1 Problem 1

1.a)

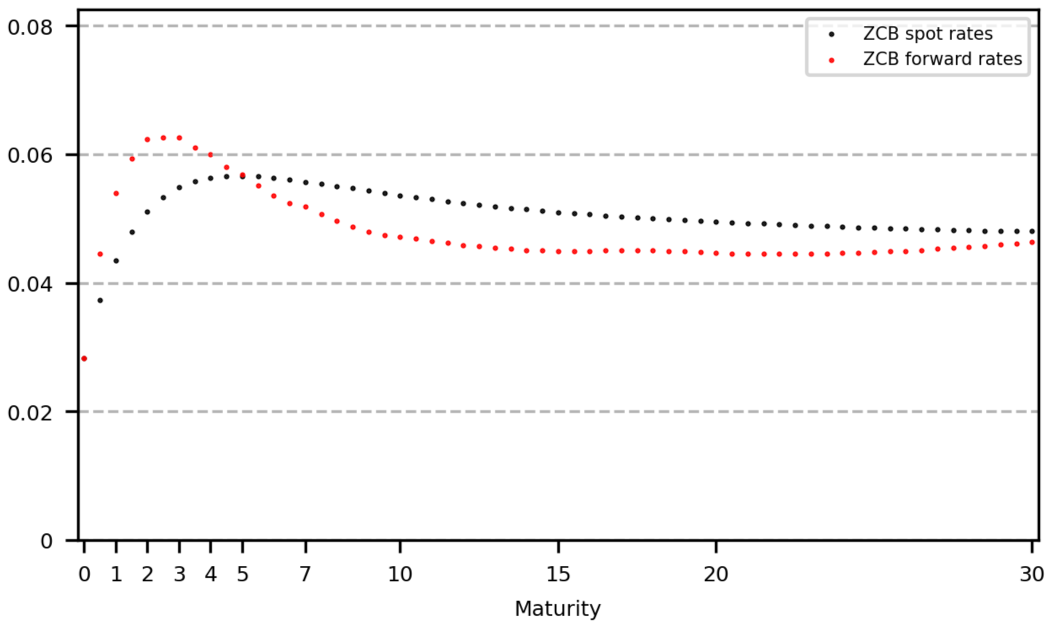

Figure 1: Calibrated spot and forward rates

In figure 1, the calibrated spot and forward rates are plotted. Moreover, the calibrated spot rates for different maturities are shown in Table 1. The sum of squared erros(SSE) are computed at 0.000367. Therefore, we have an very good fit.

Maturities	0.5	1	2	5	10	15	20	30
Spot Rates	0.0374	0.0435	0.0512	0.0567	0.0536	0.0510	0.0495	0.0480

Table 1: Spot rates for different maturities

1.b)

To guarantee consistent pricing of interest rate derivatives with closely aligned maturities, the instantaneous forward rates must be continuous. This continuity is achieved by ensuring that the prices of zero-coupon bonds (ZCBs) are differentiable. By employing Hermite interpolation for ZCB prices, we achieve a fit that is not only smooth but also differentiable, thereby satisfying the necessary conditions for forward rate continuity and reliable derivative pricing

1.c)

