Roots of Polynomials, Integer Partitions, and L-Functions

Joshua Hunt

Larry Rolen, Advisor

A thesis submitted in partial fulfillment of

of the requirements for the

Degree of Bachelor of Arts

in Mathematics

VANDERBILT UNIVERSITY

Nashville, Tennessee

April 21, 2023

1. Introduction

In this thesis, we explore diverse topics within modern number theory, which all converge in their applications to recent results. In particular, within the first few sections, we review the topics of integer partitions and Jensen polynomials, as well as classical results relating to each. We then investigate the Riemann Zeta and Xi functions and their applications to analytic number theory. From here, we investigate the Laguerre-Pólya class, listing its relevant properties and results. Based on recent work of Wagner, we define the shifted Laguerre-Pólya class, a generalization of the Laguerre-Pólya class, and similarly give some examples and properties. Next, we review Dirichlet L-functions, including their classical results and, following this, we similarly review general L-functions.

From here, we state the recent developments which inspired the research of the author and their collaborators. Firstly, we review the work of Griffin et al. proving for sequences with suitable asymptotics that, for each fixed degree, at most finitely many Jensen polynomials over that sequence are not hyperbolic (i.e. have non-real roots). This is achieved by renormalizing the Jensen polynomials and showing that as the shift tends towards infinity, they asymptotically approach the Hermite polynomials, which are known to be hyperbolic. In particular, we provide a proof of this fact which was given in presentations by the authors. We also investigate a paper by Kim and Lee, which refines the results of Griffin et al. to provide a bound on when these Jensen polynomials become hyperbolic. These results provide evidence for the Riemann Hypothesis.

We then discuss the research contributions of the author. Firstly, the author generalizes the methods of Kim and Lee to Dirichlet L-functions to find a bound on when the Jensen polynomials over these functions become hyperbolic. This bound provides evidence for the Generalized Riemann Hypothesis for Dirichlet L-functions. Also included in this section is an alternative proof that Dirichlet L-functions have order 1 which does not yet exist in the literature, to the author's knowledge. Next, the author extends the methods of Kim and Lee to a wider class of L-functions which satisfy certain "nice" properties, obtaining a bound on the hyperbolicity of Jensen polynomials over these L-functions.

Finally, we discuss the primary research topic of the author and their collaborators. The author and their collaborators sought to generalize the methods of Kim and Lee to functions in the shifted Laguerre-Pólya class. The section contains descriptions of why this generalization is natural but not trivial, as well as some preliminary results which give an idea for how such a generalization would be proved and specific developments in the partition case.

We conclude with a summary of results, a statement of further work to be done on the topic, and a review of the author's experience learning and researching for the thesis.

1.1. **Acknowledgements.** The author would like to thank their thesis advisor Dr. Larry Rolen. The author would also like to thank their research collaborator Dr. Ian Wagner, as well as the thesis defense committee members Dr. Doug Hardin and Dr. Edward Saff.

2. Background Topics

2.1. **Partitions.** In this section, we recall the definition and some basic facts about integer partitions.

Definition 2.1. An **integer partition of** n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ such that $\lambda_1 + ... + \lambda_k = n$. The number of partitions of n is denoted by $p(n)$.

Example 2.2. The partitions of 4 are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Thus, $p(4) = 5$.

Partitions have been studied for centuries, starting primarily with the work of Leonard Euler in the 18th century. Much of Euler's work created the foundation for the modern study of the partition function. We attribute to Euler an identity about the partition generating function and a recurrence relation for $p(n)$ which allow for more detailed study of the function. Beyond the work of Euler, one of the most interesting and difficult questions about the partition function was concerning the asymptotic behavior of $p(n)$ as n tends towards infinity. This was answered first by Hardy and Ramanujan and later refined by Rademacher, the results of whom are discussed later in this section.

One particularly useful tool for understanding partitions is their generating function. The partition generating function is defined as $\sum_{n\geq 0} p(n)q^n$ where the coefficient of the n-th term is the number of partitions of n. We have the following theorem of Euler concerning the partition generating function.

Theorem 2.3. *The partition generating function has the following product expression*

$$\sum_{n\geq 0} p(n)q^n = \prod_{i\geq 1} \frac{1}{1-q^i}.$$

This series would only converge for $|q| < 1$, so we consider the product as a formal power series.

Proof. We can see that $p(n)$ counts the number of integer solutions to the equation $1x_1 + 2x_2 + 3x_3 + ... = n$ with $x_i \geq 0$ for each $i \geq 1$ by considering x_i to represent the number of i's in the sum. First, the term $1x_1$ can take values $0, 1, 2, ...$ so its possibilities can be represented by a term of $(1 + q + q^2 + ...)$ in the generating function. Next, $2x_2$ can take values of $0, 2, 4, ...$ so its possibilities can be represented by $(1+q^2+q^4+...)$. We can continue this process for all i, which implies the generating function for $p(n)$ is

$$\sum_{n\geq 0} p(n)q^n = (1 + q + q^2 + ...)(1 + q^2 + q^4 + ...)(1 + q^3 + q^6 + ...)....$$

Using the identity $1 + q^i + q^{2i} + ... = \frac{1}{1-q^i}$ we obtain

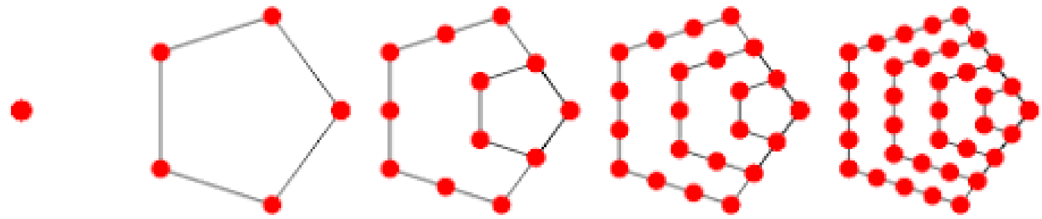
$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)... \\ &= \prod_{i\geq 1} \frac{1}{1-q^i},\end{aligned}$$

as desired. ■

Another result from Euler is the Pentagonal Number Theorem which despite its purely analytic expression, has a beautiful corollary giving a recurrence relation for the partition function.

Definition 2.4. The **nth pentagonal number**, denoted g_n, is the number of dots in the outlines of nested regular pentagons with side lengths ranging from 1 to n dots and sharing one vertex.

Example 2.5. The first few pentagonal numbers are $g_1 = 1$, $g_2 = 5$, $g_3 = 12$, $g_4 = 22$, $g_5 = 35$, and so on, which can be seen pictorially below.



It is a classical result that the pentagonal numbers satisfy the following equation and recurrence: $g_n = n(3n-1)/2$ and $g_n = g_{n-1} + 3n - 2$. Now that we have defined pentagonal numbers, we can state the Pentagonal Number Theorem.

Theorem 2.6 (Euler's Pentagonal Number Theorem)**.**

$$\prod_{j=1}^{\infty}(1-q^i) = 1 + \sum_{n=1}^{\infty}(-1)^n \left(q^{n(3n+1)/2} + q^{n(3n-1)/2}\right)$$

While originally stated by Euler, a famous proof of this fact was given by Franklin [7]. Franklin's proof used the combinatorial interpretation, noticing that the left-hand side is equal to the generating function for the number of distinct partitions with even length minus the number of distinct partitions of odd length. Franklin showed that except when the input is a pentagonal number, there is a bijection between the distinct even partitions and the distinct odd partitions, which proves the theorem. With this theorem, we can prove the following recurrence for the partition function.

Corollary 2.7. *The following recurrence relation holds*

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \ldots = \sum_{i \neq 0}(-1)^{i-1}p(n-g_i),$$

where the sum is over all nonzero integers and g_i is the i-th pentagonal number.

Proof. We recall Theorems 2.3 and 2.6 and note that $\prod_{i\geq 1}\frac{1}{1-q^i}$ and $\prod_{i\geq 1}(1-q^i)$ are inverses. These imply

$$\left(\sum_{n\geq 0}p(n)q^n\right)\left(1+\sum_{n=1}^{\infty}(-1)^n\left(q^{n(3n+1)/2}+q^{n(3n-1)/2}\right)\right) = \left(\prod_{n\geq 1}\frac{1}{1-q^n}\right)\left(\prod_{n\geq 1}(1-q^n)\right) = 1.$$

Expanding the terms we find,

$$(p(0)+p(1)q+p(2)q^2+\ldots)(1-q-q^2+q^5+q^7-q^{12}-\ldots) = 1.$$

We observe that on the right hand side of this equation the coefficient of q^n is 0 for $n > 0$. Thus, the coefficients of every term in the expansion on the left hand side must also be 0 excluding the q^0 factor. From this we find equations of sums and differences of partitions that sum to 0. Consider the term q^n in the expansion on the left hand side. We find the following sum of terms

$$p(n)q^n \cdot 1 + p(n-1)q^{n-1}\cdot(-q^1) + p(n-2)q^{n-2}\cdot(-q^2) + p(n-5)q^{n-5}\cdot q^5 + \ldots = 0q^n,$$
$$\big(p(n) - p(n-1) - p(n-2) + p(n-5) + \ldots\big)q^n = 0q^n,$$
$$\implies p(n) - p(n-1) - p(n-2) + p(n-5) + \ldots = 0.$$

Thus, we solve for $p(n)$ and obtain the desired recurrence, $p(n) = p(n-1) + p(n-2) - p(n-5) - ...$ for all $n \geq 0$. ■

Some of the most beautiful and important results about partitions come from their asymptotics. Ramanujan and Hardy were the first to find an asymptotic for $p(n)$, given by

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Here $f(n) \sim g(n)$ indicates $f(n)/g(n) \to 1$ as $n \to \infty$. Hardy and Ramanujan pioneered a technique known as the Circle Method to study this product formula. Since its inception, the Circle Method has had applications throughout analytic number theory to diverse problems such as Waring's problem [23] and the weak Goldbach conjecture [11].

The Hardy-Ramanujan asymptotic would later be improved by Rademacher, who used a refinement of their method. The Rademacher asymptotic is an exact formula for $p(n)$ given by the infinite sum

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k \geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6k} \right),$$

where $I_{3/2}$ is a modified Bessel function of weight 3/2 given by

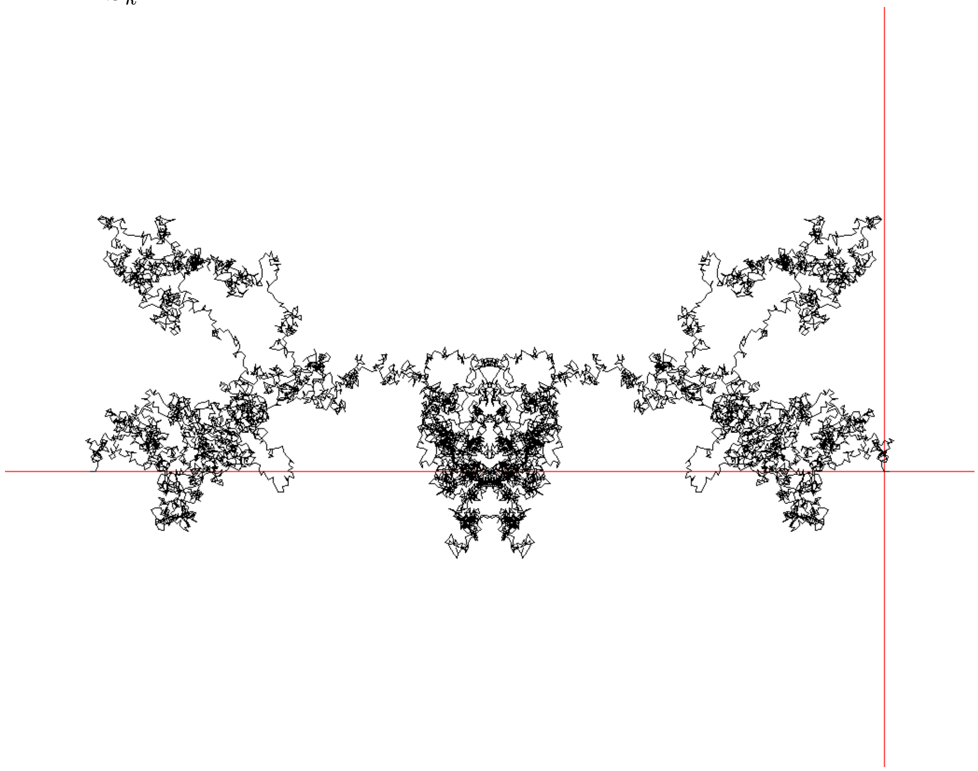
$$I_{3/2}(z) := \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right),$$

and $A_k(n)$ is a type of Kloosterman sum given by

$$A_k(n) := \sum_{\substack{0 \leq h < k \\ gcd(h,k)=1}} e^{\pi i s(h,k) - \frac{2\pi i n h}{k}}.$$

While this is an exact formula for $p(n)$ and partial sums can theoretically be taken to calculate $p(n)$ to arbitrary precision, Kloosterman sums are notoriously difficult to calculate. The graph included at the end of this section is from the Kloostermania website and plots the standard Kloosterman sum $S_k = \sum_{x=0}^{k} \exp(2i\pi \frac{ax+b\bar{x}}{p})$ for different values of k and draws lines between successive sums. This graphic gives some intuition into how sporadic and unpredictable these sums can be. Indeed, these sums may seemingly converge towards a certain values for many terms before switching and starting their convergence towards their actual limit. Since Bessel functions are generally well understood and easy to calculate, the utility of Rademacher's formula in calculating $p(n)$ is limited by our ability to calculate $A_k(n)$ which, as described, is quite difficult. This makes Rademacher's formula limited in its ability to compute $p(n)$ to any certain degree of accuracy.

Figure 1. Plot of S_k for $1 \le k \le 10007-1$ with line segments drawn between successive S_k.



2.2. **Jensen Polynomials.**

Definition 2.8. Let $a\colon \mathbb{N} \to \mathbb{R}$ be be an arithmetic function. The **Jensen polynomial of degree d and shift n associated to a** is

$$J_a^{d,n}(z) := \sum_{j=0}^{d} \binom{d}{j} a(n+j) z^j.$$

It will also be useful in some cases to associate a Jensen polynomial with an entire function using its Taylor coefficients.

Definition 2.9. Let f be an entire function with Taylor series $f(z) = \sum_{k=0}^{\infty} \frac{\gamma(k)}{k!} z^k$. Then, the **Jensen polynomial of degree d and shift n associated to f** is

$$J_f^{d,n}(z) := \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) z^j.$$

Remark. It is clear from the definition that

$$J_f^{d,n}(z) = J_{f^{(n)}}^{d,0}(z) = \sum_{j=1}^{d} \binom{d}{j} f^{(n+j)}(0) z^j.$$

Definition 2.10. A polynomial $f(x) \in \mathbb{R}[x]$ is **hyperbolic** if all of its roots are real.

One useful characteristic of Jensen polynomials over a sequence is that they can encode properties of the sequence based on their hyperbolicity.

