Behavioral Optimal Transport with an Application to Security Investment over Networks

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Abstract— **Optimal Transport (OT) is a framework that can be leveraged to guide the optimal allocation of resources from a set of sources to a set of targets. We consider the use of the OT paradigm in the security investment problem where the sources aim to allocate their constrained security resources to heterogeneous target nodes. Investing in each target makes the node less vulnerable, thus lowering its probability of a successful attack. However, humans tend to perceive such probabilities inaccurately; a phenomenon observed when facing uncertainties in their decision-making. We capture this human nature through the lens of cumulative prospect theory and establish a behavioral optimal transport framework to account for the human misperception in security investment. We analyze how this misperception behavior affects the resource allocation plan by comparing it with the original strategy designed under accurate probabilities. Knowing that the transport network could be highly complex with a large number of participants, we further develop an efficient algorithm to compute the behavioral optimal transport strategy in a fully distributed manner. Finally, we corroborate our results and algorithm through cases studies to illustrate the human behavioral impacts on the security investment scheme.**

I. Introduction

Optimal transport (OT) can be leveraged to find the most efficient allocation of resources from a set of sources to a set of targets [1]. The standard OT paradigm captures heterogeneous constraints between the sources and targets and can be used in many applications from allocation of raw materials to gradient mapping of images in machine learning [2]. OT also has the potential to guide the optimal distribution of limited security resources to targeted assets aiming to reduce their vulnerability from malicious attacks. It has been investigated that how a centralized resource owner can invest his security resources strategically to protect multiple targets from cyber adversaries [3], [4]. However, such a single-source model is insufficient to capture the emerging situations where multiple resource owners participate in securing the targets collaboratively. Thus, this paper aims to use the OT framework to investigate the security investment problem over a network with multiple sources (security resource investors) and targets (valuable assets to be protected).

Psychological studies have shown that humans often misperceive probabilities on gains and losses by over-weighting low probabilities and under-weighting high probabilities, a subject receiving significant attention in prospect theory [5], [6]. Such behavioral misperception plays an essential role in the focused security investment problem where the resource owners need to evaluate the likelihood of successful compromise of the targets under a given resource allocation scheme, a fundamental step when determining their transport strategies. To this end, we incorporate the consideration of this behavioral aspect to our model by developing a new behavioral OT framework for security investment over networks. Under this paradigm, the transport planner (i.e., resource owner) perceives the target having a relatively low chance of being compromised as more vulnerable than it is, and a target with a high probability of being attacked as less vulnerable than it is. We analyze the impact of attack success misperception on the optimal transport strategy and identify that under the behavioral OT framework, the sources will prefer more to secure those targets with higher values. In other words, the investors tend to pay more attention to higher-valued assets as they become more behavioral, yielding a discriminative transport scheme comparing with the one without behavioral consideration.

Additionally, with more and more participating nodes (sources and targets) introduced into the transport network, the computational complexity of the OT problem increases drastically [7]. Solving the security investment problem with a massive number of networked sources and targets in a centralized manner may be impractical or extremely computationally expensive. In addition, the centralized computational approach requires the transport planner to have complete information on the source and target nodes, including their utility parameters, supply and demand upper bounds, degree of misperception on attack success, and value of targets. Thus, it does not preserve a high level of privacy for participants. To this end, we propose a distributed algorithm based on the alternating direction method of multipliers (ADMM) [8], where the central transport planner is not needed and each node solves its own simpler optimization problem and communicates its decisions with the connected nodes in the network. Each pair of source and target nodes will then negotiate to reach a consensus on how many security resources should be transported. The proposed distributed algorithm converges to the same optimal solution obtained under the centralized optimization paradigm.

The contributions of this paper are summarized as follows:

1) We develop a behavioral optimal transport framework for security investment over a network. The model captures the decision-makers' misperception of security resources' effectiveness on protecting the targets, and it facilitates the analysis of behavioral impacts on the

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optimal transport plan.

2) We discover a sequential water-filling nature of the optimal security resource allocation over the targets and identify that the transport planners become more discriminative by investing in a smaller set of higher-valued targets as they tend to be more behavioral.
3) We further develop a distributed algorithm based on ADMM to compute the optimal transport strategy for large-scale transport networks. We also corroborate our algorithm and analytical results using case studies.

Related Works: Optimal security investment for defending assets has been studied extensively in literature [9]–[12]. Previous studies have also considered the behavioral impacts on security investment. For example, the authors in [13] have studied the interplay between the strategic defender and the bounded rational adversary through a game-theoretic framework. [14] has investigated the optimal investment strategies under a misperceived security risk model based on prospect theory, in which the authors have focused on a single decision maker investing on heterogeneous assets. Our work pays attention to the resource transport in a multi-source multi-target framework. Prospect theory has also been used to guide the optimal resource allocation in various applications, including water infrastructures [15], communication networks [16], and corporate project management [17]. Optimal transport is a classical research field, yet receiving growing interest recently with its applications to machine learning and data science [18]. Our work is related to the distributed discrete OT that can be applied to large-scale networks. Distributed algorithms for computing the optimal transport schemes have been proposed in different contexts, including in consideration of transport efficiency [7], fairness [19], and security [20].

The rest of this paper is organized as follows. In Section II, we formulate the security investment problem over a network based on a behavioral optimal transport framework that considers human misperception on attack success rate. We characterize the optimal security investment strategy and analyze how the behavioral consideration affects such solutions in Sections III and IV. We further develop a distributed algorithm to compute the behavioral OT strategy in Section V and corroborate our findings in Section VI.

II. Problem Formulation

In this section, we establish a behavioral optimal transport framework for security resource allocation over a network.

A. Security Resource Transport Network

In a network, we denote by $\mathscr{X} := \{1,...,|\mathscr{X}|\}$ the set of destinations/targets that receive the security resources, and $\mathscr{Y} := \{1,...,|\mathscr{Y}|\}$ the set of origins/sources that distribute security resources to the targets. Specifically, each source node $y \in \mathscr{Y}$ is connected to a number of target nodes denoted by $\mathscr{X}_y$, representing that y has choices in allocating its resources to a specific group of destinations $\mathscr{X}_y$ in the network. Similarly, it is possible that each target node $x \in \mathscr{X}$ receives resources from multiple source nodes, and this set of suppliers to node x is denoted by $\mathscr{Y}_x$. Note that $\mathscr{X}_y$, $\forall y$ and $\mathscr{Y}_x$, $\forall x$ are nonempty. It is straightforward to see that the security resources are transported over a bipartite network, where one side of the network consists of all source nodes and the other includes all destination nodes. This bipartite graph may not be complete due to constrained matching policies between participants. For convenience, we denote by $\mathscr{E}$ the set including all feasible transport paths in the network, i.e., $\mathscr{E} := \{(x,y)|x \in \mathscr{X}_y, y \in \mathscr{Y}\}$. Note that $\mathscr{E}$ also refers to the set of all edges in the established bipartite graph for security resource transportation.

