Matematik A E2020
Uge 44, Forelæsning 2

Afsnit 9.3(fortsat), 9.5

Integralregning:

Bestemte integraler (fortsat), partiel integration

Egenskaber for best. integr. (9.3)

Husk definition:

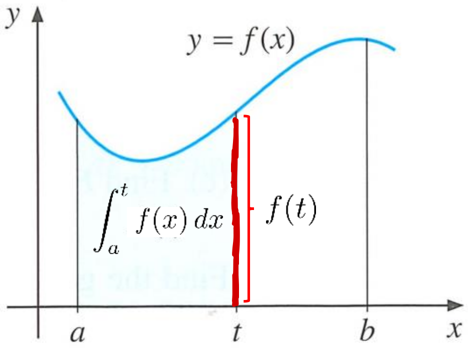
($f : [a, b] \to \mathbb{R}$ kontinuert med stamfunktion F)

$$\boxed{\int_a^b f(x)\,dx = \Big|_a^b F(x) = [F(x)]_a^b = F(b) - F(a)}$$

Heraf fås, at der for $a < t < b$ gælder:

$$\boxed{\frac{d}{dt}\int_a^t f(x)\,dx = \frac{d}{dt}\big(F(t) - F(a)\big) = F'(t) - 0 = f(t)}$$

Grafisk for ikke-negativ f:



Generalisering ($a(t)$ og $b(t)$ er differentiable funktioner):

$$\boxed{\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = f(b(t))b'(t) - f(a(t))a'(t)}$$

Bevis: Kædereglen...

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x)\,dx\right) = \frac{d}{dt}\Big(F(b(t)) - F(a(t))\Big) = b'(t)F'(b(t)) - a'(t)F'(a(t))$$

$$= \underline{b'(t)f(b(t)) - a'(t)f(a(t))} \quad \square$$

$f(x) = x^2+2$

Øvelse: Lad $g(t) = \int_0^t (x^2+2)dx$ og $h(t) = \int_0^{t^2} (x^2+2)dx$.

Bestem $g'(t)$ og $h'(t)$. ← (kun $h'(t)$)

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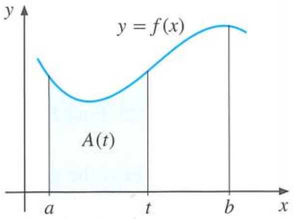
$g'(t) = f(t) = t^2+2$

$h'(t) = f(t^2)\cdot 2t - 0 = \left((t^2)^2+2\right)2t = \underline{2t^5+4t}$

$a(t) = 0$

$b(t) = t^2$

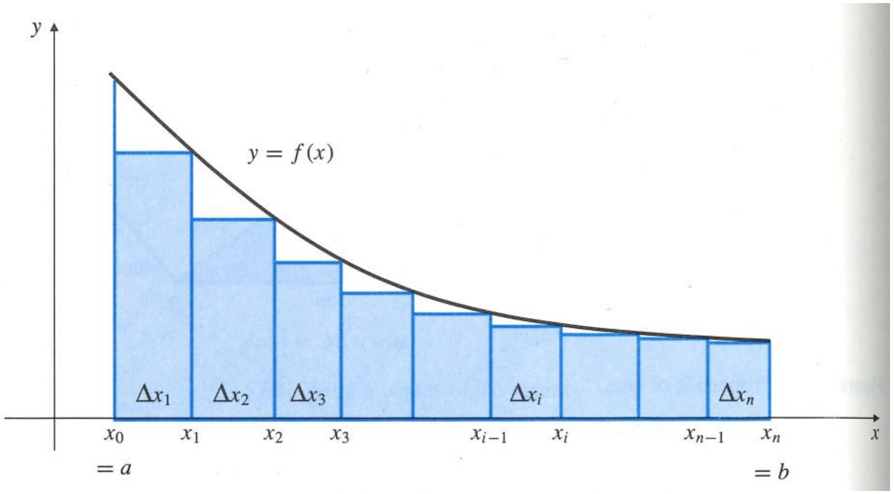
Har enhver kontinuert fkt $f : [a,b] \to \mathbb{R}$ en stamfunktion F?
Og kan vi altid finde en forskrift for F?



1) Ja (brug arealfunktionen)

2) Nej (fx. $f(x) = e^{-x^2}$)

Kort om "Riemann-integralet":



$$\int_a^b f(x)\,dx$$

defineres som grænseværdien af "det blå areal" når $n \to \infty$ og $\max_{i=1,\dots,n} \Delta x_i \to 0$

Partiel integration (9.5)

Vigtig metode til at bestemme både bestemte og ubestemte integraler!

