Advanced Financial and Macroeconometrics Exam

Exam number: 17

May 26, 2025

1 Portfolio Selection

We let $R = (R_1, R_2)' \in R^2$ denote a vector iid stochastic returns distributed as $N(\mu, \Omega)$ with:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where $a, b > 0$ and $a^2 > b^2$

1.1 Derive portfolio weights

We must derive a solution for $v = (v_1, v_2)'$ with $v_1 + v_2 = 1$. Since $v_1 + v_2 = 1$ is a constraint, we know that the whole wealth of the investor is invested. We start with:

$$\min_v v'\Omega v$$

subject to the condition $v'\iota = 1$. We then solve this by Lagrange multiplier with the λ.

$$L(v, \lambda) = v'\Omega v + \lambda[1 - v'\iota]$$

We then take the first derivative w.r.t to v

$$\frac{\partial L}{\partial v} = 2\Omega v - \lambda\iota = 0$$

We then isolate for v

$$= 2\Omega v = \lambda\iota$$

$$= v = \iota\frac{\lambda}{2}\Omega^{-1}$$

We then insert this back into the $v'\iota = 1$

$$v'\iota = 1$$

$$\iota'\frac{\lambda}{2}\Omega^{-1}\iota = 1$$

We then want to isolate for $\frac{\lambda}{2}$ as this matches the term from above. Therefore, we have:

$$\frac{\lambda}{2} = \frac{1}{\iota\Omega^{-1}\iota}$$

Which we then can insert back into the expression for v:

$$v = \frac{\lambda}{2}\Omega^{-1}\iota$$

$$v = \frac{1}{\iota'\Omega^{-1}\iota}\Omega^{-1}\iota$$

We then may insert the different vectors:

$$v^{\star}_{GMV} = (\iota'\Omega^{-1}\iota)^{-1}\Omega^{-1}\iota$$

We then calculate the individual terms:

$$\Omega^{-1}\iota = \frac{1}{a^2-b^2}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{a^2-b^2}\begin{pmatrix} a\cdot 1 & (-b)\cdot 1 \\ (-b)\cdot 1 & a\cdot 1 \end{pmatrix}$$

$$= \frac{1}{a^2-b^2}\begin{pmatrix} a-b \\ a-b \end{pmatrix} = \frac{a-b}{a^2-b^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{a-b}{a^2-b^2}\iota$$

We now turn to $\iota'\Omega^{-1}\iota$. Here we may use the result that:

$$\iota'\Omega^{-1}\iota = (\Omega^{-1}\iota)\iota = \frac{a-b}{a^2-b^2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}' = \frac{a-b}{a^2-b^2}(1+1) = \frac{2(a-b)}{a^2-b^2}$$

$$\iota'\Omega^{-1}\iota = \frac{2(a-b)}{a^2-b^2}$$

We plug this back in:

$$v^\star_{GMV} = (\iota'\Omega^{-1}\iota)^{-1}\Omega^{-1}\iota$$

$$v^\star_{GMV} = \frac{\Omega^{-1}\iota}{\iota'\Omega^{-1}\iota} = \frac{\frac{a-b}{a^2-b^2}\iota}{\frac{2(a-b)}{a^2-b^2}} = \frac{1}{2}\iota$$

Therefore, the optimal weights under the Global Minimum Variance portfolio is $v^\star_{GMV} = (\frac{1}{2}, \frac{1}{2})'$ i.e. $v_1 = v_2 = \frac{1}{2}$. The expected return is then given as:

$$\bar{\mu}_v = v'_{GMV}\star\mu = \frac{1}{2}(\mu_1 + \mu_2)$$

While we find the variance as:

$$\sigma^2_v = v'^\star_{GMV}\Omega v^\star_{GMV} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}a + \frac{1}{2}b, & \frac{1}{2}a + \frac{1}{2}b \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}\frac{a+b}{2} + \frac{1}{2}\frac{a+b}{2} = \frac{a+b}{2}$$

Therefore, the weights are given as $v^\star_{GMV} = (\frac{1}{2}, \frac{1}{2})'$ with the GMV portfolio mean $\bar{\mu}_v = \frac{1}{2}(\mu_1\mu_2)$ and GMV portfolio variance $\bar{\sigma}^2_v = \frac{a+b}{2}$