We next denote by $\pi_{xy} \in \mathbb{R}_+$ the amount of security resources transported from the origin node $y \in \mathscr{Y}$ to the destination node $x \in \mathscr{X}$, where $\mathbb{R}_+$ is the set of nonnegative real numbers. Let $\Pi := \{\pi_{xy}\}_{x\in\mathscr{X}_y, y\in\mathscr{Y}}$ be the designed transport plan. Furthermore, the security resources at each source node $y \in \mathscr{Y}$ is upper bounded by $\bar{q}_y \in \mathbb{R}_+$, i.e., $\sum_{x\in\mathscr{X}_y} \pi_{xy} \leq \bar{q}_y$.

B. Behavioral Optimal Transport

Each target node in the network faces threats and could be compromised by an attacker. If target node $x \in \mathscr{X}$ is attacked, the induced loss is $U_x > 0$. The attacker's probability of successfully compromising the target node is related to the amount of security resources received. For each target node $x \in \mathscr{X}$, defined by $p_x : \mathbb{R}_+^{|\mathscr{Y}_x|} \to [0,1]$ a function that maps the received security resources Π_x to a successful attack probability. It is natural to see that such probability should be related to the aggregated resource received by node x captured by $\sum_{y\in\mathscr{Y}_x} \pi_{xy}$. Thus, with a slightly abuse of notation, $p_x(\Pi_x)$ can be expressed by $p_x(\sum_{y\in\mathscr{Y}_x} \pi_{xy})$, where the later one shows more explicitly the relationship between the successful attack probability and the total received resources at node $x \in \mathscr{X}$.

Each target node $x \in \mathscr{X}$ minimizes its cost $U_x p_x(\Pi_x)$. To this end, the central transport planner aims to minimize the following aggregated loss $L(\Pi)$ at all targets under attacks:

$$L(\Pi) = \sum_{x\in\mathscr{X}} U_x p_x(\Pi_x). \tag{1}$$

It has been shown that humans tend to misperceive probabilities by over-weighing low probabilities and under-weighting high probabilities during decision-making under uncertainties. For a true probability $p \in [0,1]$, humans will perceive it as $w(p) \in [0,1]$, where w is a probability weighting function. One such commonly used weighting function is given by [21]

$$w(p) = \exp(-(-\log(p))^\gamma), \ p \in [0,1], \tag{2}$$

where $\gamma \in (0,1]$ is a parameter capturing the degree of misperception. When γ is more close to 0, it leads to a larger distortion of the probability function p. In comparison, when $\gamma = 1$, $w(p) = p$, indicating there is no probability misperception.

Under the perceived probability, the target node x's cost function becomes $U_x w(p_x(\Pi_x))$. Thus, the cost function for the transport planner under the perceived attack probability is

$$\tilde{L}(\Pi) = \sum_{x\in\mathscr{X}} U_x w(p_x(\Pi_x)). \tag{3}$$

The security resource allocation under the behavioral optimal transport framework can be obtained by solving the following optimization problem:

$$\begin{aligned}(\mathrm{OT-A}):\quad \min_{\Pi}\ & \sum_{x\in\mathscr{X}} U_x w(p_x(\Pi_x))\\ \text{s.t. } & 0 \le \sum_{x\in\mathscr{X}_y} \pi_{xy} \le \bar{q}_y,\ \forall y\in\mathscr{Y}, \\ & \pi_{xy}\ge 0,\ \forall\{x,y\}\in\mathscr{E}.\end{aligned} \tag{4}$$

III. Preliminary Analysis

The successful attack probability function p_x should capture the fact that a larger security investment lowers the likelihood of attack. In addition, the marginal benefit of security resource decreases for each target node. To this end, we have the following assumption.

Assumption 1: The successful attack probability function $p_x(\Pi_x)$ satisfies the following: 1) $p_x(\Pi_x)\in[0,1]$ with $\lim_{||\zeta||_1\to\infty} p_x(\zeta) = 0$, where $||\cdot||_1$ denotes the l_1 norm, and is twice differentiable, 2) $p_x(\Pi_x)$ is strictly monotonic decreasing and log-convex with respect to π_{xy}, for $y\in\mathscr{Y}_x$, and 3) $\frac{\partial p_x}{\partial \pi_{xy}}/p_x$ is bounded with respect to π_{xy}, for $y\in\mathscr{Y}_x$.

There are a number of functions of interest that satisfy the properties in Assumption 1. For example,

$$p_x(\Pi_x) = \exp(-\sum_{y\in\mathscr{Y}_x}\pi_{xy} - r_x), \tag{5}$$

where $r_x > 0$ represents the existing security investment at node x before resource transport.

As another example, $p_x(\Pi_x) = \frac{1}{\sum_{y\in\mathscr{Y}_x}\pi_{xy}+r_x}$, where $r_x > 1$ has a similar meaning as in the previous case. Both examples indicate that the security resources can effectively decrease the attack likelihood.

Lemma 1: Under Assumption 1, the perceived probability of successful attack at node x, $w(p_x(\Pi_x))$, is strictly convex in π_{xy}, $\forall x\in\mathscr{X},\ y\in\mathscr{Y}_x$.

Proof: Here, we show that the second derivative of $w(p_x(\Pi_x))$ with respect to π_{xy}, $\forall y\in\mathscr{Y}_x$, is positive. Using the probability function in (2), we have

$$\begin{aligned}\frac{\partial^2 w(p_x(\Pi_x))}{\partial\pi_{xy}^2} &= -\gamma(\gamma-1)(-\log(p_x(\Pi_x)))^{\gamma-2} w(p_x(\Pi_x))\\ &\cdot\left(\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\right)^2 + \gamma(-\log(p_x(\Pi_x)))^{\gamma-1} w(p_x(\Pi_x))\\ &\cdot\frac{p_x(\Pi_x)\cdot\frac{\partial^2 p_x(\Pi_x)}{\partial\pi_{xy}^2} - \left(\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\right)^2}{(p_x(\Pi_x))^2} + \Big(\gamma(-\log(p_x(\Pi_x)))^{\gamma-1}\\ &\cdot\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\Big)^2 w(p_x(\Pi_x)).\end{aligned} \tag{6}$$

The first and third terms multiply out to be positive, because of Assumption 1. The second term may or may not be positive depending on $(p_x(\Pi_x)\cdot\frac{\partial^2 p_x(\Pi_x)}{\partial\pi_{xy}^2} - (\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}})^2)/(p_x(\Pi_x))^2$. If the second term term is positive than the second derivative is positive and thus the function is convex. If the term is negative we need to show that:

$$\begin{aligned}&\Big[-\gamma(\gamma-1)(-\log(p_x(\Pi_x)))^{\gamma-2}\left(\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\right)^2\\ &+\Big(\gamma(-\log(p_x(\Pi_x)))^{\gamma-1}\cdot\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\Big)^2\Big]\\ &\cdot w(p_x(\Pi_x)) > \gamma(-\log(p_x(\Pi_x)))^{\gamma-1} w(p_x(\Pi_x))\\ &\cdot\frac{p_x(\Pi_x)\cdot\frac{\partial^2 p_x(\Pi_x)}{\partial\pi_{xy}^2} - \left(\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\right)^2}{(p_x(\Pi_x))^2}.\end{aligned}$$

With cancellation and some algebraic manipulation, it is easy to see that the statement is true. Thus, the objective function is strictly convex. ■

IV. Analysis of Behavioral Optimal Transport Strategies

This section characterizes the behavioral OT strategies and analyzes the impacts of behavioral considerations on the optimal transport plan.

A. Security Resource Allocation Preferences

We first have the following assumption to facilitates the analysis.

Assumption 2: Assume that the values of induced loss due to successful attack are ordered as follows: $U_1 > U_2 > \ldots > U_{|\mathscr{X}|} > 0$. Furthermore, each target node admits a same successful attack probability function, i.e., p_x, $\forall x\in\mathscr{X}$, share a same form.

We next have the following result on the marginal cost associated with the target nodes.

Lemma 2: The following inequality holds for each pair of target nodes $i\in\mathscr{X}$ and $j\in\mathscr{X}$ with $i<j$,

$$U_i\frac{\partial w(p_i(\Pi_i))}{\partial\pi_{iy}} < U_j\frac{\partial w(p_j(\Pi_j))}{\partial\pi_{jy}},\ \forall y\in\mathscr{Y}_i\cap\mathscr{Y}_j. \tag{7}$$

And the marginal cost $U_i\frac{\partial w(p_i(\Pi_i))}{\partial\pi_{iy}}$ is negative and continuously increasing to 0 in π_{iy}, $\forall i\in\mathscr{X}$.

Proof: First, we have, $\forall x\in\mathscr{X}$ and $\forall y\in\mathscr{Y}_x$,

$$\begin{aligned}U_x\frac{\partial w(p_x(\Pi_x))}{\partial\pi_{xy}} =& U_x\gamma(-\log(p_x(\Pi_x)))^{(\gamma-1)}\\ &\cdot\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\cdot w(p_x(\Pi_x)).\end{aligned} \tag{8}$$

Based on Assumption 1, $\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}$ is negative. In addition, $-\log(p_x(\Pi_x))$, $p_x(\Pi_x)$ and $w(p_x(\Pi_x))$ are all positive. Thus, $U_x\frac{\partial w(p_x(\Pi_x))}{\partial\pi_{xy}} < 0$. Lemma 1 also shows that $\frac{\partial}{\partial\pi_{xy}}(\frac{\partial w(p_x(\Pi_x))}{\partial\pi_{xy}}) > 0$, indicating that the marginal cost is monotonically increasing. In addition,

$$\begin{aligned}&\lim_{||\Pi_x||_1\to\infty}\left|U_x\cdot\frac{\partial w(p(\Pi_x))}{\partial\pi_{xy}}\right|\\ &=\lim_{||\Pi_x||_1\to\infty}\left|\gamma U_x(-\log(p_x(\Pi_x)))^{\gamma-1} w(p_x(\Pi_x))\right|\\ &\cdot\left|\frac{\partial p_x(\Pi_x)}{\partial\pi_{xy}}\Big/ p_x(\Pi_x)\right|.\end{aligned}$$

From Assumption 1, we know that as $p_x(\Pi_x)=0$ and thus $w(p_x(\Pi_x))\to 0$ and $-\log(p_x(\Pi_x))\to\infty$ as $||\Pi_x||_1\to\infty$. Since $\frac{\partial p_x}{\partial \pi_{xy}}/p_x$ is bounded, $\lim_{||\Pi_x||_1\to\infty}|U_x\cdot\frac{\partial w(p(\Pi_x))}{\partial \pi_{xy}}|=0$. Finally, to show the inequality between the marginals, we note that based on Assumption 2, $U_i>U_j$ and thus

$$U_i(-\log(p(\Pi_i)))^{\gamma-1}w(p(\Pi_i))\frac{\partial p_i(\Pi_i)}{\partial \pi_{iy}}\Big/p_i(\Pi_i)$$
$$<U_j(-\log(p(\Pi_i)))^{\gamma-1}w(p(\Pi_i))\frac{\partial p_j(\Pi_i)}{\partial \pi_{jy}}\Big/p_j(\Pi_i),$$

for $\forall \Pi_x\in\mathbb{R}_+^{|\mathscr{Y}_x|},\forall x\in\mathscr{X}$ which yields (7). ■

The following proposition characterizes the total amount of security resources received by the target nodes from sources under some general assumptions.

Proposition 1: Under Assumption 2 and a complete transport network, the optimal transport plan $\{\Pi_x^*\}_{x\in\mathscr{X}}$ satisfies the following inequality $||\Pi_1^*||_1\geq||\Pi_2^*||_1\geq\ldots\geq||\Pi_{|\mathscr{X}|}^*||_1$, and the l_1 norm $||\Pi_x||_1=\sum_{y\in\mathscr{Y}_x}\pi_{xy}$ is the total amount of security resources received by the target node $x\in\mathscr{X}$.

Proof: As each source can transport resources to every target and the qualities of resources are the same supplied by all source nodes, it is equivalent to aggregate all the source nodes as a single super node that has a capacity $\sum_{y\in\mathscr{Y}}\bar{q}_y$ managed by a central planner. Thus, the transport network can be seen consisting of a single source connected to a set of targets, i.e., $\mathscr{Y}=\{1\}$, $\mathscr{Y}_x=\mathscr{Y}$, and $\Pi_x=\pi_{x1}$, $\forall x\in\mathscr{X}$. Based on the KKT condition, for each pair of target nodes $i\in\mathscr{X}$ and $j\in\mathscr{X}$ receiving nonzero security resources from the source node, we have $U_i\frac{\partial w(p_i(\Pi_i))}{\partial \pi_{i1}}|_{\pi_{i1}=\pi_{i1}^*}=U_j\frac{\partial w(p_i(\Pi_j))}{\partial \pi_{j1}}|_{\pi_{j1}=\pi_{j1}^*}$. Assumptions 1 and 2 indicate that $\gamma U_i(-\log(p_i(\pi_{i1}^*)))^{\gamma-1}w(p_i(\pi_{i1}^*))\frac{\partial p_i(\pi_{i1}^*)}{\partial \pi_{i1}}/p_i(\pi_{i1}^*)=\gamma U_j(-\log(p_j(\pi_{j1}^*)))^{\gamma-1}w(p_j(\pi_{j1}^*))\frac{\partial p_j(\pi_{j1}^*)}{\partial \pi_{j1}}/p_j(\pi_{j1}^*)$, which yields

$$\begin{aligned}&(-\log(p_i(\pi_{i1}^*)))^{\gamma-1}w(p_i(\pi_{i1}^*))\frac{\partial p_i(\pi_{i1}^*)}{\partial \pi_{i1}}\frac{1}{p_i(\pi_{i1}^*)}\\&=\frac{U_j}{U_i}(-\log(p_j(\pi_{j1}^*)))^{\gamma-1}w(p_j(\pi_{j1}^*))\frac{\partial p_j(\pi_{j1}^*)}{\partial \pi_{j1}}\frac{1}{p_j(\pi_{j1}^*)}\\&>(-\log(p_j(\pi_{j1}^*)))^{\gamma-1}w(p_j(\pi_{j1}^*))\frac{\partial p_j(\pi_{j1}^*)}{\partial \pi_{j1}}\frac{1}{p_j(\pi_{j1}^*)}.\end{aligned}$$

