Estimation

Joshua Jun Hwan Ryu; jhryu25@stanford.edu

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1 Generative model

We are interested in the following variables:

- x_t := external state of the world (x,y position, velocity and force for stimulus and tracker)
- y_t := (latent) obsever'snoisy measurements of the state (position of the stimulus and tracker)
- z_t := (latent) obsever's perceptual estimation of the state (position estimates)
- u_t := observer's exerted force

We can describe the using the following dynamical system:

$$
\begin{array}{ll}
\text{World} & \boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + B\boldsymbol{u}_t + \boldsymbol{\xi}_t \\
\text{Measurement} & \boldsymbol{y}_t = H\boldsymbol{x}_t + \boldsymbol{\omega}_t \\
\text{Perception} & \boldsymbol{z}_{t+1} = A\boldsymbol{z}_t - BL(t,\theta,\phi)z_t + K(t+1,\theta)(y_t - H\boldsymbol{z}_t) + \epsilon_t \\
\text{Control} & \boldsymbol{u}_t = -L(t,\theta,\phi)z_t + \boldsymbol{\varepsilon}_t
\end{array}
$$

The noise terms are all stationary, independent and distributed as follows:

- Dynamics noise (determined by experimenter) $\boldsymbol{\xi}_t \sim \mathcal{N}(0, \Omega_x)$
- Measurement noise $\boldsymbol{\omega}_t \sim \mathcal{N}(0, \Omega_y)$
- Perceptual estimation noise $\boldsymbol{z}_t \sim \mathcal{N}(0, \Omega_z)$
- Control noise $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, \Omega_u)$

The dynamics are determined by:

- Dynamics: A, B determine the stimulus dynamics and control gain
- Measurment: H determine the sensory gain
- Perception: Perception is determined by the dynamics A, B and the Kalman filter $K(t, \theta)$. This is the optimal linear filter for perception, given that that observer has full access to their intended control and the external dynamics.
- Control: Control is determined by the optimal linear filter $L(t, \phi)$ from the linear-quadratic-gaussian (LQG) controller. The controller minimizes an expected tracking error subject to control costs.

We can convert the dynamical systems to conditional distributions on the variables of interest:

$$
\begin{aligned}
p(u_t \mid z_t; \theta, \phi) &= \mathcal{N}(u_t| - L(t, \theta, \phi) z_t, \Omega_u) \\
p(\boldsymbol{x}_{t+1} \mid \boldsymbol{z}_t, \boldsymbol{x}_t; \theta, \phi) &= \int_{u_t} p(\boldsymbol{x}_t \mid u_t, \boldsymbol{z}_t, \boldsymbol{x}_t; \theta, \phi) p(u_t \mid \boldsymbol{z}_t; \theta, \phi) \\
&\sim \int_{u_t} \mathcal{N}(x_{t+1}|Ax_t + Bu_t, \Omega_x) \mathcal{N}(u_t| - L(t, \theta, \phi) z_t, \Omega_u) \\
&\sim \mathcal{N}(x_{t+1}|Ax_t - BL(t, \theta, \phi) z_t, \Omega_x + B\Omega_u B^T) \\
p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}; \boldsymbol{x}_t, \theta, \phi) &= \int_{y_t} p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}, \boldsymbol{y}_t; \boldsymbol{x}_t, \theta, \phi) p(\boldsymbol{y}_t; \boldsymbol{x}_t, \theta, \phi) \\
&\sim \int_{y_t} \mathcal{N}\left(z_t \mid Az_{t-1} - BL(t-1, \theta, \phi) z_{t-1} + K(t, \theta)(y_t - H\boldsymbol{z}_{t-1}), \Omega_z\right) \mathcal{N}(y_t \mid Hx_t, \Omega_y) \\
&\sim \mathcal{N}\left(Az_{t-1} - BL(t-1, \theta, \phi) z_{t-1} + K(t, \theta) H(\boldsymbol{x}_t - \boldsymbol{z}_{t-1}), \Omega_z + K(t, \theta)\Omega_y K(t, \theta)^T\right) \\
&\triangleq \mathcal{N}\left(a_{z_{t|t-1}} z_{t-1} + b_{z_{t|t-1}}, \Omega_{z_{t|t-1}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{z_{t|t-1}} &:= A - BL(t-1, \theta, \phi) - K(t, \theta) H \\
b_{z_{t|t-1}} &:= K(t, \theta) H\boldsymbol{x}_t \\
\Omega_{z_{t|t-1}} &:= \Omega_z + K(t, \theta)\Omega_y K(t, \theta)^T
\end{aligned}
$$

