# HW #2
# Math/CS 471, Fall 2021

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# Contents

**Note**: the initial guess of $-0.5$ was not used in calculating the errors and the convergence rates (as in $x_0$ started with the approximate root after the first iteration of Newton's method).

## What:

Implement Newton's method on $f(x) = x^2$, $f(x) = x$, and $f(x) = sin(x) + cos(x^2)$. Determine the rate of convergence in each case.

Max number of iterations: 50

Tolerance: $10^{-10}$

## How:

We modified the files `drive_newton.py` and `newton.py` in order to modify the existing code to add a maximum iteration limit as well as a tolerance. We also added the `savetxt()` functionality in order to export latex tables from the code itself.

## Why:

This project is done in the effort to better understand the convergence of Newtons method at the root of functions with different multiplicities. Through this project, we also gained more experience in `Python`.

**Note**: The following tables displaying the linear and quadratic convergences of the three functions will have both the iteration number (starting at 0) and error ratio displayed in scientific notation – couldn't find a simple way to format the iterations separately besides creating a separate array and concatenating it at the end.

# 1 Function $f(x) = x^2$

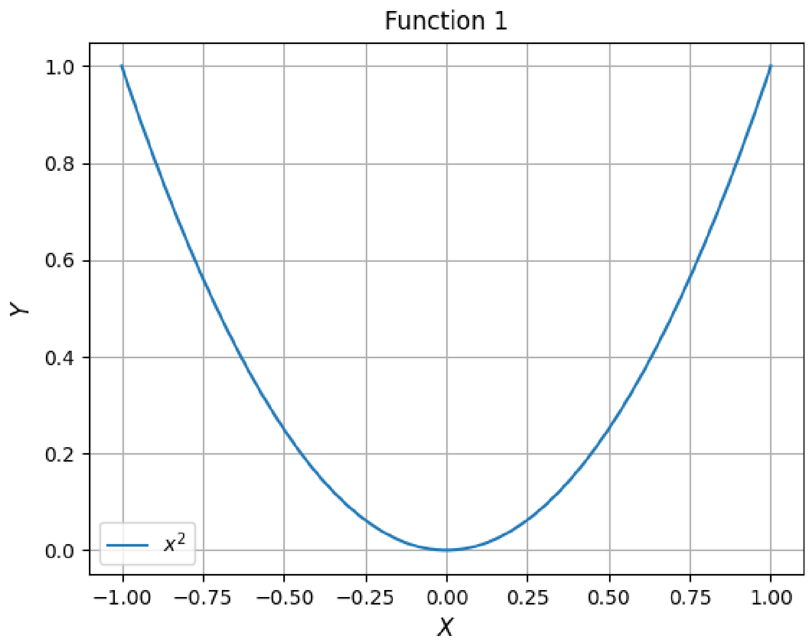


Figure 1: Plot of the function $f(x) = x^2$

In the case of the function $f(x) = x^2$, the test for linear converges in Table 1 is shown to converge to 0.5, signifying that Newton's method in this case converges linearly. Further, from Figure 1, it is apparent that $f'(x)$ near the root approaches zero. This also hints that the root has a multiplicity grater than one and therefore would require modified Newtons method in order to converge quadratically.

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 5.00000e-01 |
| 1.00000e+00 | 5.00000e-01 |
| 2.00000e+00 | 5.00000e-01 |
| 3.00000e+00 | 5.00000e-01 |
| 4.00000e+00 | 5.00000e-01 |
| 5.00000e+00 | 5.00000e-01 |
| 6.00000e+00 | 5.00000e-01 |
| 7.00000e+00 | 5.00000e-01 |
| 8.00000e+00 | 5.00000e-01 |
| 9.00000e+00 | 5.00000e-01 |
| 1.00000e+01 | 5.00000e-01 |
| 1.10000e+01 | 5.00000e-01 |
| 1.20000e+01 | 5.00000e-01 |
| 1.30000e+01 | 5.00000e-01 |
| 1.40000e+01 | 5.00000e-01 |
| 1.50000e+01 | 5.00000e-01 |
| 1.60000e+01 | 5.00000e-01 |
| 1.70000e+01 | 5.00000e-01 |
| 1.80000e+01 | 5.00000e-01 |
| 1.90000e+01 | 5.00000e-01 |
| 2.00000e+01 | 5.00000e-01 |
| 2.10000e+01 | 5.00000e-01 |
| 2.20000e+01 | 5.00000e-01 |
| 2.30000e+01 | 5.00000e-01 |
| 2.40000e+01 | 5.00000e-01 |
| 2.50000e+01 | 5.00000e-01 |
| 2.60000e+01 | 5.00000e-01 |
| 2.70000e+01 | 5.00000e-01 |
| 2.80000e+01 | 5.00000e-01 |
| 2.90000e+01 | 5.00000e-01 |
| 3.00000e+01 | 5.00000e-01 |

Table 1: Linear Convergence for $f(x) = x^2$ (stopped at 33 iterations with tolerance $10^{-10}$)

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 4.00000e+00 |
| 1.00000e+00 | 8.00000e+00 |
| 2.00000e+00 | 1.60000e+01 |
| 3.00000e+00 | 3.20000e+01 |
| 4.00000e+00 | 6.40000e+01 |
| 5.00000e+00 | 1.28000e+02 |
| 6.00000e+00 | 2.56000e+02 |
| 7.00000e+00 | 5.12000e+02 |
| 8.00000e+00 | 1.02400e+03 |
| 9.00000e+00 | 2.04800e+03 |
| 1.00000e+01 | 4.09600e+03 |
| 1.10000e+01 | 8.19200e+03 |
| 1.20000e+01 | 1.63840e+04 |
| 1.30000e+01 | 3.27680e+04 |
| 1.40000e+01 | 6.55360e+04 |
| 1.50000e+01 | 1.31072e+05 |
| 1.60000e+01 | 2.62144e+05 |
| 1.70000e+01 | 5.24288e+05 |
| 1.80000e+01 | 1.04858e+06 |
| 1.90000e+01 | 2.09715e+06 |
| 2.00000e+01 | 4.19430e+06 |
| 2.10000e+01 | 8.38861e+06 |
| 2.20000e+01 | 1.67772e+07 |
| 2.30000e+01 | 3.35544e+07 |
| 2.40000e+01 | 6.71089e+07 |
| 2.50000e+01 | 1.34218e+08 |
| 2.60000e+01 | 2.68435e+08 |
| 2.70000e+01 | 5.36871e+08 |
| 2.80000e+01 | 1.07374e+09 |
| 2.90000e+01 | 2.14748e+09 |
| 3.00000e+01 | 4.29497e+09 |

Table 2: Quadratic Convergence for $f(x) = x^2$ (stopped at 33 iterations with tolerance $10^{-10}$)

## 2 Function $f(x) = x$

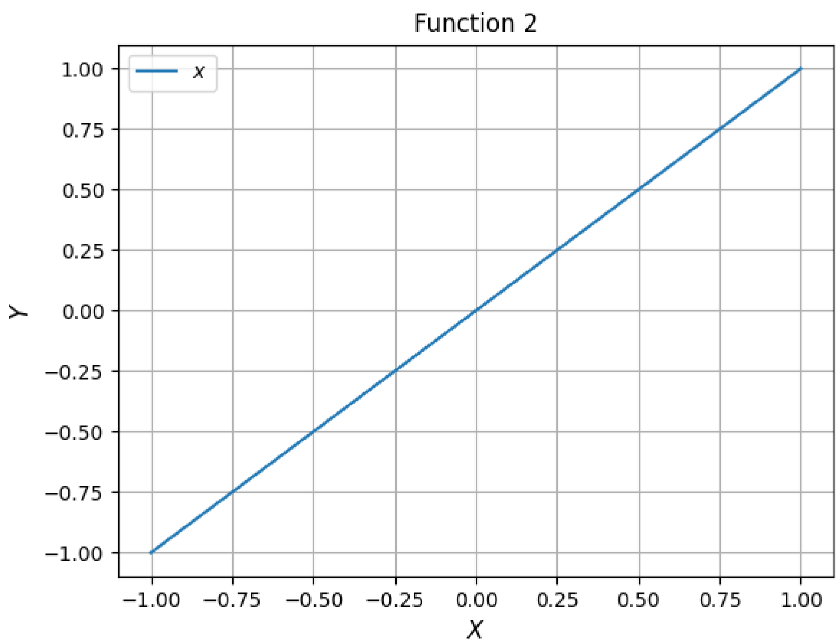


Figure 2: Plot of the function $f(x) = x$

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 0.00000e+00 |
| 1.00000e+00 | 0.00000e+00 |

Table 3: Linear Convergence for $f(x) = x$ (stopped at 2 iterations with tolerance $10^{-10}$)

Newtons method in the case of $f(x) = x$ converges in one step as any line tangent to a point on this function will find the root exactly in one step. For this reason, there are no errors to calculate linear or quadratic convergence (as shown by the 0 values returned from both Table 3 for linear convergence and Table 4 for quadratic convergence). If we look at $f'(x)$ in Figure 2 near and at the root of $x = 0$, we can see that the derivative of $f(x)$ is constant, which might help to explain that the rate of convergence is constant at every step, regardless of the initial guess.

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 0.00000e+00 |
| 1.00000e+00 | 0.00000e+00 |

Table 4: Quadratic Convergence for $f(x) = x$ (stopped at 2 iterations with tolerance $10^{-10}$)

# 3 Function $f(x) = sin(x) + cos(x^2)$

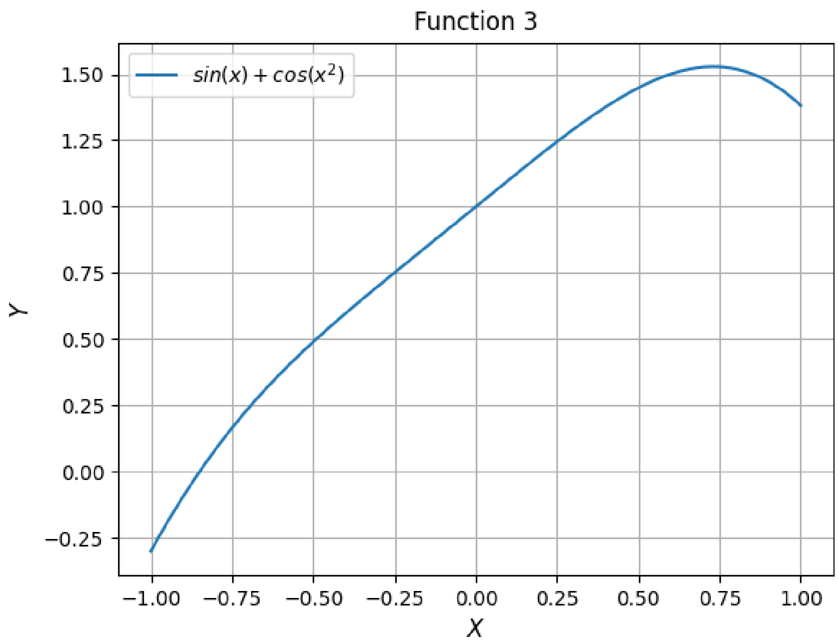


Figure 3: Plot of the function $f(x) = sin(x) + cos(x^2)$

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 6.52655e-02 |
| 1.00000e+00 | 4.05088e-03 |
| 2.00000e+00 | 1.63324e-05 |
| 3.00000e+00 | 3.19544e-07 |

Table 5: Linear Convergence for $f(x) = sin(x) + cos(x^2)$ (stopped at 6 iterations with tolerance $10^{-10}$)

In the case of the function $f(x) = sin(x) + cos(x^2)$, the function appears to converge quadratically in the few steps before the tolerance was reached. In this case, $f'(x) = cos(x) - 2xsin(x)$ and at the root $x = -0.849369$, $f'(x) = 1.782401$. Because the derivative at the root is not zero, the multiplicity of the function is one and it converges quadratically.

| iteration | error ratio |
|---|---|
| 0.00000e+00 | 8.11123e-01 |
| 1.00000e+00 | 7.71383e-01 |
| 2.00000e+00 | 7.67751e-01 |
| 3.00000e+00 | 9.19708e+02 |

Table 6: Quadratic Convergence for $f(x) = sin(x) + cos(x^2)$ (stopped at 6 iterations with tolerance $10^{-10}$)

# 4 Modified Newton's method to regain the quadratic convergence for the case where you observe linear convergence

It is possible to modify Newton's method to regain the quadratic convergence for the case where you observe linear convergence, but we must know the multiplicity of the root of the function we're trying to find beforehand. If we let the multiplicity of the root be $m$, then the modified Newton's method would be the following:

$$x_i = x_{i-1} - m\frac{f(x_{i-1})}{f'(x_{i-1})} \tag{1}$$

For the case where $f(x) = x^2$, $m$ would be 2 since $f(x) = f'(x) = 0$ at the root $x = 0$. Therefore, by using $m = 2$ in equation 1, the quadratic convergence can be obtained. We tried implementing it for $f(x) = x^2$, but if we plug in $m = 2$ in equation 1, we would get $x_i = x_{i-1} - 2\frac{x_{i-1}^2}{2x_{i-1}} = x_{i-1} - x_{i-1} = 0$, resulting in an immediate termination of Newton's method.