Mathematical Methods to Capture Shape

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1 Abstract

This essay is an overview of mathematical methods to capture the shape of a Riemannian or Lorentzian manifold. In particular, we look into the metric tensor, and the affine connection for determining the shape of a manifold. We establish the unique relationship between the metric and the affine connection in the case of spacetime manifold under the suitable assumptions stemming from General Relativity. Then, we look beyond General Relativity to describe a plausible case in which we need to consider non-metric compatible affine connection with non-zero torsion. We conclude that the shape of the spacetime for such a case can in fact be described via variational principle, however its shape has a fuzzy resolution.

2 Motivation

We all have learned that the sum of the angles in a triangle is 180 degrees. However, if one is a bit more curious and proceeds to test this idea further, then he/she will find out this is only true on a flat surface. For example, the sum of the angles in a triangle on a sphere is greater than 180 degrees. Hence, in general, if the underlying space is not flat, the shapes that live in such non-trivial geometry would have properties that are different from the shapes living in a flat space.

You may wonder why we need to consider non-trivial geometries. There are two reasons. One is simply because we can. If you are a Mathematician, then this is a good enough of a reason. Mathematicians enjoy generalizing known mathematical axioms to a much broader class. By doing so, one can reach an understanding that previously has not known. For example, in the 19th century, there were many unsatisfying things about Riemann integrations. There were no good characterizations for Riemann integrable functions, and there were a lot of functions that seemed to have an integral, but did not. However, much to Mathematician's surprise, Physicists at the time would come up with these numbers associated with those integrations which puzzled Mathematicians since those were not integrable functions. It was only at the start of the 20th century that the theory of integrations was well understood. Henri Lebesgue introduced the much broader class of integration called Lebesgue Integration. By doing so, the Mathematicians finally had a good characterization of integrable functions, and were able to integrate functions that were previously not integrable.

Another reason is because we actually live in a space that has a non-trivial geometry. Einstein's theory of General Relativity, in short, says that mass curves spacetime. Due to the presence of this curvature, particles travelling through such spacetime would have a non-trivial trajectory. This is why Physicists are interested in studying the mathematics of non-trivial geometry.

3 Mathematical Methods

3.1 Violation of Pythagoras' Law

One obvious way to describe a shape is to come up with a function that can measure distance between any two points. In General Relativity, we introduce a rank 2 tensor called a metric and this tells us what the distance between any two points on the manifold is.

On a Euclidean space, $\mathbb{R}^n$, the spatial distance between x and y is given by a generalized Pythagoras' law

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x^i - y^i)^2} = \sqrt{\eta_{ij}(x^i - y^i)(x^j - y^j)}$$

where the metric is given by

$$\eta_{ij} = \delta_{ij}$$

Then, for such distance function, we see that it satisfies the Pythagoras' law.

However, in a general Riemmanian manifold, the metric is not given by η_{ij}. For such a case, the Pythagoras' law is violated and the distance between p and q can be given by the shortest path between them. Let $\gamma(t)$ be a path such that $\gamma(0) = p$ and $\gamma(1) = q$. Then the distance function is given instead by [1]

$$d(p,q) = \inf_{\gamma} \ell(\gamma)$$
$$\ell(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt$$

With this definition, if $\mathcal{M}$ is path-connected, then $(\mathcal{M}, d)$ is a metric space. This allows us to employ many of the techniques from Real Analysis. Hence, this gives us a very firm handle on the shape of the manifold.

3.2 Non-triviallity of Parallel Transport

Say you are standing on the North pole, and imagine you have a vector pointing out from your belly button. Now, you walk straight down to the equator of the Earth, then walk along the equator and finally going back up to the North pole while keeping your vector parallel to the trajectory. You will notice that this imaginary vector will be pointing at a different direction than when you started. However, if the Earth is flat, then the vector will remain in the same direction even after completing the travel. This is because vector on a sphere goes through a non-trivial parallel transportation.

