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Part IV
Risk management with GARCH models

In this chapter we consider applications of GARCH models in relation to risk management. We define the *Value-at-Risk* and *Expected Shortfall* risk measures. The measures are important, for instance, from a regulatory perspective, as banks are obliged to disclose their estimates of such risk measures in relation to their holdings of risky assets (Basel Committee on Banking Supervision, 2013). We discuss how these measures are computed and estimated in the case that returns are $i.i.d.$ Gaussian or are generated from (G)ARCH processes. One may note that the computation of the risk measures boils down to computing a quantile (Value-at-Risk) or a conditional expectation (Expected Shortfall) of some (conditional) distribution. In the $i.i.d.$ Gaussian case these distributions are known, which makes the computations of the risk measures easy in the sense that we obtain closed-form expressions. For the GARCH models, on the other hand, conditional distributions are (in general) intractable, and we discuss how one may circumvent this issue by means of simulation-based estimation. We also discuss how the estimation uncertainty of the estimated risk measures is addressed.

IV.1 Value-at-Risk (VaR)

As in the previous chapters, we let x_{t+1} denote the log-return of some asset from t to $t+1$. We shall also need the h-period return, $h \geq 1$, which by definition is given by

$$x_{t+1,h} = \sum_{i=1}^{h} x_{t+i}.$$

A typical way of quantifying the risk of holding an asset over one period is to compute the so-called Value-at-Risk (VaR). Specifically, *the* 1*-period* VaR *at risk level* $\kappa \in (0,1)$ (or, in short, the VaR) is denoted VaR_t^κ and satisfies

$$P(x_{t+1} < -\text{VaR}_t^\kappa | \mathcal{I}_t) = \kappa, \quad \text{VaR}_t^\kappa \in \mathcal{I}_t, \tag{IV.1}$$

where $\mathcal{I}_t$ denotes some information set available at time t (e.g. the series of previous returns). Note that $P(-x_{t+1} \leq \text{VaR}_t^\kappa | \mathcal{I}_t) = 1 - \kappa$, so that the VaR measures the maximum loss $(-x_{t+1})$ not exceeded with probability $1-\kappa$, or equivalently,VaR is the $1-\kappa$ percentile of the conditional loss distribution.[1] Note that, by construction, the VaR depends on the return process, the information set $\mathcal{I}_t$, as well as the *confidence level* $1-\kappa$. Typical values of κ in applications are 1%, 2.5%, and 5%.

In the following we consider some examples of VaR computation. Throughout, we assume that the information set contains only past values of the returns, i.e. $\mathcal{I}_t = \{x_i : i \leq t\}$. We emphasize that one could include additional variables to the information set, which would lead to careful considerations, and assumptions, about how these variables are related to the return process. We first consider the VaR in the case where the returns are driven by an *i.i.d.* Gaussian process.

Example IV.1.1 *(*VaR*: The Gaussian case I) Suppose that $x_t \sim i.i.d.N(0,1)$. Then, with $\Phi(\cdot)$ the cdf of the standard normal distribution and using that x_{t+1} is independent of $\mathcal{I}_t$,*

$$P(x_{t+1} < -\text{VaR}_t^\kappa | \mathcal{I}_t) = \Phi(-\text{VaR}_t^\kappa) = \kappa$$

Hence,

$$\text{VaR}_t^\kappa = -\Phi^{-1}(\kappa),$$

i.e. the VaR *is (negative) the κ percentile of the standard normal distribution.*

Since the returns are *i.i.d.* Gaussian the previous example is straightforward to extend to an arbitrary horizon h and volatility σ. Define initially the h-period VaR as

$$P\left(x_{t+1,h} < -\text{VaR}_{t,h}^\kappa | \mathcal{I}_t\right) = \kappa. \tag{IV.2}$$

[1] Note that the definition above implicitly assumes that the VaR exists, which is indeed the case whenever the conditional return distribution is continuous. A more general definition that ensures that the VaR always exists is that $\text{VaR}_t^\kappa = \inf\{y \in \mathbb{R} : P(-x_{t+1} \leq y | \mathcal{I}_t) \geq 1-\kappa\}$. Some textbooks, such as the one by Francq and Zakoïan (2019), make the convention that the VaR must be non-negative, such that the VaR is given by $\max[0, \inf\{y \in \mathbb{R} : P(-x_{t+1} \leq y | \mathcal{I}_t) \geq 1-\kappa\}]$.

