Relative Locality

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1 Introduction

Relative locality is a proposed physical phenomenon that two different observer will differ in deciding whether or not two spacetime event coincides. One of the motivation for such proposal comes from the fact in the Einstein's theory of relativity, the observers construct spacetime coordinates by exchanging light signals. However, during that procedure, we disregard the energy of such photons in which we have no basis to do so. Hence, it is plausible to conceive that the observers may construct two different spacetime coordinates when photons with two different frequencies are used.

2 Principle of Relative Locality

We achieve relative locality by imposing non-linearity to the addition of momentum.
We consider momentum space to be the primary space in which spacetime can be constructed as a tangent space to the momentum space.

2.1 The metric

From the mass-shell condition, we get that

$$\|p\|^2 = p^\mu g_{\mu\nu} p^\nu = m^2 \tag{1}$$

Then, we see that

$$\frac{\partial}{\partial_{p_a}} = 2g^{\mu\nu} p_\mu \delta^a_\nu \tag{2}$$

$$\Rightarrow g^{\mu\nu} = \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \|p\|^2 \tag{3}$$

2.2 The algebra of interaction

In general, we denote operations that adds two momentum by

$$p' = p \oplus q \tag{4}$$

which is in general non-linear.

2.3 Connection

The connection at k is calculated by

$$\Gamma_c^{ab}(k) = -\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p \oplus_k q)_c |_{p=q=k} \tag{5}$$

where

$$p \oplus_k q = k \oplus \big((\ominus k \oplus p) \oplus (\ominus k \oplus q)\big) \tag{6}$$

which is the translation in the momentum space by k.

Then the torsion is

$$T_c^{ab}(k) = -\frac{\partial}{\partial p_a}\frac{\partial}{\partial q_b}(p \oplus_k q - q \oplus_k p)_c |_{p=q=k} \tag{7}$$

The Riemann curvature is

$$R_\sigma^{\mu\nu\rho}(k) = \frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{\nu]}}\frac{\partial}{\partial r_\rho}\big((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r)\big)_\sigma |_{p=q=r=k} \tag{8}$$

The non-metricity is

$$N^{\mu\nu\rho} = \nabla^\rho g^{\mu\nu}(k) \tag{9}$$

2.4 Variational Principle

3 Relative Simultaneity

Consider Einstein's Relativity in 1+1 dimension, the Lorentz transformation is given by

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c^2} \\ -\gamma v & \gamma \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} \tag{10}$$

where (t', x') reference frame is moving at speed v with respect to (t, x). Consider two events, $(T_L, -d)$ and (T_R, d), that happened at the same time ($\delta T = T_L - T_R = 0$) in the reference frame of (t, x). Then, the times for these two events with respect to the primed reference frame are

$$
\begin{aligned}
T'_L &= \gamma\left(T_L + \frac{vd}{c^2}\right) && (11)\\
T'_R &= \gamma\left(T_R - \frac{vd}{c^2}\right) && (12)\\
\Rightarrow \Delta T' = T'_L - T'_R &= 2\gamma\frac{vd}{c^2} && (13)\\
&= \frac{2vd}{c^2\sqrt{1-\frac{v^2}{c^2}}} && (14)\\
&= \frac{v\Delta x}{c^2\sqrt{1-\frac{v^2}{c^2}}} && (15)\\
X'_L &= \gamma\left(-T_L v + X_L\right) && (16)\\
X'_R &= \gamma\left(-T_R v + X_R\right) && (17)\\
\Delta x' &= \gamma\Delta x && (18)\\
\Delta T' &= \gamma\frac{v\Delta x}{c^2} && (19)\\
&= \frac{v\Delta x'}{c^2} && (20)\\
&= \frac{v}{c}\Delta T' && (21)
\end{aligned}
$$

Hence, we see that these two events do not happen at the same in the primed reference frame.
Say $T_L \neq T_R$. Then,

$$\Delta T' = \gamma\left(\Delta T + \frac{v\Delta x}{c^2}\right) \tag{22}$$

3.1 Connections on velocity space

Addition of velocities is given by

$$u \oplus v = \frac{v+u}{1+\frac{vu}{c^2}} \tag{23}$$

The translation by k is given by

$$u \oplus_k v = k \oplus \big((\ominus k \oplus u) \oplus (\ominus k \oplus v)\big) \tag{24}$$

Let (note that $\ominus k = -k$)

$$
\begin{aligned}
\ell &= \ell_1 \oplus \ell_2 && (25)\\
\ell_1 &= (-k) \oplus u && (26)\\
\ell_2 &= (-k) \oplus v && (27)
\end{aligned}
$$

The connections is given by

$$\Gamma\left(k\right)=\frac{\partial}{\partial u}\frac{\partial}{\partial v}\left(u\oplus_{k}v\right)|_{u=v=k} \tag{28}$$

Now, let's calculate

$$\begin{aligned}
\partial_{v}\left(k\oplus\ell\right) &= \partial_{v}\frac{k+\ell}{1+\frac{k\ell}{c^{2}}} && (29)\\
&= \frac{\left(1+\frac{k\ell}{c^{2}}\right)\partial_{v}\left(k+\ell\right)-\left(k+\ell\right)\partial_{v}\left(1+\frac{k\ell}{c^{2}}\right)}{\left(1+\frac{k\ell}{c^{2}}\right)^{2}} && (30)\\
&= \frac{\left(1+\frac{k\ell}{c^{2}}\right)\partial_{v}\ell-\left(k+\ell\right)\partial_{v}\frac{k\ell}{c^{2}}}{\left(1+\frac{k\ell}{c^{2}}\right)^{2}} && (31)\\
&= \frac{\left(1+\frac{k\ell}{c^{2}}\right)\partial_{v}\ell-\frac{1}{c^{2}}\left(k+\ell\right)k\partial_{v}\ell}{\left(1+\frac{k\ell}{c^{2}}\right)^{2}} && (32)
\end{aligned}$$

And

$$\begin{aligned}
\partial_{v}\ell &= \partial_{v}\left(\ell_{1}\oplus\ell_{2}\right) && (33)\\
&= \partial_{v}\frac{\ell_{1}+\ell_{2}}{1+\frac{\ell_{1}\ell_{2}}{c^{2}}} && (34)\\
&= \frac{\left(1+\frac{\ell_{1}\ell_{2}}{c^{2}}\right)\partial_{v}\left(\ell_{1}+\ell_{2}\right)-\left(\ell_{1}+\ell_{2}\right)\partial_{v}\frac{\ell_{1}\ell_{2}}{c^{2}}}{\left(1+\frac{\ell_{1}\ell_{2}}{c^{2}}\right)^{2}} && (35)\\
&= \frac{\left(1+\frac{\ell_{1}\ell_{2}}{c^{2}}\right)\left(\partial_{v}\ell_{1}+\partial_{v}\ell_{2}\right)-\frac{1}{c^{2}}\left(\ell_{1}+\ell_{2}\right)\left(\ell_{1}\partial_{v}\ell_{2}+\ell_{2}\partial_{v}\ell_{1}\right)}{\left(1+\frac{\ell_{1}\ell_{2}}{c^{2}}\right)^{2}} && (36)
\end{aligned}$$

Then

$$\partial_v \ell_1 = \partial_v (-k) \oplus u \tag{37}$$
$$= \partial_v \frac{u-k}{1-\frac{uk}{c^2}} \tag{38}$$
$$= \frac{\left(1-\frac{uk}{c^2}\right) \partial_v (u-k) - (u-k)\, \partial_v \left(1-\frac{uk}{c^2}\right)}{\left(1-\frac{uk}{c^2}\right)^2} \tag{39}$$
$$= 0 \tag{40}$$
$$\partial_v \ell_2 = \partial_v (-k) \oplus v \tag{41}$$
$$= \frac{\left(1-\frac{vk}{c^2}\right) \partial_v (v-k) - (v-k)\, \partial_v \left(1-\frac{vk}{c^2}\right)}{\left(1-\frac{vk}{c^2}\right)^2} \tag{42}$$
$$= \frac{\left(1-\frac{vk}{c^2}\right) + (v-k) \left(\frac{k}{c^2}\right)}{\left(1-\frac{vk}{c^2}\right)^2} \tag{43}$$

Setting $v = k$, we get

$$\partial_v \ell_2|_{v=k} = \frac{\left(1-\frac{k^2}{c^2}\right)}{\left(1-\frac{k^2}{c^2}\right)^2} = 1 \tag{44}$$
$$\ell_1|_{v=k} = \frac{u-k}{1-\frac{uk}{c^2}} \tag{45}$$
$$\ell_2|_{v=k} = 0 \tag{46}$$
$$\partial_v \ell|_{v=k} = 1 - \left(\frac{\ell_1}{c}\right)^2 \tag{47}$$
$$\ell|_{v=k} = \ell_1 \tag{48}$$

Finally,

$$\partial_v k \oplus \ell|_{v=k} = \frac{\left(1+\frac{k\ell_1}{c^2}\right)\left(1-\frac{\ell_1^2}{c^2}\right) - \frac{1}{c^2}(k+\ell_1)\, k \left(1-\frac{\ell_1^2}{c^2}\right)}{\left(1+\frac{k\ell_1}{c^2}\right)^2} \tag{49}$$