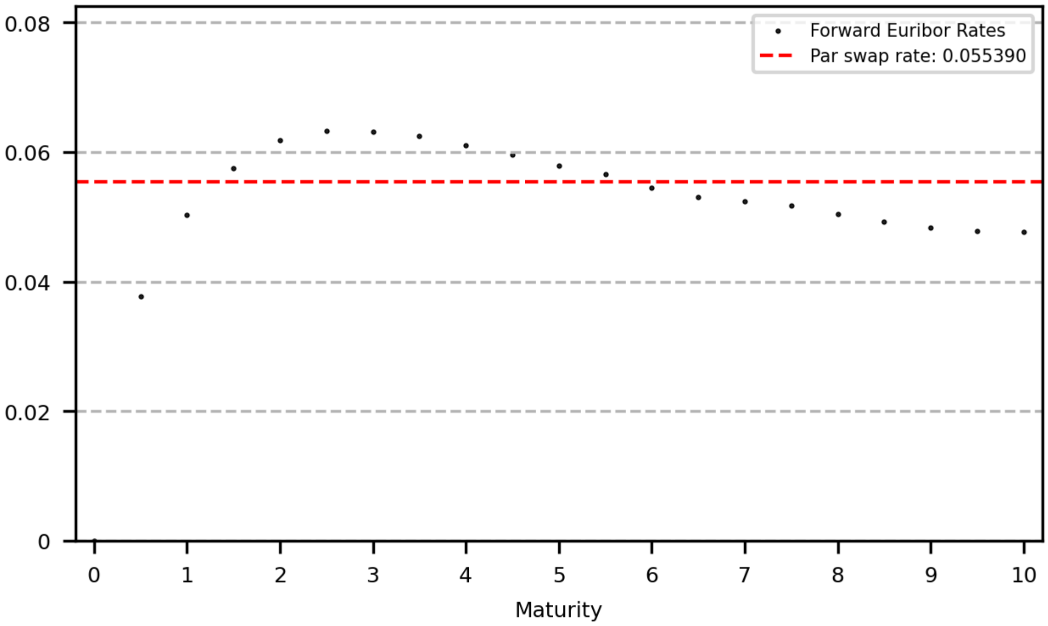

Figure 2: Forward Euribor Rates and 10Y Par Swap Rate

The 10-year par swap rate is a single fixed interest rate that balances the value of the fixed payments and the floating payments in a swap over a 10-year period. It reflects the average level of the 6-month forward EURIBOR rates, weighted by the time value of money. Essentially, it summarizes market expectations about future short-term rates into one fixed rate for a long-term agreement

2 Problem 2

2a)

Using the initial guesses given, I minimize the function using "nelder-mead" to fit the Vasicek model to the spot rates from problem 1. By optimizing with a standard error(SE) of 0.000602, I get the following values:

Estimates	$\hat{r}_0$	$\hat{a}$	$\hat{b}$
Values	0.027797	5.243720	0.271143

Table 2: Parameters for the estimated Vasicek Model

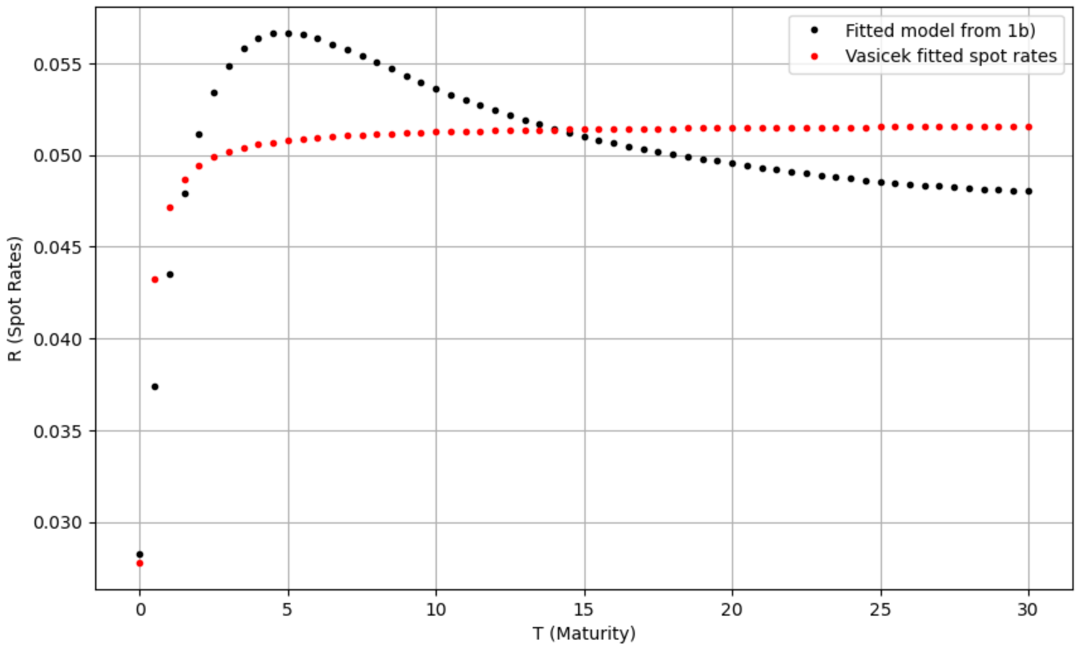

Figure 3: The fitted Vasicek Model

2.a)

By just observing the SE and the estimates, one could argue that the Vasicek model is able to fit the term structure properly. Looking at the visuals, the Vasicek model has some trouble capturing the initial bump and the afterwards sloping nature of the term structure. The Vasicek model is limited by its mean-reverting nature which makes it suitable for for smooth term structures, but it will suffer against more irregular shapes. The assumption of mean-reversion speed restricts the ability to capture non-linear or other more varying spot rates.

