# The Dicke Model

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## 1 Introduction

The Dicke model describes a system of $N$ two-level (spin-$\frac{1}{2}$) atoms within a single-model bosonic (photon) field. The atoms do not interact with each other directly; however, they do interact with the bosonic field via dipole interactions with a coupling strength of $\lambda$.

## 2 The Hamiltonian of Dicke Model

Note that we use natural unit, hence we set $\hbar = 1$. Also, our principal axis is $z$-axis.
The hamiltonian for the system of interest is

$$H = \omega_0 J_z + \omega a^* a + \frac{\lambda}{\sqrt{2j}}(a^* + a)(J_+ + J_-) \tag{1}$$

The first, the second, and the third term of Eq.(1) is the hamiltonian of $N$ number of two-level atoms, single-mode bosonic field, and the interaction between the atoms and the field respectively.
Following subsections explain where each of these terms come from in detail.

### 2.1 Derivation of Dicke Model

We start out by defining a hamiltonian known as Pauli-Fierz Hamiltonian which describes a non-relativistic charged particle coupled to an electromagnetic field. The Hilbert space of this system is

$$\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f = L^2(\mathbb{R}^3) \otimes F_s(L^2(\mathbb{R}^3)) \tag{2}$$

And the hamiltonian is defined by

$$H = \frac{1}{2m}\left(p + \frac{e}{c}A_\varphi(x)\right)^2 + H_f + e\phi_{ext}(x) \tag{3}$$

, where $p, e, c$ are momentum operator, electric charge, and speed of light respectively. And

$$H_f = \sum_{\lambda=1,2} \int d^3k \omega(k) \hbar a^*(k,\lambda) a(k,\lambda) \tag{4}$$

is the free field energy for the field, and $\lambda, k, \omega, \hbar, a, a^*$ denotes polarization, angular wavenumber, angular frequency, and reduced Planck's constant, annihilation, and creation operator respectively.

$$\phi_{ext}(x) \tag{5}$$

is an exterior potential to confine the particle, and

$$A_\varphi(x) = \sum_{\lambda=1,2} \int d^3kc \sqrt{\frac{\hbar}{2\omega(k)}} e_\lambda(k) \left( \hat{\varphi}(k) e^{ik\cdot x} a(k,\lambda) + \overline{\hat{\varphi}}(k) a^{-ik\cdot x} a^*(k,\lambda) \right) \tag{6}$$

is a quantized vector potential, where $e_\lambda(k)$ is the unit vector for the vector potential for a given angular wavenumber, and $\hat{\varphi}(k)$ is a quantized Fourier coefficient. Note that the first term Eqn.(3) is the interaction term for a particle coupled to the field.

The hamiltonian is well-defined due to the following theorem:

***Theorem***: For $e$ sufficiently small and if

$$\int d^3k |\hat{\varphi}(k)|^2 \left( \omega(k) + \frac{1}{\omega(k)^2} \right) < \infty \tag{7}$$

then, $H$ is a well-defined self-adjoint operator.

We introduce a conjugate field, which is a quantized electric field, to the quantized vector potential.

$$E_\varphi(x) = \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{\hbar\omega}{2}} e_\lambda(k) i \left( \hat{\varphi}(k) e^{ik\cdot x} a(k,\lambda) - \overline{\hat{\varphi}(k)} e^{-ik\cdot x} a^*(k,\lambda) \right) \tag{8}$$

Now, we employ Dipole approximation by substituting $A_\varphi(x)$ by $A_\varphi(0)$ assuming small confinement of the particle induced by $\phi_{ext}(x)$.
In order to obtain the more of the usual form, we perform an unitary transformation

$$U = e^{i\frac{e}{\hbar c} x \cdot A_\varphi(0)} \tag{9}$$

And one can compute that

$$U^* p U = p + \frac{e}{\hbar c} A_\varphi(0) \tag{10}$$

$$U^* x U = x \tag{11}$$

$$U^* a(k,\lambda) U = a(k,\lambda) + ix \cdot e_\lambda(k) e \hat{\varphi}(k) \sqrt{\frac{1}{2\hbar\omega(k)}} \tag{12}$$

Hence

$$\widetilde{H} = U^* H U = \frac{p^2}{2m} - ex \cdot E_\varphi(0) + \frac{m}{2}\nu^2 x^2 + H_f + e\phi_{ext}(x) \tag{13}$$

, where

$$\nu^2 = \frac{2}{3m}\int d^3k e^2 |\hat{\varphi}(k)|^2 \tag{14}$$

Now, we take $\phi_{ext}(x)$ to be harmonic, then combine it with the first and the third term in the hamiltonian which we let $\sim \frac{1}{2}m\omega_0^2 x^2$.
And we transform from position-momentum $(x, p)$ to a creation-annihilation representation $(b, b^*)$ which gives

$$\widetilde{H} = H_f + \hbar\omega_0 b^* b + \sqrt{\frac{\hbar}{2m\omega_0}}(b^* + b)eE_\varphi(0) \tag{15}$$

Lastly, we make a final assumption by only considering one mode in the cavity, hence only two levels of the oscillator:

$$F_s(L^2(\mathbb{R}^3)) \to F_s(\mathbb{C}^2) \quad \text{with } a, a^* \tag{16}$$

$$L^2(\mathbb{R}^3) \to \mathbb{C} \quad \text{with } \sigma_-, \sigma_+ \tag{17}$$

After, adjusting for the units and removing an overall constants, we get

$$\widetilde{H} = \omega a^* a + \omega_0 \frac{\sigma_z}{2} + i\lambda(\sigma_- + \sigma_+)(a - a^*) \tag{18}$$

Finally, we make the final unitary transformation via $a \mapsto ia$ which gives us

$$\widetilde{H} = \omega a^* a + \omega_0 \frac{\sigma_z}{2} + \lambda(\sigma_- + \sigma_+)(a + a^*) \tag{19}$$

or in the Rotating Wave Approximation

$$\widetilde{H}_{RWA} = \omega a^* a + \omega_0 \frac{\sigma_z}{2} + \lambda(\sigma_- a^* + \sigma_+ a) \tag{20}$$

Note that this is a hamiltonian for a single particle. Now, if we were to consider $N$ number of particles, then

$$\widetilde{H}_N = \omega a^* a + \sum_{i=1}^{N}\left\{\omega_0 \frac{\sigma_z(i)}{2} + \frac{\lambda}{\sqrt{N}}(\sigma_-(i) + \sigma_+(i))(a + a^*)\right\} \tag{21}$$

$$= \omega_0 J_z + \omega a^* a + \frac{\lambda}{\sqrt{N}}(a^* + a)(J_+ + J_-) \tag{22}$$

$$= \omega_0 J_z + \omega a^* a + \frac{\lambda}{\sqrt{2j}}(a^* + a)(J_+ + J_-) \tag{23}$$

, since $2j = N$ as we will see in the next section

Note that we moved from $\lambda \to \frac{\lambda}{\sqrt{N}}$, technically they are different. This is because the dipole interaction is inversely proportional to square root of the volume. And after absorbing constants into $\lambda$. we can let it be inversely proportional to $\sqrt{N}$.

### 2.1.1 N two-level atoms

For the sake of simplicity, assume there is only one two-level atom, then associated hamiltonian of that is

$$\frac{\omega_0}{2}\sigma_z \tag{24}$$

, where $\sigma_z$ is a Pauli matrix along $z$-axis, and $\omega_0$ is the energy gap between the ground state and the excited state.
Now, consider N number of identical but non-interacting atoms. Then, we need Pauli matrix for each of the atom separately. Hence we let $\sigma_z(i) = \mathbb{I} \otimes \cdots \otimes \sigma_z \otimes \cdots \otimes \mathbb{I}$, where $\mathbb{I}$ is an identity matrix, and $\sigma_z$ is at the $i$th position. Then the hamiltonian for $N$ number of such atoms is

$$\frac{\omega_0}{2}\sum_{i=1}^{N}\sigma_z(i) \tag{25}$$

Note that the Hilbert space in which this operator is $\mathbb{C}^{2N}$ We further simplify this by considering

$$J_z := \frac{1}{2}\sum_{i=1}^{N}\sigma_z(i) \tag{26}$$

$$\tag{27}$$

, where $J_z$ is the usual rotational angular momentum operator. The Hilbert space in which this operator acts on is spanned by

$$\{|j,m\rangle\,; m = -j, -j+1, \ldots, j-1, j\} \tag{28}$$

, where $j$ can take the values of $\frac{1}{2}, \frac{3}{2}, \frac{N}{2}$ for odd $N$ and $0, 1, \ldots, \frac{N}{2}$ for even $N$. We take $j$ to be its maximal value which is $\frac{N}{2}$. Hence, our Hilbert space is reduced to $\mathbb{C}^{N+1}$. In other words, the collection of $N$ number of two-level atoms is viewed as a single $(N+1)$-level atom.

### 2.1.2 Single-mode bosonic field

Note that in general, there are many modes to the bosonic field. However, as a first order approximation, we only consider the mode that matches the energy gap of our two-level atom because such mode is what allows the energy of the atoms to be excited or released. The hamiltonian of a single mode bosonic field is given by

$$\omega a^* a \tag{29}$$

, where $a^*$, $a$ are bosonic creation and annihilation operators.
The reason for such hamiltonian is because we can think of the field as a infinite number of harmonic oscillators. Now, Then the hamiltonian has the form

$$H = \sum_{1}^{\infty}\omega_j(a^*a + \frac{1}{2}) \tag{30}$$

But such series diverges. Since the *the constant term does not affect the dynamics*, we throw out $\sum_1^\infty \frac{\omega_j}{2}$, and we are left with renormlized hamiltonian. And, we are only considering a single mode, so we get back Eqn.(29).

### 2.2 Parity Symmetry

Define a parity operator

$$\Pi = \exp\{i\pi\hat{N}\}, \quad \hat{N} = a^*a + J_z + j \tag{31}$$

, where $\hat{N}$ is the "excitation number" which counts the total number of excited quanta. This makes sense since $a^*a\,|n\rangle = n\,|n\rangle$ counts the number of excited quanta in the photon field, and since $J_z\,|m\rangle = m\,|m\rangle$, $(J_z+j)\,|m\rangle = (m+N/2)\,|m\rangle$ counts the number of excited quanta in the atoms. Since $\hat{N}$ has integer-valued eigenvalue, $\Pi$ has a eigenvalue 1 for even eigenvalue of $\hat{N}$, and $-1$ for odd eigenvalue of $\hat{N}$. It is intuitive to see that $[H,\Pi]=0$. The interaction term in the hamiltonian has four terms $(a^*+a)(J_+ + J_-) = a^*J_+ + aJ_- + aJ_+ + a^*J_-$, where the first two terms conserve the total excitation number, but the last two terms could only changes the total excitations by two, if any. Hence, we see that the parity of the total excitation is conserved.

