Numerical Explorations of the Dynamics of FRW Cosmologies

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1 Introduction

Einstein's theory of general relativity is used extensively in the field of Cosmology. However, in order to solve the evolution of the universe, it is necessary to make certain approximation to be able to solve for the dynamics of the universe. In this paper, we will briefly go over the assumptions that we make of the universe and the equations that governs the evolution of the universe. Moreover, we will numerically solve the differential equations to see how the dynamics of the universe change with respect to equation of state and curvature.

2 FRW Cosmology

We assume that the universe is spatially isoptropic and homogenous (Friedmann-Lemanitre cosmological solutions). This is a very good approximation because we only care about the large scale evolution of the universe. More concretely, we assume that we can model the universe as [**?**]

$$\begin{aligned} (\mathcal{M}, g) \\ \mathcal{M} = J \times \sum \\ g = -dt^2 + a^2(t)\,\overline{g} \end{aligned}$$

where

- $J \subseteq \mathbb{R}$
- $a(t)$ is the scale factor.
- $\left(\sum, \overline{g}\right)$ is an isotropic and homogeneous 3-dimensional Riemannian manifold.

Then, we have that the Riemann tensor on spatial manifold must be of the form

$$\overline{R}_{ijk\ell} = k\left(\overline{g}_{ik}\overline{g}_{j\ell} - \overline{g}_{i\ell}\overline{g}_{jk}\right)$$

Note that Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

From the assumption, Einstein tensor must be

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$
$$G_{ii} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}$$
$$G_{\mu\nu} = 0 \ \text{ for } \ \mu \neq \nu$$

Then, from Einstein's field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

we have that the energy-momentum tensor must be diagonal. Under F.L. spacetime, energy-momentum tensor is always of the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} - \frac{1}{8\pi G}\Lambda \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, combining these two results, we get

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G\rho + \Lambda \quad \text{(Friedmann equation)}$$
$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 8\pi G p - \Lambda$$

Note that these two equations yield

$$\ddot{a} = -\frac{4\pi G a}{3}\left(\rho + 3p\right) + \frac{1}{3}a\Lambda$$

We have the for the equation os state

$$p_\Lambda\left(\rho_\Lambda\right) = -\rho_\Lambda \quad \text{for pure vacuum energy}$$
$$p_r\left(\rho_r\right) = \frac{1}{3}\rho_r \quad \text{for pure radiation}$$
$$p_m\left(\rho_m\right) = 0 \quad \text{for pure dust}$$

Einstein's field equation under F.L. spacetime imply that

$$\frac{d}{da}\left(\rho a^3\right) = -3pa^2 \quad \text{(continuity equation)}$$
$$\Rightarrow \frac{d\rho}{da} = -\frac{3}{a}\left(p + \rho\right)$$

3 Numerical Analysis

We would like to first compare known analytical solutions to different epoch with the numerical simulations.

3.1 Different Epochs in Flat Universe

Note that we approximate the equation of state via

$$p(\rho) = w\rho \quad \text{with} \;\; w = \text{const.}$$

Then, from the continuity equation

$$\rho(a) = \rho_0 a^{-3(w+1)}$$

we get that

$$\rho(a) = \begin{cases} \rho_m a^{-3} & \text{matter dominated epoch} \\ \rho_r a^{-4} & \text{radiation dominated epoch} \\ \rho_v & \text{dark energy dominated epoch} \end{cases}$$

Then, the scale factor is [**?**]

$$a(t) = \begin{cases} \left(t/t_r\right)^{1/2} & \text{radiation dominated epoch} \\ \left(t/t_r\right)^{2/3} & \text{radiation dominated epoch} \\ e^{t/t_v} & \text{dark energy dominated epoch} \end{cases}$$

where

$$t_r = \sqrt{\frac{3}{32\pi G\rho_r}}$$
$$t_m = \frac{1}{\sqrt{6\pi G\rho_m}}$$
$$t_v = \sqrt{\frac{3}{8\pi G\rho_v}}$$

We now compare the results of numerical solutions versus analytical solutions of $\rho(a)$. Note that we set $c = G = 1$ and $\rho_m = \rho_r = \rho_v = 1$.