Definition 2.11. An arithmetic sequence $\{a_n\}_{n\geq 0}$ is called **log-concave** if $(a_{n+1})^2 \geq a_{n+2}\cdot a_n$ for all $n \geq 0$.

The property of a sequence, $\{a_n\}_{n\geq 0}$ being log-concave is encoded by the hyperbolicity of $J_a^{2,n}(z)$. To see why this is the case, consider the roots of $J_a^{2,n}(z) = a_{n+2}z^2 + 2a_{n+1}z + a_n$, which are given by the quadratic formula

$$\begin{aligned} z &= \frac{-2a_{n+1} \pm \sqrt{4(a_{n+1})^2 - 4a_{n+2}a_n}}{2a_{n+2}} \\ &= \frac{-a_{n+1} \pm \sqrt{(a_{n+1})^2 - a_{n+2}a_n}}{a_{n+2}}. \end{aligned}$$

This implies that $J_a^{2,n}(z)$ is hyperbolic if and only if the discriminant is positive or, equivalently, if $(a_{n+1})^2 \geq a_{n+2}\cdot a_n$.

If we consider the case where $\{a_n\}_{n\geq 0} = p(n)$ is the partition function, we have the following results. Firstly, $J_p^{2,n}(z)$ is hyperbolic for $n \geq 26$ so $p(n)$ is log-concave for $n \geq 26$. This was first shown by Nicolas [17] and was proven again, more recently, by DeSalvo and Pak [6]. However, the case of $d = 2$ is not the only one which provides inequalities for the partition function. In general, for each value of d, we could obtain an inequality concerning subsequent partitions if we could show that $J_p^{d,n}(z)$ is eventually hyperbolic. Chen, Jia, and Wang [4] proved that $J_p^{3,n}(z)$ is hyperbolic for $n \geq 94$. This inspired them to conjecture that there exists an $N(d) \in \mathbb{N}$ such that $J_p^{d,n}(z)$ is hyperbolic for all $n \geq N(d)$. Most recently, Griffin, Ono, Rolen, and Zagier [10] proved the existence of such a $N(d)$ for all $d \in \mathbb{N}$.

3. The Riemann Zeta and Xi Functions

3.1. **The Zeta Function.** The Riemann zeta function is one of the most well-known and important functions in number theory and math in general. It was first described by Euler, who calculated the value of the function at positive even integers and even knew the functional equation for integral inputs! However, Riemann would deeply investigate its properties, being the first to input complex numbers into the function and discover its connections to prime numbers.

The content of this section is based on information found in [5]. Whenever we discuss the zeta function or its generalizations, we will use the standard notation of $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ for complex numbers.

Definition 3.1. For $s \in \mathbb{C}$ with $\sigma > 1$, we define the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We verify that the series $\sum_{n\geq 1} \frac{1}{n^s}$ converges absolutely for $\sigma > 1$. Indeed, for any $N \in \mathbb{N}$, we have

$$\left| \sum_{n=1}^{N} \frac{1}{n^s} \right| \leq \sum_{n=1}^{N} \frac{1}{|n^s|} = \sum_{n=1}^{N} \frac{1}{n^\sigma},$$

and the series

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

is convergent for $\sigma > 1$.

Theorem 3.2 (Euler Product Representation)**.** *We have the following representation of ζ as an infinite product over the prime numbers,*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Proof. We have that

$$\frac{1}{1-z} = 1 + z + z^2 + ... = \sum_{n=0}^{\infty} z^n$$

holds for all complex numbers with $|z| < 1$. We can apply this to the Euler product, and letting $z = p^{-s}$ gives

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...\right).$$

If we expand this product, then the general form of any term is

$$\frac{1}{p_1^{\alpha_1 s} ... p_k^{\alpha_k s}} = \frac{1}{(p_1^{\alpha_1} ... p_k^{\alpha_k})^s}.$$

By the Fundamental Theorem of Arithmetic, every integer $n \geq 2$ has a unique factorization into a product of prime numbers so each term in the expansion will uniquely determine some $\frac{1}{n^s}$ and cover all $n \in \mathbb{N}$ with $n \geq 1$. Thus, we have the desired result that

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

∎

We have so far defined $\zeta(s)$ for $s \in \mathbb{C}$ with $\sigma > 1$, however many of the results about ζ, including the Riemann Hypothesis, concern the values of $\zeta(s)$ for s outside of this region. Thus, we first want to extend this definition to a meromorphic function defined for all $s \in \mathbb{C}$ with $\sigma > 0$, except at $s = 1$.

Theorem 3.3. *The formula*

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - \lfloor x \rfloor}{x^{s+1}} dx \tag{3.1}$$

is valid for $\sigma > 0$, with a pole at $s = 1$.

Proof. Let x be a positive real number and first assume $\sigma > 1$. For a sequence $\{a_n\}_{n \geq 1}$ in $\mathbb{C}$ and function $f \colon [1, +\infty) \to \mathbb{C}$ with a continuous derivative, we have the Abel summation formula

$$\sum_{n \leq x} a_n f(n) = A(x) f(x) - \int_1^x A(u) f'(u) du,$$

where $A(x) = \sum_{n \leq x} a_n$. We can let $a_n = 1$ and $f(t) = t^{-s}$ so that $A(x) = \sum_{n \leq x} 1 = \lfloor x \rfloor$. Using Abel summation, we get that

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor u \rfloor}{u^{s+1}} du.$$

If we let $x \to \infty$, it follows that

$$\begin{aligned}
\zeta(s) = 0 + s\int_1^x \frac{\lfloor u \rfloor}{u^{s+1}}du &= s\int_1^x \frac{u-(u-\lfloor u \rfloor)}{u^{s+1}}du \\
&= s\int_1^x \frac{1}{u^s}du - s\int_1^x \frac{u-\lfloor u \rfloor}{u^{s+1}}du \\
&= s\left(\frac{u^{1-s}}{1-s}\Big|_{u=1}^{u=\infty}\right) - s\int_1^x \frac{u-\lfloor u \rfloor}{u^{s+1}}du \\
&= \frac{s}{1-s}(0-1) - s\int_1^x \frac{u-\lfloor u \rfloor}{u^{s+1}}du \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{u-\lfloor u \rfloor}{u^{s+1}}du.
\end{aligned}$$

Thus, we have established (3.1) in the case of $\sigma > 1$.

In order to show that (3.1) holds for $s \in \mathbb{C}$, $s \neq 1$ with $\sigma > 0$, it is sufficient to show that the integral in the definition converges absolutely for $\sigma > 0$ due to analytic continuation. We can see that for any $N \in \mathbb{N}$

$$\begin{aligned}
\left|\int_1^N \frac{x-\lfloor x \rfloor}{x^{s+1}}dx\right| &\leq \int_1^N \left|\frac{x-\lfloor x \rfloor}{x^{s+1}}\right| dx \\
&\leq \int_1^N \frac{1}{|x^{s+1}|}dx \\
&= \int_1^N \frac{1}{x^{\sigma+1}}dx \\
&= \left(\frac{1}{-\sigma}\cdot x^{-\sigma}\right)\Big|_{x=1}^N \\
&= \frac{-1}{\sigma}\left(\frac{1}{N^\sigma}-1\right),
\end{aligned}$$

which tends to $\frac{1}{\sigma}$ as $N \to \infty$. Thus, the integral converges absolutely for $\sigma > 0$, as desired. ∎

Now that $\zeta(s)$ has been extended to a function defined for $s \in \mathbb{C}$ with $\sigma > 0$, we can state the Riemann Hypothesis–one of the most famous unsolved problems in mathematics.

Conjecture 3.4 (Riemann Hypothesis)**.** If $\zeta(s) = 0$ for some $s \in \mathbb{C}$ with $\sigma > 0$ then $\sigma = 1/2$.

While we have showed and proved a simple extension of the Riemann zeta function to the region $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, $\zeta(s)$ can be extended to a function which is meromorphic over all of $\mathbb{C}$. This function is known as the completed Riemann zeta function and is usually denoted by $\xi(s)$. The function ξ is given by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and it satisfies the functional equation $\xi(s) = \xi(1-s)$, so from these two definitions, we can find values of $\xi(s)$ over all of $\mathbb{C}$. One interesting property of ξ is that it has zeros at the negative even integers, which are known as the trivial zeros.

3.2. **The Xi Function.** The Riemann Xi function is an entire function that maps the zeros of $\zeta(s)$ from the line with real part $\frac{1}{2}$ to the real axis. This function is denoted by $\Xi(z)$ and is given by

$$\Xi(z) := \xi\left(-iz+\frac{1}{2}\right) = \frac{1}{2}\left(-z^2-\frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

Since Ξ maps the line with real part $\frac{1}{2}$ to the real axis, it is clear that the Riemann Hypothesis is equivalent to Ξ having all real roots. We have discussed this property, and we have found that Jensen polynomials over Taylor coefficients of a function are hyperbolic if and only if the original function is hyperbolic. As such, we define Taylor coefficients $\gamma(n)$ with $z = -x^2$

$$\Xi_1(x) = \frac{1}{8}\cdot\Xi\left(\frac{i}{2}\sqrt{x}\right) =: \sum_{n\geq 0}\frac{\gamma(n)}{n!}\cdot x^n.$$

Then, a result of Pólya [18] implies that the Riemann Hypothesis is equivalent to the hyperbolicity of all of the $J_\gamma^{d,n}(X)$.

Next, we will define a way to formalize the growth of a function and determine the rates of growth of ζ and Ξ.

Definition 3.5. The **order** of a function f, denoted by $\rho(f)$ or sometimes just ρ, is given by

$$\rho(f) = \limsup_{r\to\infty}\frac{\log\log M(f;r)}{\log r},$$

where $M(f;r) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus function

It is a classical result that Ξ is an order 1 function. In this thesis we will use a known but less standard approach found in [12] that was chosen for its ability to be generalized. For this proof, we define the relevant notion of the genus of an entire function and we use the following results based on information found in [1].

Definition 3.6. The **genus** of an entire function f is the smallest integer h such that f can be represented as

$$f(z) = e^{g(z)}\prod_k\left(1-\frac{z}{z_k}\right)\exp\left\{z/z_k + \frac{1}{2}(z/z_k)^2 + \ldots + \frac{1}{h}(z/z_k)^h\right\}$$

where $g(z)$ is a polynomial with degree $\deg g \leq h$. If no such representation exists, the genus in infinite.

Remark. We are implicitly requiring in the definition that if the genus is finite then the reciprocals of the zeros of the function converge.

Lemma 3.7. If an entire function f has $\rho(f) < 1$ then f has genus 0.

Proof. In [1], the authors prove Hadamard's Theorem, namely, that the order and genus of an entire function f satisfy $h \leq \rho \leq h+1$. Since $\rho(f) < 1$, we get that $0 \leq \rho(f) < 1$ so we conclude $h = 0$. ■

We now require an estimate on the number of zeros of ζ with bounded height. The following is a recent example of a strong estimate on this number. We do not need the full strength of this result for our purposes, we only need the asymptotic behavior of the number of zeros as the height increases.

Lemma 3.8 (Corollary 1 of [22])**.** Let $N(T)$ be the number of zeros of ζ with imaginary parts in the range $(0, T)$. For $T \geq T_0 \geq e$ we have the following bound

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq 0.111 \log T + 0.275 \log \log T + 2.450 + \frac{0.2}{T_0}.$$

Theorem 3.9. *The function Ξ has order 1.*

Remark. It is clear from the definition of $\Xi(z)$ that it has the same order as $\zeta(s)$ since the Gamma function and z^2 terms have order at most 1. Using similar reasoning, we have that $(s-1)\zeta(s)$ has the same order as $\zeta(s)$. Thus, we will instead show that $\rho((s-1)\zeta(s)) = 1$ which is equivalent to $\rho(\Xi) = 1$.

Proof. We first consider the Laurent series for $\zeta(s)$ at $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n,$$

where γ_n are the Laurent-Stieltjes constants near $s = 1$. We can now multiply by $s-1$ which yields

$$(s-1)\zeta(s) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^{n+1}.$$

If an entire function can be represented by a power series, $f(z) = \sum a_n z^n$, then Theorem 2.2.2 of [3] expresses the order of f in terms of the coefficients

$$\rho(f) = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}.$$

Applying this theorem to the Laurent series for $(s-1)\zeta(s)$ gives

$$\rho = \rho((s-1)\zeta) = \limsup_{n\to\infty} \frac{n \log n}{\log\left(\left| \frac{n!}{(-1)^n \gamma_n} \right| \right)}$$

$$= \limsup_{n\to\infty} \frac{n \log n}{-\log\left(\frac{|\gamma_n|}{n!} \right)}.$$

From here, we can use following approximations for $|\gamma_n|$ due to Matsuoka [16]

$$|\gamma_n| \leq \frac{\exp(n \log \log n)}{10000}$$

which gives the inequality

$$\rho \leq \limsup_{n\to\infty} \frac{n \log n}{-\log\left(\frac{\exp(n \log \log n)}{10000 n!} \right)}$$

or

$$\rho \leq \limsup_{n\to\infty} \frac{n \log n}{\log(10000 n!) - n \log \log n}.$$

Finally, we can use Stirling's approximation, $\log(n!) \approx n \log n - n$, to find that

$$\rho \leq \limsup_{n\to\infty} \frac{n \log n}{\log(10000) + (n \log n - n) - n \log \log n}.$$

Here we can see in the limit that the denominator is dominated by the $n \log n$ term. Thus, we evaluate the limit superior as $\rho \leq 1$.

Now we must show $\rho(\Xi) \geq 1$. Assume by contradiction that $\rho(\Xi) < 1$. Then, by Lemma 3.7, this would imply Ξ has genus 0. To be genus 0, Ξ must have a decomposition of the form

$$\Xi(z) = e^{g(z)} \prod_k \left(1 - \frac{z}{z_k}\right)$$

which implies that $\sum_{k\geq 1} \frac{1}{|z_k|}$ must converge. Now we consider the bound on the number of zeros of ζ given in Lemma 3.8. This lemma says that the number of zeros of height T is approximately $N(T) \sim T \cdot \log(T)$. The number of zeros, z, with $\operatorname{Im} z \in [n-1, n]$ is given by $N(n) - N(n-1)$. We can notice that $N(n) - N(n-1)$ is approximately the derivative of $T \cdot \log(T)$ at n which, when evaluated, is $\log(n) + 1 \sim \log(n)$. Thus, the number of zeros in this box $[0,1] \times [n-1, n]$ is about $\log(n)$. We can now apply this back to our decomposition. The absolute value of each of these roots' reciprocal is at least $1/n$, so the sum of these reciprocals grows like $\log(n)/n$ as n grows. However, $\sum \log(n)/n$ diverges as $n \to \infty$, which contradicts that $\sum_{k\geq 1} \frac{1}{|z_k|}$ must converge. Thus, we conclude that $\rho(\zeta) \geq 1$.

