Simulating Quantum Field Fluctuations

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1 Abstract

This project goes over the theory of quantum field on flat spacetime. We use Klein-Gordon equation to achieve this. In order to more easily calculate the quantum field fluctuations, we introduced the concept of Fourier Transform, in particular, the discrete version with Dirichlet boundary condition. From this, we then talk about the methods through which we simulate the vacuum quantum fluctuations on flat spacetime. Then, different k-modes are excited to see how these differ from vacuum fluctuations, and what kind of pattern these exhibit.

2 Introduction

In the early 1900s, Quantum Mechanics was highly successful in explaining the laws of physics at its microscopic scale. However, one of the major problem was that Quantum Mechanics was inconsistent with special relativity. In an attempt to generalize the Schrodinger's equation covariantly, Dirac Equation and Klein-Gordon Equation were introduced. However, there were still a few issues such as the fact that these equations could not explain particle annihilation and creation. Luckily, an old idea that worked with Quantum Mechanics saved the day ! It was the somewhat mystical process called quantization. For this particular stage, it is called second quantization since we are quantizing the wavefunction itself. As Klein-Gordon is the easiest and the only case of cosmological significance [1], this project is interested in this equation to simulate quantum fluctuations in flat spacetime.

3 Theory

3.1 Klein-Gordon Equation

Consider the Klein Gordon equation [1]

$$(\partial_t^2 - \triangle + m^2)\psi = 0$$

where ψ is the usual wavefunction. In order to simulate quantum fields, we need to second-quantize the Klein Gordon equation since it is the most straightforward method that has significance in cosmology. We simply replace ψ with ϕ which is called a quantum field. Hence, the second-quantized Klein Gordon equation then becomes

$$(\partial_t^2 - \triangle + m^2)\hat{\phi}(\boldsymbol{x}, t) = 0$$

satisfying the quantization conditions

$$\begin{aligned}
&[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar\delta^3(x - x') \\
&[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \\
&[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0
\end{aligned}$$

And the equations of motion

$$\begin{aligned}
&\dot{\hat{\phi}}(x,t) = \hat{\pi}(x,t) \\
&\dot{\hat{\pi}}(x,t) = -(-\triangle + m^2)\hat{\phi}(x,t)
\end{aligned}$$

Note that

$$\hat{\phi}^*(x,t) = \phi(x,t) \Rightarrow \hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t)$$

Then, Hamiltonian for the second-quantization is

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}^2(x,t) + \frac{1}{2}\hat{\phi}(x,t)(m^2 - \triangle)\hat{\phi}(x,t)d^3x$$

Then, these satisfy equations of motion [1]

$$i\hbar\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}]$$

And

$$\begin{aligned}
[\hat{\pi}(x,t), \hat{H}] &= \left[\hat{\pi}(x,t), \int d^3x' \frac{1}{2}\hat{\pi}^2(x',t) + \frac{1}{2}\hat{\phi}(x',t)(m^2 - \triangle)\hat{\phi}(x',t)\right] \\
&= \frac{1}{2}\left[\hat{\pi}(x,t), \int d^3x' \hat{\phi}(x',t)(m^2 - \triangle)\hat{\phi}(x',t)\right] \\
&= \frac{1}{2}\int d^3x'(-i\hbar)\delta^3(x'-x)(-\dot{\hat{\pi}}(x',t)) + \hat{\phi}(x',t)(m^2 - \triangle_{x'})[\hat{\pi}(x,t), \hat{\phi}(x',t)] \\
&= \frac{1}{2}\Big(i\hbar\dot{\hat{\pi}}(x,t) + \int d^3x' \hat{\phi}(x',t)(m^2 - \triangle_{x'})(-i\hbar)\delta^3(x'-x)\Big) \\
&= \frac{i\hbar}{2}\Big(\dot{\hat{\pi}}(x,t) - m^2\hat{\phi}(x,t) + \int_{\partial\mathbb{R}^3} \hat{\phi}(x',t)\triangle_{x'}\delta(x-x') - \int d^3x' \nabla_{x'}\hat{\phi}(x',t)\cdot\nabla_{x'}\delta(x-x')\Big) \\
&= \frac{i\hbar}{2}\Big(\dot{\hat{\pi}}(x,t) - m^2\hat{\phi}(x,t) + 0 + \int d^3x' \triangle_{x'}\hat{\phi}(x',t)\delta(x-x')\Big) \\
&= \frac{i\hbar}{2}(\dot{\hat{\pi}}(x,t) - (m^2 - \triangle)\hat{\phi}(x,t)) \\
&= i\hbar\dot{\hat{\pi}}(x,t)
\end{aligned}$$