1.2 Discuss portfolio weights based on eigenvectors

Step 1
First we solve the eigenvalue problem:

$$|\Omega - \lambda I_2| = 0$$

$$|\Omega - \lambda I_2| = det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = 0$$

$$= (a-\lambda)^2 - b^2 = 0$$

$$= a^2 - 2\lambda a + \lambda^2 - b^2 = 0$$

$$= \lambda^2 - 2\lambda a + (a^2 - b^2) = 0$$

This is a quadratic equation, which we may solve ala:

$$\lambda = \frac{2a \pm \sqrt{4b^2}}{2} = \frac{2a \pm 2b}{2} = a \pm b$$

Which will yield λ_1 and λ_2:

$$\lambda_1 = a + b, \quad \lambda_2 = a - b$$

We then turn to find the eigen vectors. First for λ_1, we solve $det(\Omega - \lambda_1 I)q = 0$

$$det(\Omega - \lambda_1 I)q = 0 \Rightarrow \begin{pmatrix} a-(a+b) & b \\ b & a-(a+b) \end{pmatrix} q = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} q = 0$$

Therefore, we have $-bq_1 + bq_2 = 0 \Rightarrow q_1 = q_2$. Therefore, we can take the q_1 as:

$$q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We then turn to λ_2 and do the same, but for $det(\Omega - \lambda_2 I)q = 0$.

$$det(\Omega - \lambda_2 I)q = 0 \Rightarrow \begin{pmatrix} a-(a-b) & b \\ b & a-(a-b) \end{pmatrix} q = \begin{pmatrix} b & b \\ b & b \end{pmatrix} q = 0$$

Therefore, we have $bq_1 + bq_2 = 0 \Rightarrow q_1 = -q_2$, which gives us the following vector:

$$q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since we are subject to the normalization $Q'Q = I_2$, we can normalize the eigenvectors q_1 and q_2. This can be shown through a sinus/cosinus transformation jf. Lecture notes section 6.5.

$$q_1 = \begin{pmatrix} cos(1) \\ sin(1) \end{pmatrix}, \quad q_2 = \begin{pmatrix} -sin(1) \\ cos(-1) \end{pmatrix}$$

This will yield us the following matrix Q

$$Q = \begin{pmatrix} cos(1) & -sin(1) \\ sin(1) & cos(-1) \end{pmatrix}$$

Which if we set $Q'Q$, then we will have:

$$Q'Q = \begin{pmatrix} cos(1) & sin(1) \\ -sin(1) & cos(-1) \end{pmatrix} \begin{pmatrix} cos(1) & -sin(1) \\ sin(1) & cos(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Therefore, our result holds.

Step 2

We are then asked to rescale the vectors, q_1 and q_2. We do this by taking taking the sum of the vector and divide through the vector for it. This yields the following

$$q_1 = \begin{pmatrix} cos(1) \\ sin(1) \end{pmatrix} \frac{1}{cos(1) + sin(1)} = \begin{pmatrix} 0.3910 \\ 0.6090 \end{pmatrix}, \quad q_2 = \begin{pmatrix} -sin(1) \\ cos(-1) \end{pmatrix} \frac{1}{-sin(1) + cos(-1)} = \begin{pmatrix} 2.7941 \\ -1.7940 \end{pmatrix}$$

These calculations have been made in excel. But we can check, that both weights hold as for $q_1 = 0.3910 + 0.6090 = 1$ and $q_2 = 2.7941 - 1.7940 = 1$, i.e they both satisfy the budget constraint.

Step 3

We then use the entries in q_2 as the portfolio weights for our portfolio. This means, that we will go long in the first asset by 2.7940 and short in the second asset by -1.7940. This is based on the fact, that as the two asset have covariance. This means, that if asset a increases, then b will increase too and vice versa. Furthermore, this means that if the price in asset 1 decreases, then the short position in asset 2 will yield a positive return and vice versa. It shall be noted, that the private investor may struggle to hold this portfolio as short-selling may not be available for the private investor. In contrast, the institutional investor, e.g pension funds or equivalent financial institutions do not have the same restrictions.