## 3 Thermodynamic Limit

We let $N \to \infty$, hence $j \to \infty$. By doing so, we are able to use Holstein-Primakoff representation of the angular momentum operator,

$$J_+ = b^*\sqrt{2j - b^*b} \tag{32}$$

$$J_- = \sqrt{2j - b^*b}b \tag{33}$$

$$J_z = (b^*b - j) \tag{34}$$

, where $b^*, b$ are Bose creation and annihilation operators respectively. It is easy to see that such transformation is exact for $J_z$.
Now, then the hamiltonian has the form

$$H = \omega_0(b^*b - j) + \omega a^*a + \lambda(a^*+a)\left(b^*\sqrt{1-\frac{b^*b}{2j}} + \sqrt{1-\frac{b^*b}{2j}}b\right) \tag{35}$$

and the parity operator becomes

$$\Pi = \exp\{i\pi(a^*a + b^*b)\} \tag{36}$$

### 3.1 Normal Phase

Since $j \to \infty$, the hamiltonian can be approximated by

$$H^{(1)} = \omega_0(b^*b - j) + \omega a^*a + \lambda(a^*+a)\,(b^*+b) \tag{37}$$

$$= \omega_0 b^*b + \omega a^*a + \lambda(a^*+a)(b^*+b) - j\omega_0 \tag{38}$$

For mathematical convenience, we use momentum and position operators

$$x = \frac{1}{\sqrt{2\omega}}(a^* + a), \quad p_x = i\frac{\omega}{2}(a^* - a) \tag{39}$$

$$y = \frac{1}{\sqrt{2\omega_0}}(b^* + b), \quad p_y = i\frac{\omega_0}{2}(b^* - b) \tag{40}$$

$$\tag{41}$$

Then

$$a^* = \sqrt{\frac{\omega}{2}}x - ip_x\sqrt{\frac{1}{2\omega}}, \quad a = \sqrt{\frac{\omega}{2}}x + ip_x\sqrt{\frac{1}{2\omega}} \tag{42}$$

$$b^* = \sqrt{\frac{\omega_0}{2}}y - ip_y\sqrt{\frac{1}{2\omega_0}}, \quad b = \sqrt{\frac{\omega}{2}}y + ip_y\sqrt{\frac{1}{2\omega_0}} \tag{43}$$

So the hamiltonian becomes

$$H^{(1)} = \frac{1}{2}\{\omega^2 x^2 + p_x^2 + \omega_0^2 y^2 + p_y^2 + 4\lambda\sqrt{\omega\omega_0}xy - \omega_0 - \omega\} - j\omega_0 \tag{44}$$

From here, we proceed to diagonalize the hamiltonian by rotating our coordinate

$$x = q_1 \cos\gamma + q_2 \sin\gamma, \quad y = -q_1 \sin\gamma + q_2 \cos\gamma \tag{45}$$

What this is trying to is if say $v$ is a vector in 2-dimension, and that $x$ and $y$ are horizontal and vertical component, respectively, of $v$ in the "old" Cartesian coordinate system, then, $q_1, q_2$ are horizontal and vertical component, respectively, of $v$ in the "new" coordinate system where we rotated the "old" coordinate system by $\gamma_1$ in clock-wise direction.
Now, using this substitution, if we let

$$\frac{\sin 2\gamma}{\cos 2\gamma} = \tan 2\gamma = \frac{4\lambda\sqrt{\omega\omega_0}}{\omega_0^2 - \omega^2} \tag{46}$$

Then, we can get rid of the cross terms. (*Please refer to Appendix A for the derivation.*)
For instance, if we consider $\omega = \omega_0$, then

$$\cos 2\gamma = 0 \ \Rightarrow 2\gamma = \frac{\pi}{2} \ \Rightarrow \gamma = \frac{\pi}{4} \tag{47}$$

Then

$$x = \frac{q_1 + q_2}{\sqrt{2}}, \quad y = \frac{-q_1 + q_2}{\sqrt{2}} \tag{48}$$

And

$$H^{(1)} = \frac{1}{2}\{\omega_0^2 q_1^2 + \omega_0^2 q_2^2 + p_1^2 + p_2^2 + 2\lambda\omega_0(-q_1^2 + q_2^2) - 2\omega_0\} - j\omega_0 \tag{49}$$

$$= \frac{1}{2}\{q_1^2[\omega_0^2 - 2\lambda\omega_0] + q_2^2[\omega_0^2 + 2\lambda\omega_0] + p_1^2 + p_2^2 - 2\omega_0\} - j\omega_0 \tag{50}$$

Now, in general, we get that

$$\begin{aligned}H^{(1)} =&\frac{1}{2}\{q_1^2[\omega^2\cos^2\gamma+\omega_0^2\sin^2\gamma-4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma]\\ &+q_2^2[\omega^2\sin^2\gamma+\omega_0^2\cos^2\gamma+4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma]\\ &+p_1^2+p_2^2-\omega_0-\omega\}-j\omega_0\end{aligned} \tag{51}$$

Which equals to (*Refer to Appendix B*)

$$H^{(1)}=\frac{1}{2}(q_1^2\varepsilon_-^{(1)^2}+q_2^2\varepsilon_+^{(1)^2}+p_1^2+p_2^2-\omega_0-\omega)-j\omega_0 \tag{52}$$

,where

$$\varepsilon_\pm^{(1)^2}=\frac{1}{2}\left(\omega^2+\omega_0^2\pm\sqrt{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}\right) \tag{53}$$

Define two new bosonic operators by

$$q_1=\frac{1}{\sqrt{2\varepsilon_-^{(1)}}}(c_1^*+c_1),\quad p_1=i\sqrt{\frac{\varepsilon_-^{(1)}}{2}}(c_1^*-c_1) \tag{54}$$

$$q_2=\frac{1}{\sqrt{2\varepsilon_+^{(1)}}}(c_2^*+c_2),\quad p_2=i\sqrt{\frac{\varepsilon_+^{(1)}}{2}}(c_2^*-c_2) \tag{55}$$

Then

$$\varepsilon_-^{(1)^2}q_1^2+p_1^2=\varepsilon_-^{(1)}(2c_1^*c_1+1),\quad \varepsilon_+^{(1)^2}q_2^2+p_2^2=\varepsilon_+^{(1)}(2c_2^*c_2+1) \tag{56}$$

Hence, we get that

$$H^{(1)}=\varepsilon_-^{(1)}c_1^*c_1+\varepsilon_+^{(1)}c_2^*c_2+\frac{1}{2}(\varepsilon_+^{(1)}+\varepsilon_-^{(1)}-\omega-\omega_0)-j\omega_0 \tag{57}$$

Note that only we require

$$\omega^2+\omega_0^2\geq\sqrt{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0} \tag{58}$$

for $\varepsilon_-^{(1)}$ to be a real number, otherwise it would be a complex number. That is

$$\lambda\leq\frac{\sqrt{\omega\omega_0}}{2}=\lambda_c \tag{59}$$

is the region at which the normal phase is valid. And we see that the ground state energy is $-j\omega_0$. Note that $H^{(1)}$ commutes with the parity operator. This means that the eigenstate of $H^{(1)}$ preserves its parity in the normal phase.

### 3.2 Super-Radiant Phase

In super-radian phase, both the field and the atoms acquire macroscopic occupations. That is, all of its energies are lifted up by some amount. In order to describe such phenomena, we modify the creation operators,

$$a^* \;\to\; c^* + \sqrt{\alpha}, \quad b^* \;\to\; d^* - \sqrt{\beta} \tag{60}$$

or

$$a^* \;\to\; c^* - \sqrt{\alpha}, \quad b^* \;\to\; d^* + \sqrt{\beta} \tag{61}$$

We assume the parameters $\alpha, \beta$ are of $\mathcal{O}(j)$.
Say, we picked $a^* \to c^* + \sqrt{\alpha}$, then $a \to c - \sqrt{\alpha}$. Then, let's see what this does to our ground state. Assume $|\psi\rangle$ is a coherent ground state. Then

$$(c + \sqrt{\alpha})\,|\psi\rangle = 0 \tag{62}$$

$$\Rightarrow \big((c + \sqrt{\alpha})\,|\psi\rangle\big)^* = \langle\psi|\,(c^* + \sqrt{\alpha}) = 0 \tag{63}$$

Now, consider the expectation value of $c^*c$ which is a number operator, then

$$\langle\psi|\, c^*c\,|\psi\rangle = \big(\langle\psi|\,(c^* + \sqrt{\alpha})\big)\, c\,|\psi\rangle - \big(\langle\psi|\,\sqrt{\alpha}\big)\, c\,|\psi\rangle \tag{64}$$

$$= \cancelto{0}{\big(\langle\psi|\,(c^* + \sqrt{\alpha})\big)}c\,|\psi\rangle - \big(\langle\psi|\,\sqrt{\alpha}\big)\, c\,|\psi\rangle \tag{65}$$

$$= -\big(\langle\psi|\,\sqrt{\alpha}\big)\,(c + \sqrt{\alpha})\,|\psi\rangle + \big(\langle\psi|\,\sqrt{\alpha}\big)\,(\sqrt{\alpha}\,|\psi\rangle) \tag{66}$$

$$= -\big(\langle\psi|\,\sqrt{\alpha}\big)\,\cancelto{0}{(c + \sqrt{\alpha})\,|\psi\rangle} + \big(\langle\psi|\,\sqrt{\alpha}\big)\,(\sqrt{\alpha}\,|\psi\rangle) \tag{67}$$

$$= \alpha\,\langle\psi|\psi\rangle = \alpha \tag{68}$$

Hence, we see that ground state is lifted up by $\alpha$.
Choose Eqn.(60), and substitute this into Eqn.(35), then we that

$$H^{(2)} = \omega_0 \left\{ d^*d - \sqrt{\beta}\,(d^* + d) + \beta - j \right\} + \omega\left\{c^*c + \sqrt{\alpha}\,(c^* + c) + \alpha\right\}$$
$$+\lambda\sqrt{\frac{k}{2j}}\,(c^* + c + 2\sqrt{\alpha})\left(d^*\sqrt{\xi} + \sqrt{\xi}d - 2\sqrt{\beta}\sqrt{\xi}\right) \tag{69}$$

, where

$$\xi = 1 - \frac{d^*d - \sqrt{\beta}(d^* + d)}{k} \tag{70}$$

Now, to take the thermodynamic limit, we taylor expand $\sqrt{\xi}$, then substitute that back into the Eqn.(69). Then, we discard all the terms with its denominator $\sim \mathcal{O}(j^n)$ where $n > 0$ because we expect $a^*a$ to be increasing like $\mathcal{O}(j)$. (*Refer to Appendix C*)