Now, we calculate

$$\partial_u \left(\partial_v k \oplus \ell\right)|_{u=v=k} = \partial_u \frac{1 - \frac{\ell_1^2}{c^2} + \frac{k\ell_1}{c^2} - \frac{k\ell_1^3}{c^4} + \frac{k}{c^2}\left(k - \frac{k\ell_1^2}{c^2} + \ell_1 - \frac{\ell_1^3}{c^2}\right)}{\left(1 + \frac{k\ell_1}{c^2}\right)^2} \tag{50}$$

$$\begin{aligned} &= \frac{\partial_u \left(1 - \frac{\ell_1^2}{c^2} + \frac{k\ell_1}{c^2} - \frac{k\ell_1^3}{c^4} + \frac{k}{c^2}\left(k - \frac{k\ell_1^2}{c^2} + \ell_1 - \frac{\ell_1^3}{c^2}\right)\right)}{\left(1 + \frac{k\ell_1}{c^2}\right)^2} \\ &\quad - 2\left(1 - \frac{\ell_1^2}{c^2} + \frac{k\ell_1}{c^2} - \frac{k\ell_1^3}{c^4} + \frac{k}{c^2}\left(k - \frac{k\ell_1^2}{c^2} + \ell_1 - \frac{\ell_1^3}{c^2}\right)\right)\left(1 + \frac{k\ell_1}{c^2}\right)^{-3} \partial_u \left(1 + \frac{k\ell_1}{c^2}\right) \end{aligned} \tag{51}$$

Note that

$$\partial_u \ell_1|_{u=k} = 1 \tag{52}$$

$$\ell_1|_{u=k} = 0 \tag{53}$$

We have that

$$\Gamma(k) = \partial_u \partial_v \left(u \oplus_k v\right)|_{u=v=k} \tag{54}$$

$$= \partial_u \left(\frac{\left(c^2 - k^2\right)\left(c^2 - u^2\right)}{\left(c^2 + uv - k\left(u + v\right)\right)^2}\right)|_{u=v=k} \tag{55}$$

$$= \partial_u \left(\frac{\left(c^2 - k^2\right)\left(c^2 - u^2\right)}{\left(c^2 + uk - k\left(u + k\right)\right)^2}\right)|_{u=k} \tag{56}$$

$$= \partial_u \left(\frac{\left(c^2 - k^2\right)\left(c^2 - u^2\right)}{\left(c^2 - k^2\right)^2}\right)|_{u=k} \tag{57}$$

$$= -\frac{2u}{c^2 - k^2}|_{u=k} \tag{58}$$

$$= -\frac{2k}{c^2 - k^2} \tag{59}$$

Then, the torsion is

$$T(k) = 0 \tag{60}$$

The Riemann tensor is zero since

$$(u \oplus_k v) \oplus_k w = k \oplus \left(\left(\ominus k \oplus \left(u \oplus_k v\right)\right) \oplus \left(\ominus k \oplus w\right)\right) \tag{61}$$

$$= \frac{k + \left(\left(\ominus k \oplus \left(u \oplus_k v\right)\right) \oplus \left(\ominus k \oplus w\right)\right)}{1 + \frac{k\left(\left(\ominus k \oplus \left(u \oplus_k v\right)\right) \oplus \left(\ominus k \oplus w\right)\right)}{c^2}} \tag{62}$$

Note

$$\left(\ominus k \oplus (u \oplus_k v)\right) = \frac{-k + (u \oplus_k v)}{1 - \frac{k(u \oplus_k v)}{c^2}} \tag{63}$$

$$u \oplus_k v = \frac{k + \left((\ominus k \oplus u) \oplus (\ominus k \oplus v)\right)}{1 + \frac{k\left((\ominus k \oplus u) \oplus (\ominus k \oplus v)\right)}{c^2}} \tag{64}$$

$$(\ominus k \oplus u) \oplus (k \oplus v) = \frac{(\ominus k \oplus u) + (\ominus k \oplus v)}{1 + \frac{(\ominus k \oplus u)(\ominus k \oplus v)}{c^2}} \tag{65}$$

$$= \frac{\frac{-k+u}{1-\frac{ku}{c^2}} + \frac{-k+v}{1-\frac{kv}{c^2}}}{1 + \frac{\left(\frac{-k+u}{1-\frac{ku}{c^2}}\right)\left(\frac{-k+v}{1-\frac{kv}{c^2}}\right)}{c^2}} \tag{66}$$

$$= \frac{(k+u)\left(1 - \frac{kv}{c^2}\right) + (-k+v)\left(1 - \frac{ku}{c^2}\right)}{\left(1 - \frac{ku}{c^2}\right)\left(1 - \frac{kv}{c^2}\right) + (k+u)(k+v)} \tag{67}$$

Then

$$u \oplus_k v = \frac{k + \frac{(k+u)\left(1-\frac{kv}{c^2}\right)+(-k+v)\left(1-\frac{ku}{c^2}\right)}{\left(1-\frac{ku}{c^2}\right)\left(1-\frac{kv}{c^2}\right)+(k+u)(k+v)}}{1 + \frac{1}{c^2}\frac{(k+u)\left(1-\frac{kv}{c^2}\right)+(-k+v)\left(1-\frac{ku}{c^2}\right)}{\left(1-\frac{ku}{c^2}\right)\left(1-\frac{kv}{c^2}\right)+(k+u)(k+v)}} \tag{68}$$

$$= c^2 \left(\frac{k\left[\left(1 - \frac{ku}{c^2}\right)\left(1 - \frac{kv}{c^2}\right) + (k+u)(k+v)\right] + (k+u)\left(1 - \frac{kv}{c^2}\right) + (-k+v)\left(1 - \frac{ku}{c^2}\right)}{c^2\left[(k+u)\left(1 - \frac{kv}{c^2}\right) + (-k+v)\left(1 - \frac{ku}{c^2}\right)\right] + (k+u)\left(1 - \frac{kv}{c^2}\right) + (-k+v)\left(1 - \frac{ku}{c^2}\right)} \right) \tag{69}$$

$$= \frac{-kuv + c^2(-k+u+v)}{c^2 + uv - k(u+v)} \tag{70}$$

Then

$$\left(\ominus k \oplus (u \oplus_k v)\right) = \frac{-k + \frac{-kuv+c^2(-k+u+v)}{c^2+uv-k(u+v)}}{1 - \frac{k}{c^2}\frac{-kuv+c^2(-k+u+v)}{c^2+uv-k(u+v)}} \tag{71}$$

$$= \frac{c^2\left(c^2(-2k+u+v) + k\left(-2uv + k(u+v)\right)\right)}{c^4 + k^2uv + c^2\left(k^2 + uv - 2k(u+v)\right)} \tag{72}$$

Then

$$\left(\ominus k \oplus \left(u \oplus_k v\right)\right) \oplus \left(\ominus k \oplus w\right) = \frac{\frac{c^2\left(c^2(-2k+u+v)+k\left(-2uv+k(u+v)\right)\right)}{c^4+k^2uv+c^2\left(k^2+uv-2k(u+v)\right)} + \frac{-k+w}{1-\frac{kw}{c^2}}}{1+\frac{c^2(-2k+u+v)+k\left(-2uv+k(u+v)\right)}{c^4+k^2uv+c^2\left(k^2+uv-2k(u+v)\right)}\left(\frac{-k+w}{1-\frac{kw}{c^2}}\right)} \tag{73}$$

$$= \frac{A}{B} \tag{74}$$

where

$$A = c^2\left(c^4\left(-3k+u+v+w\right)+c^2\left(-k^2\right)+uvw+3k^2\left(u+v+w\right)-3k\left(vw+u\left(v+w\right)\right)\right) \\ - k^2\left(-3uvw+k\left(vw+u\left(v+2\right)\right)\right) \tag{75}$$

$$B = c^6 - k^3uvw + c^4\left(3k^2+vw+u\left(v+w\right)-3k\left(u+v+w\right)\right) \\ - c^2k\left(3uvw+k^2\left(u+v+w\right)-3k\left(vw+u\left(v+w\right)\right)\right) \tag{76}$$

And finally, after some serious simplification, we get that

$$\left(u \oplus_k v\right) \oplus_k w = \frac{k^2uvw + c^4\left(-2k+u+v+w\right)+c^2\left(uvw+k^2\left(u+v+w\right)-wk\left(vw+u\left(v+w\right)\right)\right)}{c^4+c^2\left(k^2+vw+u\left(v+w\right)-2k\left(u+v+w\right)+k\left(-2uvw+k\left(vw+u\left(v+w\right)\right)\right)\right)} \tag{77}$$

Now, we consider the following expression

$$u \oplus_k \left(v \oplus_k w\right) = u \oplus_k \left(\frac{-kvw+c^2\left(-k+v+2\right)}{c^2+vw-k\left(v+w\right)}\right) \tag{78}$$

$$= \frac{k^2uvw + c^4\left(-2k+u+v+w\right)+c^2\left(uvw+k^2\left(u+v+w\right)-wk\left(vw+u\left(v+w\right)\right)\right)}{c^4+c^2\left(k^2+vw+u\left(v+w\right)-2k\left(u+v+w\right)+k\left(-2uvw+k\left(vw+u\left(v+w\right)\right)\right)\right)} \tag{79}$$

Hence, we see that

$$\left(u \oplus_k v\right) \oplus_k w = u \oplus_k \left(v \oplus_k w\right) \tag{80}$$

