Macroeconomic Models

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Introduction

This book introduces the workhorse models of modern macroeconomics. Its target audience are master students or beginning doctoral students in economics who are familiar with microeconomics and calculus at the undergraduate level.

The book aims at being both concise and rather comprehensive. Conciseness is helped by simple formal arguments which are augmented by verbal explanations emphasizing economic intuition. Most formal results are derived in the text; missing derivations typically are the subject of an exercise in the (companion) exercise manual. Notes on the literature at the end of each chapter point to classic articles and further readings.

The book is about concepts and frameworks, not data. Insightful models are abstract parables—they rely on dramatic simplifications and tell internally consistent stories. Confronting models with historical data or using them for predictions is hard, especially in macroeconomics with its concern for economic aggregates. In addition to a thorough understanding of macroeconomic theory, it requires expert knowledge of real world circumstances and data as well as familiarity with econometric methods. This book develops the theory part, it has nothing to say about empirics.

A word on notation: Additions like "for all i," "at all dates t," or "$t = 0, 1, 2, \ldots$" typically are omitted unless there is danger of confusion. When a statement needs qualification, however, (for example, because it only applies for specific dates t) then the qualification is included.

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Chapter 1

Microeconomic Foundations

Modern macroeconomic models are micro founded. They derive results from assumptions about microeconomic primitives, specifically preferences and technology, in contrast to frameworks such as the IS-LM model which often rely on ad-hoc assumptions, for example about the relationship between consumption and income. Accordingly, modern macroeconomics uses the concepts and tools of *general equilibrium* theory. It describes economic outcomes as the result of optimizing choices by households and firms that interact on markets, over time, subject to affordability and feasibility constraints; the government may affect these constraints. In contrast to *partial equilibrium* analyses, general equilibrium theory accounts for feedback effects across all markets. Only those influences which are considered to be non-economic are treated as exogenous.

To prepare for the subsequent micro founded macroeconomic analysis, we review key microeconomic concepts and introduce assumptions about the primitives.

1.1 Microeconomics

1.1.1 Allocation, Feasibility, Optimality

An *allocation* consists of a consumption vector for each household and a net production vector for each firm. For example, an allocation in an economy with one household, one firm and three goods could be $\{(1,2,1),(-1,1,-1)\}$: The household consumes one unit each of the first and third good and two units of the second good, while the firm uses one unit each of the first and third good as inputs and supplies one unit of the second good. In a model with government the allocation also includes a consumption and production vector for the government. In an open economy model the allocation also includes a consumption and production vector for the "rest of the world."

An allocation is *feasible* if for each good, total consumption does not exceed the endowment plus net production. For example, the allocation given above is feasible if the endowment vector equals $(2,1,2)$ but it is not feasible if the endowment vector equals $(2,2,1)$.

A feasible allocation Pareto dominates another feasible allocation if at least one household strictly prefers the former and no household strictly prefers the latter. A feasible allocation is *Pareto optimal* or Pareto efficient if it is not Pareto dominated by any other feasible allocation. The set of Pareto optimal allocations traces the Pareto frontier.

1.1.2 Competitive Equilibrium

The consumption set of a household contains all consumption vectors that the household conceivably could consume in the absence of budgetary restrictions. For example, the consumption set might exclude negative quantities of apples.

The budget set of a household contains all consumption vectors in the household's consumption set that the household can afford to consume. The budget set is determined by household endowments, the prices of all goods, and firm profits which are distributed to households according to prespecified ownership rights.

The production set of a firm contains all production vectors that are feasible given the firm's technology.

A *competitive equilibrium* or *Walrasian equilibrium* is an allocation and a set of prices satisfying three conditions, conditional on endowments, firm production sets, and household preferences:

i. The allocation is feasible.

ii. Taking prices as given, each firm's production choice is profit maximizing.

iii. Taking prices and firm profits as given, each household's consumption choice is utility maximizing in the household's budget set.

The equilibrium is "competitive" because firms and households take prices as given. Alternative, non-competitive equilibria might exist as well where agents perceive their choices to affect prices or firm profits, and they exploit this feature. For example, a firm might want to reduce output in order to raise the equilibrium price of its product. We mostly abstract from non-competitive behavior and focus on competitive equilibria.

A competitive equilibrium with lump-sum transfers between households that sum to zero is referred to as price equilibrium with transfers.

1.1.3 Walras' Law

Let p denote the vector of prices across goods. Let $z_g^h(p)$ denote household h's *net demand function* for good g—the household's desired consumption net of its endowment and share of firm profits—as a function of p. Let $z^h(p)$ denote the vector of household h's net demand functions across goods. Let $z_g(p) \equiv \sum_h z_g^h(p)$ denote the aggregate excess demand function for good g. Finally, let $z(p) \equiv \sum_h z^h(p)$ denote the vector of excess demand functions across goods. If all households satisfy their budget constraints, then $p \cdot z^h(p) = 0$ for all h. By implication, *Walras' Law* holds: The values of excess demands sum to zero, $p \cdot z(p) = 0$.

Walras' Law has two important consequences. First, in an equilibrium with strictly positive prices all markets clear. To see this, note that the equilibrium requirements optimization and feasibility (subject to free disposal) imply $z(p) \leq 0$. If a good has strictly positive price, excess demand for that good therefore must be zero (otherwise, $p \cdot z(p) \neq 0$). Second, with strictly positive prices, market clearing in all markets but one implies market clearing in the remaining market. To see this, suppose all markets except market j clear, $z_g(p) = 0$ for all $g \neq j$, such that $\sum_{g \neq j} p_g z_g(p) = 0$. Since $p_j > 0$, $z_j(p)$ also must equal zero (otherwise, $p \cdot z(p) \neq 0$).

1.1.4 Fundamental Theorems of Welfare Economics

The fundamental theorems of welfare economics relate equilibrium allocations and Pareto optimal allocations.

The *first fundamental theorem of welfare economics* formalizes the notion of an "*invisible hand*." It states that, if an allocation and price system constitute a price equilibrium with transfers (in particular, a competitive equilibrium) and certain conditions are satisfied, then the allocation is Pareto optimal. Decentralized choices by price taking individuals thus are consistent with Pareto optimality, and this holds true even if lump-sum transfers occur before market transactions take place.

Let x^h, e^h, and y^f denote household h's consumption and endowment vectors as well as firm f's net production vector, respectively. An allocation $\{\{x^h\}_h, \{y^f\}_f\}$ is feasible if $\sum_h (x^h - e^h) \leq \sum_f y^f$. Let $\{\{x^{h\star}\}_h, \{y^{f\star}\}_f, p^\star\}$ be a competitive equilibrium with equilibrium price vector $p^\star$. The theorem claims that no feasible allocation Pareto dominates $\{\{x^{h\star}\}_h, \{y^{f\star}\}_f\}$. The proof by contradiction proceeds in three steps:

i. Suppose that a feasible Pareto dominating allocation, $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$, exists such that some household strictly prefers the consumption vector in the $\bullet$ allocation over the vector in the $\star$ allocation. Impose the condition that preferences are locally non-satiated; households and firms are competitive; and markets are complete (all goods are traded). Since the household chose optimally, the consumption vector in the $\bullet$ allocation then must be unaffordable under $p^\star$.

ii. Impose the condition that the market value of endowments be finite, for example because the number of households is finite. Summing over all households then implies
$$\sum_h p^\star \cdot (x^{h\bullet} - e^h) > \sum_f p^\star \cdot y^{f\star}.$$

iii. Since firms maximize profits, $\sum_f p^\star \cdot y^{f\star} \geq \sum_f p^\star \cdot y^{f\bullet}$. Combining the inequalities yields
$$\sum_h p^\star \cdot (x^{h\bullet} - e^h) > \sum_f p^\star \cdot y^{f\bullet}.$$

But feasibility of $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$ implies the reverse inequality. We have therefore arrived at a contradiction.

The *second fundamental theorem of welfare economics* formalizes the notion that Pareto optimal allocations can be decentralized through markets and lump-sum transfers. It states that for every Pareto optimal allocation, there exists a set of prices such that the allocation and the prices constitute a price equilibrium with transfers. Necessary assumptions of the theorem include convex production sets as well as convex and locally non-satiated preferences.

1.2 Primitives

The *primitives* of a modern macroeconomic model—the objects we take as given—are those of a microeconomic model, specifically preferences and technology. Since macroeconomic models typically feature dynamic and stochastic environments we review the event tree and discuss how preferences can be specified in such environments. We also review the neoclassical production function.

1.2.1 Event Tree

Time is denoted by t. It runs from zero, the initial date, to some final date T or to infinity. To represent exogenous risk we let ϵ^t denote the *history* of realizations of the "*state of nature*" up to and including date t. From the perspective of date $t = 0$, history ϵ^0 is known but history $\epsilon^t, t \geq 1$, is a random variable unless there is no risk.

The *event tree* in figure 1.1 illustrates a three-period example. At date $t = 1$, one of two possible realizations occurs, "up" (for example promotion) or "down" (demotion). The same happens at date $t = 2$. History ϵ^1 thus takes two values, (up) or (down), and history ϵ^2 four, (up, up), (up, down), (down, up), or $(\text{down}, \text{down})$.

Except for deterministic settings, variables need to be indexed by history to avoid ambiguity. Consider for instance a variable c, consumption of fruit say. Indexing c by ϵ^t accounts for the fact that fruit in different histories at the same date constitute different commodities. In the environment of figure 1.1, fruit consumption at date $t = 2$ can take four values, $c_2(\text{up}, \text{up})$, $c_2(\text{up}, \text{down})$, $c_2(\text{down}, \text{up})$, or $c_2(\text{down}, \text{down})$.

A variable may be indexed by date t and history $\epsilon^s, s < t$. This indicates that the variable takes the same value across all histories at date t that are continuation histories of history ϵ^s. For example, in the environment of figure 1.1, $c_2(\text{up})$ would indicate that fruit consumption at date $t = 2$ in history (up, up) and in history (up, down) are the same.

1.2.2 Preferences

In the simplest dynamic model, *households* consume a single good (which might represent a composite), c, in each history. They trade off consumption across time and histories, in parallel to the static trade-off between "apples" and "oranges" in a simple microeconomic model. As we will see in chapters 2 and 4, the optimality conditions

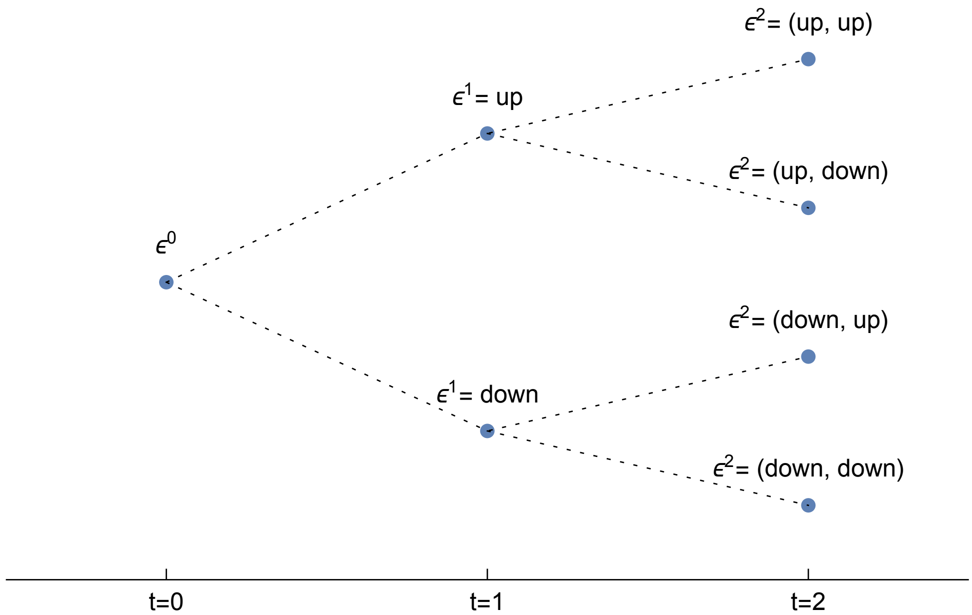

Figure 1.1: Event tree: An economy with three periods and two states of nature each at dates $t = 1, 2$, "up" or "down."

in dynamic and static settings are isomorphic under specific assumptions about the market structure.

Preferences map a sequence of history-contingent consumption into utility,

$$U\left(c_0, \{c_1(\epsilon^1)\}_{\epsilon^1}, \ldots, \{c_T(\epsilon^T)\}_{\epsilon^T}\right).$$

The *lifetime utility function* U is increasing in all its arguments. The notation $\{c_t(\epsilon^t)\}_{\epsilon^t}$ indicates that consumption at date t takes multiple values, depending on history. For example, in the environment of figure 1.1, we have $\{c_1(\epsilon^1)\}_{\epsilon^1} = \{c_1(\text{up}), c_1(\text{down})\}$.

We often assume that preferences are *additively separable* across time and histories that is, U is a weighted sum. The weight attached to date t equals β^t where $\beta \in [0, 1)$ denotes the *psychological discount factor* which measures the degree of patience. The weight attached to a particular history equals the probability that this history occurs. A consumption sequence thus is evaluated according to the discounted *expected utility* that it generates,

$$U = \mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t u(c_t(\epsilon^t))\right].$$

Here, $\mathbb{E}_s$ denotes the mathematical expectation conditional on information available at date s, history ϵ^s. Separability across time and histories implies that the marginal utility of consumption at a date and history is independent of consumption at other dates and histories. This property often simplifies the analysis.

The *period utility function* or *felicity function* u exhibits strictly positive, decreasing marginal utility. Unless otherwise noted, we assume that it is continuously differen-

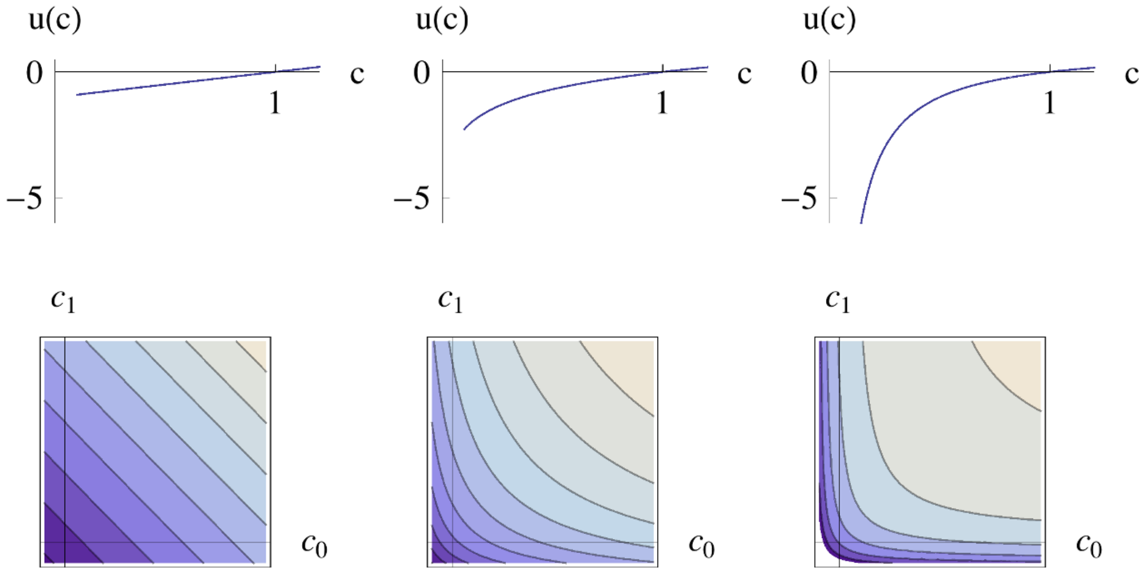

Figure 1.2: CIES preferences: Period utility function and indifference curves for $\sigma = 0.01, 0.99, 2.00$ (from left to right).

tiable; marginal utility is strictly decreasing; and consumption at each date is essential, $\lim_{c\downarrow 0} u'(c) = \infty$.

We sometimes restrict u to be of the *constant intertemporal elasticity of substitution* (CIES) or equivalently, *constant relative risk aversion* form. Function u then is given by

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ for } \sigma > 0,\ \sigma \neq 1.$$

This functional form is not defined for $\sigma = 1$, but applying L'Hôpital's rule implies $\lim_{\sigma\to 1} u(c) = \ln(c)$; the logarithmic function thus constitutes a limiting case.

As the name suggests, CIES preferences exhibit constant relative risk aversion and a constant intertemporal elasticity of substitution. To see the former, recall that the *coefficient of relative risk aversion* is defined as $-u''(c)c/u'(c)$; with CIES preferences this reduces to σ. To see the latter, recall that the *elasticity of substitution* measures how strongly a change of relative price affects relative demand. With CIES preferences, the elasticity of the ratio c_{t+1}/c_t with respect to the relative price of c_{t+1} and c_t reduces to $1/\sigma$. As we will see in chapter 2, CIES preferences simplify the equilibrium conditions in dynamic models.

Figure 1.2 illustrates the role of the elasticity of substitution. The top row of the figure plots $u(c)$ and the bottom row plots indifference curves of the utility function $U = u(c_0) + \beta u(c_1)$. As σ increases the indifference curves gain curvature; the tangency point of an indifference curve and the price line therefore moves less in response to a given change of relative price.

1.2.3 Technology

Firms employ a *production function*, f, that maps inputs of physical capital, K, and labor, L, into output. Unless otherwise noted, we assume that the production function is *neoclassical*. That is, f exhibits strictly positive and diminishing marginal products as well as *constant returns to scale*,

$$f_K(K,L), f_L(K,L) > 0; \;\; f_{KK}(K,L), f_{LL}(K,L) < 0; \;\; \phi f(K,L) = f(\phi K, \phi L), \; \phi > 0.$$

Here, subscripts denote partial derivatives, $f_K(K,L) \equiv \partial f(K,L)/\partial K$ and $f_{KK}(K,L) \equiv \partial^2 f(K,L)/(\partial K)^2$; we use this notation throughout the book.

Due to constant returns to scale, output coincides with total factor payments to the suppliers of K and L if the rental rates of K and L equal the respective marginal products. On competitive factor markets this is the case. Also due to constant returns to scale, output per worker as well as marginal products only depend on the capital-labor ratio, $k \equiv K/L$, not on K and L individually. Both these facts follow from Euler's homogeneous function theorem.

The production function satisfies the *Inada conditions* when

$$\lim_{K \downarrow 0} f_K(K,L) = \lim_{L \downarrow 0} f_L(K,L) = \infty, \;\; \lim_{K \to \infty} f_K(K,L) = \lim_{L \to \infty} f_L(K,L) = 0.$$

The Inada conditions help guarantee interior equilibria.

The *constant elasticity of substitution* (CES) production function,

$$f(K,L) = \left(\alpha K^{1-\frac{1}{\theta}} + (1-\alpha) L^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \; \theta > 0, \; \alpha \in (0,1),$$

constitutes a tractable example of a neoclassical production function. The elasticity of substitution between K and L is constant in this case and equals θ. For $\theta \to \infty$, the CES production function approaches a linear production function, and for $\theta \to 0$ it approaches the Leontief production function.

For $\theta \to 1$, the CES production function converges to the *Cobb-Douglas production function*,

$$f(K,L) = K^{\alpha} L^{1-\alpha}.$$

When production factors are paid their marginal products the Cobb-Douglas production function implies constant factor shares, $K f_K(K,L)/f(K,L) = \alpha$ and $L f_L(K,L)/f(K,L) = 1-\alpha$.

1.3 Bibliographic Notes

The Walrasian equilibrium notion is due to Walras (1874) and other representatives of the (pre)marginalist school (von Thünen, Cournot, Dupuis, Gossen, Jevons, Menger). Arrow and Debreu (1954) and McKenzie (1954) use fixed point arguments to prove existence of general equilibrium and Debreu (1959) proves the welfare theorems. Bewley (1972) proves that in economies with a finite number of *infinitely* lived, impatient households an equilibrium still exists and is Pareto optimal. Arrow (1953; 1964) and Debreu (1959, 7) define commodities with reference to the event tree. Cobb and Douglas (1928) discuss the production function named after them.

Mas-Colell, Whinston and Green (1995) provide a comprehensive review of microeconomic theory.

The notions of macroeconomic "equilibrium" have changed over the years. Hicks (1939) envisions markets that operate sequentially and he distinguishes between *temporary equilibrium* in spot markets (conditional on expectations about the future), and *equilibrium over time* when expected and actual prices coincide. Stigum (1969) relaxes the restriction on expectations; see also Grandmont (1977) who discusses the notion of a *sequence of temporary equilibria*, possibly including learning. Common price expectations across agents and consistent plans based on these expectations form an *equilibrium of plans, prices and price expectations* (Hahn, 1971; Radner, 1972). When agents have heterogeneous information sets market prices may reveal some of the unobserved information and agents may use a model of the relationship between the non-price information and equilibrium prices to infer the information; in a *rational expectations equilibrium* the model corresponds, in equilibrium, to the correct model (Lucas, 1972). Barro and Grossman (1971) study a general "disequilibrium" model where excess supplies and demands in different markets affect each other.

More broadly, the methodological approach to studying macroeconomic questions has changed profoundly over the last 60 years. In the late 1950s, microeconomic reasoning coexisted with Keynesian arguments (Keynes, 1936; Hicks, 1939) that often rested on weaker choice theoretic foundations. During the 1960s, empirical macroeconomics suffered setbacks, not least because the *Phillips curve* relationship between observed unemployment and (wage) inflation rates proved less stable than expected. Friedman's (1968, p. 8) dictum that the Phillips curve presumed "a world in which everyone anticipated that nominal prices would be stable . . . whatever happened" stimulated the search for models that better reconcile micro- and macroeconomics (see for example Phelps, 1970). Building on Muth (1961), Sargent (1971) and Lucas (1972) promoted the "rational expectations" consistency requirement.

By the late 1970s, the profession had lost faith in the "*neoclassical synthesis*" (Samuelson, 1955) and in particular, in policy analysis based on large-scale macroeconometric models. It had become clear that optimizing behavior also concerns expectation formation; reduced form relationships in models without micro foundations are not policy-invariant (Lucas, 1976); and "claims for identification in [large-scale statistical macroeconomic] models" are unfounded (Sims, 1980, p. 1). Early micro founded dynamic general equilibrium models in the 1980s lacked plausible frictions and abstracted from heterogeneity which limited their relevance. Modern macroeconomic models for applied purposes—*dynamic stochastic general equilibrium (DSGE) models*—feature heterogeneity, risk, and diverse frictions.

For an overview over the history of economic thought until 1980, see for example Niehans (1994).

Chapter 2

Consumption And Saving

The consumption-saving tradeoff of households constitutes the backbone of most modern macroeconomic models. In this chapter, we study the household's dynamic utility maximization problem and the induced demand functions. Throughout, we abstract from risk and assume that leisure does not enter preferences.

2.1 Consumption Smoothing

Consider a household that owns a (possibly negative) stock of assets, a_t. The household receives (or pays) interest income on the assets, $a_t(R_t - 1)$, where R_t denotes the gross interest rate, and receives exogenous wage income, w_t. The stock of assets and the two incomes fund consumption, c_t, or can be carried into the next period, a_{t+1}. The *dynamic budget constraint*

$$a_{t+1} = a_t R_t + w_t - c_t$$

represents the resulting *law of motion* for assets. Rearranging terms, the dynamic budget constraint states that the change in the asset position, $a_{t+1} - a_t$, equals *saving* or income minus consumption.

2.1.1 Two Periods

With two periods, the household's objective function is given by

$$u(c_0) + \beta u(c_1).$$

When the household has no assets to start with ($a_0 = 0$), the dynamic budget constraints at the two dates read

$$\begin{aligned} a_1 &= w_0 - c_0, \\ a_2 &= a_1 R_1 + w_1 - c_1. \end{aligned}$$

Since the household "dies" at the end of date $t = 1$ nobody will lend it resources at that date; terminal household assets therefore must be non-negative, $a_2 \geq 0$. Moreover, carrying strictly positive assets into date $t = 2$ would be wasteful because the household cannot consume after its death. The optimal saving choice at date $t = 1$ thus is $a_2 = 0$.

Using this result and combining the two dynamic budget constraints, we arrive at the *intertemporal budget constraint*

$$c_0 + \frac{c_1}{R_1} = w_0 + \frac{w_1}{R_1}.$$

The terms on the left-hand side represent total spending on the two goods, consumption in the first and second period, c_0 and c_1 respectively. Consumption in the first period is the *numeraire*, its price is normalized to unity. The relative price of consumption in the second period is given by the inverse of the gross interest rate, $1/R_1$. Intuitively, reducing consumption in the first period by one unit raises saving and increases consumption in the second period by R_1 units. One unit of first-period consumption therefore buys R_1 units of second-period consumption, or one unit of second-period consumption costs $1/R_1$ units of first-period consumption.

The terms on the right-hand side of the intertemporal budget constraint represent the household's wealth, that is the date $t = 0$ market value of first- and second-period wage income. Note that the intertemporal budget constraint is isomorphic to the budget constraint in a static model of consumer choice.

To find the household's optimal level of saving in the first period, we may solve the dynamic budget constraints for consumption and substitute the resulting expressions into the objective function. The household's program then reads[1]

$$\max_{a_1} u(w_0 - a_1) + \beta u(a_1 R_1 + w_1).$$

An interior solution to this program satisfies the first-order condition or *Euler equation*

$$u'(c_0) = \beta R_1 u'(c_1) \text{ or } \frac{u'(c_0)}{\beta u'(c_1)} = R_1$$

where we re-introduce the variables c_0 and c_1 for ease of notation.

The second representation of the Euler equation states that the marginal rate of substitution between current and future consumption is equated with the relative price between the two goods. This is the same condition as in a static model with "apples" and "oranges" where the price line is tangent to the highest indifference curve.

The Euler equation characterizes optimal second-period consumption relative to first-period consumption and thus, the slope of the optimal consumption path but not its level. Three factors determine whether and how strongly consumption increases or decreases over time. First, β. More patience increases the weight given to future utility and thus, the slope of the optimal consumption path. Formally, a higher β implies a higher ratio $u'(c_0)/u'(c_1)$ and thus (if u is strictly concave), a higher c_1/c_0. Second, R_1. A higher interest rate renders second-period consumption cheaper, also implying

[1] Throughout, we abstract from non-negativity constraints on consumption unless they may be relevant.

a higher c_1/c_0. Third, the curvature of the marginal utility function (recall figure 1.2). It determines how strongly a change of β or R_1 translates into a steeper or flatter consumption profile. More curvature implies a stronger *consumption smoothing* motive that is, less willingness to intertemporally substitute.

To solve for the equilibrium consumption levels in terms of the exogenous variables we combine the Euler equation and the intertemporal budget constraint (or the two dynamic budget constraints). If the period utility function is of the CIES type (such that the Euler equation reads $c_0^{-\sigma} = \beta R_1 c_1^{-\sigma}$) this yields

$$c_0 = \left(w_0 + \frac{w_1}{R_1}\right) \Big/ \left(1 + \frac{(\beta R_1)^{1/\sigma}}{R_1}\right).$$

From the budget constraint, we may also solve for a_1 and c_1.

We have completely characterized optimal consumption conditional on β, u, R_1, w_0, and w_1. Two important results emerge. First, optimal consumption depends on wealth, $w_0 + w_1/R_1$, or *permanent income*, not only on contemporaneous income as with a Keynesian consumption function. This is a consequence of the household's desire to smooth consumption over the life cycle if marginal utility is decreasing ($u'' < 0$), and its ability to do so by means of saving or borrowing.

Second, a change of interest rate affects optimal consumption threefold, through two *income* or *wealth effects* and one *substitution effect*. First, if $w_1 > 0$, an increase in the interest rate reduces wealth because it lowers the date $t = 0$ market value of future labor income. This leads the household to consume less in both periods. Second, for given quantities c_0 and $c_1 > 0$, an increase in the interest rate lowers the cost of the bundle (c_0, c_1) expressed in terms of the numeraire. The higher purchasing power leads the household to consume more of both goods. Finally, from the Euler equation, it is optimal to substitute towards the cheaper good. An increase in the interest rate thus leads the household to increase c_1 relative to c_0. The strength of the substitution effect depends on the intertemporal elasticity of substitution, $1/\sigma$.

