Data Analysis. Group Assignment 1

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1 Lemma 1.14.

Given a probability triple $(\Omega, \mathfrak{F}, P)$ then for $A \in \mathfrak{F}$ with $P(A) \neq 0$,

$$P(\cdot|A) : \mathfrak{F} \to [0, 1]$$

is a probability measure as in Definition 1.10 over $(\Omega,\mathfrak{F})$.

Proof:
We need to check two properties of the probability measure:

1) $P(\Omega) = 1$.
According to the definition of conditional probability:

$$P(\Omega|A) = \frac{P(A \cap \Omega)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

2) Let $B, C \in \mathfrak{F}$ and $B \cap C = \emptyset$. Again, according to the definition of conditional probability:

$$P(B \cup C|A) = \frac{P((B \cup C) \cap A)}{P(A)} = \frac{P((B \cap A) \cup (C \cap A))}{P(A)} = \frac{P(B \cap A) + P(C \cap A)}{P(A)} =$$

$$= \frac{P(B \cap A)}{P(A)} + \frac{P(C \cap A)}{P(A)} = P(B|A) + P(C|A).$$

2 Exercise 2.8.

Given a probability triple (Ω, F, P) and an event $A \in F$ the following properties hold:

1.

$\mathbb{1}_A = 1 - \mathbb{1}_{A^c}$, (complementation behaves like the probability)

Proof:
Let's say w denote an event.

(1) If $w \in A$:

$$\mathbb{1}_A = 1 = 1 - 0 = 1 - \mathbb{1}_{A^c}, \text{ since } \mathbb{1}_{A^c} = 0$$

(2) If $w \notin A$:

$$\mathbb{1}_A = 0 = 1 - 1 = 1 - \mathbb{1}_{A^c} \text{ , since } \mathbb{1}_{A^c} = 1$$

Hence the proof is complete.

2.

$$\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$$

Proof:
Let w denote an event.

(1) If $\mathbb{1}_{A \cap B} = 1$:

$\Rightarrow w \in (A \cap B)$
$\because w \in (A \cap B)$
$\therefore w \in A, w \in B$
$\Rightarrow \mathbb{1}_A = \mathbb{1}_B = 1$
$\Rightarrow \mathbb{1}_{A \cap B} = 1 = 1 \cdot 1 = \mathbb{1}_A \mathbb{1}_B$

(2) If $\mathbb{1}_{A \cap B} = 0$:

$\Rightarrow w \in (A \cap B)^c$
$\because w \in (A \cap B)^c$
$\therefore$ w can either belongs to $(A \cap B^c), (B \cap A^c)$ or $(A \cup B)^c$

a. $w \in (A \cap B^c)$:

$\Rightarrow \mathbb{1}_A = 1, \mathbb{1}_B = 0$
$\Rightarrow \mathbb{1}_{A \cap B} = 0 = 1 \cdot 0 = \mathbb{1}_A \mathbb{1}_B$

b. $w \in (B \cap A^c)$:

$\Rightarrow \mathbb{1}_A = 0, \mathbb{1}_B = 1$
$\Rightarrow \mathbb{1}_{A \cap B} = 0 = 0 \cdot 1 = \mathbb{1}_A \mathbb{1}_B$

c. $w \in (A \cup B)^c$:

$\Rightarrow \mathbb{1}_A = 0, \mathbb{1}_B = 0$

$$\Rightarrow \mathbb{1}_{A\cap B} = 0 = 0 \cdot 0 = \mathbb{1}_A \mathbb{1}_B$$

Hence the proof is complete.

3.

$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$ (union becomes addition - intersection)

First, we'll prove that for any $B, C \in A$ such that $B \cap C = \emptyset$ holds, $\mathbb{1}_{B\cup C} = \mathbb{1}_B + \mathbb{1}_C$ (Statement 1)

Let w denote an event:

(1) If $w \in B \cup C$:
$\Rightarrow \mathbb{1}_{B\cup C} = 1$
$\because B \cap C = \emptyset$
$\therefore$ w belongs to either B or C
$\Rightarrow \mathbb{1}_{B\cup C} = 1 = 1 + 0$ or $0 + 1 = \mathbb{1}_B + \mathbb{1}_C$

(1) If $w \notin B \cup C$:
$\Rightarrow \mathbb{1}_{B\cup C} = 0$
$\because B \cap C = \emptyset$
$\therefore w \notin B$ and $w \notin C$
$\Rightarrow \mathbb{1}_{B\cup C} = 0 = 0 + 0 = \mathbb{1}_B + \mathbb{1}_C$

Hence the proof is complete for the Statement 1.

Second, we'll start to prove this problem.