3.2 Fourier Transform

We define the Fourier transform of $\hat{\phi}(x,t)$ as [2]

$$\hat{\phi}_k(t) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot k} \hat{\phi}(x,t) d^3x$$

And the inverse Fourier transform of $\hat{\phi}_k(t)$ as

$$\hat{\phi}(x,t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ix\cdot k} \hat{\phi}_k(t) d^3k$$

3.3 Back to Klein-Gordon under Fourier Transform

One can show that the Hamiltonian can be written as

$$\hat{H} = \int_{\mathbb{R}^3} \frac{1}{2}\hat{\pi}_k(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^\dagger(t)(k^2+m^2)\hat{\phi}_k(t) d^3k$$

And the commutation relation can be written as

$$\begin{aligned}
[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] &= \left[(2\pi)^{-3/2}\int e^{-ix\cdot k}\hat{\phi}(x,t)d^3x,\ (2\pi)^{-3/2}\int e^{-ix'\cdot k'}\hat{\pi}(x',t)d^3x'\right] \\
&= (2\pi)^{-3}\iint d^3x d^3x' e^{-ix\cdot k} e^{-ix'\cdot k'} [\hat{\phi}(x,t), \hat{\pi}(x',t)] \\
&= (2\pi)^{-3}\iint d^3x d^3x' e^{-ix\cdot k} e^{-ix'\cdot k'} i\hbar\delta^3(x-x') \\
&= i\hbar(2\pi)^{-3}\int d^3x e^{-ix\cdot(k+k')} \\
&= i\hbar\delta(k+k')
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0 \\
[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0
\end{aligned}$$

And the equations of motion becomes

$$\begin{aligned}
\dot{\hat{\phi}}_k(t) &= \hat{\pi}_k(t) \\
\dot{\hat{\pi}}_k(t) &= -(k^2+m^2)\hat{\phi}_k(t)
\end{aligned}$$

So for each k, we have a harmonic oscillator with $\omega_k = \sqrt{k^2+m^2}$. Now, in order to utilize the discrete Fourier transform, we consider quantum field in a box $[0,L]\times[0,L]\times[0,L]$. Then Each mode satisfies

$$\ddot{\hat{\phi}}_k(t) + \left(\left(\frac{n\pi}{L}\right)^2 + m^2\right)\hat{\phi}_n(t) = 0$$

Then

$$\hat{\phi}(x,t) = \sum_{k=1}^{\infty} \hat{\phi}_k \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$(k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3), \quad n_1, n_2, n_3 \in \mathbb{Z}$$
$$V = L^3$$

And the Hamiltonian is

$$\hat{H} = \sum_k \frac{1}{2}\hat{\pi}_k^\dagger \hat{\pi}_k + \frac{1}{2}\omega_k^2 \hat{\phi}_k^\dagger \hat{\phi}_k$$

where $\omega_k^2 = k^2 + m^2$. And these satisfy

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\hbar\delta_{k,-k'}$$
$$[\hat{\phi}_k(t), \hat{\phi}_{k'}(t)] = 0$$
$$[\hat{\pi}_k(t), \hat{\pi}_{k'}(t)] = 0$$

$$\dot{\hat{\phi}}_k(t) = \hat{\pi}_k(t)$$
$$\dot{\hat{\pi}}_k(t) = -(-k^2 + m^2)\hat{\phi}_k(t)$$

$$\hat{\phi}_k^\dagger(t) = \hat{\phi}_{-k}(t), \qquad \hat{\pi}_k^\dagger(t) = \hat{\pi}_{-k}(t)$$

4 Method

4.1 1 Dimensional Case

Now we are ready to implement these into our code to generate quantum fields in flat spacetime. We will be using Python [3]. The equation that is being calculated is

$$\phi(x) = \sum_{n=1}^{N} \sqrt{\frac{2}{L}} \sin(\pi n x/L)\phi_k$$

where $L = 1, \quad k = \frac{2\pi}{L}n$ and N is some finite integer. Since you are extracting values ϕ_k from $\exp(-\omega_k \phi_k^2)$ for each k, we draw a random value from a normal distribution with mean 0 and standard deviation of $\frac{1}{\sqrt{2\omega_k}}$.