Therefore, the Riemann curvature is

$$R = 0 \tag{81}$$

3.2 Connection and Lorentz Factor

We can finally relate Lorentz Factor to Connection via

$$\gamma\left(v\right)=\sqrt{\frac{-c^{2}\Gamma\left(v\right)}{2v}} \tag{82}$$

Or,

$$\Gamma\left(v\right)=\frac{-2v\gamma^{2}\left(v\right)}{c^{2}} \tag{83}$$

Therefore (previously, $2d=\Delta x$),

$$\begin{aligned}\Delta T' &= \sqrt{\frac{-c^{2}\Gamma\left(v\right)}{2v}}\frac{v\Delta x}{c^{2}} && (84)\\ &= \sqrt{\frac{-\Gamma\left(v\right)}{2v}}\frac{v\Delta x}{c} && (85)\end{aligned}$$

3.3 Parallel Transport

We have that

$$\begin{aligned}u\oplus dv &= \frac{u+dv}{1+\frac{udv}{c^{2}}} && (86)\\ &= \left(u+dv\right)\left(1-\frac{udv}{c^{2}}+\left(\frac{udv}{c^{2}}\right)^{2}+\ldots\right) && (87)\\ &= u-\frac{u^{2}dv}{c^{2}}+dv+\ldots && (88)\\ &= u+\left(1-\frac{u^{2}}{c^{2}}\right)dv && (89)\\ &= u+\tau\left(u\right)dv && (90)\end{aligned}$$

Hence, the parallel transport is

$$\tau\left(u\right)=1-\frac{u^{2}}{c^{2}} \tag{91}$$

Then, we see that

$$\begin{aligned}\tau\left(u\right) &= \frac{1}{\gamma^{2}\left(u\right)} && (92)\\ \Gamma\left(u\right) &= \frac{-2u}{\tau\left(u\right)c^{2}} && (93)\end{aligned}$$

So,

$$\Delta T'=\frac{v\Delta x}{\sqrt{\tau\left(v\right)}c^{2}} \tag{94}$$

4 Relative Locality in κ-Poincaré

For simplicity, we consider $1+1$ dimensional version of the κ-Poincaré algebra. The coalgebra is given by

$$\Delta E = E \otimes \mathbb{1} + \mathbb{1} \otimes E \tag{95}$$
$$\Delta P = P \otimes \mathbb{1} + e^{-E/\kappa} \otimes P \tag{96}$$

The line element of the de Sitter in comoving coordinates is

$$ds^2 = dE^2 - e^{2E/\kappa} dp^2 \tag{97}$$

We can use the coalgebra given earlier to compute

$$p \oplus_k q = k \oplus \big((\ominus k \oplus p) \oplus (\ominus k \oplus q) \big) \tag{98}$$

Note

$$\Delta E \, |p, q\rangle = (p_0 + q_0) \, |p, q\rangle \tag{99}$$

and

$$\Delta E \, |p, \ominus p\rangle = 0 \tag{100}$$
$$\Rightarrow 0 = \big(p_0 + (\ominus p)_0 \big) \, |p, \ominus p\rangle \tag{101}$$
$$\Rightarrow (\ominus p)_0 = -p_0 \tag{102}$$

Hence,

$$(p \ominus_k q)_0 = \Big(k \oplus \big((\ominus k \oplus p) \oplus (\ominus k \oplus q) \big) \Big)_0 \tag{103}$$
$$= k + \big((-k + p) + (-k + q) \big) \tag{104}$$
$$= p + q - k \tag{105}$$

And, we see that

$$\Delta P \, |p, q\rangle = p_1 + e^{-p_0/\kappa} q_1 \tag{106}$$

And

$$\Delta P \, |p, \ominus p\rangle = 0 \tag{107}$$
$$\Rightarrow 0 = p_1 + e^{-p_0/\kappa} \, (\ominus p)_1 \tag{108}$$
$$\Rightarrow (\ominus p)_1 = -p_1 e^{p_0/\kappa} \tag{109}$$

Then,

$$
\begin{aligned}
(p \oplus_k q)_1 &= \Big(k \oplus \big((\ominus k \oplus p) \oplus (\ominus k \oplus q)\big)\Big)_1 &(110)\\
&= k_1 + e^{-k_0/\kappa} \big((\ominus k \oplus p) \oplus (\ominus k \oplus q)\big)_1 &(111)\\
&= k_1 + e^{-k_0/\kappa} \Big((\ominus k \oplus p)_1 + e^{-(-k_0+p_0)/\kappa} (\ominus k \oplus q)_1\Big) &(112)\\
&= k_1 + e^{-k_0/\kappa} \Big[(\ominus k)_1 + e^{k_0/\kappa} p_1 + e^{(k_0-p_0)/\kappa} \Big((\ominus k)_1 + e^{k_0/\kappa} q_1\Big)\Big] &(113)\\
&= k_1 + e^{-k_0/\kappa} \Big[-k_1 e^{k_0/\kappa} + e^{k_0/\kappa} p_1 + e^{(k_0-p_0)/\kappa} \Big(-k_1 e^{k_0/\kappa} + e^{k_0/\kappa} q_1\Big)\Big] &(114)\\
&= k_1 - k_1 + p_1 + e^{-p_0/\kappa} \Big(-k_1 e^{k_0/\kappa} + q_1 e^{k_0/\kappa}\Big) &(115)\\
&= p_1 + e^{(k_0-p_0)/\kappa} (q_1 - k_1) &(116)
\end{aligned}
$$

Now, we can calculate the connection.

$$
\Gamma^{\mu\nu}_{\rho}(k) = -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus_k q)_\rho |_{p=q=k} \tag{117}
$$

Consider

$$
\begin{aligned}
\Gamma^{\mu\nu}_{0}(k) &= -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus_k q)_0 |_{p=q=k} &(118)\\
&= -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p_0 + q_0 - k_0) |_{p-q-k} &(119)\\
&= 0 &(120)
\end{aligned}
$$

And

$$
\begin{aligned}
\Gamma^{\mu\nu}_{1}(k) &= -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus_k q)_1 |_{p=q=k} &(121)\\
&= -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}\Big(p_1 + e^{(k_0-p_0)/\kappa}(q_1 - k_1)\Big) |_{p=q=k} &(122)\\
&= -\left(\frac{\partial}{\partial p_\mu}\Big(e^{(k_0-p_0)/\kappa}\delta^\nu_1\Big)\right) |_{p=q=k} &(123)\\
&= -\left(-\delta^\mu_0 \frac{1}{\kappa}\delta^\nu_1\right) |_{p=q=k} &(124)\\
&= \frac{1}{\kappa}\delta^\mu_0 \delta^\nu_1 &(125)
\end{aligned}
$$

Since, the connection is the same everywhere, we will drop where the connection is defined. Hence we can simply write

$$
\Gamma^{\mu\nu}_{\rho} = \frac{1}{\kappa}\delta^1_\rho \delta^\mu_0 \delta^\nu_1 \tag{126}
$$

The torsion is

$$
\begin{aligned}
T_{\rho}^{\mu\nu} &= -\Gamma_{\rho}^{[\mu\nu]} && (127)\\
&= \frac{1}{\kappa}\left(\delta_{\rho}^{1}\delta_{0}^{\mu}\delta_{1}^{\nu} - \delta_{\rho}^{1}\delta_{0}^{\nu}\delta_{1}^{\mu}\right) && (128)\\
&= \frac{1}{\kappa}\delta_{\rho}^{1}\delta_{0}^{[\mu}\delta_{1}^{\nu]} && (129)
\end{aligned}
$$

Note, we expect Riemann curvature to be zero since the composition rule is associative. Let's calculate it !

$$
R_{\sigma}^{\mu\nu\rho} = \frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{q]}}\frac{\partial}{\partial r_{\rho}}\left((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r)\right)_{\sigma}|_{p=q=r=k} \quad (130)
$$

We see that

$$
\begin{aligned}
\left((p \oplus_k q) \oplus_k r\right)_0 &= (p \oplus_k q)_0 + r - k && (131)\\
&= (p + q - k) + r - k && (132)\\
&= p + (q + r - k) - k && (133)\\
&= p + (q \oplus_k r)_0 - k && (134)\\
&= \left(p \oplus_k (q \oplus_k r)\right)_0 && (135)
\end{aligned}
$$

And,

$$
\begin{aligned}
\left((p \oplus_k q) \oplus_k r\right)_1 &= (p \oplus_k q)_1 + e^{\left(k_0 - (p_0 + q_0 - k_0)\right)/\kappa}(r_1 - k_1) && (136)\\
&= p_1 + e^{(k_0 - p_0)/\kappa}(q_1 - k_1) + e^{(2k_0 - p_0 - q_0)/\kappa}(r_1 - k_1) && (137)\\
&= p_1 + e^{(k_0 - p_0)/\kappa}(q_1 - k_1) + e^{(k_0 - p_0)/\kappa}e^{(k_0 - q_0)/\kappa}(r_1 - k_1) && (138)\\
&= p_1 + e^{(k_0 - p_0)/\kappa}\left(q_1 + e^{(k_0 - q_0)/\kappa}(r_1 - k_1) - k_1\right) && (139)\\
&= p_1 + e^{(k_0 - p_0)/\kappa}\left((q \oplus_k r)_1 - k_1\right) && (140)\\
&= \left(p \oplus_k (q \oplus_k r)\right)_1 && (141)
\end{aligned}
$$

Therefore, we see that

$$
(p_{\oplus k} q) \oplus_k r = p \oplus_k (q \oplus_k r) \quad (142)
$$

And we get that Riemann curvature is zero as expected.

