# Fair and Distributed Optimal Transport for Resource Allocation over Networks

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***Abstract*—Optimal transport is a framework that facilitates the most efficient allocation of a limited amount of resources. However, the most efficient allocation scheme does not necessarily preserve the most fairness. In this paper, we establish a framework which explicitly considers the fairness of resource allocation over a network with heterogeneous participants. As computing the transport strategy in a centralized fashion requires significant computational resources, it is imperative to develop computationally light algorithm that can be applied to large scale problems. To this end, we develop a fully distributed algorithm with fairness with provable convergence using alternating method of multipliers. In the designed algorithm, each corresponding pair of resource supplier and receiver compute their own solutions and update the schemes through negotiation which do not require a central planner. The distributed algorithm can yield a fair and efficient resource allocation mechanism over a network. We corroborate the obtained results through case studies.**

## I. Introduction

Optimal transport (OT) is a centralized framework that enables the design of efficient schemes for distributing resources by considering heterogeneous constraints between the resource suppliers and the receivers [1]. Efficiency in transporting and distributing resources has long been sought out. For example, if there were four mines and two factories in a town the owners would want to know how they can either distribute the raw material to the two factories while minimizing overall cost, and also take in material from the four different mines while minimizing their overall cost. OT designs a transport strategy that minimizes the overall cost for the whole operation. Other than the distribution of material, the optimal transport framework can be used for other applications, including the resource allocation of medical supplies and matching between employees and tasks in an enterprise network to maximize the profit.

Under the standard OT paradigm, the resource distribution scheme maximizes the aggregated utilities of all participants in a centralized way, regardless whether that distribution is fair for the suppliers or receivers [2], [3]. As in the mines and factories example, it may be that the most optimal distribution of the raw material is to have one factory receive 80% of the material while the other receives only 20%. This is not fair to the factory receiving the smaller amount of material, and could lead to that factory shutting down. Thus, it is necessary to incorporate fairness during the transport mechanism design for constrained resource allocation, especially in the scenarios that promote social equity.

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The transport network over which the resources are distributed becomes more complex with the inclusion of massive number of suppliers and receivers. This large-scale feature of the OT problem gives rise to another concern on the centralized computation of the transport plan. The required computation for centralized planning grows exponentially with the number of participants in the framework. To this end, we aim to develop a distributed algorithm for fair and efficient resource allocation where the centralized planner is not necessary. The distributed algorithm is obtained by leveraging alternating method of multipliers (ADMM) approach [4].

To enable a fair resource allocation, we include a fairness measure to the objective function in the OT framework. Therefore, the resulting transport plan will have a balance between the efficiency and fairness. In the designed ADMM-based distributed algorithm, each participant (resource supplier or receiver) only needs to solve its own problem and exchange the results with the corresponding connected agents, which enables parallel updates on the solution. The algorithm terminates when the solution computed at each pair of supplier and receiver coincides, at which point the transport strategy given by our developed distributed algorithm is the same as the one under centralized design.

Our designed distributed algorithm offers insights for fair and efficient resource distribution over networks. First, the updates of transport strategies at both the supplier side and the receiver side can be seen as bargaining for the resources transferred. The bargaining process ends when both parties reach an agreement. Furthermore, during each update, each receiver node in the network proposes a solution that explicitly considers the fairness. In comparison, the supplier nodes solely focus on maximizing their payoff by selling their resources. At the next round of updates, each pair of supplier and receiver will propose a distribution scheme that is closer to the average of their previous solutions. It indicates that, as the bargaining processes, the resource suppliers will also consider the fairness and the receivers will take into account the efficiency of the transport plan to have a consensus. Our designed distributed algorithm can also be implemented online conveniently which is adaptive to the changes in the resource allocation network and participants' preferences mid calculation.

The contributions of this paper are summarized as follows. First, we establish a framework that can yield fair and efficient resource transportation over networks. Second, we develop a distributed algorithm based on ADMM to compute the transport strategy in which the resource suppliers and receivers negotiate iteratively. Third, we use case studies to corroborate the effectiveness and applicability to changing environment of the algorithm.

*Related Works:* Optimal resource allocation/matching has been investigated vastly in various fields and applications, including communication networks [2], energy systems [5], critical infrastructure [6] and cyber systems [7]. To compute the optimal transport strategy efficiently, a number of techniques have been developed, such as simultaneous approximation [8], population-based optimization [9], and distributed algorithms [3], [10]. Our work is also related to fair allocation of constrained resources [11], [12]. In this work, we leverage ADMM to develop a fast computational mechanism for fair and efficient resource matching over large-scale networks.

*Organization of the Paper:* The rest of the paper is organized as follows. Section II formulates a framework for fair resource allocation over a network. Section III develops a distributed algorithm to compute the optimal solution. Section IV discusses the interpretations of the designed algorithm. Section V corroborates the results using extensive case studies, and finally Section VI concludes the paper.

## II. Problem Formulation

In this section, we first present a standard OT framework for limited resource allocation over a network. Then, we extend the framework to a fair OT setting by considering the fairness in the transport strategy design.