4.2 2 Dimensional Case

For this, we use

$$\phi(x) = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \frac{2}{L} \sin(\pi n_1 x/L) \sin(\pi n_2 y/L) \phi_k$$

where $k = (k_1, k_2) = \frac{2\pi}{L}(n_1, n_2)$. Now, we are extracting ϕ_k from $\exp(-\omega_k \phi_k^2)$ where

$$\omega_k = \sqrt{k_1^2 + k^2 + m^2} = \sqrt{\frac{4\pi^2}{L^2}(n_1^2 + n_2^2) + m^2}$$

4.3 3 Dimensional Case

Similar to the other dimensions, we use

$$\phi(x) = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \sum_{n_3=1}^{N_3} \left(\frac{2}{L}\right)^{3/2} \sin(\pi n_1 x/L) \sin(\pi n_2 y/L) \sin(\pi n_3 z/L) \phi_k$$

where $k = (k_1, k_2, k_3) = \frac{2\pi}{L}(n_1, n_2, n_3)$. Again, we are extracting ϕ_k from $\exp(-\omega_k \phi_k^2)$ where

$$\omega_k = \sqrt{k_1^2 + k^2 + k_3^2 + m^2} = \sqrt{\frac{4\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2) + m^2}$$

4.4 Excited States

Now, we are going to implement quantum field fluctuations involving excited states. Note that for any given nth excited eigenstate for harmonic oscillator, the energy eigen-function is given by [4]

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x$$

where $H_n(\xi)$ is Hermite polynomials. Hence, for the quantum field fluctuation, we will use the same method we have been using, except for nth excited state for k-mode, we will extract its value from probability distribution:

$$H_n^2(\xi) e^{-\xi^2}$$

where $\xi = \sqrt{\omega_k} \phi_k$.

5 Results

5.1 Massless Cas: m = 0

We first take a look at a case where we consider mass $m = 0$. We let $L = 1$ for our results.

5.1.1 1-Dimensional Case

Now, let's see how the behaviour of the vacuum fluctuations change as we increase N. We first note that these quantum fields, in Fig. 1, satisfy the boundary condition which is good. More importantly, we see that as we increase the number of N, there is more fluctuations so the graphs look more sharp. So, as we increase N to infinity, we expect more and more of these sharp jumps in the quantum fields in space.

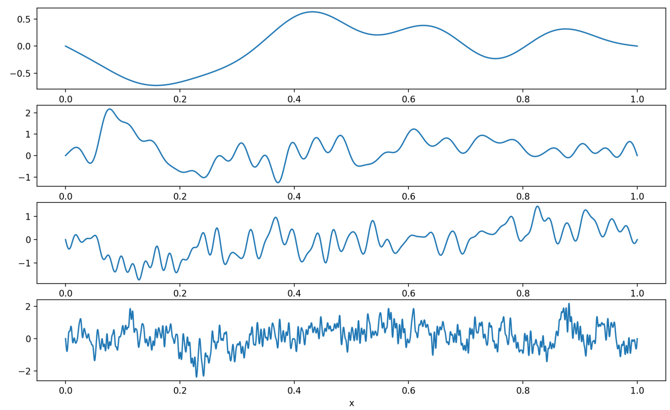

Figure 1: N=10, 50, 100, 500 from top to bottom

5.1.2 2-Dimensional Case

We now show the plots for the vacuum fluctuations in 2-dimension.