After appropriate approximation, the hamiltonian is given by

$$\begin{aligned} H^{(2)} \approx \omega c^* c &+ \left\{ \omega_0 + \frac{2\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}} \right\} d^* d - \left\{ 2\lambda\sqrt{\frac{\beta k}{2j}} - \omega\sqrt{\alpha} \right\} (c^* + c) \\ &+ \left\{ \frac{4\lambda}{k}\sqrt{\frac{\alpha k}{2j}}(j-\beta) - \omega_0\sqrt{\beta} \right\} (d^* + d) + \left\{ \frac{\lambda}{2k^2}\sqrt{\frac{\alpha\beta k}{2j}}(2k+\beta) \right\} (d^* + d)^2 \\ &+ \left\{ \frac{2\lambda}{k}\sqrt{\frac{k}{2j}}(j-\beta) \right\} (c^* + c)(d^* + d) + \left\{ \omega_0(\beta - j) + \omega\alpha - \frac{\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}}(1+4k) \right\} \end{aligned} \tag{71}$$

Now, we eliminate the terms in the hamiltonian that are linear in the bosonic operators,

$$2\lambda\sqrt{\frac{\beta k}{2j}} - \omega\sqrt{\alpha} = 0 \tag{72}$$

Then

$$\sqrt{\alpha} = \frac{2\lambda}{\omega}\sqrt{\frac{\beta k}{2j}} \tag{73}$$

We also want

$$\frac{4\lambda}{k}\sqrt{\frac{\alpha k}{2j}}(j-\beta) - \omega_0\sqrt{\beta} = 0 \tag{74}$$

Substitute Eqn.(73) to Eqn.(74), then we get that

$$\left\{ \frac{4\lambda^2}{\omega j}(j-\beta) - \omega_0 \right\} \sqrt{\beta} = 0 \tag{75}$$

The non-trivial solution is

$$\sqrt{\beta} = \sqrt{j(1-\mu)},\ \sqrt{\alpha} = \frac{2\lambda}{\omega}\sqrt{\frac{j}{2}(1-\mu^2)} \tag{76}$$

, where

$$\mu = \frac{\omega\omega_0}{4\lambda^2} = \frac{\lambda_c^2}{\lambda^2} \tag{77}$$

After eliminating the linear terms and rearranging, the hamiltonian becomes (*Refer to Appendix D*)

$$\begin{aligned} H^{(2)} &= \omega c^* c + \frac{\omega_0}{2\mu}(1+\mu) d^* d + \frac{\omega_0(1-\mu)(3+\mu)}{8\mu(1+\mu)}(d^* + d)^2 \\ &+ \lambda\mu\sqrt{\frac{2}{1+\mu}}(c^* + c)(d^* + d) - j\left\{ \frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2} \right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned} \tag{78}$$

Now, in order to diagonalize this, we introduce

$$X = \frac{1}{\sqrt{2\omega}}(c^* + c), \quad P_X = i\sqrt{\frac{\omega}{2}}(c^* - c) \tag{79}$$

$$Y = \frac{1}{\sqrt{2\tilde{\omega}}}(d^* + d), \quad P_Y = i\sqrt{\frac{\tilde{\omega}}{2}}(d^* - d) \tag{80}$$

, where

$$\widetilde{\omega} = \frac{\omega_0}{2\mu}(1 + \mu) \tag{81}$$

Then

$$c^*c = \frac{\omega}{2}X^2 + \frac{1}{2\omega}P_X^2 - \frac{1}{2} \tag{82}$$

$$d^*d = \frac{\tilde{\omega}}{2}Y^2 + \frac{1}{2\tilde{\omega}}P_Y^2 - \frac{1}{2} \tag{83}$$

$$(d^* + d)^2 = 2\tilde{\omega}Y^2 \tag{84}$$

$$(c^* + c)(d^* + d) = 2\sqrt{\omega\tilde{\omega}}XY \tag{85}$$

So the hamiltonian become

$$\begin{aligned} H^{(2)} = &\left\{\frac{\omega^2}{2}X^2 + \frac{1}{2}P_X^2 - \frac{1}{2}\omega\right\} + \left\{\frac{\tilde{\omega}^2}{2}Y^2 + \frac{1}{2}P_Y^2 - \frac{1}{2}\tilde{\omega}\right\} \\ &+ \frac{\omega_0^2(1-\mu)(3+\mu)}{8\mu^2}Y^2 + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}XY \\ &- j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned} \tag{86}$$

As before, we rotate our coordinate system via

$$X = Q_1 \cos\gamma_2 + Q_2 \sin\gamma_2 \tag{87}$$

$$Y = -Q_1 \sin\gamma_2 + Q_2 \cos\gamma_2 \tag{88}$$

We solve for $\gamma_2$ such that it eliminates the terms with $Q_1Q_2$ (*Refer to Appendix E*).

$$\tan 2\gamma_2 = \frac{2\omega\omega_0\mu^2}{\omega_0^2 - \mu^2\omega^2} \tag{89}$$

Then the diagonalized hamiltonian is

$$\begin{aligned} H^{(2)} = &\frac{1}{4}\left\{\left(-\frac{(\omega\mu^2 - \omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2 - \mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2}\right\} Q_1^2 \\ &\frac{1}{4}\left\{\left(\frac{(\omega\mu^2 - \omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2 - \mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2}\right\} Q_2^2 \\ &+ \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned} \tag{90}$$

Note that

$$\cos 2\gamma_2 = \frac{1}{\sqrt{1+\tan^2 2\gamma_2}} \tag{91}$$

$$= \frac{\omega_0^2 - \mu^2\omega^2}{\sqrt{(\omega_0^2-\mu^2\omega^2)^2 + (2\omega\omega_0\mu^2)^2}} \tag{92}$$

After substituting Eqn.(92) into Eqn. (90), we get that (*Refer to Appendix F*)

$$H^{(2)} = \frac{1}{2}\left(\varepsilon_-^{(2)^2} Q_1^2 + \varepsilon_+^{(2)^2} Q_2^2 + P_1^2 + P_2^2 - \omega - \widetilde{\omega}\right) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \tag{93}$$

, where

$$\varepsilon_\pm^{(2)^2} = \frac{1}{2}\left\{\frac{\omega_0^2}{\mu^2} + \omega^2 \pm \sqrt{\left[\frac{\omega_0^2}{\mu^2} - \omega^2\right]^2 + 4\omega^2\omega_0^2}\right\} \tag{94}$$

Define two new bosonic operators

$$Q_1 = \frac{1}{\sqrt{2\varepsilon_-^{(2)}}}(e_1^* + e_1), \quad P_1 = i\sqrt{\frac{\varepsilon_-^{(2)}}{2}}(e_1^* - e_1) \tag{95}$$

$$Q_2 = \frac{1}{\sqrt{2\varepsilon_+^{(2)}}}(e_2^* + e_2), \quad P_2 = i\sqrt{\frac{\varepsilon_+^{(2)}}{2}}(e_2^* - e_2) \tag{96}$$

Then, the hamiltonian in terms of the newly defined bosonic operators is

$$H^{(2)} = \varepsilon_-^{(2)} e_1^* e_1 + \varepsilon_+^{(2)} e_2^* e_2 + \frac{1}{2}\left\{\varepsilon_-^{(2)} + \varepsilon_+^{(2)} - \omega - \frac{\omega_0}{2\mu}(1+\mu) + \frac{2\lambda^2}{\omega}(1-\mu)\right\} - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} \tag{97}$$

In order for $\varepsilon_-^{(2)}$ to be real number, we require

$$\frac{\omega_0^2}{\mu^2} + \omega^2 \geq \sqrt{\left[\frac{\omega_0^2}{\mu^2} - \omega^2\right]^2 + 4\omega^2\omega_0^2} \tag{98}$$

$$\Rightarrow \lambda \geq \frac{\sqrt{\omega\omega_0}}{2} = \lambda_c \tag{99}$$

That is the super-radiant phase is valid for $\lambda \geq \lambda_c$. We see that the ground state energy is $-j\left(\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right)$. If we have chosen Eqn.(60) instead, we would have still gotten the same hamiltonian and $\alpha, \beta$. Note that unlike the normal phase, $H^{(2)}$ does not commute with the parity operator. Hence, the parity symmetry is broken at the phase transition. However, $H^{(2)}$ commutes with

$$\Pi^{(2)} = \exp\{i\pi[c^*c + d^*d]\} \tag{100}$$

## 3.3 Quantum Phase Transition

We see that $\varepsilon_\pm$ are the excitations for our hamiltonian. Here we graph these excitations in terms of $\lambda$ and we will investigate its order of quantum phase transitions.

### 3.3.1 Normal Phase: Order of Quantum Phase Transition

Here, we show $\varepsilon_-(\lambda \to \lambda_c)$ from the normal phase.

$$\begin{aligned}
\varepsilon_-^2(\lambda \to \lambda_c) &= \frac{1}{2}\left\{\omega^2 + \omega_0^2 - \sqrt{(\omega_0^2 - \omega^2)^2 + 16\lambda^2\omega\omega_0}\right\} && (101)\\
&= \frac{1}{2}\left\{\omega^2 + \omega_0^2 - \sqrt{(\omega^2 + \omega_0^2)^2 - 4\omega\omega_0(\omega\omega_0 - 4\lambda^2)}\right\} && (102)\\
&= \frac{1}{2}(\omega^2 + \omega_0^2)\left\{1 - \sqrt{1 - \frac{4\omega\omega_0(\omega\omega_0 - 4\lambda^2)}{(\omega^2 + \omega_0^2)^2}}\right\} && (103)\\
&= \frac{1}{2}(\omega^2 + \omega_0^2)\left\{1 - 1 + \frac{2\omega\omega_0(\omega\omega_0 - 4\lambda^2)}{(\omega^2 + \omega_0^2)^2}\right\} + \mathcal{O}(|\lambda - \lambda_c|^2) \text{ , by Taylor} && (104)\\
&= \frac{\omega\omega_0(\omega\omega_0 - 4\lambda^2)}{\omega^2 + \omega_0^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (105)\\
&= \frac{4\omega\omega_0\left(\frac{\omega\omega_0}{4} - \lambda^2\right)}{\omega^2 + \omega_0^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (106)\\
&= \frac{4\omega\omega_0(\lambda_c^2 - \lambda^2)}{\omega^2 + \omega_0^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (107)\\
&= \frac{4\omega\omega_0(\lambda_c + \lambda)(\lambda_c - \lambda)}{\omega^2 + \omega_0^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (108)\\
&= \frac{4\omega\omega_0(2\lambda_c + \lambda - \lambda_c)(\lambda_c - \lambda)}{\omega^2 + \omega_0^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (109)\\
&= \frac{8\omega\omega_0\lambda_c(\lambda_c - \lambda)}{\omega^2 + \omega_0^2} + O(|\lambda - \lambda_c|^2) && (110)\\
&= \frac{32\lambda_c^3\omega^2}{\omega^4 + 16\lambda_c^4}|\lambda_c - \lambda| + \mathcal{O}(|\lambda - \lambda_c|^2) && (111)
\end{aligned}$$

$$\therefore \varepsilon_-(\lambda \to \lambda_c) = \sqrt{\frac{32\lambda_c^3\omega^2}{\omega^4 + 16\lambda_c^4}}|\lambda_c - \lambda|^{1/2} + \mathcal{O}(|\lambda - \lambda_c|) \qquad (112)$$

We see that this is a second-order phase transition.

