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Part V
Option Pricing Based on GARCH Models

V.1 Introduction

In this note, we consider pricing of financial derivatives based on GARCH models. We focus on the pricing of European call options, but emphasize that some of the outlined methods can be applied to other derivatives, in particular ones with payoff at a known future date T. Similar to the Expected Shortfall, discussed in the previous lecture note, the computation of the derivatives prices – in essence – relies on computing the mean of a (conditional) payoff distribution. For some special cases we obtain closed-form pricing formulas, but in general we rely on simulation-based estimation.

A European call option gives the right - but not the obligation - to buy a stock for a specified strike price K on a specified date T. Hence, with S_T the price of the stock at time T, the payoff of the option is given by

$$\pi_T^{\text{Call}}(K) = \begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}, \tag{V.1}$$

i.e. $\pi_T^{\text{Call}}(K) = \max(S_T - K, 0)$. Due to its nonnegative payoff, the option must have a nonnegative price at time $t \leq T$. Techniques for determining the prices of options is a vast research field dating back to seminal work by Bachelier (1900) as well as the work of Black, Scholes, and Merton (1973). As mentioned, we will discuss how such options may be priced under the more realistic assumption that the returns on the underlying stock follows a GARCH process.

V.1.1 On derivatives pricing

We recall some important concepts from finance theory. Let $r \geq 0$ denote the (continuously compounded) risk-free interest rate. The price, B_t, at time

$t \in \mathbb{N}$ of one monetary unit invested at time $t = 0$ at the risk-free rate is by definition given by

$$B_t = B_{t-1}e^r, \quad t \in \mathbb{N}, \quad B_0 = 1. \tag{V.2}$$

Let $\mathcal{I}_t$ denote some information set available at time t, and let P_t denote the price at time t of some asset with payoff π_{t+1} at time $t+1$. The absence of arbitrage in a market is equivalent to the existence of a so-called stochastic discount factor (SDF) :

Definition V.1.1 (One-period SDF) *The stochastic discount factor (SDF),* $m_{t,t+1} \geq 0$*, satisfies*

$$P_t = E[m_{t,t+1}\pi_{t+1}|\mathcal{I}_t]. \tag{V.3}$$

For additional details, we refer to Back (2017).[1] Ideally, equation (V.3) holds for all assets, and hence an SDF can be used to price all assets in a market. We give a few important examples: Let S_t denote the price at time t of the stock that does not pay out dividends at time $t+1$. By definition, the payoff at time $t+1$ of this stock is simply the stock price at time $t+1$ ($P^S_{t+1} = S_{t+1}$), so that

$$S_t = E\left[m_{t,t+1}S_{t+1}|\mathcal{I}_t\right]. \tag{V.4}$$

Moreover, in light of the price of the risk-free bank account in (V.2), we have that

$$B_t = E[m_{t,t+1}B_{t+1}|\mathcal{I}_t] \quad \Leftrightarrow \quad 1 = E[m_{t,t+1}\underbrace{(B_{t+1/}B_t)}_{e^r}|\mathcal{I}_t],$$

so that

$$E[m_{t,t+1}|\mathcal{I}_t] = e^{-r}. \tag{V.5}$$

Lastly, we have that the European call option expiring at time $t+1$ has price

$$P_t^{\text{Call}}(t+1, K) = E[m_{t,t+1}\pi^{\text{Call}}_{t+1}(K)|\mathcal{I}_t] = E[m_{t,t+1}\max(S_{t+1} - K, 0)|\mathcal{I}_t].$$

Note that, conveniently, we may also define the $(T-t)$ period SDF:

Definition V.1.2 (Multi-period SDF) *The* $(T-t)$ *period SDF* $m_{t,T}$ *is given by*

$$m_{t,T} = \prod_{i=t}^{T-1} m_{i,i+1}, \tag{V.6}$$

where $m_{i,i+1}$ *is the one-period SDF defined in (V.3) for* $i = t, \ldots, T-1$*.*

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[1] Note that if investors were risk neutral one would simply discount payoffs according to the risk-free rate, i.e. $S_t = e^{-r}E[S_{t+1}|\mathcal{I}_t]$, as done in Section V.3. This kind of discounting only takes into account the time value of money – but it does not account for risk. The SDF is intended to capture both the time value of money and the discounting due to risk.

The SDF is sometimes refered to as the *pricing kernel*.

A straightforward application of the law of iterated expectations and (V.4)-(V.5) imply that

$$E[m_{t,T}|\mathcal{I}_t] = e^{-r(T-t)} \quad \text{and} \quad E\left[m_{t,T}S_T|\mathcal{I}_t\right] = S_t. \tag{V.7}$$

Likewise, the price (at time t) of the European call option with payoff (V.1) is

$$P_t^{\text{Call}}(T,K) = E[m_{t,T}\pi_T^{\text{Call}}(K)|\mathcal{I}_t] = E[m_{t,T}\max(S_T - K, 0)|\mathcal{I}_t].$$

and we consider the computation of this quantity under various assumptions about the joint dynamics of the SDF $(m_{t,t+1})$ and the stock price (S_t).

V.2 The normal and log-normal distributions

In this section we provide a few results for the normal and log-normal distributions that will show up to be useful for the derivations of the option prices.

By definition, a positive real-valued variable $X > 0$ is log-normal, if $Y = \log(X)$ is Gaussian. Thus with Y $\mathrm{N}(\mu, \sigma^2)$-distributed, then $X = \exp(Y)$ has density[2]

$$f(x) = \frac{1}{x}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(\log x - \mu)^2\right), \qquad x > 0. \tag{V.8}$$

We say that X is log-normal with parameters (μ, σ^2), with density $f(x)$ in (V.8). The kth moment of X is

$$EX^k = \exp\left(\sigma^2\frac{k^2}{2} + k\mu\right). \tag{V.9}$$

In particular,

$$EX = \exp\left(\frac{\sigma^2}{2} + \mu\right) \text{ and } VX = \left(\exp\left(\sigma^2\right) - 1\right)\exp\left(\sigma^2 + 2\mu\right),$$

and moreover, as will be useful for computation of kurtosis and skewness of returns later,

$$EX^2 = \exp\left(2\sigma^2 + 2\mu\right), \quad EX^4 = \exp\left(8\sigma^2 + 4\mu\right) \text{ and } EX^3 = \exp\left(\sigma^2\frac{9}{2} + 3\mu\right).$$

The following lemma states an important property of a log-normal random variable.

