

Lecture 3: Integer representation

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September 3, 2021

601.229 Computer Systems Fundamentals



Integer representation

Representing integers

- ▶ We've seen how to represent unsigned (nonnegative) integers
 - ▶ Bit string interpreted as a binary (base 2) number
- ▶ How to represent signed integers?
 - ▶ Sign magnitude
 - ▶ Ones' complement
 - ▶ Two's complement
- ▶ In examples that follow, we'll use 4-bit words
 - ▶ Ideas will generalize to larger word sizes

Desired features for signed representation

What we want in a representation for signed integers:

- ▶ About half of encoding space used for negative values
- ▶ Each represented integer has a unique encoding as bit string
- ▶ Straightforward way to do arithmetic

Sign magnitude representation

Let most significant bit be a sign bit: **0**→positive, **1**→negative

| Bit string | value | Bit string | value |
|--------------|-------|--------------|-------|
| 0 000 | 0 | 1 000 | -0 |
| 0 001 | 1 | 1 001 | -1 |
| 0 010 | 2 | 1 010 | -2 |
| 0 011 | 3 | 1 011 | -3 |
| 0 100 | 4 | 1 100 | -4 |
| 0 101 | 5 | 1 101 | -5 |
| 0 110 | 6 | 1 110 | -6 |
| 0 111 | 7 | 1 111 | -7 |

Downsides: two representations of 0, arithmetic complicated by sign bit

Ones' complement

Ones' complement: to represent $-x$, invert all of the bits of x

| Bit string | value | Bit string | value |
|------------|-------|------------|-------|
| 0000 | 0 | 1000 | -7 |
| 0001 | 1 | 1001 | -6 |
| 0010 | 2 | 1010 | -5 |
| 0011 | 3 | 1011 | -4 |
| 0100 | 4 | 1100 | -3 |
| 0101 | 5 | 1101 | -2 |
| 0110 | 6 | 1110 | -1 |
| 0111 | 7 | 1111 | -0 |

Downsides: two representations of 0, slightly complicated arithmetic

Sign magnitude and ones' complement are obsolete

- ▶ Sign magnitude and ones' complement representations are not used for integer representation by modern computers
 - ▶ But, sign magnitude is used in floating point representation
- ▶ The rest of this lecture will discuss *two's complement*

Two's complement

Two's complement: in w -bit word, the most significant bit represents -2^{w-1}

E.g., when $w = 4$,

| Representation | Bit 3 | Bit 2 | Bit 1 | Bit 0 |
|------------------|-------|-------|-------|-------|
| Unsigned | 8 | 4 | 2 | 1 |
| Two's complement | -8 | 4 | 2 | 1 |

Given bit string 1011,

- ▶ Unsigned, 1011 is $8 + 2 + 1 = 11$
- ▶ Two's complement, 1011 is $-8 + 2 + 1 = -5$

Two's complement

Two's complement: in w -bit word, the most significant bit represents -2^{w-1}

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|------------|-------|------------|-------|
| 0000 | 0 | 1000 | -8 |
| 0001 | 1 | 1001 | -7 |
| 0010 | 2 | 1010 | -6 |
| 0011 | 3 | 1011 | -5 |
| 0100 | 4 | 1100 | -4 |
| 0101 | 5 | 1101 | -3 |
| 0110 | 6 | 1110 | -2 |
| 0111 | 7 | 1111 | -1 |

Note asymmetry of negative and positive ranges: -8 is represented, 8 isn't

Thinking about two's complement

Useful way to think about a w -bit two's complement representation:

- ▶ Bit $w - 1$ is the sign bit, 0→positive, 1→negative
- ▶ If sign bit is 0, usual unsigned interpretation
- ▶ If sign bit is 1, bits $w - 2 \dots 0$ indicate the “offset” from -2^{w-1}

Two's complement example

Given $w = 4$, example bit string is 1011

- ▶ Sign bit is 1
- ▶ Offset from -2^3 is 011, which is 3 ($2+1$)
- ▶ $-8 + 3 = -5$

So, 1011 represents -5

Clicker quiz

Clicker quiz omitted from public slides

Why two's complement?

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Unsigned addition yields correct
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Wow!

Trying it out

Add two 8 bit integer values:

00101101

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$$\begin{array}{r} 00101101 \\ + 11111100 \\ \hline \end{array}$$

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$$\begin{array}{r} 00101101 \\ + 11111100 \\ \hline 100101001 \end{array}$$

Trying it out

As unsigned values:

$$\begin{array}{r} 00101101 \quad 45 \\ + \quad 11111100 \quad 252 \\ \hline 100101001 \quad 297 \end{array} \quad (\text{truncated to } 41)$$

Trying it out

As signed two's complement values:

$$\begin{array}{r} 00101101 \quad 45 \\ + \quad 11111100 \quad -4 \\ \hline 100101001 \quad 41 \end{array}$$

Subtraction via addition

- ▶ Two's complement negation: invert all bits, then add 1
- ▶ Example, negating 5
 - ▶ Original value: 00000101
 - ▶ Invert bits: 11111010
 - ▶ Add one: 11111011
 - ▶ Value is $-128 + 64 + 32 + 16 + 8 + 2 + 1 = -5$
- ▶ $a - b$ can be computed as $a + -b$
 - ▶ I.e., invert b , then add to a

Sign extension

- ▶ Sometimes it is necessary to increase the number of bits in the representation of a signed integer
 - ▶ E.g., type cast or implicit conversion of a 16 bit `short` value to a 32 bit `int` value
- ▶ In two's complement, this can be accomplished by *sign extension*: replicate the original sign bit as many times as necessary
 - ▶ This preserves the numeric value!
 - ▶ Processors typically have dedicated instructions to perform sign extension

Sign extension example

Example: extend 4 bit two's complement values 1011 and 0011 to 8 bits

| Number of bits | Bit string | Meaning |
|----------------|--------------------------|--|
| 4 | <u>1</u> 011 | $-8 + 2 + 1 = -5$ |
| 8 | 1111 <u>1</u> 011 | $-128 + 64 + 32 + 16 + 8 + 2 + 1 = -5$ |
| 4 | <u>0</u> 011 | $2 + 1 = 3$ |
| 8 | 0000 <u>0</u> 011 | $2 + 1 = 3$ |

Sign extension example program

```
#include <stdio.h>

void printbits(int x, int n) {
    for (int i = n-1; i >= 0; i--) {
        putchar(x & (1 << i) ? '1' : '0');
    }
    putchar('\n');
}

int main(void) {
    short s = -27987;
    int i = (int) s;           // <-- sign extension occurs here
    printf("%*c", 16, ' ');
    printbits(s, 16);
    printbits(i, 32);
    return 0;
}
```


Sign extension example program (output)

```
$ gcc signext.c  
$ ./a.out  
1001001010101101  
11111111111111111001001010101101
```

Clicker quiz!

Clicker quiz omitted from public slides

Extending unsigned values

Extending the representation of an unsigned value is straightforward:
unconditionally pad with 0 bits

Example: 4 bit unsigned value $1011 = 8 + 2 + 1 = 11$

As an 8 bit unsigned value, **0000** $1011 = 8 + 2 + 1 = 11$

General observation

In general, increasing the number of bits in the representation of an integer (signed or unsigned) will preserve its value

Truncation

- ▶ Truncation: *reducing* the number of bits in the representation of an integer
 - ▶ In general, this will lose information and potentially change the value
- ▶ Truncation is done by chopping off bits from the left side of the bit string
 - ▶ Whatever remains is the new representation

Truncation example

Example: convert signed 8 bit integer -14 to a 4 bit signed integer

| Number of bits | Bit string | Meaning |
|----------------|------------|---------------------------------|
| 8 | 11110010 | $-128 + 64 + 32 + 16 + 2 = -14$ |
| 4 | 0010 | 2 |

Truncation example program

```
#include <stdio.h>

void printbits(int x, int n) {
    for (int i = n-1; i >= 0; i--) {
        putchar(x & (1 << i) ? '1' : '0');
    }
    putchar('\n');
}

int main(void) {
    short s = -129;
    char c = s;          // <-- truncation occurs here
    printf("s=%d, c=%d\n", s, c);
    printbits(s, 16);
    printf("%*c", 8, ' ');
    printbits(c, 8);
    return 0;
}
```

Truncation example program (output)

```
$ gcc truncate.c
$ ./a.out
s=-129, c=127
1111111110111111
      01111111
```

Explanation:

- ▶ `short` is a 16 bit signed type, `char`¹ is a signed 8 bit type
- ▶ After truncation from 16 to 8 bits, the sign bit was 0, so the resulting value became positive
- ▶ Look at the bit representations — convince yourself the values output by `printf` make sense!

¹Compiler-dependent, tested with gcc 7.4.0 on x86-64 Linux

Conversions between signed and unsigned

- ▶ Another important type of conversion is between signed and unsigned values
- ▶ Fundamentally, data in the computer's memory has *no inherent meaning*
- ▶ It is up to the *program* to decide how to interpret data
- ▶ Conversions between signed and unsigned (without changing the number of bits) *do not change the underlying representation as bits*

Signed/unsigned conversion examples

Example: bit pattern 10010110 as signed and unsigned 8 bit integer values

Signed: $-128 + 16 + 4 + 2 = -106$

Unsigned: $128 + 16 + 4 + 2 = 150$

Signed/unsigned conversion example program

```
#include <stdio.h>

unsigned char parsebits(const char *s) {
    unsigned char val = 0;
    char c;
    while ((c = *s++)) {
        val <<= 1;
        if (c == '1') { val |= 1; }
    }
    return val;
}

int main(void) {
    unsigned char uc = parsebits("10010110");
    char c = (char) uc;      // <-- conversion from unsigned to signed
    printf("%u %d\n", uc, c);
    return 0;
}
```

Signed/unsigned conversion example program (output)

```
$ gcc convert.c  
$ ./a.out  
150 -106
```

Considerations for writing programs

Programming considerations

- ▶ Semantics of integer values and data types can be surprisingly subtle
- ▶ C and C++ further complicate matters in several ways:
 - ▶ Data type sizes vary
 - ▶ Integer representation not actually specified by the language!
 - ▶ Some operations the program could perform have semantics that are implementation-defined or (worse) *undefined*
- ▶ Recommendation: be very careful!

Implicit conversions

- ▶ In C, there are many contexts in which *implicit conversions* will occur
 - ▶ Including ones where information can be lost!
- ▶ It's important to know where implicit conversions happen and to understand their effects
- ▶ It's not a bad idea to use explicit type casts so that conversions are *explicit*, even if they aren't strictly necessary
 - ▶ Semantics of program are more obvious, avoid unintended behaviors

Sign extension

- ▶ Sign extension can sometimes have surprising consequences (bits that you thought would be 0 become 1)
- ▶ Values belonging to unsigned types (`unsigned char`, `unsigned short`, etc.) are never sign extended