Combining $\rho(\zeta) \leq 1$ with $\rho(\zeta) \geq 1$, it must be the case that $\rho(\zeta) = \rho(\Xi) = 1$, as desired. ■

4. The Laguerre-Pólya Class and Its Generalizations

4.1. **The Laguerre-Pólya Class.**

Definition 4.1. A real entire function $\psi(z) := \sum_{k\geq 0} \frac{\gamma_k}{k!} z^k$ belongs to the **Laguerre-Pólya class**, denoted $\mathcal{L} - \mathcal{P}$, if it can be represented in the form

$$\psi(z) = C z^m e^{bx - ax^2} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k}\right) e^{-\frac{z}{z_k}},$$

where $b, C, z_k \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $a \geq 0$ and $\sum_{k\geq 1} x_k^{-2} < \infty$.
If for some $\psi(z) \in \mathcal{L} - \mathcal{P}$, either $\psi(z)$ or $\psi(-z)$ can be represented as

$$\psi(z) = C z^m e^{\sigma x} \prod_{k=1}^{\infty} \left(1 + \frac{z}{z_k}\right),$$

with $C \in \mathbb{R}$, $m \in \mathbb{Z}_{\geq 0}$, $\sigma \geq 0$, $z_k > 0$, and $\sum_{k\geq 1} z_k^{-1} < \infty$ then we say ψ is **type I** and we denote $\psi \in \mathcal{L} - \mathcal{P}I$.
Finally, if $\gamma_k \geq 0$ for all $k \geq 0$ for some $\psi \in \mathcal{L} - \mathcal{P}$ we write $\psi \in \mathcal{L} - \mathcal{P}^+$.

While the definition of this class may seem based on a somewhat arbitrary representation condition, the functions in this class have useful properties and definitions. One of the most important results concerning the Laguerre-Pólya class and a historical motivation for its study is that the Riemann Hypothesis is equivalent to Ξ being in the Laguerre-Pólya class, as shown by Pólya [18].

To understand why this is the case, we first make the following definition so we may state a primary theorem of Pólya. For a sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ we define the linear operator $\Gamma_\gamma \in L(\mathbb{R}[[x]])$ by $\Gamma_\gamma(x^k) = \gamma_k x^k$.

Definition 4.2. A sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ is a **multiplier sequence of type I** if $\Gamma_\gamma(p(x))$ has only real zeros whenever the real polynomial $p(x)$ has only real zeros.
A sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ is a **multiplier sequence of type II** if $\Gamma_\gamma(p(x))$ has only real zeros whenever $p(x)$ has only real zeros with the same sign.

We now have the following result, relating the concepts of functions in the Laguerre-Pólya class and multiplier sequences with Jensen polynomials and power series.

Theorem 4.3 ([19])**.** *If $\{\gamma_k\}_{k\geq 0}$ is a sequence of nonnegative real numbers, then the following are equivalent:*

(1) *$\{\gamma_k\}_{k\geq 0}$ is a multiplier sequence.*
(2) *For each d, the polynomial $J_\gamma^{d,0}(z)$ has all real non-positive roots. Equivalently, $J_\gamma^{d,0}(z) \in \mathcal{L}-\mathcal{P}I$.*
(3) *The formal power series $\phi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z_k$ defines a function in the Laguerre-Pólya class of type I or, equivalently, $\Gamma_\gamma(e^z) \in \mathcal{L}-\mathcal{P}I$.*

From this result, we can see the relationship with the Riemann Hypothesis. We discussed earlier the result of Pólya that the Riemann Hypothesis is equivalent to Jensen polynomials over Taylor coefficients of Ξ being hyperbolic, which when taken with (3), is equivalent to saying that $\Xi \in \mathcal{L}-\mathcal{P}$.

4.2. **The Shifted Laguerre-Pólya Class.** Recent results of Griffin, Ono, Rolen, and Zagier [10] showed there is a class of sequences $\{\gamma_k\}_{k\geq 0}$ for which the Jensen polynomials $J_\gamma^{d,n}$ are all hyperbolic for fixed d but only for large enough n. We will examine these results closely later in this thesis, but we mention them now as these sequences seemingly "satisfy" the second condition of Theorem 4.3 after a certain large enough n so it is natural to consider how these sequences could fit into the Laguerre-Pólya framework. Using this as motivation, Wagner defined a generalization of the Laguerre-Pólya class in [25], beginning with the following definitions.

Definition 4.4. A real entire function $\phi(x)$ belongs to the **shifted Laguerre-Pólya class of degree** d, denoted $\mathcal{SL}-\mathcal{P}(d)$, if it is the uniform limit of polynomials $\{\phi_k\}_{k\geq 0}$ with the property that there exists an $N(d)$ such that $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots for any $n \geq N(d)$. We say $\phi \in \mathcal{SL}-\mathcal{P}(d)$ is of **type I** and write $\phi \in \mathcal{SL}-\mathcal{P}I(d)$ if all of the roots of $\phi_n^{(deg(\phi_n)-d)}(x)$ have the same sign.
A real entire function $\phi(x) = \sum_{k\geq 0} \frac{\gamma_k}{k!} z^k$ belongs to the **shifted Laguerre-Pólya class**, denoted by $\mathcal{SL}-\mathcal{P}$, if $\phi \in \mathcal{SL}-\mathcal{P}(d)$ for every $d \in \mathbb{N}$. In particular, $\phi \in \mathcal{SL}-\mathcal{P}$ if it is the uniform limit on compact subsets of $\mathbb{C}$ of a sequence of real polynomials $\{\phi_k(x)\}_{k\geq 0}$ such that for each $d \in \mathbb{N}$, there exists an $N(d)$ such that $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots for any $n \geq N(d)$. If, for an entire real function $\phi \in \mathcal{SL}-\mathcal{P}$ the sequence of polynomials has the property that $\phi_n^{(deg(\phi_n)-d)}(x)$ has all real roots of the same sign for any $n \geq N(d)$ then we say it is of **type I** and write $\phi \in \mathcal{SL}-\mathcal{P}I$. Finally, if $\gamma_k \geq 0$ for large enough k for a function $\phi \in \mathcal{SL}-\mathcal{P}I$, then we write $\phi \in \mathcal{SL}-\mathcal{P}^+$.

From this definition, we can immediately see some important properties of the shifted Laguerre-Pólya class. First, we have the inclusion $\mathcal{SL}-\mathcal{P}(d) \subset \mathcal{SL}-\mathcal{P}(d-1)$ for all $d \in \mathbb{N}$. Next, relating the shifted case back to the regular, we note that if $\phi \in \mathcal{L}-\mathcal{P}$ then $\phi \in \mathcal{SL}-\mathcal{P}(d)$ for all $d \leq \deg\phi$ or d is any nonnegative integer if ϕ is transcendental. Thus, in this case we can take $N(d) = 0$ and consider the Laguerre-Pólya class as the shift 0 case of the shifted Laguerre-Pólya class. Additionally, Wagner [25] showed that functions in the shifted Laguerre-Pólya class have order at most 2 and that functions in shifted Laguerre-Pólya class of type I have order at most 1.

To further the generalizations of the Laguerre-Pólya class, we now want to define shifted multiplier sequences with the goal of finding an analogue to Theorem 4.3.

Definition 4.5. For a nonnegative integer d, a sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ is called an **order d multiplier sequence of type I** if, for each $n \in \mathbb{N}$, $\Gamma_{\{\gamma_{k+n}\}}(p(x))$ has only real zeros whenever $p(x)$ has only real zeros and $deg(p(x)) \geq d$. A real sequence $\{\gamma_k\}_{k\geq 0}$ is called an **order d multiplier sequence of type II** if, for each $n \in \mathbb{N}$, $\Gamma_{\{\gamma_{k+n}\}}(p(x))$ has only real zeros whenever $p(x)$ has only real zeros of the same sign and $deg(p(x)) \geq d$.
A sequence of real numbers $\{\gamma_k\}_{k\geq 0}$ is called a **shifted multiplier sequence of type I** (**type II** respectively) if for each $d \in N$, there exists an $N(d)$ such that $\{\gamma_{k+n}\}_{k\geq 0}$ is an order d multiplier sequence of type I (type II respectively) for all $n \geq N(d)$.

In [25], Wagner proved the following theorem, which serves to generalize Theorem 4.3, and formalize the behavior of sequences which define functions in the shifted Laguerre-Pólya class.

Theorem 4.6 (Theorem 1.1 of [25])**.** *If $\{\gamma_k\}_{k\geq 0}$ is a sequence of nonnegative real numbers, then the following are equivalent:*

(1) *The sequence $\{\gamma_k\}_{k\geq 0}$ is a shifted multiplier sequence of type I.*
(2) *For each $d \in \mathbb{N}$, there exists an $N_2(d)$ such that $J_\gamma^{d,n}(x)$ has all real non-positive roots for all $n \geq N_2(d)$ or, equivalently, $J_\gamma^{d+n,0}(x) \in \mathcal{SL}-\mathcal{P}I(d)$ for all $n \geq N_2(d)$.*
(3) *The formal power series $\phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ defines a function in the shifted Laguerre Pólya class of type I, namely, $\Gamma_\gamma(e^z) \in \mathcal{SL}-\mathcal{P}I$.*

Example 4.7. One of the primary examples of a sequence which defines a function in the shifted Laguerre-Pólya class is the partition function $p(n)$. We discussed the existence of shifts $N(2) = 25$ and $N(3) = 94$ which make $J_p^{2,n}(z)$ and $J_p^{3,n}(z)$ hyperbolic respectively. We will discuss later in this thesis a proof that $N(d)$ exists for all d in the partition case. This implies that all $J_p^{d,n}(z)$ are eventually hyperbolic so by Theorem 4.6, we have both that $\{p(i)\}_{i\geq 0}$ is a shifted multiplier sequence of type I and $\phi_p(x) = \sum_{k=0}^{\infty} \frac{p(k)}{k!} z^k \in \mathcal{SL}-\mathcal{P}I$.

5. Generalizations of the Zeta Function

5.1. **Dirichlet L-Functions.** After working with $\zeta(s)$, it is a natural extension to consider replacing the 1 in the sum of $\frac{1}{n^s}$ with sequences or functions. Indeed, we can consider $\zeta(s)$ as the case when the sequence is identically 1, but it is unclear whether taking any other sequences will yield interesting functions. We will see that there are such sequences, but to maintain the nice properties of $\zeta(s)$, these generalizations must be picked carefully. The content of this section is based on information found in [5].

Definition 5.1. A **Dirichlet character modulo** k is a function $\chi\colon \mathbb{N} \to \mathbb{C}$ satisfying

(i) $\chi(1) = 1$;
(ii) $\chi(n_1) = \chi(n_2)$ if $n_1 \equiv n_2 \pmod{k}$;
(iii) $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$;
(iv) $\chi(n) = 0$ if and only if $(n, k) > 1$.

Example 5.2. For any $n \in \mathbb{N}$,

$$\chi_0(n) = \begin{cases} 0 & \text{if } (n,k) > 1 \\ 1 & \text{if } (n,k) = 1, \end{cases}$$

is a Dirichlet character modulo k and is called the **principal character**.

Example 5.3. There are four Dirichlet characters modulo 5, namely

$n \pmod 5$	1	2	3	4	0
$\chi_0(n)$	1	1	1	1	0
$\chi_1(n)$	1	i	$-i$	-1	0
$\chi_2(n)$	1	-1	-1	1	0
$\chi_3(n)$	1	$-i$	i	-1	0

We will define a few common terms which classify Dirichlet characters and are used in the statements of theorems later in this thesis. In particular, we note that if χ is a character modulo a prime power then χ is also a character modulo any higher power and the values will be periodic.

Definition 5.4. For a Dirichlet character χ, the smallest prime power for which χ is periodic is called the **conductor** of χ. The character χ is **primitive** if the modulus and conductor are equal.

Now we can begin to define a Dirichlet L-function.

Definition 5.5. Let χ be any character modulo k. The **Dirichlet series for** χ is

$$L(s,\chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s},$$

for any real $s > 1$.

Remark. Dirichlet L-series have an Euler product representation analogous to that of $\zeta(s)$, which follows from unique factorization and the multiplicative property of χ:

$$L(s,\chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \ldots\right) = \prod_p \frac{1}{1 - (\chi(p)/p^s)}.$$

Example 5.6. The Riemann zeta function is directly related to Dirichlet L-series. In particular, when χ is the principal character modulo k then

$$L(s,\chi_0) = \sum_{n \geq 1} \frac{\chi_0(n)}{n^s} = \sum_{\substack{n \geq 1 \\ (n,k)=1}} \frac{1}{n^s} = \prod_{p \nmid k} \frac{1}{1 - (1/p^s)} \tag{5.1}$$

$$= \prod_p \frac{1}{1 - (1/p^s)} \cdot \prod_{p \mid k} \left(1 - \frac{1}{p^s}\right) = \zeta(s) \prod_{p \mid k} \left(1 - \frac{1}{p^s}\right).$$

We can complete Dirichlet L-series in a similar way to ζ, extending their definition to all of $\mathbb{C}$. Let χ be a primitive character modulo q. Then, for the Dirichlet L-series $L(s,\chi)$, we define its completed form as

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

where $a = 0$ if $\chi(-1) = 1$ or $a = 1$ if $\chi(-1) = -1$. Some authors use the term Dirichlet L-function to refer specifically to this completed form of the Dirichlet L-series whereas other authors use the terms interchangeably. In this thesis, we will primarily use "function" over "series" unless we are specifically talking about the non-completed version.

5.2. **L-Functions.** We have considered Dirichlet L-functions which, by the specific choice of using Dirichlet characters, have many of the same properties as ζ. However, we may want to try to find further generalizations which have nice properties but may not preserve as many as in the Dirichlet L-function case. As such, we define the following.

Definition 5.7. A **Dirichlet series** is a series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $s \in \mathbb{C}$ and $\{a_n\}_{n \geq 1}$ is a sequence of complex numbers.

Using Dirichlet series, we can now define L-series and L-functions.

Definition 5.8. If a Dirichlet series $L(s)$ admits an meromorphic continuation, it is called an **L-series**, and this continuation is called an **L-function**.

Many of the properties we would want to understand for L-functions, such as the locations of zeros and poles, functional equations, etc., are either conjectural and completely unknown in general. For this thesis, we will consider a certain class of L-functions, which has the properties we need for our applications. Based on [24], we define the following aptly named condition.

Definition 5.9. A Dirichlet series $L(s)$ is **good** if the following hold

(1) The series $L(s)$ has a completed form,

$$\Lambda(s) = N^{\frac{s}{2}} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(s) \prod_{m=1}^{M} \Gamma_{\mathbb{C}}(s) \cdot L(s),$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, and $\Gamma(s)$ is the gamma function. Additionally, $\Lambda(s)$ has an integral representation given by

$$\Lambda(s) = N^{\frac{s}{2}} \int_0^{\infty} [f(t) - f(\infty)] t^s \frac{dt}{t}.$$

where f is of the form $f(t) = \alpha(0) + \sum_{n \geq n_0} \alpha(n) e^{-\pi n t}$ with $f(\infty) = \alpha(0)$.

(2) The function $f(t)$ satisfies

$$f\left(\frac{1}{Nt}\right) = \epsilon N^{\frac{k}{2}} t^k f(t),$$

where $\epsilon \in \{\pm 1\}$. This gives rise to an analytic continuation and a functional equation $\Lambda(s) = \epsilon \Lambda(k - s)$ for some $k \in \mathbb{Q}$.