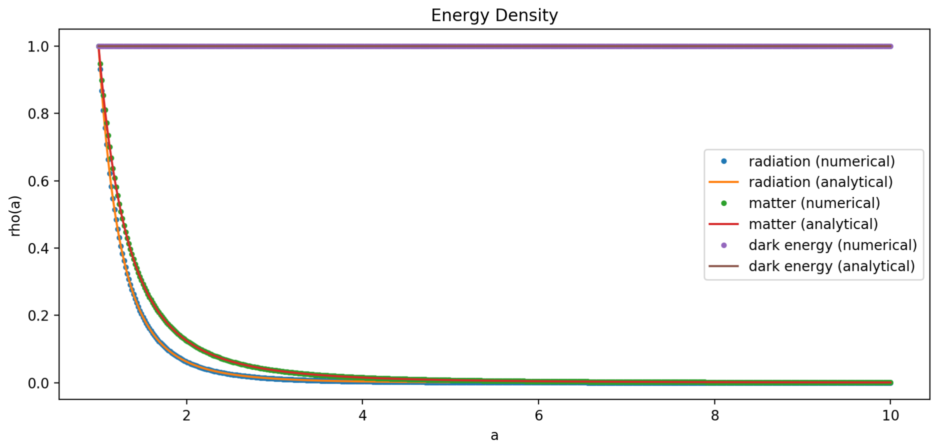

Figure 1: Numerical v.s. analytical

As expected, numerical plots versus analytical plot of $\rho\left(a\right)$ for all of the three epochs match up perfectly. We see that radiation decays a lot quicker than that of matter. Just from the plot, it may seem as tho radiation energy density is always lower than matter energy density. However, this is only because the initial value of of scale factor is $a=1$. If we were to go to where $a<1$, radiation plot will have higher energy density. For dark energy, we see that energy density stays constant as expected. Now, we move on to the scale factor.

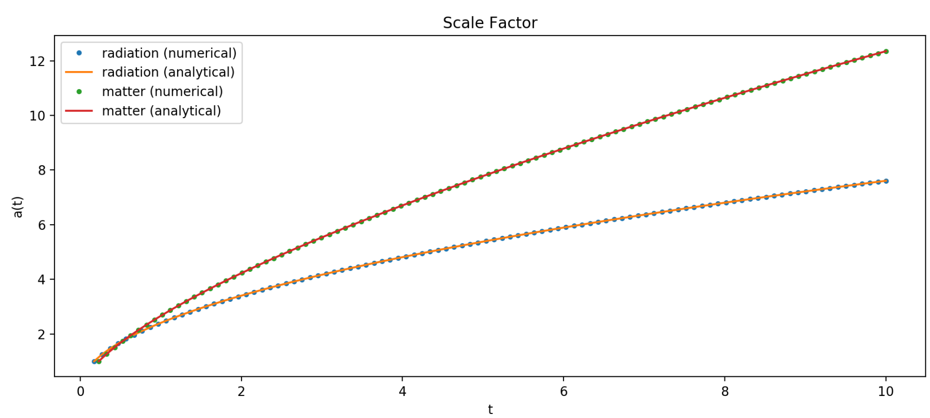

Figure 2: Numerical v.s. analytical

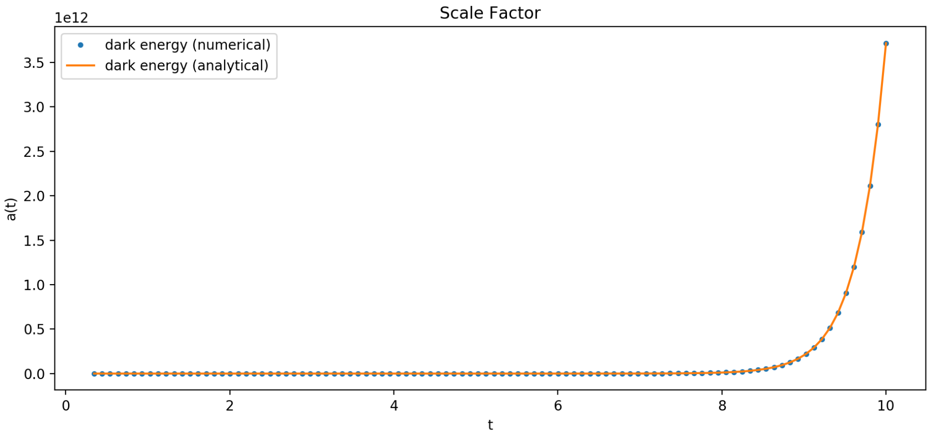

Figure 3: Numerical v.s. analytical

We see that scale factor solutions using numerical and analytical methods match up. From these plots, we see that radiation epoch causes the universe to expand at a slower rate than that of matter. The fastest expansion is caused by the dark energy contribution which is an exponential expansion. This is because the pressure of dark energy inversely related to its energy density.