A model, which better suits the term structure, could be the Ho-Lee, as this offers a time-dependent drift, which provides a greater level of flexibility to match observed yield curves. Furthermore, it does not rely on mean reversion, which allows it to fit a wider variety of term structure shapes.

2.b)

I have written an objective function called ***objective_ hwev*** which I minimize using the "nelder-mead"-algorithm. This returns the SSE between the given market prices and model prices. Furthermore, the sum of squared errors are computed as 0.000000 and the estimates are provided in the Table 3 while the term structure is plotted in Figure 4 Based on these results, it seems like the HWEV is well-equipped to fit the observed caplet prices very well. To make sure the HWEV model fits the spot rates, we must recalibrate the drift term, $\Theta(t)$, to match the observed term structure. Therefore, it is necessary as

Estimates	$\hat{\sigma}$	$\hat{a}$
Values	2.230046	0.022052

Table 3: Parameters for the estimated Hull-White Extended Vasicek

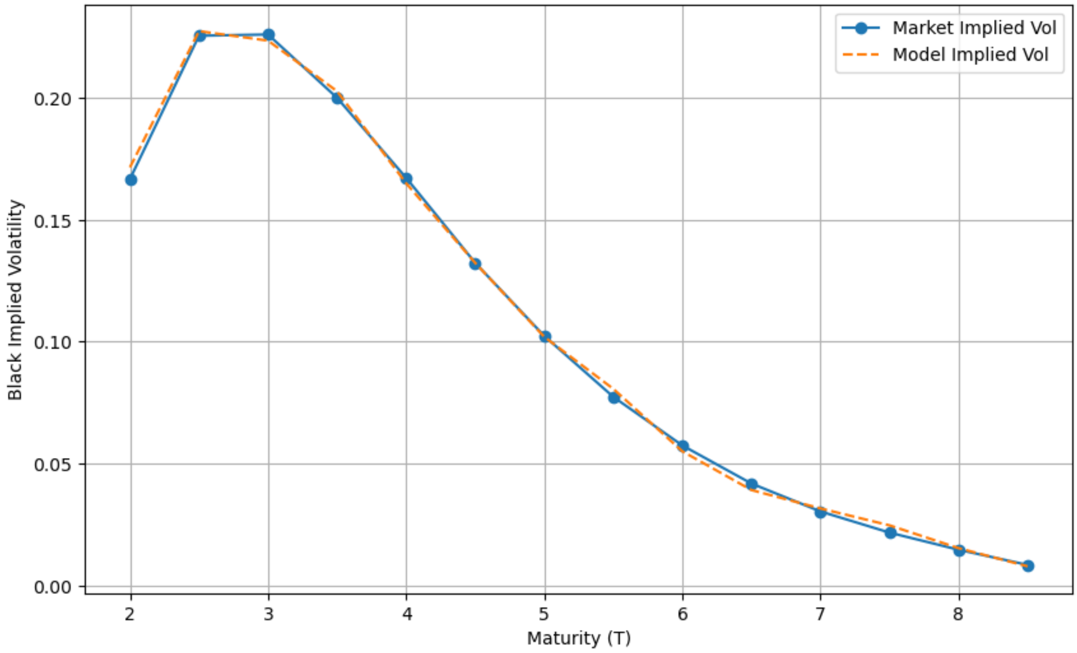

Figure 4: Caplet prices: Market and model

ZCB spot rates are the building blocks for deriving forward rates we use to price caplets. This provides consistency with the given market data, and without this, the ability of the model to price caplets accurately would suffer.

2c

The simulations for the models are provided in Figure 5 and 6. The mean-reverting pattern in the Vasicek model is clearly on display as it hovers around its long term mean $\frac{b}{a}$. We see quite rapid readjustments, which is in line with the quite high mean reverting value of a. On the other hand, the simulated HWEV model gives a much more dynamic behaviour. Especially, the mean does a much better job of capturing the term structure of the observed interest rates. Based on these finding, the HWEV model is likely the model, which will produce the more fair price, because of the dynamics of the model.

2d

In order to compute the price of the 10Y interest cap I will use the HWEV model from before, because I think this will be more likely to calculate a fair price. Therefore, I will

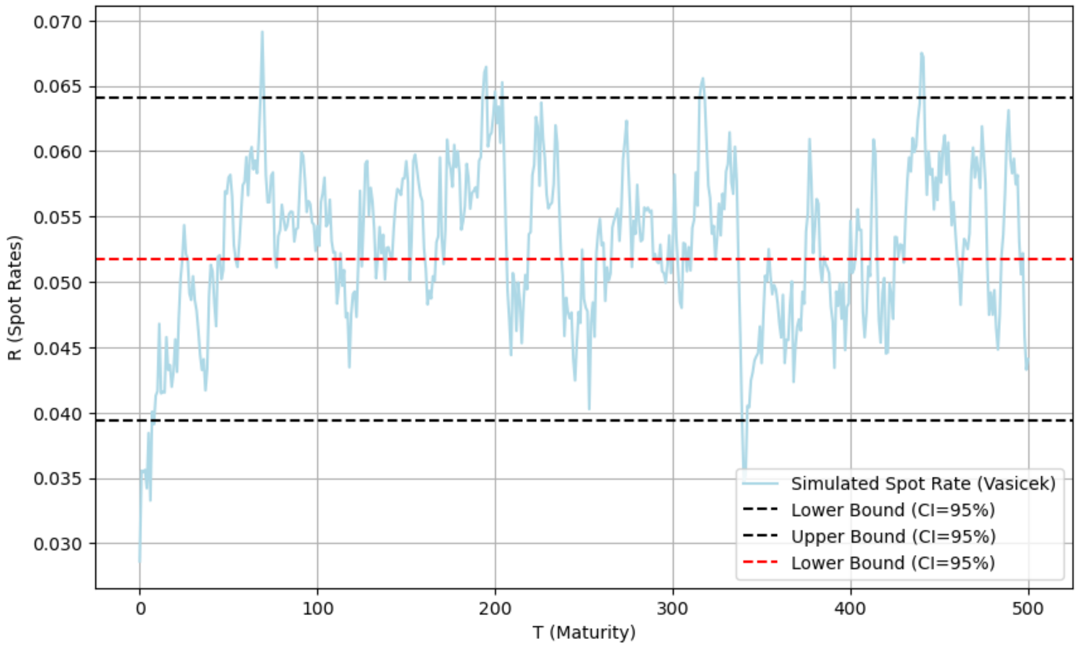

Figure 5: Simulated spot rates in the Vasicek

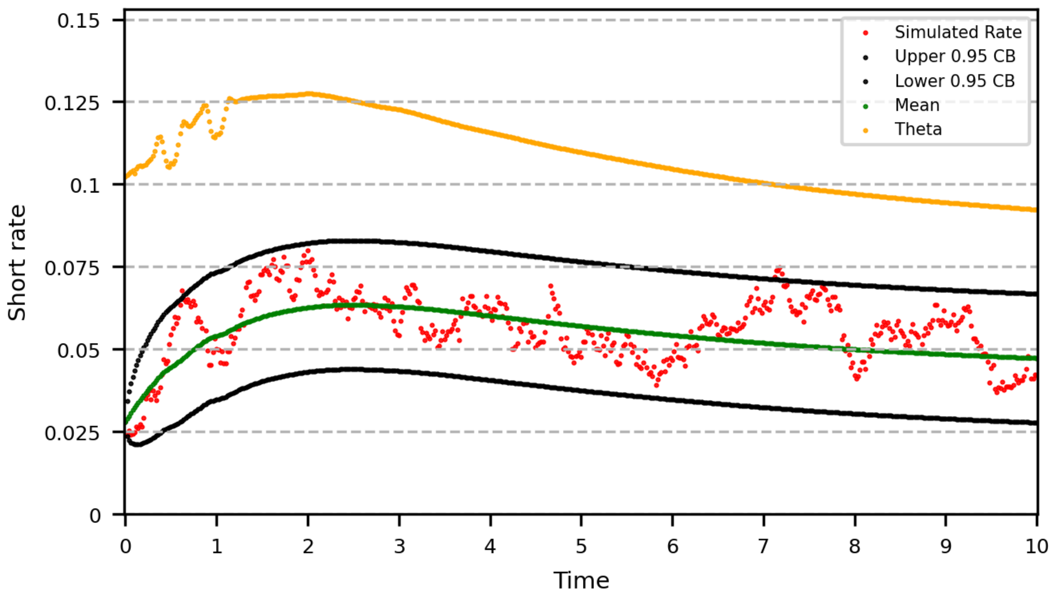

Figure 6: Simulated spot rates in the HWEV

be using the parameters I found i 2b). In the table below prices of 6M EURIBOR caplets are provided in Table 4. The price of the 10Y interest cap is simply the caplets summed up. Therefore, the price of the 10Y interest cap is given as: 120.320760 bps or as the semiannual premium priced at: 7.431333 bps