Assume other prior distributions:

$$
\begin{aligned}
p(z_0; \theta) &\triangleq \mathcal{N}(0, \Omega_{z_0}) \\
p(\theta) &\triangleq \mathcal{N}(someval, largeval\Omega_\theta) \\
p(\phi) &\triangleq \mathcal{N}(someval, largeval\Omega_\phi)
\end{aligned}
$$

2 Inference

We are interested in finding the parameters that maximize the joint probability of seeing the obsever control with the parameters $\arg\max_{\{\theta,\phi\}} \log P(\{\boldsymbol{u}_t\}, \theta, \phi|\{x_t\})$. In order to do the inference, we need to marginalize the latent variables and then optimize the objective. I adopt the EM algorithm for this, and alternate between approximating the marginal distribution on the latent variables (E-step) and optimizing the evidence lower bound (elbo):

$$
\begin{aligned}
\log P(\{\boldsymbol{u}_t\}, \theta, \phi) &= \log P(\{\boldsymbol{u}_t\}|\theta, \phi) + \log P(\theta, \phi) \\
&\geq \mathbb{E}_{q(z)} \left[\log P(\{\boldsymbol{u}_t, z_t\}|\theta, \phi) + \log q(\{z_t\})\right] + \log P(\theta, \phi) \\
&\triangleq L(q, \theta, \phi)
\end{aligned}
$$

(I omit the dependence on the stimulus x_t, and write them only as needed.)

The lower bound is tight when the variational distribution on the latents $q(z_{1:T})$ is equal to the posterior. The posterior on the latents is intractable. However, for the E-step, it suffices to calculate the latent marginals needed for the M-step. Since the joint distribution given the parameters is:

$$
P(\{z_t\}, \{\boldsymbol{u}_t\}) := P(\{z_t\}, \{\boldsymbol{u}_t\}|\theta, \phi, \{x_t\})
$$

$$= P(z_0)\prod_{t=0}^{T-1} P(z_{t+1} \mid z_t)\prod_{t=0}^{T} P(u_t \mid z_t)$$

The maximization of the elbo over the parameters amounts to:

$$\begin{aligned}
\arg\max_{\theta,\phi} L(q,\theta,\phi) &= \arg\max_{\theta,\phi} \left[\mathbb{E}_{q(z_{1:T})}\left[\log P(\{\boldsymbol{u}_t, z_t\}|\theta,\phi) + \log q(\{z_t\})\right] + \log P(\theta,\phi)\right] && (2.1)\\
&= \arg\max_{\theta,\phi} \left[\mathbb{E}_{q(z_{1:T})}\left[\log P(\{\boldsymbol{u}_t, z_t\}|\theta,\phi)\right] + \log P(\theta,\phi)\right] && (2.2)\\
&= \arg\max_{\theta,\phi} \Bigg[\mathbb{E}_{q(z_{1:T})}\left[\log P(z_0) + \sum_{t=0}^{T-1}\log P(z_{t+1} \mid z_t) + \sum_{t=0}^{T}\log P(u_t \mid z_t)\right] && (2.3)\\
&\qquad + \log P(\theta,\phi)\Bigg] && (2.4)\\
&= \arg\max_{\theta,\phi} \Bigg[\mathbb{E}_{q(z_0)}\log P(z_0) + \sum_{t=0}^{T-1}\mathbb{E}_{q(z_{t+1},z_t)}\log P(z_{t+1} \mid z_t) && (2.5)\\
&\qquad + \sum_{t=0}^{T}\mathbb{E}_{q(z_t)}\log P(u_t \mid z_t) + \log P(\theta,\phi)\Bigg] && (2.6)
\end{aligned}$$

where in the last step, we have used linearity of expectations, and take expectation over the uncaptured variables.