1.3 Discuss the results

Portfolios built directly from eigenvectors from the Ω matrix and the global-minimum-variance(GMV) portfolio solves two different optimization problems with two different sets of constraints. We derive the eigenvector portfolio from solving:

$$\min_w w'\Omega w, \quad \text{s.t } w'w = 1$$

This yields the eigenvector. Here we have chosen the eigenvector, q_2 as this corresponds to the smallest eigen-value, λ_2. By this approach, we impose a unit-form contraint on the weights. At the same time, this method does not require the investor to invest all the wealth of the investor. Therefore, this solution will not satisfy the condition $w'\iota = 1$

Moreover, we derive the Global Minimum variance portfolio by solving:

$$\min_{v'\iota=1} v'\Omega v, \quad \text{s.t } v'\iota = 1$$

This imposes the investor to invest their whole wealth. The resulting vector v will lie on the $v\iota = 1$ line, which is not the same direction as for the eigenvector problem. This would theoretically be possible, if both constraints hold. Often, this is not the case.

We see that the eigenvector portfolio is associated with short selling in order for the constraint to hold. This can be seen in Figure [1]. Here we see, that the two different portfolios lies on different paths/directions, where we that the eigenvector portfolio has introduced short-selling. As the eigenvector portfolios are found through unit-norm solutions and not simply wealth constraints, the eigenvector will often leads to unrealistic allocations of wealth for the private investor. Furthermore, indulging in high levels of short-selling could lead to losses greater than the invested wealth. This often introduces further uncertainty for the investor, especially for the private investor. Institutional investors may be able to capture the full utility of the eigenvector portfolio.

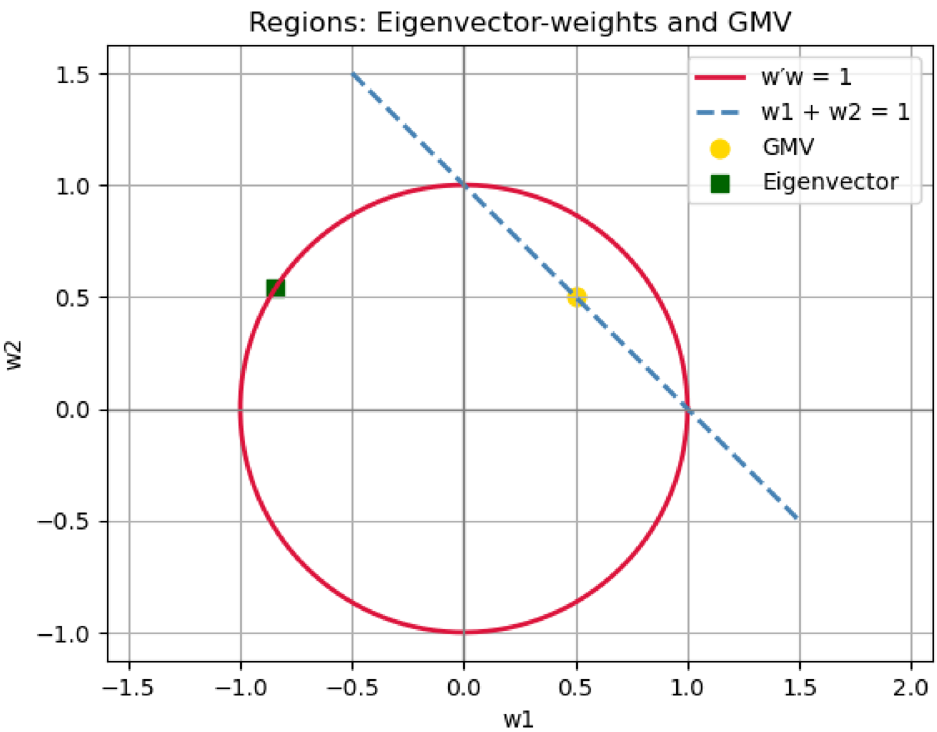

Figure 1: GMV vs Eigenvectors

2 Price discovery

We consider the VAR model for $X_t = (x_{1t}, ..., x_{pt})' \in R^p$ which is given by:

$$\Delta X_t = \alpha\beta' x_{t-1} + \varepsilon_t$$

Here with $\text{rank}(\alpha\beta') = p - 1$ and ε_t iid $N(0, \Omega)$.

2.1 Show the expression

We can then think of the efficient price, π as the cumulative sum of our ε_i shocks scaled by some underlying common trend. Under the assumptions ε_i is iid $N(0, \Omega)$ and that it can be shown, that the process is a martingale, as

$$E[\varepsilon_t|\mathcal{F}_{t-1}] = 0, \quad \text{Var}(\varepsilon_t) = \Omega$$

Since we both have iid and that the process is a martingale difference sequence, we may use Donsker's theorem.

$$T^{-\frac{1}{2}} \sum_{i=1}^{[Tu]} \varepsilon_i \xrightarrow{D} \mathcal{B}(u), \quad 0 \leq u \leq 1$$

We use Donsker's theorem to go from discrete sums to continuous Brownian motions. This can be done, as the ε_t is piecewise constant. I.e. on the space, $U \in [0,1]$, the time-scaled sums behaves like a p-dimensional Brownian motion, $\mathcal{B}(u)$, with covariance Ω.