Maturities	1	2	4	6	8	10
Values	0.000070	0.001611	0.001251	0.000258	0.000064	0.000064

Table 4: Some EURIBOR Caplets price

3 Problem 3

3a)

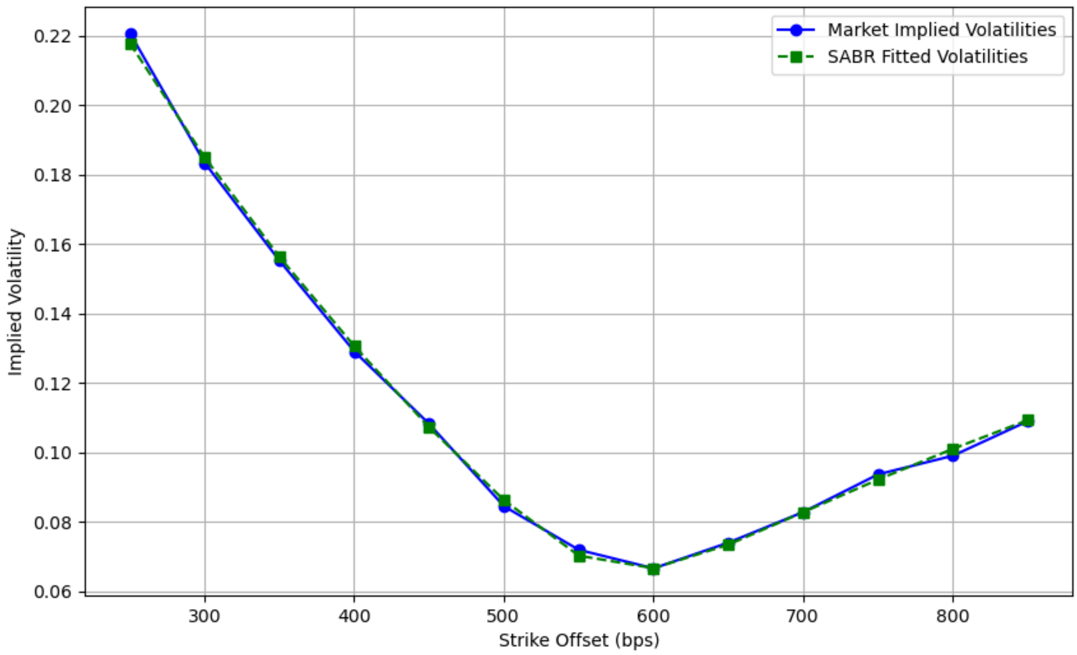

Figure 7: Implied volatilities

In Figure 7 both the market IV and the fitted IV are plotted. Clearly, the graph exhibits the infamous smirk, since the implied volatilities are higher on the left side. The smirk indicates a distribution with fatter tails, which implies higher probabilities for extreme movements in the par swap rate.

Furthermore, I calculate the 3Y7Y par swap rate. Here I find the par swap rate to be $R_{3Y7Y} = 0.5503$ with the corresponding accrual factor as $S_n = 4.7834$. I then calculate the price of the $3Y7Y$ payer swaption using black's formula with an offset of 50 bp, as this is the closest to a strike of $K = 0.06$. From this I calculate the price of the swaption to be 41.52 bps

Lastly, fitting the SABR model, I obtain a very good fit with a SSE of with an value of 0.000028 and the estimated parameters are provided as given that $\beta = 0.55$

Estimates	$\hat{\sigma}$	$\hat{\upsilon}$	$\hat{\rho}$
Values	0.02	-0.58	0.34

Table 5: Parameters for the estimated SABR-model

3c

A payer swaption is comparable to a call option as it grants the holder the right, but not the obligation, to enter into an interest rate swap as the fixed-rate payer. This allows the holder to profit when the underlying par swap rate exceeds the strike. Just like a call when the underlying exceeds the strike. In contrast, a receiver swaption benefits the holder when the underlying par swap rate falls below the strike. Just as option on stocks

I calculate the value of the strangle by Black's formula. This is done as the sum of the payer and receiver swaption for the new strikes. I then compute the Black IV's using the parameters found in fit from *b*). Based on this, I find the value of the strangle to be 47.4746 bps. The results are provided in Table 6

Metric	**Value**	**Profit/Loss**
Strangle Value	0.00474746	-
Strangle Value with F_0 up	0.00474821	+0.00000075
Strangle Value with F_0 down	0.00474784	-0.00000038
Strangle Value with σ_0 up	0.00536019	+0.00061273
Strangle Value with σ_0 down	0.00416857	-0.00057889

Table 6: Sensitivity Analysis Results for Strangle

These results show that the strangle is more sensitive to changes in the volatility, then to changes in the forward par swap rate. The direction of the exposure is positive with increasing volatility, which indicates that a higher volatility increases the value of the strangle. The changes in the par swap rate is nearly insensitive to small changes as the positions in the payer and receiver swaptions offsets each other.

