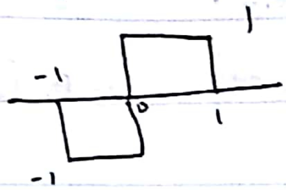
$\sum_{k=1}^{\infty} \frac{1}{k^2} \swarrow$ in closed form



$x(t)$ with period $T = 2$

$\omega_0 = \frac{2\pi}{T} = \pi$

$a_k \overset{F.S.}{\longleftrightarrow} x(t)$

$a_0 = 0$

$$a_k = \frac{1}{2}\int_{-1}^{0} -1\, e^{-jk\pi t}\, dt + \frac{1}{2}\int_{0}^{1} 1\, e^{-jk\pi t}$$

$$= \frac{1}{2}\left[\frac{e^{-jk\pi t}}{jk\pi}\Bigg|_{-1}^{0} + \frac{-e^{-jk\pi t}}{jk\pi}\Bigg|_{0}^{1} \right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{jk\pi}}{jk\pi} + \frac{-e^{-jk\pi} + 1}{jk\pi} \right] = \cancel{\frac{1 - (\cos k\pi)}{jk\pi}}$$

$$= \frac{1}{2}\left[\frac{2 - (e^{jk\pi} + e^{-jk\pi})}{jk\pi} \right] = \frac{1}{2}\left[\frac{2 - 2\cos k\pi}{jk\pi} \right]$$

$$= \frac{1 - \cos k\pi}{jk\pi} = \frac{1 - (-1)^k}{jk\pi} = \begin{cases} 0, & k \text{ even} \\ \frac{2}{jk\pi}, & k \text{ odd} \end{cases}$$

By Parsevals Thm: $\frac{1}{T}\int_T |x(t)|^2\, dt = 1 = \sum_{k=-\infty}^{\infty} |a_k|^2$

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{-\infty}^{\infty} \frac{(1-(-1)^k)^2}{(k\pi)^2} = 2\sum_{k=1}^{\infty} \frac{(1-(-1)^k)^2}{k^2\pi^2}$$

$$= 2\sum_{k=0}^{\infty} \frac{2^2}{(2k+1)^2\pi^2} = 1 \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

$$A = \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{4}A$$

$$\Rightarrow A = \frac{4}{3} \cdot \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \boxed{\frac{\pi^2}{6}}$$

a) $\phi(t)$ is an eigenfunction of S if
$\phi(t) \rightarrow \lambda\,\phi(t)$, where $\lambda \in \mathbb{C}$ is the eigenvalue
of $\phi(t)$

$\{\phi_k\}$ are eigenfunctions of S
λ_k are the corresponding eigenvalues

$$x(t) = \sum_{k=1}^{n} c_k \phi_k(t) \rightarrow y(t) = \sum_{k=1}^{n} c_k \lambda_k \phi_k(t)$$

b) Consider the following system:

$$y(t) = t^2 \frac{d^2 x(t)}{dt^2} + t\frac{dx(t)}{dt}$$

Is S linear? Yes. Is it time invariant? No.

c) Show that $\phi_k = t^k$ for $k \in \mathbb{Z}$ are eigenfunctions of S. What are the corresponding eigenvalues?

$$y(t) = t^2 \frac{d^2(t^k)}{dt^2} + t\frac{d(t^k)}{dt}$$

$$= t^2 \cdot (k)(k-1)\,t^{k-2} + t \cdot k \cdot t^{k-1}$$

$$= (k)(k-1)\,t^k + k\,t^k$$

$$= t^k k(k-1+1) = k^2 t^k$$

d) Determine output when input is $x(t) = 10\,t^{-10} + 3t$

soln: $y(t) = 1000\,t^{-10} + 3t$

1. $$\sum_{k=1}^{\infty} \frac{1}{n(n+1)} = \sum_{k=1}^{\infty} \frac{n+1-n}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \quad \text{(telescoping)}$$

$$= 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \tfrac{1}{4} \qquad = 1$$

2. $$x_1(t) = \sum_{k=0}^{40} \left(\frac{1}{3}\right)^k e^{jk\frac{2\pi t}{20}}$$

where $T = 20$

$$x_2(t) = \sum_{k=-40}^{40} \cos(k\pi)\, e^{jk\frac{2\pi t}{20}}$$

$$x_3(t) = \sum_{k=-40}^{40} j\cos\left(\frac{k\pi}{2}\right) e^{jk\frac{2\pi t}{20}}$$