(3) The coefficients of $\Lambda(s)$ are real.

(4) The completed form satisfies $\rho(\Lambda(s)) < 2$.

It is important to note that our definition of a good L-function differs slightly than the definition used by Wagner in [24]. We added an additional condition that $\rho(\Lambda(s)) < 2$, which is necessary for the purposes of this thesis. While this does slightly restrict the class from what Wagner considered, our class of good L-functions is still large and worth consideration, as many of the most famous and well-studied L-functions like $\zeta(s)$ or Dirichlet L-functions have order strictly less than 2. The definition of a good L-function used in this thesis is somewhat dense, but in essence it states that if an Dirichlet series has all of the properties we want–completed form, integral representation, functional equation, real coefficients, and order strictly less than 2–then it is given the appropriate moniker: good.

6. Motivating Results

6.1. **Proving the Hyperbolicity of Jensen Polynomials.** This section is based on results in [10]. In particular, we focus on an alternative proof of one of their theorems based on material presented in [20].

Definition 6.1. A real sequence $a(n)$ has **appropriate growth** if for each j we have

$$a(n+j) = a(n)E(n)^j e^{-\delta(n)^2(j^2/4+o(1))},$$

as $n \to +\infty$ for some real numbers $E(n) > 0$ and $\delta(n) \to 0$.

Remark. A sequence $a(n)$ with an asymptotic formula has appropriate growth if

$$\log\left(\frac{a(n+j)}{a(n)}\right) = A(n)j - B(n)j^2 + o(\delta(n)^2),$$

where $A(n) > 0$ and $0 < B(n) \to 0$.

We now need to define a class of famous polynomials which have special properties, including hyperbolicity, as we will show that we can renormalize Jensen polynomials over sequences with appropriate growth to approach these polynomials.

Definition 6.2. The **Hermite polynomials** $\{H_d(X) : d \geq 0\}$ are the polynomials defined by

$$H_d(X) = (-1)^d e^{x^2} \frac{d^d}{dx^d} e^{-x^2}$$

Example 6.3. The first few Hermite polynomials are

$$\begin{aligned}
H_0(X) &= 1,\\
H_1(X) &= 2X,\\
H_2(X) &= 4X^2 - 2,\\
H_3(X) &= 8X^3 - 12X,\\
H_4(X) &= 16X^4 - 48X^2 + 12.
\end{aligned}$$

The Hermite polynomials have many interesting properties, but for this proof, we only need the following classical results, the proofs of which we will omit.

Theorem 6.4. *Each $H_d(X)$ is hyperbolic with d distinct real roots.*

Lemma 6.5. We have the following exponential generating function for the Hermite polynomials

$$\sum_{d=0}^{\infty} H_d(X) \cdot \frac{Y^d}{d!} := e^{2XY-Y^2}.$$

Now we can define the renormalization and state the main result of this section.

Definition 6.6. If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{2^d}{\delta(n)^d \cdot a(n)} \cdot J_a^{d,n}\left(\frac{\delta(n)X-1}{E(n)}\right).$$

Theorem 6.7 (Theorem 3 of [10]). *Suppose $a(n)$ has appropriate growth. For each degree $d \geq 1$ we have*

$$\lim_{n\to+\infty} \widehat{J}_a^{d,n}(X) = H_d(X).$$

Thus, for each d, all but (possibly) finitely many $J_a^{d,n}(X)$ are hyperbolic.

The general idea of the proof is to show that for large fixed n

$$\sum_{d=0}^{\infty} \widehat{J}_a^{d,n}(X) \cdot \frac{Y^d}{d!} \approx e^{2XY-Y^2},$$

which is equivalent to the theorem by Lemma 6.5.

Proof of Theorem 6.7. To show this, we first have a generating function for Jensen polynomials with fixed n

$$\mathcal{J}_a(n;X,Y) := \sum_{d\geq 0}\sum_{j=0}^{d} \binom{d}{j} \frac{a(n+j)}{a(n)} \cdot X^j \cdot \frac{Y^d}{d!},$$

which implies that it suffices to show

$$\lim_{n\to+\infty} \mathcal{J}\left(n; \frac{\delta(n)X-1}{E(n)}, \frac{2Y}{\delta(n)}\right) = e^{2XY-Y^2}.$$

For the sake of brevity, from this point on we will omit the variable n of $E(n)$ and $\delta(n)$. For this proof we need a "master generating function" that works for all $\mathcal{J}_{a,n}(X,Y)$. We use the definition for appropriate growth, so we can replace the $\frac{a(n+j)}{a(n)}$ term in the definition of the Jensen polynomial generating function with $E(n)^j \cdot e^{-\delta(n)^2(j^2/4+C(j;n))}$ from the definition of the appropriate growth of $a(n)$. We also know $C(j;n) = o(\delta(n)^2)$ for fixed j as $n \to \infty$. Thus, we let $C(j) := C(j;n)$ and perform the substitutions to obtain

$$\mathcal{J}(n;X,Y) = \sum_{d\geq 0}\sum_{j=0}^{d} \frac{d!}{j!(d-j)!}\left(E^j \cdot e^{-\frac{\delta^2 j^2}{4}+C(j)}\right) \cdot X^j \cdot \frac{Y^d}{d!},$$

$$= \sum_{d\geq 0}\sum_{j=0}^{d} \left(e^{-\frac{\delta^2 j^2}{4}+C(j)}\right) \cdot \frac{(EXY)^j}{j!} \cdot \frac{Y^{d-j}}{(d-j)!}.$$

We can immediately notice that the sum of $\frac{Y^{d-j}}{(d-j)!}$ as $d \to \infty$ is just e^Y and that this factor is the only part of the sum that depends on d. We can factor this out with the appropriate changing of sums to find that

$$\mathcal{J}(n; X, Y) = e^Y \sum_{j \geq 0} \left(e^{-\frac{\delta^2 j^2}{4} + C(j)} \right) \cdot \frac{(EXY)^j}{j!}.$$

Now, we evaluate this "master generating function" with the inputs above, giving

$$\begin{aligned}
\mathcal{J}\left(n; \frac{\delta X - 1}{E}, \frac{2Y}{\delta}\right) &= e^{2Y\delta^{-1}} \sum_{j \geq 0} \left(e^{-\frac{\delta^2 j^2}{4} + C(j)} \right) \cdot \frac{[E\big(E^{-1}(\delta X - 1)\big)\big(2\delta^{-1} Y\big)]^j}{j!}, \\
&= e^{2Y\delta^{-1}} \sum_{j \geq 0} \left(e^{-\frac{\delta^2 j^2}{4} + C(j)} \right) \cdot \frac{[(\delta X - 1)\big(2\delta^{-1} Y\big)]^j}{j!}, \\
&= e^{2Y\delta^{-1}} \sum_{j \geq 0} \left(e^{-\frac{\delta^2 j^2}{4} + C(j)} \right) \cdot \frac{\big(2XY - 2\delta^{-1} Y\big)^j}{j!}.
\end{aligned}$$

We then apply the binomial theorem, which gives

$$\mathcal{J}\left(n; \frac{\delta X - 1}{E}, \frac{2Y}{\delta}\right) = e^{2Y\delta^{-1}} \sum_{j \geq 0} \left(e^{-\frac{\delta^2 (j)^2}{4} + C(j)} \right) \cdot \frac{\sum_{\ell=0}^{j} \binom{j}{\ell} (2XY)^{j-\ell} (-2\delta^{-1} Y)^\ell}{j!},$$

and if we let $h = j - \ell$ then this becomes

$$\begin{aligned}
&= e^{2Y\delta^{-1}} \sum_{\ell + h \geq 0} \sum_{\ell = 0}^{\ell + h} \left(e^{-\frac{\delta^2 (\ell + h)^2}{4} + C(\ell + h)} \right) \cdot \frac{(\ell + h)!}{\ell!((\ell + h) - \ell)!} \cdot \frac{(2XY)^{\ell + h - \ell} (-2\delta^{-1} Y)^\ell}{(\ell + h)!}, \\
&= e^{2Y\delta^{-1}} \sum_{\ell, h \geq 0} \left(e^{-\frac{\delta^2 (\ell + h)^2}{4} + C(\ell + h)} \right) \cdot \frac{(2XY)^h}{h!} \cdot \frac{(-2\delta^{-1} Y)^\ell}{\ell!}.
\end{aligned}$$

We first consider the exponential factor within the sum and we see the following equality in the limiting case

$$\begin{aligned}
e^{-\frac{\delta^2 (\ell + h)^2}{4} + C(\ell + h)} &= \sum_{m \geq 0} \left(\frac{\delta^2}{4} \right)^m \frac{(\ell + h + o(1))^{2m}}{m!}, \\
&= \sum_{m \geq 0} \sum_{0 \leq a \leq 2m} \left(\frac{\delta^2}{4} \right)^m \binom{2m}{a} \frac{\ell^a (h + o(1))^{2m - a}}{m!}.
\end{aligned}$$

We have the following formula for ℓ^a

$$\ell^a = \sum_{b=0}^{a} \sum_{c=0}^{b} \binom{\ell}{b} \binom{b}{c} c^a (-1)^{b-c},$$

which we can substitute to obtain

$$\begin{aligned}
e^{-\frac{\delta^2 (\ell + h)^2}{4} + C(\ell + h)} &= \sum_{\substack{m, a, b, c \geq 0 \\ 2m \geq a \geq b \geq c}} \left(\frac{\delta^2}{4} \right)^m \binom{2m}{a} \frac{(h + o(1))^{2m - a}}{m!} \binom{\ell}{b} \binom{b}{c} c^a (-1)^{b-c}, \\
&= \sum_{\substack{m, a, b, c \geq 0 \\ 2m \geq a \geq b \geq c}} \left(\frac{\delta^2}{4} \right)^m \binom{2m}{a} \frac{(h + o(1))^{2m - a}}{m!} \frac{\ell!}{b!(\ell - b)!} \cdot \frac{b!}{c!(b - c)!} c^a (-1)^{b-c},
\end{aligned}$$

$$= \sum_{\substack{m,a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!} \cdot \frac{\ell! c^a (-1)^{b-c}}{c!(\ell-b)!(b-c)!}.$$

Now we can substitute this back in which yields

$$\begin{aligned}
&\mathcal{J}\left(n; \frac{\delta X - 1}{E}, \frac{2Y}{\delta}\right)\\
&= e^{2Y\delta^{-1}} \sum_{\substack{h,m\geq 0\\ 0\leq \ell\leq N\\ a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!} \cdot \frac{\ell! c^a (-1)^{b-c}}{c!(\ell-b)!(b-c)!} \cdot \frac{(2XY)^h}{h!} \cdot \frac{(-2\delta^{-1}Y)^\ell}{\ell!},\\
&\quad = e^{2Y\delta^{-1}} \sum_{\substack{h,m\geq 0\\ 0\leq \ell\leq N\\ a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!c!(b-c)!} \cdot c^a (-1)^{b-c} \cdot \frac{(2XY)^h}{h!} \cdot \frac{(-2\delta^{-1}Y)^\ell}{(\ell-b)!}.
\end{aligned}$$

We immediately notice the $\frac{(-2\delta^{-1}Y)^\ell}{(\ell-b)!}$ factor inside the sum and the $e^{2Y\delta^{-1}}$ outside the sum. These can cancel, leaving $(-2\delta^{-1}Y)^b$ inside the sum with an error on the order of Y^N. Thus, we have

$$\begin{aligned}
&\mathcal{J}\left(n; \frac{\delta X - 1}{E}, \frac{2Y}{\delta}\right)\\
&\qquad = \sum_{\substack{h,m\geq 0\\ a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!c!(b-c)!} c^a (-1)^{b-c} \frac{(2XY)^h}{h!} (-2\delta^{-1}Y)^b + O(Y^N).
\end{aligned}$$

We note that as $n \to \infty$, most of the terms vanish. In order to aid in the cancellation, we consider the approximate equality using $a = b = 2m$

$$\begin{aligned}
\mathcal{J}\left(n; \frac{\delta X - 1}{E}, \frac{2Y}{\delta}\right) &\approx \sum_{\substack{h,m\geq 0\\ 0\leq c\leq 2m}} \left(\frac{\delta^2}{4}\right)^m \frac{c^{2m}(-1)^{2m-c}}{m!c!(2m-c)!} \cdot \frac{(2XY)^h}{h!} (-2\delta^{-1}Y)^{2m} + O(Y^N),\\
&= \sum_{\substack{h,m\geq 0\\ 0\leq c\leq 2m}} \left(\frac{\delta^2}{4}\right)^m \frac{c^{2m}(-1)^{2m-c}}{m!c!(2m-c)!} \cdot \frac{(2XY)^h}{h!} \left(\frac{4}{\delta^2}\right)^m \cdot Y^{2m} + O(Y^N),\\
&= \sum_{\substack{h,m\geq 0\\ 0\leq c\leq 2m}} \frac{c^{2m}(-1)^{2m-c}}{m!c!(2m-c)!} \cdot \frac{(2XY)^h}{h!} \cdot Y^{2m} + O(Y^N),\\
&= e^{2XY} \sum_{\substack{m\geq 0\\ 0\leq c\leq 2m}} \frac{c^{2m}(-1)^{2m-c}}{m!c!(2m-c)!} \cdot Y^{2m} + O(Y^N).
\end{aligned}$$

Finally, we use one more formula from [10], namely

$$\sum_{0\leq c\leq 2m} \frac{c^{2m}(-1)^c}{(2m)!} \binom{2m}{c} = 1,$$

to find that

$$\lim_{n\to+\infty}\mathcal{J}\left(n;\frac{\delta X-1}{E},\frac{2Y}{\delta}\right)=e^{2XY}\sum_{\substack{m\geq 0\\ 0\leq c\leq 2m}}\frac{c^{2m}(-1)^{2m-c}}{(2m)!}\cdot\frac{(2m)!}{c!(2m-c)!}\cdot\frac{Y^{2m}}{m!}+O(Y^N),$$

$$=e^{2XY}\sum_{\substack{m\geq 0\\ 0\leq c\leq 2m}}\frac{c^{2m}(-1)^{c}}{(2m)!}\binom{2m}{c}\cdot\frac{(-1)^{2m}Y^{2m}}{m!}+O(Y^N),$$

$$=e^{2XY}\sum_{m\geq 0}1\cdot\frac{(-Y^2)^m}{m!}+O(Y^N),$$

$$=e^{2XY}\cdot e^{-Y^2}+O(Y^N),$$

$$=e^{2XY-Y^2}+O(Y^N).$$

Thus, letting $N\to\infty$, we have that

$$\lim_{n\to+\infty}\mathcal{J}(n;E^{-1}(\delta X-1),2Y\delta^{-1})=e^{2XY-Y^2},$$

which is is exactly the generating function for the Hermite polynomials. Since the Hermite polynomials are hyperbolic, we conclude that all but possibly finitely many Jensen polynomials over a sequence with appropriate growth are hyperbolic for each d. ■

Example 6.8. The partition function has appropriate growth which can be seen as an immediate result from its asymptotics. Thus, for any d, all but finitely many $J_p^{d,n}(z)$ are hyperbolic, proving the conjecture of Chen, Jia, and Wang.

