Notes on

A CONCISE INTRODUCTION TO PURE MATHEMATICS

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1 Sets and Proofs

1.1 Sets

A *set* is a collection of objects called *elements* or *members*. In simple cases, we can enumerate all members with this notation: $\{1, 3, 5\}$. Often we cannot list all of the elements. For example, "the set of all real numbers whose square is less than 2" can be expressed as

$$\{x|x \text{ a real number}, x^2 < 2\}$$

where | is read as "such that."

The *empty set*, $\emptyset$, is a set with no objects.

If s is in set S, we say "s belongs to S" and denote it as $s \in S$. Conversely, if s is not in S, it is denoted as $s \notin S$.

Two sets are *equal* when they contain exactly the same elements. T is a subset of S if all elements of T also belong to S; this relationship is denoted as $T \subseteq S$, or $T \nsubseteq S$ if T is not a subset of S. The empty set is a subset of all sets.

1.2 Proofs

The purpose of a **proof** is to provide a convincing, logical justification of an opinion.

The notation $P \implies Q$ translates to "statement P implies statement Q." It can also be read as

- "if P, then Q"
- "Q if P"
- "P only if Q"

Note that $P \implies Q$ does not necessitate that $Q \implies P$. If this were the case, use the notation $P \iff Q$ which translates to "P if and only if Q".

The notation for negation, e.g. "not P," is $\bar{P}$.

With this notation, the following is logically true: if $P \implies Q$, then $\bar{Q} \implies \bar{P}$.

Example 1.2

Proving the following statement: "The square of an odd integer is odd."

Proof. Let n be an odd integer. Then n is 1 more than an even integer, so $n = 1 + 2m$ for some integer m (odd or event). Then $n^2 = (1 + 2m)^2 = 1 + 4m + 4m^2 = 1 + 4(m + m^2)$. $4(m + m^2)$ is even, therefore that $+1$ is odd, showing n^2 is odd. $\square$

This could have been written as:

$$n \text{ odd} \implies n = 1 + 2m \;\forall\; m \in \mathbb{N} \implies n^2 = 1 + 4(m + m^2) \implies n^2 \text{ odd}$$

but it is more difficult to read. Therefore, it is common (encouraged?) to include explanatory English text in a proof.

This is an example of a *direct proof* because it proceeds from assumptions to conclusions via a series of implications.

References