4 Problem 4

a+b)

Throughout this, I have calculated the following.

- A **10Y interest rate swap** priced at the par swap rate 0.055390
  - In this case, the payments are minimized if future rates stay below the fixed rate. Therefore, the client will benefit from predictable, low fixed payments while avoiding higher floating rates. However, payments are maximized if future interest rates rise above the fixed rate. In this case, the client is locked into the fixed rate and will miss the opportunity to pay the lower floating rates at the start of the swap.
- A **10Y interest rate cap** priced at 120.320760 bps or as the semi-annual premium at 7.431333
  - In this case, the payments are minimized when the future interest rates remain below the strike rate. Therefore, the client will only incur the semi-annual premium payments. However, payments are maximized if interest rates rise above the strike price, as the client pays the capped floating rate in addition to the premium. Albeit at a higher cost, this gives some protection against rate increases.
- A **3Y7Y payer swaption** priced at 41.52 bps
  - In this case, the client does not exercise the swaption and will only incur the upfront premium. However, payments are maximized if interest rates rise above the strike. Because then the swaption will be exercised, which commits the client to the fixed rate payments for the next 7 years, which might exceed the prevailing floating rates.

c+d)

The swap requires no upfront cost, which offers predictable payments but with limited flexibility, which makes it suitable for clients wanting certainty. The cap involves premium payments, which provides flexibility by capping the maximum rates while allowing for benefits from falling rates. This is good for individuals looking to limit rate risk. The swaption has an upfront payment but offers the most flexibility by allowing the client to hedge if rates rise above the strike. This makes it great for clients who are uncertain about future rate movements. To choose the best suitable instrument depends on the objective of the client. For predictability in cash flows, the swap would be the best product, while the cap is better for a client wanting to hedge against rising rates without giving up potential savings, should the rates stay low. Lastly, the swaption is best for client wanting to be flexible. .

5 Problem 5

a)

We are given the following dynamics:

$$dX_t = -\gamma X_t dt + \phi dW_T^{(1)}$$

$$dY_t = (b - aY_t)dt + \phi dW_t^{(2)}$$

Furthermore, the short rate is given as:

$$r_t = X_T + Y_T$$

We consider the function:

$$f(t, X, Y) = e^{-\gamma(T-t)}X + e^{-\alpha(T-t)}Y$$

The SDE for X_t is:

$$dX_t - \gamma X_t dt + \phi dW_t^{(1)}$$

This is a Ornstein-Uhlenbeck process, which yields the following:

$$X_t = X_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\phi dW_s^{(1)}$$

and when $t = T$, we arrive at:

$$X_T = x_0 e^{-\gamma T} + \int_0^t e^{-\gamma(T-s)}\phi dW_s^{(1)}$$

We do the same for Y_t

$$dY_t = (b - aY_t)dt + \sigma dW_t^{(2)}$$

$$Y_t = Y_o e^{-at} + \frac{b}{a}(1 - e^{-at}) + \int_0^t e^{-a(t-s)}\sigma dW_s^{(2)}$$

$$Y_T = Y_o e^{-aT} + \frac{b}{a}(1 - e^{-aT}) + \int_0^T e^{-a(T-s)}\sigma dW_s^{(2)}$$

We then plug this into the definition of the short rate and take the mean and then we take the conditional expectations for both x_0 and y_0.

$$r_T = X_T + Y_T$$

$$r_T = x_0 e^{-\gamma T} + \int_0^t e^{-\gamma(T-s)}\phi dW_s^{(1)} s + y_o e^{-aT} + \frac{b}{a}(1-e^{-aT}) + \int_0^T e^{-a(T-s)}\sigma dW_s^{(2)}$$

$$E[r_T|x_0,y_0] = E[x_0 e^{-\gamma T} + \int_0^t e^{-\gamma(T-s)}\phi dW_s^{(1)} s + y_o e^{-aT} + \frac{b}{a}(1-e^{-aT}) + \int_0^T e^{-a(T-s)}\sigma dW_s^{(2)}|x_0,y_0]$$

$$E[r_T|x_0,y_0] = E[x_0 e^{-\gamma T} + \int_0^t e^{-\gamma(T-s)}\phi dW_s^{(1)} s|x_0] + E[y_o e^{-aT} + \frac{b}{a}(1-e^{-aT}) + \int_0^T e^{-a(T-s)}\sigma dW_s^{(2)}|y_0]$$