Example IV.1.2 *(*VaR*: The Gaussian case II) Suppose that* $x_t \sim i.i.d.N(0, \sigma^2)$*. It follows that* $x_{t+1,h} = \sum_{i=1}^{h} x_{t+i} \overset{D}{=} \sigma\sqrt{h}z$ *where* $z \sim N(0,1)$ *and independent of* $\mathcal{I}_t$*. Hence,*

$$\begin{aligned} P\left(x_{t+1,h} < -\mathrm{VaR}^{\kappa}_{t,h}|\mathcal{I}_t\right) &= P\left(\sigma\sqrt{h}z < -\mathrm{VaR}^{\kappa}_{t,h}\right) \\ &= P\left(z < -\mathrm{VaR}^{\kappa}_{t,h}/(\sigma\sqrt{h})\right) \\ &= \Phi\left(-\mathrm{VaR}^{\kappa}_{t,h}/(\sigma\sqrt{h})\right) = \kappa, \end{aligned}$$

such that

$$\mathrm{VaR}^{\kappa}_{t,h} = -\sigma\sqrt{h}\Phi^{-1}(\kappa).$$

In practice, the volatility σ is typically unknown but may be estimated based on a sample of returns; see Section IV.3.

IV.2 Expected Shortfall (ES)

For several years, the VaR was viewed as an industry standard for quantifying the risk of asset portfolios. Unfortunately, VaR has some shortcomings.

In particular, VaR lacks the property of being subadditive: Let $x^{(1)}_{t+1}$ and $x^{(2)}_{t+1}$ denote the returns of two assets (Asset 1 and 2), and let $\mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1})$ and $\mathrm{VaR}^{\kappa}_{t}(x^{(2)}_{t+1})$ denote their respective VaR. Then it does not necessarily hold that

$$\mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1} + x^{(2)}_{t+1}) \leq \mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1}) + \mathrm{VaR}^{\kappa}_{t}(x^{(2)}_{t+1}).$$

For instance, for certain distributions of $x^{(1)}_{t+1}$ and $x^{(2)}_{t+1}$, with $x^{(1)}_{t+1}$ and $x^{(2)}_{t+1}$ independent and identically distributed, it holds that

$$\mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1}+x^{(2)}_{t+1}) > \mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1})+\mathrm{VaR}^{\kappa}_{t}(x^{(2)}_{t+1}) = 2\mathrm{VaR}^{\kappa}_{t}(x^{(1)}_{t+1}) = \mathrm{VaR}^{\kappa}_{t}(2x^{(1)}_{t+1}),$$

that is, one can reduce the overall risk (as measured by VaR) by holding two quantities of Asset 1 instead of holding one of each asset, even though the losses of the assets are independent. Hence, one may argue that the use of VaR may discourage diversification (see e.g. Ibragimov, 2009).

Moreover, recall that VaR is the maximum loss not exceeded with a given probability $1-\kappa$. The risk measure does not tell us how much we lose (or may expect to lose) given that the loss exceeds the VaR, and hence the VaR does not tell us anything about the risk exposure in the presence of a "tail-event" (that happens with probability κ). This has led to the so-called Expected Shortfall (ES) risk measure that, by definition, quantifies the expected loss given that the loss exceeds the VaR: *The* 1*-period* ES *at risk level* $\kappa \in (0, 1)$ is given by

$$\mathrm{ES}_t^\kappa = E[-x_{t+1} | x_{t+1} < -\mathrm{VaR}_t^\kappa, \mathcal{I}_t].$$

It follows that

$$\mathrm{ES}_t^\kappa = \kappa^{-1} \int_0^\kappa \mathrm{VaR}_t^u du,$$

and that $\mathrm{ES}_t^\kappa \geq \mathrm{VaR}_t^\kappa$.