In a network, we denote by $\mathscr{X} := \{1,...,|\mathscr{X}|\}$ the set of destinations/targets that receive the resources, and $\mathscr{Y} := \{1,...,|\mathscr{Y}|\}$ the set of origins/sources that distribute resources to the targets. Specifically, each source node $y \in \mathscr{Y}$ is connected to a number of target nodes denoted by $\mathscr{X}_y$, representing that $y$ has choices in allocating its resources to a specific group of destinations $\mathscr{X}_y$ in the network. Similarly, it is possible that each target node $x \in \mathscr{X}$ receives resources from multiple source nodes, and this set of suppliers to node $x$ is denoted by $\mathscr{Y}_x$. Note that $\mathscr{X}_y$, $\forall y$ and $\mathscr{Y}_x$, $\forall x$ are nonempty. Otherwise, the corresponding nodes are isolated in the network and do not play a role in the considered optimal transport strategy design. It is also straightforward to see that the resources are transported over a bipartite network, where one side of the network consists of all source nodes and the other includes all destination nodes. This bipartite graph may not be complete due to constrained matching policies between participants. Another reason yielding incomplete bipartite graph in practice can be the infeasible transport of resources between certain pairs of source and destination nodes incurred by long transport distance. For convenience, we denote by $\mathscr{E}$ the set including all feasible transport paths in the network, i.e., $\mathscr{E} := \{(x,y)|x \in \mathscr{X}_y, y \in \mathscr{Y}\}$. Note that $\mathscr{E}$ also refers to the set of all edges in the established bipartite graph for resource transportation.

We denote by $\pi_{xy} \in \mathbb{R}_+$ the amount of resources transported from the origin node $y \in \mathscr{Y}$ to the destination node $x \in \mathscr{X}$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. For convenience, let $\Pi := \{\pi_{xy}\}_{x\in\mathscr{X}_y, y\in\mathscr{Y}}$ be the transport plan designed for the considered network. To this end, the centralized optimal transport problem can be formulated as follows:

$$\begin{aligned}\max_{\Pi}\ & \sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}) + \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}\left(s_{xy}(\pi_{xy}) - c_{xy}(\pi_{xy})\right)\\ \text{s.t.}\quad & \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi_{xy} \leq \bar{p}_x,\ \forall x \in \mathscr{X},\\ & \underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi_{xy} \leq \bar{q}_y,\ \forall y \in \mathscr{Y},\\ & \pi_{xy} \geq 0,\ \forall (x,y) \in \mathscr{E},\end{aligned} \tag{1}$$

where $t_{xy} : \mathbb{R}_+ \to \mathbb{R}$ and $s_{xy} : \mathbb{R}_+ \to \mathbb{R}$ are utility functions for target node $x$ and source node $y$, respectively; $c_{xy} : \mathbb{R}_+ \to \mathbb{R}$ is a cost function of source node $y$ for transporting resources to target node $x$. Furthermore, $\bar{p}_x \geq \underline{p}_x \geq 0$, $\forall x \in \mathscr{X}$ and $\bar{q}_y \geq \underline{q}_y \geq 0$, $\forall y \in \mathscr{Y}$. The constraints $\underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi_{xy} \leq \bar{p}_x$ and $\underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi_{xy} \leq \bar{q}_y$ capture the limitations on the amount of requested and transferred resources at the target $x$ and source $y$, respectively.

We have the following assumption on the utilities functions $t_{xy}$ and $s_{xy}$ and the cost function $\pi_{xy}$.

**Assumption 1.** *The utility functions $t_{xy}$ and $s_{xy}$ are concave and monotonically increasing, and the transport cost function $c_{xy}$ is convex and monotonically increasing on $\pi_{xy}$, $\forall x \in \mathscr{X}, \forall y \in \mathscr{Y}$.*

There are a number of functions of interest that satisfy the properties in Assumption 1. For example, the utility functions $t_{xy}$ and $s_{xy}$ can adopt a linear form, indicating a linear growth of payoff on the amount of transferred and consumed resources. $t_{xy}$ and $s_{xy}$ can also take a logarithmic form on the argument, representing the marginal utility decreases with the amount of transported resources. The cost function $c_{xy}$ can admit linear and quadratic forms, capturing the flat and increasing growth of transport costs on the resources, respectively.

In the above formulation, there is no consideration of fairness in resource allocation. The central planner devises an optimal transport strategy by maximizing the social welfare. In practice, some target nodes may not contribute as significant as other nodes to the social objective by receiving a certain amount of resources from the sources. This efficient resource allocation plan yields a larger objective values. However, it is not fair for some nodes if their requests for resources are ignored. For example, in energy systems, the resilience planing should take into account these generally under considered communities which are hit heavily by natural disasters. Though from the central planner's perspective, the resilience planning in these areas may not contribute as significant as other areas to the system's utility by cost-benefit analysis.

Therefore, it is urgent to incorporate the equity consideration during resource allocation. One possible way to achieve this goal is to introduce a fairness measure to the objective function in the optimal transport framework as follows:

$$\begin{aligned}&\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}) + \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}\left(s_{xy}(\pi_{xy}) - c_{xy}(\pi_{xy})\right)\\ &\qquad + \sum_{x\in\mathscr{X}}\omega_x f_x\Big(\sum_{y\in\mathscr{Y}_x}\pi_{xy}\Big),\end{aligned} \tag{2}$$

where $\omega_x \geq 0$ is a weighting constant for fairness, and $f_x : \mathbb{R}_+ \to \mathbb{R}$. Note that $\sum_{y\in\mathscr{Y}_x} \pi_{xy}$ is total amount of resources received for the target node $x$. Thus, $f_x(\sum_{y\in\mathscr{Y}_x} \pi_{xy})$ quantifies the level of fairness by allocating $\sum_{y\in\mathscr{Y}_x} \pi_{xy}$ resources to each target $x$. To facilitate a fair transport strategy, the central planner needs to devise $f_x$ strategically. One consideration is that the marginal utility of the fairness term $f_x$ should decrease. Otherwise, it will lead to an unfair distribution of resources, i.e., some target nodes receive most of the resources in the network as the central planner aims to maximize $\sum_{x\in\mathscr{X}} \omega_x f_x(\sum_{y\in\mathscr{Y}_x} \pi_{xy})$.

We have the following assumption on the property of fairness function.

