Jacob Hultman
10/23/2017

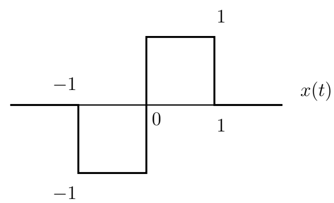
Recitation 7

1) Compute the quantity:

$$A = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution:

First, consider



with period $T = 2$. Let's find the Fourier series coefficients:

$$\omega_0 = \frac{2\pi}{T} = \pi$$

$$a_k = \frac{1}{2}\int_{-1}^{0} e^{-jk\pi t}\,dt + \frac{1}{2}\int_{0}^{1} e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-jk\pi t}}{jk\pi}\bigg|_{-1}^{0} + \frac{-e^{-jk\pi t}}{jk\pi}\bigg|_{0}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{jk\pi}}{jk\pi} + \frac{-e^{-jk\pi}+1}{jk\pi}\right]$$

$$= \frac{1}{2}\left[\frac{2-(e^{jk\pi}+e^{-jk\pi})}{jk\pi}\right] = \frac{1}{2}\left[\frac{2-2\cos(k\pi)}{jk\pi}\right]$$

$$= \frac{1-\cos(k\pi)}{jk\pi} = \frac{1-(-1)^k}{jk\pi} = \begin{cases} 0, & k \text{ even} \\ \frac{2}{jk\pi}, & k \text{ odd} \end{cases}$$

By Parseval's Theorem:

$$\frac{1}{T}\int_T |x(t)|^2\,dt = 1 = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-\infty}^{\infty} \frac{(1-(-1)^k)^2}{(k\pi)^2} = 2\sum_{k=1}^{\infty} \frac{(1-(-1)^k)^2}{k^2\pi^2}$$

$$= 2\sum_{k=0}^{\infty}\frac{2^2}{(2k+1)^2\pi^2} = 1 \Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$A = \sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{4}A$$

$$\Rightarrow A = \frac{4}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

Eigenfunctions

$\phi(t)$ is an eigenfunction of S if

$$\phi(t) \to \lambda\phi(t),$$

where $\lambda \in \mathbb{C}$ is the eigenvalue of $\phi(t)$.

$$\{\phi_k\} \text{ are eigenfunctions of } S$$

$$\lambda_k \text{ are the corresponding eigenvalues}$$

$$X(t) = \sum_{k=1}^{n} c_k\phi_k(t) \quad \to \quad y(t) = \sum_{k=1}^{n} c_k\lambda_k\phi_k(t)$$

2) Consider the following system S:

$$y(t) = t^2\frac{d^2x(t)}{dt^2} + t\frac{dx(t)}{dt}$$

a) Is S linear? **Yes.** Is it time invariant? **No.**

b) Show that $\phi_k = t^k$ for $k \in \mathbb{Z}$ are eigenfunctions of S. What are the corresponding eigenvalues?

Solution:

$$\begin{aligned}
y(t) &= t^2\frac{d^2(t^k)}{dt^2} + t\frac{d(t^k)}{dt} \\
&= t^2(k)(k-1)t^{k-2} + tkt^{k-1} \\
&= (k)(k-1)t^k + kt^k \\
&= t^k k(k-1+1) = k^2t^k
\end{aligned}$$

c) Determine output when input is $x(t) = 10t^{-10} + 3t$

Solution:

$$y(t) = 1000t^{-10} + 3t$$

3) Compute the value of:

$$\sum_{k=1}^{\infty} \frac{1}{n(n+1)}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{1}{n(n+1)} = \sum_{k=1}^{\infty} \frac{n+1-n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1$$

3) Let $T = 20$. Not sure what the question was.

$$x_1(t) = \sum_{k=0}^{40} \left(\frac{1}{3}\right)^k e^{jk\frac{2\pi t}{20}}$$
$$x_2(t) = \sum_{k=-40}^{40} \cos(k\pi) e^{jk\frac{2\pi t}{20}}$$
$$x_3(t) = \sum_{k=-40}^{40} j\cos(k\pi) e^{jk\frac{2\pi t}{20}}$$