MA1522 Put It All Together!

Vector Space

2014-2015-Semester 2 Find subspace

Question 2 [25 marks]

(a) [11 marks]

Let $V = \{(a, a, a, 0) \mid a \in \mathbb{R}\}$.

(i) Find a basis for V and determine $\dim(V)$.

(ii) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 1$. Justify your answer.

(iii) Let $U = \{\boldsymbol{u} \in \mathbb{R}^4 \mid \boldsymbol{u} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in V\}$. Find a basis for and determine the dimension of U.

2015-2016-Semester 2

Let $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be a vector space such that $\boldsymbol{v}_i$ are unit vectors for all i and $\boldsymbol{v}_i \cdot \boldsymbol{v}_j < 0$ if $i \neq j$.

(i) Show that no two vectors among $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ are linearly dependent.

(ii) Prove that $\dim V \geq 3$.

Clearly $\boldsymbol{v}_i \neq 0$.

(i) Assume that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent.

Let $\boldsymbol{v}_1 = c\boldsymbol{v}_2$. Then $0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = c(\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) = c$. We would have

$$0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = c(\boldsymbol{v}_2 \cdot \boldsymbol{v}_3) > 0,$$

which is a contradiction.

Hence, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. Similarly for any other two vectors.

(ii) Assume that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly dependent. Then any vector in $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a linear combination of the other two. Write $\boldsymbol{v}_3 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$.

$$0 > \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = c_1 + c_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \quad \text{and} \quad 0 > \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = c_1(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) + c_2.$$

If $c_1 > 0$, then $c_2(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) < -c_1 < 0$, and thus $c_2 > 0$.

By Cauhy-Schwarz inequality, $0 < -\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \leq \|\boldsymbol{v}_1\| \, \|\boldsymbol{v}_2\| = 1$. We would have

$$c_1 < c_2(-\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \leq c_2 \quad \text{and} \quad c_2 < c_1(-\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) \leq c_1$$

which is a contradiction (based on our assumption that $c_1 > 0$).

Therefore, $c_1 < 0$ and hence $c_2 < 0$.

However, this would imply that $\boldsymbol{v}_3 \cdot \boldsymbol{v}_4 = c_1(\boldsymbol{v}_1 \cdot \boldsymbol{v}_4) + c_2(\boldsymbol{v}_2 \cdot \boldsymbol{v}_4) > 0$, a contradiction.

Hence, $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent. So $\dim V \geq 3$.

Rank/Column Space/Row Space

2014-2015-Semester 2

(iii) Without performing Gaussian elimination, can you tell whether the system

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \\ 3 \end{pmatrix}$$

has no solution, exactly one solution, or infinitely many solutions? Why?

2015-2016-Semester 2

Let $\boldsymbol{A}$ be a 3×2 matrix and $\boldsymbol{B}$ be a 2×3 matrix such that

$$\boldsymbol{AB} = \begin{pmatrix} -2 & -14 & 14 \\ 5 & 15 & -10 \\ 4 & 8 & -3 \end{pmatrix}.$$

(i) [3 marks] Find a basis for the row space of $\boldsymbol{AB}$ and state the rank of $\boldsymbol{AB}$.

(ii) [2 marks] Show that $(\boldsymbol{AB})^2 = 5\boldsymbol{AB}$.

(iii) [2 marks] What is the rank of $\boldsymbol{BA}$? Justify your answer.

(iv) [3 marks] Find $\boldsymbol{BA}$. Show clearly how you derive your answer.

2016-2017-Semester 2

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same order.

(i) Prove that the nullspace of $\boldsymbol{B}$ is a subspace of the nullspace of $\boldsymbol{AB}$.

(ii) Using (i) prove that

$$\text{nullity}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{B}) \geq \text{nullity}(\boldsymbol{AB}).$$

If $\boldsymbol{Bv} = \boldsymbol{0}$, then $\boldsymbol{ABv} = \boldsymbol{0}$. So

$$\text{nullspace of } \boldsymbol{B} \subseteq \text{nullspace of } \boldsymbol{AB}.$$

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a basis for the nullspace of $\boldsymbol{B}$. Extend it to a basis for the nullspace of $\boldsymbol{AB}$:

$$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_m\}.$$

Then $\boldsymbol{Bv}_{k+1}, \ldots, \boldsymbol{Bv}_m \neq \boldsymbol{0}$, and they belong to the nullspace of $\boldsymbol{A}$. Suppose

$$c_{k+1}\boldsymbol{Bv}_{k+1} + \cdots + c_m\boldsymbol{Bv}_m = \boldsymbol{0}.$$