### 3.3.2 Super-Radiant Phase: Order of Quantum Phase Transition

We show $\varepsilon_-(\lambda \to \lambda_c)$ from the super-radiant phase.

$$
\begin{aligned}
2\varepsilon_-^2 &= \frac{\omega_0^2}{\mu^2} + \omega^2 - \sqrt{\left[\frac{\omega_0^2}{\mu^2} - \omega^2\right]^2 + 4\omega^2\omega_0^2} && (113)\\
&= \left(\frac{\omega_0^2}{\mu^2} + \omega^2\right)\left\{1 - \sqrt{1 - \frac{4\omega_0^2\omega^2\left(\frac{1}{\mu^2} - 1\right)}{\left(\frac{\omega_0^2}{\mu^2} + \omega^2\right)^2}}\right\} && (114)\\
&= \left(\frac{\omega_0^2}{\mu^2} + \omega^2\right)\frac{2\omega_0^2\omega^2\left(\frac{1}{\mu^2} - 1\right)}{\left(\frac{\omega_0^2}{\mu^2} + \omega^2\right)^2} + \mathcal{O}(|\lambda - \lambda_c|^2), \text{ by Taylor} && (115)\\
&= \frac{2\omega_0^2\omega^2\left(\frac{1}{\mu^2} - 1\right)}{\left(\frac{\omega_0^2}{\mu^2} + \omega^2\right)} + \mathcal{O}(|\lambda - \lambda_c|^2) && (116)\\
&= \frac{2\omega_0^2\omega^2\left(\frac{16\lambda^4}{\omega^2\omega_0^2} - 1\right)}{\frac{16\lambda^4}{\omega^2} + \omega^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (117)\\
&= \frac{\omega^2(32\lambda^4 - 2\omega_0^2\omega^2)}{16\lambda^4 + \omega^4} + \mathcal{O}(|\lambda - \lambda_c|^2) && (118)\\
&= \frac{32\omega^2(\lambda^4 - \lambda_c^4)}{16\lambda^4 + \omega^4} + \mathcal{O}(|\lambda - \lambda_c|^2) && (119)\\
&= \frac{32\omega^2}{16\lambda^4 + \omega^4}\left[4\lambda_c^3(\lambda - \lambda_c) + (\lambda - \lambda_c)^2 + (\lambda^2 - \lambda_c^2)\right] + \mathcal{O}(|\lambda - \lambda_c|^2) && (120)\\
&= \frac{128\omega^2\lambda_c^3(\lambda - \lambda_c)}{16\lambda_c + \omega^2} + \mathcal{O}(|\lambda - \lambda_c|^2) && (121)
\end{aligned}
$$

$$
\boxed{\therefore \varepsilon(\lambda \to \lambda_c)_- = \sqrt{\frac{64\lambda_c^3\omega^2}{16\lambda_c + \omega^4}}|\lambda_c - \lambda|^{1/2} + \mathcal{O}(|\lambda - \lambda_c|) \qquad (122)}
$$

We see that this is a second-order phase transition, same as the normal phase.

### 3.3.3 Analysis

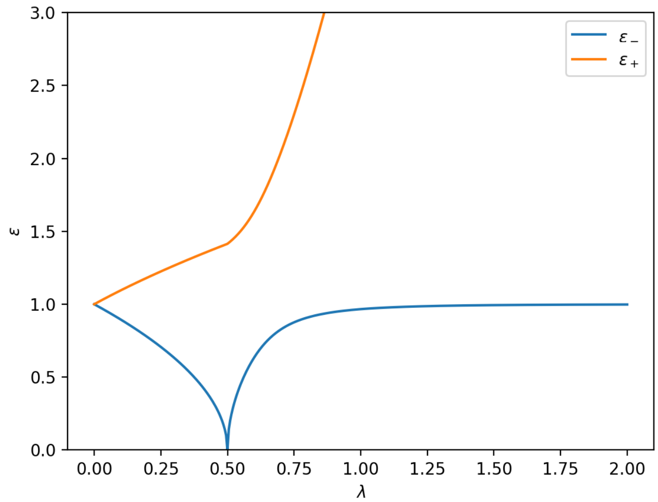



Figure 1: Excitation energies as a function of $\lambda$. Here we set $\omega = \omega_0 = 1$

We see the quantum phase transition point at $\lambda_c = 0.5$, where $\varepsilon_-$ vanishes. Note that $\varepsilon_\pm^{(1)}$ before $\lambda = 0.5$ in the figure are the fundamental excitations for normal phase, $\varepsilon_\pm$, and after $\lambda = 0.5$ in the figure are $\varepsilon_\pm^{(2)}$ which are in the super-radiant phase. We see that they agree on the values at 0.5 !

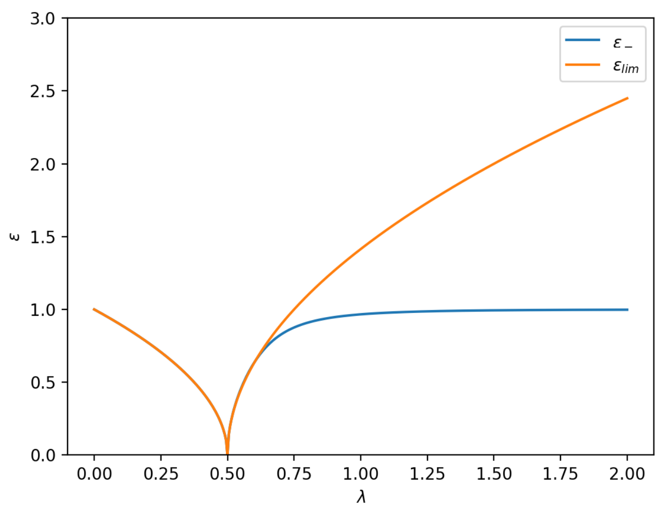



Figure 2: Comparison of $\varepsilon_-$ and $\varepsilon_-(\lambda \to \lambda_c)$ as a function of $\lambda$. We set $\omega = \omega_0 = 1$

We see that the limiting excitation energy, $\varepsilon_-(\lambda \to \lambda_c)$, agrees very well with $\varepsilon_-$ where

$|\lambda - \lambda_c|$ is small. Again, $\varepsilon_-$ and $\varepsilon_-(\lambda \to \lambda_c)$ before $\lambda = 0.5$ are in the normal phase and are in the super-radiant phase after $\lambda = 0.5$.

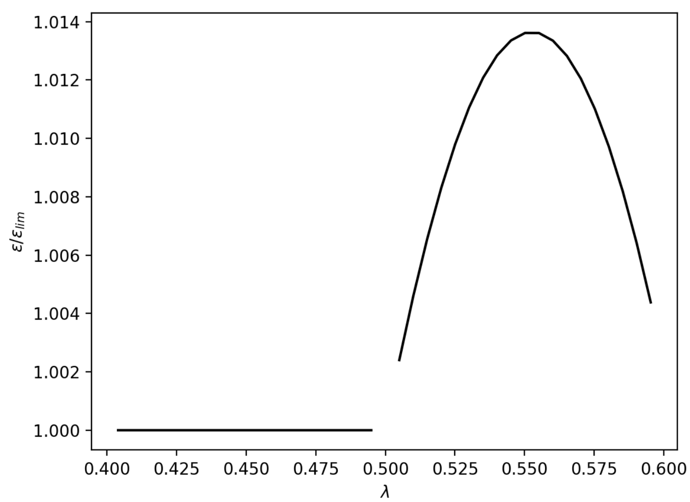



Figure 3: $\epsilon_-/\epsilon_-(\lambda \to \lambda_c)$ as a function of $\lambda$

Here, we see that the ratio of $\varepsilon_-/\varepsilon_-(\lambda \to \lambda_c)$ gets very close to one as $\lambda$ approaches $\lambda_c$.

### 3.4 Rotating Wave Approximation: Normal Phase

There is another approximation that we can use which is a Rotating Wave Approximation in which the hamiltonian takes the form of

$$H^{(1)}_{RWA} = \omega_0 J_z + \omega a^* a + \frac{\lambda}{\sqrt{2j}} \left(a^* J_- + a J_+\right) \tag{123}$$

Using the Holstein-Primakoff transformation, we get

$$H^{(1)}_{RWA} = \omega_0(b^* b - j) + \omega a^* a + \frac{\lambda}{\sqrt{2j}}(a^* \sqrt{2j - b^* b}\, b^* + a b^* \sqrt{2j - b^* b}) \tag{124}$$

We let $j \to \infty$, then

$$\begin{aligned} H^{(1)}_{RWA} &= \omega_0(b^* b - j) + \omega a^* a + \lambda(a^* b + a b^*) && (125) \\ &= \omega_0 b^* b + \omega a^* a + \lambda(a^* b + a b^*) - \omega_0 j && (126) \end{aligned}$$

We make following transformation to diagonalize the hamiltonian

$$a \to -c_1 \sin\beta + c_2 \cos\beta, \quad b \to c_1 \cos\beta + c_2 \sin\beta \tag{127}$$

And we proceed to find $\beta$ such that it eliminates the terms other than $c_1^* c_1$ and $c_2^* c_2$ (*Refer to Appendix G*).

$$\tan 2\beta = \frac{2\lambda}{\omega - \omega_0} \tag{128}$$

Then, the diagonalized hamiltonian becomes

$$H_{RWA}^{(1)} = \varepsilon_- c_1^* c_1 + \varepsilon_+ c_2^* c_2 - j\omega_0 \tag{129}$$

, where

$$\varepsilon_\pm = \frac{1}{2}\left\{\omega_0 + \omega \pm \sqrt{(\omega-\omega_0)^2 + 4\lambda^2}\right\} \tag{130}$$

We see that for $\varepsilon_-$ to makes sense, we require

$$\omega_0 + \omega \geq \sqrt{(\omega-\omega_0)^2 + 4\lambda^2} \tag{131}$$

$$\therefore \lambda \leq \sqrt{\omega\omega_0} = \lambda_{RWA,c} \tag{132}$$

There are a few differences between non-RWA and RWA, namely $\lambda_{RWA,c} = 2\lambda_c$. Also,

$$2\varepsilon_- = \omega_0 + \omega - \sqrt{(\omega-\omega_0)^2 + 4\lambda^2} \tag{133}$$

$$= (\omega_0 + \omega)\left\{1 - \sqrt{1 + \frac{4(\lambda^2 - \omega\omega_0)}{(\omega+\omega_0)^2}}\right\} \tag{134}$$

$$\Rightarrow 2\varepsilon_-(\lambda \to \lambda_c) = (\omega_0 + \omega)\left\{1 - \sqrt{1 + \frac{4(\lambda^2 - \lambda_c^2)}{(\omega+\omega_0)^2}}\right\} \tag{135}$$

$$= (\omega_0 + \omega)\frac{2(\lambda^2 - \lambda_c^2)}{(\omega+\omega_0)^2} + \mathcal{O}(|\lambda - \lambda_c|^2) \tag{136}$$

$$= \frac{2(2\lambda_c + \lambda - \lambda_c)(\lambda - \lambda_c)}{\omega + \omega_0} + \mathcal{O}(|\lambda - \lambda_c|^2) \tag{137}$$

$$= \frac{4\lambda_c(\lambda - \lambda_c)}{\omega + \omega_0} + \mathcal{O}(|\lambda - \lambda_c|^2) \tag{138}$$

$$\therefore \varepsilon_-(\lambda \to \lambda_c) = \frac{2\lambda_c}{\omega + \omega_0}|\lambda - \lambda_c| + \mathcal{O}(|\lambda - \lambda_c|^2) \tag{139}$$

We see that this is a zero-order phase transition whereas non-RWA is a second-order phase transition.