Example IV.2.1 *(*ES*: The Gaussian case I) Suppose that $x_t \sim i.i.d.N(0, 1)$. Recall that $E[X|X \in A] = E[X\mathbb{I}\{X \in A\}]/P(X \in A)$ (given that $P(X \in A) > 0$). Then*

$$\mathrm{ES}_t^\kappa = E[-x_{t+1} | x_{t+1} < -\mathrm{VaR}_t^\kappa, \mathcal{I}_t] = \frac{E[-x_{t+1}\mathbb{I}\{x_{t+1} < -\mathrm{VaR}_t^\kappa\}]}{P(x_{t+1} < -\mathrm{VaR}_t^\kappa)}.$$

Using that $P(x_{t+1} < -\mathrm{VaR}_t^\kappa) = \kappa$, we have that

$$\mathrm{ES}_t^\kappa = \kappa^{-1} \int_{\mathrm{VaR}_t^\kappa}^\infty x\phi(x)dx = \kappa^{-1} \int_{-\Phi^{-1}(\kappa)}^\infty x\phi(x)dx,$$

where $\phi(x) \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is the pdf of the standard normal distribution. It holds that

$$\frac{d}{dx}\phi(x) = -x\phi(x),$$

such that

$$\int x\phi(x)dx \quad \left(= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right) = -\phi(x).$$

Hence

$$\begin{aligned}
\int_{-\Phi^{-1}(\kappa)}^\infty x\phi(x)dx &= [-\phi(y)]_{-\Phi^{-1}(\kappa)}^\infty \\
&= [\lim_{y\to\infty} -\phi(y)] - [-\phi(-\Phi^{-1}(\kappa))] \\
&= 0 + \phi(-\Phi^{-1}(\kappa)),
\end{aligned}$$

and we have that

$$\mathrm{ES}_t^\kappa = \kappa^{-1}\phi(-\Phi^{-1}(\kappa)).$$

Example IV.2.2 *(*ES*: The Gaussian case II) Suppose that $x_t \sim i.i.d.N(0,\sigma^2)$. Let $z \sim N(0,1)$ and note that $x_{t+1} \overset{d}{=} \sigma z$. Then*

$$\begin{aligned}
\mathrm{ES}_t^\kappa &= E[-x_{t+1}|x_{t+1} < -\mathrm{VaR}_t^\kappa, \mathcal{I}_t] \\
&= E[-\sigma z|\sigma z < \sigma\Phi^{-1}(\kappa)] \\
&= \sigma E[(-z)|z < \Phi^{-1}(\kappa)] \\
&= \sigma\kappa^{-1}\phi(-\Phi^{-1}(\kappa)),
\end{aligned}$$

where we have used the arguments from the previous example. Likewise, for the h-period ES*, defined as*

$$\mathrm{ES}_{t,h}^\kappa = E[-x_{t+1,h}|x_{t+1,h} < -\mathrm{VaR}_{t,h}^\kappa, \mathcal{I}_t]$$

we have that

$$\mathrm{ES}_{t,h}^\kappa = \sigma\sqrt{h}\kappa^{-1}\phi(-\Phi^{-1}(\kappa)).$$

IV.3 VaR and ES for GARCH processes

By construction, the estimator of the volatility is subject to estimation uncertainty, which in turn implies that the estimated VaR and ES are subject to estimation uncertainty. Before turning to the GARCH case, consider initially the i.i.d. case.