**Assumption 2.** *The fairness functions $f_x$, $\forall x \in \mathscr{X}$ are concave and monotonically increasing.*

There can be various choices for the fairness function. One possible choice is a proportional fairness function [12]:

$$f_x(\sum_{y\in\mathscr{Y}_x} \pi_{xy}) = \log(\sum_{y\in\mathscr{Y}_x} \pi_{xy} + 1),\ \forall x \in \mathscr{X}. \tag{3}$$

To this end, the central planner's goal is to devise a fair and efficient transport strategy that maximizes the objective function (2) while takes into account the same set of constraints on resources capacity in (1).

## III. Distributed Algorithm for Fair and Efficient Transport Strategy Design

The planner can solve the formulated optimization problem in Section II in a centralized manner. One primal concern is the computational feasibility. It can be computationally expensive in obtaining the fair and efficient resource distribution plan when the number of sources and targets becomes enormous as can be observed in a large-scale network for resource allocation. Therefore, we shift our attention in finding a fair and efficient transport strategy from a centralized way to a fully distributed fashion.

### A. Feasibility and Optimality

Before developing the distributed algorithm, we first analyze the feasibility of the formulated optimization problem.

**Lemma 1.** *It is feasible to find a fair transport plan $\Pi$ if the following conditions are satisfied:*

$$\sum_{y\in\mathscr{Y}_x} \bar{q}_y \geq \underline{p}_x, \quad \forall x \in \mathscr{X}, \tag{4}$$

$$\sum_{y\in\mathscr{Y}} \bar{q}_y \geq \sum_{x\in\mathscr{X}} \underline{p}_x. \tag{5}$$

The two inequalities in Lemma 1 have natural interpretations. (4) ensures that all the target nodes' requests can be fulfilled. (5) indicates the the total demand of resources is less than the total supply that the source nodes can provide.

We next characterize the existence of optimal solution to the formulated problem.

**Lemma 2.** *Under Assumptions 1 and 2, and the inequalities* (4) *and* (5)*, there exists a fair and efficient transport strategy that maximizes the objective* (2) *while satisfying the constraints in* (1).

The existence of the optimal solution is guaranteed by the concavity of $t_{xy}$, $s_{xy}$ and $f_x$ and the convexity of $c_{xy}$, as well as the feasibility of the problem resulting from (4) and (5).

### B. Distributed Algorithm

In this subsection, we aim to develop a distributed algorithm to solve the formulated problem. Our first step is to rewrite the optimization problem in the ADMM form by introducing ancillary variables $\pi^t_{xy}$ and $\pi^s_{xy}$. The superscripts $t$ and $s$ indicates that the corresponding parameters belong to the target node or the source node, respectively. We then set $\pi_{xy} = \pi^t_{xy}$ and $\pi_{xy} = \pi^s_{xy}$, indicating the solutions proposed by the targets and sources are consistent with the ones proposed by the central planner. This reformulation facilitates the design of a distributed algorithm which allows us to iterate through the process in obtaining the fair and efficient transport plan. To this end, the reformulated optimal transport problem under fairness consideration is presented as follows:

$$\begin{aligned}
\min_{\Pi_t\in\mathscr{F}_t,\Pi_s\in\mathscr{F}_s,\Pi} & -\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi^t_{xy}) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} (s_{xy}(\pi^s_{xy}) \\
& - c_{xy}(\pi^s_{xy})) - \sum_{x\in\mathscr{X}} \omega_x f_x(\sum_{y\in\mathscr{Y}_x} \pi^t_{xy}) \\
\text{s.t.} \quad & \pi^s_{xy} = \pi_{xy},\ \forall (x,y) \in \mathscr{E}, \\
& \pi^t_{xy} = \pi_{xy},\ \forall (x,y) \in \mathscr{E},
\end{aligned} \tag{6}$$

where $\Pi_t := \{\pi^t_{xy}\}_{x\in\mathscr{X}_y, y\in\mathscr{Y}}$, $\Pi_s := \{\pi^s_{xy}\}_{x\in\mathscr{X}, y\in\mathscr{Y}_x}$, $\mathscr{F}_t := \{\Pi_t | \pi^t_{xy} \geq 0, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x} \pi^t_{xy} \leq \bar{p}_x,\ (x,y) \in \mathscr{E}\}$, and $\mathscr{F}_s := \{\Pi_s | \pi^s_{xy} \geq 0, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y} \pi^s_{xy} \leq \bar{q}_y,\ (x,y) \in \mathscr{E}\}$.

Note that we transform the original maximization of the social utility problem to an equivalent program of minimizing the aggregated cost. Furthermore, due to the constraints, the optimal solutions of $\Pi_t$, $\Pi_s$, and $\Pi$ to (6) are the same. Our next focus is to develop a distributed algorithm to solve the problem (6). We let $\alpha^s_{xy}$ and $\alpha^t_{xy}$ be the Lagrangian multipliers associated with the constraint $\pi^s_{xy} = \pi_{xy}$ and $\pi^t_{xy} = \pi_{xy}$, respectively. The Lagrangian then facilitates the application of ADMM in the distributed algorithm design. Specifically, the Lagrangian associated with the optimization problem (6) can then be written as follows:

$$\begin{aligned}
& L\left(\Pi_t, \Pi_s, \Pi, \alpha^t_{xy}, \alpha^s_{xy}\right) = \\
& -\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi^t_{xy}) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} \left(s_{xy}(\pi^s_{xy}) - c_{xy}(\pi^s_{xy})\right) \\
& -\sum_{x\in\mathscr{X}} \omega_x f_x(\sum_{y\in\mathscr{Y}_x} \pi^t_{xy}) + \sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} \alpha^t_{xy}(\pi^t_{xy} - \pi_{xy}) \\
& +\sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} \alpha^s_{xy}(\pi_{xy} - \pi^s_{xy}) + \frac{\eta}{2}\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} (\pi^t_{xy} - \pi_{xy})^2 \\
& +\frac{\eta}{2}\sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} (\pi_{xy} - \pi^s_{xy})^2,
\end{aligned} \tag{7}$$

where $\eta > 0$ is a positive scalar constant controlling the convergence rate in the algorithm designed below.

Note that in (7), the last two terms $\frac{\eta}{2}\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x}(\pi^t_{xy} - \pi_{xy})^2$ and $\frac{\eta}{2}\sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}(\pi_{xy} - \pi^s_{xy})^2$, acting as penalization, are quadratic. Hence, the Lagrangian function $L$ is strictly convex, ensuring the existence of a unique optimal solution.

We can apply ADMM to the minimization problem in (6). The designed distributed algorithm is presented in the following proposition.

**Proposition 1.** *The iterative steps of ADMM to* (6) *are summarized as follows:*

$$\begin{aligned}\Pi_{x,t}(k+1) \in \arg\min_{\Pi_{x,t}\in\mathscr{F}_{x,t}} & -\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}^t) - \omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}^t) \\ & + \sum_{y\in\mathscr{Y}_x}\alpha_{xy}^t(k)\pi_{xy}^t + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}(\pi_{xy}^t - \pi_{xy}(k))^2,\end{aligned} \tag{8}$$

$$\begin{aligned}\Pi_{y,s}(k+1) \in \arg\min_{\Pi_{y,s}\in\mathscr{F}_{y,s}} & -\sum_{x\in\mathscr{X}_y}\left(s_{xy}(\pi_{xy}^s) - c_{xy}(\pi_{xy}^s)\right) \\ & - \sum_{x\in\mathscr{X}_y}\alpha_{xy}^s(k)\pi_{xy}^s + \frac{\eta}{2}\sum_{x\in\mathscr{X}_y}(\pi_{xy}(k) - \pi_{xy}^s)^2,\end{aligned} \tag{9}$$

$$\begin{aligned}\pi_{xy}(k+1) = \arg\min_{\pi_{xy}} & -\alpha_{xy}^t(k)\pi_{xy} + \alpha_{xy}^s(k)\pi_{xy} \\ & + \frac{\eta}{2}(\pi_{xy}^t(k+1) - \pi_{xy})^2 + \frac{\eta}{2}(\pi_{xy} - \pi_{xy}^s(k+1))^2,\end{aligned} \tag{10}$$

$$\alpha_{xy}^t(k+1) = \alpha_{xy}^t(k) + \eta(\pi_{xy}^t(k+1) - \pi_{xy}(k+1))^2, \tag{11}$$

$$\alpha_{xy}^s(k+1) = \alpha_{xy}^s(k) + \eta(\pi_{xy}(k+1) - \pi_{xy}^s(k+1))^2, \tag{12}$$

*where* $\Pi_{\tilde{x},t} := \{\pi_{xy}^t\}_{y\in\mathscr{Y}_x, x=\tilde{x}}$ *includes the represents solution at target node* $\tilde{x}\in\mathscr{X}$*, and* $\Pi_{\tilde{y},s} := \{\pi_{xy}^s\}_{x\in\mathscr{X}_y, y=\tilde{y}}$ *represents the proposed solution at source node* $\tilde{y}\in\mathscr{Y}$*. In addition,* $\mathscr{F}_{x,t} := \{\Pi_{x,t} | \pi_{xy}^t \geq 0, y\in\mathscr{Y}_x, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi_{xy}^t \leq \bar{p}_x\}$*, and* $\mathscr{F}_{y,s} := \{\Pi_{y,s} | \pi_{xy}^s \geq 0, x\in\mathscr{X}_y, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi_{xy}^s \leq \bar{q}_y\}$*.*

*Proof.* Let $\vec{x} = [\vec{\Pi}_{x,t}^T, \vec{\Pi}^T]^T$, $\vec{y} = [\vec{\Pi}^T, \vec{\Pi}_{y,s}^T]^T$, and $\alpha = [\{\alpha_{xy}^t\}^T, \{\alpha_{xy}^s\}^T]^T$, where $\vec{\cdot}$ denotes the vectorization operator. We note that these vectors are all $2N\times 1$ where $N$ is the number of connections between targets and sources. This is also the size of $\mathscr{E}$. Now we can write the constraints in matrix form such that $A\vec{x} = \vec{y}$ where $A = [\mathbf{I},\mathbf{0},\mathbf{I},\mathbf{0}]$. Here $\mathbf{I}$ and $\mathbf{0}$ denote the identity and zero matrices respectively, both of which are $N\times N$. Next, we note that $\vec{x}\in\mathscr{F}_{\vec{x},t}$ and $\vec{y}\in\mathscr{F}_{\vec{y},s}$, where $\mathscr{F}_{\vec{x},t} = \{\vec{x} | \pi_{xy}^t \geq 0, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi_{xy}^t \leq \bar{p}_x, \{x,y\}\in\mathscr{E}\}$, $\mathscr{F}_{\vec{y},s} := \{\vec{y} | \pi_{xy}^s \geq 0, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi_{xy}^s \leq \bar{q}_y, \{x,y\}\in\mathscr{E}\}$. In turn we can solve the minimization in (6) with the iterations: 1) $\vec{x}(k+1)\in\arg\min_{\vec{x}\in\mathscr{F}_{x,t}} L(\vec{x},\vec{y}(k),\alpha(k))$; 2) $\vec{y}(k+1)\in\arg\min_{\vec{y}\in\mathscr{F}_{y,s}} L(\vec{x}(k),\vec{y},\alpha(k))$; 3) $\alpha(k+1) = \alpha(k) + \eta(A\vec{x}(k+1) - \vec{y}(k+1))$, whose convergence is proved [4]. Because we have no coupling among $\Pi_{x,t}, \Pi_{y,s}, \pi_{xy}, \alpha_{xy}^t$, and $\alpha_{xy}^s$ the above iterations can be decomposed to equations (8)-(12). ■

We can simplify equations (8)-(12) down to four equations, and the results are summarized below.