[2] To see this, note that $h(y) = \exp(y) = x$, we have $h^{-1}(x) = \log x$ with derivative $\frac{1}{x}$. Hence if Y has density $g(y)$, then x by the formula for transformation of distributions, has density $f(x) = \frac{1}{x}g(\log(x))$. That is, with $h(y) = x$, $f(x) = g\left(h^{-1}(x)\right)|\partial h^{-1}(x)|$.

Lemma V.2.1 *Let X be log-normal with parameters (μ, σ^2), and let K be a non-negative constant. Then*

$$E[\max(X - K, 0)] = E[X]\Phi(-u + \sigma) - K\Phi(-u), \quad u := \frac{\log(K) - \mu}{\sigma}$$

where $\Phi(\cdot)$ is the cdf of a standard normal distribution.

Lastly, the next lemma shows up to be useful for deriving the famous Black-Scholes-Merton pricing formula.

Lemma V.2.2 *Let X,$Y \in \mathbb{R}$ be jointly normal with covariance given by σ_{XY}. Then for any continuous function $g : \mathbb{R} \to \mathbb{R}$,*

$$E[\exp(X)g(Y)] = E[\exp(X)]E[g(Y + \sigma_{XY})].$$

V.3 Non-stochastic discounting

In this section, we consider a "simple" pricing approach, where the stock price is assumed to be log-normal and the SDF is constant. Note that if the SDF, $m_{t,t+1}$, is constant and satisfies (V.4)-(V.5), then we have that

$$m_{t,t+1} = e^{-r} \tag{V.10}$$

and

$$E(S_{t+1}/S_t|\mathcal{I}_t) = e^r. \tag{V.11}$$

The condition in (V.11) holds under the assumption that S_{t+1}/S_t is conditionally log-normal with parameters (μ, σ^2) with

$$\mu + \sigma^2/2 = r, \tag{V.12}$$

which can be seen by recalling that $E(S_{t+1}/S_t|\mathcal{I}_t) = \exp(\mu + \sigma^2/2)$ in light of (V.9). Recall that the price of the call option expiring at time $t + 1$ is

$$\begin{aligned} P_t^{\text{Call}}(t+1, K) &= E[m_{t,t+1}\max(S_{t+1} - K, 0)|\mathcal{I}_t] \\ &= S_t E[m_{t,t+1}\max(S_{t+1}/S_t - K/S_t, 0)|\mathcal{I}_t] \\ &= S_t e^{-r} E[\max(S_{t+1}/S_t - K/S_t, 0)|\mathcal{I}_t], \end{aligned}$$

Then under the aforementioned conditional log-normality of S_{t+1}/S_t, we may apply Lemma V.2.1 to evaluate $E[\max(S_{t+1}/S_t - K/S_t, 0)|\mathcal{I}_t]$. Consequently, we obtain the following theorem.

Theorem V.3.1 (One-period Black-Scholes) *Suppose that the SDF, $m_{t,t+1}$, is constant and given by (V.10), and assume that the $\mathcal{I}_t$-conditional distribution of the log-return* $\log(S_{t+1}/S_t)$ *is normal with constant mean μ and variance σ^2 satisfying (V.12). Then the price of the call option expiring at time $t+1$ is*

$$\begin{aligned} P_t^{Call}(t+1, K) &= E[m_{t,t+1} \max(S_{t+1} - K, 0)|\mathcal{I}_t] \\ &= S_t\Phi\left(-u_{t,t+1} + \sigma\right) - e^{-r}K\Phi\left(-u_{t,t+1}\right), \end{aligned}$$

where

$$u_{t,s} := \frac{\log(Ke^{-(r-\sigma^2/2)(s-t)}) - \log(S_t)}{\sigma\sqrt{s-t}}, \quad t < s. \tag{V.13}$$

The above theorem may be extended to an arbitrary time-horizon. In particular, using the definition of the SDF, it is straightforward to show that for $T \geq t+1$,

$$m_{t,T} = e^{-r(T-t)} \quad \text{and} \quad E[S_T/S_t|\mathcal{I}_t] = e^{r(T-t)}.$$

Again, the latter condition holds under the assumption that S_T/S_t is conditionally log-normal with

$$E[\log(S_T/S_t)|\mathcal{I}_t] + V[\log(S_T/S_t)|\mathcal{I}_t]/2 = r(T-t). \tag{V.14}$$

For instance, this condition holds if daily log-returns, $\{\log(S_{i+1}/S_i)\}_{i=t}^{T-1}$, are (conditional on $\mathcal{I}_t$) i.i.d. normal with mean μ and variance σ^2, so that $E[\log(S_T/S_t)|\mathcal{I}_t] = (T-t)\mu$, $V[\log(S_T/S_t)|\mathcal{I}_t] = \sigma^2(T-t)$, and (V.14) holds under the condition that $\mu + \sigma^2/2 = r$. We have the following theorem.

Theorem V.3.2 (Multi-period Black-Scholes) *Suppose that the SDF, $m_{t,t+1}$, is constant and satisfies (V.10), and let $m_{t,T}$ be given by (V.6). Assume that the $\mathcal{I}_t$-conditional distribution of the $(T-t)$-period log-return,* $\log(S_T/S_t)$, *is normal with mean $\mu(T-t)$ and variance $\sigma^2(T-t)$, where μ and σ^2 are constants satisfying (V.12). Then the price of the call option expiring at time T is*

$$\begin{aligned} P_t^{Call}(T, K) &= E[m_{t,T} \max(S_T - K, 0)|\mathcal{I}_t] \\ &= S_t\Phi\left(-u_{t,T} + \sigma\sqrt{T-t}\right) - \exp(-r(T-t))K\Phi\left(-u_{t,T}\right), \end{aligned}$$

where $u_{t,T}$ is defined in (V.13).

We note that the above pricing formulas made use of a constant SDF, which (due to no arbitrage) in turn implied that stock price was expected to grow at the risk-free rate, $E[S_T/S_t|\mathcal{I}_t] = e^{r(T-t)}$, which appears to be restrictive. In particular, we may want to price options written on stocks with other expected growth rates. In order to do so, we consider pricing under a truly random SDF, which is the topic of the next section.