Figure 2.1 illustrates the three effects. The black indifference curve, budget line and tangency point characterize the equilibrium at a low interest rate, the red counterparts the equilibrium at a higher interest rate. The substitution effect corresponds to the distance between the gray and the black point, the income effect due to increased purchasing power to the distance between the blue and the gray point, and the wealth effect due to lower discounted labor income to the distance between the red and the blue point. The figure is plotted for the logarithmic utility case, $\sigma = 1$, such that the income effect due to increased purchasing power and the substitution effect on c_0 exactly cancel.

2.1.2 More Periods

More generally, the household's program comprises a multi-period objective function and multiple dynamic budget constraints in addition to the initial and terminal condi-

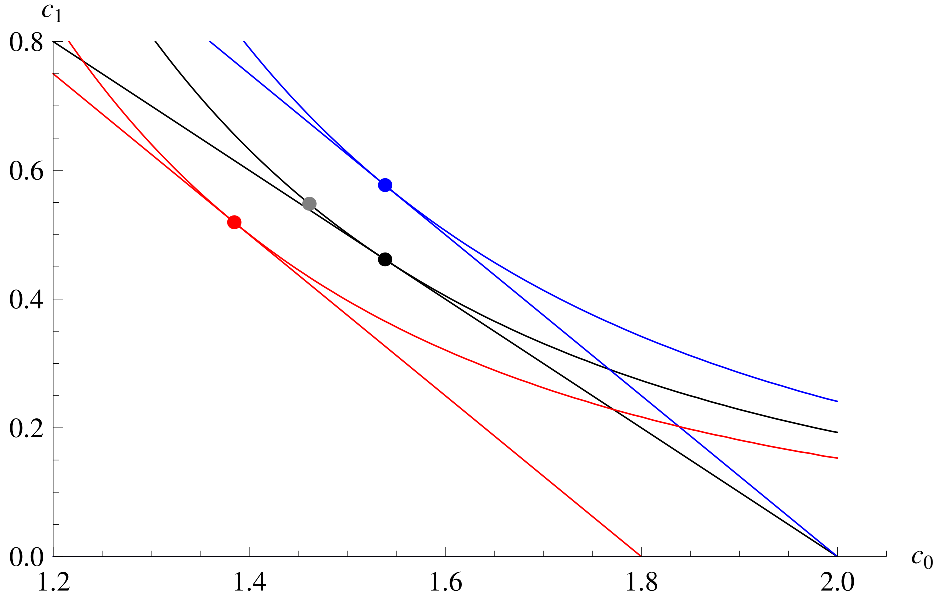

Figure 2.1: Income/wealth and substitution effects due to a change of interest rate. The black (red) tangency point indicates the equilibrium at a low (high) interest rate.

tions:

$$\max_{c_0,\dots,c_T,a_1,\dots,a_{T+1}} \sum_{t=0}^{T} \beta^t u(c_t) \text{ s.t. } a_{t+1} = a_t R_t + w_t - c_t,\ a_0 R_0 \text{ given},\ a_{T+1} \geq 0.$$

In parallel with the strategy adopted in the two-period case, we can confront this problem by solving each of the dynamic budget constraints for consumption and substituting the resulting expressions into the objective function. Conjecturing again that a_{T+1} optimally equals zero this yields

$$\max_{a_1,\dots,a_T} \sum_{t=0}^{T} \beta^t u(a_t R_t + w_t - a_{t+1}) \text{ s.t. } a_0 R_0 \text{ given},\ a_{T+1} = 0.$$

Differentiating with respect to the choice variables, we find an Euler equation for each date t.

Lagrangian

An alternative strategy uses the intertemporal budget constraint. The latter can be derived, as before, by combining the dynamic budget constraints:

$$\begin{aligned} a_{T+1} &= a_T R_T + w_T - c_T \\ &= (a_{T-1}R_{T-1} + w_{T-1} - c_{T-1})R_T + w_T - c_T \\ &= \ldots \\ &= a_0 R_0 R_1 \cdots R_T + (w_0 - c_0)R_1 R_2 \cdots R_T + (w_1 - c_1)R_2 \cdots R_T \\ &\quad + \ldots + (w_{T-1} - c_{T-1})R_T + (w_T - c_T). \end{aligned}$$

Let $q_t \equiv (R_1 R_2 \cdots R_t)^{-1}$ denote the price of date-t consumption at the initial date and define $q_0 \equiv 1$. Multiplying the intertemporal budget constraint by q_T and using the optimality condition $a_{T+1} = 0$ yields

$$q_T a_{T+1} = 0 = a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t) \text{ or } \sum_{t=0}^{T} q_t c_t = a_0 R_0 + \sum_{t=0}^{T} q_t w_t.$$

In parallel to the two-period case, the intertemporal budget constraint equalizes lifetime consumption spending and lifetime wealth at market prices.

Replacing the $T+1$ dynamic budget constraints with the single intertemporal budget constraint, the household's program can be expressed as

$$\max_{c_0,\ldots,c_T} \sum_{t=0}^{T} \beta^t u(c_t) \text{ s.t. } a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t) = 0.$$

Forming the Lagrangian (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda[a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t)]$$

and differentiating yields T first-order conditions,

$$\beta^t u'(c_t) = \lambda q_t.$$

They state that marginal utility from consumption in a period equals the price of consumption in that period, q_t, times the multiplier attached to the intertemporal budget constraint, λ. Since the multiplier measures the effect of a marginal relaxation of the constraint on the maximized objective function, λ represents the *shadow value* of wealth. Combining the first-order conditions at date t and $t+1$ yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. To solve for the optimal consumption levels, we combine the Euler equations and the intertemporal budget constraint.

Yet another approach to solving the household's program relies on forming a Lagrangian that incorporates the dynamic budget constraints and the terminal condition

$a_{T+1} \geq 0$, and deriving the optimality condition $a_{T+1} = 0$ rather than imposing it from the beginning. This Lagrangian reads (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^{T} \{\beta^t u(c_t) - \lambda_t [a_{t+1} - (a_t R_t + w_t - c_t)]\} + \mu a_{T+1}.$$

The first-order conditions with respect to c_t and a_{t+1} are given by

$$\beta^t u'(c_t) = \lambda_t,$$
$$\lambda_t = \lambda_{t+1} R_{t+1},\ t = 0, \ldots, T-1,$$

respectively. The first-order condition with respect to a_{T+1} is $\lambda_T = \mu$ and the complementary slackness condition is given by $\mu a_{T+1} = 0$.

Combining the first-order conditions with respect to consumption again yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. Moreover, non-satiation ($u' > 0$) implies $\lambda_T > 0$ and thus $\mu > 0$ which in turn implies $a_{T+1} = 0$, the *transversality condition* we had informally argued before.

Dynamic Programming

We may also solve the household's program using *dynamic programming* techniques. Recall that the *indirect utility function* gives the maximal utility as a function of the parameter(s) of the utility maximization problem. In the context of a consumption saving program, the parameters of the indirect utility function include the level of initial assets, a_t, as well as preference parameters, wages and interest rates over the remaining lifetime.

The household's *value function* is the indirect utility function and the corresponding parameters are referred to as the *state* or state variable(s) which summarize both the effects of past decisions and current information about the future. Incorporating all exogenous elements of the state into the time subscript, we can express the value function at date t as a function V_t of assets. (Alternatively, we could express it as a function of assets and the time horizon that is left, that is as a function V_{T-t} of assets.) Note that in contrast to wages and interest rates, a_t is an endogenous state variable: It constitutes a parameter in the program at date t but is a choice variable in earlier periods.

For brevity, let DBC_t denote the dynamic budget constraint at date t and let $\mathcal{C}$ denote the set of dynamic budget constraints at date $t+1$ and later as well as the terminal condition $a_{T+1} \geq 0$. The value function at date t then satisfies

$$V_t(a_t) = \max_{\{c_s, a_{s+1}\}_{s=t}^{T}} \sum_{s=t}^{T} \beta^{s-t} u(c_s) \ \text{ s.t. } \ \text{DBC}_t, \mathcal{C}, a_t \text{ given.}$$

Alternatively, it can be represented recursively as

$$V_t(a_t) = \max_{c_t, a_{t+1}} \{u(c_t) + \beta V_{t+1}(a_{t+1})\} \ \text{ s.t. } \ \text{DBC}_t, a_t \text{ given.}$$

To see this, simply rearrange terms exploiting the additive separability of preferences:

$$\begin{aligned}
V_t(a_t) &= \max_{\{c_s,a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t}u(c_s) \text{ s.t. DBC}_t, \mathcal{C}, a_t \text{ given} \\
&= \max_{c_t,a_{t+1}} u(c_t) + \left(\max_{\{c_s,a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-t}u(c_s) \text{ s.t. } \mathcal{C}, a_{t+1} \text{ given} \right) \text{ s.t. DBC}_t, a_t \text{ given} \\
&= \max_{c_t,a_{t+1}} u(c_t) + \beta \left(\max_{\{c_s,a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-(t+1)}u(c_s) \text{ s.t. } \mathcal{C}, a_{t+1} \text{ given} \right) \text{ s.t. DBC}_t, a_t \text{ given} \\
&= \max_{c_t,a_{t+1}} u(c_t) + \beta V_{t+1}(a_{t+1}) \text{ s.t. DBC}_t, a_t \text{ given.}
\end{aligned}$$

We are confronted with a system of functional equations often referred to as *Bellman equations*. These functional equations stipulate equality of functions (rather than functions evaluated at certain points). Substituting the dynamic budget constraint yields a compact representation of the Bellman equation,

$$V_t(a_t) = \max_{a_{t+1}} u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1}),$$

which has to hold at all dates and for all feasible values of a_t.

Since T is finite we can solve the system of Bellman equations by backward induction. To start the induction, note that $V_{T+1}(a_{T+1}) = 0$ for all a_{T+1}. At date T, this implies the optimal choice $a_{T+1} = 0$ and thus, the value function $V_T(a_T) = u(a_T R_T + w_T)$. Using this result and the Bellman equation at date $T-1$ we can characterize the optimal choice and the value function at date $T-1$. Proceeding backward, we can solve for all value functions V_t and *policy functions* g_t say; the latter give the optimal value of the choice variable as a function of the state, $a_{t+1} = g_t(a_t)$.[2]

To derive the Euler equation, we do not need to know the functional form of the value functions. It suffices to use the first-order and envelope conditions,

$$\begin{aligned}
u'(c_t) &= \beta V'_{t+1}(a_{t+1}), \\
V'_t(a_t) &= u'(c_t)(R_t - g'_t(a_t)) + \beta V'_{t+1}(a_{t+1})g'_t(a_t) \\
&= u'(c_t)R_t.
\end{aligned}$$

The first-order condition in the first line results from differentiating the right-hand side of the Bellman equation with respect to the choice variable. An optimal choice of a_{t+1} assures that this condition is satisfied. The *envelope condition* in the second line results from differentiating the Bellman equation with respect to the state variable. The last equality follows from substituting the first-order condition into the second condition—this is the envelope theorem at work (see below). Combining the first-order condition and the envelope condition (evaluated at $t+1$) yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. It can also be expressed as the functional equation

$$u'(a_t R_t + w_t - g_t(a_t)) = \beta R_{t+1} u'(g_t(a_t) R_{t+1} + w_{t+1} - g_{t+1}(g_t(a_t)))$$

[2] It is straightforward to write a computer program that iteratively computes approximations of V_t and g_t for $t = T, T-1, T-2, \ldots$.

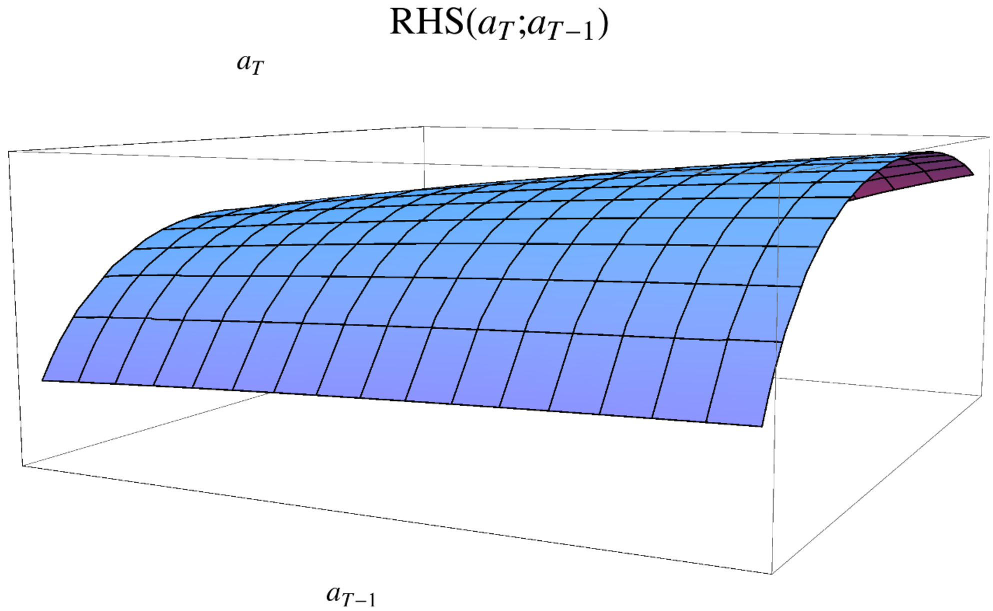

Figure 2.2: Envelope condition: $\mathrm{RHS}(a_T; a_{T-1})$ plotted against a_{T-1} and a_T.

which must hold for all feasible values of a_t.

Figure 2.2 illustrates the envelope condition at date $T-1$. The figure plots the maximand on the right-hand side of the Bellman equation at date $T-1$, $\mathrm{RHS}(a_T; a_{T-1}) \equiv u(a_{T-1}R_{T-1} + w_{T-1} - a_T) + \beta u(a_T R_T + w_T)$, against a_{T-1} (the endogenous state) and a_T (the choice variable). For a given value of the state, the maximand is a concave function of the choice variable that reaches a maximum at the optimal choice. Starting at a given value of the state and the corresponding optimal choice, a small increase in the state has two effects. First, it directly alters the maximand, corresponding to the change of RHS for a move to the right in parallel to the a_{T-1} axis. Second, it induces an adjustment of the optimal choice, a_T. But since the derivative of the maximand with respect to a_T was zero to start with, the effect of this induced change on RHS is of second order: $u'(c_t)(-g_t'(a_t)) + \beta V_{t+1}'(a_{t+1})g_t'(a_t) = 0$. The only first-order effect of an infinitesimal change of state on the value function thus is the direct one.

2.1.3 Infinite Horizon

There are several reasons to consider optimization of households (and other agents) over an *infinite horizon*, $T \to \infty$. First, because this can be interpreted as reflecting intergenerational altruism: Parents care about the utility of their children who in turn care about the utility of their children, and so on. Second, because an infinite horizon can be interpreted as reflecting a time invariant survival probability. And third, because eliminating time as a state variable makes the program simpler.

To derive the household's intertemporal budget constraint in the infinite-horizon case, we need to specify an appropriate terminal condition. This is given by the "*no-*

Ponzi-game condition" $\lim_{T\to\infty} q_T a_{T+1} \geq 0$. Note that the no-Ponzi-game condition generalizes the constraint $q_T a_{T+1} \geq 0$ from the finite horizon case, not its reduced form $a_{T+1} \geq 0$. In fact, a generalization of the latter, to $\lim_{T\to\infty} a_{T+1} \geq 0$ say, would constitute an unnecessarily tight constraint that prevents the household from holding any debt in the long run.

In contrast, the no-Ponzi-game condition $\lim_{T\to\infty} q_T a_{T+1} \geq 0$ only rules out debt positions in the long run that grow at a rate weakly higher than the interest rate. The household thus is prevented from permanently rolling over debt, including interest, and never servicing it. While the no-Ponzi-game condition guarantees that going forward, the present discounted value of debt service fully covers the outstanding debt it does not impose restrictions on the time profile of the debt service. For example, the household may once and for all pay back all outstanding debt, or it may never repay the principal but instead pay interest on the liability forever after.

Using the no-Ponzi-game condition and following the same steps as in the finite-horizon case, we can derive the infinite-horizon intertemporal budget constraint

$$a_0 R_0 + \lim_{T\to\infty} \sum_{t=0}^{T} q_t (w_t - c_t) = \lim_{T\to\infty} q_T a_{T+1} \geq 0.$$

By the same logic as in the finite-horizon case, optimality requires setting $\lim_{T\to\infty} q_T a_{T+1}$ as small as possible (see appendix B.1). The intertemporal budget constraint therefore reduces to $a_0 R_0 + \sum_{t=0}^{\infty} q_t (w_t - c_t) = 0$ and the household's program can be represented as

$$\max_{\{c_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } a_0 R_0 + \sum_{t=0}^{\infty} q_t (w_t - c_t) = 0.$$

Forming the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda [a_0 R_0 + \sum_{t=0}^{\infty} q_t (w_t - c_t)]$$

and differentiating yields the same first-order conditions as before,

$$\beta^t u'(c_t) = \lambda q_t,$$

and thus, the Euler equation.

Turn next to dynamic programming. With an infinite horizon, the household's horizon always is the same, independently of how many periods have gone by. The structure of the optimization problem therefore is independent of time—the problem is *time autonomous*—unless wages, interest rates, or preferences are time dependent. In the time autonomous case, the Bellman equation can be expressed as

$$V(a_\circ) = \max_{a_+} u(a_\circ R + w - a_+) + \beta V(a_+).$$

Note that the value functions on the left- and right-hand side are identical (the functions do not have time subscripts), in contrast with the finite-horizon case. The state variable $a_\circ$ and the choice variable a_+ are written without time subscripts to indicate the time autonomous nature of the program.

Although in the infinite-horizon case no final period exists, one can nevertheless find the value function V of the infinite-horizon problem by means of an iterative procedure that parallels the solution strategy in the finite-horizon case. This follows from mathematical results which establish that under certain conditions, (i) the value function V solving the time-autonomous Bellman equation is unique, and (ii) starting from any value function guess (for example the function that started the recursion in the finite-horizon case, $V_{T+1}(a_{T+1}) = 0$), the iterative procedure applied in the finite-horizon case yields a sequence of value functions that converges to V (see appendix A.2). When working with a computer, an approximation of the infinite-horizon value function thus can be found by running exactly the same code as in the finite horizon case except that the iterative procedure only is stopped when the sequence of value function approximations has converged.

2.2 Extensions

2.2.1 Borrowing Constraint

We have assumed that households may freely borrow as long as they satisfy the intertemporal budget constraint that is, as long as they are solvent. But borrowing against future wage income may be difficult, for example because a potential lender does not have sufficient information about the future income stream or cannot enforce repayment. This renders future wage income *illiquid* and gives rise to a new constraint—a *liquidity* or *borrowing constraint*—in addition to the intertemporal budget constraint.

The simplest possible borrowing constraint excludes all borrowing against future wage income. The household's financial assets then must be positive at all times, $a_{t+1} \geq 0$. A binding borrowing constraint is costly because it prevents consumption smoothing. To see this, consider a two-period setting with strictly concave preferences and suppose that absent a borrowing constraint, optimal consumption in the first period exceeds "cash at hand," $w_0 + a_0 R_0$, and thus requires setting $a_1 < 0$. The borrowing constraint renders this plan infeasible. Constrained optimal consumption then equals $(c_0, c_1) = (w_0 + a_0 R_0, w_1)$ that is, consumption follows income, as with a Keynesian consumption function. Note that we have identified a new saving motive: reduced borrowing (that is, increased saving) due to a binding borrowing constraint.

More formally, the Lagrangian associated with the constrained saving problem reads

$$\mathcal{L} = u(c_0) + \beta u(c_1) - \lambda \left(c_0 + \frac{c_1}{R_1} - w_0 - \frac{w_1}{R_1} - a_0 R_0 \right) + \mu(w_0 + a_0 R_0 - c_0),$$

where the non-negative multiplier μ represents the shadow cost of the borrowing constraint $w_0 + a_0 R_0 - c_0 \geq 0$. Differentiating with respect to c_0 and c_1 and combining the two conditions yields the modified Euler equation

$$u'(c_0) = \beta R_1 u'(c_1) + \mu.$$

A binding borrowing constraint, $\mu > 0$, increases the slope of the equilibrium consumption path. Moreover, $\mu > 0$ and the complementary slackness condition, $\mu(w_0 + a_0 R_0 - c_0) = 0$, imply $c_0 = w_0 + a_0 R_0$, in line with the heuristic argument above.

2.2.2 Non-Geometric Discounting and Time-Consistency

Until now, we have posited that the sequence of psychological discount factors is geometrically declining, $1, \beta, \beta^2, \beta^3, \ldots$. Under this assumption, a household that re-optimizes period by period opts to continue with the consumption plan chosen earlier in time. That is, if the household optimally chose the consumption plan $(c_t, \overrightarrow{c_{t+1}})$ with $\overrightarrow{c_{t+1}} \equiv \{c_{t+1}, c_{t+2}, \ldots\}$ at date t, then pursuing the plan $\overrightarrow{c_{t+1}}$ once time has progressed to date $t+1$ remains optimal. As a consequence, it does not matter whether we assume that the household chooses the consumption plan at date $t = 0$ once and for all, under *commitment*, or whether it re-optimizes period by period. In this sense, the initial consumption plan is *time-consistent*.

Under more general assumptions about the psychological discount factor sequence the consumption plan at date $t = 0$ need not be time-consistent. Two households, one re-optimizing period by period and the other acting under commitment, may end up with different consumption paths although their preferences at date $t = 0$ and their budget sets are identical.

Consider an extreme example in a three-period setting where the household discounts all future utility at factor β. At date $t = 0$, the household has preferences $u(c_0) + \beta(u(c_1) + u(c_2))$ while at date $t = 1$, preferences are given by $u(c_1) + \beta u(c_2)$. For simplicity, let $R_t = 1, w_t = w$. The optimal consumption path as of date $t = 0$ then solves the problem

$$\max_{c_0, c_1, c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \text{ s.t. } c_0 + c_1 + c_2 = 3w + a_0,$$

which yields $c_1 = c_2$ (assuming $u'' < 0$). A household who can commit implements this solution.

In contrast, for given a_1 the optimal consumption path as of date $t = 1$ solves

$$\max_{c_1, c_2} u(c_1) + \beta u(c_2) \text{ s.t. } c_1 + c_2 = 2w + a_1,$$

which yields $c_1 > c_2$. Absent commitment, a household re-optimizing at date $t = 1$ thus does not implement the path that is optimal from the perspective of date $t = 0$. Without commitment, the ex-ante optimal consumption plan cannot be implemented.

In equilibrium without commitment, the two "selves" of the household play a game against each other. The first self chooses c_0 and a_1. The second self chooses c_1, a_2 and c_2 conditional on a_1. Since the second self chooses $c_1 > c_2$, the first self cannot implement the ex-ante optimal plan. Anticipating the second self's "distorted" action, the first self solves

$$\max_{c_0,c_1,c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \text{ s.t. } c_0 + c_1 + c_2 = 3w + a_0, u'(c_1) = \beta u'(c_2)$$

where the second constraint, the Euler equation of the second self, reflects the "distorted" consumption choice from date $t = 1$ onward. By choosing a_1 the first self affects the state variable at date $t = 1$ and may thus influence the action taken by the second self.

2.2.3 Multiple Goods

Consider a two-period lived household that consumes two goods in each period. Their quantities, d_t and e_t, are aggregated into a CES consumption index,

$$c_t(d_t, e_t) = \left(\delta^{\frac{1}{\theta}} d_t^{1-\frac{1}{\theta}} + \varepsilon^{\frac{1}{\theta}} e_t^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \; \delta + \varepsilon = 1, \; \theta > 0,$$

with elasticity of substitution equal to θ. The price of d_t is normalized to unity and the relative price of e_t is denoted p_t. Intertemporal preferences are described by the utility function $u(c_0) + \beta u(c_1)$.

The household's consumption choice has an *intratemporal* dimension (the trade-off between d_t and e_t) and an *intertemporal* one (the trade-off between c_0 and c_1). Focusing first on the intratemporal trade-off, consider the problem of maximizing c_t subject to a given amount of spending, $z_t = d_t + p_t e_t$. The solution to this problem is given by

$$d_t = \delta \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \;\; e_t = \varepsilon p_t^{-\theta} \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \;\; c_t = \left(\delta + \varepsilon p_t^{1-\theta}\right)^{\frac{1}{\theta-1}} z_t.$$

Solving the third equation for z_t we can derive a price index, $\mathcal{P}_t$: One unit of the consumption index c_t costs

$$\mathcal{P}_t = \left(\delta + \varepsilon p_t^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

For $p_t = 1$, the price index equals unity. For $p_t \to \infty$, it increases with p_t if $\theta < 1$ but converges to a constant if $\theta > 1$. Intuitively, how strongly the relative price increase translates into a higher price index depends on the household's willingness to substitute across goods. Using the price index, we also have $d_t = \delta c_t \mathcal{P}_t^{\theta}$ and $e_t = \varepsilon c_t (\mathcal{P}_t / p_t)^{\theta}$.

Equipped with these results, we turn to the intertemporal program. The dynamic budget constraints are given by $w_0 = \mathcal{P}_0 c_0 + a_1$ and $a_1 R_1 + w_1 = \mathcal{P}_1 c_1$ where wage income and assets are expressed in terms of the numeraire d_t. The household's program therefore reads

$$\max_{c_0,c_1} u(c_0) + \beta u(c_1) \text{ s.t. } \mathcal{P}_0 c_0 + \frac{\mathcal{P}_1 c_1}{R_1} = w_0 + \frac{w_1}{R_1}$$

and the Euler equation characterizing the optimal intertemporal consumption allocation is given by

$$u'(c_0) = \beta R_1 \frac{\mathcal{P}_0}{\mathcal{P}_1} u'(c_1).$$

As usual, the marginal rate of substitution is equated with the marginal rate of transformation. But with a consumption index, the marginal rate of transformation is given by the *own rate of interest*, $R_1 \mathcal{P}_0 / \mathcal{P}_1$. The latter differs from the marginal rate of transformation for the numeraire good, R_1, whenever $\mathcal{P}_t$ (and thus, p_t) changes over time.

2.3 Bibliographic Notes

Modigliani and Brumberg (1954) and Friedman (1957) derive consumption functions based on microeconomic reasoning. Friedman (1957) emphasizes the role of permanent as opposed to current income and Modigliani and Brumberg (1954), among others, stress life cycle considerations.

Strotz (1956) analyzes time inconsistency and Laibson (1997) explores the consequences of hyperbolic rather than geometric discounting.

Dixit and Stiglitz (1977) analyze a model with a CES consumption index.

Chapter 3

Dynamic Competitive Equilibrium

We now embed the consumption-saving tradeoff in two general equilibrium models of capital accumulation: We add a firm sector and impose market clearing. In the first model, the *representative agent* or "*Ramsey*" model, we assume that households are homogeneous. This is a convenient, but strong assumption; appendix B.2 provides some discussion. In the second model, the *overlapping generations* model, we consider the interaction between households of different age. In subsequent chapters, we generalize the two models, for instance by introducing risk or a labor-leisure choice.

3.1 Representative Agent And Capital Accumulation

3.1.1 Economy

The economy is inhabited by a continuum of identical households of mass one as well as a continuum of identical firms of mass one. Both households and firms take prices as given; households also take firm profits as given. Since households and firms are homogeneous we can represent them as a *representative household* and a *representative firm*, respectively.

3.1.2 Firms

Firms solve static profit maximization problems. In each period, they rent capital, K_t, at *rental rate* r_t and labor, L_t, at *wage* w_t from households to produce *output* (the numeraire) with a neoclassical production function, f. Profits are distributed to households.