Then $\boldsymbol{B}(c_{k+1}\boldsymbol{v}_{k+1} + \cdots + c_m\boldsymbol{v}_m) = \boldsymbol{0}$, which implies

$$c_{k+1}\boldsymbol{v}_{k+1} + \cdots + c_m\boldsymbol{v}_m \in \text{nullspace of } \boldsymbol{B} = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}.$$

So $c_{k+1} = \cdots = c_m = 0$. Hence, $\boldsymbol{Bv}_{k+1}, \ldots, \boldsymbol{Bv}_m$ are linearly independent. Then

$$\text{nullity}(\boldsymbol{A}) \geq m - k = \text{nullity}(\boldsymbol{AB}) - \text{nullity}(\boldsymbol{B}).$$

So

$$\text{nullity}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{B}) \geq \text{nullity}(\boldsymbol{AB}).$$

Orthogonal Matrix

2014-2015-Semester 2

True or false: Given any 2×3 matrix $\boldsymbol{M}$ and 2×1 column vector $\boldsymbol{c}$, the linear system $\boldsymbol{Mx} = \boldsymbol{c}$ always has infinitely many least squares solutions. Justify your answer.

2015-2016-Semester 2 Use projection to extends orthogonal basis

Let $V = \text{span}\{(1,1,0,1),(3,2,1,1),(-1,0,2,-2)\}$.

(i) [3 marks] Using the Gram-Schmidt process, find an orthogonal basis for V.

(ii) [2 marks] Using the result in (i), find the projection of $(2,-2,-2,3)$ onto V.

(iii) [2 marks] Extend your basis in (i) to an orthogonal basis for $\mathbb{R}^4$.

2017-2018-Semester 2

Suppose $\mathbf{v}_1 = (1,1,1,1)$, $\mathbf{v}_2 = (1,2,4,5)$, and $\mathbf{v}_3 = (10,-30,-40,-20)$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and let $U = \text{span}(S)$.

(a) (6 points) Use the Gram-Schmidt process to find an orthonormal basis T for U.

(4 points) Find the transition matrix from S to T. You may leave the entries of your answer in terms of square roots.

Diagonalizable

2014-2015-Semester 2

Prove the following:
Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two $n \times n$ matrices. Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ is a set of linearly independent eigenvectors for both $\boldsymbol{M}$ and $\boldsymbol{N}$. Then $\boldsymbol{MN} = \boldsymbol{NM}$.

2017-2018-Semester 1

Suppose $\boldsymbol{A}$ is diagonalizable and the eigenvalues of $\boldsymbol{A}$ are λ_1, λ_2, $\ldots$, λ_k.

(i) If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$, say, $\boldsymbol{Av} = \lambda_i \boldsymbol{v}$ for some i, show that $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})(\boldsymbol{A} - \lambda_2 \boldsymbol{I}) \cdots (\boldsymbol{A} - \lambda_k \boldsymbol{I})\boldsymbol{v} = \mathbf{0}$.
(Hint: First, show that $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v} = \mathbf{0}$ and then use the result in part (b).)

Since $A\boldsymbol{v} = \lambda_i \boldsymbol{v} = \lambda_i I \boldsymbol{v}$,
$(A - \lambda_i I)\boldsymbol{v} = A\boldsymbol{v} - \lambda_i I \boldsymbol{v} = \mathbf{0}$.
Applying the result in part (b) repeatedly yields
$(A - \lambda_1 I)\ldots(A - \lambda_i I)\ldots(A - \lambda_k I) = (A - \lambda_1 I)\ldots(A - \lambda_k I)(A - \lambda_i I)$
Therefore,

$$\begin{aligned}(A - \lambda_1 I)\ldots(A - \lambda_i I)\ldots(A - \lambda_k I)\boldsymbol{v} &= (A - \lambda_1 I)\ldots(A - \lambda_k I)(A - \lambda_i I)\boldsymbol{v} \\ &= (A - \lambda_1 I)\ldots(A - \lambda_k I)(\mathbf{0}) = \mathbf{0}\end{aligned}$$

(ii) Define $S = T_{\lambda_1} \circ T_{\lambda_2} \circ \cdots \circ T_{\lambda_k}$. Prove that S is the zero transformation.

Since A is diagonalizable, by Theorem 6.2.3, A has n linearly independent eigenvectors, which will span $\mathbb{R}^n$. Let $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ be one such basis.

