$ such that all columns are pivot.
    - $$\begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
  - $S \cup \{(0,0,1,0,0), (0,0,0,0,1)\}$ is a basis for $\mathbb{R}^5$.

Find Rank

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix}$.

- A row-echelon form $\boldsymbol{R} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
- Row space of $\boldsymbol{A}$ has a basis
  - $\{(2,0,3,-1,8), (0,1,-2,-1,-3)\}$.

  Column space of $\boldsymbol{A}$ has a basis
  - $\{(2,2,-4)^{\mathrm{T}}, (0,1,-3)^{\mathrm{T}}\}$.

Then $\text{rank}(\boldsymbol{A}) = 2$.

- In particular, $\boldsymbol{C}$ is not of full rank.

Rank & Consistency of Linear System

$$\begin{cases} 2x - y = 1 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x + z = 0. \end{cases}$$

- $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & 3 & -\frac{1}{2} \\ 0 & 0 & -9 & 4 \\ 0 & 0 & 0 & \frac{7}{9} \end{array}\right).$
  - $(\boldsymbol{A} \mid \boldsymbol{b}) \to (\boldsymbol{R} \mid \boldsymbol{b}')$.
- $\operatorname{rank}(\boldsymbol{A}) = 3$ but $\operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}) = 4$.
  - So the system is inconsistent.

Remark. In general,

- $\operatorname{rank}(\boldsymbol{A}) \le \operatorname{rank}(\boldsymbol{A} \mid \boldsymbol{b}) \le \operatorname{rank}(\boldsymbol{A}) + 1.$

Determine Nullspaces and Nullities of A

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}.$$

- $(\boldsymbol{A} \mid \boldsymbol{0}) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$
- $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$
- Nullspace of $\boldsymbol{A} = \operatorname{span}\left\{(-1,1,0,0,0)^{\mathrm{T}}, (-1,0,-1,0,1)^{\mathrm{T}}\right\}$
  - $\operatorname{nullity}(\boldsymbol{A}) = 2$. Note that $\operatorname{rank}(\boldsymbol{A}) = 3$.

Dimension Theorem

Example. Let $\boldsymbol{A} = \boldsymbol{0}_{m\times n}$. $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

- Nullspace of $\boldsymbol{A} = \mathbb{R}^n$.
  - $\operatorname{nullity}(\boldsymbol{A}) = \dim(\mathbb{R}^n) = n$.
- Row space of $\boldsymbol{A} = \{\boldsymbol{0}\} \subseteq \mathbb{R}^n$, column space of $\boldsymbol{A} = \{\boldsymbol{0}\} \subseteq \mathbb{R}^m$.
  - $\operatorname{rank}(\boldsymbol{A}) = 0$.

Find $\operatorname{rank}(\boldsymbol{A})$, $\operatorname{nullity}(\boldsymbol{A})$ and $\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}})$ of the following matrices.

- Let $\boldsymbol{A}$ be a 3×4 matrix such that $\operatorname{rank}(\boldsymbol{A}) = 3$.
  - $\operatorname{nullity}(\boldsymbol{A}) = 4 - \operatorname{rank}(\boldsymbol{A}) = 4 - 3 = 1$.
  - $\boldsymbol{A}^{\mathrm{T}}$ is 4×3 and $\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = \operatorname{rank}(\boldsymbol{A}) = 3$.
    - $\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 3 - \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = 3 - 3 = 0$.
- Let $\boldsymbol{A}$ be a 7×5 matrix such that $\operatorname{nullity}(\boldsymbol{A}) = 3$.
  - $\operatorname{rank}(\boldsymbol{A}) = 5 - \operatorname{nullity}(\boldsymbol{A}) = 5 - 3 = 2$.
  - $\boldsymbol{A}^{\mathrm{T}}$ is 5×7 and $\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = \operatorname{rank}(\boldsymbol{A}) = 2$.
    - $\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 7 - \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = 7 - 2 = 5$.
- Let $\boldsymbol{A}$ be a 3×2 matrix such that $\operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 3$.
  - $\boldsymbol{A}^{\mathrm{T}}$ is 2×3 and $\operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = 3 - \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}) = 0$.
  - $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}) = 0$.
    - $\operatorname{nullity}(\boldsymbol{A}) = 2 - \operatorname{rank}(\boldsymbol{A}) = 2 - 0 = 2$.