# 4 Ground state of Dicke Hamiltonian

The Dicke hamiltonian in the RWA is

$$H_N = a^* a + \sum_{i=1}^{N}\left(\frac{\epsilon}{2}\sigma_z(i) + \frac{\lambda}{\sqrt{N}}\left\{S^+(i)a + S^-(i)a^*\right\}\right) \tag{140}$$

, where

$$S^\pm = S_x \pm iS_y = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right) \tag{141}$$

Note that this look a little different from Eqn.(123), but we just need to set $\omega = 1$ from Eqn.(123) and use slightly different notations to get the above equation. Other than that, they are equivalent.
In this section, I will prove where the ground state of Dicke hamiltonian lives in. Note that the Hilbert space for the Dicke model is

$$\mathcal{H} = \mathcal{H}_{ph} \bigotimes_{i=1}^{N} \mathbb{C} \tag{142}$$

, where

$$\mathcal{H}_{ph} = \bigoplus_{n=0}^{\infty} S_+ H^{\otimes n} = \mathbb{C} \oplus H \oplus (S_+(H \otimes H)) \oplus \dots \tag{143}$$

is called a Fock space and $S_+$ is an operator that symmetrizes a tensor. Informally, this is a direct sum of set of Hilbert spaces which represents no boson, one boson, two bosons, and so on.
It is convenient to deal with matrix representation of the hamiltonian. So, we introduce total spin operator which is defined by

$$S_N^2 = \sum_{i=1}^{N} \sum_{j=x,y,z} \left( \frac{\sigma_j(i)}{2} \right)^2 \tag{144}$$

$$\tag{145}$$

And its eigenvalue is

$$\lambda_S = S(S+1) \tag{146}$$

, where $S = 0, 1, \dots, \frac{N}{2}$ for $N$ even, and $S = \frac{1}{2}, \frac{3}{2}, \dots \frac{N}{2}$ for $N$ odd.
$H_N$ commutes with $S_N^2$. This means that we can write $H_N$ with respect to the eigenspace of $S_N^2$. That is

$$H_N = \left[ \begin{array}{c|c|c|c} H_N(S=0) & 0 & 0 & 0 \\ \hline 0 & H_N(S=1) & 0 & \ddots \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & \ddots & 0 & H_N(S=N/2) \end{array} \right] \tag{147}$$

The Hilbert space this acts on is

$$\bigotimes_{S \geq 0}^{N/2} \mathcal{H}_N(S) \tag{148}$$

In this section, we will show that the ground state is at $S = \frac{N}{2}$. But, we still need to introduce one more operator since the above Hilbert space is infinite dimensional, hence we cannot write our hamiltonian in matrix representation yet. We define a number operator by

$$C_N = a^* a + \sum_{i=1}^{N} \frac{\sigma_z(i)}{2} \tag{149}$$

This operator essentially fixes the total number of excitations in the system.
This number operators commutes with the hamiltonian and the spin operator. So, we can decompose the Hilbert space further with excitation number $C_N = C$. The decomposed Hilbert space is given by

$$\mathcal{H}_N(S) = \bigotimes_{C \geq -S}^{\infty} \mathcal{H}_N(S, C) \tag{150}$$

In $\mathcal{H}_N(S, C)$, we diagonalize in $\sum_{i=1}^{N} \frac{\sigma_z}{2}$ basis

$$\{|S, C, M\rangle_N \,,\ M = -S, -S+1, \ldots, \min\{S, C\} = [S.C]\} \tag{151}$$

Then, the matrix representation of the Dicke hamiltonian restricted to $\mathcal{H}_N(S, C)$ is given by

$$H_N(S, C) \to A$$

, where A is matrix representation of $H_N(S, C)$, and is a tridiagonal symmetric matrix such that (*Refer to Appendix H*)

$$A_{MM}(S) = C + (\epsilon - 1)M, \quad (M = -S, \ldots, [S, C] = \min\{S, C\}) \tag{152}$$

$$A_{M,M+1}(S) = \frac{\lambda}{\sqrt{N}} \sqrt{(C-M)(S(S+1) - M(M+1))} > 0, \quad (M = -S, \ldots, [S, C] - 1) \tag{153}$$

$$A_{MM'} = 0 \text{ for } \left|M - M'\right| > 1 \tag{154}$$

### 4.1 Lemma 1

*Statement*:
Let $A$ be a real symmetric matrix with $A_{ij} \geq 0$ for $i \neq j$, $\lambda$ be its maximal eigenvalue, and $v$ be the corresponding normalized eigenvector, that is $|v|^2 = 1$. Then, we can choose $v_i \geq 0$. Such construction for $A, v, \lambda$ will be used in all other remaining Lemmas and the Theorem.
*Proof:*
Note that $Av = \lambda v$. Then $v^T A v = \lambda |v| = \lambda$. That is

$$\lambda = \sum_{i,j} v_i A_{ij} v_j \leq \sum_{i,j} |v_i| A_{ij} |v_j| \tag{155}$$

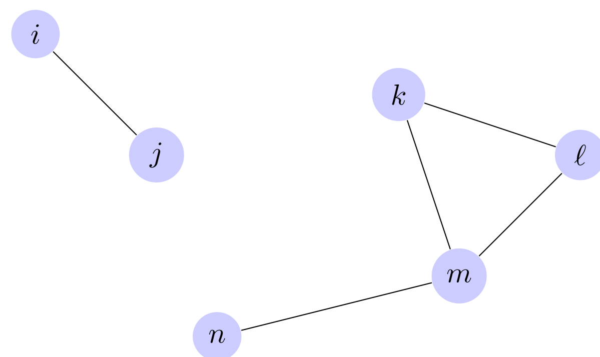
But recall that for any normalized vector $y$, we have $y^T A y \leq \lambda$. Since $|v_i|$ is a normalized vector, we have that $\sum_{i,j} |v_i| A_{ij} |v_j| \leq \lambda$.

$$\therefore \lambda = \sum_{i,j} |v_i| A_{ij} |v_j| \tag{156}$$

Hence from now on, we will assume $v$ to have only non-negative components.

### 4.2 Lemma 2

*Statement*:
Let $i \sim j$ if $A_{ij} > 0$, where $i \neq j$, or $i \sim j$ by transitively extending this notion. Note that since $A$ is assumed to be symmetric, reflexivity is satisfied. We force $i \sim i$. Hence this forms an equivalence class. Then $v_i = 0$ or $v_j > 0$ for all $i$ in the same equivalence class.
This particular construction is best described by use of graph:




In this graph, the vertices represent indices of the component of $v$. When two vertices are directly connected by an edge, it, the corresponding indices in the matrix $A$ is strictly positive. For example, $n$ and $m$ are connected by an edge, that means. $A_{mn} > 0$ and $A_{nm} > 0$. However, $A_{nk}$ and $A_{kn}$ are not strictly positive since they are not connected by an edge, even though there exists a path that connects $n$ and $k$. In this notion, we clearly see that $i, j$ forms an equivalence class, and $k, n, m, \ell$ belongs to another equivalence class. What this Lemma says is, the components of $v$ with indices that are connected by a path must all either equal to zero or greater than zero. That is, if $v_i = 0$, then $v_j = 0$, and if $v_k > 0$, then $v_n, v_m, v_\ell > 0$.
*Proof*:
Note that $\sum_j A_{ij} v_j = \lambda v_i$. Then

$$\sum_{j \neq i} A_{ij} v_j = (\lambda - A_{ii}) v_i \tag{157}$$

If $v_i = 0$, then for any $j$ such that $A_{ij} > 0$, $v_j = 0$. In other words, any vertices that are directly connected to the vertex $i$ must be zero. Going back to the above example, if $v_k = 0$, then $v_m = 0$ and $v_\ell = 0$. And since $v_m = 0$, $v_n = 0$. Hence, we see that any vertices that are connected by a path must be all zero if any one of them is zero. Similarly, we see that if $v_i > 0$, then for any $j$ such that $A_{ij} > 0$, $v_j > 0$. (Note that by Lemma 1, we assume $v_i \geq 0$).

### 4.3 Lemma 3

*Definition*: $\operatorname{supp} v = \{i \mid v_i > 0\}$
*Statement*:
Let $\widetilde{A}$ be a real symmetric matrix. If $\widetilde{A}_{ij} \geq A_{ij}$ for all $i, j \in \operatorname{supp} v$, then $\widetilde{\lambda} \geq \lambda$, where $\widetilde{\lambda}$ is a maximal eigenvalue of $\widetilde{A}$. In particular, if $\widetilde{A}_{ij} > A_{ij}$ for some $i, j \in \operatorname{supp} v$, then $\widetilde{\lambda} > \lambda$.

*Proof*:
We have that

$$\lambda = \sum_{i,j} v_i A_{ij} v_j = \sum_{i,j \in \operatorname{supp} v} v_i A_{ij} v_j \tag{158}$$

Since $\widetilde{A}_{ij} \geq A_{ij}$

$$\sum_{i,j \in \operatorname{supp} v} v_i A_{ij} v_j \leq \sum_{i,j \in \operatorname{supp} v} v_i \widetilde{A}_{ij} v_j \tag{159}$$

Also, recall that $\sum_i v_i^2 = 1$, then

$$\sum_{i,j \in \operatorname{supp} v} v_i \widetilde{A}_{ij} v_j \leq \widetilde{\lambda} \tag{160}$$

Hence

$$\lambda \leq \widetilde{\lambda} \tag{161}$$

If $\widetilde{A}_{ij} > A_{ij}$ for some $i, j \in \operatorname{supp} v$, then the above equality becomes strict. One thing to note is that $\widetilde{A}$ may not be of the same sized matrix as $A$. The proof works even if $\widetilde{A}$ acts on a vector space with higher dimension than that of a vector space in which $A$ acts on. The only requirement is that $\widetilde{A}$ be greater than or equal to $A$ in indices belonging to the support of v.