$\forall \boldsymbol{v} \in \mathbb{R}^n, \exists a_1, \ldots, a_n \in \mathbb{R}$, such that $\boldsymbol{v} = a_1\boldsymbol{v_1} + \ldots + a_n\boldsymbol{v_n} = \sum_{k=1}^n a_k\boldsymbol{v_k}$.

$$
\begin{aligned}
S(\boldsymbol{v}) &= S\left(\sum_{k=1}^n a_k\boldsymbol{v_k}\right) \\
&= \sum_{k=1}^n a_k S(\boldsymbol{v_k}) \\
&= \sum_{k=1}^n a_k\mathbf{0} && \text{from part (i)} \\
&= \mathbf{0}
\end{aligned}
$$

$\mathrm{R}(S) = \{\mathbf{0}\}$, and therefore, S is the zero transformation.

2017-2018-Semester 2

(8 points) Assume that $\mathbf{A}$ is a symmetric matrix. Show that if $m > 0$ and $\mathbf{A}^m = \mathbf{I}$, then $\mathbf{A}^2 = \mathbf{I}$.

Solution: Symmetric matrices are diagonalizable. There exist real numbers $\lambda_1, \ldots, \lambda_n$ and an invertible matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \mathbf{B}.$$

Note that each λ_i is an eigenvalue of $\mathbf{A}$. Hence by Part (b), each λ_i is either 1 or -1. Therefore ${\lambda_i}^2 = 1$, for each $1 \le i \le n$. So we have

$$\mathbf{B}^2 = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \mathbf{I}.$$

So by Part (c), $\mathbf{A}^2 = \mathbf{I}$.

Linear Transformation

2016-2017-Semester 2

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \quad T\left(\begin{pmatrix}3\\3\\2\end{pmatrix}\right) = \begin{pmatrix}-2\\1\\3\end{pmatrix}.$$

(i) Find the standard matrix for T.

(ii) Find $\mathrm{rank}(T)$ and $\mathrm{nullity}(T)$.

2015-2016-Semester 2

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that its standard matrix is diagonalisable. Prove that $\mathrm{R}(T) = \mathrm{R}(T \circ T)$ and $\mathrm{Ker}(T) = \mathrm{Ker}(T \circ T)$.

Let $\boldsymbol{A}$ be the standard matrix for T.

Then $\boldsymbol{A}$ has n linearly independent eigenvectors, say $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, associated to eigenvalues $\lambda_1, \dots, \lambda_n$, respectively.

Suppose that $\lambda_1 = \cdots = \lambda_k = 0$, and $\lambda_i \neq 0$ if $i > k$.

For each i, $\boldsymbol{A}\boldsymbol{v}_i = \lambda \boldsymbol{v}_i$, and thus $\boldsymbol{A}^2 v_i = \lambda^2 \boldsymbol{v}_i$; so $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are the eigenvectors of $\boldsymbol{A}^2$ associated to eigenvalues $\lambda_1^2, \dots, \lambda_n^2$.

Note that $\lambda_i^2 = 0$ if $i \leq k$ and $\lambda_i^2 \neq 0$ if $i > k$. Then

$$\begin{aligned}\operatorname{Ker}(T) &= \text{nullspace of } \boldsymbol{A} = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \\ &= \text{nullspace of } \boldsymbol{A}^2 = \operatorname{Ker}(T \circ T).\end{aligned}$$

$R(T \circ T) = \text{column space of } \boldsymbol{A}^2 \subseteq \text{column space of } \boldsymbol{A} = R(T)$, and

$$\dim R(T \circ T) = n - \dim \operatorname{Ker}(T) = n - \dim \operatorname{Ker}(T \circ T) = \dim R(T).$$

We conclude that $R(T \circ T) = R(T)$.

2016-2017-Semester 2

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\operatorname{Ker}(T) = \operatorname{Ker}(T \circ T)$. Prove that

$$\operatorname{Ker}(T \circ T) = \operatorname{Ker}(T \circ T \circ T).$$

If $\boldsymbol{v} \in \operatorname{Ker}(T \circ T)$, then $T \circ T(\boldsymbol{v}) = \boldsymbol{0}$; consequently $T \circ T \circ T(\boldsymbol{v}) = T(T \circ T(\boldsymbol{v})) = T(\boldsymbol{0}) = \boldsymbol{0}$, i.e., $\boldsymbol{v} \in \operatorname{Ker}(T \circ T \circ T)$.