Inhomogeneous Linear System (Relation between Null Space of A and b)

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}$.

- We have found that the nullspace of $\boldsymbol{A}$ is
  - $\operatorname{span}\left\{(-1,1,0,0,0)^{\mathrm{T}}, (-1,0,-1,0,1)^{\mathrm{T}}\right\}$.

  $\boldsymbol{Ax} = \boldsymbol{0}$ has solution space
  - $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| s,t \in \mathbb{R} \right\}$.
- One verifies that $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Projection

Example. Let $V = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{u}_1 = (1,0,1)$, $\boldsymbol{u}_2 = (1,0,-1)$.

- $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 1\cdot 1 + 0\cdot 0 + 1\cdot(-1) = 0$.

The projection of $\boldsymbol{w} = (1,1,0)$ onto V is

- $\dfrac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1 + \dfrac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\boldsymbol{u}_2 = \dfrac{1}{2}(1,0,1) + \dfrac{1}{2}(1,0,-1) = (1,0,0)$.

Orthogonal Basis Represent a vector

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $\boldsymbol{u}_1 = (1,1,1)$, $\boldsymbol{u}_2 = (1,0,-1)$, $\boldsymbol{u}_3 = (1,-2,1)$.

- $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 1\cdot 1 + 1\cdot 0 + 1\cdot(-1) = 0$.

  $\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 1\cdot 1 + 1\cdot(-2) + 1\cdot 1 = 0$.

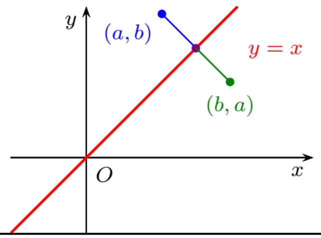
  $\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 1\cdot 1 + 0\cdot(-2) + (-1)\cdot 1 = 0$.
  - S is an orthogonal basis for $\mathbb{R}^3$.
- Let $\boldsymbol{w} = (1,-1,0) \in \mathbb{R}^3$. Then
  - $\dfrac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2} = \dfrac{1\cdot 1 + (-1)\cdot 1 + 0\cdot 1}{1^2+1^2+1^2} = 0$.
  - $\dfrac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2} = \dfrac{1\cdot 1 + (-1)\cdot 0 + 0\cdot(-1)}{1^2+0^2+(-1)^2} = \dfrac{1}{2}$.
  - $\dfrac{\boldsymbol{w}\cdot\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|^2} = \dfrac{1\cdot 1 + (-1)\cdot(-2) + 0\cdot 1}{1^2+(-2)^2+1} = \dfrac{1}{2}$.

  Then $(\boldsymbol{w})_S = (0, \frac{1}{2}, \frac{1}{2})$; and $\boldsymbol{w} = 0\boldsymbol{u}_1 + \frac{1}{2}\boldsymbol{u}_2 + \frac{1}{2}\boldsymbol{u}_3$.

Best Approximation

Example. Best approximation of (a,b) in $V = \operatorname{span}\{(1,1)\}$.

- $\boldsymbol{p} = \dfrac{(a,b)\cdot(1,1)}{\|(1,1)\|^2}(1,1) = \dfrac{a+b}{2}(1,1)$.



Find the Distance from u to plane V

Find the shortest distance from $\boldsymbol{u} = (1, 2, 3)$ to $V = \text{span}\{(1, 0, 1), (1, 1, 1)\}$.

1. Find an orthogonal basis for V:
   - $\boldsymbol{v}_1 = (1, 0, 1)$.
   - $\boldsymbol{v}_2 = (1, 1, 1) - \dfrac{(1, 1, 1) \cdot (1, 0, 1)}{\|(1, 0, 1)\|^2}(1, 0, 1) = (0, 1, 0)$.
2. Find the projection of $(1, 2, 3)$ onto V:
   - $\dfrac{(1, 2, 3) \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2} = \dfrac{(1, 2, 3) \cdot (1, 0, 1)}{\|(1, 0, 1)\|^2} = 2$
   - $\dfrac{(1, 2, 3) \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2} = \dfrac{(1, 2, 3) \cdot (0, 1, 0)}{\|(0, 1, 0)\|^2} = 2$.
   - $\boldsymbol{p} = 2\boldsymbol{v}_1 + 2\boldsymbol{v}_2 = 2(1, 0, 1) + 2(0, 1, 0) = (2, 2, 2)$.
3. Find the distance from $\boldsymbol{u}$ to $\boldsymbol{p}$
   - $d(\boldsymbol{u}, \boldsymbol{p}) = \|\boldsymbol{u} - \boldsymbol{p}\| = \|(-1, 0, 1)\| = \sqrt{2}$.