The inequality in the last step above is due to $U_j<U_i$ for $j>i$ and the negative marginal cost of target node on the received security resources. Based on Lemma 2, the marginal cost is continuously increasing. Thus, we can conclude $\pi_{i1}^*>\pi_{j1}^*$, where π_{i1}^* is the total amount of resources received by target node i. Equivalently speaking, target node i receives more security resources than target node j at the optimal solution, for $i<j\in\mathscr{X}$. ■

Proposition 1 indicates that, under some quite general conditions, target node i (higher-valued) receives more resources than target node j (lower-valued) under the optimal transport design, $\forall i<j\in\mathscr{X}$. This result is consistent with the objective of the transport planner in minimizing the aggregated expected loss of assets.

B. Sequential Water-Filling for Optimal Transport Design

The transport network is still considered to be complete. Thus, it is equivalent to combine all source nodes and regard them as a super source node with capacity $\sum_{y\in\mathscr{Y}}\bar{q}_y$. For convenience, we denote by $\tilde{\pi}_i$ the total amount of resources that target node i received from the super source node, i.e., $\tilde{\pi}_i=\sum_{y\in\mathscr{Y}_i}\pi_{iy}$ in the original framework. With a slight abuse of notation, p_x can be seen as a single-variable function on $\tilde{\pi}_i$. For all target nodes $i\in\mathscr{X}$ and $j\in\mathscr{X}$ with $i<j$, we define $\tilde{\pi}_i^{j*}$ as a quantity that satisfies

$$U_i\frac{\partial w(p_i(\tilde{\pi}_i))}{\partial \tilde{\pi}_i}\bigg|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}}=U_j\frac{\partial w(p_j(\tilde{\pi}_j))}{\partial \tilde{\pi}_j}\bigg|_{\tilde{\pi}_j=0}. \tag{9}$$

Proposition 2: Under Assumption 2 and a complete transport network, the resources received by target node j from the super source node, $\tilde{\pi}_j$, will be nonzero at the optimal solution if and only if $\sum_{y\in\mathscr{Y}}\bar{q}_y>\sum_{i=1}^{j-1}\tilde{\pi}_i^{j*}$, where $\tilde{\pi}_i^{j*}$ is defined in (9).

Proof: Suppose that $\tilde{\pi}_j^*>0$ for some target node j. Also suppose by contradiction that $\sum_{y\in\mathscr{Y}}\bar{q}_y\leq\sum_{i=1}^{j-1}\tilde{\pi}_i^{j*}$. Then, $\exists m\in\{1,\ldots,j-1\}$ such that $\tilde{\pi}_m^*<\tilde{\pi}_m^{j*}$. This indicates that it is infeasible to allocate $\tilde{\pi}_m^{j*}$ or more resources to node m without exceeding the upper bound. By definition of $\tilde{\pi}_m^{j*}$,

$$\begin{aligned}U_m\frac{\partial w(p_m(\tilde{\pi}_m))}{\partial \tilde{\pi}_m}\bigg|_{\tilde{\pi}_m=\tilde{\pi}_m^{j*}}&<U_j\frac{\partial w(p_j(\tilde{\pi}_j))}{\partial \tilde{\pi}_j}\bigg|_{\tilde{\pi}_j=0}\\&<U_j\frac{\partial w(p_j(\tilde{\pi}_j))}{\partial \tilde{\pi}_j}\bigg|_{\tilde{\pi}_j=\tilde{\pi}_j^*},\end{aligned}$$

which yields a contradiction, since the marginals must coincide at the optimal solution. Thus, $\tilde{\pi}_j^*>0$ leads to $\sum_{y\in\mathscr{Y}}\bar{q}_y>\sum_{i=1}^{j-1}\tilde{\pi}_i^{j*}$ under the optimal transport strategy.

To prove the other direction, we first suppose that $\sum_{y\in\mathscr{Y}}\bar{q}_y>\sum_{i=1}^{j-1}\tilde{\pi}_i^*$ and suppose by contradiction $\tilde{\pi}_j=0$. Then we have $\tilde{\pi}_k=0,\forall k>j$, and thus $\sum_{k=1,2,\ldots,j-1}\tilde{\pi}_k=\sum_{y\in\mathscr{Y}}\bar{q}_y$ and $\exists i\in\{1,\ldots,j-1\}$ such that $\tilde{\pi}_i>\tilde{\pi}_i^{j*}$. A sufficiently small amount of resource, $\varepsilon\in\mathbb{R}_+$ is transferred from target i to j which will lead to a net cost reduction in (4), and thus the resource allocation is no longer optimal. Starting with non-zero resource allocation to the target nodes $\{1,\ldots,j-1\}$, the total cost is $\sum_{x\in\mathscr{X}}U_xw(p_x(\tilde{\pi}_x))$. From target i that has $\tilde{\pi}_i^*=||\Pi_i^*||_1>\tilde{\pi}_i^{j*}$, remove a sufficiently small amount of resource ε and add a resource amount of ε to target j. Denote the modified transport plan as $\pi^{(\varepsilon)}$. The total cost after perturbation becomes

$$\begin{aligned}\tilde{L}(\pi^{(\varepsilon)})=\sum_{z\in\mathscr{X}\setminus\{i,j\}}U_zw(p_z(\tilde{\pi}_z^*))+U_iw(p_i(\tilde{\pi}_i-\varepsilon))\\+U_jw(p_j(\varepsilon)).\end{aligned}$$

We next define $g(\varepsilon)=U_iw(p_i(\tilde{\pi}_i-\varepsilon))+U_jw(p_j(\varepsilon))$. Then, $\tilde{L}(\pi^*)=\sum_{z\in\mathscr{X}\setminus\{i,j\}}U_zw(p_z(\tilde{\pi}_z))+g(0)$, $\tilde{L}(\pi^{(\varepsilon)})=\sum_{z\in\mathscr{X}\setminus\{i,j\}}U_zw(p_z(\tilde{\pi}_z))+g(\varepsilon)$. If $g(\varepsilon)<g(0)$, then $\tilde{L}(\pi^{(\varepsilon)})<\tilde{L}(\tilde{\pi}^*)$, which yields a positive net cost reduction, meaning the transport strategy after perturbation is worse off. It is clear that

$$\frac{dg}{d\varepsilon} = -U_i \frac{\partial w(p_i(\pi_i))}{\partial \tilde{\pi}_i}\bigg|_{\pi_i=\tilde{\pi}_i-\varepsilon} + U_j \frac{\partial w(p_j(\pi_j))}{\partial \tilde{\pi}_j}\bigg|_{\pi_j=\varepsilon}.$$