V.4 Stochastic discounting

In this section we consider pricing when the SDF is random. Since the SDF is a positive random variable, a candidate (conditionally) distribution is the log-normal. Hence, in line with the previous section, where we considered log-normal stock prices, we follow the approach of, e.g., Garcia et al. (2010, Section 2.4) and assume that the log-returns and the logarithm of the SDF are conditionally jointly normal. Specifically, we assume that

$$\begin{pmatrix} \log(S_{t+1}/S_t) \\ \log m_{t,t+1} \end{pmatrix} |\mathcal{I}_t \sim N_2 \left(\begin{pmatrix} \mu \\ \mu_m \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma_{m,r} \\ \sigma_{m,r} & \sigma_m^2 \end{pmatrix} \right). \tag{V.15}$$

At first glance, this assumption looks quite general in the sense that we have many parameters to specify the joint conditional distribution. Moreover, we may note that as we have conditioned on $\mathcal{I}_t$, the parameters may not be constant, but should simply be adapted with respect to $\mathcal{I}_t$. For now we suppress the potential time-dependence of the parameters and view them as constants. We return to the case of time-varying parameters in Section V.5. Note that the parameters cannot be chosen freely, as the no-arbitrage conditions in (V.4)-(V.5) impose some structure on the model in (V.15). First, since $m_{t,t+1}$ is (conditionally) log-normal with parameters (μ_m, σ_m^2), we have, in light of (V.9), that the first part of (V.5) implies that $\mu_m + \sigma_m^2/2 = -r$. Next, from (V.4) we have that

$$E\left[\exp\left\{\log m_{t,t+1}\right\} \times \exp\left\{\log(S_{t+1}/S_t)\right\} |\mathcal{I}_t\right] = 1, \tag{V.16}$$

and we apply the fact that $(\log(S_{t+1}/S_t), \log m_{t,t+1})$ is jointly normal and Lemma V.2.2 in order show that the no arbitrage conditions impose additional structure on the model parameters. We have the following result, which is proved in the appendix.

Lemma V.4.1 *Suppose that (V.15) holds. The no arbitrage conditions in (V.4)-(V.5) imply that*

$$\mu_m + \frac{\sigma_m^2}{2} = -r \tag{V.17}$$

and

$$\sigma_{m,r} + \mu = -\frac{\sigma^2}{2} + r. \tag{V.18}$$

We now turn to pricing of a call option under the joint normality assumption in (V.15). Recall that the price of a call option expiring at time $t+1$ is

$$\begin{aligned} P_t^{\text{Call}}(t+1, K) &= E[m_{t,t+1} \max(S_{t+1} - K, 0)|\mathcal{I}_t] \\ &= S_t E[\exp\left\{\log m_{t,t+1}\right\} \max(\exp\{\log(S_{t+1}/S_t)\} - K/S_t, 0)|\mathcal{I}_t]. \end{aligned}$$

Using (V.15) and noting that $\max(x - y, 0)$ is continuous in x for a fixed y, it follows by Lemma V.2.2, that

$$\begin{aligned} P_t^{\text{Call}}(t+1, K) &= S_t E[\exp(\log m_{t,t+1}) |\mathcal{I}_t] \\ &\quad \times E[\max(\exp\{\log(S_{t+1}/S_t) + \sigma_{m,r}\} - K/S_t, 0)|\mathcal{I}_t] \\ &= S_t E[m_{t,t+1}|\mathcal{I}_t] E[\max(e^{\sigma_{m,r}}(S_{t+1}/S_t) - K/S_t, 0)|\mathcal{I}_t] \\ &= S_t e^{-r} E[\max(e^{\sigma_{m,r}}(S_{t+1}/S_t) - K/S_t, 0)|\mathcal{I}_t], \end{aligned} \tag{V.19}$$

where the last equality follows from (V.5). An explicit formula for $P_t^{\text{Call}}(t + 1, K)$ is now obtained by (i) realizing that $e^{\sigma_{m,r}}(S_{t+1}/S_t)$ is conditionally log-normal with parameters $(\mu + \sigma_{m,r}, \sigma^2)$, (ii) applying Lemma V.2.1 to $E[\max(e^{\sigma_{m,r}}(S_{t+1}/S_t) - K/S_t, 0)|\mathcal{I}_t]$, and (iii) exploiting the parameter constraints stated in Lemma V.4.1. We have the following result.

Theorem V.4.1 *Suppose that the joint conditional distribution of the log-return and the log-SDF is given by (V.15), and suppose that the no-arbitrage conditions in (V.4)-(V.5) hold. Then the price of the call option expiring at time $t + 1$ is*

$$\begin{aligned} P_t^{Call}(t+1, K) &= E[m_{t,t+1} \max(S_{t+1} - K, 0)|\mathcal{I}_t] \\ &= S_t \Phi(-u_{t,t+1} + \sigma) - e^{-r} K \Phi(-u_{t.t+1}), \end{aligned}$$

where $u_{t,t+1}$ is defined in (V.13). That is, the price is equivalent to the one stated in Theorem V.3.1.

We refer to the appendix for a proof. Note that the pricing formula is identical to the one in Theorem V.3.1 where the SDF was assumed constant. The reason is that, under (conditional) log-normality of the SDF, the formula in Theorem V.4.1 only depends on the volatility parameter σ appearing in the dynamics equation (V.15). In particular, we note that a constant SDF is applicable with the dynamics in (V.15) by setting $\sigma_{m,r} = \sigma_m^2 = 0$ and $\mu_m = e^{-r}$, such that the log-SDF has a degenerate normal distribution with unit mass at e^{-r}. Hence, under conditionally Gaussian returns, it does not matter if the SDF is constant or log-normal when determining the price of an option.

Similar to Theorem V.3.2, we may obtain a pricing formula for the option expiring at an arbitrary time point $T \geq t + 1$. We make the following assumption about the $(T - t)$-period conditional distribution of the log-returns and the log-SDF:

$$\begin{pmatrix} \log \frac{S_T}{S_t} \\ \log m_{t,T} \end{pmatrix} |\mathcal{I}_t \sim N_2 \left((T-t) \begin{pmatrix} \mu_r \\ \mu_m \end{pmatrix}, (T-t) \begin{pmatrix} \sigma^2 & \sigma_{m,r} \\ \sigma_{m,r} & \sigma_m^2 \end{pmatrix} \right). \tag{V.20}$$

The dynamics in (V.20) hold if the joint process $\{(\log(S_{i+1}/S_i), \log m_{i,i+1})\}_{i=t}^{T-1}$ is (conditional on $\mathcal{I}_t$) i.i.d. bivariate normal with mean and covariance matrix given in (V.15). We emphasize that more advanced dynamics could be allowed, which we will consider in more detail in the next section. The assumption in (V.20) together with the no arbitrage condition in (V.7) yield the famous Black-Scholes-Merton formula stated in the following theorem. Its proof is similar to the one of Theorem V.4.1 and is left as an exercise.