We then find the conditional mean for each process. These are given as:

$$E[X_t|x_0] = x_0 e^{-\gamma T}$$

$$E[Y_T|x_0] = y_0 e^{-aT} + \frac{b}{a}(1-e^{-aT})$$

We then combine again to find the mean.

$$M[T;x_0,y_0] = x_0 e^{-\gamma T} + y_0 e^{-aT} + \frac{b}{a}(1-e^{-aT})$$

b)

The variance of r_T is the sum of the variances of X_t and Y_t this is because we know that $W_t^{(1)}$ and $W_t^{(2)}$ are independent of each other. We will start with X_T

$$E[X_T^2] = E[\left(\int_0^t e^{-\gamma(t-s)}\phi dW_s^{(1)}\right)^2] = \phi^2 \int_0^T e^{-2\gamma(T-s)} ds$$

We then evaluate the integral:

$$\phi^2 \int_0^T e^{-2\gamma(T-s)} ds = \phi^2 \frac{1}{2\gamma}[1-e^{-2\gamma T}]$$

We do the same for Y_T:

$$E[Y_T] = E[\left(\int_0^T e^{-a(T-s)}\sigma dW_s^{(2)}\right)^2] = \sigma^2 \int_0^T e^{-2a(T-s)} ds$$

$$\sigma^2 \int_0^T e^{-2a(T-s)} ds = \sigma^2 \frac{1}{2a}[1-e^{-2aT}]$$

We then combine the terms:

$$Var(r_T) = Var(X_T) + Var(Y_T) = \frac{\phi^2}{2\gamma}[1-e^{-2\gamma T}] + \frac{\sigma^2}{2a}[1-e^{-2aT}]$$

$$Var(T) = \frac{\phi^2}{2\gamma}[1-e^{-2\gamma T}] + \frac{\sigma^2}{2a}[1-e^{-2aT}]$$

To find the stationary distribution, we let $T \to \infty$, then the exponential terms given as $e^{-2\gamma T}$ and e^{-2aT} they will converge towards zero. Therefore, the variance will converge towards:

$$V_\infty = \frac{\phi^2}{2\gamma} + \frac{\sigma^2}{2a}$$

The exponentials in the mean term will also converge to 0. Therefore, the mean converges towards 0. This mean, that our stationary distribution is given as:

$$r_\infty \sim N(\frac{b}{a}, \frac{\phi^2}{2\gamma} + \frac{\sigma^2}{2a})$$

c)

Estimates	*Lower*	*Upper*
Model $T = 1$	0.013366	0.062372
Model $T = 10$	0.019561	0.080169

Table 7: Confidences intervals for the two simulated models

In the Table 7 the confidence intervals are given, while the simulated models are shown in Figure 8 and 9

