1. Let S be a causal LTI discrete time system with impulse response $h[n]$.

Suppose $h[n] = \begin{cases} \frac{1}{2}^{n-N} & , n \geq N \\ ? & , n < N \end{cases}$, where $N > 0$

Prove that S is stable.

Soln:

If S is causal, it is non-anticipative. Since $\delta[n] = 0$ for $n < 0$, we must have $h[n] = 0$ for $n < 0$. S cannot anticipate the impulse.

Now, S is BIBO stable if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$.

Pick $B = \max_{k=0,1,\dots,N-1} \{|h[k]|\}$

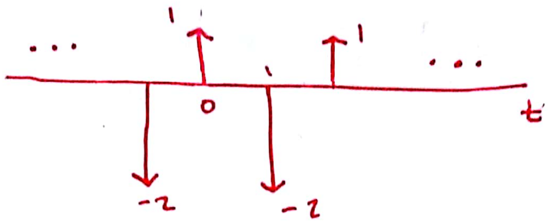
$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{N-1} |h[k]| + \sum_{k=N}^{\infty} |(1/2)^{n-N}| \leq B \cdot N + 2 < \infty$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$y[n]$ cannot depend on $x[k]$ for $k > n$, and so we see must have $h[n-k] = 0$ for $k > n$ or in other words $h[n] = 0$ for $n < 0$.

2.

Let $x_1(t)$ be as drawn below:



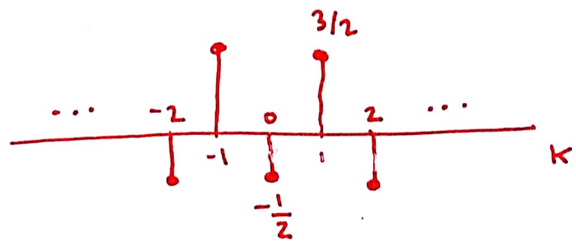
Find the F.S. coefficients a_k of $x(t)$.

Soln:

$$a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\int_{-1/2}^{3/2} x(t)\,dt = \frac{1}{2}(1-2) = -\frac{1}{2}$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_{-1/2}^{3/2} \left[\delta(t) - 2\,\delta(t-1)\right] e^{-j\frac{2\pi}{2}kt}\,dt$$

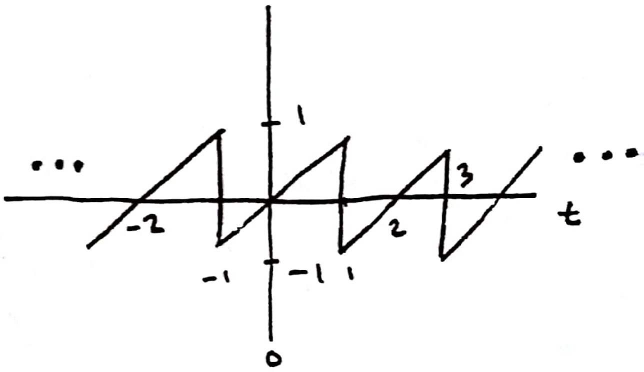
$$= \frac{1}{2} - e^{-j\frac{2\pi}{2}k(1)} = \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - (-1)^k \quad \text{for } k \neq 0$$



Impulsive in time, periodic in frequency.

Duality.

Find the F.S. coefficients a_k for the signal $x(t)$ plotted below:



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \qquad T = 2$$

$$= \frac{1}{2}\int_{-1}^{1} t e^{-jk\left(\frac{2\pi}{2}\right)t}\,dt = \frac{1}{2}\left[-\frac{te^{-jk\pi t}}{jk\pi} - \frac{e^{-jk\pi t}}{(jk\pi)^2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{-e^{-jk\pi}}{jk\pi} - \frac{e^{-jk\pi}}{(jk\pi)^2} - 1\cdot\frac{e^{jk\pi}}{jk\pi} + \frac{e^{jk\pi}}{(jk\pi)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\left(e^{jk\pi} - e^{-jk\pi}\right)}{(jk\pi)^2} - \frac{\left(e^{jk\pi} + e^{-jk\pi}\right)}{jk\pi}\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{2}{j}\frac{\sin(k\pi)}{j^2(k\pi)^2}} - \frac{2\cos(\pi k)}{jk\pi}\right] = \begin{cases} +\dfrac{j}{k\pi}, & k \text{ even} \\ -\dfrac{j}{k\pi}, & k \text{ odd} \end{cases}$$

$$= (-1)^k \frac{j}{k\pi} \quad \text{for } k \neq 0$$

and $a_0 = 0$ (DC component)

$x(t)$ real and odd <-> a_k purely imaginary

$x(t)$ real and even <-> a_k real and even