Theorem V.4.2 *(Black-Scholes-Merton formula) Suppose that the $(T-t)$ log-return and log-SDF satisfy (V.20). Moreover, assume that the no arbitrage condition in (V.7) holds. Then the price of the call option expiring at time T is*

$$\begin{aligned} P_t^{Call}(T,K) &= E[m_{t,T}\max(S_T-K,0)|\mathcal{I}_t] \\ &= S_t\Phi\left(-u_{t,T}+\sigma\sqrt{T-t}\right) - \exp(-r(T-t))K\Phi\left(-u_{t,T}\right), \end{aligned}$$

where $u_{t,T}$ is defined in (V.13).

Remark V.4.1 *An equivalent way of stating the price, often found in textbooks, is that*

$$P_t^{Call}(T,K) = S_t\Phi(d_1) - \exp(-r(T-t))K\Phi(d_2),$$

where

$$d_1 = \frac{\log(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

V.5 Pricing based on GARCH

In this section we consider option pricing when allowing for time-varying "parameters", such as time-varying conditional volatility. A natural extension of (V.15) is to allow the conditional mean and covariance matrix to be time dependent, i.e. we may assume that

$$\begin{pmatrix}\log(S_{t+1}/S_t)\\ \log m_{t,t+1}\end{pmatrix}|\mathcal{I}_t \sim N_2\left(\begin{pmatrix}\mu_{r,t+1}\\ \mu_{m,t+1}\end{pmatrix}, \begin{pmatrix}\sigma_{t+1}^2 & \sigma_{m,r,t+1}\\ \sigma_{m,r,t+1} & \sigma_{m,t+1}^2\end{pmatrix}\right), \qquad \text{(V.21)}$$

where $\sigma_{m,r,t+1}$ denotes the (potentially time-dependent) conditional covariance between $\log m_{t,t+1}$ and $\log(S_{t+1}/S_t)$. Note that the conditional distribution in (V.21) is consistent with the assumption that

$$\begin{pmatrix}\log(S_{t+1}/S_t)\\ \log m_{t,t+1}\end{pmatrix} = \begin{pmatrix}\mu_{r,t+1}\\ \mu_{m,t+1}\end{pmatrix} + \varepsilon_{t+1}, \quad \varepsilon_{t+1} = \Omega_{t+1}^{1/2} z_{t+1}, \qquad \text{(V.22)}$$

where the process $\{z_{t+1}\}$ is *i.i.d.* with a bivariate standard normal distribution, i.e.

$$z_{t+1} \sim N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right),$$

and $\Omega_{t+1}^{1/2}$ is the (positive definite) square-root of

$$\Omega_{t+1} = \begin{pmatrix}\sigma_{t+1}^2 & \sigma_{m,r,t+1}\\ \sigma_{m,r,t+1} & \sigma_{m,t+1}^2\end{pmatrix}.$$

By definition, Ω_{t+1} is a function of $\mathcal{I}_t$. In particular, this allows for the log-returns to be driven by a (G)ARCH process, e.g. $\sigma_{t+1}^2 = \omega + \alpha(x_t - \mu_{r,t+1})^2$ with $x_t := \log(S_{t+1}/S_t)$. Note that under the no arbitrage conditions in (V.4)-(V.5), the entries of the conditional mean and covariance matrix in (V.21) are subject to constraints similar to the ones given in Lemma V.4.1 above. By arguments similar to the ones that we used to prove Theorem V.4.1, we have the following result.

Theorem V.5.1 *Suppose that the log-returns and the log-SDF are conditionally jointly normal according to (V.21). Under the no arbitrage condition (V.4)-(V.5), the price of a call option expiring at time* $t+1$ *is given by*

$$P_t^{Call}(t+1, K) = S_t\Phi\left(-u_t + \sigma_{t+1}\right) - e^{-r}K\Phi\left(-u_t\right),$$

where

$$u_t := \frac{\log(Ke^{-(r-\sigma_{t+1}^2/2)}) - \log(S_t)}{\sigma_{t+1}}. \tag{V.23}$$

As mentioned, the above pricing formula for an option that expires at time $t+1$ is compatible with log-returns following a GARCH process, such as $\log(S_{t+1}/S_t)|\mathcal{I}_t \sim N(0, \sigma_{t+1}^2)$. In particular, if $\log(S_{t+1}/S_t)$ follows a GARCH process, the option price can be obtained by estimating the GARCH parameters and plugging the estimated conditional variance, $\hat{\sigma}_{t+1}^2$, into the pricing formula. Similar to the risk measures, discussed in the previous chapter, one may choose to report error bands of the estimated option price, taking into account the estimation uncertainty of the model parameters.

V.6 Simulation-based methods

As we already noted in the previous chapter, if $\log(S_{t+1}/S_t)|\mathcal{I}_t \sim N(0, \sigma_{t+1}^2)$ it will (in general) not be the case that $\log(S_{t+h}/S_t)$ is conditionally Gaussian for $h > 1$ (see also Drost and Nijman, 1993, for additional details). Hence, it is not obvious how to obtain closed-form GARCH-based prices for options expiring more than one day ahead. Instead, one may estimate the prices by means of simulation. We here consider a special case of a recent approach by Zhu and Ling (2015). Specifically, with $x_t := \log(S_t/S_{t-1})$ the log-return at time t, Zhu and Ling (2015) consider an SDF given by

$$m_{t,t+1} = \frac{\exp\left(\theta_{t+1}x_{t+1}\right)}{E[\exp\{(1+\theta_{t+1})x_{t+1}\}|\mathcal{I}_t]},$$

for some $\theta_{t+1} \in \mathcal{I}_t$ that depends on the conditional distribution of x_t and that is subject to constraints imposed by the no-arbitrage conditions in (V.4)-(V.5).[3] For instance, suppose that x_t follows a GARCH(1,1)-type process,

$$x_t = \mu_t + \sigma_t z_t, \quad z_t \sim i.i.d.N(0,1), \tag{V.24}$$

$$\sigma_t^2 = \omega + \alpha(x_{t-1} - \mu_{t-1})^2 + \beta\sigma_{t-1}^2, \tag{V.25}$$

where $\mu_t \in \mathcal{I}_{t-1}$ is the (known) conditional mean of x_t. Using that z_t is Gaussian and hence that $\exp(x_{t+1})$ is conditionally log-normal with parameters $(\mu_{t+1}, \sigma_{t+1}^2)$, we have that in light of (V.9),

$$E[\exp\{(1+\theta_{t+1})x_{t+1}\}|\mathcal{I}_t] = \exp\left\{(1+\theta_{t+1})\mu_{t+1} + \frac{(1+\theta_{t+1})^2\sigma_{t+1}^2}{2}\right\}.$$