6.2. **Bounding The Hyperbolicity.** Based on the results of [10] proving that these Jensen polynomials are hyperbolic for each degree d with at most finitely many exceptions, Kim and Lee [15] would find a bound for $N(f;d)$, the minimal integer for each d such that if $n\geq N(f;d)$ then all $J_f^{d,n}(z)$ are hyperbolic.

Let f be an entire function. We denote the zero set of f by $\mathcal{Z}(f)$ and the order of f by $\rho(f)=\rho$ as above. Additionally, we let $\mathbb{S}=\{z\in\mathbb{C}:|\operatorname{Im}z|\leq\frac{1}{2}\}$ and $S(\delta)=\{z\in\mathbb{C}:|\operatorname{Im}z|\leq\delta|z|\}$.

Theorem 6.9 (Theorem 1 of [15])**.** *Let f be a transcendental real entire function of order $\rho<2$ and $\mathcal{Z}(f)\subset\mathbb{S}$. Then, for every $c>\rho$ we have $N(f;d)=O(d^{c/2})$ as $d\to\infty$.*

Using the fact that Ξ has order 1, Kim and Lee applied this theorem to show that $N(\Xi;d)=O(d^{\frac{1}{2}+\varepsilon})$ as $d\to\infty$ for $\varepsilon>0$. The proof of the Theorem 6.9 used the following results.

Theorem 6.10 (Theorem 2 of [14])**.** *Let f be a nonconstant real entire function with $0<\rho(f)\leq 2$ and of minimal type. If there is a positive real number A such that $\mathcal{Z}(f)\subset\{z\in\mathbb{C}:|\operatorname{Im}z|\leq A\}$, then for any positive constant B there is a positive integer n_1 such that $f^{(n)}(z)$ has only real zeros in $|\operatorname{Re}z|\leq Bn^{\frac{1}{\rho}}$ for all $n\geq n_1$.*

This is the primary result which allows for us to find the bound, however the proof of this theorem is out of the scope of this paper. Kim and Lee [15] used a slight modification of Theorem 6.10 in their paper to apply it to functions with $\rho<2$ but not necessarily satisfying the minimal type condition. This was necessary for their work as Ξ does not have minimal type but it does have order less than 2. We will briefly mention how they are able to do this, as it is left implicit in their original paper.

The **type** of a function f is defined as the greatest lower bound of the set of numbers α for which f satisfies $M(f;r) < e^{\alpha r^\rho}$, where ρ is the order of f. A function has **minimal type** if it is type 0. Kim and Lee are able to remove the minimal type condition of Theorem 6.10 by changing the requirement on the order from $\rho \leq 2$ to $\rho < 2$. This is because the proof of Theorem 6.10 requires that for the function f and all $\varepsilon > 0$, there exists some r_1 such that if $r \geq r_1$ then $M(f;r) < e^{\varepsilon r^\rho}$. It is clear from the definition that if f has minimal type then it satisfies this condition. However, if we consider a function f with $\rho(f) < 2$, then by the definition of order $M(f;r) < e^{r^\rho}$. We have that e^{r^ρ} grows slower than e^{r^2} so no matter how small an ε, there must exist an r_1 such that $r \geq r_1$ makes $M(f;r) < e^{r^\rho} < e^{\varepsilon r^2}$. Thus, we can use Theorem 6.10 for functions without minimal type if they have order strictly less than 2. Kim and Lee then use this fact to state the following theorem.

Theorem 6.11 (Theorem 2 of [15])**.** *Let f be a transcendental real entire function of order $\rho(f) < 2$ and $\mathcal{Z}(f) \subset \mathbb{S}$. Then for every $c > \rho(f)$ there is a positive integer n_1 such that for all $n \geq n_1$*

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S} \,:\, |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

Proof. Since f is real, entire, with $\rho < 2$ and $\mathcal{Z}(f) \subset \mathbb{S} = \{z \in \mathbb{C} \,:\, |\operatorname{Im} z| \leq \frac{1}{2}\}$, then we can apply Theorem 6.10. Using Theorem 6.10 and choosing $B = 1$, we have that there exists a positive integer n_1 such that $\mathcal{Z}(f^{(n)}) \subset \mathbb{R}$ for $|\operatorname{Re} z| \leq n^{1/\rho}$. The zeros of $f^{(n)}$ not in $|\operatorname{Re} z| \leq n^{1/\rho}$ are still contained in $\mathbb{S}$. Thus, for any $c > \rho(f)$ we get the desired inclusion

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S} \,:\, |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

■

Theorem 6.12 (Theorem 3 of [15])**.** *Let P and Q be real polynomials, $\delta > 0$, $Z(P) \subset S(\delta)$, Q is hyperbolic, and $deg(Q) \leq \delta^{-2}$. Then the polynomial*

$$P(D)Q := \sum_{k=0}^{deg(P)} \frac{P^{(k)}(0)}{k!} Q^{(k)},$$

is hyperbolic.

The proof of this theorem is also out of the scope of this paper, so it will be omitted. In [15], Kim and Lee prove a useful corollary of this theorem. We will prove this corollary using their methods but we will provide some of the details which were left implicit in their original paper.

Corollary 6.13 (Corollary from [15])**.** *Let P be a real polynomial with $\mathcal{Z}(P) \subset S(\delta)$ for $\delta > 0$. Then, $J_P^{d,0}(z)$ is hyperbolic for $d \leq \delta^{-2}$.*

Proof. The idea for this proof is to let $Q(z) = z^d$ and $d \leq \delta^{-2}$ and show the identity

$$z^d (P(D)Q)(z^{-1}) = J_P^{d,0}(z).$$

Clearly with this choice, Q is hyperbolic and $deg(Q) = d \leq \delta^{-2}$ so we can apply Theorem 6.12 to P and z^d. This theorem gives that

$$(P(D)Q)(z) = \sum_{k=0}^{deg(P)} \frac{P^{(k)}(0)}{k!} (z^d)^{(k)}$$

$$
\begin{aligned}
&= \sum_{k=0}^{deg(P)} \frac{P^{(k)}(0)}{k!} \cdot d(d-1)...(d-k+1)(z^{d-k}) \\
&= \sum_{k=0}^{deg(P)} P^{(k)}(0) \cdot \frac{d!}{k!(d-k)!} \cdot z^{d-k} \\
&= \sum_{k=0}^{deg(P)} \binom{d}{k} P^{(k)}(0) z^{d-k}.
\end{aligned}
$$

This polynomial is hyperbolic, which implies if we evaluate $z^d(P(D)Q)(z^{-1})$ then this polynomial is also hyperbolic. Thus, we find that

$$
\begin{aligned}
z^d(P(D)Q)(z^{-1}) &= z^d \cdot \sum_{k=0}^{deg(P)} \binom{d}{k} P^{(k)}(0)(z^{-1})^{d-k} \\
&= \sum_{k=0}^{deg(P)} \binom{d}{k} P^{(k)}(0) \cdot z^d \cdot z^{-d+k} \\
&= \sum_{k=0}^{deg(P)} \binom{d}{k} P^{(k)}(0) \cdot z^k.
\end{aligned}
$$

Here we note that if $deg(P) < d$ then $P^{(k)}(0) = 0$ for $k > deg(P)$. Additionally, if $deg(P) > d$ then $\binom{d}{k} = 0$ for $k > deg(P)$. Thus, the sum above can be rewritten ranging from 0 to d, giving

$$
\begin{aligned}
z^d(P(D)Q)(z^{-1}) &= \sum_{k=0}^{d} \binom{d}{k} P^{(k)}(0) \cdot z^k \\
&= J_P^{d,0}(z).
\end{aligned}
$$

Therefore, $J_P^{d,0}(z)$ is hyperbolic when $d \leq \delta^{-2}$. ■

We now have all the necessary results to prove the main theorem of [15].

Proof of Theorem 6.9. Let $c > \rho$. By Theorem 6.11, there exists an $n \in \mathbb{N}$ such that if $n \geq n_1$ then

$$\mathcal{Z}(f^{(n)}) \subset \{z \in \mathbb{S} : |\operatorname{Re} z| \geq n^{1/c}\} \cup \mathbb{R}.$$

Now let $d, n \in \mathbb{N}$ such that

$$n \geq \max\left\{n_1, \left(\frac{d}{4}\right)^{c/2}\right\},$$

and choose $\delta = \frac{1}{2n^{1/c}} > 0$. This choice gives

$$d \leq 4n^{2/c} = (2n^{1/c})^2 = \left(\frac{1}{2n^{1/c}}\right)^{-2} = \delta^{-2}.$$

Let $P_1, P_2, ...$ be real polynomials such that $\mathcal{Z}(P_k) \subset \mathcal{Z}(f^{(n)}) \cup \mathbb{R}$ for all k and $P_k \to f^{(n)}$ uniformly on compact subsets of $\mathbb{C}$. This implies $J_{P_k}^{d,0}(z) \to J_{f^{(n)}}^{d,0}(z) = J_f^{d,n}(z)$. The existence of such polynomials is guaranteed by taking partial product of the Weierstrass factorization of f. We want to apply Corollary 6.13 to each P_k, and we have already shown the $d \leq \delta^{-2}$

condition, so now we need only show that $\mathcal{Z}(f^{(n)}) \cup \mathbb{R} \subset S(\delta)$ since $\mathcal{Z}(P_k) \subset \mathcal{Z}(f^{(n)}) \cup \mathbb{R}$. Clearly $\mathbb{R} \subset S(\delta)$ so consider the $z \in \mathcal{Z}(f^n)$ with $|\operatorname{Re} z| \geq n^{1/c}$. We can combine $|\operatorname{Im} z| \leq \frac{1}{2}$ with $n^{1/c} \leq |\operatorname{Re} z| \leq |z|$ to get

$$|\operatorname{Im} z| \cdot 1 \leq \frac{1}{2} \cdot \frac{|z|}{n^{1/c}} = \frac{1}{2n^{1/c}}|z| = \delta|z|.$$

Thus, the inequality $|\operatorname{Im} z| \leq \delta|z|$ holds for all $z \in \mathcal{Z}(f^{(n)}) \cup \mathbb{R}$ which implies $\mathcal{Z}(P_k) \subset \mathcal{Z}(f^{(n)}) \cup \mathbb{R} \subset S(\delta)$. The polynomials $P_1, P_2, ...$ then satisfy the conditions of Corollary 6.13 so $J^{d,0}_{P_k}(z)$ is hyperbolic for all k. Since $J^{d,0}_{P_k}(z) \to J^{d,0}_{f^{(n)}}(z) = J^{d,n}_f(z)$, we have that $J^{d,n}_f$ is hyperbolic with

$$N(f;d) \leq \left\lceil \max\left\{ n_1, \left(\frac{d}{4}\right)^{c/2} \right\} \right\rceil,$$

and so $N(f;d) = O(d^{c/2})$ as $d \to \infty$. ∎

7. Generalizing the Methods of Kim and Lee

7.1. **The Dirichlet L-Function Case.** We would like to extend the results of Kim and Lee [15] for Ξ to Dirichlet L-functions. This generalization is natural as $\zeta(s)$ has a close relation to $L(s,\chi_0)$, given by equation (5.1). Indeed, Dirichlet L-functions satisfy the same zeros condition where all nontrivial zeros are contained within the strip $0 \leq \operatorname{Re} z \leq 1$. Thus, we can define a $\Xi(\chi, z)$ from $L(s,\chi)$ that maps the line with real part $\frac{1}{2}$ to the real axis in the same manner as we defined $\Xi(z)$ from $\zeta(s)$. We discussed previously that Dirichlet L-functions have a completed form; for a Dirichlet L-function $L(\chi, s)$, let $\Lambda(\chi, s)$ denote its completed form. Following [24], we formally define

$$\Xi(\chi, z) := \begin{cases} \left(-z^2 - \frac{1}{4}\right) \Lambda\left(\frac{1}{2} - iz, \chi\right) & \text{if } \chi \text{ is principal} \\ \Lambda\left(\frac{1}{2} - iz, \chi\right) & \text{otherwise.} \end{cases}$$

We can note that $\Xi(\chi, z)$ is real, entire, with $\mathcal{Z}(\Xi(\chi, z)) \subset \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{1}{2}\} = \mathbb{S}$, so we may apply Theorem 6.9 if we verify that $\rho(\Xi(\chi, z)) < 2$.

Theorem 7.1. *Let $L(s,\chi)$ be a Dirichlet L-function. Then, $\rho(\Xi(\chi, z)) = 1$.*

Our approach for this proof will be analogous to the approach we used to prove that Ξ has order 1. We will introduce the following recent results concerning Dirichlet L-functions–first, an upper and lower bound on the number of roots of the functions, and secondly, a bound on the Laurent-Stieltjes constants for the functions.

Lemma 7.2 (Theorem 1.1 of [2])**.** Suppose that the Dirichlet character χ has conductor $q > 1$, and that $T \geq 5/7$. Then, the number of zeros of a Dirichlet L-function with character χ and height at most T, $N(T,\chi)$, is bounded by

$$\left| N(T,\chi) - \left(\frac{T}{\pi} \log \frac{qT}{2\pi e} - \frac{\chi(-1)}{4} \right) \right| \leq 0.22737\ell + 2\log(1+\ell) - 0.5,$$

where $\ell = \log \frac{q(T+2)}{2\pi} > 1.567$.

Next, for a Dirichlet L-function, $L(s,\chi)$, we have the following bound on the Laurent-Stieltjes constants, $\gamma_n(\chi)$, near $s = 1$.

Lemma 7.3 (Theorem 1 of [21])**.** Let χ be a primitive Dirichlet character modulo q. Then, for every $1 \leq q \leq \frac{\pi}{2} \cdot \frac{e^{(n+1)/2}}{n+1}$, we have

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-\frac{1}{2}} C(n,q) \min\left(1 + D(n,q), \frac{\pi^2}{6}\right), \tag{7.1}$$

where

$$C(n,q) = 2\sqrt{2} \exp\left\{-(n+1)\log\theta(n,q) + \theta(n,q)\left\{\log\theta(n,q) + \log\frac{2q}{\pi e}\right\}\right\}$$

and

$$\theta(n,q) = \frac{n+1}{\log\frac{2q(n+1)}{\pi}} - 1, \qquad\qquad D(n,q) = 2^{-\theta(n,q)-1}\frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

We will use this result as we used the Matsuoka bound in the proof that Ξ has order 1. Additionally, we will omit the proof of this result as it is out of the scope of this thesis.

Remark. It is clear from the definition of $\Xi(\chi, z)$ that it has the same order as $L(s,\chi)$ which has the same order as $\Lambda(s,\chi)$ since the z^2 term has order 0. Using similar reasoning, we have that $(s-1)L(s,\chi)$ has the same order as $L(s,\chi)$. Thus, we will instead show that $\rho((s-1)L(s,\chi)) = 1$ which is equivalent to $\rho(\Xi(\chi,z)) = 1$.