IV.3.1 Estimation uncertainty

Given a sample $(x_t : t = 1, \dots, T)$, with $x_t \sim i.i.d.N(0,\sigma^2)$, one may use the estimator $\hat{\sigma}_T = \sqrt{T^{-1}\sum_{t=1}^T x_t^2}$ for σ, in order to obtain an estimator of the h-period VaR:

$$\widehat{\mathrm{VaR}}_{t,h}^\kappa = -\hat{\sigma}_T\sqrt{h}\Phi^{-1}(\kappa).$$

By the LLN for i.i.d. processes, $\hat{\sigma}_T^2 \xrightarrow{p} \sigma^2$ as $T \to \infty$, and hence $\widehat{\mathrm{VaR}}_{t,h}^\kappa$ is consistent for $\mathrm{VaR}_{t,h}^\kappa$, i.e.

$$\widehat{\mathrm{VaR}}_{t,h}^\kappa \xrightarrow{p} \mathrm{VaR}_{t,h}^\kappa.$$

Next, in order to quantify the estimation uncertainty, we note that from the CLT for i.i.d. processes, $\sqrt{T}(\hat{\sigma}_T^2 - \sigma^2) \xrightarrow{d} N(0,\Sigma)$ as $T \to \infty$, where $\Sigma \equiv \mathrm{V}(x_t^2 - \sigma^2) = 2\sigma^4$. Moreover, since $x \mapsto \sqrt{x}$ is continuously differentiable on the positive real axis, we have that $\sqrt{T}(\hat{\sigma}_T - \sigma) \xrightarrow{d} N(0, \sigma^2/2)$ [by the Δ-method]. Hence,

$$\sqrt{T}\left(\widehat{\mathrm{VaR}}^{\kappa}_{t,h} - \mathrm{VaR}^{\kappa}_{t,h}\right) = -\sqrt{h}\Phi^{-1}(\kappa)\sqrt{T}(\hat{\sigma}_T - \sigma) \xrightarrow{d} N(0, h\sigma^2\Phi^{-1}(\kappa)^2/2),$$

such that

$$\widehat{\mathrm{VaR}}^{\kappa}_{t,h} \overset{a}{\sim} N(\mathrm{VaR}^{\kappa}_{t,h}, h\sigma^2\Phi^{-1}(\kappa)^2/(2T)),$$

and one may report the (approximate) 95% error bands of the VaR as $\widehat{\mathrm{VaR}}^{\kappa}_{t,h} \pm 1.96\sqrt{h/(2T)}|\Phi^{-1}(\kappa)|\hat{\sigma}_T$. In order to take into account the estimation uncertainty, or the additional "estimation risk", one may for instance use the upper band,

$$\widehat{\mathrm{VaR}}^{\kappa}_{t,h} + 1.96\sqrt{h/(2T)}|\Phi^{-1}(\kappa)|\hat{\sigma}_T,$$

as the "estimation risk-adjusted VaR measure".
As for the VaR, we may obtain an ES estimate as

$$\widehat{\mathrm{ES}}^{\kappa}_{t,h} = \kappa^{-1}\hat{\sigma}_T\sqrt{h}\phi(-\Phi^{-1}(\kappa)),$$

and, as $T \to \infty$, we have that

$$\widehat{\mathrm{ES}_{t,h}} \xrightarrow{p} \mathrm{ES}^{\kappa}_{t,h},$$

and

$$\sqrt{T}\left(\widehat{\mathrm{ES}_{t,h}} - \mathrm{ES}^{\kappa}_{t,h}\right) \xrightarrow{d} N(0, \kappa^{-2}h\phi(-\Phi^{-1}(\kappa))^2\sigma^2/2).$$

IV.3.2 GARCH

Suppose that the returns are driven by a GARCH process such that

$$x_t = \sigma_t z_t, \quad z_t \sim i.i.d.(0,1),$$

and $\sigma_t^2 > 0$ some function of past returns.

