Homework #3
CS 575: Numerical Linear Algebra
Spring 2023

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Important Notes

- `Python` 3.11 was used to run the notebook (i.e., to use the `match` statement)
- the partially completed notebook was used to complete the coding assignment, so many of the things asked (e.g., verification and testing) was done for us and they were modified for the mat-mat portion
- the `README` was done in `Markdown` but the raw text can still be viewed

Problem 1

We are asked to prove that the vector infinity norm $\|\cdot\|_\infty$ is indeed a norm. So, we need to show the three conditions that define a norm for $\|\cdot\|_\infty$:

(a) $\|x\|_\infty = 0$ if and only if $x = 0$
(b) $\|cx\|_\infty = |c|\|x\|_\infty$ for all $c \in \mathbb{R}, x \in \mathbb{R}^m,$ and
(c) $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ for all $x, y \in \mathbb{R}^m$ (the triangle inequality)

Verification of (a):

$$\begin{aligned} \|x\|_\infty = 0 &\Leftrightarrow \max_{j \in \{1,2,\ldots,m\}} |x_j| = 0 \quad \text{def. of infinity norm} \\ &\Leftrightarrow |x_j| = 0 \text{ for all } j \qquad \Leftrightarrow x = 0 \end{aligned}$$

since the max magnitude of all components is 0, all other components have magnitude ≤ 0 and magnitude only deals with non-negative quantities, so all other components also have magnitude 0

Verification of (b):

$$\begin{aligned} \|cx\|_\infty &= \max_{j \in \{1,2,\ldots,m\}} |cx_j| \quad \text{def. of infinity norm} \\ &= |c| \max_{j \in \{1,2,\ldots,m\}} |x_j| \quad \text{linearity of max} \\ &= |c|\|x\|_\infty \quad \text{def. of infinity norm on } x \end{aligned}$$

Verification of (c):

$$\begin{aligned}
\|x+y\|_\infty &= \max_{j\in\{1,2,\ldots,m\}}|x_j+y_j| \quad \text{def. of infinity norm} \\
&\le \max_{j\in\{1,2,\ldots,m\}}|x_j| + \max_{j\in\{1,2,\ldots,m\}}|y_j| \quad \text{see below} \\
&= \|x\|_\infty + \|y\|_\infty \quad \text{def. of infinity norm on } x,y
\end{aligned}$$

where the second step holds true intuitively if we let $\max_{j\in\{1,2,\ldots,m\}}|x_j| = x_i$ and $\max_{j\in\{1,2,\ldots,m\}}|y_j| = y_k$ for some indices i,k in $1,2,\ldots,m$ where equality holds true when in max of $|x_j+y_j|$, $i=k$ with the same index for x_i, y_k and different indices hold the less than inequality since other components of x,y cannot be greater than the max value (or equality if the same max value appears multiple times in x,y)

Problem 2

We are asked to show the 2-norm and vector infinity norms satisfy

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\|x\|_\infty$$

For the fist inequality $\|x\|_\infty \le \|x\|_2$,

$$\begin{aligned}
\|x\|_\infty^2 &= \left(\max_{j\in\{1,2,\ldots,m\}}|x_j|\right)^2 \\
&= \max_{j\in\{1,2,\ldots,m\}}|x_j|^2 \\
&\le \sum_{j=1}^{m}|x_j|^2 \\
&= \|x\|_2^2
\end{aligned}$$

where we can use the trick with the 2-norm and try squaring the expression initially. Then, we can see the squaring of the max and finding the max of the square would yield the same value because of the absolute value (i.e., same mapping) and it naturally follows that the sum of the squares of the components would be greater or equal to the square of the max component (just one component). The equality holds when all other components are 0 besides the max and the less than inequality holds otherwise(since any contributions are non-negative). When we take square roots, the first inequality is satisfied.

For the second inequality $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$,

$$\begin{aligned}
\|x\|_2^2 &= \sum_{j=1}^{m} |x_j|^2 \\
&\leq \sum_{j=i}^{m} \max_{j \in \{1,2,\ldots,m\}} |x_j|^2 \\
&= m \cdot \max_{i=1,2,\ldots,m} |x_j|^2 \\
\|x\|_2 &\leq \sqrt{m \cdot \max_{j \in \{1,2,\ldots,m\}} |x_j|^2} \\
&= \sqrt{m} \cdot \sqrt{\max_{j \in \{1,2,\ldots,m\}} |x_j|^2} \\
&= \sqrt{m} \cdot \max_{j \in \{1,2,\ldots,m\}} \sqrt{|x_j|^2} \\
&= \sqrt{m} \cdot \max_{j \in \{1,2,\ldots,m\}} |x_j| \\
&= \sqrt{m}\|x\|_\infty
\end{aligned}$$

where the first part holds intuitively (as explained in the first inequality) and the index of the summation was changed to i so not to be confused with the index j in the sum. Each component squared is less than or equal to the max component squared and since it does not depend on index i, we just have m times the max component squared. From there, we can take the square root on both sides and see that the square root on the max squared term is the same as the max of the square root squared term (same mapping again). The rest follows through and the second inequality is satisfied.

Problem 3

We are asked to show that the induced matrix norm is indeed a norm. Again, we have to show the three conditions (same as Problem 1) that define a norm for $\|A\| = \max_{\|x\|=1} \|Ax\|$ (properties that apply to vectors will not be derived, but they will be mentioned):

Verification of (a):

$$\|A\| = \max_{\|x\|=1} \|Ax\| = 0 \Leftrightarrow A = 0$$

where in order to prove this, we can use a trick by looking at a possibly non-maximal choice for x (i.e., does not result in the max of $\|Ax\|$). Also, we assume A is considered 0 if ***all*** elements are 0 (thus, A is not considered 0 if at least one element is nonzero, which also means at least one column in A is non the 0 vector). We can do a proof by contradiction and let a_i be the nonzero column vector in A and let our non-maximal choice for x be e_i (ith canonical basis vector, grabbing the (nonzero) ith column of A):

$$\begin{aligned}
\|A\| &= \max_{\|x\|=1} \|Ax\| \\
&\geq \|Ae_i\| \\
&= \|a_i\| \\
&> 0
\end{aligned}$$

We can see that for a non-maximal choice of x when A is not 0, the induced norm is strictly greater than 0 (given from the fact that the norm of a vector is 0 only if the vector itself is 0, which under our premise for a_i is not the case). So, this does not satisfy our initial condition that the induced norm is 0 so A has to be 0 for the induced norm to be 0.

Verification of (b):

$$\begin{aligned}
\|cA\| &= \max_{\|x\|=1} \|cAx\| \\
&= \max_{\|x\|=1} |c| \|Ax\| \quad \text{linearity of vector norm} \\
&= |c| \max_{\|x\|=1} \|Ax\| \quad \text{linearity of max} \\
&= |c| \|A\|
\end{aligned}$$

Verification of (c):

$$
\begin{aligned}
\|A+B\| &= \max_{\|x\|=1} \|(A+B)x\| \\
&= \max_{\|x\|=1} \|Ax+Bx\| \\
&\le \max_{\|x\|=1} (\|Ax\|+\|Bx\|) \quad \text{triangle inequality of vectors} \\
&\le \max_{\|x\|=1} \|Ax\| + \max_{\|x\|=1} \|Bx\| \text{triangle inequality of max} \\
&= \|A\| + \|B\|
\end{aligned}
$$

where the proof was adapted from the book for (c).

Problem 4

Let $A \in \mathbb{R}^{m \times n}$. We are asked to show that

$$
\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|
$$

Note: the proof was adapted from the book (solution provided at the back).

In order to do so, we first need to show two things

(i) $\|Ax\|_\infty \le \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$ for $\|x\|_\infty = 1, x \in \mathbb{R}^n$

(ii) there exists a $y \in \mathbb{R}, \|y\|_\infty = 1$ such that $\|Ay\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$

For (i), let $k \in 1, 2, \ldots, m$ represent the kth row index of A and kth component of Ax:

$$\begin{aligned}
|(Ax)_k| &= \left|\sum_{j=1}^{n} a_{kj}x_j\right| \quad \text{def. of maxt-vec mult.} \\
&\leq \sum_{j=1}^{n} |a_{kj}||x_j| \quad \text{inequality for magnitude mult.} \\
&\leq \sum_{j=1}^{n} |a_{kj}| \quad \text{from } \|x\|_\infty = 1 \text{ where the max component } x_j = 1 \\
&\leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|
\end{aligned}$$

where in the last step, we have reached our end goal for (i) – an arbitrary row sum is bounded by the sum of the max row sum of the magnitudes. Since we are dealing with the infinity norm, the last step is valid because using the definition, we need to find the magnitude of the max component of the vector Ax, which is what we have shown above.

Then for (ii), we have to look at the first line above and find some y where equality holds for our result. For simplicity, let k be the index where the max occurs:

$$|(Ax)_k| = \left|\sum_{j=1}^{n} a_{kj}x_j\right|$$

Since the infinity norm will take care of the max component for us, we just need to make sure each individual product $a_{kj}x_j$ is equal to $|a_{kj}|$ (replaced i with k from above). The simplest way to ensure the product is non-negative (absolute value) is to match the sign of a_{kj} for our choice of x_j. Then for the magnitude itself, we can keep the magnitude for all x_j to be 1 which satisfies $\|x\|_\infty = 1$ since the magnitude of all components of x is 1. For example, if a_{kj} is positive, x_j is positive and a similar case for when a_{kj} is negative. When a_{kj} is 0, then the sign for x_j does not matter.

Problem 5

Note: running out of time so rushed the proof (should be fairly straightforward)

We are given that $\|\cdot\|$ denotes a norm on $\mathbb{R}^m$, and a norm on $\mathbb{R}^{m\times m}$ that the vector norm induces. Also, we are given the identity matrix $I \in \mathbb{R}^{m\times m}$. We are asked to show that

- $\|I\| = 1$
- $\|Ax\| \leq \|A\|\|x\| \quad$ for all $A \in \mathbb{R}^{m\times m}, x \in \mathbb{R}^m$
- $\|AB\| \leq \|A\|\|B\| \quad$ for all $A, B \in \mathbb{R}^{m\times m}$

Before we start, we can recall the definition of an induced matrix norm (from in class):

$$\begin{aligned}
\|A\| &= \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \\
&= \max_{x\neq 0} \left\| A \frac{x}{\|x\|} \right\| \\
&= \max_{\|y\|=1} \|Ay\|
\end{aligned}$$

For $\|I\| = 1$, we can use the last line and the fact that any vector multiplied with the identity is the vector itself: $Ix = x, x \in \mathbb{R}^m$. So, we have

$$\begin{aligned}
\|I\| &= \max_{\|x\|=1} \|Ax\| \\
&= \max_{\|x\|=1} \|x\| \\
&= 1
\end{aligned}$$

where the last equality naturally follows given the constraint on the norm of x as 1, reducing the induced vector norm on I to a simple vector norm.

Then for $\|Ax\| \leq \|A\|\|x\|$, we can instead use the first line of the definition of the induced and arranging $\|Ax\| \leq \|A\|\|x\|$:

$$
\begin{aligned}
\|Ax\| &\leq \|A\|\|x\| \\
&\Leftrightarrow \\
\frac{\|Ax\|}{\|x\|} &\leq \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \\
&\Leftrightarrow \\
\frac{\|Ax\|}{\|x\|} &\leq \|A\| \quad \text{divide both sides by} \|x\|
\end{aligned}
$$

where the second equality for the max condition is a simple application of the max, leading to the norm of A by the definition of the induced norm.

Finally, for $\|AB\| \leq \|A\|\|B\| \quad$ for all $A, B \in \mathbb{R}^{m\times m}$,

$$
\begin{aligned}
\|ABx\| &= \|A(Bx)\| \\
&\leq \|A\|\|Bx\| \\
&\leq= \|A\|\|B\|\|x\|
\end{aligned}
$$

and using our result for the induced norm of AB,

$$
\begin{aligned}
\|AB\| &= \max_{x\neq 0} \frac{\|ABx\|}{\|x\|} \\
&= \max_{x\neq 0} \frac{\|A\|\|B\|\|x\|}{\|x\|} \\
&\leq= \max_{x\neq 0} \|A\|\|B\| \\
&= \|A\|\|B\|
\end{aligned}
$$

Problem 6

Note: running out of time so please see the comments in the code for implementation details

For this problem, we are doing the CS 575 version and writing an implementation of Gaussian Elimination without Row-Swapping for $N \times N$ tri-diagonal matrices. In terms of setting up the tri-diagonal matrix, we adapated the code from the `tridiagonal_matrix_fun.ipynb` file. Afterwards, we adapted the code we submitted last homework to use the tri-diagonal format of A in our implementation.

To verify that our code works for $N = 4$, we used an `assert` statement to make sure the norm of the difference was within some tolerance (which would spit out an exception if that was the case, but our code ran successfully). Afterwards, our code produce the plot below for various values of N (again, see the code for more details).

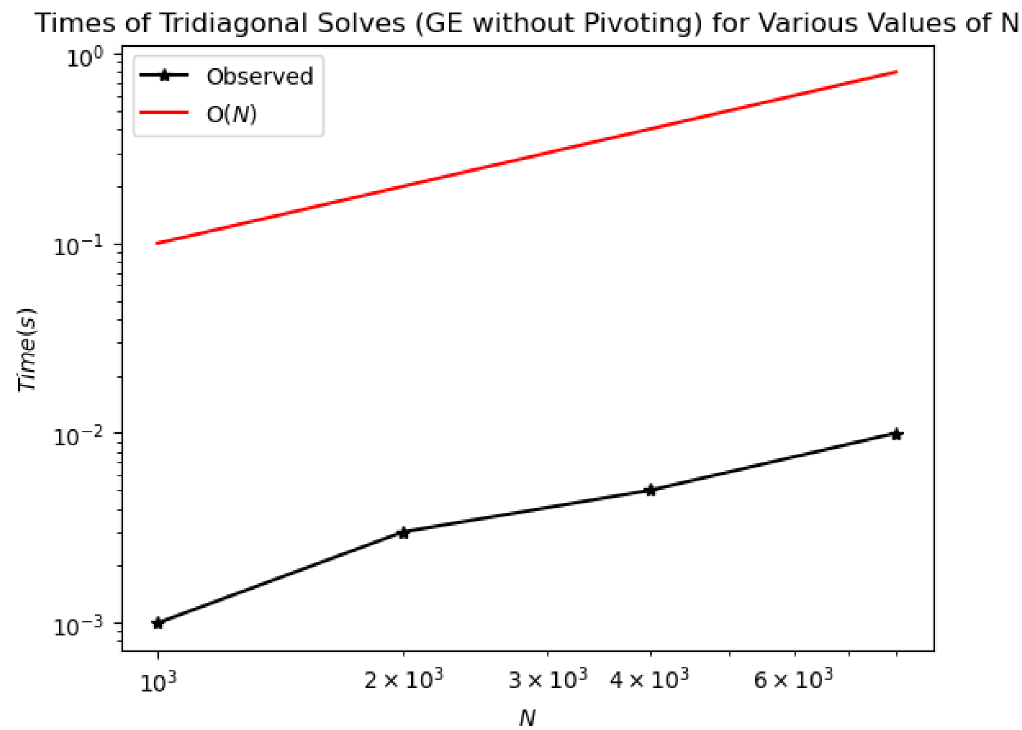

Figure 1

As we can see, our plot is approximately linear to the reference curve $O(n)$. We chose values that were just beyond the threshold where the timings would give nonzero or timings that we would expect (besides overhead, caching, etc.).