3.2 Big Rip

There has been some experimental evidence that the current value of w could be -1.19 [**?**]. We would like to explore what the behaviour of energy density and scale factor is under this equation of state. We see that for $w = -1.19$, it is very closely approximated via $\rho\left(a\right) \approx a^{1/1.75}$. According

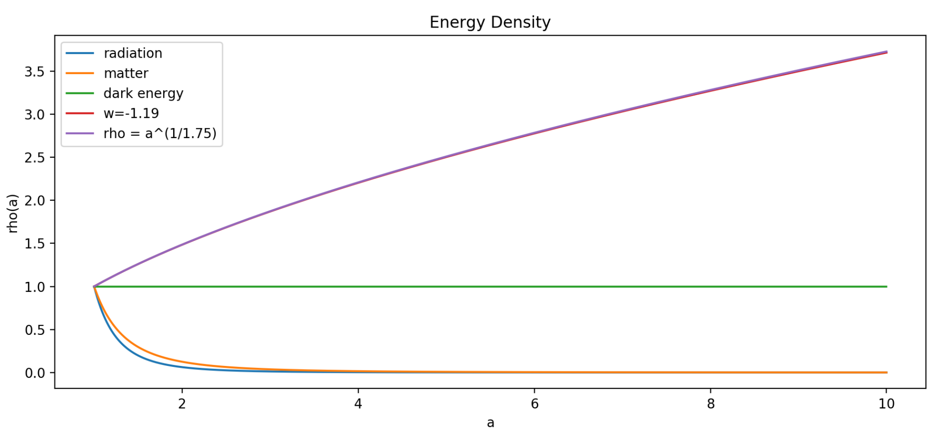

Figure 4: $w = -1.19$ plotted with other epochs

to this result, if our universe is indeed has such value for w, then the energy density will continue to rise up as the scale factor goes up which is an interesting find. According to the analysis, when

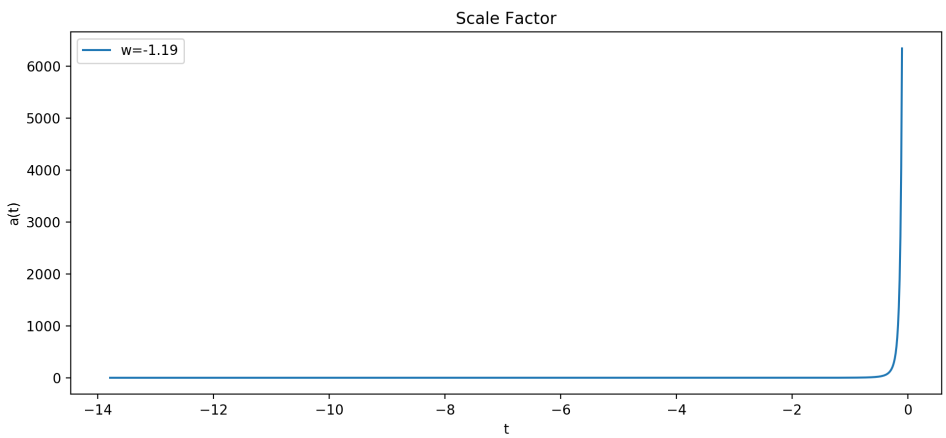

Figure 5: $w = -1.19$

$t \to 0$, $a \to \infty$. So we see that for most of the time, a stays relatively unchanged, but within a finite amount of time, the scale factor blows up to infinity. This is a catastrophic event for the universe since the universe will rip apart due in finite amount of time under extreme expansion.

3.3 Different Epochs in the Curved Universe

So far, we have looked into the case in which the universe is flat. Now, we consider cases when $k = 1, -1$ for three main epochs. For radiation epoch (Figure 6), we see that the negatively curved universe expand faster compared to the flat universe. However, the universe expand slower in positively curved universe. In fact, the scale factor flattens out eventually for positively curved universe, whereas the universe expand forever for the other two cases.

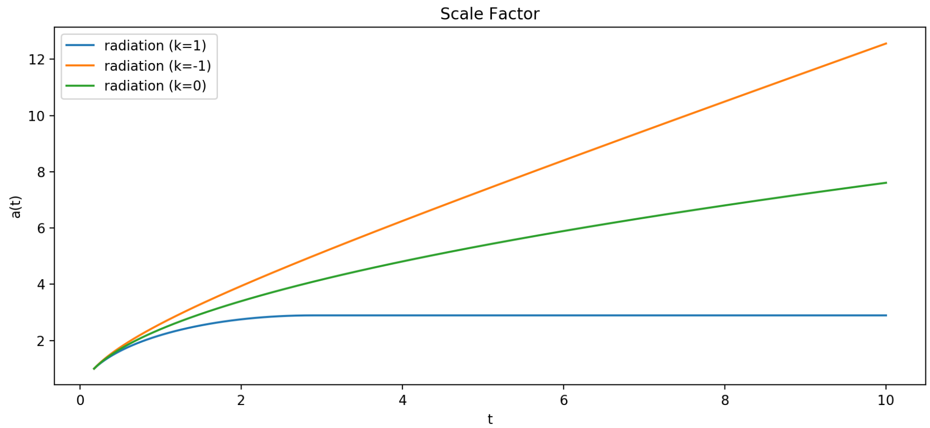

Figure 6: Radiation dominated epoch for $k = -1, 0, 1$

For the matter dominated epoch (Figure 7), we see the same trend as the radiation dominated epoch. However, the rate at which the scale factor flattens out in positively curved universe is slower than the radiation dominated epoch.