### 4.4 Theorem

*Statement*:
The ground state of $H_N$ which acts on $\mathcal{H}_N = \bigotimes_{S \geq 0}^{N/2} n_N(S) \mathcal{H}_N(S)$ is only in $S = \frac{N}{2}$. ($n_N(S) = \frac{N!(2S+1)}{(N/2+S+1)!(N/2-S)!}$ is a multiplicity factor).
*Proof*:
We prove for each term in $\mathcal{H}_N(S) = \bigotimes_{C \geq -S} \mathcal{H}_N(S, C)$
Note :
1) The diagonal terms do not depend on $S$.

$$A_{MM}(S) = C + (\epsilon - 1)M, \ (M = -S, \ldots, [S, C]) \tag{162}$$

2) The off-diagonal terms are strictly positive.

$$A_{M,M+1}(S) = \frac{\lambda}{\sqrt{N}} \sqrt{(C - M)(S(S+1) - M(M+1))} > 0, \quad (M = -S, \ldots, [S, C] - 1) \tag{163}$$

3) $A_{ij}$ is increasing as $S$ increases on $\operatorname{supp} v$ for $i \neq j$.
4) By Lemma 2, since the off-diagonal terms are all strictly positive, support of $v$ is $\{-S, \ldots, [S, C]\}$ because at least one of the component of maximal eigenvector $v$ for $A$ should have non-zero component.

5) For any unitary matrix $U$, if $B = -U^*AU$, then ground state energy of $A$ equals to negative of top state energy of $B$.

Let

$$U_{k\ell} = (-1)^k \delta_{k\ell} \tag{164}$$

$$\Rightarrow U = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & (-1)^{S+[S,C]+1} \end{bmatrix} \tag{165}$$

Then if we let $B = -U^*AU$, we get that

$$B_{ii}(S) = -A_{ii}(S) \tag{166}$$

$$B_{ij}(S) = A_{ij}(S), \ i \neq j \tag{167}$$

Denote $\lambda(S)$ to be a maximal eigenvalue of $B(S)$.
By 1) and 2) we have that

$$B_{ij}(S+1) > B_{ij}(S) \quad \text{for } i, j \in \operatorname{supp} v \tag{168}$$

Then by Lemma 3,

$$\lambda(S+1) > \lambda(S) \tag{169}$$

Hence, the largest eigenvalue is when $S = \frac{N}{2}$. But by 5), negative of such eigenvalue is the smallest eigenvalue of $A(S)$. Hence, we see that the ground state energy of $A$ is when $S = \frac{N}{2}$. In conclusion, the ground state of the hamiltonian is in $\mathcal{H}_N\left(S = \frac{N}{2}, C\right)$ for all $C$.

# 5 Appendices

## 5.1 Appendix A

The hamiltonian is

$$H^{(1)} = \frac{1}{2}\{\omega^2 x^2 + p_x^2 + \omega_0^2 y^2 + p_y^2 + 4\lambda\sqrt{\omega\omega_0}xy - \omega_0 - \omega\} - j\omega_0 \tag{170}$$

From here, we proceed to diagonalize the hamiltonian by rotating our coordinate

$$x = q_1 \cos\gamma_1 + q_2 \sin\gamma_1, \quad y = -q_1 \sin\gamma_1 + q_2 \cos\gamma_1 \tag{171}$$

Now, after substitution, we get

$$\begin{aligned} H^{(1)} =& \frac{1}{2}\{\omega^2(q_1^2 \cos^2\gamma + 2q_1q_2\cos\gamma\sin\gamma + q_2^2\sin^2\gamma) + p_x^2 \\ &+ \omega_0^2(q_1^2\sin^2\gamma - 2q_1q_2\cos\gamma\sin\gamma + q_2^2\cos^2\gamma) + p_y^2 \\ &+ 4\lambda\sqrt{\omega\omega_0}(-q_1^2\cos\gamma\sin\gamma + q_1q_2\cos^2\gamma - q_1q_2\cos^2\gamma - q_1q_2\sin^2\gamma + q_2^2\cos\gamma\sin\gamma) \\ &- \omega_0 - \omega\} - j\omega_0 \end{aligned} \tag{172}$$

We want to eliminate the term with $q_1q_2$, that is

$$\begin{aligned} 0 &= 2q_1q_2\omega^2\cos\gamma\sin\gamma - 2q_1q_2\omega_0^2\cos\gamma\sin\gamma + 4\lambda\sqrt{\omega\omega_0}q_1q_2(\cos^2\gamma - \sin^2\gamma) && (173) \\ &= 2(\omega^2 - \omega_0^2)\cos\gamma\sin\gamma + 4\lambda\sqrt{\omega\omega_0}(\cos^2\gamma - \sin^2\gamma) && (174) \\ &= (\omega^2 - \omega_0^2)\sin 2\gamma + 4\lambda\sqrt{\omega\omega_0}\cos 2\gamma, \quad \text{using trig identities} && (175) \end{aligned}$$

Hence

$$\boxed{\frac{\sin 2\gamma}{\cos 2\gamma} = \tan 2\gamma = \frac{4\lambda\sqrt{\omega\omega_0}}{\omega_0^2 - \omega^2}} \tag{176}$$

## 5.2 Appendix B

The hamiltonian is

$$\begin{aligned} H =& \frac{1}{2}\{q_1^2[\omega^2\cos^2\gamma + \omega_0^2\sin^2\gamma - 4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma] \\ &+ q_2^2[\omega^2\sin^2\gamma + \omega_0^2\cos^2\gamma + 4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma] \\ &+ p_x^2 + p_y^2 - \omega_0 - \omega\} - j\omega_0 \end{aligned} \tag{177}$$

Note that

$$\begin{aligned} \cos 2\gamma &= \frac{1}{\sqrt{1 + \tan^2 2\gamma}} && (178) \\ &= \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 16\lambda^2\omega\omega_0}} && (179) \end{aligned}$$

Consider

$$
\begin{aligned}
&\omega^2\cos^2\gamma + \omega_0^2\sin^2\gamma - 4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma \\
&= \omega^2\left(\frac{1+\cos 2\gamma}{2}\right) + \omega_0^2\left(\frac{1-\cos 2\gamma}{2}\right) - 2\lambda\sqrt{\omega\omega_0}\sin 2\gamma && (180)\\
&= \omega^2\left(\frac{1+\cos 2\gamma}{2}\right) + \omega_0^2\left(\frac{1-\cos 2\gamma}{2}\right) - 2\lambda\sqrt{\omega\omega_0}\,\frac{4\lambda\sqrt{\omega\omega_0}}{\omega_0^2-\omega^2}\cos 2\gamma && (181)\\
&= \frac{1}{2}\left\{\omega^2+\omega_0^2 - \cos 2\gamma\left(\frac{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}{\omega_0^2-\omega^2}\right)\right\} && (182)\\
&= \frac{1}{2}\left\{\omega^2+\omega_0^2 - \sqrt{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}\right\} && (183)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\omega^2\sin^2\gamma + \omega_0^2\cos^2\gamma + 4\lambda\sqrt{\omega\omega_0}\cos\gamma\sin\gamma \\
&= \omega^2\left(\frac{1-\cos 2\gamma}{2}\right) + \omega_0^2\left(\frac{1+\cos 2\gamma}{2}\right) + \frac{8\lambda^2\omega\omega_0}{\omega_0^2-\omega^2}\cos 2\gamma && (184)\\
&= \frac{1}{2}\left\{\omega^2+\omega_0^2 + \cos 2\gamma\left(\frac{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}{\omega_0^2-\omega^2}\right)\right\} && (185)\\
&= \frac{1}{2}\left\{\omega^2+\omega_0^2 + \sqrt{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}\right\} && (186)
\end{aligned}
$$

Hence

$$
\boxed{H^{(1)} = \frac{1}{2}(q_1^2\varepsilon_-^{(1)^2} + q_2^2\varepsilon_+^{(1)^2} + p_1^2 + p_2^2 - \omega_0 - \omega) - j\omega_0 \qquad (187)}
$$

,where

$$
\varepsilon_\pm^{(1)^2} = \frac{1}{2}\left(\omega^2+\omega_0^2 \pm \sqrt{(\omega_0^2-\omega^2)^2+16\lambda^2\omega\omega_0}\right) \tag{188}
$$

### 5.3 Appendix C

We would like to take the thermodynamic limit. So, we taylor expand $\sqrt{\xi}$, then substitute that back into the Eqn.(69). Then, we discard all the terms with its denominator $\sim \mathcal{O}(j^n)$ where $n > 0$.

$$
\begin{aligned}
\sqrt{\xi} &\approx 1 + \left(-\frac{d^*d - \sqrt{\beta}(d^*+d)}{2k}\right) - \frac{(d^*d-\sqrt{\beta}(d^*+d))^2}{8k^2} && (189)\\
&\approx 1 - \frac{d^*d - \sqrt{\beta}(d^*+d)}{2k} - \frac{(d^*d)^2 + \beta(d^*+d)^2 - \sqrt{\beta}d^*d(d^*+d) - \sqrt{\beta}(d^*+d)d^*d}{8k^2} && (190)\\
&= 1 + \frac{\sqrt{\beta}(d^*+d)}{2k} - \left[\frac{d^*d}{2k} + \frac{\beta(d^*+d)^2}{8k^2}\right] + \frac{\sqrt{\beta}(d^*d(d^*+d) + (d^*+d)d^*d)}{8k^2} - \frac{(d^*d)^2}{8k^2} && (191)
\end{aligned}
$$

This is quite a cumbersome calculation, so I will show the computation step by step. We first consider

$$\begin{aligned}(c^* + c + 2\sqrt{\alpha})(d^*\sqrt{\xi} + \sqrt{\xi}d - 2\sqrt{\beta}\sqrt{\xi}) &= (c^* + c)d^*\sqrt{\xi} + (c^* + c)\sqrt{\xi}d \\ &-2\sqrt{\beta}(c^* + c)\sqrt{\xi} + 2\sqrt{\alpha}d^*\sqrt{\xi} + 2\sqrt{\alpha}\sqrt{\xi}d - 4\sqrt{\alpha\beta}\sqrt{\xi} \qquad (192)\end{aligned}$$

We further subdivide this by considering,

$$(c^* + c)d^*\sqrt{\xi} \approx (c^* + c)d^* \qquad (193)$$

$$(c^* + c)\sqrt{\xi}d \approx (c^* + c)d \qquad (194)$$

$$-2\sqrt{\beta}(c^* + c)\sqrt{\xi} \approx -2\sqrt{\beta}(c^* + c)\left(1 + \frac{\sqrt{\beta}(d^* + d)}{2k}\right) \qquad (195)$$

$$2\sqrt{\alpha}d^*\sqrt{\xi} \approx 2\sqrt{\alpha}d^*\left(1 + \frac{\sqrt{\beta}(d^* + d)}{2k}\right) \qquad (196)$$

$$2\sqrt{\alpha}\sqrt{\xi}d \approx 2\sqrt{\alpha}\left(1 + \frac{\sqrt{\beta}(d^* + d)}{2k}\right)d \qquad (197)$$

$$-4\sqrt{\alpha\beta}\sqrt{\xi} \approx -4\sqrt{\alpha\beta}\left(1 + \frac{\sqrt{\beta}(d^* + d)}{2k} - \left[\frac{d^*d}{2k} + \frac{\beta(d^* + d)^2}{8k^2}\right]\right) \qquad (198)$$