Based on $\tilde{\pi}_i^* > \tilde{\pi}_i^{j*}$ and Lemma 2,

$$U_i \frac{\partial w(p_i(\tilde{\pi}_i))}{\partial \tilde{\pi}_i}\bigg|_{\tilde{\pi}_i=\tilde{\pi}_i^*} > U_j \frac{\partial w(p_j(\tilde{\pi}_j))}{\partial \tilde{\pi}_j}\bigg|_{\tilde{\pi}_j=0}.$$

Thus, $\lim_{\varepsilon\to 0} \frac{dg}{d\varepsilon}$ is negative, indicating that $g(\varepsilon)$ is decreasing for a sufficiently small ε. Therefore, we obtain $\tilde{L}(\pi^{(\varepsilon)}) < \tilde{L}(\tilde{\pi}^*)$ which is a contradiction. ∎

Proposition 2 implies that the super source node first allocates $\tilde{\pi}_1^{2*}$ security resources to target node 1, and then starts to transfer resources to both target nodes 1 and 2 while maintaining a same marginal cost until reaching $\tilde{\pi}_1^{3*}$ and $\tilde{\pi}_2^{3*}$, respectively. Afterward, in addition to target nodes 1 and 2, target node 3 starts to receive resources, and the marginal costs at all nodes are kept the same during security resource investment. The resource allocation scheme will follow this fashion until all resources are transferred. The above discussion leads to *sequential water-filling* of security resource transport over networks.

As the original transport network includes multiple source nodes, we need to determine the strategy for each of them. The above discussion indicates that the optimal transport plan can be obtained sequentially, i.e., each source node completes allocating its security resources to the targets in sequential order. Specifically, source node 1 will first transfer its resource to target node 1. If $\bar{q}_1 < \tilde{\pi}_1^{2*}$, then the next source nodes (node 2, 3, etc) will continue allocate resource to target node 1 until it receives $\tilde{\pi}_1^{2*}$ amount of resources. If $\bar{q}_1 > \tilde{\pi}_1^{2*}$, then source node 1 first allocates $\tilde{\pi}_1^{2*}$ amount of resources to the target node 1, and then start transferring the remaining resources to both targets 1 and 2 while maintaining a same marginal cost at both nodes. After source node 1 completes its resource transport, source node 2 starts to transfer its resources to the appropriate targets in a similar manner. This process terminates until all source nodes finish their resource allocation to the targets.

C. Behavioral Impacts on Optimal Transport Plan

The impact of incorporating the behavioral element to probability perception is captured by the parameter γ. Clearly, there is no behavioral consideration when $\gamma = 1$, and the probabilities are perceived as their actual values. When $\gamma \in (0,1)$, we have the following result on the behavioral impacts on the transport strategies.

Proposition 3: Under Assumptions 1 and 2, $p_x(0) < \frac{1}{e}$, $\forall x \in \mathscr{X}$, and a complete transport network, $d\tilde{\pi}_i^{j*}/d\gamma < 0$ for $i < j$, $\forall i, j \in \mathscr{X}$, with $\tilde{\pi}_i^{j*}$ defined in (9).

Proof: Based on (2) and (9), we have

$$\begin{aligned} &U_i\gamma(-\log(p_i(\tilde{\pi}_i^{j*})))^{(\gamma-1)} w(p_i(\tilde{\pi}_i^{j*})) \frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\bigg|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} \frac{1}{p_i(\tilde{\pi}_i^{j*})} \\ &= U_j\gamma(-\log(p_j(0)))^{(\gamma-1)} w(p_j(0)) \frac{\partial p_j(\tilde{\pi}_j)}{\partial \tilde{\pi}_j}\bigg|_{\tilde{\pi}_j=0} \frac{1}{p_j(0)}. \end{aligned} \tag{10}$$

Based on (10), we can characterize the sensitivity of the amount of security resources transported to each target over the behavioral parameter γ. Taking log of each side of (10) and differentiating with respect to γ yield

$$\begin{aligned} \frac{d\tilde{\pi}_i^{j*}}{d\gamma} &= \frac{((-\log(p_i(\tilde{\pi}_i^{j*})))^\gamma - 1)\log(-\log(p_i(\tilde{\pi}_i^{j*})))}{\Lambda_i^j} \\ &\quad - \frac{((-\log(p_j(0)))^\gamma - 1)\log(-\log(p_j(0)))}{\Lambda_i^j}, \end{aligned} \tag{11}$$

where $\Lambda_i^j = (\gamma - 1 - \gamma(-\log(p_i(\tilde{\pi}_i^{j*})))^\gamma) \frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\Big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} \cdot \frac{1}{p_i(\tilde{\pi}_i^{j*})}$ $\log(p_i(\tilde{\pi}_i^{j*})) + \frac{p_i(\tilde{\pi}_i^{j*}) \frac{\partial^2 p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i^2}\big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} - \left(\frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}}\right)^2}{\frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} \cdot p_i(\tilde{\pi}_i^{j*})}$. Under the assumption that $p_j(0) < \frac{1}{e}$ and $p_i(\tilde{\pi}_i^{j*}) < p_j(0)$ for $\tilde{\pi}_i^{j*} > 0$, we have $-\log(p_i(\tilde{\pi}_i^{j*})) > -\log(p_j(0)) > 1$ and thus $\log(-\log(p_i(\tilde{\pi}_i^{j*}))) > \log(-\log(p_j(0))) > 0$ and $(-\log(p_i(\tilde{\pi}_i^{j*})))^\gamma - 1 > (-\log(p_j(0)))^\gamma - 1$. Hence, the numerator of (11) is positive. From Assumption 1, we have $\frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\Big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} < 0$ and because $p_i(\tilde{\pi}_i)$ is log-convex, $p_i(\tilde{\pi}_i^{j*}) \frac{\partial^2 p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i^2}\big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}} \geq \left(\frac{\partial p_i(\tilde{\pi}_i)}{\partial \tilde{\pi}_i}\big|_{\tilde{\pi}_i=\tilde{\pi}_i^{j*}}\right)^2$. Thus, the denominator of (11) negative, which yields $d\tilde{\pi}_i^{j*}/d\gamma < 0$. ∎

Remark: The above analysis, together with Proposition 1 indicate that when the behavioral misperception on the attack success probability is considered, the sources will supply security resources to fewer target nodes than the transport plan obtained under the non-behavioral counterpart. In other words, the behavioral security resource owners prefer to secure higher-valued assets while paying less attention to those relatively lower-valued targets, and it leads to a *discriminative transport scheme* comparing with the one developed under no behavioral consideration.

V. Distributed Algorithm for Behavioral OT

In the behavioral OT framework, other than the cost of target nodes, the objective function can also incorporate the preferences of the source nodes in the transport design. The utility function of source node y on transferring π_{xy} security resources to target node x is denoted by $s_{xy}: \mathbb{R}_+ \to \mathbb{R}$. In addition, to balance the security resource transportation, the OT planner considers an upper bound of each target node $x \in \mathscr{X}$ on the received security resources from connected sources, captured by $\bar{p}_x \in \mathbb{R}_+$, i.e., $\sum_{y\in\mathscr{Y}_x} \pi_{xy} \leq \bar{p}_x$. To this end, the transport planner aims to address the following problem:

$$\begin{aligned} (\mathrm{OT-B}): \quad \min_{\Pi} \ & \sum_{x\in\mathscr{X}} U_x w(p_x(\Pi_x)) - \sum_{y\in\mathscr{Y}} \sum_{x\in\mathscr{X}_y} \tau_y s_{xy}(\pi_{xy}) \\ \text{s.t. } & 0 \leq \sum_{y\in\mathscr{Y}_x} \pi_{xy} \leq \bar{p}_x, \ \forall x \in \mathscr{X}, \\ & 0 \leq \sum_{x\in\mathscr{X}_y} \pi_{xy} \leq \bar{q}_y, \ \forall y \in \mathscr{Y}, \\ & \pi_{xy} \geq 0, \ \forall \{x,y\} \in \mathscr{E}, \end{aligned}$$

where $\tau_y \in \mathbb{R}_+$ is a positive weighting factor balancing the loss of the targets and the utility of the sources under a given security allocation strategy. It is straightforward to observe that as $\tau_y \to 0\ \forall y$, the solution to (OT-B) will degenerated to the problem in (OT-A), given that the targets have no constraint on the maximum received security resources.

As the transport network becomes complex with a large number of participating nodes, a centralized scheme to compute the optimal solution to (OT-B) can be computationally expensive. In addition, the centralized optimization paradigm requires the planner to collect heterogeneous information from all source and target nodes, including their utility parameters, supply and demand upper bounds, degree of misperception on attack success, and value of targets, which does not preserve a desirable level of privacy. Due to the above two concerns, it is necessary to devise a distributed and privacy-preserving scheme to obtain the behavioral transport strategy over a large-scale network. To facilitate the development of such an algorithm, we first introduce two ancillary variables π^t_{xy} and π^s_{xy}. The superscripts t and s indicate that the corresponding parameter belongs to a target and source node, respectively. We then set $\pi_{xy} = \pi^t_{xy}$ and $\pi_{xy} = \pi^s_{xy}$, indicating that the solutions proposed by the targets and sources are consistent. This reformulation facilitates the design of a distributed algorithm which allows us to iterate to obtain the optimal behavioral transport plan. To this end, the reformulated behavioral optimal transport problem is presented as follows:

$$\begin{aligned}\min_{\Pi^t\in\mathscr{F}^t,\Pi^s\in\mathscr{F}^s,\Pi} & \sum_{x\in\mathscr{X}} U_x w(p_x(\Pi^t_x)) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} \tau_y s_{xy}(\pi^s_{xy}) \\ \text{s.t.}\quad & \pi^t_{xy} = \pi_{xy},\ \forall\{x,y\}\in\mathscr{E}, \\ & \pi^s_{xy} = \pi_{xy},\ \forall\{x,y\}\in\mathscr{E},\end{aligned} \tag{12}$$

where $\Pi^t := \{\pi^t_{xy}\}_{x\in\mathscr{X}_y, y\in\mathscr{Y}}$, $\Pi^s := \{\pi^s_{xy}\}_{x\in\mathscr{X}, y\in\mathscr{Y}_x}$, $\mathscr{F}^t := \{\Pi^t | \pi^t_{xy} \geq 0, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x} \pi^t_{xy} \leq \bar{p}_x,\ \{x,y\}\in\mathscr{E}\}$, and $\mathscr{F}^s := \{\Pi^s | \pi^s_{xy} \geq 0, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y} \pi^s_{xy} \leq \bar{q}_y,\ \{x,y\}\in\mathscr{E}\}$.

We resort to alternating direction method of multipliers (ADMM) [8] to develop a distributed computational algorithm. First, let α^s_{xy} and α^t_{xy} be the Lagrangian multipliers associated with the constraint $\pi^s_{xy} = \pi_{xy}$ and $\pi^t_{xy} = \pi_{xy}$, respectively. The Lagrangian function associated with the optimization problem (12) can then be written as follows:

$$\begin{aligned}&\mathscr{L}(\Pi^t,\Pi^s,\Pi,\alpha^t_{xy},\alpha^s_{xy}) \\ &\quad= \sum_{x\in\mathscr{X}} U_x w(p_x(\Pi^t_x)) - \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} \tau_y s_{xy}(\pi^s_{xy}) \\ &\quad+ \sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x} \alpha^t_{xy}\left(\pi^t_{xy} - \pi_{xy}\right) + \sum_{y\in\mathscr{Y}}\sum_{x\in\mathscr{X}_y} \alpha^s_{xy}\left(\pi_{xy} - \pi^s_{xy}\right) \\ &\quad+ \frac{\eta}{2}\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x}\left(\pi^t_{xy} - \pi_{xy}\right)^2 + \frac{\eta}{2}\sum_{x\in\mathscr{X}}\sum_{y\in\mathscr{Y}_x}\left(\pi_{xy} - \pi^s_{xy}\right)^2,\end{aligned} \tag{13}$$

where η is a positive constant controlling the convergence. We have the following result on the distributed algorithm.

Proposition 4: The iterative steps of ADMM to solve (OT-B) are summarized as follows:

$$\begin{aligned}\Pi^t_x(k+1) \in \arg\min_{\Pi^t_x\in\mathscr{F}^t_x} & U_x w(p_x(\Pi^t_x)) \\ &+ \sum_{y\in\mathscr{Y}_x} \alpha^t_{xy}(k)\pi^t_{xy} + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}(\pi^t_{xy} - \pi_{xy}(k))^2,\end{aligned} \tag{14}$$

$$\begin{aligned}\Pi^s_y(k+1) \in \arg\min_{\Pi^s_y\in\mathscr{F}^s_y} & -\sum_{x\in\mathscr{X}_y} \tau_y s_{xy}(\pi^s_{xy}) \\ &- \sum_{x\in\mathscr{X}_y} \alpha^s_{xy}(k)\pi^s_{xy} + \frac{\eta}{2}\sum_{x\in\mathscr{X}_y}(\pi_{xy}(k) - \pi^s_{xy})^2,\end{aligned} \tag{15}$$

$$\begin{aligned}\pi_{xy}(k+1) = \arg\min_{\pi_{xy}} & -\alpha^t_{xy}(k)\pi_{xy} + \alpha^s_{xy}(k)\pi_{xy} \\ &+ \frac{\eta}{2}(\pi^t_{xy}(k+1) - \pi_{xy})^2 + \frac{\eta}{2}(\pi_{xy} - \pi^s_{xy}(k+1))^2,\end{aligned} \tag{16}$$

$$\alpha^t_{xy}(k+1) = \alpha^t_{xy}(k) + \eta(\pi^t_{xy}(k+1) - \pi_{xy}(k+1))^2, \tag{17}$$

$$\alpha^s_{xy}(k+1) = \alpha^s_{xy}(k) + \eta(\pi_{xy}(k+1) - \pi^s_{xy}(k+1))^2, \tag{18}$$

where $\Pi^t_{\tilde{x}} := \{\pi^t_{xy}\}_{y\in\mathscr{Y}_x, x=\tilde{x}}$ represents the solution at target node $\tilde{x}\in\mathscr{X}$, and $\Pi^s_{\tilde{y}} := \{\pi^s_{xy}\}_{x\in\mathscr{X}_y, y=\tilde{y}}$ represents the proposed solution at source node $\tilde{y}\in\mathscr{Y}$. In addition, $\mathscr{F}^t_x := \{\Pi^t_x | \pi^t_{xy} \geq 0, y\in\mathscr{Y}_x, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi^t_{xy} \leq \bar{p}_x\}$, and $\mathscr{F}^s_y := \{\Pi^s_y | \pi^s_{xy} \geq 0, x\in\mathscr{X}_y, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi^s_{xy} \leq \bar{q}_y\}$.