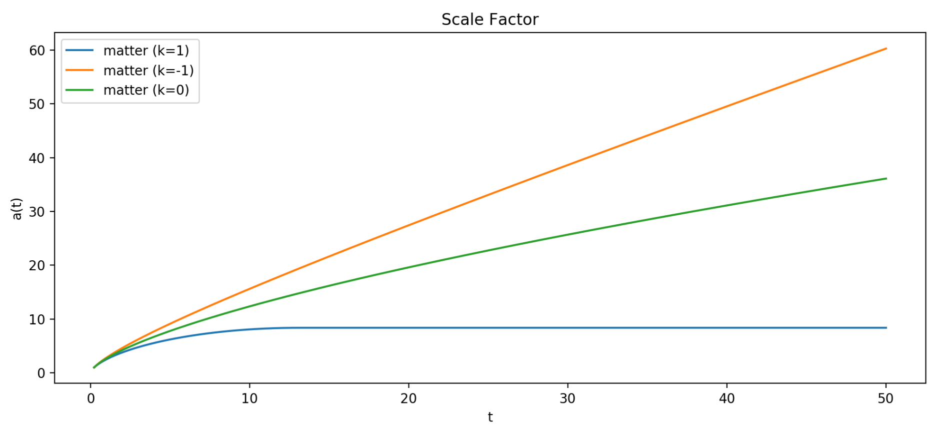

Figure 7: Matter dominated epoch for $k = -1, 0, 1$

We see that for the dark energy dominated epoch (Figure 8), all three universes basically grows exponentially. This means that the curvature of the universe does not play a significant role in the fate of the universe under dark energy dominated epoch.

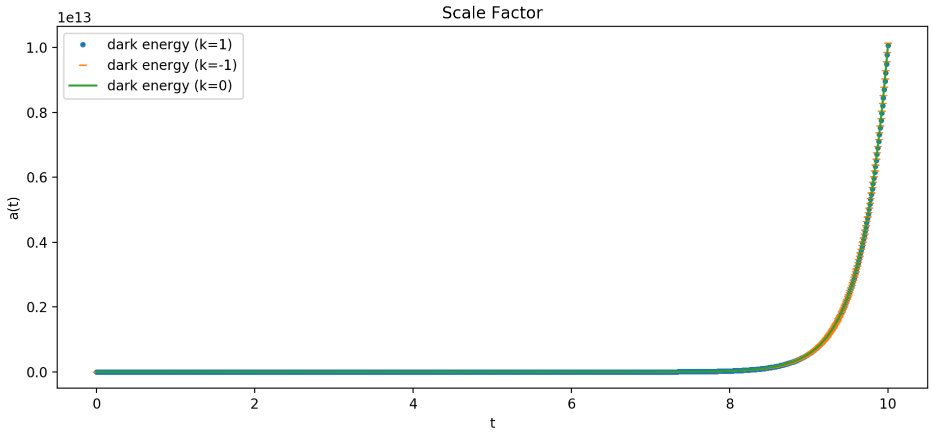

Figure 8: Dark energy dominated epoch for $k = -1, 0, 1$

Overall, we see that for negatively curved universe, the expansion of the universe is faster than that of flat universe. So, the negatively curved universe will go through the same radiation to matter to dark energy epoch and eventually will end up with the exponential expansion of the universe. However, for the positively curved universe, the rate of expansion under radiation and matter dominated epoch is much slower, and it fact the scale factor seems to not grow at certain point in time. However, the fate of the $k = +1$ universe under dark energy dominated epoch still result in the exponential expansion. Therefore, if we were to live in a universe in which dark energy is present, then even the positively curved universe will eventually be dominated by dark energy resulting in an exponential expansion.

4 Discussion

In this paper, we see that for $w < -1$, the universe will result in a Big Rip where the universe expands infinitely in a finite amount of time. Moreover, regardless of the universe's curvature, if there is dark energy present in the universe, then the universe will result in an exponential expansion. Hence, in all cases, the universe's fate seems to be that it will expand forever. We only hope that the equation of state is not of the form $p = w\rho$ where $w < -1$ which will be very catastrophic.