$$
\begin{aligned}
R_{\sigma}^{\mu\nu\rho} &= \frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{q]}}\frac{\partial}{\partial r_{\rho}}\left((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r)\right)_{\sigma}|_{p=q=r=k} && (143)\\
&= R_{\sigma}^{\mu\nu\rho} && (144)\\
&= \frac{\partial}{\partial p_{[\mu}}\frac{\partial}{\partial q_{q]}}\frac{\partial}{\partial r_{\rho}}(0)_{\sigma}|_{p=q=r=k} && (145)\\
&= 0 && (146)
\end{aligned}
$$

Next, we calculate non-metricity.

$$\nabla^{\rho} g^{\mu\nu} = \partial^{\rho} g^{\mu\nu} + \Gamma^{\mu\rho}_{\sigma} g^{\sigma\nu} + \Gamma^{\nu\rho}_{\sigma} g^{\mu\sigma} \tag{147}$$

$$= \partial^{\rho}\left(\delta^{\mu}_{0}\delta^{\nu}_{0} - \delta^{\mu}_{1}\delta^{\nu}_{1} e^{2E/\kappa}\right) + \left(\frac{1}{\kappa}\delta^{1}_{\sigma}\delta^{\mu}_{0}\delta^{\rho}_{1}\right)\left(\delta^{\sigma}_{0}\delta^{\nu}_{0} - \delta^{\sigma}_{1}\delta^{\nu}_{1} e^{2E/\kappa}\right) + \left(\frac{1}{\kappa}\delta^{1}_{\sigma}\delta^{\nu}_{0}\delta^{\rho}_{1}\right)\left(\delta^{\mu}_{0}\delta^{\sigma}_{0} - \delta^{\mu}_{1}\delta^{\sigma}_{1} e^{2E/\kappa}\right) \tag{148}$$

$$= -\delta^{\mu}_{1}\delta^{\nu}_{1}\delta^{\rho}_{0}\frac{2}{\kappa}e^{2E/\kappa} - \frac{1}{\kappa}\delta^{\mu}_{0}\delta^{\rho}_{1}\delta^{\nu}_{1}e^{2E/\kappa} - \frac{1}{\kappa}\delta^{\nu}_{0}\delta^{\rho}_{1}\delta^{\mu}_{1}e^{2E/\kappa} \tag{149}$$

$$= -\left(2\delta^{\mu}_{1}\delta^{\nu}_{1}\delta^{\rho}_{0} + \delta^{\mu}_{0}\delta^{\rho}_{1}\delta^{\nu}_{1} + \delta^{\nu}_{0}\delta^{\rho}_{1}\delta^{\mu}_{1}\right) e^{2E/\kappa} \tag{150}$$

Now, we use the connection calculated above to compute the parallel transport. We se that

$$(p \oplus_k q)_{\mu} = (p_0 + q_0 - k_0)\,\delta^{0}_{\mu} + \left(p_1 + e^{-p_0/\kappa}(q_1 - k_1)\right)\delta^{1}_{\mu} \tag{151}$$

Then

$$(p \oplus dq)_{\mu} = (p_0 + dq_0)\,\delta^{0}_{\mu} + \left(p_1 + e^{(k_0 - p_0)/\kappa} dq_1\right)\delta^{1}_{\mu} \tag{152}$$

$$= p_{\mu} + dq_0\delta^{0}_{\mu} + e^{(k_0-p_0)/\kappa} dq_1 \delta^{1}_{\mu} \tag{153}$$

$$= p_{\mu} + dq_{\nu}\left(\delta^{\nu}_{0}\delta^{0}_{\mu} + e^{-p_0/\kappa}\delta^{1}_{\mu}\delta^{\nu}_{1}\right) \tag{154}$$

So, the parallel transport is

$$\tau^{\mu}_{\nu}(p) = \delta^{\nu}_{0}\delta^{0}_{\mu} + e^{-p_0/\kappa}\delta^{1}_{\mu}\delta^{\nu}_{1} \tag{155}$$

Now, we obtain the coordinates z^{μ}_{j} from the coordinates of the j-th particle with momentum p^{j} by parallel transporting them along a geodesic toward the orgisin of the momentum space.

$$z^{\mu}_{j} = x^{\nu}_{j}\tau^{\mu}_{\nu}\left(p^{j}\right) \tag{156}$$

Then, we get that

$$z^{0} = x^{0} \tag{157}$$

$$z^{1} = x^{1} e^{-p_0/\kappa} \tag{158}$$

Now, let's calculate the Poisson brackets

$$\left\{x^{\mu}_{j}, p^{k}_{\nu}\right\} = \sum_{i}\left(\frac{\partial x^{\mu}_{j}}{\partial x^{i}_{j}}\frac{\partial p^{k}_{\nu}}{\partial p^{j}_{i}} - \frac{\partial x^{\mu}_{j}}{\partial p^{\mu}_{j}}\frac{\partial p^{k}_{\nu}}{\partial x^{i}_{j}}\right) \tag{159}$$

$$= \sum_{i}\left(\delta^{\mu}_{i}\delta^{i}_{\nu}\delta^{k}_{j}\right) \tag{160}$$

$$= \delta^{\nu}_{\mu}\delta^{k}_{j} \tag{161}$$

And

$$\left\{z_j^1, z_k^0\right\} = \sum_i \left(\frac{\partial z_j^1}{\partial x_j^i} \frac{\partial z_k^0}{\partial p_i^j} - \frac{\partial z_j^1}{\partial p_i^j} \frac{\partial z_k^0}{\partial x_j^i} \right) \tag{162}$$

$$= - \sum_i \left(-\frac{1}{\kappa} e^{-p_0/\kappa} x_j^1 \delta_0^i \delta_{jk} \delta_i^0 \right) \tag{163}$$

$$= \frac{1}{\kappa} e^{-p_0/\kappa} x_j^1 \delta_{jk} \tag{164}$$

$$= \frac{1}{\kappa} z_j^1 \delta_{jk} \tag{165}$$

5 SU(2)

Addition of momentum is given by

$$(p \oplus q)_i = q_i \sqrt{1 - (\ell_p p)^2} + p_i \sqrt{1 - (\ell_p q)^2} + \ell_p \epsilon_i^{jk} p_j q_k \tag{166}$$

, where $p^2 = p_i p^i$. For the sake of simplicity, we set $\ell_p = 1$ from now on.

$$\ominus k \oplus k = 0 \tag{167}$$

$$\Rightarrow 0 = k_i \sqrt{1 - (\ominus k)^2} + (\ominus k)_i \sqrt{1 - k^2} + \epsilon_i^{jk} (\ominus k)_j k_k \tag{168}$$

For $k = -k$, we see that this satisfies

$$k_i \sqrt{1 - (-k)^2} + (-k)_i \sqrt{1 - k^2} + \epsilon_i^{jk} (-k)_j k_k = 0 \tag{169}$$

, as required. As the inverse is unique, $\ominus k = -k$.

$$(p \oplus_k q)_i = \big(k \oplus (\ominus k \oplus p) \oplus (\ominus k \oplus q)\big)_i \tag{170}$$

$$= \ell_i \sqrt{1 - k^2} + k_i \sqrt{1 - \ell^2} + \epsilon_i^{jk} k_j \ell_k \tag{171}$$

where

$$\ell_i = \big((-k \oplus p) \oplus (-k \oplus q)\big)_i \tag{172}$$

$$= (-k \oplus q)_i \sqrt{1 - (-k \oplus p)^2} + (-k \oplus p)_i \sqrt{1 - (-k \oplus q)^2} + \epsilon_i^{jk} (-k \oplus p)_j (-k \oplus q)_k \tag{173}$$

$$(-k \oplus p)_i = p_i \sqrt{1 - k^2} - k_i \sqrt{1 - p^2} - \epsilon_i^{jk} k_j p_k \tag{174}$$

$$(-k \oplus q)_i = q_i \sqrt{1 - k^2} - k_i \sqrt{1 - q^2} - \epsilon_i^{jk} k_j q_k \tag{175}$$

Connection is given by

$$\Gamma_i^{\mu\nu}(k) = \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q)_i \,|_{p=q=k} \tag{176}$$

For the sake of simplicity, let's consider connection at the origin, then

$$\Gamma_i^{\mu\nu}(0) = \frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)_i\,|_{p=q=0} \tag{177}$$

Note that since we are dealing with a flat metric ($g^{\mu\nu} = \delta^{\mu\nu}$),

$$\frac{\partial}{\partial q_\nu}q^2 = \frac{\partial}{\partial q_\nu}q_j q^j \tag{178}$$

$$= \frac{\partial}{\partial q_\nu}q_j\delta^{jk}q_k \tag{179}$$

$$= 2\delta^{jk}q_j\delta_k^\nu \tag{180}$$

$$= 2\delta^{jk}q_j\delta_k^\nu \tag{181}$$

$$= 2q^\nu \tag{182}$$

We see that

$$\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)_i = \frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}\left(q_i\sqrt{1-p^2} + p_i\sqrt{1-q^2} + \ell_p\epsilon_i^{jk}p_j q_k\right) \tag{183}$$

$$= \frac{\partial}{\partial p_\mu}\left(\delta_i^\mu\sqrt{1-p^2} - p_i\frac{q^\nu}{\sqrt{1-q^2}} + \epsilon_i^{jk}p_j\delta_k^\nu\right) \tag{184}$$

$$= -\delta_i^\mu\frac{p^\mu}{\sqrt{1-p^2}} - \delta_i^\mu\frac{q^\nu}{\sqrt{1-q^2}} + \epsilon_i^{\mu\nu} \tag{185}$$