Proof of Theorem 7.1. We first consider the Laurent series for $L(s,\chi)$ at $s = 1$,

$$L(s,\chi) = \frac{\delta_\chi}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^n,$$

and multiplying by $s-1$ yields

$$(s-1)L(s,\chi) = \delta_\chi + \sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n(\chi)}{n!}(s-1)^{n+1},$$

If an entire function can be represented by a power series, $f(z) = \sum_{n\geq 0} a_n z^n$, then Theorem 2.2.2 of [3] expresses the order of f in terms of the coefficients

$$\rho(f) = \limsup_{n\to\infty}\frac{n\log n}{\log(1/|a_n|)}.$$

Applying this theorem to the Laurent series for $L(s,\chi)$ gives

$$\begin{aligned}\rho = \rho(L(s,\chi)) &= \limsup_{n\to\infty}\frac{n\log n}{\log\left(\left|\frac{(-1)^n\gamma_n(\chi)}{n!}\right|^{-1}\right)} \\ &= \limsup_{n\to\infty}\frac{n\log n}{-\log\left(\frac{|\gamma_n(\chi)|}{n!}\right)}.\end{aligned}$$

From here, we can use the bounds from Lemma 7.3 to find that

$$\begin{aligned}\rho &\leq \limsup_{n\to\infty}\frac{n\log n}{-\log\left(q^{-\frac{1}{2}}C(n,q)\min\left\{1+D(n,q),\frac{\pi^2}{6}\right\}\right)} \\ &= \limsup_{n\to\infty}\frac{n\log n}{-\log\left(q^{-\frac{1}{2}}\cdot\min\left\{1+D(n,q),\frac{\pi^2}{6}\right\}\right) - \log(C(n,q))}.\end{aligned}$$

We can first note that the term $\log\left(q^{-\frac{1}{2}}\cdot\min\left\{1+D(n,q),\frac{\pi^2}{6}\right\}\right)$ is bounded above by the constant term $\log\left(q^{-1/2}\cdot\frac{\pi^2}{6}\right)$. Thus, we may consider this term $O(1)$ in the limiting case. We then find

$$\begin{aligned}\log\theta=\log\left(\frac{n+1}{\log\frac{2q(n+1)}{\pi}}-1\right)&\le\log\left(\frac{n+1}{\log\frac{2q(n+1)}{\pi}}\right)\\&=\log(n+1)-\log\log\left(\frac{2q}{\pi}(n+1)\right).\end{aligned}$$

This implies

$$\begin{aligned}\log(C)&=\log\left(2\sqrt{2}\exp\left\{-(n+1)\log\theta+\theta\left(\log\theta+\log\left(\frac{2q}{\pi e}\right)\right)\right\}\right)\\&=\log(2\sqrt{2})+\left(-(n+1)\log\theta+\theta\left(\log\theta+\log\left(\frac{2q}{\pi e}\right)\right)\right)\end{aligned}$$

$$\begin{aligned}&=\log(2\sqrt{2})-(n+1)\log(n+1)-(n+1)\log\log\left(\frac{2q}{\pi}(n+1)\right)\\&\qquad+\left(\frac{n+1}{\log\frac{2q(n+1)}{\pi}}-1\right)\left[\log(n+1)-\log\log\left(\frac{2q}{\pi}(n+1)\right)+\log\left(\frac{2q}{\pi e}\right)\right]\\&\sim\log(2\sqrt{2})-n\log(n)-n\log\log(n)+\left(\frac{n}{\log n}\right)\left[\log(n)-\log\log(n)+\log\left(\frac{2q}{\pi e}\right)\right]\\&=\log(2\sqrt{2})-n\log(n)-n\log\log(n)+\left[n-\frac{n\log\log(n)}{\log(n)}+n\frac{\log\left(\frac{2q}{\pi e}\right)}{\log(n)}\right].\end{aligned}$$

Thus

$$\log(C)=-n\log n+O(n\log\log n).$$

Now, we can apply these in our limit superior

$$\begin{aligned}\rho&\le\limsup_{n\to\infty}\frac{n\log n}{O(1)-(-n\log n+O(n\log\log n))}\\&=\limsup_{n\to\infty}\frac{n\log n}{n\log n+O(n\log\log n)+O(1)}\\&=1.\end{aligned}$$

We have shown $\rho(L(s,\chi))=\rho(\Xi(\chi,z))\le 1$, so now we need only show $\rho(L(s,\chi))\ge 1$.

Assume by contradiction that $\rho(L(s,\chi))<1$. Then, by Lemma 3.7, this would imply $L(s,\chi)$ has genus 0. To be genus 0, $L(s,\chi)$ must have a decomposition of the form

$$L(s,\chi)=e^{g(z)}\prod_k\left(1-\frac{z}{z_k}\right)$$

which implies that $\sum_{k\ge1}\frac{1}{|z_k|}$ must converge. Now we consider the bound on the number of zeros of $L(s,\chi)$ given in Lemma 7.2. This lemma says that the number of zeros of height T is approximately $N(\chi,T)\sim T\cdot\log(T)$. The number of zeros, z, with $\operatorname{Im}z\in[n-1,n]$ is given by $N(n)-N(n-1)$. We can notice that $N(n)-N(n-1)$ is approximately the derivative of $T\cdot\log(T)$ at n which, when evaluated, is $\log(n)+1\sim\log(n)$. Thus, the number of zeros in this box $[0,1]\times[n-1,n]$ is about $\log(n)$. We can now apply this back to our decomposition. The absolute value of each of these roots' reciprocal is at least $1/n$, so the sum of these reciprocals grows like $\log(n)/n$ as n grows. However, $\sum \log(n)/n$ diverges as $n\to\infty$, which contradicts that $\sum_{k\geq 1}\frac{1}{|z_k|}$ must converge. Thus, we conclude that $\rho(L(s,\chi))\geq 1$.

Combining $\rho(L(s,\chi))\leq 1$ with $\rho(L(s,\chi))\geq 1$, it must be the case that $\rho(L(s,\chi)) = \rho(\Xi(\chi,z)) = 1$, as desired. ∎

Theorem 7.4. *Let $L(\chi,s)$ be a Dirichlet L-function and let $\Xi(\chi,z)$ be defined as above. Then, $N(\Xi(\chi,z);d)=O(d^{\frac{1}{2}+\varepsilon})$.*

Proof. We first note that $\Xi(\chi,z)$ is a transcendental real entire function with both $\rho(\Xi(\chi,z)) = 1 < 2$ from Theorem 7.3 and $\mathcal{Z}(\Xi(\chi,z))\subset\mathbb{S}$ by definition. Thus, we can apply the main theorem of Kim and Lee. Choose $c = 1+\varepsilon > \rho(\Xi(\chi,z))$ where $\varepsilon>0$. Then, by Theorem 6.9 we have $N(\Xi(\chi,z);d) = O(d^{\frac{1+\varepsilon}{2}}) = O(d^{\frac{1}{2}+\varepsilon})$. ∎

7.2. **The General L-Function Case.** We would also like to extend the results of Kim and Lee [15] for $\Xi(z)$ to a more general class of sufficiently "nice" completed L-functions. Based on [24] we have the following definition.

For a good Dirichlet $L(s)$ series with completed form $\Lambda(s)$, we define

$$\Xi_L(z) := \begin{cases} (-z^2-\frac{k^2}{4})\Lambda(\frac{k}{2}-iz) & \text{if } \Lambda(s) \text{ has a pole at } s=k \\ \Lambda(\frac{k}{2}-iz) & \text{otherwise.}\end{cases}$$

This construction is analogous to how the Ξ function rotates $\zeta(s)$ so the axis of symmetry for its functional equation is mapped to the real line.

The function Ξ_L is real, entire, and transcendental with $\rho(\Xi_L)<2$ by definition so we can potentially apply the methods of Kim and Lee [15]. However, we must modify them slightly, as the results are specifically stated for functions whose zeros satisfy $|\operatorname{Im} z|\leq\frac{1}{2}$. This choice is based on the functional equation for $\Xi(z)$, as all the nontrivial zeros of $\Xi(z)$ are contained within this strip. We have an analogous property for $\Xi_L(z)$, using its functional equation, which says that the nontrivial zeros of $\Xi_L(z)$ are contained within the strip $|\operatorname{Im} z|\leq\frac{k}{2}$. Denote this strip $\mathbb{S}_k$.

Theorem 7.5. *Let $L(s)$ be a good Dirichlet series. Then $N(\Xi_L;d)=O(d)$ as $d\to\infty$.*

Proof. We recall the strategy of Kim and Lee. We first want to show that Theorem 6.11 applies to $\Xi_L(z)$ as it is only stated for functions satisfying $\mathcal{Z}(f)\subset\mathbb{S}$. In the same manner as Theorem 6.11 is a special case of of Theorem 6.10, we can take a slightly different special case. Theorem 6.10 is stated for functions satisfying $\mathcal{Z}(f)\subset\{z\in\mathbb{C} : |\operatorname{Im} z|\leq A\}$ for some $A\in\mathbb{R}$. Instead of choosing $\frac{1}{2}$, we can let $A=\frac{k}{2}$ and conclude that Theorem 6.10 applies. Then, letting $B=1$ we get an exact analogy for Theorem 6.11. Thus we have that for any $c>\rho(\Xi_L)$, there exists an n_1 such that for $n\geq n_1$

$$Z(\Xi_L^{(n)})\subset\{z\in\mathbb{S}_k : |\operatorname{Re} z|\geq n^{1/c}\}.$$

We can let $n,d\in\mathbb{N}$ such that

$$n\geq\max\left\{n_1, \left(\frac{k^2}{4}\cdot d\right)^{c/2}\right\},$$

and choose $\delta = \frac{k}{2}n^{-1/c}$, noting the factor of k that differentiates this case from the original. From this choice of δ, we have

$$d \leq \left(\frac{4}{k^2}n^{2/c}\right) = \left(\frac{2}{k}n^{1/c}\right)^2 = \left(\frac{k}{2n^{1/c}}\right)^{-2} = \delta^{-2}.$$

Additionally, we see that $|\operatorname{Im} z| \leq \delta|z|$ holds for $z \in \mathcal{Z}(\Xi_L^{(n)})$ from combining $|\operatorname{Im} z| \leq \frac{k}{2}$ with $|z| \geq |\operatorname{Re} z| \geq n^{1/c}$. This implies $\mathcal{Z}(\Xi_L^{(n)}) \cup \mathbb{R} \subset S(\delta)$. Now let $P_1, P_2, ...$ be real polynomials such that $\mathcal{Z}(P_k) \subset \mathcal{Z}(\Xi_L^{(n)}) \cup \mathbb{R}$ for all $k \geq 1$ and P_k uniformly converges to $\Xi_L^{(n)}$ on compact sets in $\mathbb{C}$. Then,

$$J_{P_k}^{d,0}(z) \to J_{\Xi_L^{(n)}}^{d,0}(z)$$

uniformly on compact sets in $\mathbb{C}$. Additionally, we can apply Corollary 6.13 which says that $J_{P_k}^{d,0}(z)$ is hyperbolic for all k. Thus, $J_{\Xi_L^{(n)}}^{d,0}(z) = J_{\Xi_L}^{d,n}(z)$ is hyperbolic and we can conclude that

$$N(\Xi_L; d) \leq \left\lceil \max\left\{n_1, \left(\frac{k^2}{4} \cdot d\right)^{c/2}\right\}\right\rceil.$$

Finally, using the fact that $\rho(\Xi_L) < 2$ for any good L-function, we let $c = 2 > \rho$ and obtain the bound $N(\Xi_L; d) = O(d^{\frac{2}{2}}) = O(d)$ as $d \to \infty$. ■

8. Further Generalizations

8.1. **Generalizing to the Shifted Laguerre-Pólya Class.** This section contains the preliminary results of the ongoing joint work between the author, Larry Rolen, and Ian Wagner. Based on the previous section and considering that $\Xi_L \in \mathcal{SL}-\mathcal{P}$, it is natural to consider whether the methods of obtaining a bound on $N(f; d)$ used by Kim and Lee [15] can be generalized to the entire shifted Laguerre-Pólya class. The author and their collaborators have found that such a generalization, should it exist, is not simple. Indeed, we face the following problems with a direct application.

The first problem which arises is that in general, for $\phi_\gamma \in \mathcal{SL}-\mathcal{P}$, the zeros of ϕ_γ can have arbitrarily large imaginary part. This implies that we cannot use Lemma 6.10 or Theorem 6.11, as they may only be applied to functions whose zeros have finitely bounded imaginary part. However, it is important to note that Kim and Lee use this theorem to show that for a function f with the specified properties, then $\mathcal{Z}(f^{(n)}) \subset S(\delta)$ for large enough n. They only prove this because it directly implies the desired inclusion $\mathcal{Z}(P_k) \subset S(\delta)$ since, by their choice of P_k, $\mathcal{Z}(P_k) \subset \mathcal{Z}(f^{(n)})$. Thus, in applying the methods to the $\mathcal{SL}-\mathcal{P}$, it may be easier to directly prove that $\mathcal{Z}(P_k) \subset S(\delta)$ since it is not the case in general that $\mathcal{Z}(\phi_\gamma) \subset S(\delta)$ for some $\delta > 0$.

However, we recall the following about the general philosophy of functions in the shifted Laguerre-Pólya class: the Taylor coefficients γ_k should act more and more like Taylor coefficients of a function in the Laguerre-Pólya class as k grows. This philosophy gives a heuristic for how this generalization should work. In particular, since the results hold for functions in the Laguerre-Pólya class, there should be a point for any function in the shifted Laguerre-Pólya class where the results start to hold. The author and his collaborators have not yet formalized this notion for generalizing the methods of Kim and Lee, but we believe for any $\phi \in \mathcal{SL}-\mathcal{P}$ there may be some term n_δ which is akin to the n_1 term described by Kim and Lee. This hypothetical n_δ is such that if $n \geq n_\delta$ then zeros of of $\phi^{(n)}$ satisfy a similar condition to Theorem 6.11.

Alternatively, in attempting to directly prove the inclusion $\mathcal{Z}(P_k) \subset S(\delta)$, we have made progress. When choosing these polynomials P_k, Kim and Lee are able to use partial products of the Weierstrass factorization of a function $f \in \mathcal{L}-\mathcal{P}$ as their sequence of polynomials that uniformly converge to f. They are able to make this choice as they have implicitly the inclusions $\mathcal{Z}(P_k) \subset \mathcal{Z}(f) \subset \mathbb{S}$. Despite functions in $\mathcal{SL}-\mathcal{P}$ having a Weierstrass factorization, since no such bound exists for $\mathcal{Z}(\phi_\gamma)$, we must choose our polynomials more carefully. There is evidence to support that the correct choice of polynomials for any $\phi_\gamma \in \mathcal{SL}-\mathcal{P}$ is

$$P_{k,n}(z) := J_\gamma^{k,n}(z/k) = \sum_{j=0}^{k} \binom{k}{j} \gamma_{n+j}(z/k)^j.$$

Wagner proved in [25] that $P_{k,n}(z) \to \phi_\gamma^{(n)}$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{C}$. Thus, the polynomials $P_{k,n}$ satisfy one of the necessary conditions. By above, we would want to show $\mathcal{Z}(P_{k,n}) \subset S(\delta)$, as this would imply that the Jensen polynomials $J^{d,0}_{\phi_\gamma^{(n)}}(z) = J^{d,n}_{\phi_\gamma}(z)$ are hyperbolic. Thus, our first step is to find a bound on the size of the roots of $P_{k,n}(z)$.