Taking rental rates and wages as given, the representative firm maximizes

$$\max_{K_t, L_t} f(K_t, L_t) - K_t r_t - L_t w_t.$$

The first-order conditions

$$\begin{aligned} f_K(K_t, L_t) &= r_t, && (3.1) \\ f_L(K_t, L_t) &= w_t && (3.2) \end{aligned}$$

define demand functions for capital and labor. The budget constraint of the representative firm reads

$$f(K_t, L_t) = K_t r_t + L_t w_t + z_t \tag{3.3}$$

where z_t denotes profits. In equilibrium, profits equal zero, due to constant returns to scale and price taking.

3.1.3 Households

The representative household maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The dynamic budget constraint and Euler equation, respectively, are given by

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t - c_t + z_t, \\ u'(c_t) &= \beta R_{t+1} u'(c_{t+1}). \end{aligned}$$

Since all households are alike (and the economy is closed and there is no government sector) they do not hold claims vis-a-vis each other or third parties. Accordingly, household assets correspond to the physical capital stock in the economy: The capital stock per worker, k_t, equals a_t. Capital *depreciates* at rate δ per period. The gross return on a unit of saving thus equals the rental rate of capital paid by firms, r_t, plus the unit of capital net of depreciation: $R_t = 1 + r_t - \delta$.

Combining these conditions yields

$$\begin{aligned} k_{t+1} &= k_t(1 + r_t - \delta) + w_t - c_t + z_t, && (3.4) \\ u'(c_t) &= \beta(1 + r_{t+1} - \delta) u'(c_{t+1}). && (3.5) \end{aligned}$$

Households also satisfy the transversality condition $\lim_{T\to\infty} q_T k_{T+1} = 0$ or equivalently, using the Euler equation, $\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$.[1] The initial capital stock, k_0, is given.

3.1.4 Market Clearing

There are three goods in each period: Labor, capital (inherited from the last period), and output which can be used for consumption and investment (accumulation of new capital). Since there is one representative household whose time endowment per period equals unity, labor and capital market clearing requires firms to demand one unit of labor and k_t units of capital:

$$\begin{aligned} K_t &= k_t, && (3.6) \\ L_t &= 1. && (3.7) \end{aligned}$$

By Walras' Law, market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. In the economy considered here, this can be seen by combining (3.3), (3.4), (3.6), and (3.7) to find

$$k_{t+1} = k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1 w_t$$

[1] With a finite horizon, the transversality condition reduces to $k_{T+1} = 0$.

which simplifies to the *resource constraint*

$$k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_t.$$

The resource constraint states that the market for the output good clears. Equivalently, *gross investment* plus consumption equals output, or saving equals *net investment*. The condition represents the GDP identity in a closed economy without government sector.

3.1.5 General Equilibrium

In general equilibrium, the transversality condition as well as conditions (3.1)–(3.7) and thus, the resource constraint hold at all dates. The equilibrium conditions can be reduced to (i) the transversality condition; (ii) two core equations in capital and consumption; and (iii) five remaining conditions that determine r_t, w_t, z_t, K_t, and L_t. The two core equations are given by the resource constraint and the Euler equation with the rental rate of capital expressed in terms of the marginal product of capital:

$$\begin{aligned} k_{t+1} &= k_t(1-\delta) + f(k_t, 1) - c_t, &(3.8)\\ u'(c_t) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)u'(c_{t+1}). &(3.9) \end{aligned}$$

Note that, conditional on k_t and c_t, these two equations pin down k_{t+1} and c_{t+1}.

For a given initial capital stock, k_0, conditions (3.8) and (3.9) completely pin down the equilibrium sequences for capital and consumption once a starting value for consumption, c_0, is specified. This starting value cannot freely be chosen, however, because the sequences also must satisfy the transversality condition,

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0.$$

As we will see below, there is a unique c_0 such that the paths implied by (k_0, c_0) as well as (3.8) and (3.9) satisfy the transversality condition.

3.1.6 Social Planner Allocation and Pareto Optimality

We have characterized the equilibrium conditions in the *decentralized economy* with firms and households. Alternatively, we can characterize the *social planner allocation* in a *"Robinson Crusoe" economy*. This economy is inhabited by a single consumer-producer who operates the production function f and saves in the form of capital. Robinson Crusoe solves

$$\max_{\{c_t, k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_t,\ k_0 \text{ given},\ k_{t+1} \geq 0.$$

The non-negativity constraint on capital is not binding if f satisfies the Inada conditions. Solving this program yields exactly the same conditions as those characterizing the decentralized equilibrium, namely conditions (3.8) and (3.9) and the transversality condition (see appendix B.3).

Since the social planner allocation is the feasible allocation preferred by the representative household it necessarily is Pareto optimal. By implication, the decentralized equilibrium allocation is Pareto optimal as well. This is not surprising since the economy satisfies the conditions of the first welfare theorem.

3.1.7 Analysis

Phase Diagram

Since (3.8) and (3.9) constitute non-linear first-order difference equations the model cannot generally be solved in closed form. However, we may qualitatively characterize equilibrium by means of a *phase diagram* which illustrates the system dynamics. The phase diagram is constructed based on the relations

$$\begin{aligned} c_t &= f(k_t, 1) - \delta k_t, && (3.10) \\ 1 &= \beta(1 + f_K(k_{t+1}, 1) - \delta). && (3.11) \end{aligned}$$

Condition (3.10) follows from (3.8) when the capital stock is constant over time, $k_t = k_{t+1}$. In this case, consumption plus *replacement investment* equals output. Condition (3.11) follows from (3.9) when $c_t = c_{t+1}$. For consumption to be constant over time, βR_{t+1} must equal unity.

In figure 3.1, relations (3.10) and (3.11) are represented by the black concave schedule and the black vertical line, respectively. Their intersection defines the *steady state* of the system, (k, c), where all equilibrium conditions are satisfied and all variables do not change over time.

From (3.10), consumption is maximized subject to a time invariant capital stock when the latter equals the "*golden-rule*" capital stock, k^{gr}, which satisfies

$$f_K(k^{\text{gr}}, 1) = \delta.$$

The steady-state or "*modified-golden-rule*" capital stock is lower than the golden-rule capital stock, $k < k^{\text{gr}}$, because from (3.11)

$$f_K(k, 1) = \delta + \beta^{-1} - 1 > \delta.$$

Capital stock dynamics outside of steady state are determined by the resource constraint. Suppose that $c_t > f(k_t, 1) - \delta k_t$ that is, c_t lies above the concave schedule. Gross investment then falls short of the replacement investment necessary to maintain the capital stock, and as a consequence $k_{t+1} < k_t$. Conversely, a choice of c_t below the concave schedule implies $k_{t+1} > k_t$.

Consumption dynamics outside of steady state are determined by the Euler equation. If the capital stock is smaller than k then the marginal product of capital and thus, the gross rate of return on capital are higher than in steady state and consumption rises, $c_{t+1} > c_t$. Conversely, a capital stock larger than k implies $c_{t+1} < c_t$.

The system dynamics therefore differ across the four regions separated by (3.10) and (3.11): If $k_t < k$ and $c_t < f(k_t, 1) - \delta k_t$ then both the capital stock and consumption rise over time. If $k_t > k$ and $c_t < f(k_t, 1) - \delta k_t$ then the capital stock rises and consumption falls. If $k_t < k$ and $c_t > f(k_t, 1) - \delta k_t$ then the capital stock falls and consumption rises. Finally, if $k_t > k$ and $c_t > f(k_t, 1) - \delta k_t$ then both the capital stock and consumption fall.

The paths indicated by dots in figure 3.1 illustrate the system dynamics. Consider a low initial capital stock, $k_0 = 0.5k$ say. The figure illustrates three candidate adjustment paths that start at different initial consumption levels, c_0. All these candidate paths satisfy (3.8) and (3.9) but only one—the blue path—satisfies (3.8) and (3.9) only in future periods and meets the transversality condition. Too low an initial consumption level implies non-convergent dynamics to the "bottom right" (in red) where the interest rate is negative and thus, the transversality condition violated. Too high an initial consumption level implies non-convergent dynamics to the "top left" (in red) where the Euler equation prescribes consumption growth but household assets tend to zero. Only an intermediate initial consumption level implies convergent dynamics (in blue) towards the steady state, and only this path satisfies (3.8), (3.9), and the transversality condition.

Similarly, for a high initial capital stock, $k_0 = k^{gr}$ say, too low or too high an initial consumption value implies non-convergent dynamics, indicated by the red paths starting above k^{gr}, while an appropriate intermediate starting value implies convergent dynamics, indicated by the blue path.

As a function of the initial capital stock, k_0, the associated consumption level guaranteeing convergent equilibrium dynamics, $c_0(k_0)$, traces out the *saddle path*. The saddle path gives the equilibrium initial consumption level for an initial capital stock, and it indicates the path along which convergent equilibrium dynamics occur. The blue dots in figure 3.1 illustrate the segment of the saddle path between $0.5k$ and k^{gr}.

Based on the phase diagram we may not only analyze equilibrium dynamics in environments with constant technology (and preferences) but also the response to changing fundamentals. Suppose, for example, that the production function is known to change from f to g say at some future date $t = T$. From that future date onwards, system dynamics then are governed by the "new" Euler equation and resource constraint (with f replaced by g). Before date $t = T$, in contrast, the "old" Euler equation and resource constraint (with technology f) determine the dynamic behavior.

Unlike what one might expect at first sight, there is no "jump" of (k_t, c_t) at date $t = T$ from the old to the new saddle path—such a jump would violate the Euler equation or the resource constraint. Instead, for $t < T$, (k_t, c_t) moves *off* the old saddle path towards the new saddle path; at date $t = T$, (k_t, c_t) smoothly meets the new saddle path; and for $t > T$, (k_t, c_t) moves along the new saddle path towards the long-run steady state which is determined by the new Euler equation and resource constraint.

Solution Methods

Beyond the phase diagram, several strategies may be used to solve the model. One, described below, is based on a *linear approximation* of the difference equation system

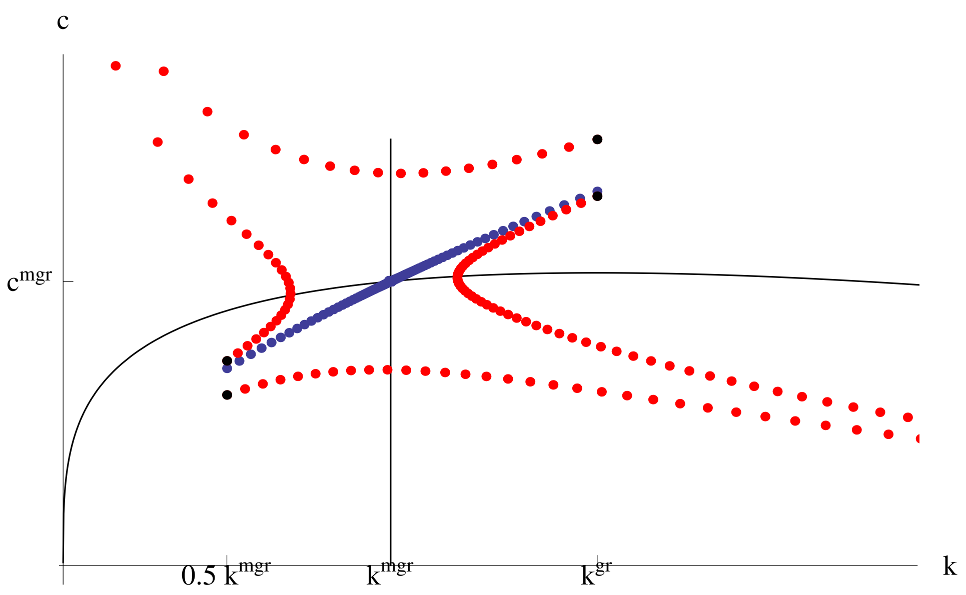

Figure 3.1: Dynamics in the representative agent model: Steady-state resource constraint and Euler equation (in black) as well as (k_t, c_t)-paths for different values of (k_0, c_0) (in red or blue).

(3.8) and (3.9); it involves eigenvalue and eigenvector operations. Another strategy is based on simply trying different starting values c_0 conditional on k_0 and checking whether the induced system dynamics are convergent. Finally, one may solve the social planner's program numerically, using dynamic programming methods.

The approximation strategy rests on linearizing (3.8) and (3.9) about the steady state. For example, totally differentiating (3.8) and evaluating at the steady state yields

$$\begin{aligned} & dk_{t+1} + dc_t = (f_K(k,1) + 1 - \delta) dk_t, \\ \Rightarrow \quad & \hat{k}_{t+1} + \hat{c}_t \frac{c}{k} = \beta^{-1} \hat{k}_t, \end{aligned}$$

where a circumflex denotes infinitesimal relative deviations from the corresponding steady state value, e.g., $\hat{c}_t \equiv (c_t - c)/c$. Similarly, taking logarithms in (3.9), totally differentiating and evaluating at the steady state yields (letting $\sigma \equiv -u''(c)c/u'(c)$)

$$\begin{aligned} & \ln(u'(c_t)) = \ln(\beta) + \ln(1 + f_K(k_{t+1}, 1) - \delta) + \ln(u'(c_{t+1})), \\ \Rightarrow \quad & -\sigma \frac{dc_t}{c} = \frac{f_{KK}(k,1) dk_{t+1}}{1 + f_K(k,1) - \delta} - \sigma \frac{dc_{t+1}}{c}, \\ \Rightarrow \quad & \hat{c}_t = -\frac{\beta}{\sigma} f_{KK}(k,1)\, k\, \hat{k}_{t+1} + \hat{c}_{t+1}. \end{aligned}$$

Approximating the original system to the first order means that we apply the linearized system to deviations from steady state even if they are larger than infinitesimal.

Next, we represent the linearized equations in vector and matrix form as

$$M_1 \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M_0 \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}, \ M_1 \equiv \begin{bmatrix} 1 & -\frac{\beta}{\sigma} f_{KK}(k,1)k \\ 0 & 1 \end{bmatrix}, \ M_0 \equiv \begin{bmatrix} 1 & 0 \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Multiplying by the inverse of M_1 yields

$$\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}, \ M \equiv M_1^{-1} M_0 = \begin{bmatrix} 1 - \frac{c}{\sigma}\beta f_{KK} & \frac{f_{KK}k}{\sigma} \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Finally, we express the linearized system in terms of the eigenvalues ρ_1 and ρ_2 as well as the corresponding eigenvectors v_1 and v_2 of the matrix M. An eigenvalue of M satisfies $\det(M - \rho I) = 0$ that is, it solves the characteristic equation $\mathcal{C}(\rho) = 0$ with $\mathcal{C}(\rho) \equiv \rho^2 - \rho(1 + \beta^{-1} - c\beta f_{KK}(k,1)/\sigma) + \beta^{-1}$. The latter is a continuous quadratic function satisfying $\mathcal{C}(0) > 0$, $\mathcal{C}(1) < 0$, and $\lim_{\rho\to\infty} \mathcal{C}(\rho) = \infty$. It follows that $0 < \rho_1 < 1 < \rho_2$. In fact, $\rho_2 = 1/(\beta\rho_1)$. Using standard results (see appendix A.3), we have

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \varphi_1 \rho_1^t v_1 + \varphi_2 \rho_2^t v_2,$$

where φ_1, φ_2 are arbitrary constants that remain to be determined. The requirement that system dynamics be stable implies that φ_2 must equal zero since ρ_2^t grows without bound as $t \to \infty$. The second constant, φ_1, is pinned down by the initial condition for the capital stock, $\hat{k}_0 = \varphi_1 v_{1[2]}$, where $v_{1[2]}$ denotes the second element of the eigenvector v_1. In conclusion,

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \rho_1^t v_1 \frac{\hat{k}_0}{v_{1[2]}}.$$

The saddle path of the linearized system is given by the function

$$\hat{c}_0(\hat{k}_0) = \frac{\hat{k}_0}{v_{1[2]}} v_{1[1]}.$$

Its slope in (k, c)-space satisfies

$$\frac{dc_0}{dk_0} = \frac{c}{k} \frac{v_{1[1]}}{v_{1[2]}}.$$

For $\sigma \to \infty$ or $f_{KK}k/f_K \to 0$, the slope approaches $(1-\beta)/\beta$. Lower values of σ (a higher intertemporal elasticity of substitution) or more negative elasticities of the marginal product with respect to k increase dc_0/dk_0.

The *speed of convergence* to the steady state is determined by the stable eigenvalue, ρ_1. The higher this eigenvalue, the slower the convergence.

3.1.8 Population Growth

Suppose the number of household members grows at gross rate ν per period and the household's objective thus equals $\sum_{t=0}^{\infty} \beta^t \nu^t u(c_t)$ where c_t denotes per-capita consumption as before. Normalizing the population size at date $t = 0$ to unity, the resource constraint now is given by

$$\nu^{t+1} k_{t+1} = \nu^t k_t (1 - \delta) + f(\nu^t k_t, \nu^t) - \nu^t c_t,$$

where k_t continues to denote the capital stock per capita. Equivalently,

$$\nu k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

Intuitively, positive population growth implies that the capital-labor ratio at date $t + 1$ is smaller (by the factor ν) than the per-capita resources not consumed at date t. Except for this difference, the conditions characterizing the centralized or decentralized equilibrium are not affected by population growth.

3.2 Overlapping Generations And Capital Accumulation

3.2.1 Economy

Households live for two periods rather than over an infinite horizon. In each period, a continuum of young and old households of mass one each inhabit the economy, see figure 3.2. Young households are born without assets; they work, consume, and save for retirement. Old households retire, consume the return on their saving (i.e., households leave no bequests), and die. The assets held by the retirees correspond to the capital stock in the economy. At date $t = 0$ the old cohort is endowed with the initial capital stock, k_0.

3.2.2 Firms

The firm sector is identical to the one in the representative agent model and conditions (3.1)–(3.3) apply. Profits are distributed to old households.

3.2.3 Households

The dynamic budget constraints of a worker and a retiree at date t as well as the Euler equation of a young household are given by

$$\begin{aligned} k_{t+1} &= w_t - c_{1,t}, & (3.12)\\ c_{2,t} &= k_t(1 + r_t - \delta) + z_t, & (3.13)\\ u'(c_{1,t}) &= \beta(1 + r_{t+1} - \delta) u'(c_{2,t+1}), & (3.14) \end{aligned}$$

respectively. Here, $c_{1,t}$ and $c_{2,t}$ denote consumption at date t of a young and old household, respectively.

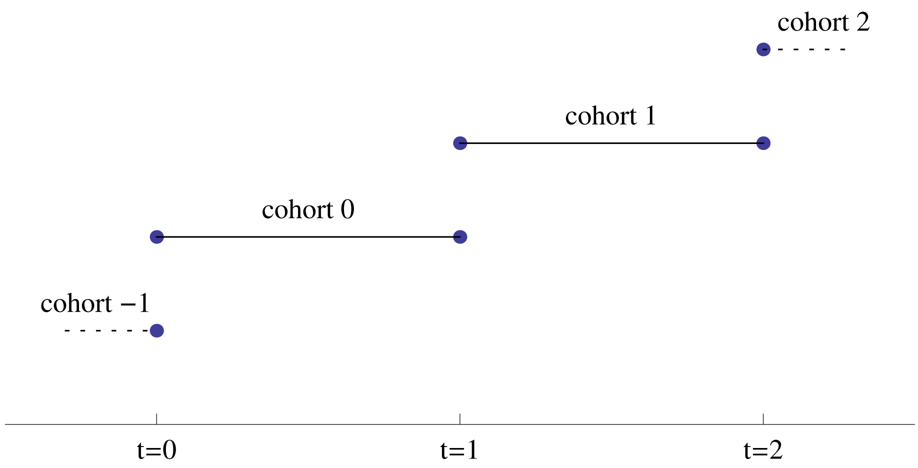

Figure 3.2: Overlapping generations.

3.2.4 Market Clearing

Labor and capital market clearing requires that firms demand one unit of labor and k_t units of capital, implying the equilibrium conditions (3.6) and (3.7). By Walras' Law, market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. Combining (3.3), (3.6), (3.7), (3.12), and (3.13) and letting $c_t \equiv c_{1,t} + c_{2,t}$ yields

$$\begin{aligned} k_{t+1} &= w_t - c_{1,t} + k_t(1 + r_t - \delta) - c_{2,t} + z_t \\ &= k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1 w_t. \end{aligned}$$

This simplifies to the same resource constraint as in the representative agent model,

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

3.2.5 General Equilibrium

In general equilibrium, conditions (3.1)–(3.3), (3.6)–(3.7), and (3.12)–(3.14) (and thus, the resource constraint) hold simultaneously. These equilibrium conditions can be reduced to three core equations,

$$\begin{aligned} k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_{1,t} - c_{2,t}, \\ c_{2,t} &= k_t(1 + f_K(k_t, 1) - \delta), \\ u'(c_{1,t}) &= \beta(1 + f_K(k_{t+1}, 1) - \delta) u'(c_{2,t+1}), \end{aligned}$$

as well as five remaining conditions that determine r_t, w_t, z_t, K_t, and L_t. Compared with the representative agent model, an additional budget constraint is present; it determines how consumption is split between workers and retirees. The Euler equation characterizes the slope of the consumption profile over the household's life cycle.

Conditional on k_t, the second core equation pins down $c_{2,t}$. Moreover, since $c_{2,t+1} = k_{t+1}(1 + f_K(k_{t+1}, 1) - \delta)$, the first and third condition pin down $c_{1,t}$ and k_{t+1}. For an initial capital stock, k_0, the core equations therefore completely determine the equilibrium paths of capital and consumption over the infinite horizon.

An alternative representation of equilibrium uses the *saving function*. Let $a_{t+1} = a(w_t, R_{t+1})$ denote equilibrium saving of a worker. The saving function a combines the Euler equation and the intertemporal budget constraint which extends over two periods; it depends on wealth (given by the wage) and the interest rate. Combined with the equilibrium relations between factor prices and the capital-labor ratio, the saving function defines a *law of motion* for capital,

$$k_{t+1} = a(w_t, R_{t+1}) \text{ where } w_t = f_L(k_t, 1),\ R_{t+1} = 1 - \delta + f_K(k_{t+1}, 1). \tag{3.15}$$

Under certain functional form assumptions this law of motion can be solved in closed form.

Depending on preferences and technology the function $k_{t+1}(k_t)$ defined by (3.15) may intersect the 45 degree line never, once, or multiple times; accordingly, no steady state with a strictly positive capital stock, a unique such steady state, or multiple steady states may exist. A steady state is stable and non-oscillating if in a neighborhood around it, k_{t+1} increases in k_t, but by less than one-to-one. Writing (3.15) as $k_{t+1} = \tilde{a}(k_t, k_{t+1})$ and totally differentiating implies

$$\frac{dk_{t+1}}{dk_t} = \frac{\partial \tilde{a}(k_t, k_{t+1})/\partial k_t}{1 - \partial \tilde{a}(k_t, k_{t+1})/\partial k_{t+1}}.$$

A steady state thus is stable and non-oscillating if the value of the expression on the right-hand side, evaluated at steady state, lies between zero and one.

3.2.6 Analysis

In contrast to the representative agent model, households in the overlapping generations model are heterogeneous. As a consequence, average saving in the economy differs from the saving of young or old households, and the slope of the consumption profile of a young household need not match the slope of the aggregate consumption profile.

This has important implications for the steady state. While the first steady-state condition of the representative agent model, condition (3.10), also applies in the overlapping generations model, the second one, condition (3.11), does not. In the representative agent model, this second condition follows from the requirement that aggregate and thus, individual consumption is constant over time. In the overlapping generations model, in contrast, constancy of aggregate consumption (or of young-age consumption or old-age consumption) does not imply that the consumption profile of an individual household is flat over the life cycle. The steady-state capital stock, the fixed point of (3.15), therefore need not satisfy the condition $\beta(1-\delta+f_K(k,1))=1$. In fact, depending on preferences and the production function, it can be smaller or—unlike in the representative agent model—larger than the golden-rule capital stock.

Figure 3.3 illustrates the transition dynamics from a low initial capital stock, assuming either a high ($\alpha=0.3$, top panel) or low ($\alpha=0.2$, bottom panel) capital share. (We posit Cobb-Douglas technology and logarithmic preferences and let $\beta=0.98^{25}$ and $\delta=1-(1-0.05)^{25}$, such that one period in the model corresponds to 25 years.) The concave black schedules depict the steady-state resource constraint; the red and blue dots represent $c_{1,t}$ and $c_{2,t}$ respectively; and the black dots represent aggregate consumption, c_t. The top panel of the figure illustrates the transition when the capital share is high. Both young-age consumption, old-age consumption, and aggregate consumption increase during the transition. Over each life cycle, however, consumption does not increase. The young cohort at the beginning of the transition faces a gross interest rate approximately equal to β^{-1} and accordingly chooses a flat consumption profile. Subsequent young cohorts are richer because they earn higher wages but they face lower interest rates and thus choose downward sloping consumption profiles. The economy converges to a steady state satisfying $k<k^{\text{gr}}$.

The lower panel illustrates the transition when the capital share and the golden-rule capital stock are lower. Again, all consumption components increase during the transition but over each life cycle, consumption decreases. Moreover, wages relative to interest rates are higher than in the previous case, generating a stronger motive for life-cycle saving. The economy now converges to a steady state satisfying $k>k^{\text{gr}}$.

3.2.7 Pareto Optimality

Since the steady-state capital stock in the overlapping generations model may exceed the golden-rule capital stock, the steady-state interest rate need not satisfy $R>1$. This contrasts sharply with the situation in the representative agent model where the steady-state net interest rate always is positive, and it has important efficiency implications. In fact, a steady state with $R<1$ is *Pareto inefficient*, for two reasons.

First, because the economy *over accumulates capital* or is *dynamically inefficient*. When $R<1$ the marginal unit of capital contributes negatively to steady-state total consumption: The marginal contribution to output, $f_K(k,1)$ which equals r in equilibrium, falls short of the marginal replacement investment to compensate for depreciation, δ. When $R<1$, a reduction of the capital stock therefore does not only free resources for contemporaneous consumption; when the capital stock is permanently reduced, it also frees resources in all future periods because the fall in output is more than compensated by lower replacement investment. Accordingly, a steady-state allocation with $R<1$ (that is, $r<\delta$) cannot be Pareto optimal.

To take a stark example, suppose that capital does not contribute to production at all ($f_K(K_t,L_t)=0$, capital accumulation amounts to storage) and depreciates at a positive rate (a fraction of the stored goods spoils). The condition for dynamic inefficiency then is met and from a social perspective, which only takes feasibility restrictions into

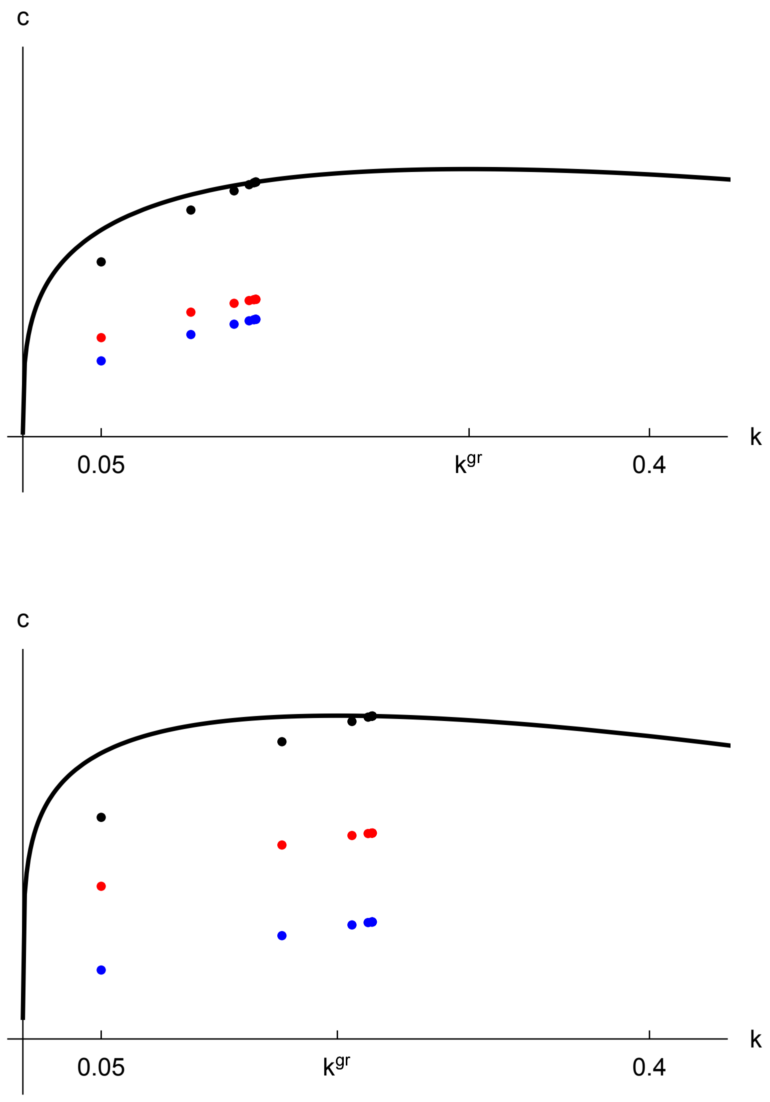

Figure 3.3: Transition dynamics in the OLG model: High (top) and low (bottom) capital share.

account, capital accumulation is wasteful. Nevertheless, households do accumulate capital to finance old-age consumption because they must satisfy their budget constraints.