**Proposition 2.** *The iterations* (8)-(12) *can be simplified as follows:*

$$\begin{aligned}\Pi_{x,t}(k+1) \in \arg\min_{\Pi_{x,t}\in\mathscr{F}_{x,t}} & -\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi_{xy}^t) - \omega_x f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}^t) \\ & + \sum_{y\in\mathscr{Y}_x}\alpha_{xy}(k)\pi_{xy}^t + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}\left(\pi_{xy}^t - \pi_{xy}(k)\right)^2,\end{aligned} \tag{13}$$

$$\begin{aligned}\Pi_{y,s}(k+1) \in \arg\min_{\Pi_{y,s}\in\mathscr{F}_{y,s}} & -\sum_{x\in\mathscr{X}_y}\left(s_{xy}(\pi_{xy}^s) - c_{xy}(\pi_{xy}^s)\right) \\ & - \sum_{x\in\mathscr{X}_y}\alpha_{xy}(k)\pi_{xy}^s + \frac{\eta}{2}\sum_{x\in\mathscr{X}_y}\left(\pi_{xy}(k) - \pi_{xy}^s\right)^2,\end{aligned} \tag{14}$$

$$\pi_{xy}(k+1) = \frac{1}{2}\left(\pi_{xy}^t(k+1) + \pi_{xy}^s(k+1)\right), \tag{15}$$

$$\alpha_{xy}(k+1) = \alpha_{xy}(k) + \frac{\eta}{2}\left(\pi_{xy}^t(k+1) - \pi_{xy}^s(k+1)\right). \tag{16}$$

*Proof.* As (10) is strictly concave, we can solve it by first-order condition: $\pi_{xy}(k+1) = \frac{1}{2\eta}(\alpha_{xy}^t(k) - \alpha_{xy}^s(k)) + \frac{1}{2}(\pi_{xy}^t(k+1) + \pi_{xy}^s(k+1))$. By substituting the above equation into (11) and (12) we get: $\alpha_{xy}^t(k+1) = \frac{1}{2}(\alpha_{xy}^t(k) + \alpha_{xy}^s(k)) + \frac{\eta}{2}(\pi_{xy}^t(k+1) - \pi_{xy}^s(k+1))$, $\alpha_{xy}^s(k+1) = \frac{1}{2}(\alpha_{xy}^t(k) + \alpha_{xy}^s(k)) + \frac{\eta}{2}(\pi_{xy}^t(k+1) - \pi_{xy}^s(k+1))$. We can see that $\alpha_{xy}^t = \alpha_{xy}^s$ during each update. Hence, $\pi_{xy}(k+1)$ can be further simplified as $\pi_{xy}(k+1) = \frac{1}{2}(\pi_{xy}^t(k+1) + \pi_{xy}^s(k+1))$. In addition, we can achieve (11) and (12) from $\alpha_{xy}^t = \alpha_{xy}^s = \alpha_{xy}$ represented in (16). ■

We can iterate through equations (13)-(16) to obtain a fair and efficient resource transport strategy until getting a convergence. Note that the fairness is explicitly considered during the solution updates, which can be seen from the ADMM iteration step (13). For convenience, we summarize the iterations in Proposition 2 in Algorithm 1.

**Algorithm 1** Distributed Algorithm

**while** $\Pi_{x,t}$ and $\Pi_{y,s}$ not converging **do**
  Compute $\Pi_{x,t}(k+1)$ using (13), for all $x\in\mathscr{X}_y$
  Compute $\Pi_{y,s}(k+1)$ using (14), for all $y\in\mathscr{Y}_x$
  Compute $\pi_{xy}(k+1)$ using (15), for all $\{x,y\}\in\mathscr{E}$
  Compute $\alpha_{xy}(k+1)$ using (16), for all $\{x,y\}\in\mathscr{E}$
**end while**
**return** $\pi_{xy}(k+1)$, for all $\{x,y\}\in\mathscr{E}$

## IV. Discussions on the Distributed Algorithm

In this section, we discuss several crucial aspects of the proposed distributed algorithm for fair and efficient resource allocation mechanisms.

### *A. Fairness and Efficiency Tradeoff*

The fairness of the transport scheme is ensured during the updates of solutions. As shown in (13), the level of fairness is regulated by the parameter $\omega_x$, $x\in\mathscr{X}$. Specifically, $\omega_x$ trades off between the efficiency and fairness of the transport strategy. With a larger $\omega_x$, the fairness term has a more significant impact on the solution, yielding a fairer resource allocation plan. For every target $x$, it maximizes $f_x(\sum_{y\in\mathscr{Y}_x}\pi_{xy}^t)$ at each step. The concavity of $f_x$ guarantees that it is impossible for a single target in the network receiving all the resources. Together with the penalization terms $\sum_{y\in\mathscr{Y}_x}\alpha_{xy}(k)\pi_{xy}^t + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}\left(\pi_{xy}^t - \pi_{xy}(k)\right)^2$, it also ensures that the request for resources from each target $x$ will not be arbitrarily large.

Another interesting observation is that after both the target node $x$ and source node $y$ proposing their strategy (based on

(13) and (14)), the central manager will mediate both requests by taking an average of the fair solution $\pi^t_{xy}$ and the efficient solution $\pi^s_{xy}$ as shown in (15). Hence, the final solution yielded by Algorithm 1 will be both fair and efficient.