Let $\boldsymbol{v} \in \operatorname{Ker}(T \circ T \circ T)$. Set $\boldsymbol{w} = T(\boldsymbol{v})$. Then

$$T \circ T(\boldsymbol{w}) = T \circ T(T(\boldsymbol{v})) = T \circ T \circ T(\boldsymbol{v}) = \boldsymbol{0}.$$

So $\boldsymbol{w} \in \operatorname{Ker}(T \circ T) = \operatorname{Ker}(T)$. We thus have

$$T \circ T(\boldsymbol{v}) = T(T(\boldsymbol{v})) = T(\boldsymbol{w}) = \boldsymbol{0};$$

that is, $\boldsymbol{v} \in \operatorname{Ker}(T \circ T)$.

Subspace

2014-2015-Sem2

(i)

$$V = \{(a,a,a,0) \mid a \in \mathbb{R}\} = \{a(1,1,1,0) \mid a \in \mathbb{R}\} = \text{span}\{(1,1,1,0)\}.$$

Thus $\{(1,1,1,0)\}$ is a basis for V and $\dim(V) = 1$.

(ii) Let $W = \text{span}\{(1,1,1,0),(0,1,0,0),(0,0,1,0)\}$. Since $V = \text{span}\{(1,1,1,0)\}$, we have $V \subseteq W$ and $W \cap V = V$ and thus $\dim(W \cap V) = 1$. On the other hand,

$$a(1,1,1,0) + b(0,1,0,0) + c(0,0,1,0) = (0,0,0,0) \Rightarrow a = b = c = 0.$$

Thus $\{(1,1,1,0),(0,1,0,0),(0,0,1,0)\}$ is a linearly independent set and $\dim(W) = 3$.

(iii) Since $V = \text{span}\{(1,1,1,0)\}$, $\boldsymbol{u} = (u_1,u_2,u_3,u_4) \in U$ if and only if $(u_1,u_2,u_3,u_4) \cdot (1,1,1,0) = 0$, which implies $u_1 + u_2 + u_3 = 0$. Solving, we have

$$\begin{cases} u_1 &= -s-t \\ u_2 &= s \\ u_3 &= t \\ u_4 &= r, \qquad s,t,r \in \mathbb{R}. \end{cases}$$

So

$$U = \text{span}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}.$$

A basis for U is $\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ and $\dim(U) = 3$.

Rank Problem

2014-2015-sem2

Yes. The system has exactly one solution. First of all, observe that the 3 columns of the coefficient matrix of the system are precisely $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and the constant matrix is $\boldsymbol{w}$. From (ii), we know $\boldsymbol{w}$ belongs to the column space $V = \text{span}(S)$ of the matrix, and hence the system is consistent. Furthermore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is a basis for V, hence $\boldsymbol{w}$ can only be expressed as a linear combination of this basis in exactly one way. Hence the system has exactly one solution.

2015-2016-Semester 2

(i) We check that

$$\begin{pmatrix}-2 & -14 & 14\\ 5 & 15 & -10\\ 4 & 8 & -3\end{pmatrix} \xrightarrow[\frac{1}{5}R_2]{-\frac{1}{2}R_1} \begin{pmatrix}1 & 7 & -7\\ 1 & 3 & -2\\ 4 & 8 & -3\end{pmatrix} \xrightarrow[R_3-4R_1]{R_2-R_1} \begin{pmatrix}1 & 7 & -7\\ 0 & -4 & 5\\ 0 & -20 & 25\end{pmatrix} \xrightarrow{R_3-5R_2} \begin{pmatrix}1 & 7 & -7\\ 0 & -4 & 5\\ 0 & 0 & 0\end{pmatrix}$$

So a basis for the row space is given by $\{(1,7,-7),(0,-4,5)\}$ and $\text{rank}(\boldsymbol{AB}) = 2$.

(ii) One verifies that:

$$(\boldsymbol{AB})^2 = \begin{pmatrix}-10 & -70 & 70\\ 25 & 75 & -50\\ 20 & 40 & -15\end{pmatrix} = 5\boldsymbol{AB}.$$

(iii) $\text{rank}(\boldsymbol{BA}) \geq \text{rank}(\boldsymbol{A}(\boldsymbol{BA})\boldsymbol{B}) = \text{rank}((\boldsymbol{AB})^2) = 2$.

Since $\boldsymbol{BA}$ is 2×2, so $\text{rank}(\boldsymbol{BA}) = 2$.

(iv) From (ii),

$(\boldsymbol{BA})^3 = \boldsymbol{BABABA} = \boldsymbol{B}(\boldsymbol{AB})^2\boldsymbol{A} = \boldsymbol{B}(5\boldsymbol{AB})\boldsymbol{A} = 5(\boldsymbol{BA})^2$.