Second, a steady state with $R < 1$ also is inefficient because it entails a suboptimal allocation of consumption over the life cycle. To see this, consider an endowment economy with a fixed total endowment in each period and suppose that the steady-state interest rate satisfies $R < 1$. A transfer scheme that takes Δ units of the good from each young household and gives them to each old household then makes everybody better off: The old in the period when the scheme is introduced gain because they receive Δ without having to contribute; and the young in this and all subsequent periods gain because the effective gross "interest rate" of unity which they receive on their contribution exceeds the market interest rate. (See also the discussions in subsection 5.3.2 and section 9.2.)

When households save both sources of inefficiency are present. Going back to the example of storage with depreciation assume that storage k pays a rate of return $1 - \delta < 1$. With a young-age endowment w and transfers Δ, young-age consumption is given by $w - k - \Delta$ and old-age consumption equals $k(1 - \delta) + \Delta$. A marginal increase of Δ improves welfare of current and future generations as long as $-u'(w - k - \Delta) + \beta u'(k(1 - \delta) + \Delta) \geq 0$ (using the envelope condition). But from the Euler equation, $-u'(w - k - \Delta) + \beta(1 - \delta)u'(k(1 - \delta) + \Delta) = 0$ as long as households store. We conclude that an expansion of the transfer scheme improves welfare of all generations as long as the Euler equation holds with equality that is, as long as households store.

The possibility of Pareto improving transfers in an inefficient economy hinges on the assumption of an infinite horizon. If there existed a last period then transferring resources from the young to the old would hurt the young in the last period and the transfer scheme would not lead to a Pareto improvement. Related, in an inefficient economy the market value of endowments is infinite (since $R < 1$ and the endowment sequence has infinite length), reflecting the "*double infinity*" of households and commodities. The proof of the first welfare theorem which relies on a finite market value of endowments therefore does not go through. Even if all cohorts could trade with each other (i.e., all cohorts were infinitely lived but for all $t \geq 0$, cohort t only valued consumption at dates t and $t + 1$) a steady state with $R < 1$ still could arise—and still would be inefficient.

3.2.8 Population Growth

Suppose the number of young households grows at gross rate ν per period and normalize the cohort size at date $t = 0$ to unity. Maintaining the definition of k_t as capital stock per worker as well as $c_{1,t}$ and $c_{2,t}$ as per-capita consumption, the budget constraints in equilibrium now read

$$\begin{aligned} c_{1,t} &= w_t - k_{t+1}\nu, \\ c_{2,t} &= k_t(1 + r_t - \delta)\nu \end{aligned}$$

and the resource constraint is given by

$$\nu^{t+1}k_{t+1} = \nu^t k_t(1-\delta) + f(\nu^t k_t, \nu^t) - \nu^t c_{1,t} - \nu^{t-1}c_{2,t}$$

or

$$\nu k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_{1,t} - c_{2,t}/\nu.$$

The condition for inefficiency generalizes to

$$f_K(k,1) < \delta + \nu - 1 \ \text{ or } \ R < \nu,$$

relating the net marginal product of capital or the interest rate to the net growth rate of the economy.

3.3 Bibliographic Notes

The representative agent, Ramsey, or neoclassical growth model is due to Ramsey (1928), Cass (1965), and Koopmans (1965).

The overlapping generations model builds on Allais (1947) and is due to Samuelson (1958); Diamond (1965) introduces capital in the model. Modigliani and Brumberg (1954) discuss life cycle saving as well as the aggregation of heterogeneous consumption profiles. Shell (1971) and Balasko and Shell (1980) analyze inefficiency in deterministic overlapping generations endowment economies; Chattopadhyay and Gottardi (1999) study the case with endowment risk. Malinvaud (1953), Phelps (1965), and Cass (1972) analyze capital over accumulation in deterministic environments, and Zilcha (1990) studies the case with stochastic production; see also Barbie, Hagedorn and Kaul (2007).

Beyond the material covered in the chapter, Yaari (1965) and Blanchard (1985) analyze models of "perpetual youth" where households face a constant probability of death while new cohorts enter the economy.

Chapter 4

Risk

With risk, income and consumption streams may be random. This introduces new elements in the household's consumption-saving tradeoff. We study this tradeoff in two environments, with *incomplete markets* and *complete markets* respectively, depending on the number of assets with linearly independent returns. Thereafter, we analyze risk sharing and study how uninsurable income risk affects capital accumulation. Throughout, we assume that households evaluate random consumption streams according to the expected utility criterion.

4.1 Consumption, Saving, and Insurance

4.1.1 Incomplete Markets

Suppose that there are two periods. In the second period, one of two histories is realized, $\epsilon^1 = h$ or $\epsilon^1 = l$, with probability $\eta(h)$ and $\eta(l)$ respectively. (At date $t = 1$, the history and the state of nature are identical.) The wage in the second period, $w_1(\epsilon^1)$, depends on the history; it equals $w_1(h)$ or $w_1(l)$. The household has access to one asset, a_1, and solves

$$\begin{aligned} \max_{a_1, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\eta(h)u(c_1(h)) + \eta(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1 = w_0 - c_0, \;\; c_1(\epsilon^1) = a_1 R_1 + w_1(\epsilon^1). \end{aligned}$$

Note that the return on saving is not history-contingent, in contrast to the wage. This assumption is not important; what is crucial is that fewer assets than states of nature are available. Note also that consumption in the second period is history-contingent.

Substituting the second-period dynamic budget constraints into the first-period dynamic budget constraint, we find one intertemporal budget constraint for each history,

$$c_0 + \frac{c_1(h)}{R_1} = w_0 + \frac{w_1(h)}{R_1} \;\text{ and }\; c_0 + \frac{c_1(l)}{R_1} = w_0 + \frac{w_1(l)}{R_1}.$$

If the second-period wage assumes the high value, then the intertemporal budget constraint restricts the present value of lifetime consumption expenditures to not exceed

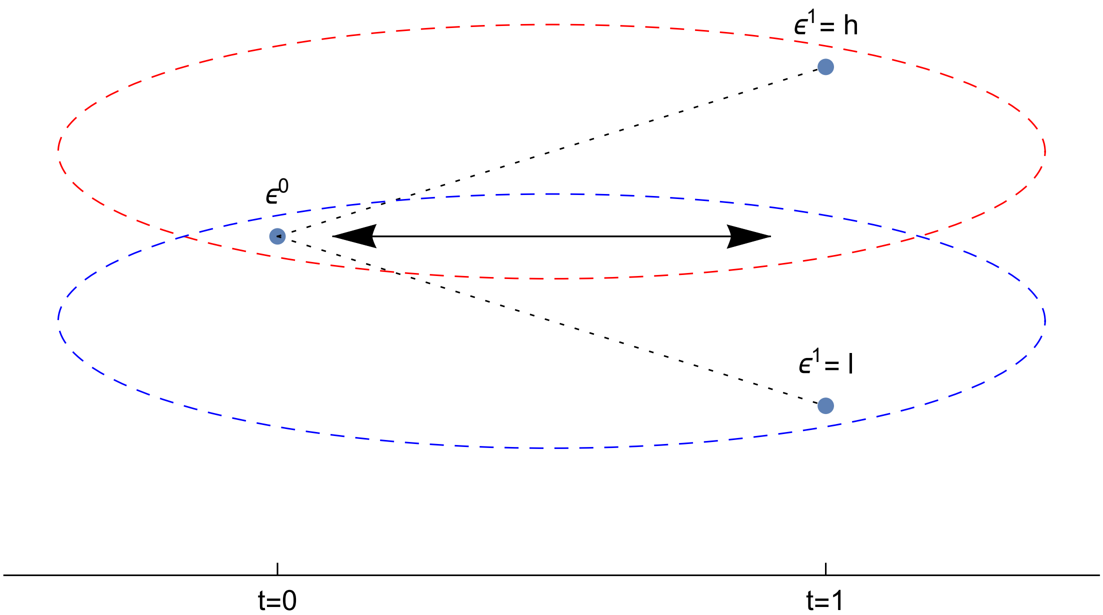

Figure 4.1: Incomplete markets: Two intertemporal budget constraints and one adjustment margin.

$w_0 + w_1(h)/R_1$. If, in contrast, it assumes the low value then the present value must not exceed $w_0 + w_1(l)/R_1$.

The household faces incomplete markets because it cannot exchange consumption in history h against consumption in history l. Figure 4.1 illustrates this. The dashed ellipses indicate the range of the two intertemporal budget constraints: One connects the initial period and the high state in the second period, the other the initial period and the low state. The arrows indicate the household's single margin of adjustment, corresponding to the choice of a_1: Resources can be shifted over time—saving reduces c_0 and increases both $c_1(h)$ and $c_1(l)$—but not to a specific node in the event tree.

To characterize the solution to the household's problem we can substitute the dynamic budget constraints into the objective function and differentiate with respect to a_1. This yields the *stochastic Euler equation*

$$u'(c_0) = \beta R_1 \mathbb{E}_0 \left[u'(c_1(\epsilon^1)) \right].$$

Intuitively, the cost of saving represented on the left-hand side is balanced with the average benefit across histories represented on the right-hand side.

Precautionary Saving

Assume that $\beta R_1 = 1$ such that the Euler equation reduces to $u'(c_0) = \mathbb{E}_0 \left[u'(c_1(\epsilon^1)) \right]$. Without risk, the equilibrium consumption profile would be flat in this case. With risk, in contrast, it cannot be flat for all ϵ^1 because $w_1(\epsilon^1)$ is stochastic; in fact, the consumption profile generally is not even flat on average. To see this, assume that

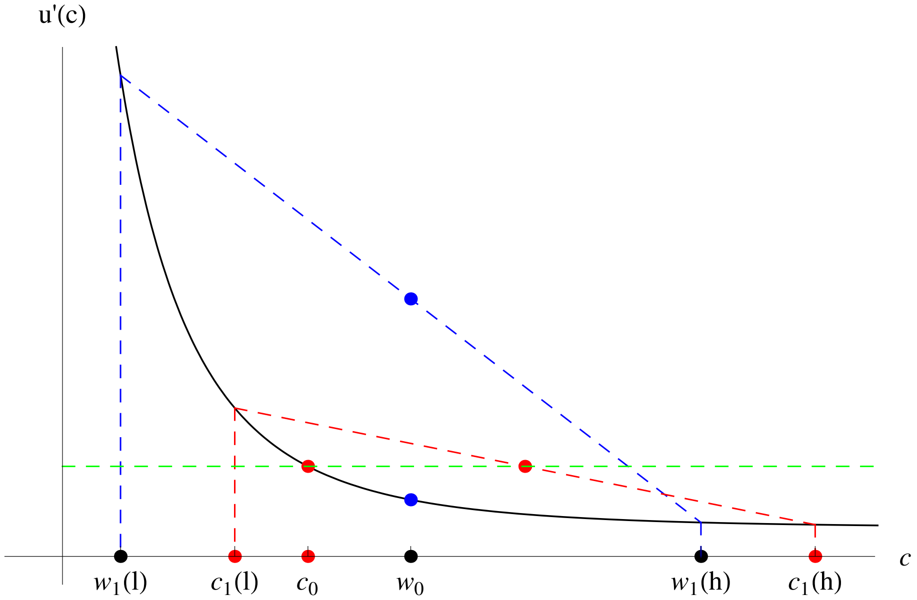

Figure 4.2: Convex marginal utility and income risk imply precautionary saving.

preferences are not only strictly concave, $u' > 0, u'' < 0$, as usual, but marginal utility also is convex, $u''' > 0$. Most plausible period utility functions satisfy this condition. By Jensen's inequality we then have $\mathbb{E}_0\left[u'(c_1(\epsilon^1))\right] > u'(\mathbb{E}_0\left[c_1(\epsilon^1)\right])$ and the Euler equation therefore stipulates $u'(c_0) > u'(\mathbb{E}_0\left[c_1(\epsilon^1)\right])$ or $c_0 < \mathbb{E}_0\left[c_1(\epsilon^1)\right]$. We conclude that convex marginal utility implies strictly positive average consumption growth, in spite of $\beta R_1 = 1$; saving is higher than in the absence of risk, reflecting a *precautionary saving* motive or *prudence*. In contrast, linear marginal utility implies $c_0 = \mathbb{E}_0\left[c_1(\epsilon^1)\right]$ and concave marginal utility implies $c_0 > \mathbb{E}_0\left[c_1(\epsilon^1)\right]$.

Figure 4.2 illustrates the precautionary saving motive. The figure plots the marginal utility function (in black) against consumption. Suppose first that the wage is deterministic and equal to w_0 in both periods. Since $\beta R_1 = 1$ optimal consumption then equals w_0 in both periods and saving equals zero. Consider next the case of interest with a risky wage in the second period, indicated by black dots. If the household continues not to save then $c_0 = w_0$, $c_1(h) = w_1(h)$, and $c_1(l) = w_1(l)$. Due to the convexity of the marginal utility function, expected marginal utility of second-period consumption, indicated by the upper blue dot, exceeds marginal utility of first-period consumption, indicated by the lower blue dot, and the Euler equation is violated. Intuitively, the "downside" risk for consumption affects average marginal utility more strongly than the "upside" risk. To satisfy the Euler equation, saving must rise, first-period consumption must fall to c_0, and history-contingent second-period consumption must rise to $c_0(l)$ or $c_0(h)$, all indicated by the lower red dots. In equilibrium, marginal utility in the first period and expected marginal utility in the second period, indicated by the upper red dots, coincide.

Note that the effect of risk on average marginal utility falls with household wealth.

As a consequence, a richer household engages in less precautionary saving.

Certainty Equivalence

While strict convexity of the marginal utility function is plausible it renders solving the model difficult. Linear marginal utility, which is associated with a quadratic utility function, simplifies the analysis—at the cost of abstracting from precautionary saving. It implies $\mathbb{E}_0[u'(c_1(\epsilon^1))] = u'(\mathbb{E}_0[c_1(\epsilon^1)])$ and thus, that the Euler equation reduces to $c_0 = \mathbb{E}_0[c_1(\epsilon^1)] + \phi$ where ϕ equals zero when $\beta R_1 = 1$. This is an instance of *certainty equivalence* that is, the optimality conditions only depend on the expected value of the variable of interest (here, consumption).

To appreciate the gain in tractability due to certainty equivalence, consider a three-period setting with quadratic utility and $\beta R_t = 1$ at all times. The intertemporal budget constraint then reads

$$c_0 + \beta c_1(\epsilon^1) + \beta^2 c_2(\epsilon^2) = w_0 + \beta w_1(\epsilon^1) + \beta^2 w_2(\epsilon^2)$$

and the Euler equations reduce to $c_0 = \mathbb{E}_0[c_1(\epsilon^1)]$ and $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Equilibrium consumption thus follows a *random walk*. Using the law of iterated expectations, we can combine these results to find

$$c_0(1 + \beta + \beta^2) = w_0 + \beta \mathbb{E}_0[w_1(\epsilon^1)] + \beta^2 \mathbb{E}_0[w_2(\epsilon^2)].$$

At date $t = 1$, history ϵ^1 the intertemporal budget constraint conditional on saving in the initial period ($a_1 = w_0 - c_0$) reads

$$c_1(\epsilon^1) + \beta c_2(\epsilon^2) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta w_2(\epsilon^2),$$

and the Euler equation is given by $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Taking expectations and combining the two conditions yields

$$c_1(\epsilon^1)(1 + \beta) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta \mathbb{E}_1[w_2(\epsilon^2)].$$

Comparing the results for c_0 conditional on information at date $t = 0$ and for $c_1(\epsilon^1)$ conditional on information at date $t = 1$, history ϵ^1 we note that

$$(c_1(\epsilon^1) - c_0)(1 + \beta) = (\mathbb{E}_1 - \mathbb{E}_0)[w_1(\epsilon^1) + \beta w_2(\epsilon^2)].$$

That is, the sign and magnitude of the innovation $c_1(\epsilon^1) - \mathbb{E}_0[c_1(\epsilon^1)]$ reflects how the expected present discounted value of income in and after date $t = 1$ changes as the information set changes from date $t = 0$ to date $t = 1$, history ϵ^1.

Risk of Binding Borrowing Constraint

A binding borrowing constraint reduces consumption. It also affects consumption earlier in time, before the constraint binds. In a stochastic environment, this effect is present whenever a borrowing constraint may bind with strictly positive probability. We illustrate this in a three-period setting with stochastic income $w_1(\epsilon^1)$ in the second period and non-stochastic income w_0 and w_2 otherwise. For simplicity, we let $\beta = 1$ and assume that gross returns also equal unity. Only borrowing at date $t = 1$ is prohibited, $a_2(\epsilon^1) \geq 0$.

We start by deriving the value function at date $t = 1$, history ϵ^1, when uncertainty is resolved. In states where $a_1 + w_1(\epsilon^1) \geq w_2$ the preferred level of a_2 is positive and the borrowing constraint does not bind. Consumption in the second and third period is equal in this case and given by $(a_1 + w_1(\epsilon^1) + w_2)/2$. In states where $a_1 + w_1(\epsilon^1) < w_2$, in contrast, the borrowing constraint does bind and consumption in the second and third period equals $a_1 + w_1(\epsilon^1)$ and w_2, respectively. The value function thus equals

$$V_1(a_1 + w_1(\epsilon^1)) = \begin{cases} u(a_1 + w_1(\epsilon^1)) + u(w_2) & \text{if } w_1(\epsilon^1) < w_2 - a_1 \\ 2 \cdot u\left(\frac{a_1 + w_1(\epsilon^1) + w_2}{2}\right) & \text{if } w_1(\epsilon^1) \geq w_2 - a_1 \end{cases} .$$

Note that the derivative of the value function has a kink at the critical value $a_1 + w_1(\epsilon^1) = w_2$ below which consumption smoothing is infeasible: $\lim_{\delta \downarrow 0} V_1''(w_2 - \delta) = u''(w_2)$ whereas $V_1''(w_2) = u''(w_2)/2$. That is, the derivative is convex around the critical level, independently of whether marginal utility is convex or not; all that is required for the convexity of V' is that preferences are strictly concave.

Consider now the effect of the potentially binding borrowing constraint at date $t = 1$ on saving in the initial period, a_1. While the household's program

$$\max_{a_1} \; u(w_0 - a_1) + \mathbb{E}_0[V_1(a_1 + w_1(\epsilon^1))]$$

yields the usual Euler equation, $u'(c_0) = \mathbb{E}_0[V_1'(a_1 + w_1(\epsilon^1))]$, the convexity of V' leads the household to save more at date $t = 0$ than if no risk of a binding borrowing constraint were present. The intuition mirrors the one for precautionary saving although it is the risk of a binding borrowing constraint in combination with strictly concave preferences—not convexity of marginal utility—which drives the result.

Buffer Stock Saving

Consider an impatient household in an environment with constant interest rates, $\beta R < 1$. Absent risk, this household would choose a declining consumption path. With risk, in contrast, the precautionary saving motive or the risk of a future binding borrowing constraint work in the opposite direction and encourage saving.

The net effect on saving may depend on household wealth. If the marginal propensity to consume falls with household wealth then the two motives encouraging saving gain in strength as the household becomes poorer: For given future income risk, less wealth translates into higher consumption risk and thus, a stronger saving motive. The wealth dependent saving motive on the one hand and impatience on the other give rise to a target ratio of financial assets to average income. During good times, the household builds up a *buffer stock* of financial assets from which it draws during bad times.

4.1.2 Complete Markets

Turning to complete markets, consider again the environment with two periods and two histories. In contrast to the incomplete market setting, the household now has access to two assets with linearly independent returns. For simplicity, we assume that these two assets are *Arrow securities* that is, securities that only pay off in one history each (we relax this assumption later). We denote by a_1^1 the quantity of the first Arrow security that pays off if and only if $\epsilon^1 = h$, and we denote the history-dependent return on this security by $R_1^1(\epsilon^1)$ with $R_1^1(l) = 0$. Similarly, a_1^2 denotes the quantity of the second Arrow security that pays off if and only if $\epsilon^1 = l$, and its return is denoted $R_1^2(\epsilon^1)$ with $R_1^2(h) = 0$. The household's program reads

$$\begin{aligned}\max_{a_1^1, a_1^2, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\eta(h)u(c_1(h)) + \eta(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1^1 + a_1^2 = w_0 - c_0,\ c_1(\epsilon^1) = a_1^1 R_1^1(\epsilon^1) + a_1^2 R_1^2(\epsilon^1) + w_1(\epsilon^1).\end{aligned}$$

As in the incomplete-market setting, three dynamic budget constraints bind. In contrast to the incomplete-market setting, however, these three constraints can be combined into a single intertemporal budget constraint rather than separate ones for each history:

$$c_0 + \frac{c_1(h)}{R_1^1(h)} + \frac{c_1(l)}{R_1^2(l)} = w_0 + \frac{w_1(h)}{R_1^1(h)} + \frac{w_1(l)}{R_1^2(l)}.$$

The situation is akin to a static environment where the household can exchange all goods ($c_0, c_1(h)$, and $c_1(l)$) against each other—the household faces complete markets. In particular, and in contrast to the incomplete-market setting, the two assets do not only allow the household to shift resources across time (that is, exchange c_0 against a *bundle* of $c_1(h)$ and $c_1(l)$) but also to specific nodes in the event tree. Equivalently, they allow to shift resources across histories at date $t = 1$, by buying less of one Arrow security and more of the other. Since consumption in the two histories can be chosen independently of each other the household may achieve full insurance ($c_1(h) = c_1(l)$) although $w_1(h) \neq w_1(l)$, unlike in the incomplete-market case.

Figure 4.3 illustrates the complete-market setting. The dashed ellipse indicates the range of the single intertemporal budget constraint that connects the initial period and both states in the second period. The arrows indicate the two margins of adjustment, corresponding to the choices of a_1^1 and a_1^2.

The first-order conditions of the program with Arrow securities are given by the Euler equations

$$u'(c_0) = \beta R_1^1(h)\eta(h)u'(c_1(h)) \text{ and } u'(c_0) = \beta R_1^2(l)\eta(l)u'(c_1(l)).$$

These conditions do not contain an expectation operator because the choice of a_1^1 or a_1^2 affects second-period consumption only in one history each. If returns are actuarially fair that is, return differentials compensate for risk such that $R_1^1(h)/R_1^2(l) = \eta(l)/\eta(h)$, then it is optimal to smooth consumption across states, $c_1(h) = c_1(l)$. If, moreover, $\beta R_1^1(h)\eta(h) = 1$, then consumption is perfectly smoothed over time as well, unlike in the incomplete-market economy.

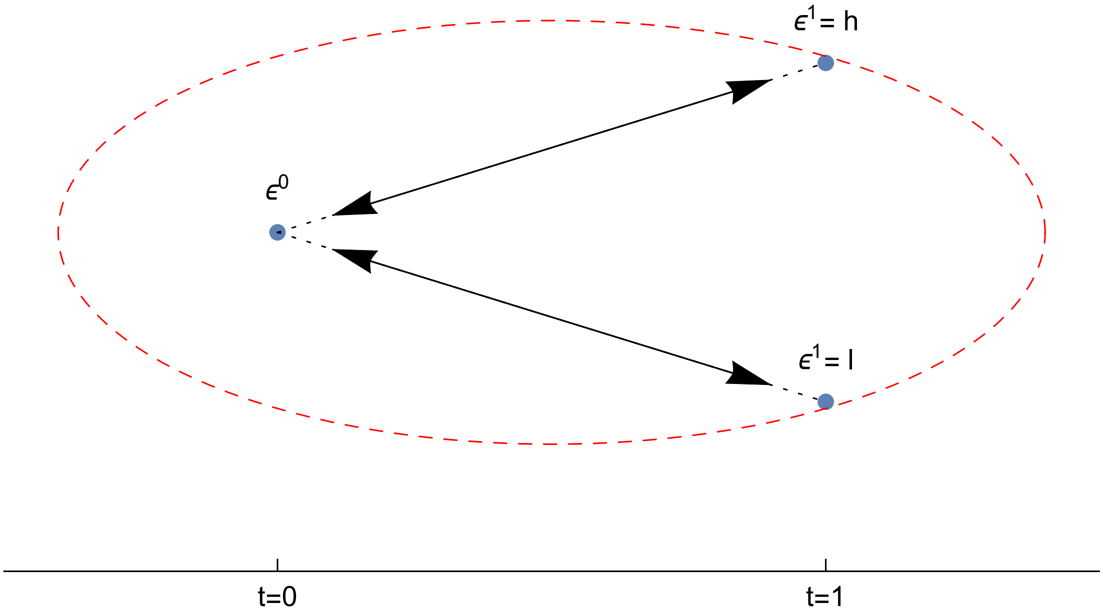

Figure 4.3: Complete markets: One intertemporal budget constraint and two adjustment margins.

Generalizations

Market completeness does not require the existence of Arrow securities. It only requires as many assets with linearly independent returns as states of nature (which is guaranteed with a complete set of Arrow securities). To understand the independence requirement, consider a general return structure with $R_1^i(\epsilon^1) \geq 0, i = 1, 2; \epsilon^1 = h, l$. (With Arrow securities, $R_1^1(l) = R_1^2(h) = 0$.) The dynamic budget constraints in the second period can be expressed as

$$\begin{bmatrix} c_1(h) - w_1(h) \\ c_1(l) - w_1(l) \end{bmatrix} = \begin{bmatrix} R_1^1(h) & R_1^2(h) \\ R_1^1(l) & R_1^2(l) \end{bmatrix} \begin{bmatrix} a_1^1 \\ a_1^2 \end{bmatrix}.$$

If the return vectors $R_1^1(\epsilon^1)$ and $R_1^2(\epsilon^1)$ are linearly independent then the matrix on the right-hand side has full rank and its determinant, $D = R_1^1(h)R_1^2(l) - R_1^1(l)R_1^2(h)$, differs from zero. The equation thus can be solved for a_1^1 and a_1^2. Substituting the resulting expressions into the first-period dynamic budget constraint yields the single intertemporal budget constraint

$$w_0 - c_0 + (w_1(h) - c_1(h))\frac{R_1^2(l) - R_1^1(l)}{D} + (w_1(l) - c_1(l))\frac{R_1^1(h) - R_1^2(h)}{D} = 0.$$

(When $R_1^1(l) = R_1^2(h) = 0$, this reduces to the constraint in the case with Arrow securities.)

The term $(R_1^2(l) - R_1^1(l))/D$ in the intertemporal budget constraint represents the price of second-period consumption in history h, expressed in terms of first-period consumption. To see this, note that purchasing ϕ units of the first asset and $-\phi R_1^1(l)/R_1^2(l)$

units of the second yields a return of $\phi(R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))$ in history h and zero in history l. To secure one additional unit of consumption in history h, the household thus must acquire $\phi = (R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))^{-1} = R_1^2(l)/D$ units of the first asset and $-R_1^1(l)/D$ units of the second, at a cost of $(R_1^2(l) - R_1^1(l))/D$.[1] Similarly, $(R_1^1(h) - R_1^2(h))/D$ represents the price of consumption in history l.

