Semester Project: Auto-Parametric Resonance

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Abstract

Computationally simulating an elastic pendulum system with auto-parametric resonance by employing the Euler-Cromer method found that the energy of the system is not conserved.

I Introduction

The system this project aims to simulate is a combination of a spring-mass system and a string-mass, or pendulum, system. Starting with the principles, a spring-mass system has three main parameters, the mass attached to the spring (m measured in kilograms), the relaxed length of the spring with no mass on it (l_0 measured in meters), and the spring constant k (measured in Newtons per meter). When the spring is extended with the mass attached and let go, the system will oscillate with a period dependent on the mass and spring constant. In physics, this is often referred to as a harmonic oscillator.

The other system, the simple pendulum, is also a harmonic oscillator. This system is characterized by a mass on the end of string that swings. The only parameter of a pendulum that has any consequence is the length of the string, measured in meters (gravity is also a factor, but that is not likely to change so it is neglected for our purposes).

Looking at each simple system independently, their quantitative analysis is simple. For the spring-mass system, the kinematics of the system are relatively simple and are a staple in elementary physics classes. When the mass is pulled and the spring is stretched, and the spring exerts a force equal to $F_s = -k * s$, where s is the change in the spring's length from its rest length with the mass attached. Its kinetic energy can be evaluated by $KE = \frac{1}{2}mv^2$, and its potential energy is evaluated by $U_s = \frac{1}{2}k(l - l_0)^2$ where l is the instantaneous length of the spring. The pendulum system, when released from a height with the string taught, has only gravity acting on it so the force is modeled by $F_g = -g * m$, where g is the acceleration due to gravity ($9.8\frac{m}{s^2}$). The kinetic energy is the same as the spring model, and the potential is evaluated by $U_p = m * g * h$, where h is the difference height from its rest state, where the mass is motionless.

Together, the arithmetic differs slightly, but the modelling of forces is the same. The combined system, typically referred to as an elastic pendulum, has the same kinetic energy and spring potential energy as before, but the pendulum potential energy differs slightly – the potential energy of the elastic pendulum is calculated with the equation $U_{ep} = m*g*l^2*cos(\theta)$ where θ is the angle between the perpendicular in the $-y$ direction and the spring [2].

Now for the fun part, auto-parametric resonance. Auto-parametric resonance can best be described with the elastic pendulum system, hence its implementation, it is characterized by the the stitch between oscillatory modes. This can be observed from the elastic pendulum system when the mass is roughly a tenth of the spring constant, when this condition is met, the system can be observed starting with an oscillation in the spring, straight up and down, then slowly transitioning to swinging back and forth with little stretch in the spring. This occurs because the simple pendulum's length is changing by increasing when it swing upward so that the spring stretches then the spring decreases in length as the mass swings past the rest position. The accumulation of kinetic energy from changing the length can cause the mass to swing violently [1].

II Methods

The algorithm employed to model this system for analysis is the Euler-Cromer algorithm. This algorithm is popular for its simplicity and relatively low error accumulation. This method has inputs of initial conditions like initial time, final time, number of time-steps, initial velocity, and then loops over each time-step and updates the value previous and adds the new value to the array or list that it then outputs.

The simulation ran with a mass of $10kg$, a spring constant $k = 100\frac{N}{m}$, and relaxed length $l_0 = 5m$.

The simulation ran for 10 seconds with 1000 time-steps. The simulation ran and generated a graph with the energies, potential and kinetic, as well as their sum to demonstrate if the system was energy-conservative. The momentum with respect to time was also recorded and graphed to show if the system was instead momentum-conservative.

III Data

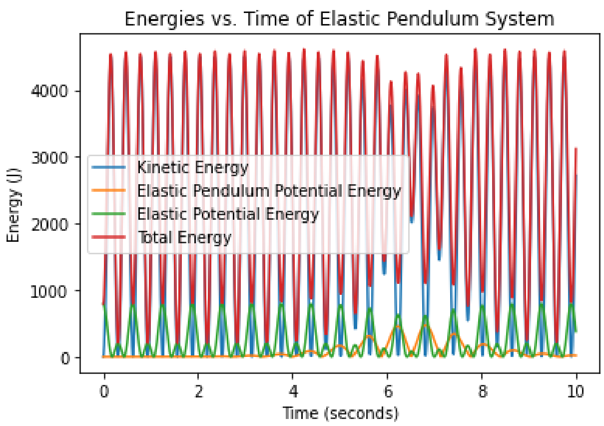

Figure 1: Graph of kinetic, total, and potential energies with respect to time.

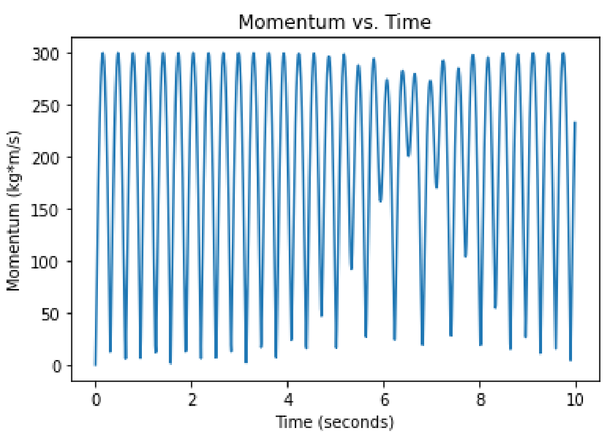

Figure 2: Graph of momentum with respect to time.

IV Results & Discussion

From Figure 1, the system demonstrates autoparametric resonance from the dip in kinetic energy and increase in Elastic Pendulum potential energy, which indicated a switch from spring motion to pendulum-like swinging. However, the most disappointing aspect of this graph is the total energy curve. The kinetic energy is the dominating term and ultimately drives the total energy to oscillate with it. The change in total energy proves that the system is not energy conservative. Additionally, in Figure 2, the momentum also oscillated which indicates that it is neither energy or momentum conservative.

This result comes as a surprise as the system was picked and calculated carefully. It was expected that the simulation would show that the energy is conserved in the transition between oscillatory modes, but not demonstrate momentum-conservation due to the swinging having a stop before it changes direction. This error is likely due to an arithmetic error. We see that the magnitude of the kinetic energy is much much bigger than the potential energy, though the arithmetic gets hairy, the specific cause is unknown. The equations for kinetic energy we unaltered and should be correct. Further investigation is required to determine the error and obtain concrete, conclusive results.

References

[1] Steve Mould. What if swings had springs instead of ropes: Autoparametric resonance, 02 2022.

[2] Qisong Xiao, Shenghao Xia, Corey Zammit, Nirantha Balagopal, and Zijun Li. Dynamics of the elastic pendulum.