Motivated by this example, we see that this non-triviality of parallel transport can carry information about the underlying geometry of the space. In order to define a parallel transport, one needs to introduce covariant derivative, ∇, which is a generalization of directional derivative on $\mathbb{R}^n$. Covariant derivative is defined as [2]

$$\begin{aligned}\nabla : T^1(\mathcal{M}) \times T^1(\mathcal{M}) &\to T^1(\mathcal{M})\\ \nabla(\eta, \xi) &= \nabla_\xi \eta\end{aligned}$$

satisfying

$$\begin{aligned}\nabla_{f\xi}\eta &= f\nabla_\xi \eta \quad \forall f \in C^\infty(\mathcal{M})\\ \nabla_\xi(f\eta) &= \xi(f)\eta + f\nabla_\xi\eta\end{aligned}$$

In a canonical chart, covariant derivative is given by

$$\begin{aligned}\nabla_\xi \eta &= \xi^i \eta^k_{;i} \frac{\partial}{\partial x^k}\\ \eta^k_{;i} &= \eta^k_{,i} + \eta^j \Gamma^k_{ij}\end{aligned}$$

We say that a tangent vector field η is auto-parallel along a path γ when ($\dot{\gamma}(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$)

$$\begin{aligned}\nabla_{\dot{\gamma}}\eta &= 0\\ \Rightarrow \frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt}\Gamma^i{}_{kj} &= 0\end{aligned}$$

where $\dot{\gamma} = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$.

Let $\gamma(t)$ be a path. Then, the parallel transport τ at $\gamma(t)$ to $\gamma(s)$ applied to an autoparallel tangent vector field η along the path γ is given by

$$\begin{aligned}\tau_{t\to s} : T_{\gamma(t)}(\mathcal{M}) &\to T_{\gamma(s)}(\mathcal{M})\\ \tau_{t\to s}(\eta(\gamma(t))) &= \eta(\gamma(s))\end{aligned}$$

Proposition 3.1. *The relationship between covariant derivative and parallel transport is*

$$\nabla_{\dot{\gamma}}\eta(\gamma(t)) = \frac{d}{ds}\Big|_{s=t} \tau_{s\to t}(\eta(\gamma(s)))$$

Proof. [3] We use a canonical chart. Note that,

$$\begin{aligned}\eta(\gamma(t)) &= \tau_{s\to t}\eta(\gamma(s))\\ \frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt}\Gamma^i{}_{kj} &= 0\end{aligned}$$

Then

$$
\begin{aligned}
\frac{d(\tau_{s\to t})^i_j\eta(\gamma(s))^j}{dt} + \eta(s)^j\frac{dx^k}{dt}{\Gamma^i}_{kj} = 0 \\
\Rightarrow \frac{d}{dt}[(\tau_{s\to t})^i_j\eta(\gamma(s))^j] = -\eta(\gamma(t))^j\frac{dx^k}{dt}{\Gamma^i}_{kj} \\
\Rightarrow \frac{d}{dt}\Big|_{t=s}[(\tau_{s\to t})^i_j\eta(\gamma(s))^j] = [\frac{d}{dt}\Big|_{t=s}(\tau_{s\to t})^i_j]\eta(\gamma(t))^j = -\eta(\gamma(t))^j\frac{dx^k}{dt}{\Gamma^i}_{kj} \\
\Rightarrow \frac{d}{dt}\Big|_{t=s}(\tau_{s\to t})^i_j = -\frac{dx^k}{dt}{\Gamma^i}_{kj}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{ds}\Big|_{s=t}(\tau_{s\to t}\eta(\gamma(s)))^i &= \frac{d}{ds}\Big|_{s=t}((\tau_{t\to s})^{-1}\eta(\gamma(s)))^i \\
&= -\frac{d}{ds}\Big|_{s=t}(\tau_{t\to s})^i_j\eta^j + \frac{d}{ds}\Big|_{s=t}\eta^i(\gamma(s)) \\
&= \frac{d}{ds}\Big|_{s=t}\eta^i(\gamma(s)) + \eta(\gamma(t))^j\frac{dx^k}{dt}{\Gamma^i}_{kj} \\
&= \frac{d}{dt}\eta^i(\gamma(t)) + \eta(\gamma(t))^j\frac{dx^k}{dt}{\Gamma^i}_{kj} = (\nabla_{\dot\gamma}\eta(\gamma(t)))^i
\end{aligned}
$$

□

Hence, covariant derivative defines the parallel transport. This allow us to detect if the geometry is flat or not.