In an interior equilibrium, the Euler equations now read

$$\begin{aligned} u'(c_0) &= \beta(R_1^1(h)\eta(h)u'(c_1(h)) + R_1^1(l)\eta(l)u'(c_1(l))), \\ u'(c_0) &= \beta(R_1^2(h)\eta(h)u'(c_1(h)) + R_1^2(l)\eta(l)u'(c_1(l))). \end{aligned}$$

Linear combinations of these equations recover the Euler equations for the Arrow securities. For example, multiplying the first equation by $R_1^2(l)$ and the second by $-R_1^1(l)$ and summing yields

$$u'(c_0) = \beta \frac{D}{R_1^2(l) - R_1^1(l)} \eta(h)u'(c_1(h)).$$

Market completeness does not require that all date- and history-contingent goods can be traded in the initial period (either by means of Arrow securities or combinations of assets with linearly independent returns). It suffices if there are securities to sequentially transfer purchasing power between all nodes of the event tree and if in each node, all goods can be traded on spot markets.

To see this, consider an economy with three periods, $t = 0, 1, 2$; S states of nature in both the second and the third period (i.e., S^2 histories at date $t = 2$); G goods at each node of the event tree in the second and third period; and one good at date $t = 0$. At each node, one good serves as numeraire. A complete set of Arrow securities involves $(S + S^2)G$ securities, namely SG for the delivery of history-contingent goods at date $t = 1$ and S^2G for delivery at date $t = 2$. Note that the returns on the Arrow securities implicitly define relative prices between the goods at each node, and between the numeraire good at different nodes.

Consider next an alternative market structure where only S contingent securities are traded in the first period; each security delivers a specific amount of the numeraire good in one specific state at date $t = 1$ (and each security delivers in a different state). Once uncertainty is resolved at date $t = 1$, the numeraire good is traded against the other goods on spot markets and S contingent securities for history-contingent date-$t = 2$ numeraire goods are traded. Finally, once uncertainty is resolved at date $t = 2$, all goods again are traded on spot markets. This alternative market structure only uses $S + G + S + G$ markets but it provides the same trading possibilities as the complete set of Arrow securities. It also generates the same budget set when the spot market

[1] When $R_1^1(l) = R_1^2(l)$ but $D \neq 0$ then the return on one asset strictly dominates the return on the other: In history l both assets generate the same return, but in history h one generates a strictly higher return than the other (if $D > 0$ the first asset returns more, if $D < 0$ the second does). Buying the asset with the strictly higher return and selling the one with the lower return allows to increase consumption in history h without having to give up consumption in the initial period or in history l; the price of consumption in state h therefore equals zero. See also section 5.3.

prices and the returns on the contingent securities correspond to the implied relative prices in the environment with Arrow securities.

4.1.3 General Case

Consider a two-period setup with a finite number of histories and a finite number of assets indexed by i. Markets may be complete or incomplete. The household's program reads

$$\max_{c_0, \{a_1^i\}_i, \{c_1(\epsilon^1)\}_{\epsilon^1}} \quad u(c_0) + \beta \mathbb{E}_0 \left[u(c_1(\epsilon^1)) \right]$$
$$\text{s.t.} \quad \sum_i a_1^i = w_0 - c_0, \; c_1(\epsilon^1) = \sum_i a_1^i R_1^i(\epsilon^1) + w_1(\epsilon^1).$$

For each asset i that the household purchases or sells the corresponding Euler equation

$$u'(c_0) = \beta \mathbb{E}_0[u'(c_1(\epsilon^1)) R_1^i(\epsilon^1)]$$

holds. Expressed differently, $1 = \mathbb{E}_0[m_1(\epsilon^1) R_1^i(\epsilon^1)]$ where $m_1(\epsilon^1) \equiv \beta u'(c_1(\epsilon^1))/u'(c_0)$ denotes the household's marginal rate of substitution. Note that $\sum_i a_1^i = \mathbb{E}_0[m_1(\epsilon^1) \sum_i a_1^i R_1^i(\epsilon^1)]$ because the household either is invested in the asset i in which case $1 = \mathbb{E}_0[m_1(\epsilon^1) R_1^i(\epsilon^1)]$, or it is not invested in which case $a_1^i = 0$.

Multiplying the dynamic budget constraints at date $t = 1$ by $m_1(\epsilon^1)$ and taking expectations yields

$$\mathbb{E}_0[m_1(\epsilon^1) c_1(\epsilon^1)] = \mathbb{E}_0 \left[m_1(\epsilon^1) \sum_i a_1^i R_1^i(\epsilon^1) \right] + \mathbb{E}_0[m_1(\epsilon^1) w_1(\epsilon^1)].$$

Adding the dynamic budget constraint at date $t = 0$, we arrive at the equilibrium condition

$$c_0 + \mathbb{E}_0[m_1(\epsilon^1) c_1(\epsilon^1)] = w_0 + \mathbb{E}_0[m_1(\epsilon^1) w_1(\epsilon^1)]. \tag{4.1}$$

Condition (4.1) holds independently of whether markets are complete or incomplete. When markets are complete, (4.1) incorporates all equilibrium restrictions imposed by the intertemporal budget constraint and the Euler equations (except possibly a restriction that rates of return are given exogenously). When markets are incomplete, in contrast, condition (4.1) represents these equilibrium conditions only partially because it is an average of the multiple intertemporal budget constraints which bind individually.

To see this in the two special cases considered earlier, recall that in the saving problem with Arrow securities the intertemporal budget constraint and Euler equations are given by

$$w_0 - c_0 + \frac{w_1(h) - c_1(h)}{R_1^1(h)} + \frac{w_1(l) - c_1(l)}{R_1^2(l)} = 0, \; \frac{1}{R_1^1(h)} = \eta(h) m_1(h), \; \frac{1}{R_1^2(l)} = \eta(l) m_1(l).$$

Substituting the latter into the former yields (4.1). In the saving problem with incomplete markets and a safe return, in contrast, the history specific intertemporal budget constraint and the Euler equation are given by

$$w_0 - c_0 + \frac{w_1(\epsilon^1) - c_1(\epsilon^1)}{R_1} = 0, \ \frac{1}{R_1} = \mathbb{E}_0 \left[m_1(\epsilon^1) \right].$$

Substituting the latter into the former does not yield (4.1). But summing the intertemporal budget constraints for the two histories, weighted by the respective probabilities and marginal rates of substitution, does yield condition (4.1) once the Euler equation is imposed.

4.2 Risk Sharing

4.2.1 Borch Rule

Consider an economy with heterogeneous groups of representative households who may buy and sell assets with contingent returns. In equilibrium at date t,

$$1 = \mathbb{E}_t[m^h_{t+1}(\epsilon^{t+1}) R^i_{t+1}(\epsilon^{t+1})]$$

for all assets i and all (representative) households h where m^h_{t+1} denotes h's marginal rate of substitution. For households from two different groups, l and n say, it follows that

$$\mathbb{E}_t[(m^l_{t+1}(\epsilon^{t+1}) - m^n_{t+1}(\epsilon^{t+1})) R^i_{t+1}(\epsilon^{t+1})] = 0.$$

When markets are complete this condition simplifies. To see this most directly, assume without loss of generality that the assets include a complete set of Arrow securities. (Recall that the return of an Arrow security equals zero in all histories except a single one.) The condition then reduces to $m^l_{t+1}(\epsilon^{t+1}) = m^n_{t+1}(\epsilon^{t+1})$. If both households share the same utility function this implies

$$\frac{u'(c^n_t(\epsilon^t))}{u'(c^l_t(\epsilon^t))} = \frac{u'(c^n_{t+1}(\epsilon^{t+1}))}{u'(c^l_{t+1}(\epsilon^{t+1}))}. \tag{4.2}$$

Condition (4.2), which is referred to as *Borch rule*, states that households *share risk*—whenever marginal utility of one household is high or low the same holds true for the other. The ratio of marginal utilities reflects differences in wealth. When households have homothetic preferences (i.e., preferences generating linear Engel curves) then condition (4.2) simplifies to

$$\frac{c^n_t(\epsilon^t)}{c^l_t(\epsilon^t)} = \frac{c^n_{t+1}(\epsilon^{t+1})}{c^l_{t+1}(\epsilon^{t+1})}.$$

Risk sharing is Pareto optimal. To see this, consider the problem of maximizing the welfare of household l subject to the other households attaining given levels of welfare, $\{\bar{U}^h\}_h$. The Lagrangian reads

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_0\left[\sum_{t=0}^{\infty}(\beta^l)^t u(c_t^l(\epsilon^t))\right] + \sum_{h\neq l}\lambda^h \mathbb{E}_0\left[\sum_{t=0}^{\infty}(\beta^h)^t u(c_t^h(\epsilon^t)) - \bar{U}^h\right] \\ &\quad + \sum_{t,\epsilon^t}\mu_t(\epsilon^t)\left\{\ldots - c_t^l(\epsilon^t) - \sum_{h\neq l} c_t^h(\epsilon^t) + \ldots\right\},\end{aligned}$$

where we assume that the number of households is the same in each group; λ^h denotes the multiplier associated with the reservation utility requirement for household h; and $\mu_t(\epsilon^t)$ denotes the multiplier associated with the resource constraint. We do not need to be specific about the production side of the economy, thus the dots. Differentiating with respect to consumption yields

$$u'(c_t^l(\epsilon^t)) = \lambda^h u'(c_t^h(\epsilon^t))$$

which implies the risk sharing condition (4.2).

4.2.2 Aggregate and Idiosyncratic Risk

Suppose that households have endowments, $w_t^h(\epsilon^t) \equiv w_t(\epsilon^t) + \iota_t^h(\epsilon^t)$, with an *aggregate* and an *idiosyncratic* or household specific component. The former, $w_t(\epsilon^t)$, is the same across all groups while the latter, $\iota_t^h(\epsilon^t)$, varies across groups. We assume that the sum of the idiosyncratic components equals zero in all histories. Markets are complete and in equilibrium, households thus share risk.

Let φ^h denote wealth of household h relative to average wealth. With identical and homothetic preferences condition (4.2) then implies $c_t^h(\epsilon^t) = \varphi^h c_t(\epsilon^t)$ where $c_t(\epsilon^t)$ denotes average consumption. Using the resource constraint, this yields

$$c_t^h(\epsilon^t) = \varphi^h w_t(\epsilon^t).$$

Note that consumption of all households is proportional to the average endowment in the economy. In other words, with complete markets consumption of households only reflects aggregate shocks and idiosyncratic risk is fully diversified.

4.3 Uninsurable Income Risk And Capital Accumulation

In stark contrast to the environment with risk sharing we now consider a setting where insurance is ruled out. We assume that there is only idiosyncratic risk and analyze the consequences of missing insurance markets for capital accumulation.

4.3.1 Economy

The structure of the economy is the same as in the representative agent model of section 3.1, except for one difference: The time endowment of each household is random. Formally, there is a continuum of measure one of infinitely lived households, indexed by $h \in [0,1]$. The time endowment of household h at date t is given by $1+\iota_t^h$. It is strictly positive, bounded, and i.i.d. across households with minimum value $1+\underline{\iota}$ and mean $\mathbb{E}_t[1+\iota_{t+1}^h]=1$. As a consequence, aggregate labor supply equals unity at all times. We assume that the time endowment follows a *Markov process* that is, the probability distribution of the endowment in a period only depends on its realized value in the preceding period.

Households have access to a risk-free asset with gross interest rate R_t. Net financial assets of households correspond to the economy's capital stock, k_t. Firms rent labor at the competitive wage w_t per unit of time, and capital at the competitive rate r_t. Capital depreciates at rate δ and thus, $R_t = 1 + r_t - \delta$.

We consider a *stationary equilibrium*: While the time endowment and assets of an individual household change from period to period, the joint distribution of time endowments and assets across the population is time invariant. Accordingly, the aggregate capital stock and thus, interest rates and wages are time invariant as well.

4.3.2 Households

Household h maximizes $\mathbb{E}_0[\sum_{t=0}^{\infty}\beta^t u(c_t^h(\epsilon^t))]$. Its dynamic budget constraint is given by

$$a_{t+1}^h(\epsilon^t) = a_t^h(\epsilon^{t-1})R + w(1+\iota_t^h(\epsilon^t)) - c_t^h(\epsilon^t).$$

Consumption must be non-negative. As debt must be serviced under all circumstances this implies a *natural borrowing limit* equal to the market value of future labor income in the worst possible history,

$$a_{t+1}^i(\epsilon^t) \geq \underline{a} \equiv -w(1+\underline{\iota})\frac{1}{R-1}.$$

In addition, a tighter borrowing constraint may bind. For example, if households are excluded from borrowing at all then the natural borrowing limit is replaced by the restriction $a_{t+1}^h(\epsilon^t) \geq 0$.

At date t, history ϵ^t the state variables in the household's program are $(a_t^h(\epsilon^{t-1}), \iota_t^h(\epsilon^t))$ as well as the constant wage and interest rate, and the control variables are $(a_{t+1}^h(\epsilon^t), c_t^h(\epsilon^t))$. Let $(a_\circ^h, \iota_\circ^h)$ denote household assets and the time endowment in the current period, and let (a_+^h, ι_+^h) denote those objects in the subsequent period. Since the program is time autonomous the Bellman equation for household h reads

$$V(a_\circ^h, \iota_\circ^h; w, R) = \max_{a_+^h} u(a_\circ^h R + w(1+\iota_\circ^h) - a_+^h) + \beta\mathbb{E}\left[V(a_+^h, \iota_+^h; w, R)|\iota_\circ^h\right]$$

subject to the borrowing constraint; the expectation is conditional on $\iota_\circ^h$ because the current time endowment may contain information about the probability distribution of next period's endowment (the Markov assumption implies that $\iota^h_\circ$ contains all such information).

If ι risk were absent (or equivalently, if markets were complete and households insured each other), the household would face a risk-free, constant labor income stream. With $\beta R = 1$, its optimal consumption would be time invariant and equal to $a^h_\circ(R-1)+w$. With risk, in contrast, optimal consumption cannot be constant and finite. If it were, its value would have to be consistent with the worst case scenario of minimum time endowments forever after. But after a more favorable time endowment realization the household could increase consumption and this implies a contradiction.

When $\beta R \geq 1$, the household's first-order and envelope conditions yield

$$V_a(a^h_\circ, \iota^h_\circ; w, R) = Ru'(c^h_\circ) \geq \mathbb{E}\left[V_a(a^h_+, \iota^h_+; w, R)|\iota^h_\circ\right] = R\mathbb{E}\left[u'(c^h_+)|\iota^h_\circ\right].$$

With $u' > 0$ and $u'' < 0$, optimal consumption stochastically converges to infinity in this case as the household accumulates more and more assets to *self insure* against low future time endowment realizations. Formally, this result follows from three observations. First, the Euler equation $u'(c^h_\circ) \geq \mathbb{E}\left[u'(c^h_+)|\iota^h_\circ\right]$ implies that marginal utility follows a submartingale, which converges. Second, marginal utility only converges if consumption converges, due to $u'' < 0$. Third, with finite asset holdings an income shock translates into a change of consumption. Convergence of marginal utility therefore requires asset holdings to converge to infinity.

We conclude that for households to accumulate a finite level of assets the interest rate must satisfy $\beta R < 1$.

4.3.3 General Equilibrium

The stationary equilibrium in the hypothetical economy without risk (or with insurance) would satisfy $R = \beta^{-1} = 1 + r - \delta$, $f_K(k, 1) = r$, and $f_L(k, 1) = w$. Aggregate consumption would equal $k(R-1)+w$.

In the stationary equilibrium in the economy with risk, in contrast, R must be strictly smaller than β^{-1} to clear the market for capital; otherwise the supply of capital would grow without bound while firms' demand would be bounded. Households accumulate assets when their time endowment is high and run them down when it is low. The capital stock in the economy is constant and since $R < \beta^{-1}$, it is strictly larger than in the economy without risk, and so are wages. Although the risk is purely idiosyncratic and washes out in the aggregate, self insurance gives rise to a higher capital stock.

Suppose for simplicity that the time endowment can assume m possible values and the asset holdings of a household n values; both m and n are finite. The $m \times m$ transition matrix Π^ι whose rows sum to unity contains the transition probabilities of the time endowment; the (i, j) element of Π^ι gives the probability that ι_+ takes the j-th of the m possible values conditional on ι taking the i-th such value.

The state (a^h, ι^h) of household h then can take mn values. Together with the transition matrix Π^ι, the decision rules of households define a transition matrix for this state,

Π say which is of size $mn \times mn$. Let $\Pi^\top$ denote the transpose of Π, and let d of size $mn \times 1$ denote the probability distribution of households over the possible states; the elements of d sum to one. Note that conditional on $d_\circ$, the probability distribution in the subsequent period is given by $d_+ = \Pi^\top d_\circ$. A *stationary distribution* therefore satisfies $d = \Pi^\top d$; it is the normalized eigenvector associated with the unit eigenvalue of $\Pi^\top$.

4.4 Bibliographic Notes

von Neumann and Morgenstern (1944) introduce expected utility. Modigliani and Brumberg (1954) discuss the "precautionary motive" for saving and Friedman (1957, p. 16) discusses saving as a "reserve for emergencies." Leland (1968) relates the precautionary saving motive to the convexity of marginal utility, and Sandmo (1970) analyzes the differences between return and labor income risk. The analysis of the saving problem with quadratic utility is due to Hall (1978). Kimball (1990) defines prudence as the sensitivity of an optimal choice (here saving) to risk. Zeldes (1989*b*) simulates optimal consumption choices in the presence of a precautionary motive and finds that the marginal propensity to consume transitory income varies with the level of household assets. Zeldes (1989*a*) emphasizes that the risk of a binding borrowing constraint affects equilibrium consumption. Deaton (1991) analyzes the role of assets as "buffer stock" in a model with borrowing constraints, precautionary motive, and impatience. Carroll (1997) analyzes buffer stock saving in a model with precautionary motive and impatience and shows that households target a wealth-to-permanent-income ratio. Arrow (1953; 1964) and Radner (1972) analyze equilibrium with sequential trading. Borch (1962) derives the risk sharing condition. Aiyagari (1994) analyzes idiosyncratic risk and capital accumulation in stationary equilibrium, building on Bewley (1977; 1980; 1986) and Huggett (1993). Chamberlain and Wilson (2000) prove that consumption grows without bound if $\beta R \geq 1$.

Beyond the material covered in the chapter, Cass and Shell (1983) analyze the implications of risk and limited participation for efficiency; they show that even with complete markets and in a finite horizon economy, extrinsic risk (sunspots) can cause Pareto inefficient allocations when market participation in an overlapping generations economy is limited to those households that are alive.

Chapter 5

Asset Returns and Asset Prices

If a household invests in multiple assets it is indifferent between the investments at the margin. We derive the implications of this indifference and of market clearing for asset returns and prices. Throughout we assume that households maximize expected utility. In appendix B.4 we discuss a model that relaxes this assumption.

5.1 Asset Pricing Kernel

Consider the equilibrium in an economy with two periods and risk. Markets may be complete or incomplete. We saw earlier (see subsection 4.1.3) that for each asset i and each household h that purchases or sells the asset, an Euler equation

$$u'(c_0^h) = \beta \mathbb{E}_0[u'(c_1^h(\epsilon^1))R_1^i(\epsilon^1)] \text{ or } 1 = \mathbb{E}_0[m_1^h(\epsilon^1)R_1^i(\epsilon^1)]$$

holds where $m_1^h(\epsilon^1) \equiv \beta u'(c_1^h(\epsilon^1))/u'(c_0^h)$ denotes household h's marginal rate of substitution.

This has two important implications. First, when asset i is held by different households, l and n say, then the marginal rates of substitution of these households satisfy $\mathbb{E}_0[m_1^l(\epsilon^1)R_1^i(\epsilon^1)] = \mathbb{E}_0[m_1^n(\epsilon^1)R_1^i(\epsilon^1)]$ that is, the return weighted average marginal rates of substitution coincide. When l and n hold multiple assets then this imposes multiple cross-household restrictions on their marginal rates of substitution. When markets are complete, the cross-household restrictions imply that along each history, both households have the same marginal rate of substitution, $m_1^l(\epsilon^1) = m_1^n(\epsilon^1)$, (see subsection 4.2.1). More generally, the equilibrium marginal rates of substitution, $\{m_1(\epsilon^1)\}_{\epsilon^1}$, which also are referred to as the *asset pricing kernel* or the *stochastic discount factor*, are the same for all households when markets are complete, independently of whether the households are homogeneous or not.

The second implication concerns average return differentials across assets, or excess returns to which we turn next.

5.2 Excess Returns

5.2.1 C-CAPM

When a household with marginal rates of substitution $\{m_1(\epsilon^1)\}_{\epsilon^1}$ purchases or sells multiple assets, j and k say, then the returns on these assets satisfy $\mathbb{E}_0[m_1(\epsilon^1)R_1^j(\epsilon^1)] = \mathbb{E}_0[m_1(\epsilon^1)R_1^k(\epsilon^1)]$ that is, the kernel weighted average returns on the assets coincide. Expressing the equality as $\mathbb{E}_0[m_1(\epsilon^1)(R_1^j(\epsilon^1) - R_1^k(\epsilon^1))] = 0$ and using the definition of covariance yields

$$\mathbb{E}_0[m_1(\epsilon^1)]\mathbb{E}_0[R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] + \mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] = 0.$$

Equilibrium therefore imposes restrictions on the *expected returns* and *return covariances* of assets. The latter reflect how strongly asset returns covary with the marginal rate of substitution.

Suppose that asset f is risk-free, $R_1^f(\epsilon^1) = R_1^f$, and the household holds the risk-free asset such that $1 = \mathbb{E}_0[m_1(\epsilon^1)]R_1^f$. The *excess return* of asset i that is, the expected return net of the risk-free return then satisfies

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = -\frac{\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} = -\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]R_1^f.$$

According to this *consumption capital asset pricing model* (C-CAPM) result, the excess return is proportional to the covariance between the asset's return and the marginal rate of substitution. Note that the excess return compensates for covariation of the asset return with marginal utility, not for return volatility per se.

Since β and $u'(c_0)$ in $m_1(\epsilon^1)$ are constants, the sign of the excess return depends on the covariance between $R_1^i(\epsilon^1)$ and $u'(c_1(\epsilon^1))$. The asset pays zero excess return if this covariance is zero, for example because utility is linear (risk neutrality) or consumption is deterministic (full insurance). If the asset return covaries negatively with the marginal rate of substitution and thus (if utility is strictly concave) positively with $c_1(\epsilon^1)$, then the excess return is positive. Intuitively, the asset is a bad hedge in this case; it tends to pay more when the marginal benefit from additional resources is small. To induce the household to nevertheless hold the asset its return must be high. If the asset return covaries negatively with $c_1(\epsilon^1)$, in contrast, then the excess return is negative; when the asset is a good hedge it need not pay a high average return.

The C-CAPM implies a *mean-variance frontier* that bounds the absolute value of an asset's excess return given the standard deviation of its return:

$$\begin{aligned} \mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f &= -\frac{\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} \\ \Rightarrow \quad |\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f| &\leq \frac{\mathbb{S}\text{td}_0[m_1(\epsilon^1)]\mathbb{S}\text{td}_0[R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]}. \end{aligned}$$

Here, we use the fact that the covariance equals the product of the standard deviations and the correlation coefficient, which lies between minus and plus one.

5.2.2 CAPM

The C-CAPM establishes a linear relation between the equilibrium excess return on an asset and the covariance between the asset return and the asset pricing kernel. The *capital asset pricing model* (CAPM), which precedes the C-CAPM, similarly establishes such a linear relation; but in the case of the CAPM it is the covariance between the asset return and the return on the *market portfolio* encompassing all risky assets—not the pricing kernel—which enters the relation.

The CAPM follows from the C-CAPM under the assumption that consumption is a linear function of the return on the market portfolio—the *market return* $R_1^m(\epsilon^1)$—and the marginal utility function is accurately approximated to the first order. The asset pricing kernel $m_1(\epsilon^1)$ then is a linear function of $R_1^m(\epsilon^1)$ and letting ϕ denote a factor of proportionality, we have

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \mathbb{C}\mathrm{ov}_0\left[R_1^m(\epsilon^1), R_1^i(\epsilon^1)\right]\phi R_1^f.$$

In particular, $\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f = \mathbb{C}\mathrm{ov}_0[R_1^m(\epsilon^1), R_1^m(\epsilon^1)]\phi R_1^f$. It follows that

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \frac{\mathbb{C}\mathrm{ov}_0[R_1^m(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{V}\mathrm{ar}_0[R_1^m(\epsilon^1)]}(\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f).$$

The ratio on the right-hand side of the last equality represents asset i's "*beta*," the normalized covariation between the asset return and the market return. (Formally, beta equals the projection of $R_1^i(\epsilon^1)$ on $R_1^m(\epsilon^1)$.) According to the CAPM, the excess return equals the product of the asset's beta and the market's excess return.

Originally, the CAPM was derived under the alternative assumption that a representative household values the mean return on its portfolio (positively) as well as the return variance (negatively). The optimal *portfolio choice* then implies a linear relation between an asset's excess return and the covariance between the asset and portfolio returns. Moreover, market clearing requires that the household's portfolio choice corresponds to the market portfolio and thus, the portfolio return to the market return.

Formally, let e denote the $n \times 1$ vector of expected returns on n risky assets; and V the $n \times n$ variance-covariance matrix of the returns. The household chooses the portfolio shares invested in the risky assets, represented by the $n \times 1$ vector x, and maximizes the expected portfolio return minus γ times the portfolio variance, where γ reflects risk aversion. Letting a T superscript denote "transposed" and $\bar{x}$ the portfolio share $1 - \sum_{i=1}^n x_i$ invested in the risk-free asset, the household's problem reads

$$\max_x \ x^T e + (1 - \bar{x})R^f - \gamma x^T V x$$

and yields the first-order condition

$$e^T - R^f = 2\gamma\, x^T V.$$

Post multiplying the last condition by x and combining the result with the first-order condition yields

$$e^T - R^f = \frac{x^T V}{x^T V x}[(e^T - R^f)x].$$

Market clearing implies that the shares chosen by the household, x, correspond to the shares of the risky assets in the market portfolio (the former are $1-\bar{x}$ times the latter). The ratio on the right-hand side of the equation thus corresponds to the vector of betas; and the expression in brackets corresponds to the excess return on the market portfolio. The difference on the left-hand side represents the vector of excess returns.

5.3 Asset Prices

To derive the implications of the C-CAPM for *asset prices*, we use the definition of a return: The gross rate of return between date t and $t+1$, $R^i_{t+1}(\epsilon^{t+1})$, equals the payoff at date $t+1$ relative to the asset price at date t, $p^i_t(\epsilon^t)$; and the payoff consists of the asset price, $p^i_{t+1}(\epsilon^{t+1})$, and the dividend, $d^i_{t+1}(\epsilon^{t+1})$:

$$R^i_{t+1}(\epsilon^{1+1}) \equiv \frac{p^i_{t+1}(\epsilon^{t+1}) + d^i_{t+1}(\epsilon^{t+1})}{p^i_t(\epsilon^t)}. \tag{5.1}$$

We can therefore rewrite the return condition, $1 = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})R^i_{t+1}(\epsilon^{t+1})]$, as

$$p^i_t(\epsilon^t) = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})(p^i_{t+1}(\epsilon^{t+1}) + d^i_{t+1}(\epsilon^{t+1}))].$$

Conditional on an asset pricing kernel and a probability distribution over histories, any asset with specified payoffs thus can be priced by computing the expectation of the kernel times the payoff.