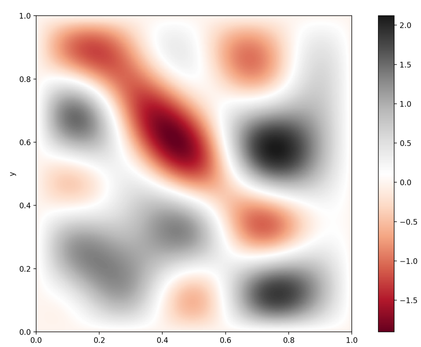

Figure 2: $N_1 = N_2 = 5$

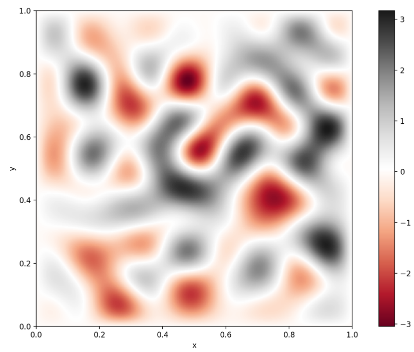

Figure 3: $N_1 = N_2 = 10$

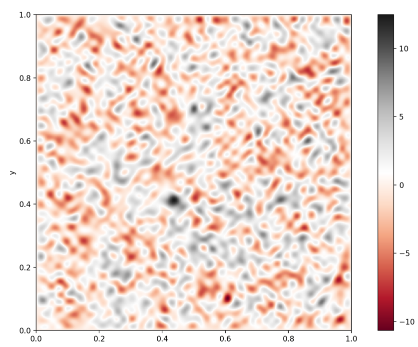

Figure 4: $N_1 = N_2 = 50$

As you can see in Fig. 2, Fig. 3, and Fig. 4, similarly to 1-dimensional case, as we include more k-modes, the fluctuations become more prevalent. Hence, we also expect that the plots will be filled more with the fluctuations as we include more k-modes.

5.1.3 3-Dimensional Case

Now, let's move onto 3-dimensional case. Note that the plots are shown in x, y axis at different z values. Similar to $1, 2$-dimensions, we again observe that the behaviour of the fluctuations become more prominent as we increase N, going from Fig. 5 to Fig. 6, and to Fig. 7. However, as we are adding more dimensions to our cases, the rate at which the fluctuations become more rapid with respect to the number of modes is a lot stronger. This is due to the fact that as we increase the number of modes in all three dimensions, we are in fact adding N^3 number of modes in 3-dimension, whereas in other dimensions, we are adding N^m where m is the number of dimensions.