Then,

$$\Gamma_i^{\mu\nu}(0) = -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus q)_i\,|_{p=q=0} = -\epsilon_i^{\mu\nu} \tag{186}$$

Now, let's consider

$$\Gamma_i^{\mu\nu}(k) \tag{187}$$

Then, we need to calculate

$$-\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}(p \oplus_k q) \tag{188}$$

$$= -\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial q_\nu}\left(\ell_i\sqrt{1-k^2} + k_i\sqrt{1-\ell^2} + \epsilon_i^{jk}k_j\ell_k\right) \tag{189}$$

$$= -\frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell_i}{\partial q_\nu}\right)\sqrt{1-k^2} + \frac{k_i}{2\sqrt{1-\ell^2}}\frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell^2}{\partial q_\nu}\right) - \frac{k_i}{4\,(1-\ell)^{3/2}}\frac{\partial \ell^2}{\partial p_\mu}\frac{\partial \ell^2}{\partial q_\nu} - \epsilon_i^{jk}k_j\frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell_k}{\partial q_\nu}\right) \tag{190}$$

$$= -A_i^{\mu\nu}\sqrt{1-k^2} + \frac{k_i}{2\sqrt{1-\ell^2}}B^{\mu\nu} - \epsilon_i^{jk}k_jC_k^{\mu\nu} - \frac{k_i}{4\,(1-\ell)^{3/2}}D^{\mu\nu} \tag{191}$$

where

$$A_i^{\mu\nu} = \frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell_i}{\partial q_\nu}\right) \tag{192}$$

$$B^{\mu\nu} = \frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell^2}{\partial q_\nu}\right) \tag{193}$$

$$C_k^{\mu\nu} = \frac{\partial}{\partial p_\mu}\left(\frac{\partial \ell_k}{\partial q_\nu}\right) \tag{194}$$

$$D^{\mu\nu} = \frac{\partial \ell^2}{\partial p_\mu}\frac{\partial \ell^2}{\partial q_\nu} \tag{195}$$

Now, let's compute $A_i^{\mu\nu}$.

$$\frac{\partial \ell_i}{\partial q_\nu} = \frac{\partial}{\partial q_\nu}\left(\left(-k\oplus q\right)_i\sqrt{1-\left(-k\oplus p\right)^2} + \left(-k\oplus p\right)_i\sqrt{1-\left(-k\oplus q\right)^2} + \epsilon_i^{jk}\left(-k\oplus p\right)_j\left(-k\oplus q\right)_k\right) \tag{196}$$

$$= \frac{\partial\left(-k\oplus q\right)_i}{\partial q_\nu}\sqrt{1-\left(-k\oplus p\right)^2} - \frac{\left(-k\oplus p\right)_i}{2\sqrt{1-\left(-k\oplus q\right)^2}}\frac{\partial\left(-k\oplus q\right)^2}{\partial q_\nu} + \epsilon_i^{jk}\left(-k\oplus p\right)_j\frac{\partial\left(-k\oplus q\right)_k}{\partial q_\nu} \tag{197}$$

Then (from now on we let $\frac{\partial}{\partial p_\mu} = \partial_{p_\mu}$),

$$A_i^{\mu\nu} = \frac{\partial}{\partial p_\mu}\frac{\partial \ell_i}{\partial q_\nu} \tag{198}$$

$$= -\partial_{q_\nu}\left(-k\oplus q\right)_i\frac{\partial_{p_\mu}\left(-k\oplus p\right)^2}{2\sqrt{1-\left(-k\oplus p\right)^2}} - \partial_{p_\mu}\left(-k\oplus p\right)_i\frac{\partial_{q_\nu}\left(-k\oplus q\right)^2}{2\sqrt{1-\left(-k\oplus q\right)^2}} + \epsilon_i^{jk}\partial_{p_\mu}\left(-k\oplus p\right)_j\partial_{q_\nu}\left(-k\oplus q\right)_k \tag{199}$$

where

$$\partial_{q_\nu}\left(-k\oplus q\right)_i = \partial_{q_\nu}\left(q_i\sqrt{1-k^2} - k_i\sqrt{1-q^2} - \epsilon_i^{jk}k_jq_k\right) \tag{200}$$

$$= \delta_i^\nu\sqrt{1-k^2} + k_i\frac{q^\nu}{\sqrt{1-q^2}} - \epsilon_i^{jk}k_j\delta_k^\nu \tag{201}$$

$$= \delta_i^\nu\sqrt{1-k^2} + k_i\frac{q^\nu}{\sqrt{1-q^2}} - \epsilon_i^{j\nu}k_j \tag{202}$$

$$\partial_{p_\mu}\left(-k\oplus p\right)_i = \delta_i^\mu\sqrt{1-k^2} + k_i\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_i^{j\mu}k_j \tag{203}$$

$$\partial_{p_\mu}\left(-k\oplus p\right)^2 = 2\left(-k\oplus p\right)^\sigma\partial_{p_\mu}\left(-k\oplus p\right)_\sigma \tag{204}$$

$$= 2\left(-k\oplus p\right)^\sigma\left(\delta_\sigma^\mu\sqrt{1-k^2} + k_\sigma\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_\sigma^{j\mu}k_j\right) \tag{205}$$

$$= 2\left(p^\sigma\sqrt{1-k^2} - k^\sigma\sqrt{1-p^2} - \epsilon^{\sigma jk}k_jp_k\right)\left(\delta_\sigma^\mu\sqrt{1-k^2} + k_\sigma\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_i^{j\mu}k_j\right) \tag{206}$$

Then

$$\partial_{p_\mu}\left(-k\oplus p\right)^2|_{p=k} = 2\left(p^\sigma\sqrt{1-p^2} - p^\sigma\sqrt{1-p^2} - \epsilon^{\sigma jk}p_jp_k\right)\left(\delta^\mu_\sigma\sqrt{1-p^2} + p_\sigma\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_i^{j\mu}p_j\right) \tag{207}$$

$$= -2\left(\epsilon^{\sigma jk}p_jp_k\right)\left(\delta^\mu_\sigma\sqrt{1-p^2} + k_\sigma\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_i^{j\mu}p_j\right) \tag{208}$$

$$= 0 \tag{209}$$

Similarly,

$$\partial_{q_\nu}\left(-k\oplus q\right)^2|_{q=k} = 0 \tag{210}$$

Therefore,

$$A_i^{\mu\nu}|_{p=q=k} = \epsilon_i^{jk}\left(\delta_j^\mu\sqrt{1-p^2} + p_j\frac{p^\mu}{\sqrt{1-p^2}} - \epsilon_j^{\rho\mu}p_\rho\right)\left(\delta_k^\nu\sqrt{1-p^2} + p_k\frac{p^\nu}{\sqrt{1-p^2}} - \epsilon_k^{\sigma\nu}p_\sigma\right) \tag{211}$$

$$= \epsilon_i^{jk}\Bigg[\delta_j^\mu\delta_k^\nu\left(1-p^2\right) + \frac{p_jp_kp^\mu p^\nu}{1-p^2} + \epsilon_j^{\rho\mu}\epsilon_k^{\sigma\nu}p_\rho p_\sigma + \delta_j^\mu p_kp^\nu - \epsilon_k^{\sigma\nu}\delta_j^\mu p_\sigma\sqrt{1-p^2}$$
$$+\delta_k^\nu p_jp^\mu - \frac{\epsilon_k^{\sigma\nu}p_\sigma p_jp^\mu}{\sqrt{1-p^2}} - \epsilon_j^{\rho\mu}p_\rho\delta_k^\nu\sqrt{1-p^2} - \frac{\epsilon_j^{\rho\mu}p_\rho p_kp^\nu}{\sqrt{1-p^2}}\Bigg] \tag{212}$$

$$= \epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{jk}\epsilon_j^{\rho\mu}\epsilon_k^{\sigma\nu}p_\rho p_\sigma + \epsilon_i^{\mu k}p_kp^\nu - \epsilon_i^{\mu k}\epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2} + \epsilon_i^{j\nu}p_jp^\mu - \frac{\epsilon_i^{jk}\epsilon_k^{\sigma\nu}p_\sigma p_jp^\mu}{\sqrt{1-p^2}}$$
$$- \epsilon_i^{j\nu}\epsilon_j^{\rho\mu}p_\rho\sqrt{1-p^2} - \frac{\epsilon_i^{jk}\epsilon_j^{\rho\mu}p_\rho p_kp^\nu}{\sqrt{1-p^2}} \tag{213}$$

$$= \epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{\mu k}\left(p_kp^\nu - \epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_i^{j\nu}p_jp^\mu + \epsilon_i^{jk}\epsilon_j^{\rho\mu}\left(\epsilon_k^{\sigma\nu}p_\rho p_\sigma - \frac{p_\rho p_kp^\nu}{\sqrt{1-p^2}}\right)$$
$$- \frac{\epsilon_i^{jk}\epsilon_k^{\sigma\nu}p_\sigma p_jp^\mu}{\sqrt{1-p^2}} - \epsilon_i^{j\nu}\epsilon_j^{\rho\mu}p_\rho\sqrt{1-p^2} \tag{214}$$