Example IV.3.1 *(*VaR*: ARCH(1), Gaussian errors) Suppose that $z_t \sim N(0,1)$, and $\sigma_t^2 = \omega + \alpha x_{t-1}^2$. Then, similar to Example IV.1.2,*

$$\begin{aligned} P\left(x_{t+1} < -\mathrm{VaR}^{\kappa}_{t}|\mathcal{I}_t\right) &= P\left(\sigma_{t+1}z_{t+1} < -\mathrm{VaR}^{\kappa}_{t,1}|\mathcal{I}_t\right) \\ &= P\left(z_{t+1} < -\mathrm{VaR}^{\kappa}_{t,1}/\sigma_{t+1}|\mathcal{I}_t\right) \\ &= \Phi(-\mathrm{VaR}^{\kappa}_{t,1}/\sigma_{t+1}), \end{aligned}$$

where we have used that $\sigma_{t+1} \in \mathcal{I}_t$ and that z_{t+1} is independent of $\mathcal{I}_t$. We hence obtain that

$$\mathrm{VaR}_t^{\kappa} = -\sigma_{t+1}\Phi^{-1}(\kappa).$$

As discussed in Part III, we may obtain an estimator for the parameters $\theta = (\omega, \alpha)'$ by quasi-maximum likelihood given a sample of returns, $\hat{\theta}_T = (\hat{\omega}_T, \hat{\alpha}_T)'$. Based on this estimator we obtain an estimator for the conditional volatility, given by

$$\hat{\sigma}_{t+1} = \sqrt{\hat{\omega}_T + \hat{\alpha}_T x_t^2},$$

and we have the estimated VaR*,*

$$\widehat{\mathrm{VaR}}_t^{\kappa} = -\hat{\sigma}_{t+1}\Phi^{-1}(\kappa).$$

Notice that (unlike the case of Gaussian returns in the previous examples) VaR_t^{κ} *is random as it depends on x_t. In order to analyze the statistical properties of the* VaR *estimator, it is customary to consider x_t as fixed and setting it equal to some fixed value, $x_t = x$. We then have that*

$$\widehat{\mathrm{VaR}}_t^{\kappa} - \mathrm{VaR}_t^{\kappa} = -(\hat{\sigma}_{t+1} - \sigma_{t+1})\Phi^{-1}(\kappa),$$

with $\hat{\sigma}_{t+1}^2 = \hat{\omega}_T + \alpha_T x^2$ and $\sigma_{t+1}^2 = \omega + \alpha x^2$. Recall that under certain conditions discussed in Part III, as $T \to \infty$,

$$\hat{\theta}_T \xrightarrow{p} \theta \quad \text{and} \quad \sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, \Omega_I^{-1}\Omega_S\Omega_I^{-1}).$$

This implies that, by a first-order Taylor expansion (up to a negligible remainder term),

$$\hat{\sigma}_{t+1} - \sigma_{t+1} = \frac{1}{2\sqrt{\omega + \alpha x^2}}(1, x^2)(\hat{\theta}_T - \theta),$$

and we conclude that

$$\widehat{\mathrm{VaR}}_t^{\kappa} \xrightarrow{p} \mathrm{VaR}_t^{\kappa},$$

and

$$\sqrt{T}(\widehat{\mathrm{VaR}}_t^{\kappa} - \mathrm{VaR}_t^{\kappa}) \xrightarrow{d} N\left(0, \frac{\Phi^{-1}(\kappa)^2}{4(\omega + \alpha x^2)}(1, x^2)\Omega_I^{-1}\Omega_S\Omega_I^{-1}(1, x^2)'\right).$$

Similar to Example IV.3.1, one may use this result to construct error bands for the estimated VaR*.*

Example IV.3.2 *(*VaR*: ARCH(1), non-Gaussian errors) Suppose that z_t has some unknown distribution with cdf F_z. Then as in the previous example, we have that*

$$\mathrm{VaR}_t^\kappa = -\sigma_{t+1} F_z^{-1}(\kappa),$$

where F_z^{-1} is the (generalized) inverse of F_z. In addition to estimating σ_{t+1}, we need an estimate of $F_z^{-1}(\kappa)$. We may obtain such an estimate by computing the empirical κ percentile of the standardized residuals. Specifically, we let

$$\hat{z}_t \equiv \frac{x_t}{\hat{\sigma}_t}, \quad \text{with } \hat{\sigma}_t^2 = \hat{\omega}_T + \hat{\alpha}_T x_t^2, \quad t = 1, \ldots, T,$$

and consider the ordered residuals $\hat{z}_{(1)} \leq \ldots \leq \hat{z}_{(T)}$ in order to obtain

$$\hat{F}_z^{-1}(\kappa) = \hat{z}_{(\max\{\lfloor T\kappa \rfloor, 1\})}, \tag{IV.3}$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. The VaR *estimate is given by*

$$\widehat{\mathrm{VaR}_t^\kappa} = -\hat{\sigma}_{t+1} \hat{F}_z^{-1}(\kappa).$$