This, combined with the no-arbitrage condition in (V.5), implies that

$$m_{t,t+1} = \frac{\exp\left(\theta_{t+1}x_{t+1}\right)}{\exp\left\{(1+\theta_{t+1})\mu_{t+1} + \frac{(1+\theta_{t+1})^2\sigma_{t+1}^2}{2}\right\}} \quad \text{with } \theta_{t+1} = \frac{r-\mu_{t+1}}{\sigma_{t+1}^2} - \frac{1}{2}. \tag{V.26}$$

We refer to Zhu and Ling (2015) for many more details and derivations. Recall that $m_{t,T} = \prod_{i=t+1}^{T} m_{i-1,i}$ and that

$$P_t^{\text{Call}}(T,K) = E\left[m_{t,T}\max(S_T - K, 0)|\mathcal{I}_t\right]. \tag{V.27}$$

One may estimate $P_t^{\text{Call}}(T,K)$ by the following algorithm.

Algorithm V.6.1 *Suppose that the returns on the stock follows the GARCH(1,1) process in (V.24)-(V.25), and that the SDF is given by (V.26). Moreover assume that $\mu_{t+1} = g(x_t, \mu_t, \sigma_t^2)$ for some known function $g : \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$.*[4]

[3] Note that this choice of SDF is mathematically tractable. As such, this SDF does not contain any deep financial-economic insight. In particular, note that the no-arbitrage condition in (V.4) is always satisfied for this choice of SDF, since the condition is equivalent to $E[m_{t,t+1}\exp\{x_{t+1}\}|\mathcal{I}_t] = 1$.

[4] Note that this choice of function $g(\cdot)$ is quite flexible, in the sense that is allows for GARCH-in-mean ($\mu_{t+1} = \gamma\sigma_{t+1}^2 = \gamma(\omega+\alpha(x_t-\mu_t)^2+\beta\sigma_t^2)$), auto-regressive ($\mu_{t+1} = \phi x_t$), and moving average ($\mu_{t+1} = \varphi(x_t - \mu_t)$) terms, or a combination hereof. Obviously, one could also allow for additional lags of the quantities (x_t, μ_t, σ_t^2).

1. *For $i = 1, \ldots, M$ (for some large M) draw $z_{t+1}^{(i)}, \ldots, z_T^{(i)}$ independently from $N(0,1)$, and compute for $j = t+1, \ldots, T$,*

$$\begin{aligned}
\mu_j^{(i)} &= g(x_{j-1}^{(i)}, \mu_{j-1}^{(i)}, \sigma_{j-1}^{(i)2}), \\
\sigma_j^{(i)2} &= \omega + \alpha(x_{j-1}^{(i)} - \mu_{j-1}^{(i)}) + \beta\sigma_{j-1}^{(i)2}, \\
x_j^{(i)} &= \mu_j^{(i)} + z_j^{(i)}\sqrt{\sigma_j^{2(i)}},
\end{aligned}$$

where $\mu_t^{(i)} = \mu_t \in \mathcal{I}_t$, $\sigma_t^{(i)2} = \sigma_t^2 \in \mathcal{I}_t$ and $x_t^{(i)} = x_t \in \mathcal{I}_t$. Moreover, compute

$$\xi_T^{(i)} := m_{t,T}^{(i)} \max(S_T^{(i)} - K, 0),$$

where

$$m_{t,T}^{(i)} = \prod_{j=t+1}^{T} m_{j-1,j}^{(i)},$$

with, for $j = t+1, \ldots, T$,

$$m_{j-1,j}^{(i)} := \frac{\exp(\theta_j^{(i)} x_j^{(i)})}{\exp\left[(1+\theta_j^{(i)})\mu_j^{(i)} + (1+\theta_j^{(i)})^2\sigma_j^{(i)2}/2\right]}, \quad \theta_j^{(i)} := \frac{r - \mu_j^{(i)}}{\sigma_j^{(i)2}} - \frac{1}{2},$$

and

$$S_T^{(i)} = S_t \exp(\sum_{j=t+1}^{T} x_j^{(i)}).$$

2. *The price, $P_t^{Call}(T, K)$, in (V.27) may be approximated by*

$$P_t^{Call,sim}(T, K) = \frac{1}{M}\sum_{i=1}^{M} \xi_T^{(i)}.$$

V.7 Data on option prices

Table 1 contains prices of European call options written on the S&P 500 Index (SPX) for different strikes and maturities.

The data are retrieved from Bloomberg. The options are traded at the Chicago Board Options Exchange (CBOE), and we refer to the CBOE webpage for additional details about the contracts. The table contains mid prices obtained after the closing of the exchange on September 17th 2015. The first row of the table contains the expiration dates of the options, whereas the second row states the number of trading days between September 17th, 2015

Table 1: S&P 500 Call Option Prices, September 17, 2015

Strike	18-09-2015 1	16-10-2015 21	20-11-2015 45	19-12-2015 64	15-01-2016 82	18-03-2016 125
1600			384.2	385.4	389.05	392.25
1650		335.2	336.3	338.5	370.5	347.9
1700	285.7	286.2	289.05	292.5	297.8	304.65
1750	236.45	237.8	242.9	247.6	253.8	266.78
1800	186.45	190.5	198.2	209	211.5	222.55
1850	136.7	144.95	155.5	162.95	171.3	184.25
1900	86.85	102.1	115.5	124.3	139.25	165
1950	37	63.45	79.05	90.6	104.35	114.85
2000	2.45	31.3	47.65	57	67.47	90.8
2050	0.08	10	23.5	33	45.3	63.75
2100	0.05	1.6	8.6	14.85	24	43.5
2150		0.3	1.85	5.79	10.1	24.1
2200		0.1	0.6	2.15	3.4	12.5
2250			0.37	0.6	0.9	5.75
2300			0.1	0.15	0.5	2.1
2350				0.1		1.35
2400						0.7

and the expiration days. It should be mentioned that the liquidity (measured in terms of the trading volume) is low for some of the contracts, meaning that the quoted prices may not be that precise. In particular, this is the case for certain contracts with long maturity and for "deep out of the money contracts" (i.e. contracts with K much greater than S_t) with short maturity. The closing price (S_t) of the SPX Index was 1990.20 on September 17th 2015.