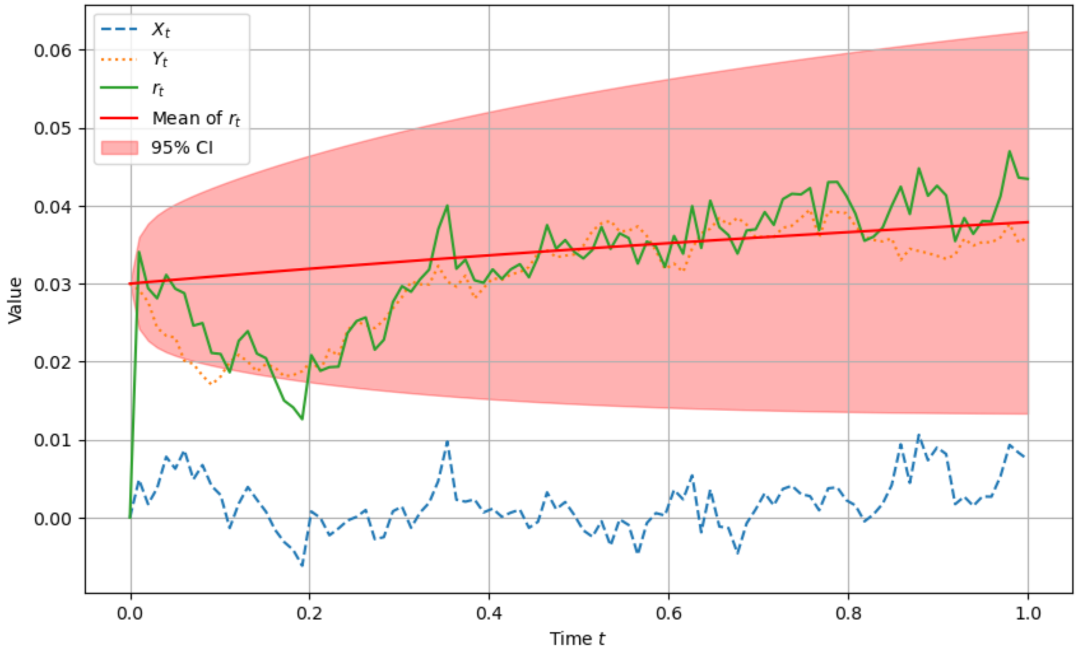

Figure 8: Trajectories of X_t, Y_t, and r_t for $T = 1$

d)

Given that the mean reverting factor in X_t, given at $\gamma = 32$, means that fluctuations will quickly be dampened. On the other hand, the slower mean reversion in Y_T lets the trajectory exhibit greater persistence over time, which is because of the slower rate of mean reversion, at $a = 0.5$.

Furthermore, the fluctuations in r_t arise from the combined distributions from X_T and Y_T. Where X_T has higher mean reversion and a volatility parameter set at 0.03, X_T shows a lot of short-lived fluctuations, which drive is rapid up and down pattern notable seen in the $T = 10$ trajectory. Looking at Y_T both with slower mean reversion and lower volatility. This means, that Y_T fluctuations will persist longer, this will be more predominant seen in r_t over the long term. Lastly, while it can not be seen in the short term in Figure 8,

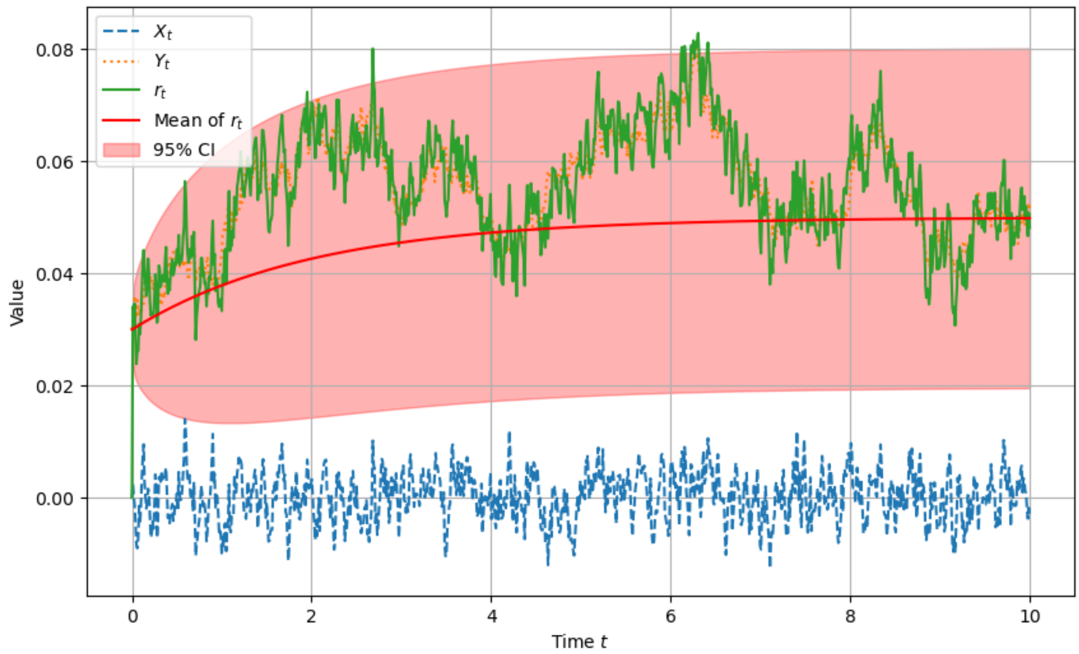

Figure 9: Trajectories of X_t, Y_t, and r_t for $T = 10$

when looking at the longer period, we see that X_T has less influence over the process we go towards ∞ and it settles around $t = 8$. At this point in time, it seems that the process has settled around it stationary distribution at $\frac{b}{a}$, which solely comes from the Y_T process.