Example IV.3.3 *(*ES*, ARCH(1)) Suppose that $z_t \sim i.i.d.N(0,1)$. Note that $x_{t+1} = \sigma_{t+1} z_{t+1}$, and similar to Example IV.2.2 we have that*

$$\begin{aligned}
\mathrm{ES}_{t,1}^\kappa &= E[-x_{t+1} | x_{t+1} < -\mathrm{VaR}_t^\kappa, \mathcal{I}_t] \\
&= E[-\sigma_{t+1} z_{t+1} | \sigma_{t+1} z_{t+1} < \sigma_{t+1} \Phi^{-1}(\kappa), \mathcal{I}_t] \\
&= \sigma_{t+1} E[(-z_{t+1}) | z_{t+1} < \Phi^{-1}(\kappa)] \\
&= \kappa^{-1} \sigma_{t+1} \phi(-\Phi^{-1}(\kappa)),
\end{aligned}$$

If we, as in Example IV.3.2, instead assume that z_t has an unknown distribution,

$$\begin{aligned}
\mathrm{ES}_t^\kappa &= \sigma_{t+1} E[(-z_{t+1}) | z_{t+1} < F_z^{-1}(\kappa)] \\
&= \kappa^{-1} \sigma_{t+1} E[(-z_{t+1}) \mathbb{I}\{z_{t+1} < F_z^{-1}(\kappa)\}]
\end{aligned}$$

Note that the quantity $E[(-z_{t+1})\mathbb{I}\{z_{t+1} < F_z^{-1}(\kappa)\}]$ may be estimated based on the standardized residuals and the estimator for $F_z^{-1}(\kappa)$ defined in (IV.3):

$$\hat{E}[-z_{t+1} \mathbb{I}\{z_{t+1} < F_z^{-1}(\kappa)\}] = \frac{1}{T} \sum_{t=1}^{T} (-\hat{z}_{t+1}) \mathbb{I}\{\hat{z}_{t+1} < \hat{F}_z^{-1}(\kappa)\}.$$

IV.4 Simulation-based estimation

Until now, we have focused on the computation of one-period ahead VaR and ES for GARCH processes. In this section, we consider the multi-period ahead risk measures. Such measures are much more burdensome to compute, because the multi-period ahead conditional distributions of GARCH processes are unknown, as illustrated below.

Consider the ARCH(1) process with Gaussian errors, i.e. $x_{t+1} = \sqrt{\omega + \alpha x_t^2} z_{t+1}$ with $z_t \sim i.i.d.N(0,1)$. Since the factor $\sqrt{\omega + \alpha x_t^2}$ is known given the information set $\mathcal{I}_t$, it holds that $x_{t+1}|\mathcal{I}_t \sim N(0, \omega + \alpha x_t^2)$ which we exploited in the previous section to obtain the VaR and ES. Now, suppose that we want to compute the two-period ahead VaR. This relies on computing the κ percentile of the conditional loss distribution, i.e. the conditional distribution of $-x_{t+1,2}$ given $\mathcal{I}_t$ with $x_{t+1,2} = (x_{t+1} + x_{t+2})$. By recursions,

$$x_{t+2} = \sqrt{\omega + \alpha x_{t+1}^2} z_{t+2} = \sqrt{\omega + \alpha(\omega + \alpha x_t^2) z_{t+1}^2} z_{t+2}.$$

Clearly, $x_{t+2}|\mathcal{I}_{t+1} \sim N(0, \omega + \alpha x_{t+1}^2)$, but it is not clear what the conditional distribution of x_{t+2} (given $\mathcal{I}_t$) is, since the factor $\sqrt{\omega + \alpha(\omega + \alpha x_t^2) z_{t+1}^2}$ is unknown given the information set $\mathcal{I}_t$. In particular, one can show that the conditional distribution is non-Gaussian.