$$= \epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{\mu k}\left(p_kp^\nu - \epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_i^{j\nu}p_jp^\mu + \left(\delta_i^\mu\delta^{k\rho} - \delta_i^\rho\delta^{k\mu}\right)\left(\epsilon_k^{\sigma\nu}p_\rho p_\sigma - \frac{p_\rho p_kp^\nu}{\sqrt{1-p^2}}\right)$$
$$- \left(\delta_i^\sigma\delta^{j\nu} - \delta_i^\nu\delta^{j\sigma}\right)\frac{p_\sigma p_jp^\mu}{\sqrt{1-p^2}} - \left(\delta_i^\mu\delta^{\nu\rho} - \delta_i^\rho\delta^{\nu\mu}\right)p_\rho\sqrt{1-p^2} \tag{215}$$

$$= \epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{\mu k}\left(p_kp^\nu - \epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_i^{j\nu}p_jp^\mu + \left(p_ip^\mu - \delta_i^\mu p^2\right)\frac{p^\nu}{\sqrt{1-p^2}} - \epsilon^{\mu\sigma\nu}p_\sigma p_i$$
$$+ \left(\delta_i^\nu p^2 - p_ip^\nu\right)\frac{p^\mu}{\sqrt{1-p^2}} + \left(p_i\delta^{\nu\mu} - \delta_i^\mu p^\nu\right)\sqrt{1-p^2} \tag{216}$$

$$= \epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{\mu k}\left(p_kp^\nu - \epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_i^{j\nu}p_jp^\mu - \epsilon^{\mu\sigma\nu}p_\sigma p_i$$
$$+ \left(\delta_i^\nu p^\mu - \delta_i^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} + \left(p_i\delta^{\nu\mu} - \delta_i^\mu p^\nu\right)\sqrt{1-p^2} \tag{217}$$

$$\tag{218}$$

Now, then

$$B^{\mu\nu} = 2\left(\partial_{p_\mu}\ell^\omega \partial_{q_\nu}\ell_\omega + \ell^\lambda \partial_{p_\mu}\partial_{q_\nu}\ell_\lambda\right) \tag{219}$$

We see that

$$\partial_{q_\nu}\ell_\omega|_{p=q=k} = \frac{\partial\left(-k\oplus q\right)_\omega}{\partial q_\nu}\sqrt{1-\left(-k\oplus p\right)^2} - \frac{\left(-k\oplus p\right)_\omega}{2\sqrt{1-\left(-k\oplus q\right)^2}}\cancelto{0}{\frac{\partial\left(-k\oplus q\right)^2}{\partial q_\nu}} + \epsilon^{jk}_{\omega}\cancelto{0}{\left(-k\oplus p\right)_j}\frac{\partial\left(-k\oplus q\right)_k}{\partial q_\nu} \tag{220}$$

$$= \left(\delta^\nu_\omega\sqrt{1-p^2} + p_\omega\frac{p^\nu}{\sqrt{1-p^2}} - \epsilon^{j\nu}_{\omega}p_j\right)\sqrt{1-\left(-k\oplus p\right)^2} \tag{221}$$

Note

$$\left(-k\oplus p\right)^2|_{p=q=k} = 0 \tag{222}$$

Then

$$\partial_{q_\nu}\ell_\omega|_{p=q=k} = \delta^\nu_\omega\sqrt{1-p^2} + p_\omega\frac{p^\nu}{\sqrt{1-p^2}} - \epsilon^{j\nu}_{\omega}p_j \tag{223}$$

Similarly,

$$\partial_{p_\mu}\ell^\omega|_{p=q=k} = \partial_{p_\mu}\left(-k\oplus p\right)^\omega \tag{224}$$

$$= \delta^{\omega\mu}\sqrt{1-p^2} + \frac{p^\omega p^\mu}{\sqrt{1-p^2}} - \epsilon^{\omega j\mu}p_j \tag{225}$$

Then

$$\partial_{p_\mu}\ell^\omega\partial_{q_\nu}\ell_\omega|_{p=q=k} = \left(\delta^\nu_\omega\sqrt{1-p^2} + p_\omega\frac{p^\nu}{\sqrt{1-p^2}} - \epsilon^{j\nu}_{\omega}p_j\right)\left(\delta^{\omega\mu}\sqrt{1-p^2} + \frac{p^\omega p^\mu}{\sqrt{1-p^2}} - \epsilon^{\omega\ell\mu}p_\ell\right) \tag{226}$$

$$= \delta^{\mu\nu}\left(1-p^2\right) + \frac{p^\mu p^\nu p^2}{\sqrt{1-p^2}} + \epsilon^{j\nu}_{\omega}\epsilon^{\omega\ell\mu}p_jp_\ell + 2p^\mu p^\nu - \epsilon^{\nu\ell\mu}p_\ell\sqrt{1-p^2} - \epsilon^{\mu j\nu}p_j\sqrt{1-p^2} \tag{227}$$

$$= \delta^{\mu\nu}\left(1-p^2\right) + \frac{p^\mu p^\nu p^2}{\sqrt{1-p^2}} + \epsilon^{j\nu}_{\omega}\epsilon^{\omega\ell\mu}p_jp_\ell + 2p^\mu p^\nu - \epsilon^{\nu\ell\mu}p_\ell\sqrt{1-p^2} + \epsilon^{\nu j\mu}p_j\sqrt{1-p^2} \tag{228}$$

$$= \delta^{\mu\nu}\left(1-p^2\right) + \frac{p^\mu p^\nu p^2}{\sqrt{1-p^2}} + \epsilon^{j\nu}_{\omega}\epsilon^{\omega\ell\mu}p_jp_\ell + 2p^\mu p^\nu \tag{229}$$

$$= \delta^{\mu\nu}\left(1-p^2\right) + \frac{p^\mu p^\nu p^2}{\sqrt{1-p^2}} + \left(\delta^{j\ell}\delta^{\nu\mu} - \delta^{j\mu}\delta^{\nu\ell}\right)p_jp_\ell + 2p^\mu p^\nu \tag{230}$$

$$= \delta^{\mu\nu}\left(1-p^2\right) + \frac{p^\mu p^\nu p^2}{\sqrt{1-p^2}} + \delta^{\nu\mu}p^2 - p_\mu p_\nu + 2p^\mu p^\nu \tag{231}$$

$$= \delta^{\mu\nu} + p^\mu p^\nu\left(1 + \frac{p^2}{\sqrt{1-p^2}}\right) \tag{232}$$

Note that

$$\ell^{\lambda}|_{p=q=k} = 0 \tag{233}$$

Hence,

$$B^{\mu\nu}|_{p=q=k} = 2\left[\delta^{\mu\nu} + p^{\mu}p^{\nu}\left(1 + \frac{p^2}{\sqrt{1-p^2}}\right)\right] \tag{234}$$

Finally,

$$\begin{aligned} C_k^{\mu\nu}|_{p=q=k} &= \epsilon_k^{\mu\nu}\left(1-p^2\right) + \epsilon_k^{\mu\omega}\left(p_\omega p^\nu - \epsilon_\omega^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_k^{j\nu}p_j p^\mu - \epsilon^{\mu\sigma\nu}p_\sigma p_k \\ &\quad + \left(\delta_k^\nu p^\mu - \delta_k^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} + \left(p_k\delta^{\nu\mu} - \delta_k^\mu p^\nu\right)\sqrt{1-p^2} \end{aligned} \tag{235}$$

$$D^{\mu\nu}|_{p=q=k} = 0 \tag{236}$$

Combining all the terms, in the end we get that (note $\ell^2|_{p=q=k} = 0$)

$$\begin{aligned} \Gamma_i^{\mu\nu}(k) = &-\left[\epsilon_i^{\mu\nu}\left(1-p^2\right) + \epsilon_i^{\mu k}\left(p_k p^\nu - \epsilon_k^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_i^{j\nu}p_j p^\mu - \epsilon^{\mu\sigma\nu}p_\sigma p_i\right. \\ &\quad \left. + \left(\delta_i^\nu p^\mu - \delta_i^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} + \left(p_i\delta^{\nu\mu} - \delta_i^\mu p^\nu\right)\sqrt{1-p^2}\right]\sqrt{1-p^2} \\ &+ \left[\delta^{\mu\nu} + p^\mu p^\nu\left(1 + \frac{p^2}{\sqrt{1-p^2}}\right)\right]p_i \\ &- \epsilon_i^{jk}p_j\left[\epsilon_k^{\mu\nu}\left(1-p^2\right) + \epsilon_k^{\mu\omega}\left(p_\omega p^\nu - \epsilon_\omega^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_k^{\lambda\nu}p_\lambda p^\mu - \epsilon^{\mu\sigma\nu}p_\sigma p_k\right. \\ &\quad \left. + \left(\delta_k^\nu p^\mu - \delta_k^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} + \left(p_k\delta^{\nu\mu} - \delta_k^\mu p^\nu\right)\sqrt{1-p^2}\right] \end{aligned} \tag{237}$$