Vi starter med ubestemte integraler

Lad f og g være differentiable funktioner.
Vi differentierer $f(x)g(x)$ vha produktreglen:

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Heraf fås... $\int \left(f'(x)g(x) + f(x)g'(x)\right) dx = f(x)g(x) + C$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx + \cancel{C}$$

Partiel integration, ubestemte integraler ((9.5.1), s. 344)

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Eksempler

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

$f(x)=x,\ g'(x)=e^{ax}$

$$\int xe^{ax}\,dx = x\cdot\left(\frac{1}{a}e^{ax}\right) - \int 1\cdot\left(\frac{1}{a}e^{ax}\right)dx = \frac{x}{a}e^{ax} - \frac{1}{a}\int e^{ax}dx$$

$(a \neq 0)$

$$= \frac{x}{a}e^{ax} - \frac{1}{a}\left(\frac{1}{a}e^{ax} + C_1\right) = \underline{\frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + C}$$

(hvor C er arbitrær konst!)

$$\int \ln(x)\,dx = \int \ln(x)\cdot 1\,dx = \ln(x)\cdot x - \int \frac{1}{x}\cdot x\,dx$$

$$= \ln(x)\cdot x - \int 1\,dx = \underline{\ln(x)\cdot x - x + C}$$

$$\int x^2e^{-x}\,dx = x^2\left(-e^{-x}\right) - \int 2x\left(-e^{-x}\right)dx = -x^2e^{-x} + 2\int xe^{-x}dx$$

$(a=-1)$

$$= -x^2e^{-x} + 2\left(-xe^{-x} - e^{-x} + C_1\right)$$

$$= \underline{-e^{-x}\left(x^2 + 2x + 2\right) + C}$$

Bestemte integraler:

Da $f(x)g(x)$ er stamfunktion til $f'(x)g(x) + f(x)g'(x)$:

$$\int_a^b \left(f'(x)g(x) + f(x)g'(x)\right) dx = [f(x)g(x)]_a^b$$

Partiel integration, bestemte integraler ((9.5.2), s. 346)

$$\boxed{\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx}$$

Eksempel: $(f(x) = x+1,\ g'(x) = 2^x)$ $\quad$ [HUSK: $2^x = e^{\ln(2)\cdot x}$]

$$\int_0^1 (x+1)2^x\,dx = \left[(x+1)\cdot\frac{1}{\ln(2)}2^x\right]_0^1 - \int_0^1 1\cdot\frac{1}{\ln(2)}2^x\,dx$$

$$= \left[(x+1)\frac{1}{\ln(2)}2^x\right]_0^1 - \frac{1}{\ln(2)}\left[\frac{1}{\ln(2)}2^x\right]_0^1$$

$$= \left(\frac{2}{\ln(2)}\cdot 2 - \frac{1}{\ln(2)}\cdot 1\right) - \frac{1}{\ln(2)}\left(\frac{1}{\ln(2)}\cdot 2 - \frac{1}{\ln(2)}\cdot 1\right)$$

$$= \frac{3}{\ln(2)} - \frac{1}{(\ln(2))^2} = \frac{3\ln(2) - 1}{(\ln(2))^2}.$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Øvelse:

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$$\int 6x(1+x)^5\,dx$$

$$= 6x\left(\frac{1}{6}(1+x)^6\right) - \int 6\left(\frac{1}{6}(1+x)^6\right)dx$$

$$= x(1+x)^6 - \int (1+x)^6\,dx = x(1+x)^6 - \left(\frac{1}{7}(1+x)^7 + C_1\right)$$

$$= x(1+x)^6 - \frac{1}{7}(1+x)^7 + C$$

Ekstra: (hvis tid) $\int_0^a x3^x\,dx$

$$= \left[x\frac{1}{\ln(3)}3^x\right]_0^a - \int_0^a 1\cdot\frac{1}{\ln(3)}3^x\,dx$$

(Se eks. på slide 7)

$$= \left(\frac{a}{\ln(3)}3^a - 0\right) - \frac{1}{\ln(3)}\left[\frac{1}{\ln(3)}3^x\right]_0^a$$

$$= \frac{a\cdot 3^a}{\ln(3)} - \frac{1}{\ln(3)}\left(\frac{1}{\ln(3)}3^a - \frac{1}{\ln(3)}3^0\right)$$

$$= \frac{a\cdot 3^a}{\ln(3)} - \frac{3^a - 1}{(\ln(3))^2}.$$

Et sidste eksempel...

Bestem integralet (a er en konstant, $x > 0$)

$$\int \frac{\ln(x^a)}{x}\, dx = a \int \frac{\ln(x)}{x}\, dx = \frac{a}{2}\left(\ln(x)\right)^2 + C$$

$$\int \frac{\ln(x)}{x}\, dx = \ln(x) \cdot \ln(x) - \int \frac{1}{x} \cdot \ln(x)\, dx$$

$$\left[\begin{array}{l} f(x) = \ln(x) \\ g'(x) = \frac{1}{x} \end{array}\right]$$

Heraf:

$$2 \int \frac{\ln(x)}{x}\, dx = \left(\ln(x)\right)^2 + C_1$$

$$\Leftrightarrow \int \frac{\ln(x)}{x}\, dx = \frac{1}{2}\left(\ln(x)\right)^2 + C_2$$

Bemærk: Opgaver med partiel integration kommer i uge 46
(for at udjævne mængden af opgaver)