Example V.7.1 *Consider the European call option with strike* $K = 1950$ *and expiration date on October 16, 2015. Using a sample from January 4, 2010 - September 17, 2015 of S&P 500 log-returns, we get an estimate* $\hat{\sigma} = 0.010050$ *based on the sample variance of the returns. With the risk-free rate* $r = 0.003/251$ *and* $T - t = 21$ *trading days to expiration, we get an option price of 60.11, based on the standard Black-Scholes-Merton pricing formula in Theorem V.4.2.*
Estimating a GARCH(1,1) model as in (V.24)-(V.25) with constant $\mu_t \equiv \gamma \in \mathbb{R}\ \forall t$ *based on the same data series, we get* $(\hat{\omega}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) = (0.041367 \times 10^{-4}, 0.14645, 0.81185, 0.072782 \times 10^{-2})$. *Using Algorithm V.6.1 (with* $B = 10^6$*) we obtain an option price of 62.47, which is closer to the actual market price of 63.45.*
Considering all options expiring on October 16, 2015 with strikes ranging from 1650-2050, we find that the Black-Scholes-Merton formula yields an av-

erage absolute relative pricing error[5] *of 7.3%, whereas the GARCH-approach yields a relative error of 9.5%.*

V.8 Additional literature

The pricing of options based on GARCH models dates back to the seminal work by Duan (1995), see also Taylor (2005, Chapter 14). Typically, the GARCH-based pricing methods rely on choosing a suitable SDF in order to find the dynamics of the log-returns under the equivalent martingale measure Q, see Appendix 1. Under certain conditions, the Q-dynamics are also GARCH-type, which allows for simulating the log-returns under Q, so that

$$P_t^{\text{Call}}(T,K) = E^Q[e^{-r(T-t)}\max(S_T-K,0)|\mathcal{I}_t]$$

can be computed by simulations.

The literature on GARCH-based option pricing is huge and typically focus on various extensions of the GARCH model, such as incorporating skewness in the conditional return distributions (Christoffersen et al., 2006), jumps in underlying stock price (Christoffersen et al., 2008), and incorporating realized volatility (based on high frequency data) in the GARCH model (Christoffersen et al., 2014). A recent book on GARCH-based pricing is written by Chorro et al. (2015).

A rigorous, technical treatment of discrete-time finance is given in Föllmer and Schied (2011), see also Back (2017). For a derivation of the Black-Scholes formula in continuous time we refer to Taylor (2005) and Björk (2009).

[5] With $\{P_t^{\text{mkt}}(T,K_i) : i=1,\dots,N\}$ the actual market prices of options with strikes $K_1,\dots,K_N$ and $\{P_t^{\text{model}}(T,K_i) : i=1,\dots,N\}$ the model-based prices, the average absolute relative pricing error of the model is defined as $AARPE^{\text{model}} = N^{-1}\sum_{i=1}^N |(P_t^{\text{mkt}}(T,K_i)-P_t^{\text{model}}(T,K_i))/P_t^{\text{mkt}}(T,K_i)|$

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Appendix 1: On P and Q

In light of the assumption about the joint conditional distribution of $\lambda_{t,t+1} := \log(m_{t,t+1})$ and $x_{t+1} := \log(S_{t+1}/S_t)$ in (V.21)-(V.22), consider their joint pdf $f_P(\lambda, x)$ conditional on $\mathcal{I}_t$, where P indicates that $\lambda_{t,t+1}$ and x_{t+1} are distributed according to the distribution P. In order to ease the notation we here suppress that the conditional distribution depends on t. Then by definition of the SDF

$$\begin{aligned}
1 &= E\left[\exp(\lambda_{t,t+1})\exp(x_{t+1})|\mathcal{I}_t\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(\lambda)\exp(x) f_P(\lambda, x) d\lambda dx \\
&= \int_{-\infty}^{\infty} e^{-r}\exp(x)\left(e^r \int_{-\infty}^{\infty} \exp(\lambda) f_P(\lambda, x) d\lambda\right) dx \\
&= \int_{-\infty}^{\infty} e^{-r}\exp(x) f_Q(x) dx
\end{aligned}$$

where $f_Q(x) := \left(e^r \int_{-\infty}^{\infty} \exp(\lambda) f_P(\lambda, x) d\lambda\right)$. Notice that $f_Q(x) > 0$ for all x and

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Q(x)dx = e^r \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(\lambda) f_P(\lambda, x) d\lambda dx &= e^r E[\exp(\lambda_{t,t+1})|\mathcal{I}_t] \\
&= e^r E[m_{t,t+1}|\mathcal{I}_t] = 1.
\end{aligned}$$

Hence the function $f_Q(x)$ can be viewed as a pdf of x_{t+1} under some probability measure Q such that

$$\int_{-\infty}^{\infty} e^{-r} x f_Q(x) dx = E^Q[e^{-r} x_{t+1}|\mathcal{I}_t].$$

We have that

$$E^Q\left[e^{-r}\frac{S_{t+1}}{S_t}|\mathcal{I}_t\right] = 1 \quad \Leftrightarrow \quad E^Q[e^{-r(t+1)} S_{t+1}|\mathcal{I}_t] = e^{-rt} S_t,$$

which means that the discounted stock price $e^{-rt}S_t$ is a martingale with respect to $\mathcal{I}_t$ under the measure Q. Due to this property Q is typically referred to as the "equivalent martingale measure" (or risk-neutral measure). In a vast proportion of the financial literature people choose to work with this measure instead of the SDF.
Moreover, notice that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(\lambda)\exp(x) f_P(\lambda, x) d\lambda dx \\
&= \int_{-\infty}^{\infty} \exp(x) \left(\int_{-\infty}^{\infty} \exp(\lambda) f_P(\lambda|x) d\lambda\right) f_P(x) dx \\
&= \int_{-\infty}^{\infty} \exp(x) m(x) f_P(x) dx,
\end{aligned}$$

where $m(x) := \left(\int_{-\infty}^{\infty} \exp(\lambda) f_P(\lambda|x) d\lambda\right)$, and in light of the derivations above,

$$m(x) = e^{-r}\frac{f_Q(x)}{f_P(x)}.$$

Hence we can write the SDF as

$$m_{t,t+1} = m(x_{t+1}) = e^{-r}\frac{f_Q(x_{t+1})}{f_P(x_{t+1})}.$$

It should be mentioned that in discrete time (as considered here) the stochastic discount factor as well as the equivalent martingale measure are typically not unique, meaning that there exist several prices of the European call option compatible with absence of arbitrage (technically: markets are incomplete). Hence, given the collection of all stochastic discount factors, one could compute an interval of prices spanning the no-arbitrage bounds. A more standard way to proceed is, as here, to pick a single measure Q or SDF compatible with absence of arbitrage.