Consequently, one may view the conditional distribution as intractable, and instead one may approximate the risk-measures by means of simulations, as outlined in the following algorithm.

Algorithm IV.4.1 *(Simulation-based* VaR *for ARCH(1) with Gaussian errors) Let $(\omega, \alpha)'$ and x_t be known and fixed.*

1. *For $i = 1, \ldots, M$ (with $(1-\kappa)M \geq 1$) draw $z_{t+1}^{(i)}$ and $z_{t+2}^{(i)}$ independently from $N(0,1)$, and compute*

$$x_{t+1,2}^{(i)} = (x_{t+1}^{(i)} + x_{t+2}^{(i)}),$$

with

$$x_{t+1}^{(i)} = \sqrt{\omega + \alpha x_t^2} z_{t+1}^{(i)},$$
$$x_{t+2}^{(i)} = \sqrt{\omega + \alpha (x_{t+1}^{(i)})^2} z_{t+2}^{(i)}.$$

2. *Consider the ordered returns $x_{t+1,2}^{[M]} \leq \ldots \leq x_{t+1,2}^{[1]}$. Using the definition of* VaR *in (IV.2), obtain the approximate* VaR *as the $(1-\kappa)$ percentile of the simulated losses, i.e.*

$$\text{VaR}_{t,2}^{\kappa,\text{sim}} = -(x_{t+1,2}^{[\lfloor (1-\kappa)M \rfloor]}).$$

Typically, M is chosen to be quite large, say 10^5 or 10^6 in order to increase the precision of the VaR estimate. Note that the above algorithm may be extended or modified in several directions:

- One may extend to different GARCH models and to arbitrary horizon h.
- As is typical in practice, the values of $(\omega, \alpha)'$ are unknown, and one may in step 1 replace $(\omega, \alpha)'$ by estimates $(\hat{\omega}_T, \hat{\alpha}_T)'$.
- If the distribution of z_t is unknown, then one may in step 1 draw $z_{t+1}^{(i)}$ and $z_{t+2}^{(i)}$ with equal probability (and with replacement) from the aforementioned standardized residuals, $(\hat{z}_t : t = 1, \dots, T)$.
- In a third step, one may compute (similar to Example IV.3.3 above) the ES as

$$\mathrm{ES}_{t,2}^{\kappa,\mathrm{sim}} = \kappa^{-1} \frac{1}{M} \sum_{i=1}^{M} (-x_{t+1,2}^{(i)}) \mathbb{I}\{x_{t+1,2}^{(i)} < -\mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim}}\}.$$

Next, we consider how to address the estimation uncertainty of the above algorithm, in the case that we use estimated parameter values $(\hat{\omega}_T, \hat{\alpha}_T)'$ in step 1. Since we do not have an explicit expression for the ES in terms of the model parameters, it is not possible to apply Taylor expansions arguments as we did for the VaR estimator. Instead we address the uncertainty by means of an extra layer of simulations, where we exploit that the estimator $\hat{\theta}_T = (\hat{\omega}_T, \hat{\alpha}_T)'$ is approximately Gaussian. Recall from Part III, that (under suitable conditions), as $T \to \infty$, $\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{d} N(0, \Omega_I^{-1}\Omega_S\Omega_I^{-1})$, such that

$$\hat{\theta}_T \overset{a}{\sim} N(\theta, T^{-1}\Omega_I^{-1}\Omega_S\Omega_I^{-1}), \tag{IV.4}$$

and there exist consistent estimators, $\hat{\Omega}_{S,T}$ and $\hat{\Omega}_{I,T}$, for Ω_S and Ω_I. The following relies on, in a first step, to draw parameter values from the distribution $N(\hat{\theta}_T, T^{-1}\hat{\Omega}_{I,T}^{-1}\hat{\Omega}_{S,T}\hat{\Omega}_{I,T}^{-1})$ in order to take into account the random variation in $\hat{\theta}_T$ and hence in the estimator for the VaR (see e.g. Blasques et al. 2016, for a similar approach in relation to the computation of forecasting error bands).