### *B. Implementation of Fairness*

In the reformulated problem (6), we associated the fairness function, $f_x$, $\forall x \in \mathscr{X}$, with the corresponding target node. This leads to natural interpretations that when proposing the transport strategy, each target needs to be aware of the fairness of the resource allocation over networks. In a resource distribution market, the supplier (source node) may not care where its resources are allocated finally. However, a target cares whether it gets more or less resources than another target. For example, if a large company is distributing resources to customers, the company (the source) does not care where their product goes as long as they sell the product, while consumers care if only a few consumers are able to buy the product. This observation is consistent with the iteration steps (13) and (14), where each target $x, x \in \mathscr{X}$, aims to maximize the fairness term $\omega_x f_x(\sum_{y\in\mathscr{Y}_x} \pi^t_{xy})$, while each source $y, y \in \mathscr{Y}$, merely maximize its own utility.

Note that during the problem reformulation, the fairness term could also be applied to the source (supplier side). Then, the cost function in (6) becomes $-\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} t_{xy}(\pi^t_{xy}) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y}(s_{xy}(\pi^s_{xy}) - c_{xy}(\pi^s_{xy})) - \sum_{x\in\mathscr{X}} \omega_x f_x(\sum_{y\in\mathscr{Y}_x} \pi^s_{xy})$. We can design distributed algorithm to solve this reformulated problem by using similar techniques in Section III. When the fairness term is associated with the source side, it means that all the suppliers need to inherently consider fairness when distributing resources. It also can be interpreted that if a source enters the market, it needs to comply with the agreed fairness rules in resource allocations.

### *C. Continuous and Distributed Resource Allocation*

In Algorithm 1, all the participants (sources and targets) updates their decisions on the transferred/requested resources iteratively in a distributed fashion. In a resource distribution market, the number of participants and their preferences can vary over time. For example, some suppliers will leave the market when they finish the allocation of their resources. Similarly, new target nodes may join the market when they need to purchase resources. Hence, it is necessary to devise a continuous resource allocation mechanism that is adaptive to the changes in the market. We can extend the Algorithm 1 in this regard and implement it in an online form. Specifically, when there are changes in the market, we can continue to solve the optimal transport problem by using Algorithm 1 with necessary updates resulting from the market changes. In this way, we do not need to recompute the fair and efficient transport strategy for the new scenario from scratch. The algorithm will take into account these changes inherently and continuously update the solution in a distributed way. One remark is that depending on how quickly the resource allocation market changes, we may or may not be able to have convergence to a solution because there may not be enough

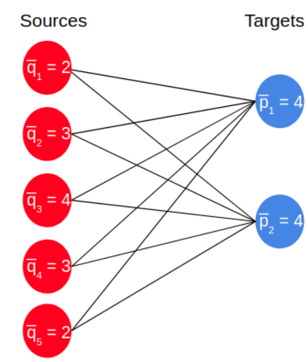



Fig. 1. Network structure for resource transportation. In this case, every source node is connected to every target node.

iterations. We will illustrate the continuous resource allocation with a case study in Section V.

## V. Case Studies

In this section, we corroborate our algorithm for distributed optimal transport with fairness consideration. We consider a scenario with five target nodes and two source nodes and a transport network structure connecting all source nodes to both target nodes. Figure 1 shows the network structure of resource transportation. We define the upper bound, $\bar{q}_y$ for source nodes $y \in \mathscr{Y} = \{1,2,3,4,5\}$ and $\bar{p}_x$ for target node $x \in \mathscr{X} = \{1,2\}$. The lower bounds, $\underline{q}_y$ and $\underline{p}_x$ are 0 for all nodes. There is utility associated for with each for both the target and the source nodes. We adopt linear utility and cost functions as follows: $t_{xy}(\pi_{xy}) = \delta_{xy}\pi_{xy}$, $s_{xy}(\pi_{xy}) = \sigma_{xy}\pi_{xy}$ and $c_{xy}(\pi_{xy}) = \zeta_{xy}\pi_{xy}$, $\forall \{x,y\} \in \mathscr{E}$. The corresponding parameters are selected as:

$$[\delta_{xy}]_{x\in\mathscr{X},y\in\mathscr{Y}} = \begin{bmatrix} 1 & 3 & 1 & 3 & 2 \\ 2 & 2 & 4 & 1 & 2 \end{bmatrix}$$

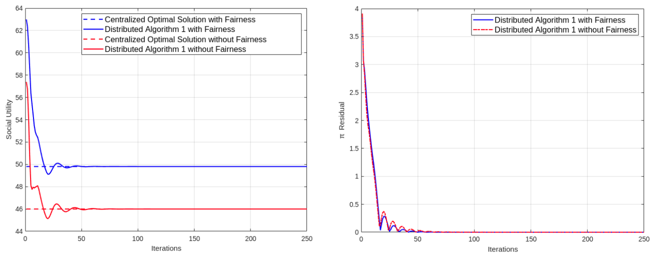
$$[\sigma_{xy}]_{x\in\mathscr{X},y\in\mathscr{Y}} = \begin{bmatrix} 3 & 3 & 5 & 4 & 5 \\ 4 & 3 & 5 & 3 & 6 \end{bmatrix}$$

$$[\zeta_{xy}]_{x\in\mathscr{X},y\in\mathscr{Y}} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 3 & 1 & 2 \end{bmatrix}$$

We consider proportional fairness in the resource allocation, i.e., the fairness function admits the form shown in (3).