Consider the third term

$$\begin{aligned} &\epsilon_i^{jk}p_j\left[\epsilon_k^{\mu\nu}\left(1-p^2\right) + \epsilon_k^{\mu\omega}\left(p_\omega p^\nu - \epsilon_\omega^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right) + \epsilon_k^{\lambda\nu}p_\lambda p^\mu - \epsilon^{\mu\sigma\nu}p_\sigma p_k\right. \\ &\quad \left. + \left(\delta_k^\nu p^\mu - \delta_k^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} + \left(p_k\delta^{\nu\mu} - \delta_k^\mu p^\nu\right)\sqrt{1-p^2}\right] \end{aligned} \tag{238}$$

$$\begin{aligned} &= \left(\delta_i^\mu\delta^{j\nu} - \delta_i^\nu\delta^{j\mu}\right)p_j\left(1-p^2\right) + \left(\delta_i^\mu\delta^{j\omega} - \delta_i^\omega\delta^{j\mu}\right)\left(p_\omega p^\nu - \epsilon_\omega^{\sigma\nu}p_\sigma\sqrt{1-p^2}\right)p_j \\ &\quad + \left(\delta_i^\lambda\delta^{j\nu} - \delta_i^\nu\delta^{j\lambda}\right)p_j p_\lambda p^\mu \cancelto{0}{- \epsilon_i^{jk}\epsilon^{\mu\rho\nu}p_\sigma p_j p_k} + \epsilon_i^{jk}p_j\left(\delta_k^\nu p^\mu - \delta_k^\mu p^\nu\right)\frac{p^2}{\sqrt{1-p^2}} \\ &\quad + \epsilon_i^{jk}p_j\left(p_k\delta^{\nu\mu} - \delta_k^\mu p^\nu\right)\sqrt{1-p^2} \end{aligned} \tag{239}$$

$$\begin{aligned} &= \left(\delta_i^\mu p^\nu - \delta_i^\nu p^\mu\right)\left(1-p^2\right) + p^\nu\left(\delta_i^\mu p^2 - p_i p^\mu\right) + \epsilon_i^{\sigma\nu}p_\sigma p^\mu\sqrt{1-p^2} \\ &\quad + p^\mu\left(p_i p^\nu - \delta_i^\nu p^2\right) + \left(\epsilon_i^{j\nu}p^\mu - \epsilon_i^{j\mu}p^\nu\right)\frac{p_j p^2}{\sqrt{1-p^2}} - \epsilon_i^{j\mu}p_j p^\nu\sqrt{1-p^2} \end{aligned} \tag{240}$$

Now, we collect like terms in $\Gamma_i^{\mu\nu}(k)$ and let's consider each of these terms with following coefficients:

$$\left(1-p^2\right)^{3/2} : -\epsilon_i^{\mu\nu} \tag{241}$$

$$\begin{aligned}\left(1-p^2\right) :\ & \epsilon_i^{\mu k}\epsilon_k^{\sigma\nu}p_\sigma - \left(p_i\delta^{\nu\mu} - \delta_i^\mu p^\nu\right) - \left(\delta_i^\mu p^\nu - \delta_i^\nu p^\mu\right) && (242)\\ &= \left(\delta_i^\sigma\delta^{\mu\nu} - \delta_i^\nu\delta^{\mu\sigma}\right)p_\sigma - p_i\delta^{\nu\mu} + \delta_i^\nu p^\mu && (243)\\ &= 0 && (244)\end{aligned}$$

$$\begin{aligned}\sqrt{1-p^2} :\ & -\epsilon_i^{\mu k}p_k p^\nu - \epsilon_i^{j\nu}p_j p^\mu + \epsilon^{\mu\sigma\nu}p_\sigma p_i - \epsilon_i^{\sigma\nu}p_\sigma p^\mu + \epsilon_i^{j\mu}p_j p^\nu && (245)\\ &= 2p_\sigma\left(p^\nu\epsilon_i^{\sigma\mu} - p^\mu\epsilon_i^{\sigma\nu}\right) + p_i\epsilon^{\mu\sigma\nu}p_\sigma && (246)\end{aligned}$$

$$\begin{aligned}\frac{p^2}{\sqrt{1-p^2}} :\ & \epsilon_i^{j\nu}p_j p^\mu - \epsilon_i^{j\mu}p_j p^\nu + p_i p^\mu p^\nu && (247)\\ &= p_\sigma\left(p^\mu\epsilon_i^{\sigma\nu} - p^\nu\epsilon_i^{\sigma\mu}\right) + p_i p^\mu p^\nu && (248)\end{aligned}$$

$$\begin{aligned}p^2 :\ & \delta_i^\nu p^\mu - \delta_i^\mu p^\nu - p^\nu\delta_i^\mu + p^\mu\delta_i^\nu && (249)\\ &= 2\left(p^\mu\delta_i^\nu - p^\nu\delta_i^\mu\right) && (250)\end{aligned}$$

$$\begin{aligned}1 :\ & p_i p^\mu p^\nu - p_i p^\mu p^\nu + p_i\delta^{\mu\nu} + p_i p^\mu p^\nu && (251)\\ &= p_i\delta^{\mu\nu} + p_i p^\mu p^\nu && (252)\end{aligned}$$

Hence, the connection is

$$\begin{aligned}\Gamma_i{}^{\mu\nu}(k) = &-\left(1-p^2\right)^{3/2}\epsilon_i{}^{\mu\nu} + \sqrt{1-p^2}\left[2p_\sigma\left(p^\nu\epsilon_i{}^{\sigma\mu} - p^\mu\epsilon_i{}^{\sigma\nu}\right) + p_i\epsilon^{\mu\sigma\nu}p_\sigma\right]\\ &+ 2p^2\left(p^\mu\delta_i^\nu - p^\nu\delta_i^\mu\right) + \frac{p^2}{\sqrt{1-p^2}}\left[p_\sigma\left(p^\mu\epsilon_i{}^{\sigma\nu} - p^\nu\epsilon_i{}^{\sigma\mu}\right) + p_i p^\mu p^\nu\right]\\ &+ p_i\delta^{\mu\nu} + p_i p^\mu p^\nu && (253)\\ = &-\left(1-p^2\right)^{3/2}\epsilon_i{}^{\mu\nu} + \sqrt{1-p^2}\left(2\epsilon_i{}^{\sigma[\mu}p^{\nu]}p_\sigma + \epsilon^{\mu\sigma\nu}p_i p_\sigma\right) + 2p^2 p^{[\mu}\delta_i^{\nu]}\\ &+ \frac{p^2}{\sqrt{1-p^2}}\left(-\epsilon_i{}^{\sigma[\mu}p^{\nu]}p_\sigma + p_i p^\mu p^\nu\right) + p_i\delta^{\mu\nu} + p_i p^\mu p^\nu && (254)\end{aligned}$$

Or more concisely,

$$\begin{aligned}\Gamma_i{}^{\mu\nu}(k) = &-\left(1-p^2\right)^{3/2}\epsilon_i{}^{\mu\nu} + \sqrt{1-p^2}\left(2\Omega_i{}^{[\mu\nu]} + \Omega^{\mu\nu}{}_i\right) + 2p^2 p^{[\mu}\delta_i^{\nu]}\\ &+ \frac{p^2}{\sqrt{1-p^2}}\left(-\Omega_i{}^{[\mu\nu]} + p_i p^\mu p^\nu\right) + p_i\delta^{\mu\nu} + p_i p^\mu p^\nu && (255)\end{aligned}$$

where

$$\Omega_i{}^{\mu\nu} = \epsilon_i{}^{\sigma\mu} p^\nu p_\sigma \tag{256}$$

Now, we see that if we set $k = 0$, we recover

$$\Gamma_i{}^{\mu\nu} \left(k = 0\right) = -\epsilon_i{}^{\mu\nu} \tag{257}$$

as desired.
The torsion is then

$$T_i{}^{\mu\nu} = -2\left(1-p^2\right)^{3/2} \epsilon_i{}^{\mu\nu} + \sqrt{1-p^2}\left(4\Omega_i{}^{[\mu\nu]} + \Omega^{[\mu\nu]}{}_i\right) + 4p^2 p^{[\mu}\,\delta_i^{\nu]} + \frac{2\Omega_i{}^{[\mu\nu]}p^2}{\sqrt{1-p^2}} \tag{258}$$

The non-metricity is

$$\begin{aligned}
\nabla^\rho g^{\mu\nu} &= \nabla^\rho \delta^{\mu\nu} &(259)\\
&= \partial^\rho \delta^{\mu\nu} + \Gamma_\sigma{}^{\mu\rho}\delta^{\sigma\nu} + \Gamma_\sigma{}^{\nu\rho}\delta^{\mu\sigma} &(260)\\
&= \Gamma^{\nu\mu\rho} + \Gamma^{\mu\nu\rho} &(261)\\
&= \Gamma^{(\mu\nu)\rho} &(262)\\
&= \sqrt{1-p^2}\left(-2p^{(\mu}\,\epsilon^{\nu)\sigma\rho} + p^{(\mu}\,\epsilon^{\nu)\sigma\rho}p_\sigma\right) + 2p^2\left(p^{(\mu}\,\delta^{\nu)\rho} - 2p^\rho\delta^{\mu\nu}\right)\\
&\quad + \frac{p^2}{\sqrt{1-p^2}}\left(p^{(\mu}\,\epsilon^{\nu)\sigma\rho}p_\sigma + p^{(\mu}\,p^{\nu)}p^\rho\right) + p^{(\mu}\,\delta^{\nu)\rho} + p^{(\mu}\,p^{\nu)}p^\rho &(263)\\
&= -p^{(\mu}\,\epsilon^{\nu)\sigma\rho}\sqrt{1-p^2} + 2p^2\left(p^{(\mu}\,\delta^{\nu)\rho} - 2p^\rho\delta^{\mu\nu}\right)\\
&\quad + \frac{p^2}{\sqrt{1-p^2}}\left(p^{(\mu}\,\epsilon^{\nu)\sigma\rho}p_\sigma + p^{(\mu}\,p^{\nu)}p^\rho\right) + p^{(\mu}\,\delta^{\nu)\rho} + p^{(\mu}\,p^{\nu)}p^\rho &(264)
\end{aligned}$$