Thus, in the E-step we need to update the marginals and the two-slice marginals:

$$\begin{aligned}
q(z_t) &:= p(z_t|u_{1:T},\theta,\phi)\\
q(z_t, z_{t+1}) &:= p(z_t, z_{t+1}|u_{1:T},\theta,\phi)
\end{aligned}$$

2.1 E-step

The marginals $p(z_t|\{u_t\})$ are calculated using an analog of forward-backward algorithm for state-space models [2, 3].

Then the distributions of interest are given by:

$$\begin{aligned}
p(z_t|u_{1:T}) &= \int_{z_{t+1}} p(z_t|z_{t+1},u_{1:T})p(z_{t+1}|u_{1:T}) && (2.7)\\
&= \int_{z_{t+1}} p(z_t|z_{t+1},u_{1:t},\cancel{u_{t+1:T}})p(z_{t+1}|u_{1:T}) && (2.8)\\
p(z_t,z_{t+1}|u_{1:T}) &= p(z_t|z_{t+1},u_{1:T})p(z_{t+1}|u_{1:T}) && (2.9)\\
&= p(z_t|z_{t+1},u_{1:t},\cancel{u_{t+1:T}})p(z_{t+1}|u_{1:T}) && (2.10)
\end{aligned}$$

The distributions of interest are Gaussian and $\forall s,t$we defined them as:

- $p(z_s|u_{1:t}) \sim \mathcal{N}(\mu_{s|t},\Sigma_{s|t})$.
- $p(z_s, z_{s+1}|u_{1:t}) \sim \mathcal{N}\left(\mu_{s,s+1|t},\Sigma_{s,s+1|t}\right)$.

$p(z_t|z_{t+1}, u_{1:t})$ is found by computing $p(z_t|u_{1:t})$ and $p(z_{t+1}|u_{1:t})$ in the forward step, then taking the Gaussian conditional of their joint distribution:

$$p(z_t, z_{t+1}|u_{1:t}) = \mathcal{N}\left(\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix} \mid \begin{bmatrix} \mu_{t|t} \\ \mu_{t+1|t} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|t} & \Sigma_{t|t} a^T_{z_{t+1|t}} \\ a_{z_{t+1|t}} \Sigma_{t|t} & \Sigma_{t+1|t} \end{bmatrix}\right) \tag{2.11}$$

$$p(z_t|z_{t+1}, u_{1:t}) = \mathcal{N}(\mu_{t|t} + J_t(z_{t+1} - \mu_{t+1|t}), \Sigma_{t|t} - J_t \Sigma_{t+1|t} J_t^T) \tag{2.12}$$

$$J_t \triangleq \Sigma_{t|t} a^T_{z_{t|t-1}} \Sigma^{-1}_{t+1|t} \tag{2.13}$$

$p(z_{t+1}|u_{1:T})$ can be computed recursively in the backward step. And similarly,

$$p(z_s, z_{s+1}|u_{1:T}) = \mathcal{N}\left(\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix} \mid \begin{bmatrix} \mu_{t|T} \\ \mu_{t+1|T} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|T} & \Sigma_{t|T} a^T_{z_{t+1|t}} \\ a_{z_{t+1|t}} \Sigma_{t|T} & \Sigma_{t+1|T} \end{bmatrix}\right)$$

2.1.1 Forward step: Kalman filtering

Forward step involves calculating the filtered marginal probability of the current latent variable given the current and past observations $p(z_t|u_{1:t})$. While I do not derive Kalman filter here, I outline them in the appendix. Details can also be found in [2] and [1].