Now, add Eqn.(193) and Eqn.(194), then we get

$$(c^* + c)(d^* + d) \qquad (199)$$

Now, add Eqn.(199) and Eqn.(195), then we get

$$\boxed{-2\sqrt{\beta}(c^* + c) + (c^* + c)(d^* + d)\left(1 - \frac{\beta}{k}\right) \qquad (200)}$$

Add Eqn.(196) and Eqn.(197).

$$\begin{aligned}&2\sqrt{\alpha}(d^* + d) + \frac{\sqrt{\alpha\beta}}{k}(d^*(d^* + d) + (d^* + d)d) \\ &= 2\sqrt{\alpha}(d^* + d) + \frac{\sqrt{\alpha\beta}}{k}((d^* + d)^2 - 1) \qquad (201)\\ &= 2\sqrt{\alpha}(d^* + d) + \frac{\sqrt{\alpha\beta}}{k}(d^* + d)^2 - \frac{\sqrt{\alpha\beta}}{k} \qquad (202)\end{aligned}$$

Now, add Eqn.(198) and Eqn.(202)

$$\frac{2\sqrt{\alpha\beta}}{k}d^*d + (d^* + d)\left(2\sqrt{\alpha} - \frac{2\sqrt{\alpha\beta}}{k}\right) + (d^* + d)^2\left(\frac{\sqrt{\alpha\beta}}{k} + \frac{\sqrt{\alpha\beta}\beta}{2k^2}\right) - 4\sqrt{\alpha\beta} - \frac{\sqrt{\alpha\beta}}{k}$$

$$\boxed{= \frac{2\sqrt{\alpha\beta}}{k}d^*d + 2\sqrt{\alpha}(d^* + d)\left(1 - \frac{\beta}{k}\right) + \frac{\sqrt{\alpha\beta}}{k}(d^* + d)^2\left(1 + \frac{\beta}{2k}\right) - \sqrt{\alpha\beta}\left(4 + \frac{1}{k}\right) \qquad (203)}$$

Now, add Eqn.(200) and Eqn. (203), then multiply the above equations by $\lambda\sqrt{\frac{k}{2j}}$. After simplifying, we get,

$$
\begin{aligned}
& d^*d\left(\frac{2\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}}\right) + (c^*+c)\left(-2\lambda\sqrt{\frac{\beta k}{2j}}\right) + (c^*+c)(d^*+d)\left(\frac{2\lambda}{k}\sqrt{\frac{k}{2j}}(j-\beta)\right) \\
& + (d^*+d)\left(\frac{4\lambda}{k}\sqrt{\frac{\alpha k}{2j}}(j-\beta)\right) + (d^*+d)^2\left(\frac{\lambda}{2k^2}\sqrt{\frac{\alpha\beta k}{2j}}(2k+\beta)\right) \\
& + \left(\frac{-\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}}(1+4k)\right)
\end{aligned} \tag{204}
$$

Then the hamiltonian under such approximation is given by

$$
\begin{aligned}
H^{(2)} = \omega c^*c &+ \left\{\omega_0 + \frac{2\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}}\right\} d^*d - \left\{2\lambda\sqrt{\frac{\beta k}{2j}} - \omega\sqrt{\alpha}\right\}(c^*+c) \\
&+ \left\{\frac{4\lambda}{k}\sqrt{\frac{\alpha k}{2j}}(j-\beta) - \omega_0\sqrt{\beta}\right\}(d^*+d) + \left\{\frac{\lambda}{2k^2}\sqrt{\frac{\alpha\beta k}{2j}}(2k+\beta)\right\}(d^*+d)^2 \\
&+ \left\{\frac{2\lambda}{k}\sqrt{\frac{k}{2j}}(j-\beta)\right\}(c^*+c)(d^*+d) + \left\{\omega_0(\beta-j) + \omega\alpha - \frac{\lambda}{k}\sqrt{\frac{\alpha\beta k}{2j}}(1+4k)\right\}
\end{aligned} \tag{205}
$$

## 5.4 Appendix D

After eliminating the linear terms, the hamiltonian is

$$
\begin{aligned}
H^{(2)} = \omega c^*c &+ \left\{\omega_0 + \frac{2\lambda^2}{\omega}(1-\mu)\right\} d^*d + \left\{\frac{\lambda^2}{2\omega(1+\mu)}(1-\mu)(3+\mu)\right\}(d^*+d)^2 \\
&+ \left\{\lambda\mu\sqrt{\frac{2}{1+\mu}}\right\}(c^*+c)(d^*+d) \\
&+ \left\{-j\left(\omega_0\mu + 2\frac{\lambda^2}{\omega}(1-\mu^2)\right) - \frac{\lambda^2}{\omega}(1-\mu)\right\}
\end{aligned} \tag{206}
$$

Note that

$$
\lambda^2 = \frac{\omega\omega_0}{4\mu} \tag{207}
$$

Substitute Eqn.(207) into the first three terms of Eqn. (206). Then we get

$$
\begin{aligned}
H^{(2)} = \omega c^*c &+ \frac{\omega_0}{2\mu}(1+\mu)d^*d + \frac{\omega_0(1-\mu)(3+\mu)}{8\mu(1+\mu)}(d^*+d)^2 \\
&+ \lambda\mu\sqrt{\frac{2}{1+\mu}}(c^*+c)(d^*+d) - j\left(\omega_0\mu + 2\frac{\lambda^2}{\omega}(1-\mu^2)\right) - \frac{\lambda^2}{\omega}(1-\mu)
\end{aligned} \tag{208}
$$

Consider,

$$\omega_0\mu + 2\frac{\lambda^2}{\omega}(1-\mu^2) = \omega_0\mu + 2\frac{\lambda^2}{\omega} - 2\frac{\lambda^2}{\omega}\mu^2 \tag{209}$$

$$= \omega_0\left(\frac{\omega\omega_0}{4\lambda^2}\right) + 2\frac{\lambda^2}{\omega} - 2\frac{\lambda^2}{\omega}\left(\frac{\omega\omega_0}{4\lambda^2}\right)^2 \tag{210}$$

$$= \frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2} \tag{211}$$

Then, we get that

$$\boxed{\begin{aligned} H^{(2)} &= \omega c^* c + \frac{\omega_0}{2\mu}(1+\mu)d^*d + \frac{\omega_0(1-\mu)(3+\mu)}{8\mu(1+\mu)}(d^*+d)^2 \\ &\quad + \lambda\mu\sqrt{\frac{2}{1+\mu}}(c^*+c)(d^*+d) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned}} \tag{212}$$

## 5.5 Appendix E

Our hamiltonian is

$$\begin{aligned} H^{(2)} &= \frac{\omega^2}{2}(Q_1^2\cos^2\gamma_2 + Q_2^2\sin^2\gamma_2 + 2Q_1Q_2\cos\gamma_2\sin\gamma_2) \\ &\quad + \left\{\frac{\tilde{\omega}^2}{2} + \frac{\omega_0^2(1-\mu)(3+\mu)}{8\mu^2}\right\}(Q_1^2\sin^2\gamma_2 + Q_2^2\cos^2\gamma_2 - 2Q_1Q_2\cos\gamma_2\sin\gamma_2) \\ &\quad + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}(-Q_1^2\cos\gamma_2\sin\gamma_2 + Q_2^2\cos\gamma_2\sin\gamma_2 + Q_1Q_2\cos^2\gamma_2 - Q_1Q_2\sin^2\gamma_2) \\ &\quad + \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned} \tag{213}$$

Note that

$$\frac{\tilde{\omega}^2}{2} + \frac{\omega_0^2(1-\mu)(3+\mu)}{8\mu^2} = \frac{\omega_0^2(1+\mu)^2}{8\mu^2} + \frac{\omega_0^2(1-\mu)(3+\mu)}{8\mu^2} \tag{214}$$

$$= \frac{\omega_0^2}{2\mu^2} \tag{215}$$

Hence,

$$\begin{aligned} H^{(2)} &= \frac{\omega^2}{2}(Q_1^2\cos^2\gamma_2 + Q_2^2\sin^2\gamma_2 + 2Q_1Q_2\cos\gamma_2\sin\gamma_2) \\ &\quad + \frac{\omega_0^2}{2\mu^2}(Q_1^2\sin^2\gamma_2 + Q_2^2\cos^2\gamma_2 - 2Q_1Q_2\cos\gamma_2\sin\gamma_2) \\ &\quad + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}(-Q_1^2\cos\gamma_2\sin\gamma_2 + Q_2^2\cos\gamma_2\sin\gamma_2 + Q_1Q_2\cos^2\gamma_2 - Q_1Q_2\sin^2\gamma_2) \\ &\quad + \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \end{aligned} \tag{216}$$

Then, in order to eliminate the terms with $Q_1Q_2$, we need to solve

$$0 = \omega^2 \cos\gamma_2 \sin\gamma_2 - \frac{\omega_0^2}{\mu^2}\cos\gamma_2\sin\gamma_2 + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}(\cos^2\gamma_2 - \sin^2\gamma_2) \tag{217}$$

$$= \left(\omega^2 - \frac{\omega_0^2}{\mu^2}\right)\frac{1}{2}\sin 2\gamma_2 + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}(1 - 2\sin^2\gamma_2) \tag{218}$$

$$= \left(\omega^2 - \frac{\omega_0^2}{\mu^2}\right)\frac{1}{2}\sin 2\gamma_2 + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\cos 2\gamma_2 \tag{219}$$

Then

$$\left(\omega^2 - \frac{\omega_0^2}{\mu^2}\right)\frac{1}{2}\tan 2\gamma_2 = -2\lambda\sqrt{\mu}\sqrt{\omega\omega_0} \tag{220}$$

$$\Rightarrow \tan 2\gamma_2 = \frac{-4\lambda\sqrt{\mu}\mu^2\sqrt{\omega\omega_0}}{\mu^2\omega^2 - \omega_0^2} \tag{221}$$

$$= \frac{2\sqrt{\omega\omega_0}\mu^2\sqrt{\omega\omega_0}}{\omega_0^2 - \mu^2\omega^2} \tag{222}$$

$$\tag{223}$$

$$\boxed{\therefore \tan 2\gamma_2 = \frac{2\omega\omega_0\mu^2}{\omega_0^2 - \mu^2\omega^2}} \tag{224}$$

## 5.6 Appendix F

$$\begin{aligned}
H^{(2)} = {} & \frac{\omega^2}{2}(Q_1^2\cos^2\gamma_2 + Q_2^2\sin^2\gamma_2) \\
& + \frac{\omega_0^2}{2\mu^2}(Q_1^2\sin^2\gamma_2 + Q_2^2\cos^2\gamma_2) \\
& + 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}(-Q_1^2\cos\gamma_2\sin\gamma_2 + Q_2^2\cos\gamma_2\sin\gamma_2) \\
& + \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu)
\end{aligned} \tag{225}$$

$$\begin{aligned}
= {} & \frac{1}{2}\left\{\omega^2\cos^2\gamma_2 + \frac{\omega_0^2}{\mu^2}\sin^2\gamma_2 - 4\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\cos\gamma_2\sin\gamma_2\right\}Q_1^2 \\
& + \frac{1}{2}\left\{\omega^2\sin^2\gamma_2 + \frac{\omega_0^2}{\mu^2}\cos^2\gamma_2 + 4\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\cos\gamma_2\sin\gamma_2\right\}Q_2^2 \\
& + \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu)
\end{aligned} \tag{226}$$