3.2.1 Torsion and Curvature

To further describe a shape of a manifold, we can introduce Torsion and Curvature [4]. Torsion map is defined as

$$
\begin{aligned}
\mathcal{T} : T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) \to T^1_p(\mathcal{M}) \\
\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]
\end{aligned}
$$

Then, we can define Torsion tensor as follows

$$
\begin{aligned}
T : T^1_p(\mathcal{M})^* \times T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) \to \mathcal{F}_p(M) \\
T(\omega, X, Y) =< \omega, \mathcal{T}(X,Y) >
\end{aligned}
$$

In a local coordinate, the components of torsion tensor is (we let $\partial_i = \frac{\partial}{\partial x^i}$)

$$
\begin{aligned}
T^k_{ij} &=< dx^k, \mathcal{T}(\partial_i, \partial_j) > \\
&=< dx^k, \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i > \\
&=< dx^k, {\Gamma^r}_{[ij]}\partial_r > \\
&= {\Gamma^k}_{[ij]}
\end{aligned}
$$

Note that we use the notation

$$T_{[ij]} = T_{ij} - T_{ji}$$

The Curvature map is given by

$$\begin{aligned}\mathcal{R} : T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) &\to T^1_p(\mathcal{M}) \\ \mathcal{R}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) &- \nabla_{[X,Y]}Z\end{aligned}$$

Then, the Curvature tensor is defined by

$$\begin{aligned}R : T^1_p(\mathcal{M})^* \times T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) \times T^1_p(\mathcal{M}) &\to \mathcal{F}_p(\mathcal{M}) \\ R(\omega, X, Y, Z) &= < \omega, \mathcal{R}(X,Y)Z >\end{aligned}$$

In a local chart,

$$\begin{aligned}R^i{}_{jk\ell} &= < dx^i, \mathcal{R}(\partial_k, \partial_\ell)\partial_j > \\ &= < dx^i, [\nabla_{\partial_k}, \nabla_{\partial_\ell}]\partial_j > \\ &= < dx^i, \nabla_{\partial_{[k}}\Gamma^s{}_{\ell]j}\partial_s > \\ &= K^i{}_{j[k\ell]}\end{aligned}$$

where

$$K^i{}_{jk\ell} = \Gamma^i{}_{\ell j,k} + \Gamma^s{}_{\ell j}\Gamma^i{}_{ks}$$

These two tensors have very good intuitive pictures. If you live in a manifold with non-zero torsion, then something very strange can happen. Consider $X, Y \in T_p\mathcal{M}$ such that $X \neq Y$. Now, if you travel from point p infinitesimally in the direction of X and then Y direction, and compare with travelling in the direction of Y then in the direction of X, you will find out that you end up at a different point on the manifold.

To see this concretely, let $\tilde{X}$ to be the vector X parallel transported along in the direction of Y, and $\tilde{Y}$ to be the vector parallel transported along in the direction of X. Let ϕ be a canonical chart. Then $\phi_* X = x^k\partial_k$, $\phi_* Y = y^k\partial_k$, and $\phi(p) = a$. Observe that

$$\begin{aligned}\widetilde{x}^k(a^i + y^i) &\approx x^k(a^i) + \dot{x}^k(a^i) \\ &= x^k(a^i) - \Gamma(a)^k_{ij}y^i x^j \\ \widetilde{y}^k(a^i + x^i) &\approx y^k(a^i) - \Gamma(a)^k_{ij}x^i y^j\end{aligned}$$

Then, the point that we reach by traveling along Y then $\widetilde{X}$ is

$$u_{Y \to \widetilde{X}} = a^k + y^k + x^k - \Gamma(a)^k_{ij}y^i x^j$$

And the point that we reach by travelling along X then along $\widetilde{Y}$ is

$$u_{X \to \widetilde{Y}} = a^k + y^k + x^k - \Gamma(a)^k_{ij}x^i y^j$$

Then, the distance between these two points is

$$\begin{aligned} u_{Y\to\widetilde{X}} - u_{X\to\widetilde{Y}} &= -\Gamma(x)^k_{ij} y^i x^j + \Gamma(a)^k_{ij} x^i y^j \\ &= \Gamma(a)^k_{[ji]} y^i x^j \\ &= \phi_* \mathcal{T}^k_{ji} Y^i X^j \end{aligned}$$

Similarly, the curvature tensor also has a very nice physical intuition. If we assume that the torsion is zero and consider a vector $X, Y, Z \in T_p(\mathcal{M})$ such that $X \neq Y \neq Z$. Then, if we parallel transport X along Y, $\widetilde{Z}$, $-\widetilde{Y}$, $-Z$ respectively, where $\widetilde{Z}$ is Z that is parallel transported along Y and $\widetilde{Y}$ is Y that is parallel transported along Z. Then, we see that under a canonical chart ϕ where $\phi(p) = a$,

$$(\widetilde{x} - x)^k \approx y^i z^j R^k_{ij\ell} x^\ell$$

where $\widetilde{x}^k \partial_k$ is a vector after the travel along the infinitesimal parallelogram made by Y and Z.