As an inductive assumption, assume that $p(z_{t-1}|x_{1:t}) \sim \mathcal{N}(\mu_{t-1|t-1}, \Sigma_{t-1|t-1})$. Then, the distributions can be computed recursively as:

$$\begin{aligned}
p(z_t|x_{1:t}) &\triangleq \mathcal{N}\left(\mu_{t|t-1}, \Sigma_{t|t-1}\right) \\
\mu_{t|t-1} &\triangleq a_{z_{t|t-1}} \mu_{t-1|t-1} + b_{z_{t|t-1}} \\
\Sigma_{t|t-1} &\triangleq a_{z_{t|t-1}} \Sigma_{t-1|t-1} a^T_{z_{t|t-1}} + \Omega_{z_{t|t-1}} \\
p(z_t|x_{1:t+1}) &\triangleq \mathcal{N}\left(\mu_{t|t}, \Sigma_{t|t}\right) \\
\mu_{t|t} &\triangleq \mu_{t|t-1} + R_t r_t \\
\Sigma_{t|t} &\triangleq (I + R_t(-BL_t))\, \Sigma_{t|t-1} \\
R_t &\triangleq \Sigma_{t|t-1}(-BL_t)^T \left(BL_t \Sigma_{t|t-1}(BL_t)^T + \Omega_x + B\Omega_u B^T\right)^{-1} && \text{(Kalman filter)} \\
r_t &\triangleq x_{t+1} - (Ax_t - BL_t \mu_{t|t-1}) && \text{(innovation)}
\end{aligned}$$

note to myself: Numerical stability: Thrun et al. 2006; Simon 2006. take care of 0 probabilities and other small probabilities properly (some of these matrices are sparse)

2.1.2 Backward step: Kalman smoothing

The backward step involves calculating the smoothed marginals $p(z_t|u_{1:T})$. Again, I omit the derivations and just outline the results here. The results follow from equations 2.8 and 2.13. Details can also be found in references [2] and [1].

As an inductive hypothesis, assume that $p(z_{t+1}|x_{1:T}) \sim \mathcal{N}(\mu_{t+1|T}, \Sigma_{t+1|T})$. Then, the smoothed marginals can be computed recursively as:

$$p(z_t|x_{1:T}) \triangleq \mathcal{N}(\mu_{t|T}, \Sigma_{t|T})$$

$$
\begin{aligned}
\mu_{t|T} &\triangleq \mu_{t|t} + J_t \left(\mu_{t+1|T} - \mu_{t+1|t}\right) \\
\Sigma_{t|T} &\triangleq \Sigma_{t|t} + J_t \left(\Sigma_{t+1|T} - \Sigma_{t+1|t}\right) J_t^T \\
J_t &\triangleq \Sigma_{t|t} a_{z_{t|t-1}}^T \Sigma_{t+1|t}^{-1}
\end{aligned}
$$

2.2 M-step

In the M-step, we want to calculate the the expected ELBO over the calculated marginals, as in equation 2.6 and then maximize it with respect to the parameters. Similar calculations can be found in [3].

$$
\begin{aligned}
\arg\max_{\theta,\phi} L(q,\theta,\phi) = \arg\max_{\theta,\phi} \Bigg[& \mathbb{E}_{q(z_0)} \log P(z_0) + \sum_{t=0}^{T-1} \mathbb{E}_{q(z_{t+1},z_t)} \log P(z_{t+1} \mid z_t) \\
& + \sum_{t=0}^{T} \mathbb{E}_{q(z_t)} \log P(u_t \mid z_t) + \log P(\theta,\phi) \Bigg]
\end{aligned}
$$