Appendix 2: Technical proofs

V.8.1 Proof of Lemma V.2.1

Note that $X \stackrel{d}{=} s\exp(\varepsilon\sigma)$, where $s := \exp(\mu)$ and $\varepsilon \stackrel{d}{=} N(0,1)$. Hence,

$$\begin{aligned}
&E[\max(X-K,0)] = E[\max\{s\exp(\varepsilon\sigma) - K, 0\}] \\
&= sE[\max\{\exp(\varepsilon\sigma) - K/s, 0\}] = s\int_{-\infty}^{\infty} \max\{\exp(u\sigma) - K/s, 0\}\phi(u) du,
\end{aligned}$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution. Next, by straightforward integration,

$$
\begin{aligned}
& s \int_{-\infty}^{\infty} \max\{\exp(u\sigma) - K/s, 0\}\phi(u)du \\
&= s \int_{\log(K/s)/\sigma}^{\infty} [\exp(\sigma u) - K/s]\phi(u)du \\
&= s \left[\int_{\log(K/s)/\sigma}^{\infty} \exp(\sigma u)\phi(u)du - \int_{\log(K/s)/\sigma}^{\infty} (K/s)\phi(u)du \right] \\
&= s \left[\int_{\log(K/s)/\sigma}^{\infty} \exp(\sigma^2/2)\phi(u-\sigma)du - \int_{\log(K/s)/\sigma}^{\infty} (K/s)\phi(u)du \right] \\
&= s \left[\exp(\sigma^2/2) \int_{\log(K/s)/\sigma-\sigma}^{\infty} \phi(u)du - (K/s) \int_{\log(K/s)/\sigma}^{\infty} \phi(u)du \right] \\
&= s\exp(\sigma^2/2) \left(\int_{-\infty}^{\infty} \phi(u)du - \int_{-\infty}^{\log(K/s)/\sigma-\sigma} \phi(u)du \right) \\
&\qquad - s(K/s) \left(\int_{-\infty}^{\infty} \phi(u)du - \int_{-\infty}^{\log(K/s)/\sigma} \phi(u)du \right) \\
&= s \left[\exp(\sigma^2/2)(1 - \Phi(\log(K/s)/\sigma - \sigma)) - (K/s)(1 - \Phi(\log(K/s)/\sigma)) \right] \\
&= s \left[\exp(\sigma^2/2)\Phi(-\log(K/s)/\sigma + \sigma) - (K/s)\Phi(-\log(K/s)/\sigma) \right]
\end{aligned}
$$

where we have used that $\exp(\sigma u)\phi(u) = \exp(\sigma^2/2)\phi(u-\sigma)$ and $\Phi(x) = 1 - \Phi(-x)$ for $x \in \mathbb{R}$. The result now follows by collecting terms and using that, by definition, $s = \exp(\mu)$, and that $E[X] = \exp(\mu + \sigma^2/2)$. $\square$

V.8.2 Proof of Lemma V.2.2

By the law of iterated expectations,

$$
\begin{aligned}
E\left[\exp(X)g(Y)\right] = E\left[E[\exp(X)g(Y)|Y]\right] &= E\left[g(Y)E[\exp(X)|Y]\right] \\
&= E\left[g(Y)\exp\left(\mu_{X|Y} + \frac{1}{2}\sigma^2_{X|Y}\right)\right],
\end{aligned}
$$

where the last equality follows by Lemma V.8.1 below, and with $\mu_{X|Y} = \mu_X + \omega(Y - \mu_Y)$, $\omega := \sigma_{XY}/\sigma_Y^2$, and $\sigma^2_{X|Y} := \sigma^2_X - \omega\sigma_{XY}$. Next, we note that

$$
\begin{aligned}
&E\left[g(Y)\exp\left(\mu_{X|Y}+\frac{1}{2}\sigma^2_{X|Y}\right)\right]\\
&=\int_{-\infty}^{\infty} g(y)\exp\left(\mu_{X|y}+\frac{1}{2}\sigma^2_{X|Y}\right) f(y)dy\\
&=\int_{-\infty}^{\infty} g(y)\exp\left(\mu_X-\omega\mu_Y+\omega y+\frac{1}{2}\sigma^2_{X|Y}\right) f(y)dy\\
&=\exp\left(\frac{1}{2}\sigma^2_{X|Y}+\mu_X-\omega\mu_Y\right)\int_{-\infty}^{\infty} g(y)\exp\left(\omega y\right) f(y)dy\\
&=\exp\left(\frac{1}{2}\sigma^2_{X|Y}+\mu_X-\omega\mu_Y\right)\int_{-\infty}^{\infty} g(y)\frac{1}{\sqrt{2\pi\sigma^2_Y}}\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma^2_Y}+\omega y\right) dy.
\end{aligned}
$$

Straightforward derivations yield that

$$
\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma^2_Y}+\omega y\right)=\exp\left(-\frac{(y-\mu_Y-\sigma_{XY})^2}{2\sigma^2_Y}\right)\exp\left(\frac{\sigma^2_{XY}+2\mu_Y\sigma_{XY}}{2\sigma^2_Y}\right),
$$

which implies that

$$
\begin{aligned}
E\left[g(Y)\exp\left(\mu_{X|Y}+\frac{1}{2}\sigma^2_{X|Y}\right)\right]&=\exp\left(\frac{1}{2}\sigma^2_{X|Y}+\mu_X-\omega\mu_Y+\frac{\sigma^2_{XY}+2\mu_Y\sigma_{XY}}{2\sigma^2_Y}\right)\\
&\quad\times\int_{-\infty}^{\infty} g(y)\frac{1}{\sqrt{2\pi\sigma^2_Y}}\exp\left(-\frac{(y-\mu_Y-\sigma_{XY})^2}{2\sigma^2_Y}\right) dy\\
&=\exp\left(\frac{1}{2}\sigma^2_X+\mu_X\right)\\
&\quad\times\int_{-\infty}^{\infty} g(y)\frac{1}{\sqrt{2\pi\sigma^2_Y}}\exp\left(-\frac{(y-\mu_Y-\sigma_{XY})^2}{2\sigma^2_Y}\right) dy\\
&=E[\exp\left(X\right)]E\left[g(Y+\sigma_{XY})\right],
\end{aligned}
$$

where the last equality holds by observing that

$$
\frac{1}{\sqrt{2\pi\sigma^2_Y}}\exp\left(-\frac{(y-\mu_Y-\sigma_{XY})^2}{2\sigma^2_Y}\right)
$$

is the pdf of $Y+\sigma_{XY}\sim\mathrm{N}(\mu_Y+\sigma_{XY},\sigma^2_Y)$, and that $E[\exp\left(X\right)]=\exp\left(\frac{1}{2}\sigma^2_X+\mu_X\right)$ by (V.9). $\square$