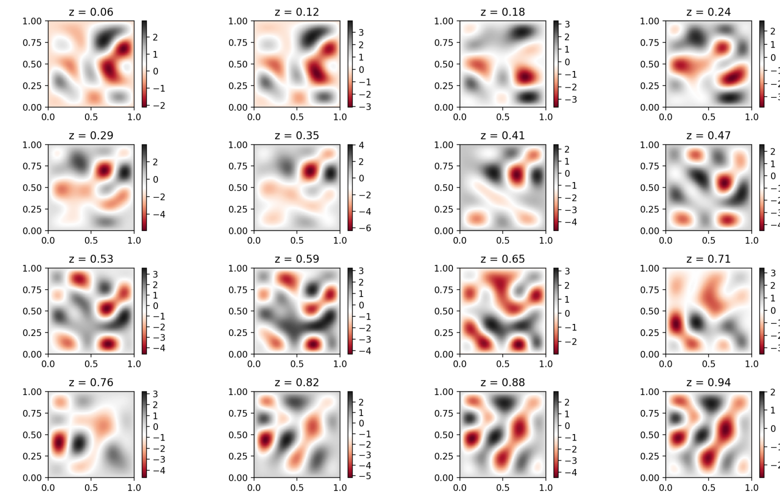

Figure 5: $N_1 = N_2 = N_3 = 5$

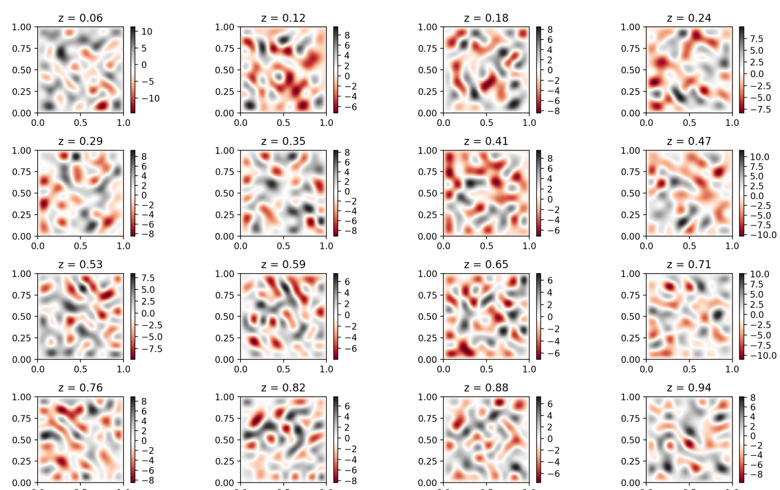

Figure 6: $N_1 = N_2 = N_3 = 10$

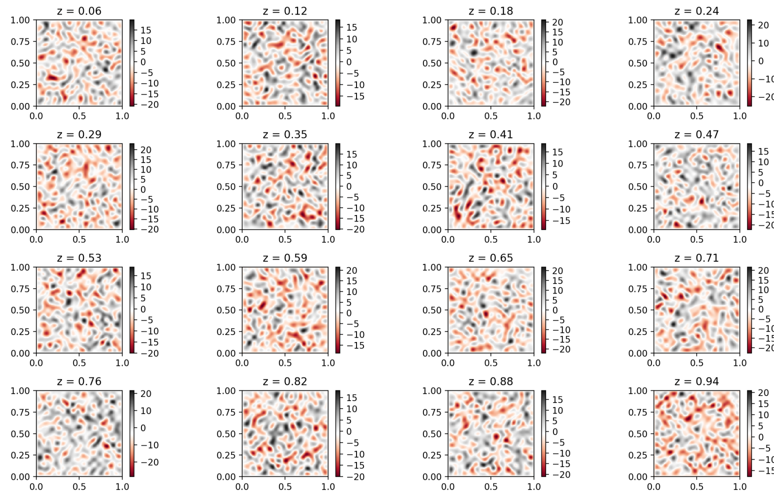

Figure 7: $N_1 = N_2 = N_3 = 20$

5.2 Non-massless Case : m $>>$ 1

Now let's take a look at what happens to the behaviour of the field fluctuations when we include mass in the calculation.