From (iii), we have $\boldsymbol{BA}$ is full rank and invertible.

It follows that $\boldsymbol{BA} = 5\boldsymbol{I} = \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}$.

Orthogonal Matrix

2014-2015-sem2

True. If $\boldsymbol{M}$ is 2×3, the rank of $\boldsymbol{M}$ is at most 2, and hence the nullity of $\boldsymbol{M}$ is at least 1. This implies the nullity of $\boldsymbol{M}^T\boldsymbol{M}$ is at least 1. Hence $\boldsymbol{M}^T\boldsymbol{M}$ is not invertible.

Since $\boldsymbol{M}^T\boldsymbol{M}\boldsymbol{x} = \boldsymbol{M}^T\boldsymbol{c}$ is always consistent, this means it will have infinitely many solutions. i.e. $\boldsymbol{Mx} = \boldsymbol{c}$ always has infinitely many least squares solutions.

2015-2016-Sem2

(i) Let $\boldsymbol{u}_1 = (1,1,0,1)$, $\boldsymbol{u}_2 = (3,2,1,1)$ and $\boldsymbol{u}_3 = (-1,0,2,-2)$.

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{u}_1 = (1,1,0,1),\\
\boldsymbol{v}_2 &= \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\boldsymbol{v}_1 = (3,2,1,1) - \frac{6}{3}(1,1,0,1) = (1,0,1,-1),\\
\boldsymbol{v}_3 &= \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3\cdot\boldsymbol{v}_2}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2}\boldsymbol{v}_2\\
&= (-1,0,2,-2) - \frac{-3}{3}(1,1,0,1) - \frac{3}{3}(1,0,1,-1) = (-1,1,1,0).
\end{aligned}$$

An orthogonal basis for V is $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$.

(ii) The projection of $\boldsymbol{v} = (2,-2,-2,3)$ onto V is

$$\begin{aligned}
\boldsymbol{p} &= \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_1)}{(\boldsymbol{v}_1\cdot\boldsymbol{v}_1)}\boldsymbol{v}_1 + \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_2)}{(\boldsymbol{v}_2\cdot\boldsymbol{v}_2)}\boldsymbol{v}_2 + \frac{(\boldsymbol{v}\cdot\boldsymbol{v}_3)}{(\boldsymbol{v}_3\cdot\boldsymbol{v}_3)}\boldsymbol{v}_3\\
&= \frac{3}{3}(1,1,0,1) + \frac{-3}{3}(1,0,1,-1) + \frac{-6}{3}(-1,1,1,0) = (2,-1,-3,2).
\end{aligned}$$

(iii) Note that $\boldsymbol{v}-\boldsymbol{p} = (0,-1,1,1)$ is orthogonal to each $\boldsymbol{v}_i$. Then

$$\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,(0,-1,1,1)\}$$

is an orthogonal basis for $\mathbb{R}^4$.

2017-2018-Semester 2

1-(b)

Solution: $[\mathbf{v}_1]_T = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{u}_1 \\ \mathbf{v}_1 \cdot \mathbf{u}_2 \\ \mathbf{v}_1 \cdot \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. $[\mathbf{v}_2]_T = \begin{pmatrix} \mathbf{v}_2 \cdot \mathbf{u}_1 \\ \mathbf{v}_2 \cdot \mathbf{u}_2 \\ \mathbf{v}_2 \cdot \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 6 \\ \sqrt{10} \\ 0 \end{pmatrix}$.

$[\mathbf{v}_3]_T = \begin{pmatrix} \mathbf{v}_3 \cdot \mathbf{u}_1 \\ \mathbf{v}_3 \cdot \mathbf{u}_2 \\ \mathbf{v}_3 \cdot \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} -40 \\ -7\sqrt{10} \\ \sqrt{910} \end{pmatrix}$. So the transition matrix from S to T is:

$$\mathbf{P} = \begin{pmatrix} 2 & 6 & -40 \\ 0 & \sqrt{10} & -7\sqrt{10} \\ 0 & 0 & \sqrt{910} \end{pmatrix}.$$

Linear Transformation

2016-2017-Sem2: Use A^{-1} to determine the standard matrix!

(i) $\boldsymbol{A}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$. $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -6 & 3 \\ -5 & 7 & -3 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 4 & -1 \\ -2 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$.

(ii) $\boldsymbol{A} \xrightarrow[R_3+\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1} \begin{pmatrix} -4 & 4 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} -4 & 4 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$.

So $\operatorname{rank}(T) = 2$ and $\operatorname{nullity}(T) = 3 - 2 = 1$.