While macroeconomists relate the asset pricing kernel to consumption, financial economists often take it as given or derive it from market prices and payoffs. Financial economists also use observed market prices and payoffs to price "new" securities such as derivatives. The *law of one price* states that portfolios with identical payoffs have the same price (unless households face portfolio restrictions). A *strong arbitrage* is a portfolio with a strictly negative price that pays off a non-negative amount in every history; it only exists if the law of one price is violated. Equilibrium rules out the existence of a strong arbitrage (except in the presence of portfolio restrictions) and no-arbitrage conditions thus impose constraints on the prices of new securities.

5.3.1 Fundamental Value

With multiple periods, iterating the pricing equation forward yields

$$\begin{aligned}
p^i_0 &= \mathbb{E}_0\left[m_1(\epsilon^1)\left(d^i_1(\epsilon^1) + \mathbb{E}_1\left[m_2(\epsilon^2)\left(d^i_2(\epsilon^2) + \ldots + \mathbb{E}_{T-1}\left[m_T(\epsilon^T)d^i_T(\epsilon^T)\right]\right)\right]\right)\right] \\
&\quad + \mathbb{E}_0\left[m_1(\epsilon^1)\mathbb{E}_1\left[m_2(\epsilon^2)\ldots\mathbb{E}_{T-1}\left[m_T(\epsilon^T)p^i_T(\epsilon^T)\right]\right]\right] \\
&= \mathbb{E}_0\left[\sum_{s=1}^{T}(m_1(\epsilon^1)\cdots m_s(\epsilon^s))d^i_s(\epsilon^s)\right] + \mathbb{E}_0\left[(m_1(\epsilon^1)\cdots m_T(\epsilon^T))p^i_T(\epsilon^T)\right] \\
&= \mathbb{E}_0\left[\sum_{s=1}^{T}\beta^s\frac{u'(c_s(\epsilon^s))}{u'(c_0)}d^i_s(\epsilon^s)\right] + \mathbb{E}_0\left[\beta^T\frac{u'(c_T(\epsilon^T))}{u'(c_0)}p^i_T(\epsilon^T)\right],
\end{aligned}$$

where we use the law of iterated expectations. The asset price has two components: The expected present discounted value of the dividend stream until date $t = T$; and the expected present discounted value of the price at this date. If T is the final period such that $p^i_T(\epsilon^T) = 0$ then the former component is the asset's *fundamental value*.

If the asset has an infinite maturity then its price satisfies

$$p^i_0 = \lim_{T\to\infty} \mathbb{E}_0 \left[\sum_{s=1}^{T} \beta^s \frac{u'(c_s(\epsilon^s))}{u'(c_0)} d^i_s(\epsilon^s) \right] + \lim_{T\to\infty} \mathbb{E}_0 \left[\beta^T \frac{u'(c_T(\epsilon^T))}{u'(c_0)} p^i_T(\epsilon^T) \right].$$

Again, it has two components, a fundamental value and a *bubble* component (the right-most term). Whether p^i_0 exceeds the fundamental value depends on whether the bubble component is strictly positive and thus, whether $p^i_T(\epsilon^T)$ grows more quickly than $\beta^T u'(c_T(\epsilon^T))/u'(c_0)$ shrinks as $T \to \infty$. We turn next to the question whether this is possible.

5.3.2 Bubbles

For simplicity, suppose that the utility function is linear and dividends are constant such that $m_t(\epsilon^t) = \beta$, $d^i_t(\epsilon^t) = d^i$, and

$$p^i_0 = \lim_{T\to\infty} \sum_{s=1}^{T} \beta^s d^i + \lim_{T\to\infty} \beta^T \mathbb{E}_0[p^i_T(\epsilon^T)].$$

One solution to this equation is a constant price equal to the fundamental value, $p^i_t = d^i\beta/(1-\beta)$; the bubble component equals zero in this case. Another candidate solution is $p^i_t = d^i\beta/(1-\beta) + \text{bubble}^i_t$ where $\{\text{bubble}^i_t\}_{t\geq 0}$ is a strictly positive sequence satisfying $\text{bubble}^i_t = \beta\,\text{bubble}^i_{t+1}$; that is, the bubble grows at the rate of interest. This candidate solution satisfies the asset pricing equation because

$$p^i_t = d^i \frac{\beta}{1-\beta} + \text{bubble}^i_t = \beta d^i + \beta d^i \frac{\beta}{1-\beta} + \beta\,\text{bubble}^i_{t+1} = \beta(d^i + p^i_{t+1}).$$

(Still other candidate solutions involve stochastic bubbles.)

To check whether the candidate solution with a bubble component is consistent with rational expectations, suppose first that the number of potential investors is finite. In this case it is impossible that all households purchasing the asset at a bubbly price expect somebody else to purchase it at an even higher bubbly price in the future. A bubbly price therefore is inconsistent with common knowledge in a rational expectations equilibrium.

Suppose next that new potential investors enter the economy as time progresses. A household may then purchase the asset at a bubbly price expecting to resell it to subsequent investors with similar expectations. When the interest rate strictly exceeds the economy's growth rate then such expectation formation cannot be rational; a bubble growing at the rate of interest would eventually outgrow the economy and newcomers would not be able to purchase the bubble any longer. But when the interest rate falls short of the growth rate, then a bubble may be sustained in rational expectations equilibrium.

Recall that the growth rate in an inefficient overlapping generations economy exceeds the interest rate. Such an environment therefore admits bubbles. In fact, a bubble can play exactly the same role as a Pareto improving inter generational transfer scheme (see subsection 3.2.7): When an initial old cohort creates a pure bubble and old households in each period sell the bubble to young ones the latter transfer resources to the former; this absorbs saving of the young and reduces or eliminates capital over accumulation. A pure bubble of this type can be interpreted as money (see section 9.2). While equilibrium imposes restrictions on the growth rate of the bubble it is consistent with infinitely many bubble sizes. That is, the equilibrium allocation with a bubble is *indeterminate*.

5.4 Term Structure of Interest Rates

The price of a risk-free one period bond that pays off unity equals

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1})\, 1 \right],$$

and the risk-free one period gross interest rate, $R_{t+1}^{f1}(\epsilon^t)$, equals the inverse of the bond price (from condition (5.1)). (Note that the risk-free interest rate is indexed by ϵ^t because the return is the same across all histories ϵ^{t+1} subsequent to ϵ^t.) More generally, a risk-free s period bond that pays off unity is priced at

$$p_t^{fs}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \cdots m_{t+s}(\epsilon^{t+s})\, 1 \right]$$

and the risk-free s period gross interest rate, $R_{t+s}^{fs}(\epsilon^t)$, equals the inverse of $p_t^{fs}(\epsilon^t)$.

Because $\{m_{t+1}(\epsilon^{t+1})\}_{\epsilon^{t+1}}$ affects both short- and longer-term interest rates these rates satisfy cross-restrictions. Consider $R_{t+1}^{f1}(\epsilon^t)$ and $R_{t+2}^{f2}(\epsilon^t)$:

$$\begin{aligned}
\left(R_{t+2}^{f2}(\epsilon^t)\right)^{-1} &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) m_{t+2}(\epsilon^{t+2}) \right] = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \mathbb{E}_{t+1} \left[m_{t+2}(\epsilon^{t+2}) \right] \right] \\
&= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right] \\
&= \left(R_{t+1}^{f1}(\epsilon^t) \right)^{-1} \mathbb{E}_t \left[\left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right] + \mathbb{C}\mathrm{ov}_t \left[m_{t+1}(\epsilon^{t+1}), \left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right].
\end{aligned}$$

Accordingly, there are two drivers of (the inverse of) the long-term interest rate, $R_{t+2}^{f2}(\epsilon^t)$. First, current and expected future (inverse) short-term rates. Second, a covariance term unless future short-term interest rates are uncorrelated with consumption. The *expectations hypothesis* abstracts from the second driver; it postulates that the *term premium* equals zero.

To compare the returns on bonds of different maturity it is useful to express them in normalized form, over time intervals of the same length (e.g., on an annual basis). The *term structure* of interest rates,

$$\left\{R^{f1}_{t+1}(\epsilon^t), \sqrt[2]{R^{f2}_{t+2}(\epsilon^t)}, \sqrt[3]{R^{f3}_{t+3}(\epsilon^t)}, \ldots\right\},$$

collects the normalized returns. The *yield curve* graphically represents the term structure. The yield curve is upward sloping when the normalized return on longer-term bonds exceeds the return on shorter-term bonds,

$$\sqrt[s-1]{R^{f,s-1}_{t+s-1}(\epsilon^t)} < \sqrt[s]{R^{fs}_{t+s}(\epsilon^t)},\ s > 1.$$

This can reflect either expected future short-term returns that exceed the current short-term return, or positive term premia of longer-term bonds.

5.5 Asset Prices in an Endowment Economy

Every model with a consumption-savings margin is a model of asset prices for equilibrium consumption implies an asset pricing kernel and this allows to price arbitrary assets. In an economy with homogeneous households the kernel is unique since all households have the same consumption process. In an endowment economy with homogeneous households the pricing is particularly straightforward because the equilibrium consumption process is exogenous, as we now show.

5.5.1 Economy

Consider an economy with a continuum of mass one of infinitely lived homogeneous households who own a fixed capital stock that consists of a continuum of mass one of "trees." Dividends—the "fruit" of the trees—are exogenous, stochastic, and cannot be stored. They are the only source of income for the households. The budget constraint of household h reads

$$c^h_t(\epsilon^t) + p^{tr}_t(\epsilon^t)\left(tr^h_{t+1}(\epsilon^t) - tr^h_t(\epsilon^{t-1})\right) = tr^h_t(\epsilon^{t-1})d^{tr}_t(\epsilon^t),$$

where $c^h_t(\epsilon^t)$ denotes consumption; $p^{tr}_t(\epsilon^t)$ the tree price; $tr^h_{t+1}(\epsilon^t)$ the household's stock of trees between t and $t+1$; and $d^{tr}_t(\epsilon^t)$ the dividend.

5.5.2 General Equilibrium

While an individual household perceives its asset holdings and consumption to be endogenous, market clearing requires that each household owns one tree, $tr^h_{t+1}(\epsilon^t) =$

1, and consumes the dividend in full. The market price of trees supports this choice. Absent bubbles, it satisfies

$$p_t^{tr}(\epsilon^t) = \mathbb{E}_t \left[\sum_{s=1}^{\infty} \beta^s \frac{u'(d_{t+s}^{tr}(\epsilon^{t+s}))}{u'(d_t^{tr}(\epsilon^t))} d_{t+s}^{tr}(\epsilon^{t+s}) \right].$$

Using the equilibrium pricing kernel we may also determine the price of arbitrary other assets, including those that are in zero net supply and not actually traded. For example, the price of a risk-free one period bond that pays off unity is given by

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t \left[\beta \frac{u'(d_{t+1}^{tr}(\epsilon^{t+1}))}{u'(d_t^{tr}(\epsilon^t))} \right],$$

and the price of an *option* that gives the right to sell a tree in the subsequent period at price $\bar{p}$ equals

$$p_t^{option}(\epsilon^t) = \mathbb{E}_t \left[\beta \frac{u'(d_{t+1}^{tr}(\epsilon^{t+1}))}{u'(d_t^{tr}(\epsilon^t))} \max \left[0, \bar{p} - p_{t+1}^{tr}(\epsilon^{t+1}) \right] \right].$$

The capital stock in an economy may be viewed as a fruit yielding tree and we may thus associate the tree price with a broad measure of stock prices. According to that interpretation the excess return on the tree equals the expected *equity premium*.

5.6 Bibliographic Notes

The CAPM is due to Sharpe (1964), Lintner (1965), and Mossin (1966); the C-CAPM is due to Lucas (1978) and Breeden (1979). For introductions to financial economics and asset pricing, see LeRoy and Werner (2014) and Cochrane (2001). Tirole (1982) proves that bubbles cannot arise in a rational expectations equilibrium with finitely many investors. Tirole (1985) analyzes bubbles in overlapping generations economies. The model of asset prices in an endowment economy is due to Lucas (1978). Mehra and Prescott (1985) analyze implications for the equity premium; see also Weil (1989).

Beyond the material covered in the chapter, Modigliani and Miller (1958) show that the value of a firm is independent of its liability structure, a consequence of "value additivity." Incentive problems and other frictions undermine the result, see Tirole (2006). Magill and Quinzii (1996) cover equilibrium in economies with incomplete financial markets and heterogeneous agents.

Chapter 6

Labor Supply, Growth, and Business Cycles

We have assumed so far that households supply labor at no cost and thus, that labor supply effectively is exogenous. To endogenize labor supply, we now introduce an opportunity cost of working: foregone utility from leisure. We first consider the problem of a household that chooses how much to work (and save). Subsequently, we embed the household's choice in extensions of the representative agent model analyzed in section 3.1. The first extension focuses on growth and the second on business cycles: we use *dynamic stochastic general equilibrium (DSGE) models* to study the economy's equilibrium response to stochastic shocks. Throughout, we normalize the household's time endowment to one unit per period.

6.1 Goods Versus Leisure Consumption

6.1.1 One Period

Intensive Margin Consider first a static setting. Letting x denote leisure and thus, $1-x$ time spent working, the program of a household reads

$$\max_{c,x} u(c,x) \text{ s.t. } c = w(1-x),$$

where w denotes the wage. Utility depends on the consumption of goods, c, and leisure. We assume that u is strictly increasing and concave in both arguments. Note that the wage represents the price of leisure relative to goods consumption. This is most evident when we express the budget constraint as

$$c + wx = w,$$

equating expenditures for goods and leisure consumption with wealth, all expressed in terms of the good.

An interior optimal choice of (c,x) satisfies the first-order condition

$$u_x(c,x) = u_c(c,x)w.$$

That is, the household equates the relative price of leisure and goods consumption with the marginal rate of substitution, $u_x(c,x)/u_c(c,x)$. Combined with the budget constraint, the first-order condition yields the solution to the household's program.

A wage change alters both the household's wealth and the price of leisure relative to goods consumption. Accordingly, it induces income and substitution effects. The *compensated (Hicksian) labor supply elasticity* equals the elasticity of $1-x$ with respect to w when utility is held constant. In contrast, the *Frisch labor supply elasticity* equals the elasticity of $1-x$ with respect to w for a fixed marginal utility of wealth. In a dynamic setting, the Frisch elasticity determines the intertemporal elasticity of substitution of labor supply.

For example, assume that

$$u(c,x) = \bar{u}(c) - \gamma\frac{(1-x)^{1+\varphi}}{1+\varphi},\ \varphi > 0,$$

for some strictly increasing and concave function $\bar{u}$. Letting λ denote the multiplier associated with the budget constraint—the marginal utility of wealth—the household's labor supply satisfies

$$\gamma(1-x)^{\varphi} = \lambda w$$

and the Frisch elasticity of labor supply equals $d\ln(1-x)/d\ln(w)|_{\lambda} = \varphi^{-1}$.

Extensive Margin Suppose next that workers may only work either a fraction $h > 0$ of their time or not at all that is, the labor supply choice occurs at the *extensive* rather than the intensive margin. The aggregate labor supply elasticity then can be substantially higher than suggested by the preference parameter φ.

To see this, consider a household with a continuum of identical members. Each member works h "hours" with probability η and does not work with probability $1-\eta$. Aggregate labor supply therefore equals ηh. The household solves

$$\max_{\eta,c^0,c^1}\ \eta\bar{u}(c^1) + (1-\eta)\bar{u}(c^0) - \gamma\frac{\eta h^{1+\varphi} + (1-\eta)0}{1+\varphi}\ \text{ s.t. }\ \eta c^1 + (1-\eta)c^0 = \eta h w,$$

where c^1 and c^0 denote consumption of working and non-working household members, respectively. Risk sharing implies $c^1 = c^0 = c$ (see section 4.2). Letting

$$\bar{\gamma} \equiv \gamma\frac{h^{1+\varphi}}{1+\varphi} > 0,\ \bar{w} \equiv hw,$$

the program can then be expressed as

$$\max_{\eta,c}\ \bar{u}(c) - \bar{\gamma}\eta\ \text{ s.t. }\ c = \eta\bar{w}$$

which corresponds to the intensive margin program of a household with preference parameter $\varphi = 0$ that faces wage $\bar{w}$. We conclude that the household labor supply elasticity at the extensive margin equals infinity.

6.1.2 More Periods

With two (and similarly, with more) periods, the household's choice at the intensive margin solves

$$\max_{c_0,c_1,x_0,x_1} u(c_0,x_0)+\beta u(c_1,x_1) \text{ s.t. } c_0+\frac{c_1}{R_1}+w_0x_0+\frac{w_1x_1}{R_1}=w_0+\frac{w_1}{R_1},$$

where c_t and x_t denotes goods and leisure consumption at date t. The budget constraint equates total spending on the left-hand side with wealth on the right-hand side. The first-order conditions yield

$$\begin{aligned} u_c(c_0,x_0) &= \beta R_1 u_c(c_1,x_1), \\ u_x(c_t,x_t) &= u_c(c_t,x_t)w_t; \end{aligned}$$

the intertemporal first-order condition can also be expressed as

$$u_x(c_0,x_0)=\beta R_1\frac{w_0}{w_1}u_x(c_1,x_1),$$

which equates the relative price of leisure in the first and second period, R_1w_0/w_1, with the corresponding marginal rate of substitution. If u is additively separable leisure consumption thus rises and labor supply falls over time if and only if $\beta R_1 w_0/w_1>1$.

6.2 Growth

6.2.1 Exogenous Growth

We now introduce the labor-leisure trade-off into the general equilibrium model analyzed in section 3.1 and we allow for exogenous, time-varying productivity as an additional determinant of output. While we maintain the assumption of constant returns to scale in capital and labor, we adopt a very general specification of productivity growth: We assume that per-capita output, y_t, depends on calendar time, in addition to the per-capita capital stock, k_t, and per-capita labor input, $1-x_t$,

$$y_t=\tilde{f}(k_t,1-x_t,t).$$

Conditional on its third argument the production function $\tilde{f}$ satisfies the Inada conditions and exhibits decreasing marginal products. We also allow for exogenous gross population growth at rate ν.

In this modified environment, the first welfare theorem continues to apply. The decentralized equilibrium therefore solves the planner problem

$$\begin{aligned} &\max_{\{c_t,x_t,k_{t+1}\}_{t\geq 0}} && \sum_{t=0}^{\infty}\beta^t\nu^t u(c_t,x_t) \qquad (6.1)\\ &\text{s.t.} && \nu k_{t+1}=k_t(1-\delta)+\tilde{f}(k_t,1-x_t,t)-c_t,\ \ k_0 \text{ given},\ k_{t+1}\geq 0. \end{aligned}$$

Before deriving the equilibrium conditions, we analyze the conditions for a *balanced growth path* along which all variables grow at constant (but possibly different) rates and are strictly positive.

Requirements for a Balanced Growth Path

Suppose that the economy starts to grow along a balanced growth path at date $t = T$ and let γ_z denote the gross growth rate of a generic variable z along the balanced growth path. Note that we must have $\gamma_x = 1$ since the time endowment is bounded. Dividing the resource constraint in program (6.1) at date $t > T$ by γ_k^{t-T} and rearranging yields

$$k_T(\nu\gamma_k - 1 + \delta) = y_T(\gamma_y/\gamma_k)^{t-T} - c_T(\gamma_c/\gamma_k)^{t-T}.$$

Since the left-hand side of this equality is independent of t the right-hand side must be time independent as well. Moreover, since $y_T, k_T, c_T > 0$, this implies $\gamma_y = \gamma_k = \gamma_c$.

Denoting the common gross growth rate of per-capita output, capital and consumption by γ we thus have established

$$k_T(\nu\gamma - 1 + \delta) = y_T - c_T \text{ and } y_T\gamma^{t-T} = \tilde{f}(k_T\gamma^{t-T}, 1 - x_T, t).$$

The latter equality and constant returns to scale imply

$$\tilde{f}(k_T, 1 - x_T, T) = y_T = \frac{1}{\gamma^{t-T}}\tilde{f}(k_T\gamma^{t-T}, 1 - x_T, t) = \tilde{f}\left(k_T, \frac{1 - x_T}{\gamma^{t-T}}, t\right).$$

Comparing the left- and right-most expressions, we conclude that the only form of technological progress consistent with a balanced growth path is *labor augmenting* technological progress at the gross growth rate γ.

Up to some normalization of the initial level of productivity, output per capita thus depends on the per-capita capital stock and per-capita labor supply in *efficiency units*, $(1 - x_t)\gamma^t$:

$$y_t = f(k_t, (1 - x_t)\gamma^t),$$

where f denotes the neoclassical production function considered in section 3.1. In the special case of a Cobb-Douglas production function, labor augmenting (or "Harrod-neutral") technological progress is isomorphic to progress that is capital augmenting ("Solow-neutral") or multiplying $f(k, 1 - x)$ ("Hicks-neutral").

Balanced growth path dynamics also impose restrictions on preferences. To see this, consider the Euler equation and intratemporal first-order condition implied by program (6.1). These conditions read

$$\begin{aligned}
\frac{u_c(c_t, x_t)}{\beta u_c(c_{t+1}, x_{t+1})} &= 1 - \delta + f_K(k_{t+1}, (1 - x_{t+1})\gamma^{t+1}), \\
\frac{u_x(c_t, x_t)}{u_c(c_t, x_t)\gamma^t} &= f_L(k_t, (1 - x_t)\gamma^t).
\end{aligned}$$

Due to constant returns to scale, the terms on the right-hand side—which in equilibrium correspond to the gross interest rate, R_{t+1}, and the wage per efficiency unit, w_t/γ^t—are constant along a balanced growth path. Consistency therefore requires that the elasticity of u_c with respect to consumption is constant (so that the ratio of marginal utilities of consumption does not change along a balanced growth path) and that u_x/u_c grows at the same rate as consumption. These two requirements imply

$$u(c,x) = \begin{cases} \frac{c^{1-\sigma}v(x)}{1-\sigma}, & \sigma > 0, \sigma \neq 1 \\ \ln(c) + v(x), & \sigma = 1 \end{cases},$$

where the function v needs to satisfy additional conditions to guarantee that u is increasing and concave.

To gain intuition for the preference restrictions, note that along a balanced growth path consumption and the wage grow at the same rate such that the intratemporal first-order condition takes the form

$$u_c(w_T\gamma^{t-T}\xi, x_T)w_T\gamma^{t-T} = u_x(w_T\gamma^{t-T}\xi, x_T)$$

for all $t > T$ where $\xi > 0$ denotes some constant. To satisfy this condition, preferences must give rise to income and substitution effects on leisure that offset each other in a static environment (see subsection 6.1.1). Along the balanced growth path, the model then generates a constant capital-output ratio, constant wage growth and interest rates, and constant factor shares in national income.

A final condition that we need to impose on the primitives concerns productivity growth: it must not be too high because otherwise, the objective is unbounded. To see this, note that

$$\sum_{t=T}^{\infty}\beta^{t-T}\nu^{t-T}\frac{(c_T\gamma^{t-T})^{1-\sigma}v(x_T)}{1-\sigma} = \sum_{t=T}^{\infty}(\beta\nu\gamma^{1-\sigma})^{t-T}\chi_1,$$

where χ_1 denotes a constant. Boundedness requires $\beta\nu\gamma^{1-\sigma} < 1$.

Representation in Detrended Form

It is useful to express program (6.1) in terms of stationary that is, detrended variables. To this end, we divide all variables except leisure by the cumulative balanced growth path growth rate, γ^t. Letting a "bar" denote detrended variables, this yields

$$\begin{aligned} \max_{\{\bar{c}_t, x_t, \bar{k}_{t+1}\}_{t\geq 0}} \quad & \sum_{t=0}^{\infty}\beta^t\nu^t\gamma^{t(1-\sigma)}u(\bar{c}_t, x_t) \qquad (6.2) \\ \text{s.t.} \quad & \nu\gamma\bar{k}_{t+1} = \bar{k}_t(1-\delta) + f(\bar{k}_t, 1-x_t) - \bar{c}_t, \ \ \bar{k}_0 \text{ given}, \ \bar{k}_{t+1} \geq 0, \end{aligned}$$

where u satisfies the restrictions discussed above. Defining $\beta^\star \equiv \beta\nu\gamma^{1-\sigma}$, the first-order conditions simplify to

$$u_x(\bar{c}_t, x_t) = f_L(\bar{k}_t, 1-x_t)u_c(\bar{c}_t, x_t), \qquad (6.3)$$

$$\nu\gamma u_c(\bar{c}_t, x_t) = \beta^\star(1-\delta+f_K(\bar{k}_{t+1}, 1-x_{t+1}))u_c(\bar{c}_{t+1}, x_{t+1}). \qquad (6.4)$$

Condition (6.3) is equivalent to the intratemporal first-order condition in the decentralized equilibrium,

$$u_x(c_t, x_t) = f_L(k_t, (1-x_t)\gamma^t)\gamma^t u_c(c_t, x_t) = w_t u_c(c_t, x_t),$$

and condition (6.4) is equivalent to the standard Euler equation

$$u_c(c_t, x_t) = \beta(1 - \delta + f_K(\bar{k}_{t+1}, 1 - x_{t+1}))u_c(c_{t+1}, x_{t+1}) = \beta R_{t+1} u_c(c_{t+1}, x_{t+1}).$$

Since along the balanced growth path, per-capita consumption grows at the gross rate γ and thus, marginal utility at rate $\gamma^{-\sigma}$, the Euler equation implies that the gross *interest rate* satisfies

$$R = \gamma^{\sigma}/\beta.$$

When $\gamma \neq 1$, the interest rate does not only reflect the psychological discount factor (as in the model without growth) but also the curvature of preferences and the growth rate. Intuitively, in a growing (or shrinking) economy, per-capita consumption grows (or shrinks) as well. Since the curvature of preferences determines the intertemporal elasticity of substitution it also affects by how much the interest rate changes in order to induce the equilibrium growth rate of consumption. A lower willingness to intertemporally substitute (higher σ) implies that the gross interest rate differs by more from β^{-1}.

6.2.2 Endogenous Growth

Long-run per-capita growth in the model of the previous subsection reflects exogenous productivity growth. The model explains why per-capita output, investment and consumption grow at the same rate as productivity, but it does not explain why productivity and thus, the economy grows.

At the root of the model's inability to endogenously generate sustained per-capita growth lies an Inada condition: As the capital intensity rises, the marginal product of capital declines and the incentive to further accumulate falls. We now relax this condition and show that models with a suitably modified production function endogenously generate sustained per-capita growth. Throughout the subsection, we abstract from productivity and population growth, $\gamma = \nu = 1$, as well as from leisure, $x_t = 0$.

Ak Technology

Consider first an extreme production function,

$$f(K, L) = AK, \ A > 0.$$

Function f exhibits constant returns to scale but does not feature decreasing marginal products; in fact, the marginal product of capital is constant. With this *Ak technology* the resource constraint reads

$$k_{t+1} = k_t(1 - \delta) + Ak_t - c_t,$$

wages equal zero, and the interest rate is constant at value $R_t = 1 - \delta + A$. The Euler equation therefore implies that marginal utility grows at a constant rate. With CIES preferences,

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1 - \delta + A),$$

and from the resource constraint (and the transversality condition), the capital stock grows at this rate as well. Dividing the resource constraint by k_t and substituting the expression for the growth rate yields the equilibrium initial consumption level, c_0, for a given initial capital stock, k_0:

$$[\beta(1-\delta+A)]^{\frac{1}{\sigma}} = 1-\delta+A-\frac{c_0}{k_0}.$$

The initial consumption-capital ratio, c_0/k_0, thus is maintained forever—the economy immediately reaches the balanced growth path.

The economy exhibits sustained positive per-capita growth if $\beta R > 1$ but household utility only is well defined if $\sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma} < \infty$, that is if $\beta(c_{t+1}/c_t)^{1-\sigma} < 1$. Both conditions are satisfied if

$$\beta(1-\delta+A) > 1 > \beta(1-\delta+A)^{1-\sigma}.$$

Note that both technology (A and δ) and preferences (β and σ) affect the equilibrium growth rate, unlike with a neoclassical production function. If policy affected the interest rate paid to investors (for example by taxing or subsidizing capital income) it would also affect the growth rate.

Two Sectors

One unappealing feature of the Ak model is that labor income equals zero. A two-sector version of the model where the first sector operates a neoclassical technology to produce the consumption good while the second produces investment goods with an Ak technology, remedies this problem. The model makes clear that endogenous growth does not require a linear technology in all sectors; it suffices for the marginal product in the sector where the accumulated factor of production (capital) is produced, to be bounded from below.

Let k_t^c denote the capital stock employed in the production of consumption goods; the remaining capital stock, $k_t - k_t^c$, is used to produce investment goods. We assume a Cobb-Douglas production function with capital share $\alpha \in (0,1)$ in the consumption goods sector and normalize labor to one. Market clearing thus requires $c_t = (k_t^c)^\alpha$. The first welfare theorem applies so that we can solve the social planner problem to characterize equilibrium:

$$\begin{aligned} \max_{\{c_t, k_t^c, k_{t+1}\}_{t\geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t u((k_t^c)^\alpha) \\ \text{s.t.} \quad & k_{t+1} = k_t(1-\delta) + A(k_t - k_t^c),\ \ k_0 \text{ given},\ k_{t+1} \geq 0,\ k_t \geq k_t^c \geq 0, \end{aligned}$$

and the first-order conditions with respect to k_t^c and k_{t+1} yield

$$u'(c_t)\alpha(k_t^c)^{\alpha-1} = u'(c_{t+1})\alpha(k_{t+1}^c)^{\alpha-1}\beta(1-\delta+A).$$

Along a balanced growth path, k_t^c and k_t grow at the same rate. Using this fact as well as the production function in the consumption goods sector and imposing CIES preferences, we find the following relation between the balanced growth path gross growth rates of consumption and capital, γ_c and γ_k respectively:

$$\gamma_c = \gamma_k^{\alpha} = [\beta(1-\delta+A)]^{\frac{\alpha}{1-\alpha+\alpha\sigma}}.$$

Consumption grows more slowly than the capital stock because the marginal product of capital in the consumption goods sector decreases. In the decentralized equilibrium the growth differential is reflected in a trend increase of the price of consumption relative to investment goods; this price differential renders investors indifferent between the two sectors although the marginal product of capital in the consumption goods sector faces a secular decline.

Externalities

Endogenous growth does not require the marginal product to be bounded from below at the level of an individual firm; it suffices to have an aggregate production function with this property, and technological spillovers may cause the marginal products of capital at the aggregate and the firm level to differ.

To see this, consider a one-sector model where the representative firm operates the technology

$$y_t = A_t f(k_t, 1) = A_t k_t^{\alpha}.$$

Labor is normalized to unity, the capital share $\alpha \in (0,1)$, and A_t denotes productivity which households and firms take as given.

Suppose that productivity depends positively on the average capital stock in the economy, which in equilibrium equals the capital stock owned by the representative household. Specifically, we assume that

$$A_t = A k_t^{1-\alpha}.$$

In equilibrium, per-capita output then is a linear function of the capital-labor ratio, $y_t = Ak_t$, while at the level of an individual firm, the production function is neoclassical.

Since investment by an individual firm increases the productivity of all firms—it generates a positive externality—the conditions of the first welfare theorem are violated. To characterize the competitive equilibrium, we therefore need to derive the household and firm optimality conditions rather than those of a social planner. The program of the representative household who takes productivity as given reads

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t\geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & k_{t+1} = k_t(1-\delta) + A_t k_t^{\alpha} - c_t, \ \ k_0 \text{ given}, \ k_{t+1} \geq 0. \end{aligned}$$

Assuming CIES preferences, the first-order conditions of this program combined with the relation between productivity and the average capital stock reduce to

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1-\delta+\alpha A).$$

Wages and interest rates are determined by firms' marginal products, taking productivity as given.

Under parameter conditions similar to those discussed earlier, equilibrium growth is strictly positive and the objective function is bounded. In contrast to the growth models considered so far, however, equilibrium growth is inefficiently low. Unlike individual households, a social planner would take into account that $A_t = Ak_t^{1-\alpha}$ when solving the program. As a consequence, the social planner would implement an allocation with a higher growth rate than in competitive equilibrium.

6.3 Business Cycles

6.3.1 Real Business Cycles

To study "real," that is productivity-driven business cycles, we modify the neoclassical growth model analyzed in subsection 6.2.1 in one respect: We assume that productivity does not only grow deterministically, at gross rate γ, but may also fluctuate stochastically. Specifically, we assume that per-capita capital stock, k_t, and per-capita labor supply, $1 - x_t$, generate per-capita output

$$y_t = f(k_t, (1 - x_t)\gamma^t) \cdot A_t,$$

where A_t—the new model element—stochastically fluctuates around a mean value of one. We assume that in period t and history ϵ^t, the relative deviation of $A_t(\epsilon^t)$ from its mean, $\hat{A}_t(\epsilon^t) \equiv A_t(\epsilon^t) - 1$, is governed by the first-order stochastic difference equation

$$\hat{A}_t(\epsilon^t) = \rho_A \hat{A}_{t-1}(\epsilon^{t-1}) + \iota_t(\epsilon^t), \;\; 0 \leq \rho_A < 1,$$

where $\iota_t(\epsilon^t)$ is i.i.d. with mean zero.

The fundamental theorems of welfare economics apply. Accordingly, we can characterize the equilibrium in the real business cycle (RBC) model by solving the planner's problem, corresponding to program (6.2) augmented by the stochastic productivity term. To simplify the notation, we adopt a recursive formulation and let $(\bar{k}_\circ, A_\circ)$ denote the state, $(\bar{c}_\circ, x_\circ, \bar{k}_+)$ the control, and ι_+ the productivity innovation in the subsequent period. The Bellman equation reads

$$\begin{aligned} & V(\bar{k}_\circ, A_\circ) = \max_{\bar{c}_\circ, x_\circ, \bar{k}_+} \left\{ u(\bar{c}_\circ, x_\circ) + \beta^\star \mathbb{E}\left[V(\bar{k}_+, A_+) | k_\circ, A_\circ \right] \right\} \qquad (6.5) \\ \text{s.t.} \quad & \nu\gamma\bar{k}_+ = \bar{k}_\circ(1-\delta) + f(\bar{k}_\circ, 1 - x_\circ)A_\circ - \bar{c}_\circ, \\ & A_+ = 1 + \rho_A(A_\circ - 1) + \iota_+, \end{aligned}$$

where $\beta^\star \equiv \beta\nu\gamma^{1-\sigma}$ and the controls are bounded. The first-order conditions and envelope condition reduce to

$$\begin{aligned} u_x(\bar{c}_\circ, x_\circ) &= f_L(\bar{k}_\circ, 1 - x_\circ)A_\circ u_c(\bar{c}_\circ, x_\circ), && (6.6) \\ \nu\gamma u_c(\bar{c}_\circ, x_\circ) &= \beta^\star \mathbb{E}[\{1 - \delta + f_K(\bar{k}_+, 1 - x_+)A_+\} u_c(\bar{c}_+, x_+) | k_\circ, A_\circ], && (6.7) \end{aligned}$$

where $(\bar{c}_+, x_+)$ denote optimal control choices in the subsequent period. These equilibrium conditions differ from (6.3) and (6.4) only insofar as marginal products are augmented by the corresponding A terms and future outcomes are weighted by their respective conditional probabilities.

In the special case with a Cobb-Douglas production function, full depreciation, and a utility function that is logarithmic in consumption and additively separable, analytical solutions for $(\bar{c}_\circ, x_\circ, \bar{k}_+)$ are available. To derive the equilibrium allocation under more general assumptions, we may solve the Bellman equation numerically or resort to an approximate solution based on the linearized equilibrium conditions.

For the latter strategy, consider first the deterministic balanced growth path of the economy which results when A_t always equals one. The steady-state values of the detrended variables, $(\bar{k}, \bar{c}, x)$, satisfy

$$\begin{aligned}
\nu\gamma\bar{k} &= \bar{k}(1-\delta) + f(\bar{k}, 1-x) - \bar{c}, \\
u_x(\bar{c}, x) &= f_L(\bar{k}, 1-x) u_c(\bar{c}, x), \\
\nu\gamma u_c(\bar{c}, x) &= \beta^\star (1 - \delta + f_K(\bar{k}, 1-x)) u_c(\bar{c}, x).
\end{aligned}$$

Linearizing the resource constraint in program (6.5) as well as the optimality conditions (6.6) and (6.7) about the steady state values $(\bar{k}, \bar{c}, x)$ yields a system of linear difference equations. In forming this system, we exploit the *certainty equivalence* property: If a non-linear function h is linearized about the value z then $\mathbb{E}_\circ[h(z_+)] \approx \mathbb{E}_\circ[h(z) + h'(z) \cdot (z_+ - z)] = h(z) + h'(z) \cdot (\mathbb{E}_\circ[z_+] - z)$; that is, in the linearized system, only the conditional mean of random variables is relevant. Using this property and substituting the linearized intratemporal first-order condition into the other two linearized equilibrium conditions, we arrive at (switching to sequence notation)

$$\begin{aligned}
\begin{bmatrix} \hat{k}_{t+1}(\epsilon^t) \\ \mathbb{E}_t[\hat{c}_{t+1}(\epsilon^{t+1})] \end{bmatrix} &= M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix} + N_1 \mathbb{E}_t[\hat{A}_{t+1}(\epsilon^{t+1})] + N_0 \hat{A}_t(\epsilon^t) \\
&= M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix} + N \hat{A}_t(\epsilon^t).
\end{aligned}$$

Here, a circumflex denotes relative deviations from the steady-state value (e.g., $\hat{c}_t \equiv (\bar{c}_t - \bar{c})/\bar{c}$), the elements of the (2×2) matrices M, N_0, and N_1 contain parameters and functions evaluated at the steady-state values, and $N \equiv \rho_A N_1 + N_0$.

This system with one predetermined (capital) and one non-predetermined endogenous variable (consumption) differs twofold from the system analyzed previously, in the context of the deterministic representative agent model (see subsection 3.1.7). First, matrices M and N do not only reflect the linearized resource constraint and Euler equation in general equilibrium but they also incorporate the intratemporal first-order condition that relates leisure to consumption, the capital stock and productivity. Second, the presence of temporary productivity variation introduces an exogenous shock process. As noted before, stochasticity of the productivity shock does not introduce any additional complication because of certainty equivalence.

We solve the system using essentially the same approach as in the deterministic environment (see subsection 3.1.7). When the matrix M has one stable and one unstable eigenvalue, the equilibrium value $\hat{c}_t(\epsilon^t)$ is uniquely determined by the requirement that conditional on the state, the difference equation system generates paths for expected consumption and capital that converge to their steady-state values, $(\bar{c}, \bar{k})$. Moreover, given $\hat{k}_t(\epsilon^{t-1})$, $\hat{A}_t(\epsilon^t)$, and $\hat{c}_t(\epsilon^t)$, the intratemporal first-order condition uniquely determines $x_t(\epsilon^t)$. Appendix B.5 contains a detailed discussion.

Figures 6.1–6.3 illustrate the response of the model economy to a productivity shock at date $t = 3$. We assume a Cobb-Douglas production function with capital share 0.3 and a depreciation rate of 5 percent, a discount factor $\beta = 0.98$, and CIES preferences of the form

$$u(c, x) = \begin{cases} \ln(c) + \frac{x^{1-\varphi}}{1-\varphi}, \ \varphi > 0, \varphi \neq 1 \\ \ln(c) + \ln(x), \ \varphi = 1 \end{cases} .$$

We compare three scenarios, distinguished by the autocorrelation coefficient, ρ_A, and the willingness of households to intertemporally substitute leisure, $1/\varphi$.

Figure 6.1 illustrates the response to a temporary productivity shock, $\rho_A = 0$, when the elasticity of substitution equals unity, $\varphi = 1$. Productivity improves at date $t = 3$, $\iota_3 > 0$, and reverts to its steady-state value, A, at date $t = 4$.

In equilibrium, all endogenous variables except for the predetermined capital stock contemporaneously respond to the productivity shock. Wages and consumption rise relative to their steady-state values and eventually revert back to the latter. Leisure falls on impact before rising in the subsequent period and similarly embarking on a path back to its steady-state value. The interest rate responds inversely, and the capital stock increases at date $t = 4$ before converging back to steady state.

To interpret these developments, consider a household in the decentralized equilibrium. Following the productivity shock, the household faces an increased wage and anticipates higher wages and lower interest rates in the future. The positive wealth effect induces higher consumption, while the lower interest rates induce a substitution effect towards present consumption (see the Euler equation (6.7)). The high wage at date $t = 3$ also induces a strong substitution effect from leisure to goods consumption (see condition (6.6)), stimulating labor supply. In subsequent periods, the wealth effect on leisure consumption dominates the substitution effect, leading to an increase of leisure consumption relative to steady state.

The higher output due to increased productivity and stronger labor supply at date $t = 3$ is not fully consumed. Part of it is saved and invested, generating a higher capital stock in subsequent periods. It is this higher capital stock from date $t = 4$ onward that keeps wages persistently elevated and interest rates subdued. From date $t = 4$ onward, negative net investment generates resources for consumption. Capital accumulation at date $t = 3$ and dissipation starting at date $t = 4$ thus allow to smooth aggregate consumption.

By definition, the equilibrium dynamics satisfy the resource constraint and the optimality conditions (6.6) and (6.7) at all times. Note, however, that the Euler equation prescribes equality of the marginal rate of substitution and the marginal rate of transformation only in expected terms. At the time of the shock, $t = 3$, expected (average) and realized values differ. Accordingly, the realized interest and consumption growth

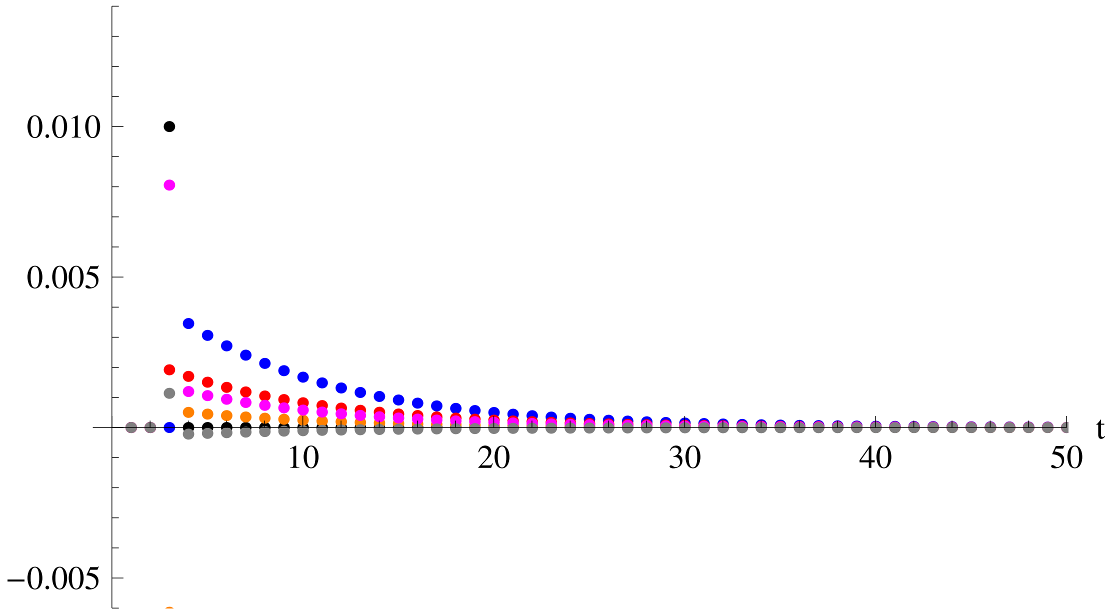

Figure 6.1: Effects of a productivity shock when $\rho_A = 0$ and $\varphi = 1$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t$, and $\hat{R}_t$ are indicated in black, blue, red, orange, magenta and gray, respectively.

rates do not satisfy the deterministic version of the Euler equation. Equilibrium goods and leisure consumption in the shock period are determined in a forward looking way, by the requirement that their choice places the economy on the saddle path subject to the expected productivity shock sequence.

Figure 6.2 illustrates the response to the same shock at date $t = 3$, $\iota_3 > 0$, but under the assumption that such a shock is highly persistent, $\rho_A = 0.99$. We keep the elasticity of substitution φ unchanged.

The persistent technological improvement gives rise to a similarly persistent rise in wages. Capital is accumulated during more than just one period because the marginal product of capital is elevated until date $t = 17$ (reflected in the interest rate), inducing households to delay consumption in spite of the strong wealth effect. Because of the long-lasting rise of wages, leisure consumption is persistently depressed and accordingly, labor supply stimulated.

Finally, figure 6.3 illustrates the response to the same shock under the assumption that it is persistent, $\rho_A = 0.9$, and the elasticity of substitution twice as high as before, $\varphi = 0.5$. The lower persistence gives rise to a faster convergence of the endogenous variables, and the higher elasticity implies a more pronounced labor supply response to the shock.

Recursive Competitive Equilibrium We have characterized equilibrium dynamics by solving the system of linear difference equations that approximates the non-linear equilibrium conditions. Alternatively, we may use numerical methods to find approximations of the value and policy functions that represent equilibrium. When the decen-

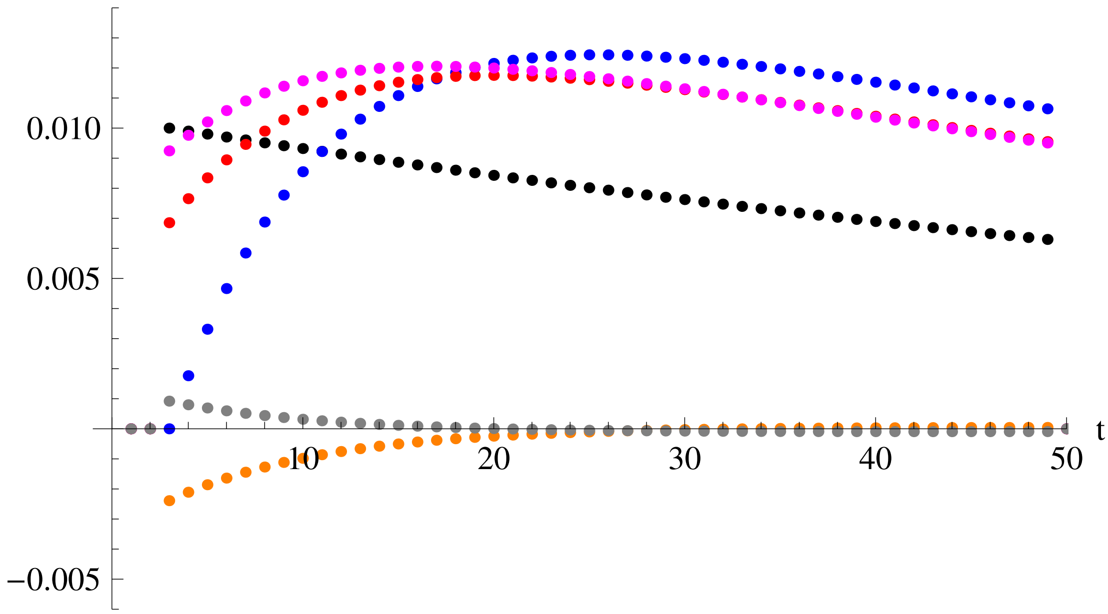

Figure 6.2: Effects of a productivity shock when $\rho_A = 0.99$ and $\varphi = 1$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t$, and $\hat{R}_t$ are indicated in black, blue, red, orange, magenta and gray, respectively.

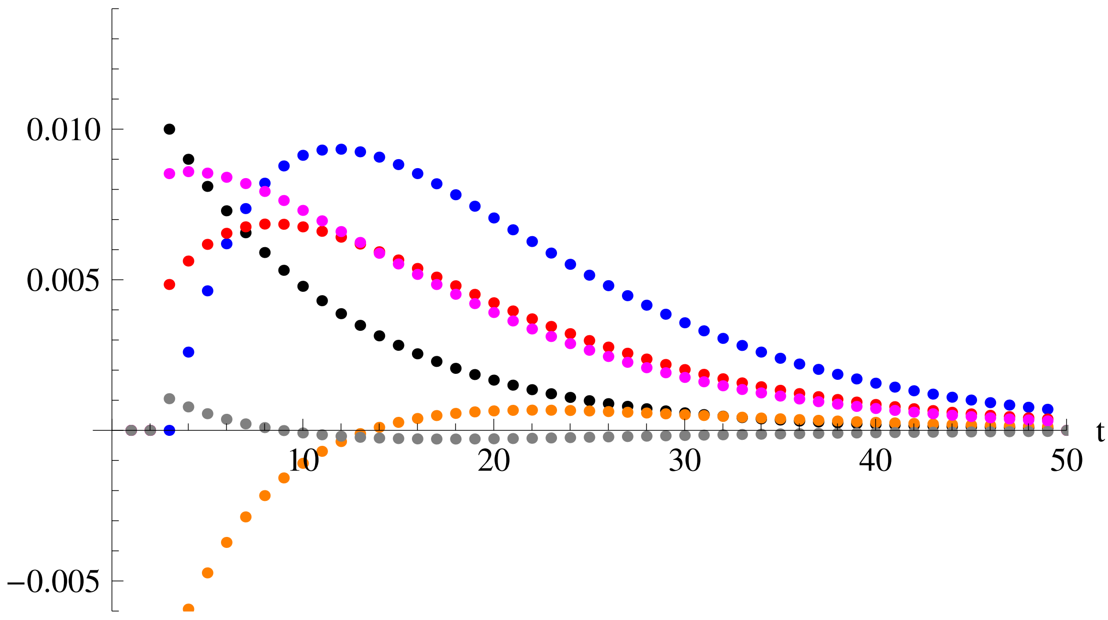

Figure 6.3: Effects of a productivity shock when $\rho_A = 0.9$ and $\varphi = 0.5$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t$, and $\hat{R}_t$ are indicated in black, blue, red, orange, magenta and gray, respectively.

tralized equilibrium allocation corresponds to the allocation chosen by a social planner then we may simply iterate the planner's value and policy functions until they converge sufficiently. In the baseline RBC model where this condition is satisfied, the state variables entering as arguments of the planner's value and policy functions are the economy's capital stock and the level of productivity.

We may also represent the decentralized equilibrium recursively, as a *recursive competitive equilibrium*, and approximate the equilibrium numerically based on this representation. This solution strategy is available independently of whether the decentralized equilibrium allocation solves a social planner problem or not. However, it requires that we carefully distinguish between the state variables of individual agents (e.g., the capital stock of an individual household) and aggregate state variables that individuals take as given (e.g., the economy-wide capital stock which determines equilibrium wages and interest rates). Without this distinction the program would characterize the equilibrium in an economy where individuals choose aggregate variables.

In parallel to the recursively formulated dynamic program of an individual decision maker the objects in a recursive competitive equilibrium are functions of the state, rather than sequences. They include value functions as well as policy functions of the optimizing agents; price functions; and laws of motion for the aggregate state variables which describe how decision makers perceive these variables to evolve over time.

In the case of the RBC model, the aggregate state variables are productivity, $A_\circ$, and the economy-wide capital stock, $\bar{K}_\circ$; the state variables of the representative household additionally include the household's capital stock, $\bar{k}_\circ$. In equilibrium, $\bar{k}_\circ = \bar{K}_\circ$. The recursive competitive equilibrium is given by a value function and policy functions, $V, \bar{k}', c, x$, respectively, which are functions of the household's state; as well as price functions and a law of motion for capital, $w, R, \bar{K}'$, respectively, which are functions of the aggregate state, such that the following conditions are satisfied: First, the value function satisfies the household's Bellman equation subject to the budget constraint, law of motion for productivity, law of motion for aggregate capital, and price taking,

$$\begin{aligned}
V(\bar{k}_\circ, A_\circ, \bar{K}_\circ) &= \max_{c_\circ, x_\circ, \bar{k}_+} \left\{ u(c_\circ, x_\circ) + \beta^\star \mathbb{E}\left[V(\bar{k}_+, A_+, \bar{K}_+) | \bar{k}_\circ, A_\circ, \bar{K}_\circ \right] \right\} \\
c_\circ &= R(A_\circ, \bar{K}_\circ)\bar{k}_\circ + w(A_\circ, \bar{K}_\circ)(1 - x_\circ) - \nu\gamma\bar{k}_+, \\
A_+ &= 1 + \rho_A(A_\circ - 1) + \iota_+, \\
\bar{K}_+ &= \bar{K}'(A_\circ, \bar{K}_\circ);
\end{aligned}$$

and the policy functions are associated with the optimal household choices. Second, the price functions reflect constant returns to scale and competitive firms, and both the price functions and the law of motion for capital are consistent with the policy rules and market clearing,

$$\begin{aligned}
w(A_\circ, \bar{K}_\circ) &= f_L(\bar{K}_\circ, 1 - x(\bar{K}_\circ, A_\circ, \bar{K}_\circ))A_\circ, \\
R(A_\circ, \bar{K}_\circ) &= 1 - \delta + f_K(\bar{K}_\circ, 1 - x(\bar{K}_\circ, A_\circ, \bar{K}_\circ))A_\circ, \\
\bar{K}'(A_\circ, \bar{K}_\circ) &= \bar{k}'(\bar{K}_\circ, A_\circ, \bar{K}_\circ).
\end{aligned}$$

More generally, in an environment with heterogeneous agents, the equilibrium is given by several value functions and associated policy functions, and the consistency and market clearing requirements account for the heterogeneous groups.

To find an approximate solution of the above system we discretize the aggregate state space as well as the state space of the representative household on grids and represent the functions by vectors whose sizes correspond to the sizes of the respective grids. (For notational simplicity, in what follows we do not distinguish between the original functions and the vectors.) Next, we guess $w, R, \bar{K}'$, and given that guess, we solve the household's problem by standard dynamic programming techniques. We check whether the consistency requirements are satisfied at each grid point. If they are not satisfied we update the guess and solve the household's problem again. This procedure is repeated until the consistency requirements are (approximately) satisfied.

6.3.2 Sunspot-Driven Business Cycles

In the RBC model, fluctuations are driven by exogenous shocks to productivity. A level of productivity and its expected future path is associated with a unique equilibrium sequence for each endogenous variable. The uniqueness reflects the saddle-path property of the dynamic system: Conditional on the predetermined capital stock, only a specific value for consumption and leisure is consistent with the equilibrium conditions including the requirement that system dynamics be stable. In turn, the saddle-path property reflects the fact that the number of unstable eigenvalues in the dynamic system equals the number of non-predetermined variables.

Consider now a different dynamic system where the number of stable eigenvalues is strictly larger than the number of predetermined variables. Conditional on the predetermined capital stock, consumption and leisure then are *indeterminate*—multiple initial values for consumption and leisure are consistent with the equilibrium conditions, see appendix B.5. As a consequence, the endogenous variables may not only respond to fundamental shocks, for instance productivity shocks, but also to non-fundamental "sunspot" shocks. The latter do not affect technology, preferences, or other fundamentals in the economy; their only role is to coordinate expectations when they are not pinned down by economic fundamentals and the equilibrium conditions. We now analyze a model where this is the case.

To render the analysis as transparent as possible we assume that there are no fundamental shocks at all. If the dynamic system exhibited the saddle-path property, all endogenous variables therefore would follow deterministic paths. However, due to a modification of the production function the system does not exhibit the saddle-path property (the number of stable eigenvalues exceeds the number of predetermined variables by one). Conditional on the capital stock the requirement that system dynamics satisfy all equilibrium conditions including stability then leaves one degree of freedom—the initial level of consumption or leisure is "free" and may reflect a non-fundamental sunspot shock.

Specifically, we assume that the production function exhibits *increasing returns to scale*, similar to the growth model with externalities (see subsection 6.2.2); output per capita is given by

$$y_t = f(k_t, (1 - x_t)\gamma^t) \cdot A_t,$$

where A_t does not exogenously fluctuate as in the RBC model, but instead is determined by aggregate production which each individual firm and household takes as given:

$$A_t = f(k_t, (1 - x_t)\gamma^t)^\chi, \; \chi \geq 0.$$

For $\chi = 0$, the model reduces to the RBC model with constant productivity.