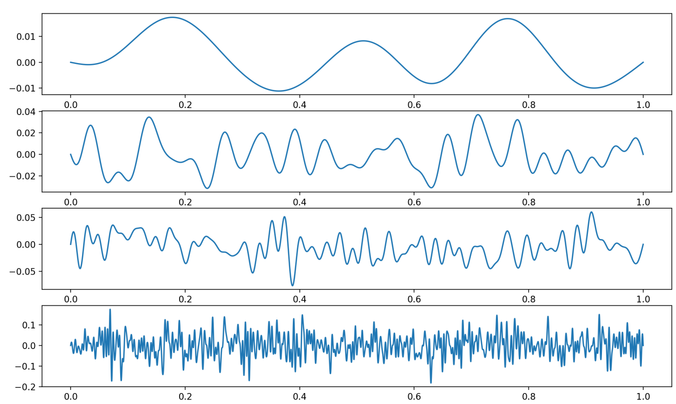

Figure 8: 1-dimensional case with $N = 10, 50, 100, 500$ from top to bottm

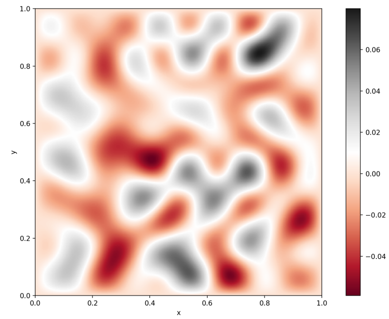

Figure 9: 2-dimensional case with $N_1 = N_2 = 10$

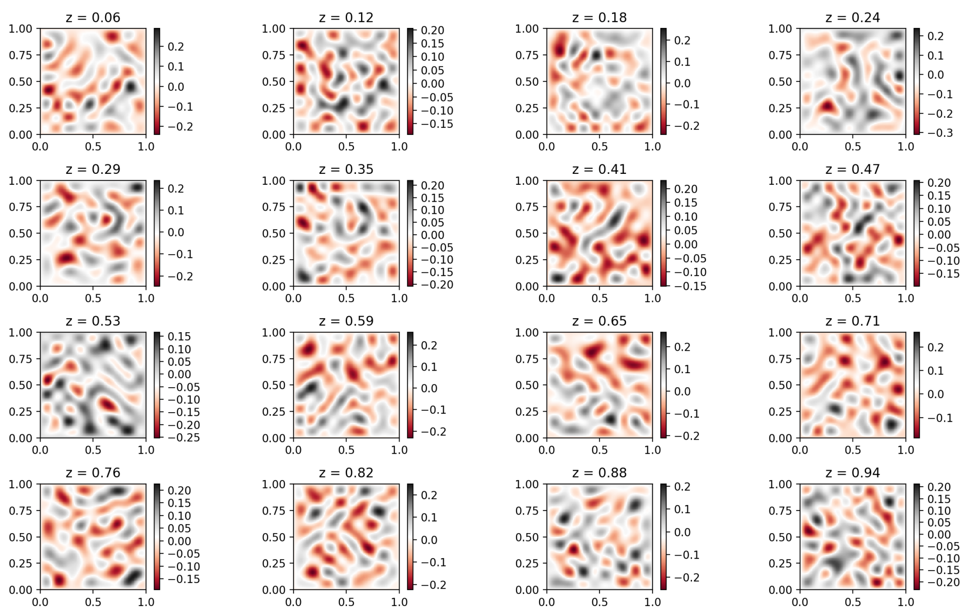

Figure 10: 3-dimensional case with $N_1 = N_2 = N_3 = 10$

We see clearly in Fig. 8, 1-dimensional case, that the strength of the vacuum fluctuation for this massive case is suppressed compared to the massless case. This is to be expected since as we increase the mass, the normal distribution from which we extract our values for the quantum field, becomes narrower and narrower towards the mean value of 0. In fact, the standard deviation for the normal distribution goes as $\sim \frac{1}{\sqrt{\omega_k}} \sim \frac{1}{m}$. Also, as you can see in Fig. 9 and Fig. 10, such a phenomenon is shown in the higher dimensions.

5.3 Excited States

Now, let's take look at what happens if we include excited states to our system. A good starting point is to first take a look at 1-dimensional case. We will from now on assume massless case.

5.3.1 1-Dimensional Case

The figures are showing quantum fluctuations in the case of exciting $k = \frac{2\pi}{L} n$ mode where $n = 3$ to 10th eigen-energy.

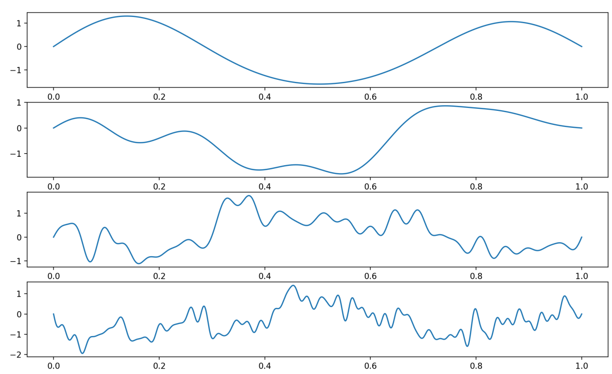

Figure 11: $N = 5,\ 10,\ 50,\ 100$ from top to bottom

We see that in Fig. 11, there is periodicity to the quantum fluctuation; however, as we increase the number of k-modes to the calculation, we see less of the periodicity due to the noise added.

5.3.2 2-Dimensional Case

Now, we move onto 2-dimensional case to better look at the behaviour of the quantum field fluctuations. If we look at Fig. 12 and Fig. 13 separately, we clearly see a consistent standing wave behaviour in the y-direction. But also, we see semi-standing waves behaviour with differing frequencies in the x-direction that grows proportionally with respect to N. This is because we selectively excited the k-modes that has a period $\frac{2L}{n'}$ with $n' = 1, 3$ for Fig. 12 and Fig. 13 respectively in the y-direction. However, this process also excited all of the other x-component of the quantum fields with the specified y-directional k-mode. Hence, as we increase the number of k-modes, there are more quantum fields with higher frequencies in the x-direction that are being excited. Hence, we expect the frequency of the standing wave in the y-direction to still remain even when we increase the number of k-modes as opposed to the fluctuations in the x-direction.