Consider

$$\begin{aligned}
&\omega^2\cos^2\gamma_2 + \frac{\omega_0^2}{\mu^2}\sin^2\gamma_2 - 4\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\cos\gamma_2\sin\gamma_2 \\
&= \omega^2\cos^2\gamma_2 + \frac{\omega_0^2}{\mu^2}\sin^2\gamma_2 - 2\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\sin 2\gamma_2 && (227)\\
&= \omega^2\left(1+\cos 2\gamma_2\right) + \frac{\omega_0^2}{\mu^2}\left(1-\cos 2\gamma_2\right) - 4\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\left(\frac{2\omega\omega_0\mu^2}{\omega_0^2-\mu^2\omega^2}\right)\cos 2\gamma_2 && (228)\\
&= \left(\omega^2 - \frac{\omega_0^2}{\mu^2} - 8\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\frac{\omega\omega_0\mu^2}{\omega_0^2-\mu^2\omega^2}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2} && (229)\\
&= \left(\omega^2 - \frac{\omega_0^2}{\mu^2} - 4\frac{\omega^2\omega_0^2\mu^2}{\omega_0^2-\mu^2\omega^2}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2} && (230)\\
&= \left(-\frac{(\omega\mu^2-\omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2-\mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2} && (231)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\omega^2\sin^2\gamma_2 + \frac{\omega_0^2}{\mu^2}\cos^2\gamma_2 + 4\lambda\sqrt{\mu}\sqrt{\omega\omega_0}\cos\gamma_2\sin\gamma_2 \\
&= \left(\frac{(\omega\mu^2-\omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2-\mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2} && (232)
\end{aligned}$$

Hence,

$$\boxed{\begin{aligned}
H^{(2)} = \frac{1}{4}&\left\{\left(-\frac{(\omega\mu^2-\omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2-\mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2}\right\}Q_1^2 \\
&\frac{1}{4}\left\{\left(\frac{(\omega\mu^2-\omega_0^2)^2 + (2\omega\omega_0\mu^2)^2}{\mu^2(\omega_0^2-\mu^2\omega^2)}\right)\cos 2\gamma_2 + \omega^2 + \frac{\omega_0^2}{\mu^2}\right\}Q_2^2 \\
&+ \frac{1}{2}(P_1^2 + P_2^2 - \omega - \tilde{\omega}) - j\left\{\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right\} - \frac{\lambda^2}{\omega}(1-\mu) \qquad (233)
\end{aligned}}$$

## 5.7 Appendix G

Note that our hamiltonian is

$$H_{RWA}^{(1)} = \omega_0 b^* b + \omega a^* a + \lambda(a^* b + a b^*) - \omega_0 j \qquad (234)$$

And that we are making the transformation

$$a \to -c_1\sin\beta + c_2\cos\beta, \quad b \to c_1\cos\beta + c_2\sin\beta \qquad (235)$$

Note that by plugging in the above transformations, we get that

$$b^*b = c_1^*c_1\cos^2\beta + c_2^*c_2\sin^2\beta + (c_1^*c_2 + c_2^*c_1)\cos\beta\sin\beta \tag{236}$$
$$a^*a = c_1^*c_1\sin^2\beta + c_2^*c_2\cos^2\beta + (-c_1^*c_2 - c_2^*c_1)\cos\beta\sin\beta \tag{237}$$
$$a^*b = -c_1^*c_1\cos\beta\sin\beta + c_2^*c_2\cos\beta\sin\beta - c_1^*c_2\sin^2\beta + c_2^*c_1\cos^2\beta \tag{238}$$
$$ab^* = -c_1^*c_1\cos\beta\sin\beta + c_2^*c_2\cos\beta\sin\beta + c_1^*c_2\cos^2\beta - c_2^*c_1\sin^2\beta \tag{239}$$

We would like to get rid of the terms that are not of the forms $c_1^*c_1$ and $c_2^*c_2$.

$$(\omega_0 - \omega)(c_1^*c_2 + c_2^*c_1)\cos\beta\sin\beta + \lambda((c_1^*c_2 + c_1c_2^*)(\cos^2\beta - \sin^2\beta)) = 0 \tag{240}$$
$$\Rightarrow (\omega_0 - \omega)\cos\beta\sin\beta + \lambda(\cos^2\beta - \sin^2\beta) = 0 \tag{241}$$
$$\Rightarrow (\omega_0 - \omega)\frac{1}{2}\sin 2\beta + \lambda\cos 2\beta = 0 \tag{242}$$
$$\tag{243}$$

$$\boxed{\therefore \tan 2\beta = \frac{2\lambda}{\omega - \omega_0}} \tag{244}$$

Hence

$$\begin{aligned} H_{RWA}^{(1)} = {} & \omega_0\left\{c_1^*c_1\cos^2\beta + c_2^*c_2\sin^2\beta\right\} + \omega\left\{c_1^*c_1\sin^2\beta + c_2^*c_2\cos^2\beta\right\} \\ & + 2\lambda\cos\beta\sin\beta\left\{-c_1^*c_1 + c_2^*c_2\right\} \\ & - j\omega_0 \end{aligned} \tag{245}$$
$$\begin{aligned} = {} & \omega_0\left\{c_1^*c_1\cos^2\beta + c_2^*c_2\sin^2\beta\right\} + \omega\left\{c_1^*c_1\sin^2\beta + c_2^*c_2\cos^2\beta\right\} \\ & + \lambda\sin 2\beta\left\{-c_1^*c_1 + c_2^*c_2\right\} \\ & - j\omega_0 \end{aligned} \tag{246}$$
$$\begin{aligned} = {} & c_1^*c_1(\omega_0\cos^2\beta + \omega\sin^2\beta - \lambda\sin 2\beta) \\ & + c_2^*c_2(\omega_0\sin^2\beta + \omega\cos^2\beta + \lambda\sin 2\beta) - j\omega_0 \end{aligned} \tag{247}$$

Now consider

$$\omega_0\cos^2\beta + \omega\sin^2\beta - \lambda\sin 2\beta = \omega_0\left(\frac{1+\cos 2\beta}{2}\right) + \omega\left(\frac{1-\cos 2\beta}{2}\right) - \lambda\sin 2\beta \tag{248}$$
$$= \frac{\omega_0}{2} + \frac{\omega}{2} + \frac{\cos 2\beta}{2}(\omega_0 - \omega) - \lambda\frac{2\lambda}{\omega\omega_0}\cos 2\beta \tag{249}$$
$$= \frac{1}{2}\left\{\omega_0 + \omega - \cos 2\beta\left(\frac{(\omega-\omega_0)^2 + 4\lambda^2}{\omega - \omega_0}\right)\right\} \tag{250}$$

Note that

$$\cos 2\beta = \frac{1}{\sqrt{1+\tan^2 2\beta}} = \frac{\omega - \omega_0}{\sqrt{(\omega-\omega_0)^2 + 4\lambda^2}} \tag{251}$$

By substituting Eqn.(251) into Eqn.(250), we get

$$\frac{1}{2}\left\{\omega_0+\omega-\sqrt{(\omega-\omega_0)^2+4\lambda^2}\right\} \tag{252}$$

Similarly,

$$\omega_0\sin^2\beta+\omega\cos^2\beta+\lambda\sin 2\beta=\frac{1}{2}\left\{\omega_0+\omega+\sqrt{(\omega-\omega_0)^2+4\lambda^2}\right\} \tag{253}$$

Hence

$$\boxed{H_{RWA}^{(1)}=\varepsilon_- c_1^* c_1+\varepsilon_+ c_2^* c_2-j\omega_0} \tag{254}$$

, where

$$\varepsilon_\pm=\frac{1}{2}\left\{\omega_0+\omega\pm\sqrt{(\omega-\omega_0)^2+4\lambda^2}\right\} \tag{255}$$

### 5.8 Appendix H

Here we will investigate how to get a matrix representation of $H_N(S,C)$ restricted to $\mathcal{H}_N(S,C)$. Let $|S,C,M\rangle_N=|M\rangle$ and $H_N(S,C)\to A$ where $A$ is matrix representation. Recall that

$$\langle M'|\,H_N(S,C)\,|M\rangle \tag{256}$$

gives a matrix element at $A_{M'M}$. Consider $\langle M|\,H_N(S,C)\,|M\rangle$. Rewrite

$$\begin{aligned} H_N(S,C) &= a^*a+\sum_{i=1}^{N}\left(\frac{\epsilon}{2}\sigma_z(i)+\frac{\lambda}{\sqrt{N}}\left\{S^+(i)a+S^-(i)a^*\right\}\right) && (257)\\ &= a^*a+\epsilon J_z+\frac{\lambda}{\sqrt{N}}\left\{J_+a+J_-a^*\right\} && (258)\end{aligned}$$

, where $J_z\,|M\rangle=M\,|M\rangle$, and $J_\pm\,|M\rangle=\sqrt{S(S+1)-M(M\pm 1)}\,|M\pm 1\rangle$. Then since

$$\begin{aligned} H_N(S,C)\,|M\rangle &= \left(a^*a+\epsilon J_z+\frac{\lambda}{\sqrt{N}}\{J_+a+J_-a^*\}\right)|M\rangle && (259)\\ &= (C+(\epsilon-1)M)a\,|M\rangle+v\,|M+1\rangle+w\,|M-1\rangle && (260)\end{aligned}$$

, where $v,w$ are some number. Then

$$\boxed{\begin{aligned} \langle M|\,H_N(S,C)\,|M\rangle &= \langle M|\left\{(C+(\epsilon-1)M)a\,|M\rangle+v\,|M+1\rangle+wa^*\,|M-1\rangle\right\} && (261)\\ &= C+(\epsilon-1)M && (262)\end{aligned}}$$

Also, consider $\langle M| H_N(S,C) |M+1\rangle$. Observe taht

$$H_N(S,C) |M+1\rangle = v |M+1\rangle + w |M+2\rangle + \sqrt{(C-M)(S(S+1)-M(M+1))} |M\rangle \tag{263}$$

, where $v, w$ are some number. Then

$$\boxed{\langle M| H_N(S,C) |M+1\rangle = \sqrt{(C-M)(S(S+1)-M(M+1))}} \tag{264}$$

Since $H_N(S,C)$ is a hermitian operator, we have that $\langle M| H_N(S,C) |M+1\rangle = \langle M+1| H_N(S,C) |M\rangle$. It is trivial to check that $\langle M'| H_N(S,C) |M\rangle = 0$ for $|M-M'| > 1$.