Calculating each summand and ignoring constants

$$
\begin{aligned}
\mathbb{E}_{q(z_0)} \log P(z_0) &\propto -\frac{1}{2} \log |\Omega_{z_0}| - \mathbb{E}_{q(z_0)} \frac{1}{2} z_0^T \Omega_{z_0}^{-1} z_0 \\
&= -\frac{1}{2} \log |\Omega_{z_0}| - \mathbb{E}_{q(z_0)} \frac{1}{2} \mathrm{Tr}(\Omega_{z_0}^{-1} z_0 z_0^T) \\
&= -\frac{1}{2} \log |\Omega_{z_0}| - \frac{1}{2} \mathrm{Tr}(\Omega_{z_0}^{-1} \mathbb{E}_{q(z_0)} (z_0 z_0^T)) \\
&\qquad \text{by linearity of trace} \\
&= -\frac{1}{2} \log |\Omega_{z_0}| - \frac{1}{2} \mathrm{Tr} \left(\Omega_{z_0}^{-1} \left(\Sigma_{0|T} + \mu_{0|T} \mu_{0|T}^T \right) \right)
\end{aligned}
$$

The state transition terms are computed as follows:

$$
\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E}_{q(z_{t+1},z_t)} \log P(z_{t+1} \mid z_t) &\propto -\frac{1}{2} \sum_{t=0}^{T-1} \left[\log |\Omega_{z_{t+1|t}}| + \mathrm{Tr} \left(\Omega_{z_{t+1|t}}^{-1} \mathbb{E}_{q(z_{t+1},z_t)} Quad \right) \right] \\
&= -\frac{1}{2} \sum_{t=0}^{T-1} \Bigg[\log |\Omega_{z_{t+1|t}}| + \mathrm{Tr} \Bigg(\Omega_{z_{t+1|t}}^{-1} \Bigg(P_{t+1} \\
&\qquad - a_{z_{t+1|t}} P_{t+1,t}^T - P_{t+1,t} a_{z_{t+1|t}}^T + a_{z_{t+1|t}} P_t a_{z_{t+1|t}}^T \\
&\qquad - \mu_{t+1|T} b_{z_{t+1|t}}^T - b_{z_{t+1|t}} \mu_{t+1|T}^T + a_{z_{t+1|t}} \mu_{t|T} b_{z_{t+1|t}}^T + b_{z_{t+1|t}} \mu_{t|T}^T a_{z_{t+1|t}}^T \\
&\qquad + b_{z_{t+1|t}} b_{z_{t+1|t}}^T \Bigg) \Bigg) \Bigg] \\
P_{t+1} &\triangleq \Sigma_{t+1|T} + \mu_{t+1|T} \mu_{t+1|T}^T \\
P_t &\triangleq \Sigma_{t|T} + \mu_{t|T} \mu_{t|T}^T \\
P_{t+1,t} &\triangleq (a_{z_{t+1|t}} \Sigma_{t|T} + \mu_{t+1|T} \mu_{t|T}^T) \\
P_{t,t+1} &\triangleq \Sigma_{t|T} a_{z_{t+1|t}}^T + \mu_{t|T} \mu_{t+1|T}^T = P_{t+1,t}^T
\end{aligned}
$$

Note that the joint expectations marginalizes out except the covariances, which we know from the joint distribution. Explicitly, the quadratic form is:

$$\begin{aligned}
Quad &= (z_{t+1} - a_{z_{t+1|t}} z_t - b_{z_{t+1|t}})(z_{t+1} - a_{z_{t+1|t}} z_t - b_{z_{t+1|t}})^T \\
&= z_{t+1} z_{t+1}^T + a_{z_{t+1|t}} z_t z_t^T a_{z_{t+1|t}}^T + b_{z_{t+1|t}} b_{z_{t+1|t}}^T \\
&\quad - z_{t+1} z_t^T a_{z_{t+1|t}}^T - z_{t+1} b_{z_{t+1|t}}^T - a_{z_{t+1|t}} z_t z_{t+1}^T - b_{z_{t+1|t}} z_{t+1}^T \\
&\quad + a_{z_{t+1|t}} z_t b_{z_{t+1|t}}^T + b_{z_{t+1|t}} z_t^T a_{z_{t+1|t}}^T \\
\mathbb{E}_{q(z_{t+1}, z_t)} Quad &= \mathbb{E}_{q(z_{t+1}, z_t)} (z_{t+1} - a_{z_{t+1|t}} z_t - b_{z_{t+1|t}})(z_{t+1} - a_{z_{t+1|t}} z_t - b_{z_{t+1|t}})^T \\
&= \mathbb{E}_{q(z_{t+1})} z_{t+1} z_{t+1}^T + \mathbb{E}_{q(z_t)} a_{z_{t+1|t}} z_t z_t^T a_{z_{t+1|t}}^T + b_{z_{t+1|t}} b_{z_{t+1|t}}^T \\
&\quad - \mathbb{E}_{q(z_{t+1}, z_t)} z_{t+1} z_t^T a_{z_{t+1|t}}^T - \mathbb{E}_{q(z_{t+1})} z_{t+1} b_{z_{t+1|t}}^T - \mathbb{E}_{q(z_{t+1}, z_t)} a_{z_{t+1|t}} z_t z_{t+1}^T - b_{z_{t+1|t}} z_{t+1}^T \\
&\quad + \mathbb{E}_{q(z_t)} a_{z_{t+1|t}} z_t b_{z_{t+1|t}}^T + \mathbb{E}_{q(z_t)} b_{z_{t+1|t}} z_t^T a_{z_{t+1|t}}^T \\
&= (\Sigma_{t+1|T} + \mu_{t+1|T} \mu_{t+1|T}^T) + a_{z_{t+1|t}} (\Sigma_{t|T} + \mu_{t|T} \mu_{t|T}^T) a_{z_{t+1|t}}^T \\
&\quad - (a_{z_{t+1|t}} \Sigma_{t|T} + \mu_{t+1|T} \mu_{t|T}^T) a_{z_{t+1|t}}^T - a_{z_{t+1|t}} (\Sigma_{t|T} a_{z_{t+1|t}}^T + \mu_{t|T} \mu_{t+1|T}^T) - \mu_{t+1|T} b_{z_{t+1|t}}^T \\
&\quad - b_{z_{t+1|t}} \mu_{t+1|T}^T + a_{z_{t+1|t}} \mu_{t|T} b_{z_{t+1|t}}^T + b_{z_{t+1|t}} \mu_{t|T}^T a_{z_{t+1|t}}^T + b_{z_{t+1|t}} b_{z_{t+1|t}}^T \\
&= (\mu_{t+1} - a_{z_{t+1|t}} \mu_t - b_{z_{t+1|t}})(\mu_{t+1} - a_{z_{t+1|t}} \mu_t - b_{z_{t+1|t}})^T + \Sigma_{t+1|T} - a_{z_{t+1|t}} \Sigma_{t|T} a_{z_{t+1|t}}^T \\
&= P_{t+1} + a_{z_{t+1|t}} P_t a_{z_{t+1|t}}^T - a_{z_{t+1|t}} P_{t,t+1} - P_{t,t+1}^T a_{z_{t+1|t}}^T \\
&\quad - \mu_{t+1|T} b_{z_{t+1|t}}^T - b_{z_{t+1|t}} \mu_{t+1|T}^T + a_{z_{t+1|t}} \mu_{t|T} b_{z_{t+1|t}}^T + b_{z_{t+1|t}} \mu_{t|T}^T a_{z_{t+1|t}}^T \\
&\quad + b_{z_{t+1|t}} b_{z_{t+1|t}}^T
\end{aligned}$$

Lastly, the likelihoods terms are summed as follows:

$$\begin{aligned}
\sum_{t=0}^{T} \mathbb{E}_{q(z_t)} \log P(u_t \mid z_t) &\propto -\frac{1}{2} \sum_{t=0}^{T} \left[\log |\Omega_\varepsilon| + \mathrm{Tr} \left(\Omega_\varepsilon^{-1} \mathbb{E}_{q(z_t)} (u_t + L(t, \theta, \phi) z_t)(u_t + L(t, \theta, \phi) z_t)^T \right) \right] \\
&= -\frac{1}{2} \sum_{t=0}^{T} \Bigg[\log |\Omega_\varepsilon| + \mathrm{Tr} \Bigg(\Omega_\varepsilon^{-1} (u_t + L(t, \theta, \phi) \mu_{t|T})(u_t + L(t, \theta, \phi) \mu_{t|T})^T \\
&\qquad + L(t, \theta, \phi) \Sigma_{t|T} L(t, \theta, \phi)^T \Bigg) \Bigg]
\end{aligned}$$

