Matematik A E2020
Uge 48, Forelæsning 2

Eksempler på optimeringsproblemer
(i en økonomisk model)

Lidt overblik

- Evt hængepartier fra mandag

- Eksempler på optimeringsproblemer!
  - En model af et duopol (exercise 13.4.5 fra bogen)
    - "Bertrand-konkurrence med differentierede produkter"
    - Løsninger til opgaven (fra Student's Manual) uploades til Absalon sammen med slides med noter

- Næste uge: Optimering med bibet./Lagrangemetoden (kapitel 14)

Exercise 13.4.5 (s. 516)

SM 5. *(Duopoly)* Each of two firms A and B produces its own brand of a commodity such as mineral water in amounts denoted by x and y, which are sold at prices p and q per unit, respectively. Each firm determines its own price and produces exactly as much as is demanded. The demands for the two brands are given by

$$x = 29 - 5p + 4q \text{ and } y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$. (Assume that the functions to be maximized have maxima, and at positive prices.)

(a) Initially, the two firms collude in order to maximize their combined profit, as one monopolist would. Find the prices (p, q), the production levels (x, y), and the profits of firms A and B.

(b) Then an anti-trust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given. If q is fixed, how will A choose p as a function $p = p_A(q)$ of q? If p is fixed, how will B choose q as a function $q = q_B(p)$ of p?

(c) Under the assumptions in part (b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

(d) Draw a diagram with p along the horizontal axis and q along the vertical axis, and draw the "reaction" curves $p_A(q)$ and $q_B(p)$. Show on the diagram how the two firms' prices change over time if A breaks the cooperation first by maximizing its profit, taking B's initial price as fixed, then B answers by maximizing its profit with A's price fixed, then A responds, and so on.

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→ Opstil profitmaksimeringsproblemet og løs det (find max-pkt)!

Profitfkt: $\pi_M(p,q) = \underbrace{xp - (5+x)}_{\pi_A} + \underbrace{yq - (3+2y)}_{\pi_B}$

$= x(p-1) + y(q-2) - 8$

$= (29 - 5p + 4q)(p-1) + (16 + 4p - 6q)(q-2) - 8$

Profitmax-problem: $\max_{p,q_i>0;} \pi_M(p,q)$

FOCs: mht p: $-5(p-1) + (29 - 5p + 4q)\cdot 1 + 4(q-2) = 0$

mht q: $4(p-1) - 6(q-2) + (16 + 4p - 6q)\cdot 1 = 0$

$$\left.\begin{aligned} -10p + 8q + 26 &= 0 \\ 8p - 12q + 24 &= 0 \end{aligned}\right\}$$ 2 lin. lign. m. 2 obek.

Løsn: $p = 9,\ q = 8$

$x = 16,\ y = 4$

$\pi_M = 144$

Er det et max-pkt? π strengt bet. for (globalt) max-pkt. [Afsnit 13.2]

$\pi''_{M\,pp} = -10 \leq 0$ $\quad \pi''_{M\,qq} = -12 \leq 0$ $\quad \pi''_{M\,pq} = 8$

$$\pi''_{M\,pp} \cdot \pi''_{M\,qq} - \left(\pi''_{M\,pq}\right)^2 = 120 - 64 \geq 0$$

$$x = 29 - 5p + 4q \text{ and } y = 16 + 4p - 6q$$

Firm A has total costs $5 + x$, whereas firm B has total costs $3 + 2y$.

(b) Then an anti-trust authority prohibits collusion, so each producer maximizes its own profit, taking the other's price as given. If q is fixed, how will A choose p as a function $p = p_A(q)$ of q? If p is fixed, how will B choose q as a function $q = q_B(p)$ of p?

→ Opstil profitmaksimeringsproblem for hhv. A og B

Løs dem og find dermed $p_A(q)$ og $q_B(p)$

A: $\max_p \; xp - (5+x) = (29 - 5p + 4q)(p-1) - 5$

B: $\max_q \; yq - (3+2y) = (16 + 4p - 6q)(q-2) - 3$

$p_A(q)$

A-FOC : $-5(p-1) + (29 - 5p + 4q) \cdot 1 = 0 \Rightarrow p = \frac{1}{5}(17 + 2q)$

B-FOC : $-6(q-2) + (16 + 4p - 6q) \cdot 1 = 0 \Rightarrow q = \frac{1}{3}(7 + p)$

$q_B(p)$

(c) Under the assumptions in part (b), what constant equilibrium prices are possible? What are the production levels and profits in this case?

Fra (b):

$$p_A(q) = \frac{1}{5}(17+2q) \qquad q_B(p) = \frac{1}{3}(7+p)$$

En *(Nash)-ligevægt* (Nash equilibrium) består af priser p^* og q^* så:

$$p_A(q^*) = p^* \quad \text{og} \quad q_B(p^*) = q^*$$

→ Find ligevægten ved at opstille to ligninger med to ubekendte og løse dem

Sammenlign med situationen fra (a)

$$\frac{1}{5}(17+2q^*) = p^* \text{ og } \frac{1}{3}(7+p^*) = q^*$$

Løs for p^* og q^* : $p^* = 5$, $q^* = 4$

$x^* = 20$, $y^* = 12$

$\pi_A^* = 75$, $\pi_B^* = 21$

(d) Draw a diagram with p along the horizontal axis and q along the vertical axis, and draw the "reaction" curves $p_A(q)$ and $q_B(p)$. Show on the diagram how the two firms' prices change over time if A breaks the cooperation first by maximizing its profit, taking B's initial price as fixed, then B answers by maximizing its profit with A's price fixed, then A responds, and so on.

Fra (b):

$$p_A(q) = \frac{1}{5}(17 + 2q) \qquad q_B(p) = \frac{1}{3}(7 + p)$$

Plot disse i (p, q)-koordinatsystem: