HW #3
Math/CS 471, Fall 2021

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Contents

In this homework, we explored numerical integration of Equation 1 by the trapezoid rule as well as by Gauss Quadrature. Here, k was taken to be π and π^2. Equation 2 was used to determine the error of the trapezoid rule as well as Gauss Quadrature for both k values in the figures below. Note that as $|I_{n+1} - I_n|$ in Equation 2 denotes the absolute difference in the approximation of Equation 1 over the entire integral, this is the global absolute error.

$$I = \int_{-1}^{1} e^{cos(kx)} dx \tag{1}$$

$$(\delta_{abs})_{n+1} = |I_{n+1} - I_n| \tag{2}$$

1 Trapezoid Rule:

$$\int_{X_L}^{X_R} f(x)dx \approx h \left(\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i) \right) \tag{3}$$

The trapezoid rule (Equation 3) for $k = \pi^2$ converges cubically as seen in figure 1. Normally the order of convergence for a Newton-Cotes method, such as the trapezoid rule, is equal to the number of points in each subinterval. However, in the case of the trapezoid rule, fortuitous cancellation (with the positive and negative errors) on the global scale leads to an additional order of convergence. This third order (cubic) convergence was predicted theoretically in class, so it was expected in Figure 1 where the cubic reference curve is near parallel to the error from the trapezoid rule at $k = \pi^2$. When it comes to the trapezoid rule for $k = \pi$, we observe an exponential convergence in figure 1. This is explained in the subsection below.

Exponential Convergence for Trapezoid Rule $k = \pi$

The linked article in the homework pdf directs us to a page on the Euler–Maclaurin formula. Part of the description of this formula describes Clenshaw–Curtis quadrature and a discrete cosine transform of the Euler-Maclaurin formula which takes advantage of the periodicity of the integral being approximated in order to dramatically increase the order of convergence. This method appears similar to the use of the Fourier transform to take advantage of periodicity as we did in class. Because of this, the exponential convergence was expected.

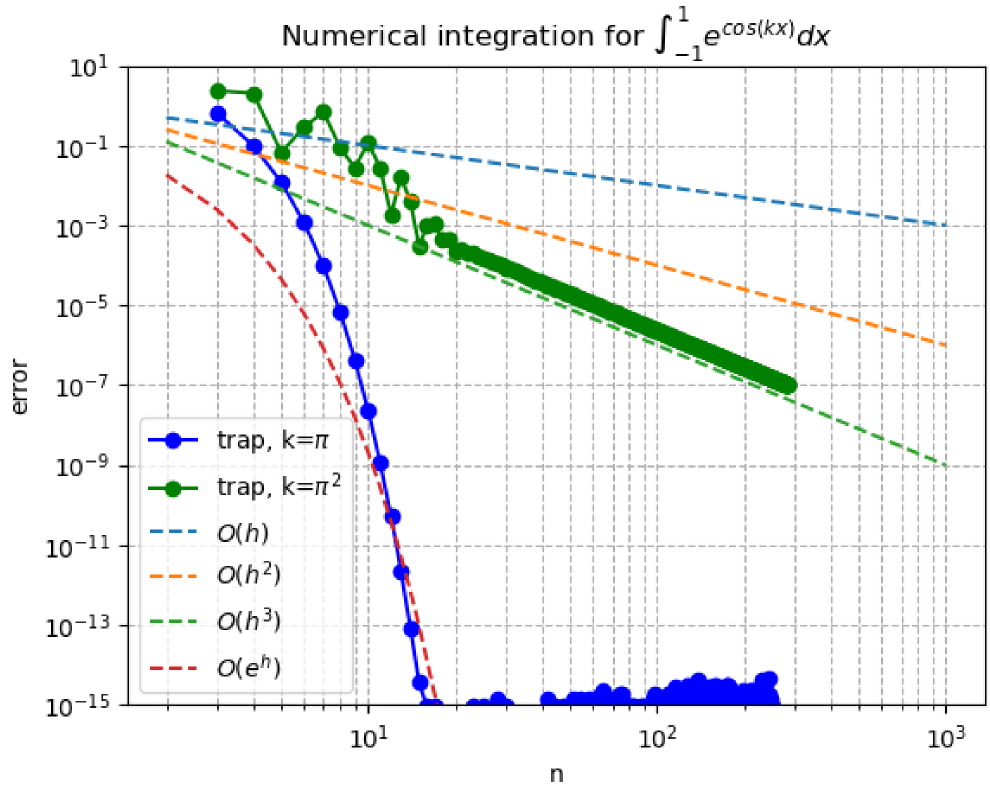

Figure 1: Error (Equation 2) in Equation 1 via the Trapezoid Rule at k = π and k = π^2. **Note:** In the legend given above, we give the reference curves in big O notation in terms of h with an assumption of e as the base of the exponential. Look at the end of Section Gauss Quadrature for more details. Also, the reference curves are meant more as a guide for visualizing the relative convergence rates and are not modified to exactly match them.

Figure 2 and in Figure 3 plot Equation 1 over the interval of interest. From these graphs, it is apparent that in Figure 2, Equation 1 completes one period within the bounds. This differs from Figure 3 where slightly more than three periods are completed over the bounds. Therefore, the errors do not all cancel out over the case for $k = \pi^2$ which may explain further why exponential convergence is not obtained in this case.

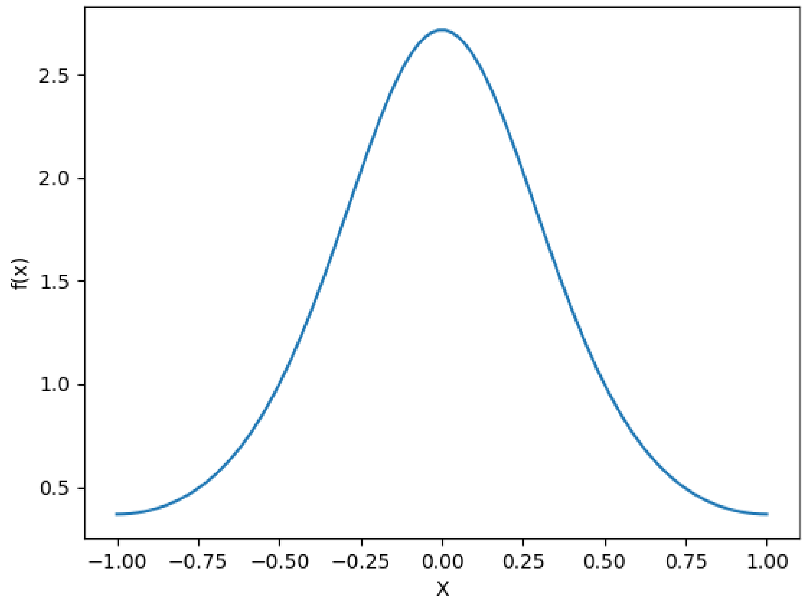

Figure 2: Graph of Equation 1 at $k = \pi$

2 Gauss Quadrature:

$$\int_{-1}^{1} f(x)\omega(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i) \tag{4}$$

Gauss Quadrature when applied to Equation 1 for both $k = \pi$ and $k = \pi^2$ was found to converge exponentially in both cases. In Figure 4, the error in the approximation of Equation 1 appears to trend closer to the exponential (red) reference curve than the linear, quadratic, or cubic reference curves. This exponential convergence was predicted theoretically in class.

Note: For the three cases where the error converges exponentially ($O(e^h)$, or in terms of n, $\epsilon(n) \propto C^{-\alpha n}$ for some constants C and α, since $h \propto \frac{1}{n}$) – every curve except the Trapezoid rule with the case $k = \pi^2$ – we can observe that a "tail"-like artifact emerges when the error reaches $\simeq 10^{-15}$. This is due to to a loss of machine precision in storing the computation and results in the oscillation as shown in Figures 1 and 4 for the described curves. We

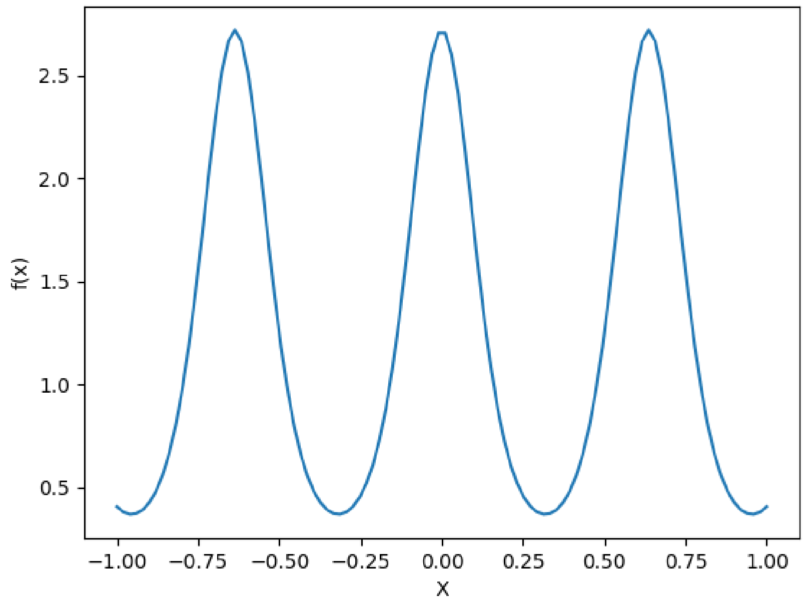

Figure 3: Graph of Equation 1 at $k = \pi^2$

spoke with Dr. Schroder about the use of e^h as the exponential reference curve. He mentioned that it is as valid as the fitted $C^{-\alpha n}$.

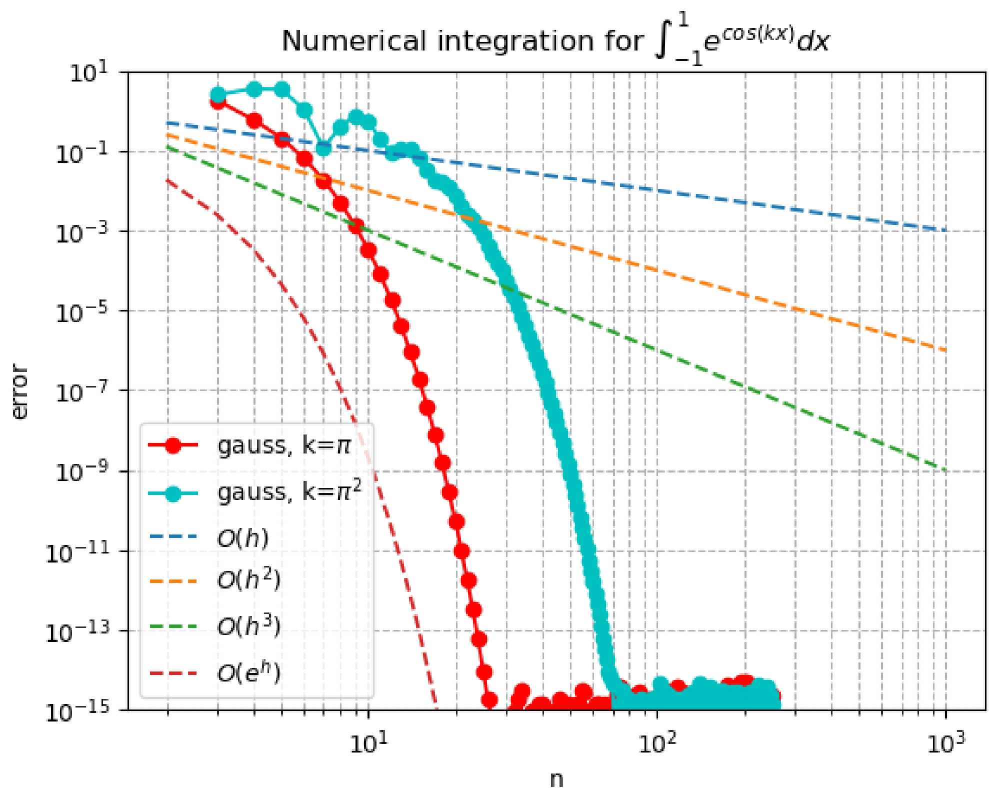

Figure 4: Error in Equation 1 via Gauss Quadrature at k = π and k = π^2. **Note:** In the legend given above, we give the reference curves in big O notation in terms of h with an assumption of e as the base of the exponential. Look at the end of Section Gauss Quadrature for more details. Also, the reference curves are meant more as a guide for visualizing the relative convergence rates and are not modified to exactly match them.

3 Cost Compassion

First, we can see that n can serve as a proxy for the number of function executions since the number of function evaluations scales with n. n represents the number of subintervals, where a function evaluation occurs at each point on the subintervals ($n + 1$ points).

The following values of n are found experimentally with a tolerance of 10^{-10}. In our computation, n started at the value of 2 and was incremented sequentially until the target tolerance was met ($n = 2, 3, \dots$). For reference, the tolerance is compared to the relative absolute error in our numerical integral Equation 1, as depicted in Equation 2.

Values of n for $k = \pi$ and $k = \pi^2$:

n for the trapezoid rule at $k = \pi$: 12 (11 iterations)
n for the trapezoid rule at $k = \pi^2$: on the order of 10^4 if we extend our reference line to an error of 10^{-10} (estimate – too computationally expensive)
n for the Gauss Quadrature at $k = \pi$: 20 (19 iterations)
n for the Gauss Quadrature at $k = \pi^2$: 54 (53 iterations)

For Gauss Quadrature, once the nodes have been computed, one function evaluation and one multiplication are needed at each step in the summation, resulting in a total of $n + 1$ multiplications and $n + 1$ evaluations of $f(x_i)$. After the function evaluations and multiplications, there are n additions. Therefore the operation count of Gauss Quadrature after the nodes have been computed is about $(n) + 2(n + 1)$. Here, I leave out the cost of computing the nodes as these can be tabulated values. In this case, $n = 2, 3, \dots, N$ where $N - 1$ is the total number of iterations required to reach the desired tolerance as n starts at 2.

For the trapezoid rule, the equation is a bit more complicated. There are n summations of $n + 1$ function evaluations, one division, and a multiplication by h. All together, the trapezoid rule has an operation count of about $n + (n + 1)$ if we ignore the constant operations at the end.

From this, it is clear that the trapezoid rule has fewer operations per iteration, but since they both have an operation cost of $O(n)$, they scale similarly with large values of n. However, because the order of convergence of Gauss Quadrature is exponential and the order of convergence of the trapezoid rule is cubic, the overall cost of Gauss Quadrature is less, requiring less iterations (n) and therefore, less function evaluations in total are required to reach the desired tolerance of 10^{-10}

In a general setting, where the trapezoid rule does not converge exponentially (as shown in Figure 1 for the case $k = \pi$), Gauss Quadrature is likely to be more efficient. This is explained above with by comparing the relative rate of convergence.