### 0.0.1 Maximum degree method

Now lets calculate the $T$ distribution for IE1-star method. Let $v^*$ be the "right" vertex. Let $I$ be the event that edge $(v^*, v^*)$ is a signal edge, and $R$ be the number of verticies $u \in V$ such that edges $(v^*, u)$ and $(u, v^*)$ are both signal edges. We will refer to this type of edge as a doubled signal edge. Then we can rewrite the probability of picking the right vertex as

$$P[\text{right vertex}] = \sum_r \Big( P[I, R=r] P[\text{right vertex}|I, R=r] + P[\bar{I}, R=r] P[\text{right vertex}|\bar{I}, R=r] \Big) \quad (1)$$

$P[I, R=r]$ can be found with simple combinitorics. First choose the $r$ doubled signal edges, next choose the vertices for remaining signal edges, and remembering that either incoming or outgoing edge and be signal. We get

$$P[I, R=r] = \frac{\binom{n-1}{r}\binom{n-1-r}{m-2r-1} 2^{m-2r-1}}{\binom{2n-1}{m}} \quad (2)$$

Notice that the support of $R$ depends on $m$. The support of $R$ is $\max(0, m-n)$ to $\lfloor (m-1)/2 \rfloor$.

Similarly for $\bar{I}$, except the support of $R$ is now $\max(0, m-n+1)$ to $\lfloor m/2 \rfloor$ if $m < n+1$.

$$P[\bar{I}, R=r] = \frac{\binom{n-1}{r}\binom{n-1-r}{m-2r} 2^{m-2r}}{\binom{2n-1}{m}} \quad (3)$$

Let $d_v$ be the estimated "degree" of vertex $v$ (includes both incoming and outgoing edges)

$$d_v = \sum_{v \in u_1, u_2} \hat{q}_{u_1, u_2}.$$

Given $R$ and $I$, $v \neq v^*$ can be divided into 3 dijoint sets: $d_v$ with 0, 1, or 2 signal edges. Specificaly there are exactly $R$ vertices with 2, $m-2R-1$ with 1, and $n-1-(m-2R-1)-R = n-m+R$ verticies with 0 signal edges. Let us denote a vertex with $i$ signal edges as $d(i)$. (Thus $d(2)_{(r:r)}$ is maximum degree of $r$ vertices with 2 signal egdes.)

$$\begin{aligned} P[\text{right vertex}|I, R=r] &= P[d_{v^*} > \max_{v \neq v^*} d_v | I, R=r] \\ &= P[d_{v^*} > d(2)_{(r:r)}, d_{v^*} > d(1)_{(m-2r-1:m-2r-1)}, d_{v^*} > d(0)_{(n-m+r:n-m+r)}] \end{aligned}$$

Similarly for $\bar{I}$,

$$P[\text{right vertex}|\bar{I}, R=r] = P[d_{v^*} > d(2)_{(r:r)}, d_{v^*} > d(1)_{(m-2r:m-2r)}, d_{v^*} > d(0)_{(n-m+r-1:n-m+r-1)}]$$

Notice that $d_{v^*} \sim$Binomial$(q, ms_1)$+Binomial$(p, s_1(2n-1-m))$, and $d_i \sim$Binomial$(q, s_1 i)$+Binomial$(p, s_1(2n-1-i))$. Let us refine our old notation of a normal random variable to be able to reflect number of verticies,

$$N_{p,n} \sim \text{Normal}(p, p(1-p)/n)$$

Using the same normal approximation as the agnostic method

$$\begin{aligned} d(0)_{(n-m+r:n-m+r)} &\approx N_{p,s_1(2n-1)_{(n-m+r:n-m+r)}} \\ d(1)_{(m-2r:m-2r)} &\approx N_{p,s_1(2n-2)_{(m-2r:m-2r)}} + N_{q,s_1} \\ d(2)_{(r:r)} &\approx N_{p,s_1(2n-3)_{(r:r)}} + N_{q,2s_1} \\ d_{v^*} &\approx N_{q,s_1 m} + N_{p,s_1(2n-1-m)}. \end{aligned}$$

When $n \to \infty$ the dependence of $d_v$ diminishes []. Thus

$$\begin{aligned}
P[\text{right vertex}|I, R=r] &= P[d_{v^*} > d(2)_{(r:r)}, d_{v^*} > d(1)_{(m-2r-1:m-2r-1)}, d_{v^*} > d(0)_{(n-m+r:n-m+r)}] \\
&\approx \int f_{d_{v^*}}(z) P[z > d(2)_{(r:r)}] P[z > d(1)_{(m-2r-1:m-2r-1)}] P[z > d(0)_{(n-m+r:n-m+r)}]\ dz \\
&= \int P[\text{Binomial}(q, ms_1) + \text{Binomial}(p, s_1(2n-1-m)) = z] \\
&\quad P[z > d(2)_{(r:r)}] P[z > d(1)_{(m-2r-1:m-2r-1)}] P[z > d(0)_{(n-m+r:n-m+r)}]\ dz
\end{aligned}$$