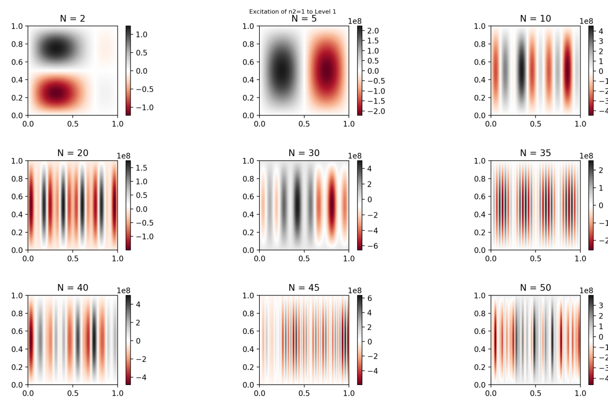

Figure 12: $n_2 = 1$ excited to energy level 1

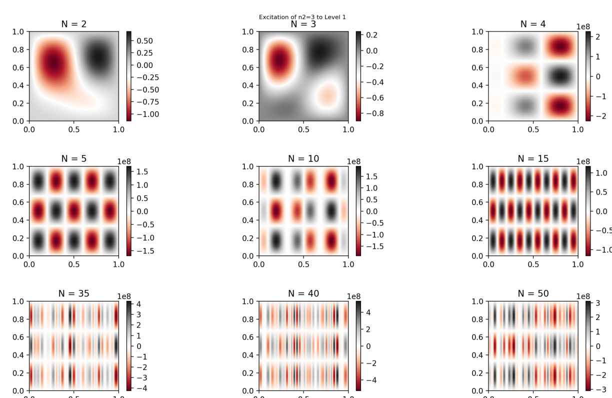

Figure 13: $n_2 = 3$ excited to energy level 1

Now, if we compare Fig. 12 and Fig. 13, we see that the period of standing wave got shortened when we moved from exciting $n_2 = 1$ to $n_2 = 3$. This is to be expected since the excited k-modes in Fig. 13 have Fourier coefficients with higher frequency compared to the excited k-modes in Fig. 13.

Also, we see an interesting behaviour with Fig. 14 with regards to $N \geq 35$ as the standing wave behaviour gets fuzzy due to the minor noises that gets added to the excitation modes. However, we see that not all N exhibits such a behaviour. This could be due to the fact that certain k-modes gets higher values for its quantum field which cancel out the standing wave behaviour. Interesting thing to note is that this standing wave behaviour somewhat reappears when we added more of the higher k-modes, with some minor noises that was added to it. This could be explained if newly added k-modes somewhat suppressed the large noises that came from the previously added modes.

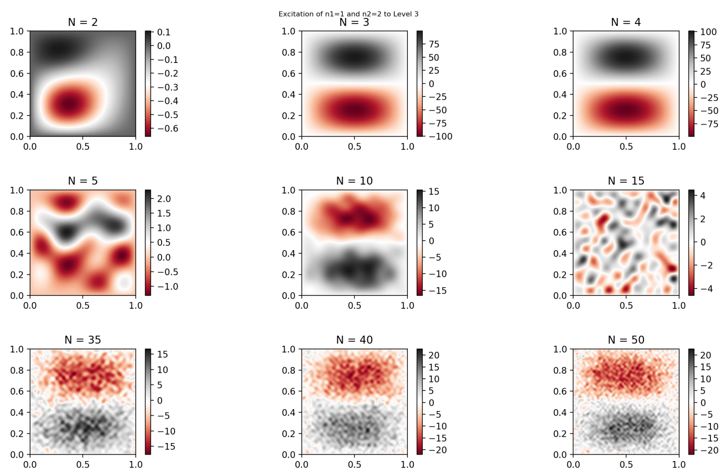

Figure 14: $(n_1, n_2) = (1, 2)$ excited to energy level 3

One potentially puzzling point is that in Fig. 12 and Fig. 13, the standing wave behaviour were always present in all N. On the other hand, such a behaviour was not consistent in Fig. 14 for different N. This is because in Fig. 12 and Fig. 13, as we increased N, we actually excited more k-modes since the newly added k-modes always had corresponding n_2 that got excited. But, in Fig. 14, we actually added only one particular k-mode; hence, newly added k-modes only added vacuum noise to our system.

Now, one might wonder as to what might happen if we excite k-modes that either have $n_1 = 1$ or $n_2 = 1$. As you might have expected, we see two competing semi-standing wave behaviour orthogonal to one another in Fig. 15. One thing to note is that the frequency of both standing waves increase as we increase the number of N. The reasoning is analogous to that of Fig. 12: we are exciting the quantum field with higher frequencies as we increase N. Now, if we excite k-modes with either $n_1 = 3$ or $n_2 = 3$, we get similar behaviour, but with higher frequencies as you can see in Fig. 16.