Lemma V.8.1 *Let $(X,Y)'$ be bivariate $N_2(\mu,\Omega)$-distributed with mean $\mu=E\,(X,Y)'$ and covariance matrix $\Omega=Var\,(X,Y)'$ given by,*

$$
\mu=\begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix},\ \Omega=\begin{pmatrix}\sigma^2_X & \sigma_{XY}\\ \sigma_{YX} & \sigma^2_Y\end{pmatrix}.
$$

Then

$$E[\exp(X)|Y=y] = \exp(\mu_{X|y} + \frac{1}{2}\sigma^2_{X|Y}),$$

where $\mu_{X|y} = \mu_X + \omega(y - \mu_Y)$, $\omega := \sigma_{XY}/\sigma^2_Y$, *and* $\sigma^2_{X|Y} := \sigma^2_X - \omega\sigma_{XY}$.

Proof: Recall from Chapter I that the conditional distribution of X given $Y = y$ is $N(\mu_{X|y}, \sigma^2_{X|Y})$. We conclude that $\exp(X)$ is conditionally (on $Y = y$) log-normal with parameters $(\mu_{X|y}, \sigma^2_{X|Y})$, and hence using a conditional version of (V.9) for $k = 1$, we have that $E[\exp(X)|Y = y] = \exp(\mu_{X|y} + \frac{1}{2}\sigma^2_{X|Y})$. $\square$

V.8.3 Proof of Theorem V.3.1

We have that

$$P_t^{\text{Call}}(t+1, K) = S_t e^{-r} E[\max(S_{t+1}/S_t - K/S_t, 0)|\mathcal{I}_t]. \qquad \text{(V.28)}$$

Note that (S_{t+1}/S_t) is conditionally log-normal with mean e^r under the no-arbitrage condition (V.12). A straightforward application of Lemma V.2.1 yields that

$$\begin{aligned} E[\max(S_{t+1}/S_t - K/S_t, 0)|\mathcal{I}_t] &= E[S_{t+1}/S_t|\mathcal{I}_t]\Phi(-u_{t,t+1} + \sigma) - \frac{K}{S_t}\Phi(-u_{t,t+1}), \\ &= e^r\Phi(-u_{t,t+1} + \sigma) - \frac{K}{S_t}\Phi(-u_{t,t+1}), \qquad \text{(V.29)} \end{aligned}$$

where $u_{t,t+1}$ is defined in (V.13). The result follows by combining (V.29) and (V.28). $\square$

V.8.4 Proof of Lemma V.4.1

Equation (V.17) is immediate. Turning to proving (V.18), recall that under no arbitrage we have that (V.16) holds. Using that (V.15) and applying Lemma V.2.2, we have that

$$\begin{aligned} & E\left[\exp\left\{\log m_{t,t+1}\right\} \times \exp\left\{\log(S_{t+1}/S_t)\right\}|\mathcal{I}_t\right] \\ &= E\left[\exp\left\{\log m_{t,t+1}\right\}|\mathcal{I}_t\right] E[\exp\left\{\log(S_{t+1}/S_t)\right\}|\mathcal{I}_t] \\ &= E\left[m_{t,t+1}|\mathcal{I}_t\right]\exp(\sigma_{m,r})E[S_{t+1}/S_t|\mathcal{I}_t] \\ &= e^{-r}\exp(\sigma_{m,r})\exp(\mu + \sigma^2/2), \end{aligned}$$

where we have used (V.5), the fact that S_{t+1}/S_t is conditionally log-normal with parameters (μ, σ^2), and (V.9). Substituting into (V.16) and taking logs yield that

$$-r + \sigma_{m,r} + \mu + \frac{\sigma^2}{2} = 0,$$

and we conclude that (V.18) holds. $\square$

V.8.5 Proof of Theorem V.4.1

From (V.16), we have that

$$P_t^{\text{Call}}(t+1, K) = S_t e^{-r} E[\max(e^{\sigma_{m,r}}(S_{t+1}/S_t) - K/S_t, 0)|\mathcal{I}_t].$$

Note that $e^{\sigma_{m,r}}(S_{t+1}/S_t)$ is conditionally log-normal with parameters $(\mu + \sigma_{m,r}, \sigma^2)$. By Lemma V.2.1 we have that

$$E[\max(e^{\sigma_{m,r}}(S_{t+1}/S_t) - K/S_t, 0)|\mathcal{I}_t] = E[e^{\sigma_{m,r}}(S_{t+1}/S_t)|\mathcal{I}_t]\Phi(-\tilde{u}+\sigma) - \frac{K}{S_t}\Phi(-\tilde{u}),$$

where

$$\tilde{u} := \log\left(\frac{K/S_t}{\exp(\mu + \sigma_{m,r})}\right)/\sigma$$

and

$$E[e^{\sigma_{m,r}}(S_{t+1}/S_t)|\mathcal{I}_t] = \exp(\mu + \sigma_{m,r} + \sigma^2/2).$$

By Lemma V.4.1, we have that

$$\tilde{u} = \log\left(\frac{K/S_t}{\exp(r - \sigma^2/2)}\right)/\sigma = \log\left(\frac{Ke^{-(r-\sigma^2/2)}}{S_t}\right)/\sigma = u_{t,t+1},$$

with $u_{t,t+1}$ defined in (V.13), and that

$$E[e^{\sigma_{m,r}}(S_{t+1}/S_t)|\mathcal{I}_t] = e^r.$$

Collecting terms gives that

$$P_t^{\text{Call}}(t+1, K) = S_t\Phi(-u_{t,t+1} + \sigma) - \frac{K}{e^r}\Phi(-u_{t,t+1}). \quad \square$$