We then remember the matrix in front of the sum, $(\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp$ We may denote this as C. We may think C as fixed weights, which turns the vector $S_T(u)$ into our price. As these weights do not change over time, we may take the Brownian motion and apply C to it. This will still give us a Brownian motion, but scaled differently. Therefore, by the continuous mapping theorem from the lecture notes, we have:

$$T^{-\frac{1}{2}} \pi_{[Tu]} = C \left(\frac{1}{\sqrt{T}} \sum_{i=1}^{[Tu]} \varepsilon_i \right) \xrightarrow{D} CB(u)$$

Where $C = (\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp$

2.2 Now argue for the proxy of the efficient price

We are now given the Granger representation with constant $X_0 = 0$. As the constant is zero, we are left with the trend and the stationary components. We must show that $f_t = \bar{\beta}_\perp X_t$ is a good proxy for the efficient price. Firstly, we may define $\bar{\beta}_\perp = \beta_\perp(\beta'_\perp \beta_\perp)^{-1}$. We then take the transposed $\bar{\beta}_\perp$, leaving us with:

$$\bar{\beta}'_\perp = (\beta_\perp(\beta'_\perp \beta_\perp)^{-1})' = (\beta'_\perp \beta_\perp)^{-1}\beta'_\perp$$

Therefore, we can start with the f_t.

$$f_t = \bar{\beta}'_\perp X_t = \bar{\beta}'_\perp \beta_\perp (\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i + \bar{\beta}'_\perp \alpha(\beta'\alpha)^{-1} S_t$$

$$f_t = (\beta'_\perp \beta_\perp)^{-1}\beta'_\perp \beta_\perp (\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i + (\beta'_\perp \beta_\perp)^{-1}\beta'_\perp \alpha(\beta'\alpha)^{-1} S_t$$

$$f_t = I_p \cdot (\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i + (\beta'_\perp \beta_\perp)^{-1} \cdot 0 \cdot (\beta'\alpha)^{-1} S_t$$

$$f_t = (\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp \sum_{i=1}^{t} \varepsilon_i$$

$$f_t = \pi$$

By the two orthogonalities from above, the stationary effects drops out, and the first collapses exactly to the definition of the efficient price. It shall be noted, that this only holds, if $\beta \in R^{p \times r}$ has full column rank and $\beta_\perp \in^{p \times (p-r)}$ is chosen such that its columns span the orthogonal complement of span(β). If this holds, then f_t is a good proxy for the efficient price.

2.3 Show that $\hat{f}_t$ is consistent

Following the same steps as just above, we see that we may use the estimator for the efficient price, if the same steps as section [2.2] goes trough. This leaves us with

$$\hat{f}_t = \hat{\bar{\beta}}'_\perp X_t$$

We need to check that replacing $\bar{\beta}_\perp$ by its estimator $\hat{\bar{\beta}}_\perp$ will not affect the limit, i.e that it is consistent. We can write:

$$\hat{f}_t = \hat{\bar{\beta}}_\perp X_t = \bar{\beta}'_\perp X_t + (\hat{\bar{\beta}}_\perp - \bar{\beta}_\perp)' X_t = \pi_t + \Delta_t$$

$$\hat{f}_t = \pi_t + \Delta_t$$

Here π_t is the efficient price as found above and $\Delta = (\hat{\bar{\beta}}_\perp - \bar{\beta}_\perp)'X_t$. Onto this we multiply $T^{-\frac{1}{2}}$

$$T^{-\frac{1}{2}}_{[Tu]}\hat{f}_t = T^{-\frac{1}{2}}_{[Tu]}(\pi_t + \Delta_t)$$

Therefore, we must show that:

$$\hat{\bar{\beta}}_\perp \xrightarrow{p} \bar{\beta}_\perp$$

This makes the $\hat{\bar{\beta}}_\perp - \bar{\beta}_\perp = o_p(1)$ the operator norm, which means that the largest singular value of the difference converges to zero in probability. Therefore, we may define the following:

$$(\hat{\bar{\beta}}_\perp - \bar{\beta}_\perp)'T^{-\frac{1}{2}}X_{[Tu]}$$

This converges to zero in probability for every u, and this convergence does not depend on u. Therefore, we may use it in our case:

$$T^{-\frac{1}{2}}\Delta_{[Tu]} = (\hat{\bar{\beta}}_\perp - \bar{\beta}_\perp)'T^{-\frac{1}{2}}X_{[Tu]} \xrightarrow{p} 0, \quad \text{uniformly in } u \in [0,1]$$

i.e. since the estimator error will go to zero in the operator-norm and the normalized partial sum process remains bounded in probability, the product will vanish. We now return to:

$$T^{-\frac{1}{2}}_{[Tu]}\hat{f}_t = T^{-\frac{1}{2}}_{[Tu]}(\pi_t + \Delta_t)$$

$$T^{-\frac{1}{2}}_{[Tu]}\hat{f}_t = T^{-\frac{1}{2}}_{[Tu]}\pi_t + T^{-\frac{1}{2}}_{[Tu]}\Delta_t$$

Earlier, we have shown that $T^{-\frac{1}{2}}_{[Tu]}\pi_t \xrightarrow{D} (\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp\mathcal{B}(u)$. At the same time, we have shown that the second term $T^{-\frac{1}{2}}\Delta_{[Tu]}$ vanishes uniformly in probability.

Slutsky's theorem for stochastic processes convergence then guarantees that adding a term, which shrinks to zero in probability will not alter the law in the limit. Therefore, our estimator $T^{-\frac{1}{2}}_{[Tu]}\hat{f}_t$ must converge to the same limit as $T^{-\frac{1}{2}}\pi_{[Tu]}$.

2.4 Show that as $T \to \infty$

From earlier, we have shown that $\hat{f}_T = \hat{\bar{\beta}}'_\perp X_T$ and $\pi_T = \bar{\beta}'_\perp X_T$. We then start by subtracting these two.

$$\hat{f}_T - \pi_T = \hat{\bar{\beta}}'_\perp X_T - \bar{\beta}'_\perp X_T = (\hat{\bar{\beta}}'_\perp - \bar{\beta}'_\perp)'X_t$$

We then multiply by $1 = \frac{\sqrt{T}}{\sqrt{T}}$

$$\hat{f}_T - \pi_T = \sqrt{T}(\hat{\bar{\beta}}'_\perp - \bar{\beta}'_\perp)'X_T\frac{1}{\sqrt{T}}$$

Here the first term is $\sqrt{T}(\hat{\bar{\beta}}'_\perp - \bar{\beta}'_\perp)'$ is the estimation error in the co-integration space and the second term $\frac{X_T}{\sqrt{T}}$ is the cumulated data at T. In distribution, they both satisfy their own CLT. Firstly, the estimation error:

$$\sqrt{T}(\hat{\bar{\beta}}'_\perp - \bar{\beta}'_\perp)' \xrightarrow{D} W \sim N(0, V_\beta)$$

Secondly, for the data:

$$\frac{X_T}{\sqrt{T}} \xrightarrow{D} M \sim N(0, \Sigma_x)$$

Moreover, if W and M are asymptotically independent we may use the continuous mapping theorem. By taking a dot-product

$$(w, m) \to w'm$$

This is just a continuous function of the pair (w, m). Therefore, by the Continuous Mapping theorem.

$$(\hat{\bar{\beta}}'_\perp - \bar{\beta}'_\perp)'X_T \xrightarrow{D} Z = W'M$$

Here Z is the random variable we get by taking the dot-product of the two independent normals W and M. Lastly, all of the errors we get from the finite T approximation, we will bundle into a single term, which vanishes in probability, $o_p(1)$. This leaves us with:

$$\hat{f}_T - \pi_T = Z + o_p(1)$$

In order to find the confidence bands. We must multiply the standard error by 1.96. This is for the 95% confidence bands. Therefore, we take the

$$SE = \sqrt{Var(\hat{f}_T - \pi_T)} \cdot 1.96$$

We then add and substract this on both sides of the estimation result, which will yield us the confidence bands. Furthermore, following theory, the closer we get to infinity, the better the estimation should be. Therefore, the estimator should converge towards its true price, but even as we collect more data, the level of uncertainty will settle into a fixed spread instead of vanishing as $T \to \infty$