Conservative Vector Fields

Notes for 12 April 2022

Review of Concepts

A vector field is a function that assigns vectors to points in space:

> $\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$

Vector fields can be used to model fluid flows, electrical fields, gravitational fields, etc.

A vector field is called conservative if it is the gradient of some scalar function.

> $\vec{F}$ is conservative if there exists a function f such that $\vec{F} = \nabla f$.

f is called a potential function.

Question: If we are given a vector field, how can we determine if it is conservative?

In two-dimensional space:

> Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field on an open, simply-connected region D, and P, Q have continuous first-order partial derivatives, and $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ throughout D. Then $\vec{F}$ is conservative.

What about three-dimensional space?

Let's first learn about the curl of a vector field and how it can be used to show the vector field is conservative.

Curl

Physically, the curl represents the rotation or circulation caused by a vector field at any given point. The curl produces another vector field that defines these rotations throughout the field. If we look at the curl at a given point, we would have a vector that defines the axis of rotation. The magnitude of the vector would be the amount of spin about that axis.

Mathematically, we define the curl of a vector function as follow:

> Let $\vec{F}(x, y, z) = P\hat{i} + Q\hat{j} + R\hat{k}$ be a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist. Then the curl of $\vec{F}$ is a vector field on $\mathbb{R}^3$ defined by curl

> $$(\vec{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

We have some special notation that can help us to generate this formula.

First, define the del operator as $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

Applying the del operator to a function f, we have $\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$.

Taking the cross-product of the del operator with a vector field $\vec{F}$ and using the determinant notation, we have

> $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \hat{\imath}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Example: Determine the curl of $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + xz\hat{k}$

> $$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$
>
> $$= \hat{\imath}\left(\frac{\partial(xz)}{\partial y} - \frac{\partial(yz)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(xz)}{\partial x} - \frac{\partial(xy)}{\partial z}\right) + \hat{k}\left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)$$
>
> $$= \hat{\imath}\,(-y) - \hat{\jmath}\,(z) + \hat{k}\,(-x)$$
>
> $$= -\left(y\hat{\imath} + z\hat{\jmath} + x\hat{k}\right)$$

Theorem:

> If f is a function of three variables and has continuous second-order partial derivatives, then curl $(\nabla f) = \vec{0}$.

Proof:

> Since $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$, replace P, Q, and R in the curl formula with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$, respectively.
>
> Then, $\text{curl}(\nabla f) = \hat{\imath}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \hat{\jmath}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$.
>
> By continuity of the f and its derivatives, the order in which the partials are taken can be exchanged. Thus, each of the components reduces to zero, and we have curl $(\nabla f) = \vec{0}$.

Notice that if $\vec{F}$ is a conservative vector field, then $\vec{F} = \nabla f$ for some f and curl $(\vec{F}) =$ curl $(\nabla f) = \vec{0}$.

Warning: The converse is not necessarily true!

But, under very specific situations, we can determine that a vector field is conservative.

Theorem:

> If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ and the components have continuous partial derivatives and curl$(\vec{F}) = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Example: Determine if the vector function $\vec{F} = \langle yz, xz - 2y, xy + 1\rangle$ is conservative.

> First, we see that $\vec{F}$ is defined on all of $\mathbb{R}^3$. That is, there are no domain restrictions on the componenents of $\vec{F}$.
>
> Next, the components of $\vec{F}$ do have continuous partial derivatives.
>
> Finally,
>
> $$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz - 2y & xy + 1 \end{vmatrix}$$
>
> $$= \hat{i}\left(\frac{\partial(xy+1)}{\partial y} - \frac{\partial(xz-2y)}{\partial z}\right) - \hat{j}\left(\frac{\partial(xy+1)}{\partial x} - \frac{\partial(yz)}{\partial z}\right) + \hat{k}\left(\frac{\partial(xz-2y)}{\partial x} - \frac{\partial(yz)}{\partial y}\right)$$
>
> $$= \hat{i}\,(x - x) - \hat{j}\,(y - y) + \hat{k}\,(z - z)$$
>
> $$= \vec{0}$$
>
> Because all three conditions are met, the given vector field is conservative.

Finding the Potential Function of a Conservative Vector Function.

You may have seen how to determine the potential function of a two-dimensional conservative vector function. Finding it for a three-dimensional conservative vector function is similar, but has an extra step.

To see this, let's take the vector function that we just showed was conservative and determine the potential function, f.

Example: Find the potential function for $\vec{F} = \langle yz, xz - 2y, xy + 1\rangle$

> Starting with the first component of \vec{F}, we have

> $f_x = yz \implies \boxed{f(x,y,z) = xyz + g(y,z)}$
>
> Taking the partial with respect to y gives us $f_y = xz + \frac{\partial g(y,z)}{\partial y}$.
>
> Setting this equal to the second component of $\vec{F}$, we have
>
> $xz + \frac{\partial g(y,z)}{\partial y} = xz - 2y$
>
> The xz terms cancel, and we are left with
>
> $\frac{\partial g(y,z)}{\partial y} = -2y \implies g(y,z) = -y^2 + h(z)$. Combining this with the previous result, we have $\boxed{f(x,y,z) = xyz - y^2 + h(z)}$.
>
> Finally, taking the partial with respect to z gives us $f_z = xy + h'(z)$.
>
> Setting this equal to the third component of $\vec{F}$, we have
>
> $xy + h'(z) = xy + 1$
>
> The xy terms cancel and we are left with
>
> $h'(z) = 1 \implies h(z) = z + K$. Putting it all together, we have $\boxed{\boxed{f(x,y,z) = xyz - y^2 + z + K}}$

A Problem for You to Try

Determine if $\vec{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ is conservative.

If it is conservative, determine its potenial function.