Theorem 8.1. *If z_0 is a root of $P_{k,n}(z)$ for some $\phi_\gamma \in \mathcal{SL}-\mathcal{P}$, then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|z_0| \leq k^2 \cdot \frac{\gamma_{n+k-1}}{\gamma_{n+k}}$.*

The motive behind this theorem is to use the following lemma originally due to Kakeya [13], but we will use it as it is stated in [8].

Lemma 8.2. If $p(z) = \sum_{j=0}^{k} a_j z^j$ is a polynomial of degree k with real and positive coefficients, then all the zeros of $p(z)$ lie in the annulus $R_1 \leq |z| \leq R_2$, where $R_1 = \min_{1\leq j\leq k-1} \frac{a_j}{a_{j+1}}$ and $R_2 = \max_{1\leq j\leq k-1} \frac{a_j}{a_{j+1}}$.

To apply this lemma, we can consider the sequence $\{a_j/a_{j+1}\}_{j=0}^{k-1}$ where a_i is the coefficient of the i-th term of $P_{k,n}$. We must also find the maximum of this sequence and to do this, we will show that the series is increasing. We will also make the standard assumption that without loss of generality, the sequence $\{\gamma_k\}$ is eventually non-negative.

Theorem 8.3. *There exists an $N \in \mathbb{N}$ such that for all $n \geq N$ the sequence $\{a_j/a_{j+1}\}_{j=1}^{k-1}$ as defined above is increasing.*

Proof. To show that $\{a_j/a_{j+1}\}_{j=1}^{k-1}$ is increasing, we want to show that $\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j}$ for all $1 \leq j \leq k-1$. Using the general form for coefficients of $P_{k,n}$, this is equivalent to

$$\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j} \iff \frac{\binom{k}{j} \cdot k^{-j} \cdot \gamma_{n+j}}{\binom{k}{j+1} \cdot k^{-(j+1)} \cdot \gamma_{n+j+1}} \geq \frac{\binom{k}{j-1} \cdot k^{-(j-1)} \cdot \gamma_{n+j-1}}{\binom{k}{j} \cdot k^{-j} \cdot \gamma_{n+j}}$$

$$\iff \frac{k!}{j!(k-j)!} \cdot \frac{(j+1)!(k-j-1)!}{k!} \cdot k \cdot \frac{\gamma_{n+j}}{\gamma_{n+j+1}} \geq \frac{k!}{(j-1)!(k-j+1)!} \cdot \frac{j!(k-j)!}{k!} \cdot k \cdot \frac{\gamma_{n+j-1}}{\gamma_{n+j}}$$

$$\iff \frac{j+1}{k-j} \cdot \frac{\gamma_{n+j}}{\gamma_{n+j+1}} \geq \frac{j}{k-j+1} \cdot \frac{\gamma_{n+j-1}}{\gamma_{n+j}}.$$

We can then rearrange terms to find that

$$\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j} \iff (\gamma_{n+j})^2 \geq \frac{j}{j+1} \cdot \frac{k-j}{k-j+1} \cdot \gamma_{n+j-1} \cdot \gamma_{n+j+1}. \tag{8.1}$$

The terms $\frac{j}{j+1}, \frac{k-j}{k-j+1}$ are always less than 1, so we find that if $\gamma_{n+j}^2 \geq \gamma_{n+j-1}\gamma_{n+j+1}$, or equivalently if the sequence $\{\gamma_n\}$ is log-concave, then $\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j}$. As discussed in an earlier section, a sequence being log-concave is an equivalent property to its degree 2 Jensen polynomials being hyperbolic. We are only considering sequences $\{\gamma_k\}$ which define functions $\phi_\gamma \in \mathcal{SL}-\mathcal{P}$; thus, by Theorem 1.1 of [25], for each $d \in \mathbb{N}$ there exists an $N(\phi_\gamma; d)$ such that $J_\gamma^{d,n}(z)$ is hyperbolic for all $n \geq N(\phi_\gamma; d)$. We can apply this theorem for $d = 2$ to get that there exists an $N(\phi_\gamma; 2)$ such that if $n \geq N(\phi_\gamma; 2)$ then $J_\gamma^{2,n}(z)$ hyperbolic and $\{\gamma_k\}$ is log-concave. Using this, let $n \geq N = N(\phi_\gamma; 2) + 1$ and we have

$$(\gamma_{n+j})^2 \geq \gamma_{n+j-1}\gamma_{n+j+1} \geq \frac{j}{j+1} \cdot \frac{k-j}{k-j+1} \cdot \gamma_{n+j-1} \cdot \gamma_{n+j+1},$$

which implies $\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j}$ for all $1 \leq j \leq k-1$. ■

Corollary 8.4. *There exists an $N \in \mathbb{N}$ such that if $n \geq N$ then*

$$\max_{0 \leq j < k} \{a_j/a_{j+1}\} = a_{k-1}/a_k,$$

where a_j is the j-th coefficient of $P_{k,n}$.

Proof. By Theorem 8.3, there exists an N such that if $n \geq N$ then the sequence $\{a_j/a_{j+1}\}_{j=0}^{k-1}$ is increasing. This implies that the maximum of the sequence occurs at $j = k-1$; thus, if $n \geq N$ then the maximum is a_{k-1}/a_k. ■

Proof of Theorem 8.1. By the Corollary, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $\max_{0 \leq j < k}\{a_j/a_{j+1}\} = a_{k-1}/a_k$. Combining this with Lemma 8.2, we get that if $n \geq N$,

$$|z_0| \leq \max_{1 \leq j \leq k-1} \frac{a_j}{a_{j+1}} = \frac{a_{k-1}}{a_k} = \frac{\binom{k}{k-1} \cdot k^{-k+1} \cdot \gamma_{n+k-1}}{\binom{k}{k} \cdot k^{-k} \cdot \gamma_{n+k}} = k^2 \cdot \frac{\gamma_{n+k-1}}{\gamma_{n+k}},$$

as desired. ■

Unfortunately, for everything to work like in the proof of the main theorem from [15], the bound must be independent of k, n. Our bound, while not necessarily a tight bound, gives evidence to the notion that a bound independent of k, n may not exist. Thus, we will alter the polynomials we are studying slightly and find ones with a bound independent of k, n. When $k \leq d$, we keep the same choice of $P_{k,n} = J_\gamma^{k,n}(z/k)$ but for $k > d$ we define

$$\widehat{P}_{d,k,n}(z) := \sum_{j=0}^{d} \binom{k}{j} \gamma_{n+j} (z/k)^j.$$

We make this choice as for $k \leq d$, $P_{k,n}(z) = \widehat{P}_{d,k,n}(z)$ and when $k > d$, $P_{k,d}(z)$ and $\widehat{P}_{d,k,n}$ have the same first d Taylor coefficients. This implies that for all k

$$J_{P_{k,n}}^{d,0}(z) = J_{\widehat{P}_{d,k,n}}^{d,0}(z).$$

Now we must bound the zeros of $\widehat{P}_{d,k,n}(z)$.

Theorem 8.5. *If z_0 is a root of $\widehat{P}_{d,k,n}(z)$ for some $\phi_\gamma \in \mathcal{SL} - \mathcal{P}$, then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then*

$$|z_0| \leq d \cdot \frac{k}{k-d+1} \cdot \frac{\gamma_{n+d-1}}{\gamma_{n+d}}.$$

Proof. It is clear that the same reasoning for the proofs of Theorems 8.1 and 8.3 hold; however, the sequence of quotients of coefficients is now $\{a_j/a_{j+1}\}_{j=0}^{d-1}$ which implies that the maximum is a_{d-1}/a_d. Then, by Lemma 8.2, for $z_0 \in \mathcal{Z}(\widehat{P}_{d,k,n}(z))$ we have

$$|z_0| \leq \max_{1\leq j\leq d-1} \frac{a_j}{a_{j+1}} = \frac{a_{d-1}}{a_d} = \frac{\binom{k}{d-1} \cdot k^{-d+1} \cdot \gamma_{n+d-1}}{\binom{k}{d} \cdot k^{-d} \cdot \gamma_{n+d}} = d \cdot \frac{k}{k-d+1} \cdot \frac{\gamma_{n+d-1}}{\gamma_{n+d}}.$$

■

Here we would like to use the fact that if $k \leq d$ then $k^2 \leq d^2$ and if $k \geq d$ then $d \geq \frac{k}{k-d+1}$ to make our bound on roots of $\widehat{P}_{d,k,n}(z)$ something of the form d^2M where M is a constant that accounts for the γ terms. However, the author has not yet found such an M that is independent of n and works for all sequences $\{\gamma_i\}$ defining a function in the shifted Laguerre-Pólya class.

If such a bound exists then we would be one step closer to apply the methods of Kim and Lee. However, we have not yet shown that $\mathcal{Z}(\widehat{P}) \subset S(\delta)$ for some choice of δ. In numerical experiments using the partition function $p(n)$ for the sequence $\{\gamma_i\}$, it does not appear to be the case that $\mathcal{Z}(\widehat{P}) \subset S(\delta)$ either as $k \to \infty$ or for large shift n. Plots of $\widehat{P}_{d,k,n}(z)$ are shown below with fixed $d = 100$, and every permutation of large and small k, n. We would expect a distribution of roots with small imaginary parts near the origin if it is true that $\mathcal{Z}(\widehat{P}) \subset S(\delta)$ for some $\delta \geq 0$.

FIGURE 2. Roots of $\widehat{P}_{d,k,n}(z)$ for $d = 100$, $k = 101$, $n = 0$

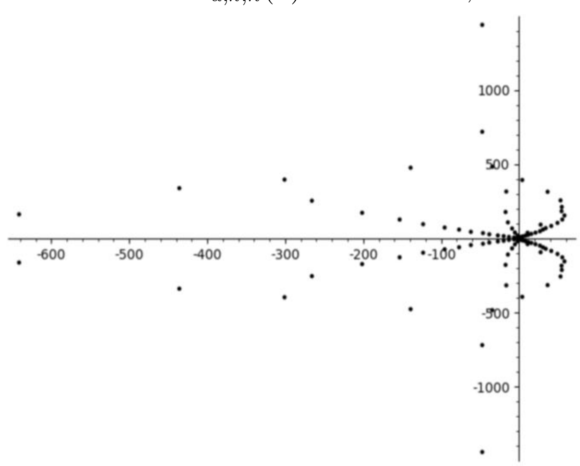


FIGURE 3. Roots of $\widehat{P}_{d,k,n}(z)$ for $d = 100$, $k = 101$, $n = 10^9$

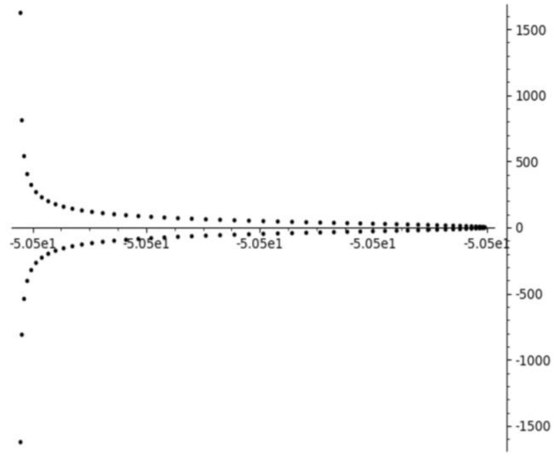


FIGURE 4. Roots of $\widehat{P}_{d,k,n}(z)$ for $d = 100$, $k = 10^9$, $n = 0$

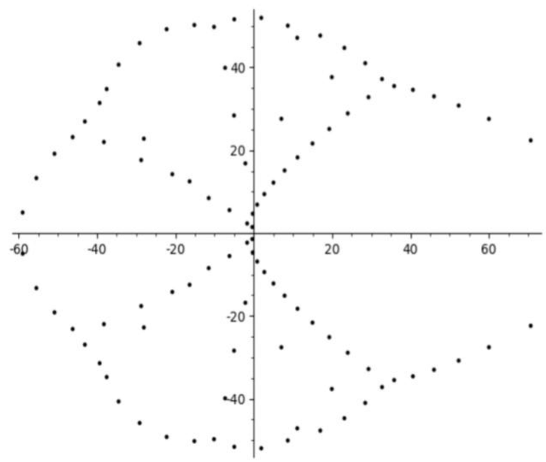
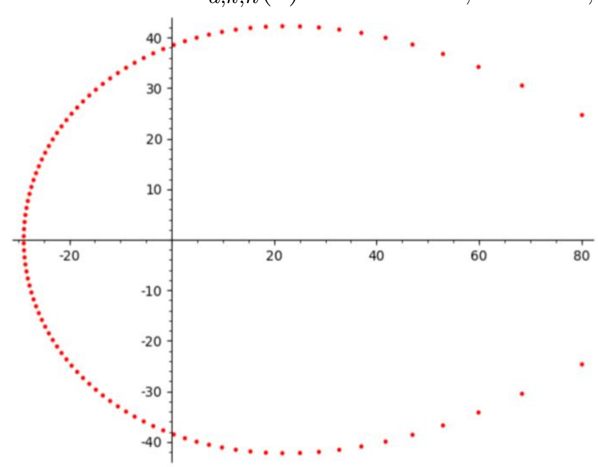


FIGURE 5. Roots of $\widehat{P}_{d,k,n}(z)$ for $d = 100$, $k = 10^9$, $n = 10^9$



These plots show that the behavior when both k, n large is not what we would expect, as the roots seem to form an ellipse/tear-drop shape around the origin. This is a mysterious result, as both the philosophy of the shifted Laguerre-Pólya class and our proofs above seem to imply the desired pseudo-hyperbolic behavior. Instead, we have a completely different geometric distribution of roots. This distribution, while unexpected, is somewhat regular and bounded so we may be able to use this fact to find a bound on $N(\phi_\gamma; d)$.

8.2. **The Partition Case.** As described in earlier sections, the formal power series

$$\phi_p(z) = \sum_{j=0}^{\infty} \frac{p(j)}{j!} z^j,$$

where $p(j)$ is the j-th partition number, was shown to be in the shifted Laguerre-Pólya class of type I [10], [25]. The partition case is one of particular interest, since $J_p^{d,n}(z)$ being hyperbolic implies identities about sequences of partitions, like log-concavity in the case of $J_p^{2,n}(z)$. The partition numbers are more "well-behaved" than a general shifted multiplier sequence, so we are able to make the independent bound discussed in the previous section. We first apply the theorems from the previous section in the partition case. Let $P_{k,n}(p(i); z)$ and $\widehat{P}_{d,k,n}(p(i); z)$ be defined as the usual $P_{k,n}(z)$ and $\widehat{P}_{d,k,n}(z)$ but evaluated over the specific sequence $\{\gamma_i\} = \{p(i)\}$.