Assuming that $K(t, \theta)$ and $L(t, \theta, \phi)$ are differentiable functions of the parameters $\{\theta, \phi\}$, we can then use gradient ascent on this objective function to calculate the optimal parameters.

3 Model comparison

1. Non-Kalman observer. We don't take into account probabilistic view
2. Nonadaptive Estimator
3. Adaptive estimator
4. Noise structures

4 Appendix: Kalman Filtering and Smoothing

4.1 Kalman Filter

Note that we bundle u_t with x_{t+1} since z_t depends on both the target and the control. We can also reformulate these equations in terms of the target and the cursor separately, and condition all the probabilities on the target. This will create a separate estimation equations for the target and the control. Otherwise, the target variance will be underestimated.

$$\begin{aligned}
p(z_t|u_{1:t}, x_{1:t+1}) &\propto p(z_t, u_{1:t}, x_{1:t+1}) \\
&= \int_{z_{t-1}} p(z_t, z_{t-1} u_{1:t-1}, u_t, x_{1:t}, x_{t+1}) \\
&= \int_{z_{t-1}} p(u_t, x_{t+1}|z_t, z_{t-1}, u_{1:t-1}, x_{1:t}) p(z_t|z_{t-1}, u_{1:t-1}, x_{1:t}) p(z_{t-1}|u_{1:t-1}, x_{1:t}) p(u_{1:t-1}, x_{1:t}) \\
&\propto p(x_{t+1}|u_t, x_t) p(u_t|z_t) \int_{z_{t-1}} p(z_t|z_{t-1}, u_{1:t-1}, x_{1:t}) \alpha_{t-1}(z_{t-1}) \\
&\propto p(u_t|z_t) \int_{z_{t-1}} p(z_t|z_{t-1}) \alpha_{t-1}(z_{t-1}) \\
&= \mathcal{N}\left(u_t \mid -L(t,\theta,\phi) z_t, \Omega_\varepsilon\right) \int_{z_t} \mathcal{N}\left(z_{t-1}|a_{z_t} z_{t-1} + b_{z_t}, \Omega_{z_{t|t-1}}\right) N(z_{t-1}|\mu_{t-1}, \Sigma_{t-1}) \\
&= \mathcal{N}\left(u_t \mid -L(t,\theta,\phi) z_t, \Omega_\varepsilon\right) \mathcal{N}\left(z_t|a_{z_t}\mu_{t-1} + b_{z_t}, \Omega_{z_{t|t-1}} + a_{z_t}\Sigma_{t-1} a_{z_t}^T\right) \\
&= \mathcal{N}\left(z_t|\mu_t, \Sigma_t\right)
\end{aligned}$$

where:

$$\begin{aligned}
\Sigma_t &= \left[\left(\Omega_{z_{t|t-1}} + a_{z_t}\Sigma_{t-1} a_{z_t}^T\right)^{-1} + L_t^T \Omega_\varepsilon^{-1} L_t\right]^{-1} \\
\mu_t &= \Sigma_t \left[-L_t^T \Omega_\varepsilon^{-1} u_t + \left[\Omega_{z_{t|t-1}} + a_{z_t}\Sigma_{t-1} a_{z_t}^T\right]^{-1} (a_{z_t}\mu_{t-1} + b_{z_t})\right]
\end{aligned}$$

4.2 Kalman Smoother

Unlike the foward backward algorithm in hidden markov models that computes $p(x_{t+1:T}|z_t)$ in the backwards step, in the backwards step, it is more common and more computationally tractable to compute $p(z_t|x_{t+1:T})$ directly [2].

References

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