One advantage of this picture versus the metric picture is that these tensors have very nice intuitive picture for the local structure of the manifold.

4 Relationship between the metric and affine connection

So far, we have two different ways of describing the shape of a manifold, using the metric and the affine connection. However, there is a relationship between these two under a suitable assumption [5].

We assume that the parallel transport of two vectors X and Y must be so that their inner product remains the same. Then, we can prove that for any path $\gamma(t)$,

$$d/dt(g(\gamma(t))_{\mu\nu} y^\mu(\gamma(t)) x^\nu(\gamma(t))) = 0 \Rightarrow \nabla_X g = 0 \quad \forall X$$

We call ∇ is metric preserving if the above condition is met. In fact, for any $(\mathcal{M}, g)$ and a tensor field $\mathcal{T}$ where $\mathcal{T}^k_{ij} = -\mathcal{T}^k_{ji}$, there exists a metric-preserving ∇ whose torsion is $\mathcal{T}$.

Note that $T^k_{ij} = \Gamma^k_{[ij]}$ is a tensor. From the equivalence principle of General Relativity, since Γ express gravitational and pseudo forces, we require that for any point $p \in \mathcal{M}$, there exists a chart for p such that $\Gamma(p) = 0$. And under such choice of chart, we see that torsion is zero. But since the torsion is a tensor, it must be true that the torsion vanishes in all coordinate system. Therefore, we can use the following theorem to establish a unique relationship between the metric g and the affine connection ∇:

Theorem 4.1. *(Fundamental Theorem of (pseudo) Riemannian Geometry) For each (pseudo) Riemanninan manifold $(\mathcal{M}, g)$, there exists a unique metric compatible and torsionless ∇. Such affine connection is called the Levi-Civita connection.*

In a chart, the Levi-Civita connection is given by

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\alpha\nu,\mu} + g_{\mu\alpha,\nu} - g_{\nu\mu,\alpha})$$

Therefore, under suitable assumptions, $(\mathcal{M}, g)$ and $(\mathcal{M}, \nabla)$ describe the same shape.

5 Non-trivial Affine Connection: Relative Locality

You may wonder if there is any physically interesting situations in which we should consider a manifold that has a non-torsionless and non-metric compatible affine connection. Such a manifold comes about when considering a momentum space in which the algebra of interactions is in general non-linear, non-commutative, and non-associative. This consideration surprisingly has a intriguing consequence on the notion of locality and the shape of spacetime.

In General Relativity, it takes a stance that spacetime is independent of the kinds of the probes used which implies that the momentum space is flat. For example, in Einstein's theory of Relativity, observers construct spacetime coordinates by sending light signals and back. In this process, they discard the information about its energies and momentum. Such a notion is called the absolute locality. On the other hand, relative locality is a a proposed physical phenomenon where spacetime is dependent on the quantum nature of the probe used such as its energy and quantum number. For example, if two world-line of particles meet at some point in a spacetime coordinate constructed by one observer, then it may not be in general true that those two world lines will meet for a different observer.

5.1 Principle of Relative Locality

In the paper [6], the author achieves relative locality by imposing non-linearity to the addition of momentum. And it considers a momentum space to be the primary space in which spacetime can be constructed as a tangent space to the momentum space.

5.1.1 The Metric, the Algebra of Interaction, and the Affine Connection

From the mass-shell condition, we get that

$$\|p\|^2 = p^\mu g_{\mu\nu} p^\nu = m^2$$

Then, we see that the metric is given by

$$\begin{aligned} \frac{\partial}{\partial_{p_a}} &= 2g^{\mu\nu} p_\mu \delta^a_\nu \\ \Rightarrow g^{\mu\nu} &= \frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}\|p\|^2 \end{aligned}$$

We denote operations that adds two momentum by

$$p' = p \oplus q$$

which is in general non-linear, non-commutative, and non-associative.

The connection at k is calculated by

$$\Gamma^{ab}_c(k) = -\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p \oplus_k q)_c|_{p=q=k}$$

where

$$p \oplus_k q = k \oplus ((\ominus k \oplus p) \oplus (\ominus k \oplus q))$$

which is the translation in the momentum space by k. Then the torsion is

$$T_c^{ab}(k) = -\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p \oplus_k q - q \oplus_k p)_c|_{p=q=k}$$

The Riemann curvature is

$$R_\sigma^{\mu\nu\rho}(k) = \frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{\nu]}}\frac{\partial}{\partial r_\rho}((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r))_\sigma|_{p=q=r=k}$$

The non-metricity is

$$N^{\mu\nu\rho} = \nabla^\rho g^{\mu\nu}(k)$$

Therefore, we arrive at the case where it has a non-metric compatible connection with non-zero torsion. Next question is then, how do we detect the shape of the spacetime from this momentum space? This can be answered by using Variational Principle.

5.2 Variational Principle

We take a stance that spacetime is an auxilary concept which emerges as a cotangent space to the momentum space. We now look into dynamics of particles. Consider the actions

$$S^{total} = \sum_J S^I_{free} + S^{int}$$

where

$$\begin{aligned} S^I_{free} &= \int ds(x^a_J \dot{k}^J_a + \mathcal{N}_J C^J(k)) \\ S^{int} &= \mathcal{K}(k(0))_a z^a \\ \mathcal{N}_J, z^a &\quad \text{are the Lagrange multipliers} \end{aligned}$$

Here S^I_{free} is simply an action for the individual particles and S^{int} is an action for the interactions part. k^I_a is the momenta of Ith particle and x^a_I is the momenta of the momenta of Ith particle which lives in a cotangent space of our momentum space at point k^I_a. $C^J(k)$ is a constraint for the mass-shell condition

$$\begin{aligned} C(p) &= D^2(p) - m^2 \\ D^2(p) &= \int_0^p ds g^{\mu\nu}\dot{p}_\mu \dot{p}_\nu \end{aligned}$$

and $\mathcal{K}(k(0))$ is a constraint for the conservation of energy and momentum where the interaction is taking place at the affine parameter $s = 0$,

$$\begin{aligned}
&\mathcal{K}_a(k) = 0 \\
&\mathcal{K}_a(k) = \sum_I k_a^I - \sum_{J \in \mathcal{J}(I)} C_{I,J} \Gamma_a^{bc} k_b^J k_a^I + \dots
\end{aligned}$$

Now, we take the variation to be zero,

$$\delta S^{total} = 0$$

Then, the equations must be

$$\begin{aligned}
x_J^a(0) &= z^b \frac{\delta \mathcal{K}_b}{\delta k_a^J} \\
&= z^a - z^b \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma_b^{ac} k_c^L + \dots
\end{aligned}$$

The interactions take place in which above equation is satisfied. We see that it is separated by intervals z^a,

$$\begin{aligned}
\Delta x_J^a(0) &= -z^b \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma_b^{ac} k_c^L + \dots \\
\Rightarrow \Delta x &\approx |z||\Gamma|k
\end{aligned}$$

Hence, using the variational principle, we were able to construct a spacetime. However, its shape is dependent on the probes used in constructing the spacetime. Therefore, the shape of the spacetime has a finite resolution $\approx \Delta x$. If in fact the the algebra of interactions in the momentum space is non-linearm then this approach of describing the spacetime is advantageous over the previous two methods of ascribing the metric and the connection to the spacetime manifold because such a approach by definition has an infinite resolution, and is not observer dependent.

6 Summary

We see that under suitable assumptions, $(\mathcal{M}, g)$ and $(\mathcal{M}, \nabla)$ produce excellent grasp in describing the shape of the spacetime manifold. However, using the ∇ to determine the shape is more intuitive compared to using the metric.

However, there are cases in which ∇ cannot be uniquely determined. One such case comes from considering non-trivial addition of momentum. For this case, the momentum space has an affine connection that is not metric compatible with non-zero torsion. Furthermore, we saw that we can describe the shape of its spacetime using the variational principle, and this method produces the shape of the spacetime which is fuzzy in a sense that it depends on the probes being used to construct the spacetime. However, if the addition of momentum in fact has the non-trivial form shown earlier, then this approach is the best candidate for describing the shape of the spacetime manifold compared to the other methods covered in this essay.

References

[1] Nicklas Persson. Shortest paths and geodesics in metric spaces. Master's thesis, Umea University, Sweden, 2012.

[2] Achim Kemf. general relativity for cosmology lecture 8, 2019.

[3] Norbert Straumann. *General Relativity with Applications to Astrophysics*. Springer, 2004.

[4] Achim Kemf. general relativity for cosmology lecture 9, 2019.

[5] Achim Kemf. general relativity for cosmology lecture 10, 2019.

[6] Giovanni Amelino-Camelia, Laurent Freidel, Jerzy Kowalski-Glikman, and Lee Smolin. Principle of relative locality. *Phys. Rev. D*, 84:084010, Oct 2011.