Proof :
$B = (A \cap B) \cup (A^c \cap B)$
Since $(A \cap B) \cap (A^c \cap B) = \emptyset$
$\Rightarrow \mathbb{1}_B = \mathbb{1}_{A\cap B} + \mathbb{1}_{A^c\cap B}$ (result from Statement 1)
$\Rightarrow \mathbb{1}_{A^c\cap B} = \mathbb{1}_B - \mathbb{1}_{A\cap B}$

$A \cup B = A \cup (A^c \cap B)$
Since $A \cap (A^c \cap B) = \emptyset$
$\Rightarrow \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_{A^c\cap B}$ (result from Statement 1)
$\Rightarrow \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B}$ (substitute $\mathbb{1}_{A^c\cap B}$ with $\mathbb{1}_B - \mathbb{1}_{A\cap B}$)
$\Rightarrow \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$ (result from Lemma 2.8.2)

Hence the proof is complete.

3 Property 4 of Theorem 2.18.

The cumulative distribution function(CDF) is defined as

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

From the properties of the distribution function, which includes $F(+\infty) = 1$, we get

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{x \to +\infty} \int_{-\infty}^{x} f(t)\,dt = \lim_{x \to +\infty} F(x) = 1$$

Hence,

$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

That is, the total area underneath a probability density function is 1.

4 Exercise 2.59.

Consider two independent fair coin tosses (2 sided), i.e. X, Y $\sim$ Bernoulli(1/2), and let 1 be heads and 0 is tails. Let Z = X + Y , what is the PMF of Z given X. Once, you have this, what is the joint PMF of (Z, X)?
When X = 0,Y could be 0 or 1,then Z could be 0 or 1,so

$$P(Z = 0|X = 0) = 1/2$$

and

$$P(Z = 1|X = 0) = 1/2$$

When X = 1,Y could be 0 or 1,then Z could be 1 or 2,so

$$P(Z = 1|X = 1) = 1/2$$

and

$$P(Z = 2|X = 1) = 1/2$$

Based on the PMF of Z given X, the joint PMF of (Z, X) is

$$P(Z = 0, X = 0) = 1/4$$

and

$$P(Z = 1, X = 0) = 1/4$$
$$P(Z = 1, X = 1) = 1/4$$

and

$$P(Z = 2, X = 1) = 1/4$$

5 Theorem 2.60: discrete RV

Tower property

$$\mathbb{E}\left(\mathbb{E}\left[\,X\,|\,Y\,\right]\right) = \mathbb{E}[X]$$

Prove the "tower property" for a discrete random variable.

Proof:
Let us assume that

$$X = \sum_{i=1}^{\infty} p_i \delta_{x_i} \text{ and } Y = \sum_{i=1}^{\infty} q_i \delta_{y_i}$$

or, in other words, X_i equals x_i with probability p_i, and Y_i equals y_i with probability q_i. If X (or Y) is finite instead of countable, then we assume that p_i (or q_i) equals 0 for i larger than the number of values a RV can take.

Following the lecture notes, we denote

$$g(y) = \mathbb{E}[X \,|\, Y = y].$$

Then the left hand side could be written in the following way:

$$\mathbb{E}\left(\mathbb{E}\left[\,X\,|\,Y\,\right]\right) = \mathbb{E}\left(g(Y)\right) = \sum_{i=1}^{\infty} q_i g(y_i) = \sum_{i=1}^{\infty} q_i \mathbb{E}\left[\,X\,|\,Y = y_i\,\right] =$$

$$= \sum_{i=1}^{\infty} q_i \sum_{j=1}^{\infty} x_j P(X = x_j \,|\, Y = y_i) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q_i x_j P(X = x_j \,|\, Y = y_i) =$$

$$= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} q_i x_j P(X = x_j \,|\, Y = y_i) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} P(Y = y_i) x_j P(X = x_j \,|\, Y = y_i)$$

Consider the inner sum for a specific j:

$$\sum_{i=1}^{\infty} P(Y = y_i) x_j P(X = x_j \,|\, Y = y_i) = x_j \sum_{i=1}^{\infty} P(Y = y_i) P(X = x_j \,|\, Y = y_i)$$

By the law of total probability

$$\sum_{i=1}^{\infty} P(Y = y_i) P(X = x_j \,|\, Y = y_i) = P(X = x_j),$$

hence, the inner sum we have considered equals simply

$$x_j P(X = x_j) = x_j p_j.$$

Therefore,

$$\mathbb{E}\left(\mathbb{E}\left[\,X\,|\,Y\,\right]\right) = \sum_{j=1}^{\infty} x_j p_j = \mathbb{E}[X].$$

That concludes the proof.