Least Squares Solutions

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. The least squares solution is $\boldsymbol{x} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{12} \end{pmatrix}$.

1. The column space of $\boldsymbol{A}$ is $V = \text{span}\{(1, 3, 5), (1, 1, 1)\}$.
2. Use Gram-Schmidt process to find an orthogonal basis for V:
   - $\boldsymbol{v}_1 = (1, 3, 5)$,
   - $\boldsymbol{v}_2 = (1, 1, 1) - \dfrac{(1, 3, 5) \cdot (1, 1, 1)}{\|(1, 3, 5)\|^2}(1, 3, 5) = \left(\dfrac{26}{35}, \dfrac{8}{35}, -\dfrac{2}{7}\right)$.
3. Find the projection of $\boldsymbol{b}$ onto V:
   - $\boldsymbol{p} = \dfrac{\boldsymbol{b} \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \dfrac{\boldsymbol{b} \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 = \left(\dfrac{5}{6}, \dfrac{7}{3}, \dfrac{23}{6}\right)$.
4. Solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$ ($\boldsymbol{p}$ is written as a column vector).
   - $\left(\begin{array}{cc|c} 1 & 1 & \frac{5}{6} \\ 3 & 1 & \frac{7}{3} \\ 5 & 1 & \frac{23}{6} \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{1}{12} \\ 0 & 0 & 0 \end{array}\right)$.

Example 2

Consider the following data:

x	1	3	5
y	1	2	4

Assume that the data satisfies a linear relation $y = ax + b$.

- What are the best choices of a and b?
  - $y = ax + b = \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.
- The least squares solution to the system:
  - $\begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{12} \end{pmatrix}$.
- The best linear function which fits the data is
  - $y = \dfrac{3}{4}x + \dfrac{1}{12}$.

Example 3

Suppose r, s and t are parameters satisfying $t = cr^2 + ds + e$.

- Find the least squares solutions to $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 4 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1.6 \\ 2.8 \\ 0.8 \\ 5.1 \\ 5.9 \end{pmatrix}$.
  - Pre-multiply by the transpose of the coefficient matrix.
    - $\begin{pmatrix} 34 & 14 & 10 \\ 14 & 10 & 6 \\ 10 & 6 & 6 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 47.6 \\ 24.1 \\ 16.7 \end{pmatrix}$.
  - Solve the system to get $\begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0.9275 \\ 0.9225 \\ 0.3150 \end{pmatrix}$ (up to 4 decimal places).
- The data is modeled by $t = 0.9275r^2 + 0.9225s + 0.3150$.

Find projection using Least Square Solutions

$V = \text{span}\{(1, -1, 1, -1), (1, 2, 0, 1), (2, 1, 1, 0)\}$ and $\boldsymbol{b} = (1, 1, 1, 1)$.

1. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.
2. Solve the system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$:
   - $\begin{pmatrix} 4 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{5} - t \\ \frac{4}{5} - t \\ t \end{pmatrix}$.
3. The projection of $\boldsymbol{b}$ onto V is
   - $\boldsymbol{p} = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} \frac{6}{5} \\ \frac{6}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$.

Orthonormal bases and Transition Matrix

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

- $\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, $\boldsymbol{u}_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$, $\boldsymbol{u}_3 = \frac{1}{\sqrt{6}}(1, -2, 1)$.
- $\boldsymbol{v}_1 = (0, 0, 1)$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}(1, -1, 0)$, $\boldsymbol{v}_3 = \frac{1}{\sqrt{2}}(1, 1, 0)$.

Both S and T are orthonormal bases for $\mathbb{R}^3$. (Verify!)