### *A. Fair and Distributed Resource Allocation*

We first show the effectiveness of the designed Algorithm 1. Specifically, we compare the optimal transport strategies with and without fairness considerations using Algorithm 1. For the algorithm with fairness, we set the weighting factor $\omega_x = 3$. We focus on comparing their induced social utility. The social utility is the aggregate of the payoffs of the sources and targets and the benefits of fairness in resource allocation. The results are shown in Fig. 2. As shown Fig. 2(a), the distributed algorithm (both with and without fairness consideration) converges to the corresponding centralized optimal solution $\pi^o_{xy}$ (i.e., problem (6) is solved directly). We also observe in Fig. 2(a) that the algorithm with fairness converges



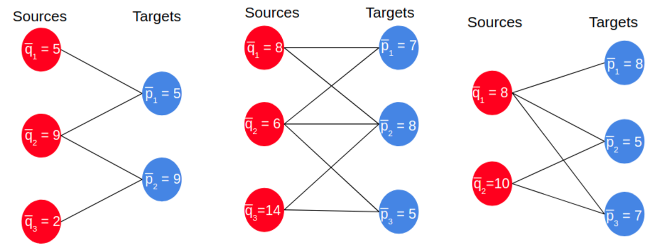

(a) Social utility under transport strategies (b) $\sqrt{\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x}(\pi_{xy}(k)-\pi^o_{xy})^2}$

Fig. 2. Impact of fairness consideration on the transport strategy design using Algorithm 1. (a) and (b) depict the trajectories of social utility and residual of transport strategy, respectively.

to a higher social utility. The increase in the social utility is due to the addition of fairness when designing the resource transport scheme. We also note that the fairness has little effect on the convergence of the algorithm. Fig. 2(b) shows the residual of transport strategy. The residual measures the difference between the strategy at the current update and the centralized optimal solution. We can observe that the residual goes to 0 around $k = 50$, which demonstrates the effectiveness of the designed distributed algorithm.

### *B. Online Distributed Resource Allocation*

Next, we investigate a case study using the discussed continuous, or online, distributed algorithm in Section IV-C. We again adopt linear forms for our utility, cost and fairness functions as in the previous scenario. In this case, the resource allocation network changes over time as shown in Fig. 4. Specifically, there are three sources and two targets at $k = 0$, and not all of which are connected, i.e., the bipartite graph is incomplete. At step $k = 250$, one target node joins the network, and hence the network has three source nodes and three target nodes. At step $k = 500$, one source node leaves the network, and hence two source nodes needs to satisfy the requests from three target nodes. When the resource allocation network is changed, the upper bounds on the amount of transferable resources at sources and the amount of sources requested at the targets are also updated, with each parameter shown in Fig. 4. The weighting constant on the fairness is chosen as $\omega_x = 3$, $\forall x \in \mathscr{X}$, throughout the case study. Other parameters are summarized in Table I. The online algorithm addresses the problem continuously without resetting the algorithm. The results are shown in Fig. 4. We can see that when the resource transport network changes (at $k = 250,\ 500$), the online algorithm will respond to these changes quickly by proposing new allocation schemes. The solutions obtained from the online distributed algorithm are consistent with the centralized optimal solutions. Thus, this online distributed algorithm is applicable to the resource distribution market with frequent changes.

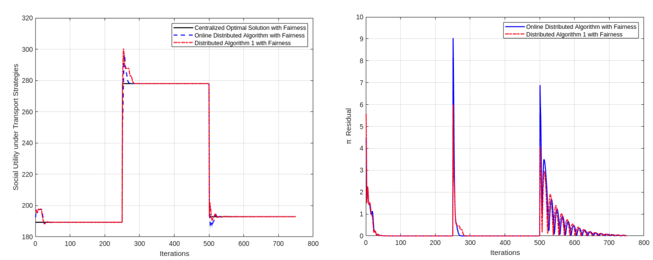



(a) $k = 0$ (b) $k = 250$ (c) $k = 500$

Fig. 3. Network structures for the continuous/online resource allocation.

TABLE I
PARAMETERS IN THE CASE STUDY OF ONLINE DISTRIBUTED RESOURCE ALLOCATION

| | | | | |
|---|---|---|---|---|
| $k=0$ | $\delta_{11}=2$<br>$\sigma_{11}=7$<br>$\zeta_{11}=2$<br>$\delta_{13}=0$<br>$\sigma_{31}=0$ | $\delta_{21}=3$<br>$\sigma_{21}=8$<br>$\zeta_{21}=4$<br>$\sigma_{13}=0$<br>$\zeta_{31}=0$ | $\delta_{22}=4$<br>$\sigma_{22}=6$<br>$\zeta_{22}=2$<br>$\zeta_{13}=0$ | $\delta_{32}=4$<br>$\sigma_{32}=5$<br>$\zeta_{32}=2$<br>$\delta_{31}=0$ |
| $k=250$ | $\delta_{11}=4$<br>$\delta_{22}=5$<br>$\delta_{33}=2$<br>$\sigma_{21}=7$<br>$\sigma_{32}=12$<br>$\zeta_{13}=0$<br>$\zeta_{31}=0$ | $\delta_{12}=2$<br>$\delta_{23}=4$<br>$\sigma_{11}=8$<br>$\sigma_{22}=10$<br>$\sigma_{33}=4$<br>$\zeta_{21}=4$<br>$\zeta_{32}=6$ | $\delta_{13}=0$<br>$\delta_{31}=0$<br>$\sigma_{12}=14$<br>$\sigma_{23}=9$<br>$\zeta_{11}=2$<br>$\zeta_{22}=5$<br>$\zeta_{33}=2$ | $\delta_{21}=3$<br>$\delta_{32}=6$<br>$\sigma_{13}=0$<br>$\sigma_{31}=0$<br>$\zeta_{12}=10$<br>$\zeta_{23}=5$ |
| $k=500$ | $\delta_{11}=3$<br>$\delta_{23}=3$<br>$\sigma_{22}=7$<br>$\zeta_{13}=1$<br>$\sigma_{21}=0$ | $\delta_{12}=2$<br>$\sigma_{11}=5$<br>$\sigma_{23}=4$<br>$\zeta_{22}=1$<br>$\zeta_{21}=0$ | $\delta_{13}=5$<br>$\sigma_{12}=7$<br>$\zeta_{11}=2$<br>$\zeta_{23}=2$ | $\delta_{22}=3$<br>$\sigma_{13}=5$<br>$\zeta_{12}=2$<br>$\delta_{21}=0$ |