Theorem 8.6. *If z_0 is a root of $\widehat{P}_{d,k,n}(p(i); z)$ and $n \geq 27$ then $|z_0| \leq d^2$.*

Proof. We may apply Theorems 8.1 and 8.5 and we note that the N in the proofs of these theorems is defined as $N = N(\phi_\gamma; 2) + 1$. In the partition case, we know that $N(\phi_p; 2) = 26$ [17]. Thus, if $n \geq 27$ then Theorem 8.1 gives the following bound for $z_1 \in \mathcal{Z}(P_{k,n}(p(i); z))$

$$|z_1| \leq k^2 \cdot \frac{p(n+k-1)}{p(n+k)},$$

and Theorem 8.5 gives a bound for $z_2 \in \mathcal{Z}(\widehat{P}_{d,k,n}(p(i); z))$ when $k > d$

$$|z_2| \leq d \cdot \frac{k}{k-d+1} \cdot \frac{p(n+d-1)}{p(n+d)}.$$

We can first note that since the partition numbers are strictly increasing, $p(i)/p(i+1) \leq 1$ for all $i \geq 0$. Next, the first bound applies for $k \leq d$ so in this case $k^2 \leq d^2$. Finally, the second bound applies for $k \geq d$ which is a sufficient condition to satisfy $\frac{k}{k-d+1} \leq d$. Combining all of these gives a bound on the size of any zeros $z_0 \in \mathcal{Z}(\widehat{P}_{d,k,n}(p(i); z))$ for all k

$$\begin{aligned} |z_0| &\leq \begin{cases} k^2 \cdot \frac{p(n+k-1)}{p(n+k)} & k \leq d \\ d \cdot \frac{k}{k-d+1} \cdot \frac{p(n+d-1)}{p(n+d)} & k > d \end{cases} \\ &\leq \begin{cases} d^2 \cdot 1 & k \leq d \\ d \cdot d \cdot 1 & k > d \end{cases} \\ &= d^2. \end{aligned}$$

Thus, $|z_0| \leq d^2$ for all $k \geq 1$ as desired. ■

We have shown that $\widehat{P}_{d,k,n}(p(i); z)$ satisfies the d^2 bound on the size of its roots for all $d \geq 1$ when $n \geq 27$. We make the following conjecture about the remaining 27 possible values of n.

Conjecture 8.7. If z_0 is a root of $\widehat{P}_{d,k,n}(p(i);z)$ then

$$|z_0| \leq d^2$$

for all $d \geq 1$, $k \geq 1$, and $n \geq 0$.

The author and their collaborators make this conjecture based on the results numerical experiments in cases with small shift n. We will show directly that $\widehat{P}_{d,k,n}(p(i);z)$ satisfies this bound for $n = 0$, $k \leq d$, and all $d \geq 1$ in this thesis. To fully prove the conjectured bound, one would need to verify the bound for $d \geq 1$ and $k \geq 1$ in the cases $n = 0, 1, 2..., 26$.

Theorem 8.8. *Let z_0 be a root of $P_{k,0}(p(i);z)$ and let $k \leq d$. Then $|z_0| \leq d^2$.*

Lemma 8.9. The sequence $\{a_j/a_{j+1}\}_{j=0}^{d-1}$ is increasing for $d \geq 1$.

Proof. Using $\gamma_i = p(i)$ and $n = 0$ in inequality (8.1), we get that

$$\frac{a_j}{a_{j+1}} \geq \frac{a_{j-1}}{a_j} \iff p(j)^2 \geq \frac{j}{j+1} \cdot \frac{d-j}{d-j+1} \cdot p(j-1) \cdot p(j+1). \tag{8.2}$$

From [17] and [6], we have that $p(j)^2 > p(j-1)p(j+1)$ for $j \geq 26$. We also may observe that if $j = 1$ then the inequality becomes $1 \geq \frac{d-1}{d}$ which is true when $d > 1$. Thus, we need only check the inequality for $2 \leq j \leq 25$ and all $d \geq 1$. To do this, we again rearrange the inequality to isolate d and evaluate this equation with j values 2 to 25. We find that,

$$p(j-1)p(j+1) \leq \frac{d-j+1}{d-j} \cdot \frac{j+1}{j} p(j)^2 \iff \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}(d-j) \leq d-j+1,$$

$$\iff \left(\frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}\right) d - \left(\frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}\right) j \leq d - j + 1,$$

$$\iff j - 1 - \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j^2}{j+1} \leq d - \left(\frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}\right) d,$$

$$\iff j - 1 - \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j^2}{j+1} \leq d\left(1 - \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}\right),$$

and so

$$\frac{a_{j-1}}{a_j} \leq \frac{a_j}{a_{j+1}} \iff \frac{j - 1 - \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j^2}{j+1}}{1 - \frac{p(j-1)p(j+1)}{p(j)^2} \cdot \frac{j}{j+1}} \leq d. \tag{8.3}$$

Thus, we use this inequality in (8.3) to find how large d must be for the inequality to hold by computing the ceiling of d for each $2 \leq j \leq 25$. The results are displayed in the table at the end of the section. From this, we can conclude that the inequality is satisfied for $2 \leq j \leq 25$ when $d \geq 4$

We manually check the remaining 3 cases, $d = 1, 2, 3$. For this, we will return to using inequality (8.2). Clearly if $d = 1$ then $\{a_i/a_{i+1}\}_{i=0}^{d-1}$ is monotonic since it only contains one element. Next, if $d = 2$ we compare two terms, $j = 0, j = 1$ and find that

$$\frac{a_0}{a_1} = \frac{(0+1)2}{2-0} \cdot \frac{p(0)}{p(0+1)} = 1 \leq \frac{a_1}{a_2} = \frac{(1+1)2}{2-1} \cdot \frac{p(1)}{p(1+1)} = 2.$$

Finally, if $d = 3$ then there are 3 terms in the sequence so we verify the following two comparisons

$$\frac{a_0}{a_1} = \frac{(0+1)3}{3-0} \cdot \frac{p(0)}{p(0+1)} = 1 \leq \frac{a_1}{a_2} = \frac{(1+1)3}{3-1} \cdot \frac{p(1)}{p(1+1)} = \frac{3}{2}.$$

$$\frac{a_1}{a_2} = \frac{3}{2} \leq \frac{a_2}{a_3} = \frac{(2+1)3}{3-2} \cdot \frac{p(2)}{p(2+1)} = 6.$$

Thus, we have that $a_{j-1}/a_j \leq a_j/a_{j+1}$ for all $1 \leq j \leq d$ and $d \geq 1$ so $\{a_j/a_{j+1}\}_{j=0}^{d-1}$ is monotonically increasing for $d \geq 1$. ■

Corollary 8.10. *We have that* $\max_{0 \leq i < k}\{a_i/a_{i+1}\} = a_{k-1}/a_k$.

Proof. By the previous lemma, we have that the sequence $\{a_i/a_{i+1}\}_{i=0}^{k-1}$ is increasing. This implies that the maximum of the the sequence occurs at $i = k-1$. Thus, the maximum is a_{k-1}/a_k. ■

Proof of Theorem 8.6. Recalling the methods from the previous section, by [13] we have that for any n-degree polynomial with positive, real coefficients a_i, $0 \leq i < n$ then the zeroes of the polynomial satisfy $|z| \leq \max_{0 \leq i < n}\{a_i/a_{i+1}\}$. We can note that the degree k Jensen polynomial satisfies these coefficient properties so, we have that every zero, z_0, of $P_{k,0}(p(i); z)$ is contained within

$$|z_0| \leq \max_{0 \leq i < k}\{a_i/a_{i+1}\} = a_{k-1}/a_k = \frac{(k)k}{k-(k)+1} \cdot \frac{p(k-1)}{p(k)} = k^2 \cdot \frac{p(k-1)}{p(k)},$$

from the corollary. Then, we can use that $p(n)/p(n+1) \leq 1$ for all $n \geq 0$ to conclude that all zeroes of $P_{k,0}(p(i); z)$ satisfy $|z| \leq k^2 \leq d^2$. ■

We can now consider how the generalization of the methods of Kim and Lee would follow in the partition case. The function ϕ_p is transcendental real entire with $\rho \leq 1 < 2$ since $\phi_p \in \mathcal{SL} - \mathcal{P}I$. Assuming there exists some $n_\delta \in \mathbb{N}$ as described in the previous section, we could use the bound on the zeros of $\widehat{P}_{d,k,n}(P(i); z)$ proved in Theorem 8.6. We would choose n, d such that

$$n \geq \max\{n_\delta, 26, d^{5c/2}\}$$

and $\delta(d) = d^2/n^{1/c}$. Then we would have $d \leq \delta(d)^{-2}$ and $\mathcal{Z}(\widehat{P}) \subset S(\delta(d))$ so Corollary 6.13 would apply and

$$J_{\widehat{P}}^{d,n} \to J_{\phi_p^{(n)}}^{d,0} = J_{\phi_p}^{d,n}$$

would be hyperbolic. We would then have

$$N(\phi_p; d) \leq \left\lceil \max\left\{n_\delta, 26, d^{5c/2}\right\}\right\rceil$$

which would imply $N(\phi_p; d) = O(d^{5 \cdot 1/2}) = O(d^{5/2})$ as $d \to \infty$. We can also note that if Conjecture 8.7 is true, then the 26 can be eliminated from the maximum.

j	LHS	$\lceil LHS \rceil$
2	0	0
3	−3	−3
4	39/41	1
5	−199/19	−10
6	65/31	3
7	−529/53	−9
8	47/21	3
9	−79/19	−4
10	79/31	3
11	−51/7	−7
12	4769/1315	4
13	−6141/1097	−5
14	69779/24511	3
15	−7091/2531	−2
16	207/65	4
17	−821/209	−3
18	269/73	4
19	−20217/6157	−3
20	2347/713	4
21	−1791/1021	−1
22	107147/34027	4
23	−21521/10021	−2
24	730689/202711	4
25	−1417191/940091	−1

Table 1. Values of the lower bound on d such that the sequence is increasing. LHS denotes the evaluation of the quotient on the left-hand side of (8.3).

9. Conclusion

There is much left to be proved and explored within the topics discussed in this thesis. In particular, proving the existence of an "n_δ" beyond which a function in the shifted Laguerre-Pólya class acts like a function in the Laguerre-Pólya class would be a great impetus in formalizing the generalizations of the methods of Kim and Lee as well as understanding the utility of this new class of functions. Additionally, understanding the mysterious behavior of the roots of $\widehat{P}$ which, despite acting different than what we might expect, still have a bounded pattern, would add a beautiful geometric aspect to this topic. I am excited to continue working with these concepts as I prepare to enter my graduate studies.

I am grateful for the opportunity to participate in mathematics research and to present my findings in this thesis. My knowledge and appreciation for number theory, and math in general, have increased tremendously over the past year. I would like to extend another special thanks to Dr. Larry Rolen, who has formally served as both my thesis and academic advisor, but has also served as a fantastic mathematical mentor during my time at Vanderbilt. I would also like to extend a special thanks to Dr. Ian Wagner, whose number theory class first sparked my interest in the field, and who has continued to encourage my interest.

References

[1] L. Alfohrs. *Complex Analysis*, International series in pure and applied mathematics, 3rd ed.

[2] M. A. Bennett, G. Martin, K. O'Bryant, A. Rechnitzer. *Counting Zeros of Dirichlet L-Functions*, arXiv:2005.02989.

[3] R. P. Boas, Jr. *Entire Functions*, Academic Press, New York.

[4] W. Y. C. Chen, D. X. Q. Jia, and L. X. W. Wang. *Higher Order Turán Inequalities for the Partition Function*, Trans. Amer. Math. Soc., accepted for publication.

[5] J-M. De Konnick, F. Luca. *Analytic Number Theory: Exploring The Anatomy of the Integers*, Graduate Studies in Mathematics, **134** (2012).

[6] S. DeSalvo, I. Pak. *Log-concavity of the partition function*, Ramanujan J. **38**, (2015), 61–73.

[7] F. Franklin. *Sur le d'eveloppement du produit infini* $(1-x)(1-x^2)(1-x^3)(1-x^4)$..., Comptes Rendus Hebdomadaires des S'eances de l'Acad'emie des Sciences (Paris) **92**, 448–450.

[8] R. Gardner and N. Govil. *Eneström-Kakeya Theorem and some of its generalizations*, Current Topics in Pure and Computational Complex Analysis, Birkhäuser, New Delhi, (2014).

[9] M. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp, and I. Wagner. *Jensen polynomials for the Riemann Xi function*, arXiv:1910.01227.

[10] M. Griffin, K. Ono, L. Rolen, and D. Zagier. *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci., USA **116**, no. 23 (2019), 11103-11110.

[11] H. A. Helfgott. *The ternary Goldbach conjecture is true*, arXiv:1312.7748.

[12] User "joriki." *Order of growth of* $(s-1)\zeta(s)$, Stack Exchange.

[13] S. Kakeya. *On the limits of the roots of an algebraic equation with positive coefficients*, Tôhoku Math. J., **2** (1912), 140-142.

[14] Y-O. Kim. *Critical points of real entire functions and a conjecture of Pólya*, Proc. Amer. Math. Soc. 124, no. 3 (1996), 819–830.

[15] Y-O. Kim and J. Lee. *A note on the zeros of Jensen polynomials*, arXiv:2105.05386.

[16] Y. Matsuoka, *Generalized Euler constants associated with the Riemann zeta function*, Number theory and combinatorics. Japan 1984 (Tokyo, Okayama and Kyoto, 1984), (1985), 279–295.

[17] J.L. Nicolas. *Sur les entiers N pour lesquels il y a beaucoup de groupes ab ´eliens d'ordre N*, Ann. Inst. Fourier, **28**, no. 4 (1978), 1–16.

[18] G. Pólya. *Über die algebraisch-funktionentheoretischen Untersuch ungen von J.L.W.V. Jensen*, Kgl Danske Vid Sel Math-Fys Medd, 7, (1927), 3–33.

[19] G. Pólya and J. Schur. *Über zwei Arten Faktorenfolgen in der Theorie der algebrais chen Gleichungen*, Journal für die reine und angewandte Mathematik, 144 (1914), 89–113.

[20] L. Rolen. *Jensen-Pólya Program for the Riemann Hypothesis and Related Problems*. Presentation.

[21] S. Saad Eddin. *Explicit upper bounds for the Stieltjes constants*, J. Number Theory, **133**, (2013), 1027–1044.

[22] T. S. Trudgian. *An improved upper bound for the argument of the Riemann zeta-function on the critical line II*, J. Number Theory 134 (2014), 280–292.

[23] R. C. Vaughan. *The Hardy-Littlewood method*, vol **125**, Cambridge University Press, 1997.

[24] I. Wagner. *The Jensen-Pólya program for various L-functions*, Forum Mathematicum, **32**, no. 2 (2020), 525–539.

[25] I. Wagner. *On a new class of Laguerre-Pólya type functions with applications in number theory*, arXiv:2108.01827v2.