Consider a mass m on a frictionless table subject to a piecewise constant force

$$f(t) = f_i \text{ for } t \in [i-1, i) \text{ and } \; i = 1, \ldots, T_n.$$

Let $y(t) = (x, v)$ denote the position and velocity of the mass. We'll use the notation $z_i = z(i)$ for $z \in \{x, v, y\}$ since with a piecewise constant force the problem discretizes. Suppose $y_0 = \mathbf{0}$, $T_n = 10$ and we seek a force program f_i which lands the mass in a desired state $y_d = (1, 0)$ at $t = T_n$.

Under the assumed initial rest, the velocity $v(t)$ can be written: $v = A^{vf} f$. Since the system is causal it is clear that $A^{vf}_{ij} = 0$ for $i < j$. (The velocity v_i cannot depend on a force f_j which has not yet been applied.) Applying Newton's first law under constant force gives $\Delta v = a\Delta t = \frac{f}{m}\Delta t$. For piecewise constant f and $\Delta t = (i+1) - (i) = 1$, the velocity at time t is simply the running sum of the forces applied so far:

$$v = A^{vf} f = \frac{1}{m} \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \ldots & 1 \end{pmatrix} f,$$

where the upper-diagonal elements are zero.

Since the applied forces are piecewise constant, the velocities are piecewise linear, and the discrete-time positions can be written as a linear function of the sampled velocities: $x = A^{xv} v$, where

$$A^{xv}_{i,j} = \begin{cases} 0 & i > j \\ \frac{1}{2} & i = j \\ 1 & i < j \end{cases}$$

Next form $A^{xf} = A^{xv} A^{vf}$.

Of particular interest to us is the final state $y = (x_{10}, v_{10})$. Defining row vectors

$$(a^{xf}_i)^T = A^{xf}_{10,i} \quad \text{and} \quad (a^{vf}_i)^T = A^{vf}_{10,i}$$

and defining:

$$A \in \mathbb{R}^{2\times 10} = \begin{pmatrix} (a^{xf})^T \\ (a^{vf})^T \end{pmatrix}, \tag{1}$$

observe $y = Af$.

As it stands, finding a force program is a feasibility problem. Let's add a regularizer to reign in the solutions. With an l_2 penalty the problem takes the form:

$$\begin{aligned} \underset{f}{\text{minimize}} \quad & \frac{1}{2} f^T f \\ \text{subject to} \quad & Af = y_d, \end{aligned} \tag{2}$$

which is the least-norm problem for an overdetermined system of equations. Let's solve it for practice. We introduce a Lagrangian,

$$\mathcal{L} = f^T f + \lambda^T (Af - y_d) \tag{3}$$

and take first order conditions (FOCs):

$$\frac{\partial \mathcal{L}}{\partial f} = f + A^T \lambda = 0 \tag{4a}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Af - y_d = 0. \tag{4b}$$

Left-multiplying equation (4a) by A we have:

$$Af - AA^T \lambda = 0, \tag{5}$$

and substituting from equation (4b) we have:

$$AA^T \lambda = -y_d. \tag{6}$$

Since A is fat and full-rank, AA^T is invertible, and so equation (6) becomes:

$$\lambda = -(AA^T)^{-1} y_d. \tag{7}$$

Substituting λ back into equation (4a), we obtain the optimal f, which we recognize as the least-norm solution:

$$f = A^T(AA^T)^{-1}y_d. \tag{8}$$

While l_2 gives us traction analytically, other penalties give rise to solutions with very nice interpretations, and in some cases yield convex formulations. In a propulsion application, an $\|\cdot\|_1$ penalty on the forces would correspond closely to constraining fuel consumption (Boyd), while $\|\cdot\|_\infty$ would penalize the max thruster.