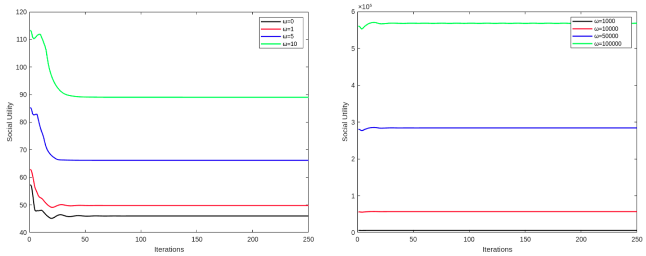



(a) Social utility under online transport strategies (b) $\sqrt{\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x}(\pi_{xy}(k)-\pi^o_{xy})^2}$

Fig. 4. Adaptive fair and efficient transport strategies design using the online algorithm. The transport network structure and participants preferences change over time at $k = 250,\ 500$. (a) and (b) depict the trajectories of social utility and residual of transport strategy, respectively.

### *C. Impacts of Degree of Fairness on Social Utility*

The weighting factor $\omega_x$ balances the efficiency and fairness of the transport strategy. It also affects what the social utility converges to. In this case study, we choose eight different values of $\omega_x = \omega$, $\forall x \in \mathscr{X}$: four are relatively small (less than 10) and four are relatively large (greater than 1000). Fig. 5 shows that $\omega_x$ has little impact on how quickly the algorithm converges, with all eight cases converging with less than 50 iterations. We can also see that a larger $\omega_x$ leads to convergence to a greater social utility. This indicates that,




(a) Small $\omega$ (b) Large $\omega$

Fig. 5. Different levels of fairness and their impacts on social utility.

in the investigated scenario, the social utility increases with the participants preferring to have a fairer than an efficient resource allocation scheme.

## VI. Conclusion

In this paper, we have investigated fair and efficient transport of limited amount of resources in a network of participants with various preferences. The designed distributed algorithm can successfully yield the identical transport plan designed under the centralized manner, making our algorithm applicable to large-scale networks. The fairness is explicitly promoted in the algorithm, through bargaining and negotiations between each pair of resource supplier (source) and resource receiver (target). Throughout the negotiation steps, the sources maximize their revenue but need to consider the fairness requests. Similarly, the targets optimize the fairness but should take into account the efficiency of resource allocation as well. The negotiation/algorithm terminates when the two parties reach a consensus. As for future work, we can investigate fair and dynamic resource transport under a moving horizon objective.

## References


[1] A. Galichon, *Optimal Transport Methods in Economics*. Princeton University Press, 2018.

[2] C. You, K. Huang, H. Chae, and B.-H. Kim, "Energy-efficient resource allocation for mobile-edge computation offloading," *IEEE Transactions on Wireless Communications*, vol. 16, no. 3, pp. 1397–1411, 2016.

[3] R. Zhang and Q. Zhu, "Consensus-based distributed discrete optimal transport for decentralized resource matching," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 5, no. 3, pp. 511–524, 2019.

[4] S. Boyd, N. Parikh, and E. Chu, *Distributed optimization and statistical learning via the alternating direction method of multipliers*. Now Publishers Inc, 2011.

[5] A. S. Awad, M. F. Shaaban, T. H. El-Fouly, E. F. El-Saadany, and M. M. Salama, "Optimal resource allocation and charging prices for benefit maximization in smart pev-parking lots," *IEEE Transactions on Sustainable Energy*, vol. 8, no. 3, pp. 906–915, 2016.

[6] L. Huang, J. Chen, and Q. Zhu, "Distributed and optimal resilient planning of large-scale interdependent critical infrastructures," in *2018 Winter Simulation Conference (WSC)*, 2018, pp. 1096–1107.

[7] J. Chen and Q. Zhu, "Security investment under cognitive constraints: A gestalt nash equilibrium approach," in *2nd Annual Conference on Information Sciences and Systems (CISS)*, 2018, pp. 1–6.

[8] V. S. Mirrokni, S. O. Gharan, and M. Zadimoghaddam, "Simultaneous approximations for adversarial and stochastic online budgeted allocation," in *Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms*. SIAM, 2012, pp. 1690–1701.

[9] K. Deb and C. Myburgh, "A population-based fast algorithm for a billion-dimensional resource allocation problem with integer variables," *European Journal of Operational Research*, vol. 261, no. 2, pp. 460–474, 2017.

[10] D. Niu and B. Li, "An efficient distributed algorithm for resource allocation in large-scale coupled systems," in *Proceedings of IEEE INFOCOM*, 2013, pp. 1501–1509.

[11] P. Coucheney, C. Touati, and B. Gaujal, "Fair and efficient user-network association algorithm for multi-technology wireless networks," in *Proceedings of IEEE INFOCOM*, 2009, pp. 2811–2815.

[12] A. Abdel-Hadi and C. Clancy, "A utility proportional fairness approach for resource allocation in 4G-LTE," in *International Conference on Computing, Networking and Communications*, 2014, pp. 1034–1040.