Proof: Let $\vec{x} = [(\vec{\Pi}^t_x)^T, \vec{\Pi}^T]^T$, $\vec{y} = [\vec{\Pi}^T, (\vec{\Pi}^s_y)^T]^T$, and $\alpha = [\{\alpha^t_{xy}\}^T, \{\alpha^s_{xy}\}^T]^T$, where $\vec{\ }$ denotes the vectorization operator. Note that these vectors are all $2|\mathscr{E}|\times 1$. The constraints can be rewritten in the matrix form $A\vec{x} = \vec{y}$, where $A = [\mathbf{I},\mathbf{0},\mathbf{I},\mathbf{0}]$ with $\mathbf{I}$ and $\mathbf{0}$ denoting the identity and zero matrices, respectively. Next, note that $\vec{x}\in\mathscr{F}_{\vec{x},t}$ and $\vec{y}\in\mathscr{F}_{\vec{y},s}$, where $\mathscr{F}^t_{\vec{x}} = \{\vec{x} | \pi^t_{xy} \geq 0, \underline{p}_x \leq \sum_{y\in\mathscr{Y}_x}\pi^t_{xy} \leq \bar{p}_x, \{x,y\}\in\mathscr{E}\}$, $\mathscr{F}^s_{\vec{y}} := \{\vec{y} | \pi^s_{xy} \geq 0, \underline{q}_y \leq \sum_{x\in\mathscr{X}_y}\pi^s_{xy} \leq \bar{q}_y, \{x,y\}\in\mathscr{E}\}$. Then, we can solve (12) using the iterations: 1) $\vec{x}(k+1) \in \arg\min_{\vec{x}\in\mathscr{F}^t_{\vec{x}}} \mathscr{L}(\vec{x},\vec{y}(k),\alpha(k))$; 2) $\vec{y}(k+1) \in \arg\min_{\vec{y}\in\mathscr{F}^s_{\vec{y}}} \mathscr{L}(\vec{x}(k),\vec{y},\alpha(k))$; 3) $\alpha(k+1) = \alpha(k) + \eta(A\vec{x}(k+1) - \vec{y}(k+1))$, whose convergence is proved in [8]. As there is no coupling among $\Pi^t_x, \Pi^t_y, \pi_{xy}, \alpha^t_{xy}$, and α^s_{xy}, the above iterations can be rewritten into (14)-(18). ■

The iterations in (14)-(18) can be further simplified to four iterations.

Proposition 5: The iterations (14)-(18) can be simplified as follows:

$$\begin{aligned}\Pi^t_x(k+1) \in \arg\min_{\Pi^t_x\in\mathscr{F}^t_x} & U_x w(p_x(\Pi^t_x)) \\ &+ \sum_{y\in\mathscr{Y}_x} \alpha_{xy}(k)\pi^t_{xy} + \frac{\eta}{2}\sum_{y\in\mathscr{Y}_x}(\pi^t_{xy} - \pi_{xy}(k))^2,\end{aligned} \tag{19}$$

$$\begin{aligned}\Pi^s_y(k+1) \in \arg\min_{\Pi^s_y\in\mathscr{F}^s_y} & -\sum_{x\in\mathscr{X}_y} \tau_y s_{xy}(\pi^s_{xy}) \\ &- \sum_{x\in\mathscr{X}_y} \alpha_{xy}(k)\pi^s_{xy} + \frac{\eta}{2}\sum_{x\in\mathscr{X}_y}(\pi_{xy}(k) - \pi^s_{xy})^2,\end{aligned} \tag{20}$$

$$\pi_{xy}(k+1) = \frac{1}{2}\left(\pi^t_{xy}(k+1) + \pi^s_{xy}(k+1)\right), \tag{21}$$

$$\alpha_{xy}(k+1) = \alpha_{xy}(k) + \frac{\eta}{2}\left(\pi^t_{xy}(k+1) - \pi^s_{xy}(k+1)\right). \tag{22}$$

Proof: As (16) is strictly concave, we can solve it by first-order condition: $\pi_{xy}(k+1) = \frac{1}{2\eta}(\alpha^t_{xy}(k) - \alpha^s_{xy}(k)) +$

$\frac{1}{2}(\pi^t_{xy}(k+1)+\pi^s_{xy}(k+1))$. Substituting the above equation into (17) and (18) yields: $\alpha^t_{xy}(k+1)=\frac{1}{2}(\alpha^t_{xy}(k)+\alpha^s_{xy}(k))+\frac{\eta}{2}(\pi^t_{xy}(k+1)-\pi^s_{xy}(k+1))$, and $\alpha^s_{xy}(k+1)=\frac{1}{2}(\alpha^t_{xy}(k)+\alpha^s_{xy}(k))+\frac{\eta}{2}(\pi^t_{xy}(k+1)-\pi^s_{xy}(k+1))$. Thus, $\alpha^t_{xy}=\alpha^s_{xy}$ during each update, and $\pi_{xy}(k+1)$ can be further simplified as $\pi_{xy}(k+1)=\frac{1}{2}(\pi^t_{xy}(k+1)+\pi^s_{xy}(k+1))$. In addition, (17) and (18) can be reduced to (22) based on $\alpha^t_{xy}=\alpha^s_{xy}=\alpha_{xy}$. ■

We summarize iterations (19)-(22) into the following Algorithm 1.

