Recitation 11 11/27/17

1. Prove by induction: $\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n}$

Soln:

Base case $n=1$: $e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega}$ (given pair)

Assume true for k:

$$\frac{t^{k-1}}{(k-1)!} e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^k}$$

Show true for $k+1$:

$$x_{k+1}(t) = \frac{t^k e^{-at} u(t)}{(k!)} = \frac{t}{k} \frac{t^{k-1} e^{-at} u(t)}{(k-1)!} = \frac{-1}{jk} \cdot -jt \frac{t^{k-1} e^{-at} u(t)}{(k-1)!}$$

And $-jt\, x(t) \longleftrightarrow \frac{d\, X(j\omega)}{d\omega}$

$$x_{k+1}(t) \longleftrightarrow \frac{-1}{jk} \cdot \frac{d}{d\omega}\left[\frac{1}{(a+j\omega)^k}\right] = \frac{-1}{jk} \cdot -jk \cdot \frac{1}{(a+j\omega)^{k+1}} = \frac{1}{(a+j\omega)^{k+1}}$$

2. Find the Fourier transform of $\frac{1}{(1+jt)^2}$

Soln: Use duality.

$$\frac{1}{(1+j\omega)^2} \longleftrightarrow te^{-t} u(t)$$

Let $f(t) = te^{-t}u(t)$ so that $F(\omega) = \frac{1}{(1+j\omega)^2}$

Then by duality $F(t) \longleftrightarrow 2\pi f(-\omega)$

$$= -2\pi e^{\omega} u(-\omega) \cdot \omega$$

3. Let $F(\omega) = \mathcal{F}\{x(t)\}$. Find an expression for $\mathcal{F}\{F(\omega)\}$.

Soln:

$$f(t) = \frac{1}{2\pi}\int F(\omega) e^{j\omega t} d\omega$$

$$2\pi f(-t) = \underbrace{\int F(\omega) e^{-j\omega t} d\omega}_{\mathcal{F}\{F(\omega)\}}$$

Interchanging role of ω and t:

$$2\pi f(-\omega) = \int F(t) e^{-j\omega t} dt$$

And so we see:

$$f(t) \longleftrightarrow F(\omega)$$

$$F(t) \longleftrightarrow 2\pi f(-\omega)$$

4. Let $x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$. Find $X(j\omega)$.

Soln:

$$\frac{\sin t}{\pi t} \longleftrightarrow \Pi(\omega)$$

$$\left(\frac{\sin t}{\pi t}\right)^2 \longleftrightarrow \frac{1}{2\pi}\Pi(\omega) * \Pi(\omega) = \frac{1}{\pi}\Lambda(\omega/2)$$

$$t\, x(t) \longleftrightarrow j\frac{d}{d\omega}\left[X(j\omega)\right]$$

And so

$$X(j\omega) = \begin{cases} \frac{j}{2\pi}, & -2 < \omega < 0 \\ \frac{-j}{2\pi}, & 0 < \omega < 2 \end{cases}$$

5. Find numerically $\displaystyle\int_{-\infty}^{\infty} \frac{(\sin t)^4}{t^2}\, dt := A$

We know $\displaystyle\frac{t(\sin t)^2}{\pi^2 t^2} \longleftrightarrow \begin{cases} \frac{j}{2\pi}, & -2 < \omega < 0 \\ \frac{-j}{2\pi}, & 0 < \omega < 2 \end{cases} := X(j\omega)$

And by parseval's theorem:

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{2\pi}\right)^2 \cdot 2 + \left(\frac{1}{2\pi}\right)^2 \cdot 2\right]$$

$$= \frac{1}{2\pi^3} = \frac{A}{\pi^4} \Rightarrow A = \frac{\pi}{2}$$