- $\boldsymbol{u}_1 = (\boldsymbol{u}_1 \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{u}_1 \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + (\boldsymbol{u}_1 \cdot \boldsymbol{v}_3)\boldsymbol{v}_3$.
  - $\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \frac{2}{\sqrt{6}}\boldsymbol{v}_3$.
- $\boldsymbol{u}_2 = (\boldsymbol{u}_2 \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{u}_2 \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + (\boldsymbol{u}_2 \cdot \boldsymbol{v}_3)\boldsymbol{v}_3$.
  - $\boldsymbol{u}_2 = -\frac{1}{\sqrt{2}}\boldsymbol{v}_1 + \frac{1}{2}\boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_3$.
- $\boldsymbol{u}_3 = (\boldsymbol{u}_3 \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{u}_3 \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + (\boldsymbol{u}_3 \cdot \boldsymbol{v}_3)\boldsymbol{v}_3$.
  - $\boldsymbol{u}_3 = \frac{1}{\sqrt{6}}\boldsymbol{v}_1 + \frac{3}{\sqrt{12}}\boldsymbol{v}_2 - \frac{1}{\sqrt{12}}\boldsymbol{v}_3$.

- The transition matrix from S to T:
  - $\boldsymbol{P} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2} & \frac{3}{\sqrt{12}} \\ \frac{2}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{\sqrt{12}} \end{pmatrix}$.
- The transition matrix from T to S:
  - $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{12}} & -\frac{1}{\sqrt{12}} \end{pmatrix}$.

QR Decomposition

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

- $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ has an orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$:

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{u}_1 = (1, -1, 2), & \|\boldsymbol{v}_1\| &= \sqrt{6}; \\ \boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = \boldsymbol{u}_2 - \frac{1}{6}\boldsymbol{v}_1 = \frac{1}{6}(11, 7, -2), & \|\boldsymbol{v}_2\| &= \sqrt{\tfrac{29}{6}}; \\ \boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 \\ &= \boldsymbol{u}_3 - \frac{2}{6}\boldsymbol{v}_1 - \frac{-1/3}{29/6}\boldsymbol{v}_2 = \frac{3}{29}(-2, 4, 3), & \|\boldsymbol{v}_3\| &= \tfrac{3}{\sqrt{29}}. \end{aligned}$$

  Then

  - $\boldsymbol{Q} = \begin{pmatrix} \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} & \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} & \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{11}{\sqrt{174}} & -\frac{2}{\sqrt{29}} \\ -\frac{1}{\sqrt{6}} & \frac{7}{\sqrt{174}} & \frac{4}{\sqrt{29}} \\ \frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{174}} & \frac{3}{\sqrt{29}} \end{pmatrix}$.

Then

- $\boldsymbol{R} = \begin{pmatrix} \|\boldsymbol{v}_1\| & \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} & \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} \\ 0 & \|\boldsymbol{v}_2\| & \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} \\ 0 & 0 & \|\boldsymbol{v}_3\| \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & \sqrt{\frac{29}{6}} & \frac{-1/3}{\sqrt{29/6}} \\ 0 & 0 & \frac{3}{\sqrt{29}} \end{pmatrix}$.

Extended to an orthonormal basis(If necessary)

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$.

- An orthonormal basis for $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$.
- It is extended to an orthonormal basis for $\mathbb{R}^3$:
  - $\boldsymbol{w}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\boldsymbol{w}_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$, $\boldsymbol{w}_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
- $\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$.
- $\boldsymbol{R} = \begin{pmatrix} \boldsymbol{w}_1 \cdot \boldsymbol{u}_1 & \boldsymbol{w}_1 \cdot \boldsymbol{u}_2 & \boldsymbol{w}_1 \cdot \boldsymbol{u}_3 \\ 0 & \boldsymbol{w}_2 \cdot \boldsymbol{u}_2 & \boldsymbol{w}_2 \cdot \boldsymbol{u}_3 \\ 0 & 0 & \boldsymbol{w}_3 \cdot \boldsymbol{u}_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & 0 \end{pmatrix}$.

Application of Diagonalization

- The **Fibonacci numbers** a_n are defined by
  - $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

  Note that $a_{n+1} = a_{n-1} + a_n$ for $n \geq 1$.
  - $\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} + a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$.

  Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Then $\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$.
  - $\boldsymbol{A}^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} & \left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} & \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{pmatrix}$.
  - $\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{pmatrix}$.

  Therefore, $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Diagonalization

Let $\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix}$.

1. Find the characteristic polynomial $\det(\lambda \boldsymbol{I} - \boldsymbol{B})$:

$$\begin{vmatrix} \lambda-1 & 1 & -1 & 1 \\ 1 & \lambda-1 & 1 & -1 \\ -1 & 1 & \lambda-3 & -1 \\ 1 & -1 & -1 & \lambda-3 \end{vmatrix} \xlongequal[\substack{R_2-R_4 \\ R_3+R_4}]{R_1-(\lambda-1)R_4} \begin{vmatrix} 0 & \lambda & \lambda-2 & -\lambda^2+4\lambda-2 \\ 0 & \lambda & 2 & -\lambda+2 \\ 0 & 0 & \lambda-4 & \lambda-4 \\ 1 & -1 & -1 & \lambda-3 \end{vmatrix}$$

$$= -\begin{vmatrix} \lambda & \lambda-2 & -\lambda^2+4\lambda-2 \\ \lambda & 2 & -\lambda+2 \\ 0 & \lambda-4 & \lambda-4 \end{vmatrix} \xlongequal{R_2-R_1} -\begin{vmatrix} \lambda & \lambda-2 & -\lambda^2+4\lambda-2 \\ 0 & -\lambda+4 & \lambda^2-5\lambda+4 \\ 0 & \lambda-4 & \lambda-4 \end{vmatrix}$$

$$\xlongequal{R_3+R_2} -\begin{vmatrix} \lambda & \lambda-2 & -\lambda^2+4\lambda-2 \\ 0 & -\lambda+4 & \lambda^2-5\lambda+4 \\ 0 & 0 & \lambda^2-4\lambda \end{vmatrix} = -\lambda(-\lambda+4)(\lambda^2-4\lambda) = \lambda^2(\lambda-4)^2.$$

Then $\boldsymbol{B}$ has eigenvalues 0 and 4.

2. $0\boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 1 & -1 & -1 & -3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1, 1, 0, 0)$$
$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = (1, -1, -1, 1).$$
$$\boldsymbol{w}_1 = \boldsymbol{v}_1 / \|\boldsymbol{v}_1\| = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0, 0\right),$$
$$\boldsymbol{w}_2 = \boldsymbol{v}_2 / \|\boldsymbol{v}_2\| = \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right).$$

3. $4\boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0\right)$$
$$\boldsymbol{v}_4 = \boldsymbol{u}_4 - \frac{\boldsymbol{u}_4 \cdot \boldsymbol{v}_3}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3 = \left(-\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1\right).$$
$$\boldsymbol{w}_3 = \boldsymbol{v}_3 / \|\boldsymbol{v}_3\| = \left(\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, 0\right),$$
$$\boldsymbol{w}_4 = \boldsymbol{v}_4 / \|\boldsymbol{v}_4\| = \left(-\tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{1}{\sqrt{12}}, \tfrac{3}{\sqrt{12}}\right).$$

4. View $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4$ as column vectors.

- Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 & \boldsymbol{w}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{1}{2} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{1}{2} & 0 & \frac{3}{\sqrt{12}} \end{pmatrix}$.
  - $\boldsymbol{P}$ is an orthogonal matrix.
  - $\boldsymbol{P}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Singular Value Decomposition

Let $\boldsymbol{A} = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$. $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}$.

1. $\det(\lambda \boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = \begin{vmatrix} \lambda - 80 & -100 & -40 \\ -100 & \lambda - 170 & -140 \\ -40 & -140 & \lambda - 200 \end{vmatrix} = (\lambda - 360)(\lambda - 90)\lambda.$

- $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ has eigenvalues $360, 90, 0$.

2. $(360\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = t\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}; \boldsymbol{v}_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

$(90\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = t\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}; \boldsymbol{v}_2 = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

$(0\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \boldsymbol{x} = t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}; \boldsymbol{v}_3 = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.

3. $\sigma_1 = \sqrt{360} = 6\sqrt{10}$ and $\sigma_2 = \sqrt{90} = 3\sqrt{10}$.

- $\boldsymbol{u}_1 = \frac{1}{\sigma_1}\boldsymbol{A}\boldsymbol{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$ and $\boldsymbol{u}_2 = \frac{1}{\sigma_2}\boldsymbol{A}\boldsymbol{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{pmatrix}$.

4. $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is already an orthonormal basis for $\mathbb{R}^2$.

$\boldsymbol{U} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{pmatrix}, \boldsymbol{V} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}$.

○ Then $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{T}}$ is the singular value decomposition.

Find Standard Matrix

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as

○ $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ 2x \\ -3y \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n$.

- $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1x + 1y \\ 2x + 0y \\ 0x - 3y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore$ T is a linear transformation.

- The standard matrix for T is $\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix}$.

Find Linear Transformation

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be a linear transformation:

- $T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}$, $T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}$, $T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$.

Let $\boldsymbol{P}=\begin{pmatrix}1&0&2\\1&1&0\\1&1&-1\end{pmatrix}\xrightarrow[R_3-R_1]{R_2-R_1}\begin{pmatrix}1&0&2\\0&1&-2\\0&1&-3\end{pmatrix}\xrightarrow{R_3-R_2}\begin{pmatrix}1&0&2\\0&1&-2\\0&0&-1\end{pmatrix}$.

- $\boldsymbol{P}$ is invertible; so $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

Every vector in $\mathbb{R}^3$ is a unique linear combination of vectors in the basis:

- $\begin{pmatrix}x\\y\\z\end{pmatrix}=c_1\begin{pmatrix}1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}1&0&2\\1&1&0\\1&1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$.
  - $\left(\begin{array}{ccc|c}1&0&2&x\\1&1&0&y\\1&1&-1&z\end{array}\right)\xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}}\left(\begin{array}{ccc|c}1&0&0&x-2y+2z\\0&1&0&-x+3y-2z\\0&0&1&y-z\end{array}\right)$.

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=c_1T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)+c_2T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)+c_3T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)$$
$$=(x-2y+2z)\begin{pmatrix}1\\3\end{pmatrix}+(-x+3y-2z)\begin{pmatrix}-1\\2\end{pmatrix}+(y-z)\begin{pmatrix}4\\-1\end{pmatrix}$$
$$=\begin{pmatrix}2x-y\\x-y+3z\end{pmatrix}=\begin{pmatrix}2&-1&0\\1&-1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Change of Bases (Find Standard Matrix)

Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be a linear transformation such that

- $T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}$, $T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}$, $T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$.

We have seen that $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

- Let $\boldsymbol{P}=\begin{pmatrix}1&0&2\\1&1&0\\1&1&-1\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}1&-1&4\\3&2&-1\end{pmatrix}$,
  - $(\boldsymbol{P}\mid\boldsymbol{I})\xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}}(\boldsymbol{I}\mid\boldsymbol{P}^{-1})=\left(\begin{array}{ccc|ccc}1&0&0&1&-2&2\\0&1&0&-1&3&-2\\0&0&1&0&1&-1\end{array}\right)$.
  - The standard matrix for T is $\boldsymbol{B}\boldsymbol{P}^{-1}=\begin{pmatrix}2&-1&0\\1&-1&3\end{pmatrix}$.

Representation of Range: Rank(T)

Example. $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}x+y\\y\\x\end{pmatrix}$.

- Standard matrix for T: $\boldsymbol{A}=\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}$.
- $\mathrm{R}(T)=\mathrm{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$; $\quad\mathrm{rank}(T)=\mathrm{rank}(\boldsymbol{A})=2$.

dim(R(T))

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

- $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix}, \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$
  - $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
  - $\mathrm{R}(T) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}\right\}$; $\mathrm{rank}(T) = \dim(\mathrm{R}(T)) = 2.$

Find Kernel 1

Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

- $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x-z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
- Find the kernel of T_1. Let
  - $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x-z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$
  - $\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $\mathrm{Ker}(T) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}.$

Find Kernel 2

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

- $T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
- Find the kernel of T_2. Let
  - $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix}.$
  - $z = y$ and $x = 0 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$
  - $\mathrm{Ker}(T_2) = \left\{\begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \middle| \, y \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$

Represent Kernel

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

- $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix}$, for $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4$.
  - $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

    Let $w = s$, $z = t$ be arbitrary parameters. Then $x = -3t$, $y = t$.
  - $\mathrm{Ker}(T) = \left\{\begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \middle| \, s, t \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix}\right\}.$
    - $\mathrm{nullity}(T) = \dim(\mathrm{Ker}(T)) = \mathrm{nullity}(\boldsymbol{A}) = 2.$