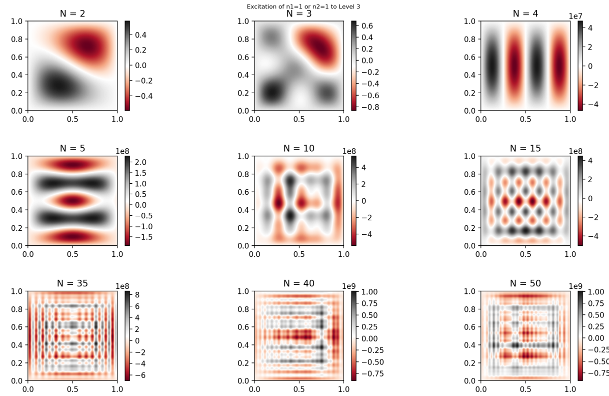

Figure 15: $n_1 = 1$ or $n_2 = 1$ excited to energy level 3

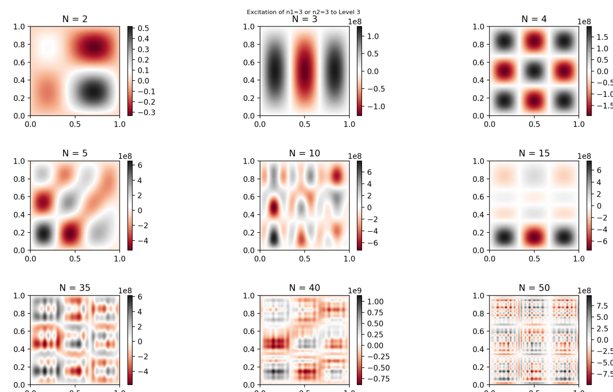

Figure 16: $n_1 = 3$ or $n_3 = 3$ excited to energy level 3

Lastly, let us try what happens to the quantum fields if we excite only the k-modes such that $n_1 = n_2$. As you can see in Fig. 17, we see interesting standing wave behaviour in the shape of X with increasing frequencies with respect to N. This is because we are exciting the k-modes in the x and y-directions that have the same phases. And due to constructive and destructive interference, these waves combined together produce diagonal waves !

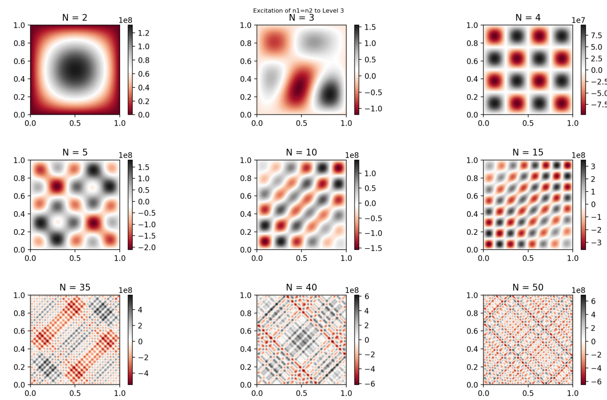

Figure 17: $n_1 = 3$ or $n_3 = 3$ excited to energy level 3

6 Discussion

There are certainly a few limiting assumptions with regards to this essay. One was the Dirichlet boundary conditions. Hence, in the future, we can explore different kinds of boundary conditions to see if it makes any significant changes to our results. Also, we dived a bit into different excitation modes. We saw that given different excitations, the quantum fluctuations produce interesting patterns. We could explore more in-depth into exciting the fields with different k-modes to see if these quantum fields exhibit any exotic patterns. Also, we could try exciting the quantum fields to a very high energy to see how this might change its behaviour. In particular, we noticed that in Fig. 14, the standing wave behaviour were slightly inconsistent. Hence, we can see if increasing the energy level could provide sharper patterns. Moreover, it will be interesting to try moving into 3-dimensional quantum fluctuations in 3D graph. This will provide even richer patterns that we could potentially explore. It will also be very exciting to make a 3D plot that can evolve these quantum fluctuations in time, and perhaps, we can even introduce varieties of conditions at different times to observe how the field fluctuations behave under these constraints.

7 Conclusions

In this essay, we explored the Klein-Gordon equation to produce quantum field fluctuations that are in vacuum and in certain excitation modes. We saw that even with a few limiting assumptions, this framework provides enough flexibility that can be very useful in studying quantum field theory in the cosmological setting. In particular, we saw how the vacuum fluctuations become more chaotic as we increase the number of k-modes, as well as increasing the number of dimensions. Also, as we expected, we saw including the mass into the quantum field, suppressed the fluctuations. We then looked into exciting different k-modes, namely the case when $n_2 = 1, 3$, $(n_1, n_2) = (1, 2)$, $n_1 = 1 \lor n_2 = 1$, $n_1 = 3 \lor n_2 = 3$, and $n_1 = n_2$. These provided us with interesting insight into how these excitation modes produce interesting patterns into our quantum field fluctuations.

References

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