Algorithm IV.4.2 *(Simulation-based* VaR *for ARCH(1) with Gaussian errors under estimation uncertainty) Let x_t be known and fixed, and suppose that estimator $\hat{\theta}_T = (\hat{\omega}_T, \hat{\alpha}_T)'$ satisfies (IV.4) and that we have some estimators, $\hat{\Omega}_{S,T}$ and $\hat{\Omega}_{I,T}$, for Ω_S and Ω_I.*

1. *For $b = 1, \ldots, B$ (say, $B = 999$), draw $\theta^{(b)} = (\omega^{(b)}, \alpha^{(b)})'$ from*
$$N(\hat{\theta}_T, T^{-1}\hat{\Omega}_{I,T}^{-1}\hat{\Omega}_{S,T}\hat{\Omega}_{I,T}^{-1}). \tag{IV.5}$$

2. *For a given b, for $i = 1, \ldots, M$ draw $z_{t+1}^{(i,b)}$ and $z_{t+2}^{(i,b)}$ independently from $N(0,1)$, and compute*
$$x_{t+1,2}^{(i,b)} = (x_{t+1}^{(i,b)} + x_{t+2}^{(i,b)}),$$
with
$$\begin{aligned} x_{t+1}^{(i,b)} &= \sqrt{\omega^{(b)} + \alpha^{(b)} x_t^2}\, z_{t+1}^{(i,b)}, \\ x_{t+2}^{(i,b)} &= \sqrt{\omega^{(b)} + \alpha^{(b)} (x_{t+1}^{(i,b)})^2}\, z_{t+2}^{(i,b)}. \end{aligned}$$

3. *For a given b, consider the ordered returns $x_{t+1,2}^{[M,b]} \leq \ldots \leq x_{t+1,2}^{[1,b]}$. Using the definition of* VaR *in (IV.2), compute the approximate* VaR *as the $(1-\kappa)$ percentile of the simulated losses, i.e.*
$$\mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim},b} = -(x_{t+1,2}^{[\lfloor (1-\kappa)M \rfloor, b]}).$$

4. *Consider the ordered B* VaR *estimates, $\mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim},(1)} \leq \ldots \leq \mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim},(B)}$. The 95% error interval of the* VaR *is given by*
$$[\mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim},(B\times 0.025)}, \mathrm{VaR}_{t,2}^{\kappa,\mathrm{sim},(B\times 0.975)}].$$

Remark IV.4.1 *In step 1, one may end up with a draw of $\omega^{(b)}$ and/or $\alpha^{(b)}$ that is negative, and hence outside of the parameter space. In this case one may make a new draw of $\theta^{(b)}$. Alternatively, one may reparametrize the model, e.g. by estimating $\tilde{\omega} = \log(\omega)$ and $\tilde{\alpha} = \log(\alpha)$ that are freely varying.*

Remark IV.4.2 *As an alternative to the algorithm outlined above, one may address the estimation uncertainty by means of bootstrap methods. In particular, instead of drawing each $\theta^{(b)}$ from the "asymptotic" distribution in (IV.5), one may estimate each $\theta^{(b)}$ from a bootstrap sample generated from the assumed data generating process with true parameter values given by $\hat{\theta}_T = (\hat{\omega}_T, \hat{\alpha}_T)'$. One advantage of such an approach is that the draws $(\theta^{(b)} : b = 1, \ldots, B)$ may be closer to mimicking draws from the unknown finite-sample distribution of $\hat{\theta}_T$. See e.g. Cavaliere, Pedersen and Rahbek (2018) for more details on bootstrap methods in relation to (G)ARCH processes.*

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