Now, let's calculate the parallel transport operator. We see that

$$\begin{aligned}
\left(p \oplus dq\right)_i &= dq_i\sqrt{1-p^2} + p_i\sqrt{1-dq^2} + \epsilon_i{}^{jk}p_j dq_k &(265)\\
&\approx p_i + dq_i\sqrt{1-p^2} + \epsilon_i{}^{jk}p_j dq_k &(266)\\
&= p_i + dq_j\left(\delta_i^j\sqrt{1-p^2} + \epsilon_i{}^{mn}p_m\delta_n^j\right) &(267)\\
&= p_i + dq_j\tau_i^j\left(p\right) &(268)
\end{aligned}$$

where

$$\tau_i^j\left(p\right) = \delta_i^j\sqrt{1-p^2} + \epsilon_i{}^{mj}p_m \tag{269}$$

The coordinates z_j^μ can be obtained from the coordinates of the j-th particle with momentum p^j.

$$\begin{aligned}
z_j^\mu &= x_j^\nu\tau_\nu^\mu\left(p^j\right) &(270)\\
&= x_j^\nu\left(\delta_\nu^\mu\sqrt{1-\left(p^j\right)^2} + \epsilon_\nu{}^{m\mu}p_m^j\right) &(271)\\
&= x_j^\mu\sqrt{1-\left(p^j\right)^2} + \epsilon_\nu{}^{m\mu}x_j^\nu p_m^j &(272)
\end{aligned}$$

The Poisson brackets are

$$\left\{x_j^\mu, p_\nu^k\right\} = \sum_i \left(\frac{\partial x_j^\mu}{\partial x_j^i}\frac{\partial p_\nu^k}{\partial p_i^j} - \frac{\partial x_j^\mu}{\partial p_j^\mu}\frac{\partial p_\nu^k}{\partial x_j^i}\right) \tag{273}$$

$$= \delta_\mu^\nu \delta_j^k \tag{274}$$

And

$$\left\{z_j^\lambda, z_k^\omega\right\} = \sum_i \left(\frac{\partial z_j^\lambda}{\partial x_j^i}\frac{\partial z_k^\omega}{\partial p_i^j} - \frac{\partial z_j^\lambda}{\partial p_i^j}\frac{\partial z_k^\omega}{\partial x_j^i}\right) \tag{275}$$

We see that

$$\frac{\partial z_k^\sigma}{\partial x_j^i} = \delta_{jk}\delta_i^\sigma \sqrt{1-\left(p^j\right)^2} + \epsilon_\nu{}^{m\sigma}\delta_i^\nu \delta_{jk} p_m^j \tag{276}$$

$$= \left(\delta_i^\sigma \sqrt{1-\left(p^j\right)^2} + \epsilon_i{}^{m\sigma} p_m^j\right)\delta_{jk} \tag{277}$$

And

$$\frac{\partial z_k^\sigma}{\partial p_i^j} = -\frac{x_k^\sigma \delta^{mn}\delta^{jk}\left(p_m^k \delta_n^i + p_n^k \delta_m^i\right)}{2\sqrt{1-\left(p^k\right)^2}} + \epsilon_\nu{}^{m\sigma} x_k^\nu \delta^{jk}\delta_m^i \tag{278}$$

$$= -\frac{x_k^\sigma \delta^{jk}\left(p_m^k \delta^{im} + p_n^k \delta^{in}\right)}{2\sqrt{1-\left(p^k\right)^2}} + \epsilon_\nu{}^{i\sigma} x_k^\nu \delta^{jk} \tag{279}$$

$$= -\frac{x_k^\sigma \delta^{jk} p_m^k \delta^{im}}{\sqrt{1-\left(p^k\right)^2}} + \epsilon_\nu{}^{i\sigma} x_k^\nu \delta^{jk} \tag{280}$$

$$= \left(-\frac{x_k^\sigma p_m^k \delta^{im}}{\sqrt{1-\left(p^k\right)^2}} + \epsilon_\nu{}^{i\sigma} x_k^\nu\right)\delta^{jk} \tag{281}$$

$$\tag{282}$$

Hence, it's trivial to see that when $j \neq k$ or $\lambda = \omega$,

$$\left\{z_j^\lambda, z_k^\omega\right\} = 0 \tag{283}$$

Consider the case $j = k$ and $\lambda \neq \omega$ (we will drop j, k for simplicity).

$$\frac{\partial z^\lambda}{\partial x^i}\frac{\partial z^\omega}{\partial p_i} = \left(\delta_i^\lambda\sqrt{1-p^2} + \epsilon_i{}^{m\lambda}p_m\right)\left(-\frac{x^\omega p_\sigma \delta^{i\sigma}}{\sqrt{1-p^2}} + \epsilon_\nu{}^{i\omega}x^\nu\right) \tag{284}$$

$$= -\delta_i^\lambda x^\omega p_\sigma \delta^{i\sigma} + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} - \frac{\epsilon_i{}^{m\lambda}p_m x^\omega p_\sigma \delta^{i\sigma}}{\sqrt{1-p^2}} + \epsilon_i{}^{m\lambda}\epsilon_\nu{}^{i\omega}p_m x^\nu \tag{285}$$

$$= -\delta^{\lambda\sigma}x^\omega p_\sigma + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} - \frac{\epsilon^{\sigma m\lambda}p_m x^\omega p_\sigma}{\sqrt{1-p^2}} + \delta^{m\omega}\delta_\nu^\lambda p_m x^\nu \tag{286}$$

$$= -\delta^{\lambda\sigma}x^\omega p_\sigma + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} + \delta^{m\omega}p_m x^\lambda \tag{287}$$

$$= -p_m\delta^{m[\lambda}x^{\omega]} + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} \tag{288}$$

And (again for $j = k$ and $\lambda \neq \omega$)

$$\frac{\partial z^\lambda}{\partial p_i}\frac{\partial z^\omega}{\partial x^i} = \left(-\frac{x^\lambda p_\sigma \delta^{i\sigma}}{\sqrt{1-p^2}} + \epsilon_\nu{}^{i\lambda}x^\nu\right)\left(\delta_i^\omega\sqrt{1-p^2} + \epsilon_i{}^{m\omega}p_m\right) \tag{289}$$

$$= p_m\delta^{m[\lambda}x^{\omega]} + \delta_i^\omega \epsilon_\nu{}^{i\lambda}x^\nu\sqrt{1-p^2} \tag{290}$$

$$= \tag{291}$$

Hence,

$$\left\{z^\lambda, z^\omega\right\} = -p_m\delta^{m[\lambda}x^{\omega]} + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} - p_m\delta^{m[\lambda}x^{\omega]} - \delta_i^\omega \epsilon_\nu{}^{i\lambda}x^\nu\sqrt{1-p^2} \tag{292}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + \delta_i^\lambda \epsilon_\nu{}^{i\omega}x^\nu\sqrt{1-p^2} - \delta_i^\omega \epsilon_\nu{}^{i\lambda}x^\nu\sqrt{1-p^2} \tag{293}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + \epsilon_\nu{}^{\lambda\omega}x^\nu\sqrt{1-p^2} - \epsilon_\nu{}^{\omega\lambda}x^\nu\sqrt{1-p^2} \tag{294}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + \left(\epsilon_\nu{}^{\lambda\omega} - \epsilon_\nu{}^{\omega\lambda}\right)x^\nu\sqrt{1-p^2} \tag{295}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + \left(\epsilon_\nu{}^{\lambda\omega} + \epsilon_\nu{}^{\lambda\omega}\right)\left(z^\nu - \epsilon_\sigma{}^{m\nu}x^\sigma p_m\right) \tag{296}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + 2\epsilon_\nu{}^{\lambda\omega}\left(z^\nu - \epsilon_\sigma{}^{m\nu}x^\sigma p_m\right) \tag{297}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + 2\left(\epsilon_\nu{}^{\lambda\omega}z^\nu - \epsilon_\nu{}^{\lambda\omega}\epsilon_\sigma{}^{m\nu}x^\sigma p_m\right) \tag{298}$$

$$= -2p_m\delta^{m[\lambda}x^{\omega]} + 2\left(\epsilon_\nu{}^{\lambda\omega}z^\nu - \left(\delta_\sigma^\lambda\delta^{\omega m} - \delta^{\lambda m}\delta_\sigma^\omega\right)x^\sigma p_m\right) \tag{299}$$

$$= 2\left(-p_m\delta^{m[\lambda}x^{\omega]} + \epsilon_\nu{}^{\lambda\omega}z^\nu - \left(x^\lambda p_m\delta^{\omega m} - x^\omega p_m\delta^{\lambda m}\right)\right) \tag{300}$$

$$= 2\left(-p_m\left(\delta^{m\lambda}x^\omega - \delta^{m\omega}x^\lambda\right) + \epsilon_\nu{}^{\lambda\omega}z^\nu - x^\lambda p^\omega + x^\omega p^\lambda\right) \tag{301}$$

$$= 2\left(-p^\lambda x^\omega + p^\omega x^\lambda + \epsilon_\nu{}^{\lambda\omega}z^\nu - x^\lambda p^\omega + x^\omega p^\lambda\right) \tag{302}$$

$$= 2\epsilon_\nu{}^{\lambda\omega}z^\nu \tag{303}$$

In general,

$$\left\{z_j^\lambda, z_k^\omega\right\} = 2\epsilon_\nu{}^{\lambda\omega}z_j^\nu\delta_{jk} \tag{304}$$

6 References

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