Applying the normal approximation for binomials,

$$\begin{aligned}
&\approx \int P[N_{q,s_1 m} + N_{p,s_1(2n-1-m)} = z] P[z > N_{p,s_1(2n-3)_{(r:r)}} + N_{q,2s_1}] \\
&\quad P[z > N_{p,s_1(2n-2)_{(m-2r-1:m-2r-1)}} + N_{q,s_1}] P[z > N_{p,s_1(2n-1)_{(n-m+r:n-m+r)}}]\ dz \\
&= \int \left( \int_{z>y} P[N_{q,s_1 m} + N_{p,s_1(2n-1-m)} = z | N_{q,s_1 m} = y] f_{N_{q,s_1 m}}(y)\ dy \right) \\
&\quad \left( \int_{z>x} P[z > N_{p,s_1(2n-3)_{(r:r)}} + N_{q,2s_1} | N_{q,2s_1} = x] f_{N_{q,2s_1}}(x)\ dx \right) \\
&\quad \left( \int_{z>w} P[z > N_{p,s_1(2n-2)_{(m-2r-1:m-2r-1)}} + N_{q,s_1} | N_{q,s_1} = w] f_{N_{q,s_1}}(w)\ dw \right) \\
&\quad P[z > N_{p,s_1(2n-1)_{(n-m+r:n-m+r)}}]\ dz \\
&= \int \left( \int_{z>y} f_{N_{q,s_1 m}}(y) f_{N_{p,s_1(2n-1-m)}}(z-y)\ dy \right) \\
&\quad \left( \int_{z>x} F^r_{N_{p,s_1(2n-3)}}(z-x) f_{N_{q,2s_1}}(x)\ dx \right) \\
&\quad \left( \int_{z>w} F^{m-2r-1}_{N_{p,s_1(2n-2)}}(z-w) f_{N_{q,s_1}}(w)\ dw \right) F^{n-m+r}_{N_{p,s_1(2n-1)}}(z)\ dz \\
&= \iiiint\limits_{z>y, z>x, z>w} F^r_{N_{p,s_1(2n-3)}}(z-x) F^{m-2r-1}_{N_{p,s_1(2n-2)}}(z-w) F^{n-m+r}_{N_{p,s_1(2n-1)}}(z) \\
&\quad f_{N_{p,s_1(2n-1-m)}}(z-y) f_{N_{q,s_1 m}}(y) f_{N_{q,2s_1}}(x) f_{N_{q,s_1}}(w)\ dw\ dx\ dy\ dz
\end{aligned}$$

Similarly when conditioning on $\bar{I}$ instead,

$$\begin{aligned}
P[\text{right vertex}|\bar{I}, R=r] &\approx \iiiint\limits_{z>y, z>x, z>w} F^r_{N_{p,s_1(2n-3)}}(z-x) F^{m-2r}_{N_{p,s_1(2n-2)}}(z-w) F^{n-m+r-1}_{N_{p,s_1(2n-1)}}(z) \\
&\quad f_{N_{p,s_1(2n-1-m)}}(z-y) f_{N_{q,s_1 m}}(y) f_{N_{q,2s_1}}(x) f_{N_{q,s_1}}(w)\ dw\ dx\ dy\ dz
\end{aligned}$$

Notice after picking the right vertex that the probability of $T$ is the same as before except instead of $n^2 - m$ edges with probability $p$ there are only $2n-1-m$ edges. If the wrong vertex is chosen, then it is still possible to have 0, 1, or 2 signal edges. For $i < 2$

$$P[i \text{ signal edges possible}|\text{wrong vertex}] = \frac{\binom{m}{i}\binom{2n-3}{m-i}}{\binom{2n-1}{m}} \tag{4}$$

The distribution of $T = t$ for $t > 2$ can be approximated as follows

$$\begin{aligned}
P[T=t] &= P[\text{right vertex}] P[T=t|\text{right vertex}] + P[\text{wrong vertex}] P[T=t|\text{wrong vertex}] && (5) \\
&\approx P[\text{right vertex}] P[N_{p_{(2n-1-m-k+t:2n-1-m)}} < N_{q_{(m-t+1:m)}}, N_{q_{(m-t:m)}} < N_{p_{(2n-1-m-k+t+1:2n-1-m)}}] && (6) \\
&\quad +0 && (7) \\
&= && (8)
\end{aligned}$$

For $t \leq 2$,

$$
\begin{aligned}
P[T = t] &= P[\text{right vertex}]P[T = 2|\text{right vertex}] + P[\text{wrong vertex}]P[T = t|\text{wrong vertex}] && (9)\\
&= P[\text{right vertex}]P[N_{p_{(2n-1-m-k+t:2n-1-m)}} < N_{q_{(m-t+1:m)}}, N_{q_{(m-t:m)}} < N_{p_{(2n-1-m-k+t+1:2n-1-m)}}] && (10)\\
&\quad +P[\text{wrong vertex}]\sum_{i=0}^{2} P[T = t|\text{wrong edge, } i \text{ possible}]P[i \text{ possible}|\text{wrong edge}] && (11)\\
&= P[\text{right vertex}]P[N_{p_{(2n-1-m-k+t:2n-1-m)}} < N_{q_{(m-t+1:m)}}, N_{q_{(m-t:m)}} < N_{p_{(2n-1-m-k+t+1:2n-1-m)}}] && (12)\\
&\quad +P[\text{wrong vertex}]\sum_{i\geq t} P[T = t|\text{wrong edge, } i \text{ possible}]P[i \text{ possible}|\text{wrong edge}] && (13)\\
&= && (14)
\end{aligned}
$$

$P[T = t|\text{wrong edge, } i \text{ possible}]$ is again similarly computable as $P[T = t]$ except there are only $2n-1-m$ total edges and $i$ signal edges.