Algorithm 1 Distributed Algorithm for Behavioral OT

while Π^t_x and Π^s_y not converging **do**
    Compute $\Pi^t_x(k+1)$ using (19), for all $x\in\mathscr{X}_y$
    Compute $\Pi^s_y(k+1)$ using (20), for all $y\in\mathscr{Y}_x$
    Compute $\pi_{xy}(k+1)$ using (21), for all $\{x,y\}\in\mathscr{E}$
    Compute $\alpha_{xy}(k+1)$ using (22), for all $\{x,y\}\in\mathscr{E}$
end while
return $\pi_{xy}(k+1)$, for all $\{x,y\}\in\mathscr{E}$

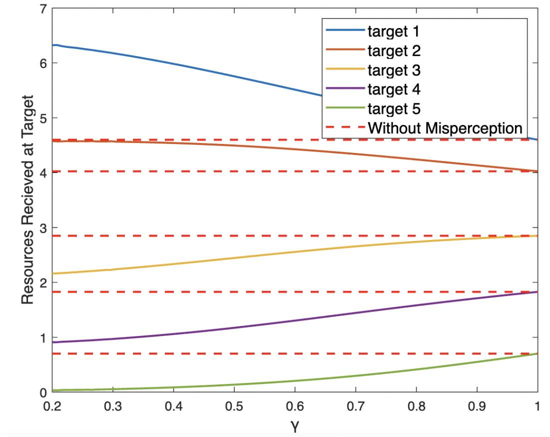
VI. Case Studies

In this section, we corroborate the developed results and the proposed algorithm focusing on the impacts of the behavioral consideration on the transport plan. We investigate a transport network consisting of two source nodes and five target nodes. We define the loss parameter at each target as $U_1=12$, $U_2=9$, $U_3=5$, $U_4=3$, and $U_5=2$. Additionally, we use Prelec's probability weighting function defined in (2) and the probability function shown in (5). We set the upper bound of security resources to $\bar{q}_1=10$ units and $\bar{q}_2=4$ units, meaning that those two source nodes can at most invest that amount of security resources to the targets.

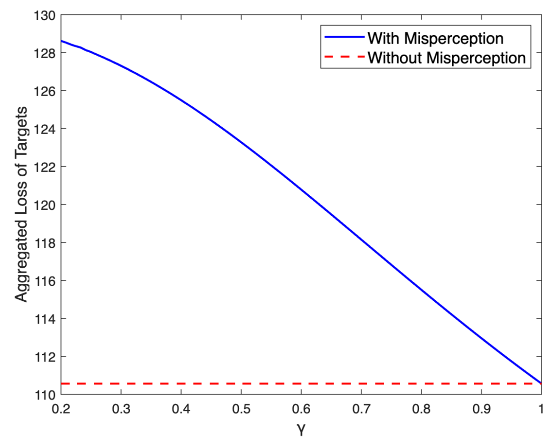
A. Impact of Behavioral Consideration

We first examine how incorporating the behavioral considerations will impact the transportation of security resources from the sources to targets. This involves looking at how the parameter γ affects the outcome of the transport plan.

We expect that as the parameter γ goes to one, the amount of resources received at each target should be the same as when misperception is not considered, i.e., the objective function would be $U_x(p_x(\Pi_x))$. Fig. 1(a) corroborates this result. We can also observe that the target node with a larger value U_x receives more resources, indicating that the transport planner prefers to secure more valuable targets under a constrained budget. The relationship between the amount of received resources at targets follows from the order of node's value U_x in Assumption 2. Additionally, as the behavioral parameter γ goes to 1, the aggregated loss at the targets under the optimal transport plan converges to the same value when misperception is not considered, as shown in Fig. 1(b). It can also be seen that the optimal behavioral strategy is not as efficient as the one under the accurate perception.



(a)



(b)

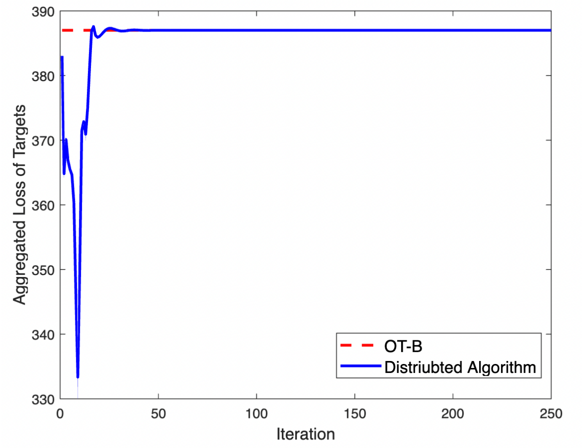
Fig. 1. Impact of behavioral misperception on the optimal transport plan. (a): Impact of γ on the amount of resources received at each target node. The solution converges to the optimal transport without misperception as γ goes to 1. (b): Aggregated loss of the targets with varying γ. A larger degree of misperception yields a less efficient transport strategy.

B. Verification of Distributed Algorithm

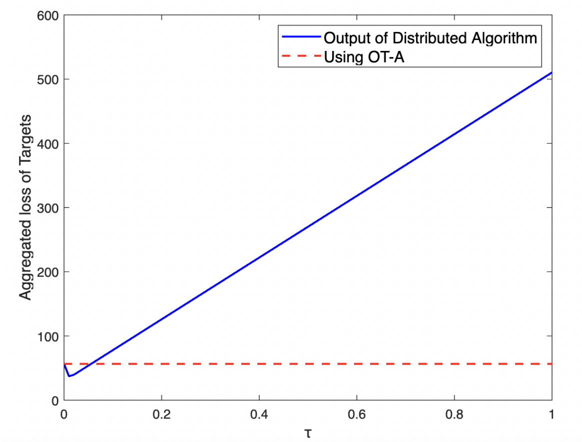
We next show the performance of the proposed distributed algorithm in Algorithm 1. We use Algorithm 1 to solve the optimization problem (OT-B). Fig. 2(a) shows that the distributed algorithm can efficiently converge to the centralized optimal solution. We also examine how the parameter τ_y influences the outcome of the transport plan. In the case study, τ_y is set to be the same at every source node, i.e., $\tau_y=\tau,\ \forall y\in\mathscr{Y}$. We leverage the developed distributed algorithm to compute the optimal behavioral transport strategy for various $\tau\in[0,1]$, and Fig. 2(b) depicts the results. We can observe that when τ goes to zero in (OT-B), the aggregated loss of target nodes under the optimal transport coincides with the one to (OT-A). This result makes sense as the utility term s_{xy} no longer plays a role in (OT-B) when $\tau=0$.

VII. Conclusion

This paper has developed a behavioral optimal transport (OT) framework for security investments over a network consisting of multiple source nodes and heterogeneous target



(a)



(b)

Fig. 2. (a): Effectiveness of the distributed algorithm 1. The distributed algorithm converges to the centralized optimal solution to (OT-B). (b): Impact of weighting constant τ on the aggregated loss of the targets. The solution degenerates to the one to (OT-A) as τ goes to 0.

nodes. The behavioral element captures human's misperception of the successful attack probabilities at targets under a given level of security investment. The analysis has shown that the optimal behavioral transport strategy admits a sequential water-filling nature. In addition, we have discovered that fewer targets will receive security resources under the behavioral OT paradigm than the solution to standard OT, revealing the sub-optimal feature of the strategy due to the behavioral misperception. We have further developed an efficient distributed algorithm to compute the transport plan with a convergence guarantee, and it enjoys advantages when the transportation network becomes enormous and complex. The case studies have corroborated that under the behavioral OT, the transport planner favors the targets with a larger value in security investment, often resulting in lower-valued targets receiving smaller amounts or no resources. Future works include extending the current framework to an adversarial setting and develop resilient security investment strategies over networks.

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