Recitation 10

11/13/17

1. Let $x(t)$ be odd and suppose $\int_0^\infty x(t)\sin(\omega t)\,dt = \frac{\omega}{1+\omega^2}$

Find $X(j\omega)$ using only the definition of the FT.

Soln:

$$\int_0^\infty x(t)\sin(\omega t)\,dt = \int_0^\infty x(t)\left[e^{j\omega t} - e^{-j\omega t}\right]dt \cdot \frac{1}{2j}$$

$$= \frac{1}{2j}\left[\int_0^\infty x(t)e^{j\omega t}\,dt - \int_0^\infty x(t)e^{-j\omega t}\,dt\right]$$

$$= \quad '' \quad + \int_0^{-\infty} x(-\tau)e^{j\omega\tau}\,d\tau\Big] \qquad \text{letting } d\tau = -dt$$

$$= \quad '' \quad + \int_0^{-\infty} -x(\tau)e^{j\omega\tau}\,d\tau\Big] \qquad (x(\tau) \text{ is odd})$$

$$= \quad '' \quad + \int_{-\infty}^{0} x(\tau)e^{j\omega\tau}\,d\tau\Big] \qquad \left(\text{swapping integration bounds}\right)$$

$$= \frac{1}{2j}\underbrace{\int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt}_{X(-j\omega)} = \frac{\omega}{1+\omega^2} \qquad (\text{combining two integrals})$$

$$\Rightarrow \quad X(j\omega) = \frac{-2j\omega}{1+\omega^2} \qquad (\text{by definition})$$

2.

Let $X(j\omega) = \dfrac{\omega^2}{1+4\omega^2}$. Find $x(t)$.

Soln: $\dfrac{\omega^2 + 3\omega^2 - 3\omega^2 + 1 - 1}{1+4\omega^2} = \dfrac{1+4\omega^2 - (1+3\omega^2)}{1+4\omega^2}$

$$= 1 - \frac{1+3\omega^2}{1+4\omega^2}$$

$$X(j\omega) = 1 - \frac{1}{1+4\omega^2} - 3X(j\omega)$$

$$\Rightarrow \quad X(j\omega) = \frac{1}{4} - \frac{\frac{1}{4}}{1+4\omega^2} = \frac{1}{4} + \frac{A}{1+2j\omega} + \frac{B}{1-2j\omega}$$

A: Let $\omega = \frac{j}{2}$

$$A = \frac{\left(\frac{j}{2}\right)^2}{1 - 2j\left(\frac{j}{2}\right)} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$$

B: Let $\omega = -\frac{j}{2}$

$$B = \frac{\left(\frac{-j}{2}\right)^2}{1 + 2j\left(\frac{-j}{2}\right)} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$$

$$X(j\omega) = \frac{1}{4} - \frac{1}{8}\cdot\frac{1}{1+2j\omega} - \frac{1}{8}\cdot\frac{1}{1-2j\omega} = \frac{1}{4} - \frac{1}{16}\cdot\frac{1}{1/2 + j\omega} - \frac{1}{16}\cdot\frac{1}{1/2 - j\omega}$$

Using table 4.2:

time reversal

$$x(t) = \frac{1}{4}\delta(t) - \frac{1}{16}\exp(-t/2)u(t) + \exp(t/2)u(-t)$$

4. $y(t) = x(t) * h(t)$

$g(t) = x(3t) * h(3t)$

$X(j\omega)$, $H(j\omega)$

Use FT properties to show that $g(t)$ has the form $g(t) = A\, y(Bt)$. Find A and B.

Soln:

$$x(3t) \leftrightarrow \frac{1}{3} X\left(\frac{j\omega}{3}\right)$$

$$h(3t) \leftrightarrow \frac{1}{3} H\left(\frac{j\omega}{3}\right)$$

$$G(j\omega) = \frac{1}{9} X\left(\frac{j\omega}{3}\right) H\left(\frac{j\omega}{3}\right) = \frac{1}{9} Y\left(\frac{j\omega}{3}\right) = \frac{1}{3} \cdot \frac{1}{3} Y\left(\frac{j\omega}{3}\right)$$

$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

$$\leftrightarrow \frac{1}{3} y(3t)$$

$$A = \frac{1}{3}, \quad B = 3$$

5. Solve 4) except in the time domain.

Soln:

$$g(t) = \int_{-\infty}^{\infty} \bar{x}(\tau)\, \bar{h}(t-\tau)\, d\tau \qquad \text{where} \quad \bar{x}(\tau) = x(3\tau)$$
$$\text{and} \quad \bar{h}(\tau) = h(3\tau)$$

$$= \int_{-\infty}^{\infty} x(3\tau)\, h(3t - 3\tau)\, d\tau \qquad \text{Let} \quad 3\, d\tau = d\bar{\tau}$$

$$= \frac{1}{3} \int_{-\infty}^{\infty} x(\bar{\tau})\, h(3t - \bar{\tau})\, d\bar{\tau}$$

$$= \frac{1}{3}\, x(\tau) * h(\tau) \Big|_{\tau = 3t}$$

$$= \frac{1}{3}\, y(3t)$$

$A = \frac{1}{3} \qquad B = 3$