Due to increasing returns to scale the economy does not satisfy the conditions of the welfare theorems. To characterize the decentralized equilibrium, we therefore use the first-order conditions of households and firms (which take productivity as given) in the RBC model (equations (6.5)–(6.7)) and replace A_t in those conditions by the expression given above. Along the balanced growth path with gross population growth rate ν, per-capita output grows at gross rate μ say. Adopting recursive notation, the equilibrium conditions in terms of detrended variables are

$$\begin{aligned}
\nu\mu\bar{k}_+ &= \bar{k}_\circ(1-\delta) + f(\bar{k}_\circ, 1 - x_\circ)^{1+\chi} - \bar{c}_\circ, \\
u_x(\bar{c}_\circ, x_\circ) &= f_L(\bar{k}_\circ, 1 - x_\circ) f(\bar{k}_\circ, 1 - x_\circ)^\chi u_c(\bar{c}_\circ, x_\circ), \\
\nu\mu u_c(\bar{c}_\circ, x_\circ) &= \beta^\star \mathbb{E}[\{1 - \delta + f_K(\bar{k}_+, 1 - x_+) f(\bar{k}_+, 1 - x_+)^\chi\} u_c(\bar{c}_+, x_+)],
\end{aligned}$$

where $\beta^\star \equiv \beta\nu\mu^{1-\sigma}$. The unique steady-state values of the detrended variables, $(\bar{k}, \bar{c}, x)$, satisfy the same conditions once the subscripts ($+$ and $\circ$) as well as the expectations operator are dropped. A Cobb-Douglas production function with capital share α then implies

$$\mu \equiv \gamma^{\frac{(1-\alpha)(1+\chi)}{1-\alpha(1+\chi)}} \nu^{\frac{\chi}{1-\alpha(1+\chi)}}.$$

Note that μ reduces to γ if $\chi = 0$.

Linearizing the system of equilibrium conditions about the steady state and reducing it to a system of two difference equations in capital and consumption yields the system (switching to sequence notation)

$$\begin{bmatrix} \hat{k}_{t+1}(\epsilon^t) \\ \mathbb{E}_t[\hat{c}_{t+1}(\epsilon^{t+1})] \end{bmatrix} = M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix}.$$

A circumflex denotes relative deviations from the steady-state value and the elements of the (2×2) matrix M contain parameters and functions evaluated at steady-state values.

If φ is sufficiently small (marginal utility of leisure is inelastic) and χ sufficiently large (output has large positive externalities), the matrix M has two stable eigenvalues. Intuitively, a small φ renders the disutility from working inelastic and a large χ implies that the marginal product of labor *increases* with labor input. Conditional on the capital stock, different combinations of consumption and leisure thus satisfy the intratemporal first-order condition.

Figure 6.4 illustrates the economy's response to a sunspot shock at date $t = 3$ which coordinates expectations to anticipate a contemporaneous rise in productivity. No additional sunspot shocks occur later in time that is, all variables assume their expected

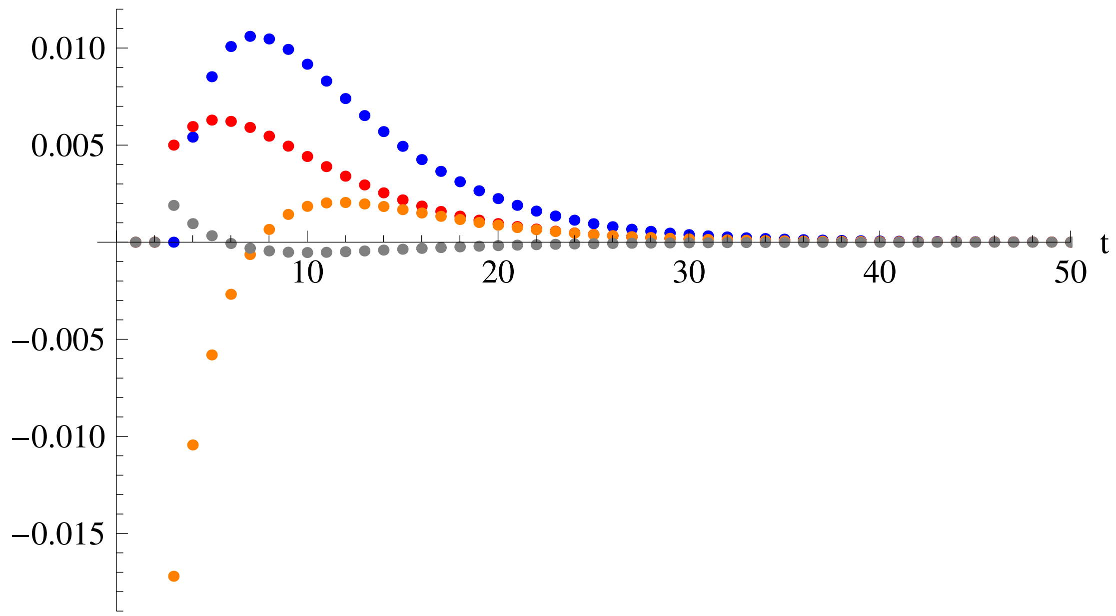

Figure 6.4: Effects of a sunspot shock when $\varphi \approx 0$. $\hat{k}_t, \hat{c}_t (= \hat{w}_t), \hat{x}_t/20$, and $\hat{R}_t$ are indicated in blue, red, orange and gray, respectively.

values from date $t = 4$ onwards. We assume logarithmic utility of consumption and very inelastic disutility of labor ($\varphi \approx 0$); the intratemporal first-order condition therefore implies $\hat{c}_t(\epsilon^t) = \hat{w}_t(\epsilon^t)$.

In response to the anticipated productivity increase, labor supply and output rise and productivity therefore rises as well, as anticipated. This is reflected in a higher wage, a higher interest rate, higher consumption, and capital accumulation. In the subsequent period, labor supply starts to revert while the capital stock continues to grow; the interest rate falls but remains elevated and consumption and wages increase further. During the following transition, labor supply and the interest rate fall below their initial levels before eventually, all variables monotonically converge to their steady-state values.

6.4 Bibliographic Notes

Becker (1965) analyzes labor supply and Lucas and Rapping (1969) study a model of intertemporal labor supply. Hansen (1985) and Rogerson (1988) study economies with indivisible labor. The restrictions on technology in the model of subsection 6.2.1 are due to Uzawa (1961); the proof follows Schlicht (2006); and King, Plosser and Rebelo (1988) derive the restrictions on preferences, see also King, Plosser and Rebelo (2002, p. 94). The balanced growth path restrictions discussed in subsection 6.2.1 imply that the model replicates *Kaldor's (1961) "stylized facts"* including a constant capital-output ratio, constant wage growth and interest rates, and constant factor shares in national income. The two-sector model in subsection 6.2.2 is due to Rebelo (1991) and the model with externalities follows Romer (1986). Barro and Sala-i-Martin (1995) and Acemoglu (2009) cover economic growth. Brock and Mirman (1972) analyze the stochastic growth model without labor-leisure choice. The RBC model is due to Kydland and Prescott (1982), Long and Plosser (1983), and King et al. (1988), see also Cooley (1995). King et al. (2002) carefully describe the solution strategy adopted in the text. Lucas and Prescott (1971) study a recursive competitive equilibrium (see also Prescott and Mehra, 1980; Stokey and Lucas, 1989). The model with sunspots follows Benhabib and Farmer (1994).

Beyond the material covered in the chapter, Krusell and Smith (1998) extend Aiyagari's (1994) model to analyze the interaction of aggregate *and* uninsurable idiosyncratic risk. Since the state in their model includes the complete wealth distribution Krusell and Smith (1998) represent it in terms of a few moments. Computed approximate equilibrium dynamics suggest that the wealth distribution has a minor effect on aggregate investment and consumption because poor households save and consume little and the rich self insure well. To assess the cost of business cycles Lucas (1987, 3) compares the utility from two consumption sequences, one following a deterministic growth path and the other fluctuating around such a path. Alvarez and Jermann (2004) relate the marginal cost of consumption fluctuations to asset prices. Azariadis (1981) analyzes sunspot-driven business cycles in overlapping generations models.

Chapter 7

The Open Economy

In the open economy, agents import and export goods and they save, borrow, and insure internationally. As a consequence, domestic production exceeds absorption by the *trade balance* and the economy accumulates or runs down *net foreign assets*. A positive trade balance implies an equal-sized increase in net foreign assets as it forces foreigners to borrow from domestic agents. Absent capital gains or losses, the *current account*—the trade balance plus income from net foreign assets—therefore equals the change in net foreign assets.

We study the determinants of net foreign assets, the trade balance, and the current account and analyze the welfare gains from intertemporal trade. Moreover, we examine factors that influence the real exchange rate as well as the consequences of international risk sharing.

7.1 Current Account and Net Foreign Assets

Consider the economy with homogeneous households, firms, and capital accumulation analyzed in section 3.1. Unlike in section 3.1 we assume that the economy is open such that the trade balance, tb_t, and the net foreign asset position, nfa_t, generally differ from zero. Letting a_t denote household assets and k_t the domestic capital stock, the trade balance and net foreign assets are given by

$$\begin{aligned} tb_t &= f(k_t, 1) - c_t - (k_{t+1} - k_t(1-\delta)), \\ nfa_t &= a_t - k_t, \end{aligned}$$

where f and c_t denote the standard neoclassical production function and consumption, respectively.

We assume that the economy is "small" (that is, domestic saving does not affect world interest rates) and open so that capital can freely move in and out of the country and the domestic and international rental rates on capital, r_t, are identical. Firms take this rate as given and optimally choose labor and capital inputs. The firms' optimality conditions combined with the labor market clearing condition imply

$$\begin{aligned} f_K(k_t, 1) &= r_t, \\ f_L(k_t, 1) &= w_t. \end{aligned}$$

The first condition pins down the capital stock and the second determines the wage, w_t. Note that with constant world rental rates and time invariant technology the domestic capital stock and the wage are constant at all times.

The remaining equilibrium conditions include the dynamic budget constraint of households and the Euler equation,

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t - c_t, \\ u'(c_t) &= \beta R_{t+1} u'(c_{t+1}), \end{aligned}$$

where the gross interest rate reflects the rental rate and depreciation, $R_t = 1 + r_t - \delta$. Households also satisfy a transversality condition. The current account, ca_t, satisfies

$$\begin{aligned} ca_t &\equiv tb_t + (R_t - 1)nfa_t = f(k_t, 1) - c_t - (k_{t+1} - k_t(1-\delta)) + (R_t - 1)nfa_t \\ &= f(k_t, 1) - c_t - k_{t+1} + k_t(1-\delta) + \left(nfa_{t+1} + k_{t+1} - k_t R_t - w_t + c_t - nfa_t\right) \\ &= f(k_t, 1) - w_t - k_t(R_t - 1 + \delta) + nfa_{t+1} - nfa_t = nfa_{t+1} - nfa_t, \end{aligned}$$

where we use the definition of net foreign assets as well as the budget constraints of households and firms.

To solve for equilibrium consumption we iterate the dynamic budget constraint forward (see subsections 2.1.2–2.1.3). Assuming for simplicity that $R_t = R = \beta^{-1}$ we find

$$c_t = \rho\, a_t R + \rho \sum_{s=0}^{\infty} R^{-s} w_{t+s}, \tag{7.1}$$

where we define the annuity factor $\rho \equiv \left(\sum_{s=0}^{\infty} R^{-s}\right)^{-1} = (R-1)/R$. As always, equilibrium consumption reflects lifetime wealth. Unlike in closed-economy general equilibrium models, however, the interest rate is exogenous and the open capital account allows for domestic saving and investment and thus, consumption and capital accumulation, to be decoupled. Specifically, under our assumption about R and β, equilibrium consumption is constant.

Now abstract from production and capital and let $\{w_{t+s}\}_{s\geq 0}$ in equation (7.1) denote a possibly time varying endowment sequence. The trade balance then equals $tb_t = w_t - c_t$ and the current account is given by

$$ca_t \equiv tb_t + (R-1)nfa_t = w_t - c_t + nfa_{t+1} - w_t + c_t - nfa_t = nfa_{t+1} - nfa_t,$$

where we use the household's budget constraint. Let $\tilde{w}_t$ denote the *permanent income* or annuity corresponding to the endowment sequence:

$$\tilde{w}_t \sum_{s=0}^{\infty} R^{-s} \equiv \sum_{s=0}^{\infty} R^{-s} w_{t+s} \;\Leftrightarrow\; \tilde{w}_t \equiv \rho \sum_{s=0}^{\infty} R^{-s} w_{t+s}.$$

From the household's dynamic budget constraint, consumption equals $nfa_t R + w_t - nfa_{t+1}$, and from optimality condition (7.1), it equals $\rho\ nfa_t R + \tilde{w}_t$. Equalizing the two expressions implies $nfa_t R(\rho - 1) + nfa_{t+1} = w_t - \tilde{w}_t$ or

$$ca_t = nfa_{t+1} - nfa_t = w_t - \tilde{w}_t. \tag{7.2}$$

Condition (7.2) states that net foreign assets increase (decrease) when the endowment exceeds (falls short of) permanent income, reflecting the usual consumption smoothing motive (see chapter 2).

Introducing endowment risk does not affect these findings (except that w_{t+s} in condition (7.1) is replaced by $\mathbb{E}_t[w_{t+s}(\epsilon^{t+2})]$) as long as certainty equivalence holds (see subsection 4.1.1).

7.2 Real Exchange Rate

Suppose now that the endowment has two components, a non-tradable component, w_t^N, and a tradable component, w_t^T. Non-tradables only can be consumed domestically while tradables can both be consumed domestically and shipped abroad at no cost. The tradable good serves as numeraire and the price of non-tradables is denoted p_t.

Household consumption is a CES aggregate of the tradable and the non-tradable good,

$$c_t = c(c_t^T, c_t^N). \tag{7.3}$$

The price of c_t, which increases in p_t, is denoted $\mathcal{P}_t$ (see subsection 2.2.3 for the formula of $\mathcal{P}_t$). The *real exchange rate* is the price of one unit of domestic consumption relative to the price of a consumption unit abroad which we normalize to unity. The real exchange rate thus equals $\mathcal{P}_t$ and it increases in p_t—a rising price of non-tradables implies a real exchange rate appreciation.

The household takes world interest rates as given and maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to (7.3), the dynamic budget constraint,

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t^T + w_t^N p_t - c_t \mathcal{P}_t \\ &= a_t R_t + w_t^T + w_t^N p_t - c_t^T - c_t^N p_t \end{aligned}$$

(assets are denominated in the numeraire), and a no-Ponzi-game condition. The first-order conditions are given by (see subsection 2.2.3)

$$u'(c_t)/\mathcal{P}_t = \beta R_{t+1} u'(c_{t+1})/\mathcal{P}_{t+1}, \tag{7.4}$$

$$u'(c_t) c_T(c_t^T, c_t^N) = \beta R_{t+1} u'(c_{t+1}) c_T(c_{t+1}^T, c_{t+1}^N), \tag{7.5}$$

$$p_t = \frac{c_N(c_t^T, c_t^N)}{c_T(c_t^T, c_t^N)}, \tag{7.6}$$

where $c_N(c_t^T, c_t^N)$ and $c_T(c_t^T, c_t^N)$ denote partial derivatives. Condition (7.4) represents the Euler equation for c_t; note that the own rate of interest for the consumption index equals $R_{t+1}\mathcal{P}_t/\mathcal{P}_{t+1}$. Condition (7.5) gives the Euler equation for tradable consumption whose own rate of interest equals R_{t+1}. Finally, condition (7.6) equalizes the price of non-tradable consumption in terms of tradable consumption and the corresponding marginal rate of substitution.

Assume that $R_t = \beta^{-1}$ and $w_t^N = w^N$ in all periods. Imposing market clearing, $c_t^N = w^N$, conditions (7.5) and (7.6) then reduce to

$$\begin{aligned} u'(c_t)c_T(c_t^T, w^N) &= u'(c_{t+1})c_T(c_{t+1}^T, w^N), \\ p_t &= \frac{c_N(c_t^T, w^N)}{c_T(c_t^T, w^N)}. \end{aligned}$$

From the first condition, the consumption index and both its components are constant over time. Domestic market clearing ($c_t^N = w^N$) and the household's intertemporal budget constraint then imply that c_t^T equals the annuity value of net foreign assets plus permanent income from the tradable endowment sequence (condition (7.1) with c_t replaced by c_t^T and w_{t+s} replaced by w_{t+s}^T). Tradable consumption thus increases in the initial net asset position and the permanent income from tradable goods. In a slightly extended model with differentiated export and import goods, it also increases in the *terms of trade*—the price of exports relative to imports—since improved terms of trade effectively increase the market value of the tradable endowment.

The second condition pins down the real exchange rate. Recall that non-tradable consumption is fixed (by domestic market clearing) while tradable consumption reflects the net asset position and the tradable endowment sequence (possibly accounting for the terms of trade). Higher net foreign assets or higher permanent income from tradables therefore increase the marginal rate of substitution on the right-hand side of the condition and thus, the price of non-tradables and the real exchange rate. Intuitively, higher household wealth raises the demand for tradables and non-tradables but with the latter in fixed supply, their equilibrium price must rise for markets to clear. We conclude that a wealthier economy (measured in terms of tradables) or one with a stronger preference for non-tradables has a more appreciated real exchange rate.

Over longer horizons, factors of production can be reallocated between sectors, implying that tradable and non-tradable output no longer are exogenous. This undermines the link between household wealth and the real exchange rate; in fact, the latter may be completely determined on the supply side.

To see this, relax the endowment assumption and suppose that competitive domestic firms employ capital and labor to produce goods. Output of tradables and non-tradables, respectively, is given by $A_t^T f^T(K_t^T, L_t^T)$ and $A_t^N f^N(K_t^N, L_t^N)$ where A_t^T and A_t^N denote productivity levels and the arguments of the constant returns to scale functions f^T and f^N denote capital and labor inputs in the two sectors. Capital is internationally mobile and earns the exogenous rental rate r_t while labor is mobile across sectors and earns the wage w_t.

For competitive firms to produce both goods the marginal value products of all inputs must equal their respective rental rates, implying

$$\begin{aligned} A_t^T f_K^T(K_t^T, L_t^T) &= r_t, \\ p_t A_t^N f_K^N(K_t^N, L_t^N) &= r_t, \\ A_t^T f_L^T(K_t^T, L_t^T) &= w_t, \\ p_t A_t^N f_L^N(K_t^N, L_t^N) &= w_t. \end{aligned}$$

Due to constant returns to scale, the marginal products are functions of the respective capital-labor ratios, k_t^T or k_t^N. Conditional on A_t^T, A_t^N, r_t (and independently of household preferences), the four conditions thus pin down k_t^T, k_t^N, w_t, and p_t.

Higher tradable-sector productivity raises the price of non-tradables. This follows from the fact that a higher A_t^T raises k_t^T (from the first condition) and thus, w_t (from the third condition); that a higher w_t raises either p_t or k_t^N (from the fourth condition); and that p_t and k_t^N adjust in the same direction (from the second condition, because A_t^N and r_t are fixed). Intuitively, higher productivity in the tradable sector at given rental rates increases equilibrium wages. To attract workers non-tradable sector firms must pay higher wages even if their productivity is unchanged. For the marginal value products of capital and labor in the non-tradable sector to remain unchanged and rise, respectively, both p_t and k_t^N must rise. Similar reasoning establishes that an increase in non-tradable-sector productivity lowers p_t.

For an alternative perspective, consider the zero-profit conditions of firms,

$$\begin{aligned} A_t^T f^T(k_t^T, 1) &= w_t + k_t^T r_t, \\ p_t A_t^N f^N(k_t^N, 1) &= w_t + k_t^N r_t, \end{aligned}$$

which are implied by constant returns to scale and competition. Totally differentiating the zero-profit condition in the tradable sector (holding the rental rate fixed) and using the first-order condition with respect to K_t^T to cancel terms yields $dA_t^T f^T(k_t^T, 1) = dw_t$. This can be expressed as $\hat{A}_t^T = \hat{w}_t \sigma_t^T$ where a circumflex denotes an infinitesimal relative deviation and $\sigma_t^T \equiv w_t/(A_t^T f^T(k_t^T, 1))$ denotes the labor share in the tradable sector. Similarly, totally differentiating the zero-profit condition in the non-tradable sector, using the first-order condition with respect to K_t^N to cancel terms and letting $\sigma_t^N \equiv w_t/(p_t A_t^N f^N(k_t^N, 1))$ yields $\hat{p}_t + \hat{A}_t^N = \hat{w}_t \sigma_t^N$.

Combining the two expressions, we find

$$\hat{p}_t = \hat{A}_t^T \frac{\sigma_t^N}{\sigma_t^T} - \hat{A}_t^N,$$

which confirms that an increase in tradable-sector productivity raises p_t while an increase in non-tradable-sector productivity decreases it. Intuitively, an increase in tradable-sector productivity lowers the unit cost of the tradable good. As a consequence, competition pushes up wages and the non-tradable sector avoids losses only if its output price rises. If productivity in both sectors rises, p_t still increases as long as tradable-sector productivity grows more quickly and the labor share in the non-tradable sector is larger than in the tradable sector. If the two conditions are satisfied, the model thus explains the *Baumol-Bowen* effect—the secular increase of the relative price of non-tradables—as well as the *Harrod-Balassa-Samuelson* effect, namely real appreciations in countries with faster productivity growth and thus, higher incomes.

A change in the world interest rate affects the real exchange rate too. Similar calculations to the previous ones show that, for constant productivity levels,

$$\hat{p}_t = \frac{dr_t}{p_t f^N}(k_t^N - k_t^T) = \frac{\hat{r}_t}{\sigma_t^T}(\sigma_t^T - \sigma_t^N).$$

A higher rental rate thus causes a real depreciation as long as the capital-labor ratio in the tradable sector is higher than in the non-tradable sector. This result mirrors the *Stolper-Samuelson theorem* according to which a price change benefits the production factor that is employed more intensively in the expanding sector.

7.3 Gains From Trade

Trade allows countries to mutually exploit *comparative advantage* that results from relative productivity or endowment differences. In addition to static gains, opening economies up generates gains from intertemporal trade (saving and borrowing) and from risk sharing.

Focusing first on the gains from intertemporal trade, consider a two-period setting. The economy is endowed with an initial stock of assets, a_0; one unit of labor in each period; and a constant returns to scale production function, f, that is the same as in the rest of the world. Domestic assets equal the domestic capital stock plus net foreign assets, $a_t = k_t + nfa_t$. The budget constraints of the economy are given by

$$\begin{aligned} c_0 &= f(k_0, 1) - (k_0 - a_0)r_0 + a_0(1-\delta) - a_1, \\ c_1 &= f(k_1, 1) - (k_1 - a_1)r_1 + a_1(1-\delta). \end{aligned}$$

When the economy is closed, $a_t = k_t$. After opening up, capital in- or outflows assure that the domestic and international rental rate of capital, r_t, are identical and the capital-labor-ratio satisfies

$$f_K(k_t, 1) = r_t.$$

Suppose first that households are homogenous. The representative agent maximizes $u(c_0) + \beta u(c_1)$ and chooses a_1. When the economy is closed this choice satisfies the Euler equation $u'(c_0) = \beta(f_K(a_1, 1) + 1 - \delta)u'(c_1)$. When it is open, in contrast, capital flows in or out, $k_t \neq a_t$, and the choice of a_1 satisfies the Euler equation $u'(c_0) = \beta(r_1 + 1 - \delta)u'(c_1)$. Using the budget constraints, a first-order approximation of the welfare effect from capital account liberalization about the closed-economy capital stocks, a_0 and a_1, yields

$$u'(c_0)(f_K(a_0, 1) - r_0)(k_0 - a_0) + \beta u'(c_1)(f_K(a_1, 1) - r_1)(k_1 - a_1).$$

(The indirect welfare effect of the induced change in a_1 equals zero, due to the envelope condition.)

Note that in each period, the product $(f_K(a_t, 1) - r_t)(k_t - a_t)$ is positive, implying that the disposable income in each period rises and the welfare effect is positive. Intuitively, when capital flows into the economy, $k_t > a_t$, the marginal product of capital falls from $f_K(a_t, 1)$ to r_t and output benefits from productive, cheap inframarginal units of foreign capital. Conversely, when capital flows out, $k_t < a_t$, the marginal product rises and domestic production falls but the inframarginal units of freed capital earn a rental rate abroad that exceeds the marginal product in autarky.

While capital in- or outflows unambiguously raise disposable income they affect the returns on capital and labor unequally. For example, if the international capital-labor ratio exceeds the domestic ratio in autarky then opening up the capital account lowers the rental rate but raises domestic wages. With homogeneous households this change of factor incomes is of no importance since all income goes to the same representative household. With heterogeneous households, in contrast, unequal effects on factor incomes imply that capital account liberalization may benefit some groups while hurting others.

To see this, consider one change to the setup studied above: The initial capital stock now is owned by an old generation that dies after the first period. The change of rental rate in the first period then exclusively affects the old generation while the wage change and the change of second-period rental rate only affects the young generation. Since the rental rate in the first period moves opposite to the capital-labor ratio we can immediately conclude that capital inflows after a capital account liberalization harm the initial old generation although the inflows increase aggregate disposable income. With sufficiently high transfers from the young to the old generation, however, all cohorts benefit.

7.4 International Risk Sharing

The (aggregate) gains from intertemporal trade reflect efficiency gains due to the international equalization of marginal rates of transformation and substitution. Similar gains arise from the equalization of the marginal rate of substitution across histories that is, from international risk sharing.

Consider the framework with tradable and non-tradable endowments analyzed in section 7.2 and suppose that endowments are risky and households can trade a complete set of Arrow securities denominated in the tradable good (the numeraire). Assuming identical psychological discount factors, the equilibrium marginal rate of substitution between consumption of the tradable good at date t, history ϵ^t, and at date $t+s$, history ϵ^{t+s},

$$\beta^s \frac{u'(c_{t+s}(\epsilon^{t+s}))}{u'(c_t(\epsilon^t))} \frac{c_T(c^T_{t+s}(\epsilon^{t+s})), w^N_{t+s}(\epsilon^{t+s}))}{c_T(c^T_t(\epsilon^t), w^N_t(\epsilon^t))},$$

then is the same for all households in all countries. Equivalently, the marginal rate of substitution between the consumption index at the two dates and histories, corrected for variation in the real exchange rate,

$$\beta^s \frac{u'(c_{t+s}(\epsilon^{t+s}))/\mathcal{P}_{t+s}(\epsilon^{t+s})}{u'(c_t(\epsilon^t))/\mathcal{P}_t(\epsilon^t)},$$

also is the same across countries. This follows from the risk-sharing result in section 4.2 (see condition (4.2)) and the equilibrium conditions (7.4) and (7.5) once we allow for risk. With complete markets, marginal utility of the consumption index thus is not perfectly correlated internationally unless the real exchange rate is constant. Marginal utility grows faster in countries whose consumer price index grows more quickly.

With incomplete markets, the correlation is even weaker. For example, when only a risk-free bond (denominated in tradables) is traded then the expected marginal rate of substitution corrected for the price index,

$$\beta^s \frac{\mathbb{E}_t[u'(c_{t+s}(\epsilon^{t+s}))/\mathcal{P}_{t+s}(\epsilon^{t+s})]}{u'(c_t(\epsilon^t))/\mathcal{P}_t(\epsilon^t)},$$

is equalized across countries. This follows from the stochastic Euler equation variant of condition (7.4).

7.5 Bibliographic Notes

Buiter (1981), Obstfeld (1982), Sachs (1981), and Svensson and Razin (1983) contain early models of the current account with optimizing agents. The Harrod-Balassa-Samuelson effect is named after Harrod (1933), Balassa (1964), and Samuelson (1964). Dornbusch, Fischer and Samuelson (1977) propose a tractable model of the terms of trade. Samuelson (1939) discusses the (static) gains from trade; and Fried (1980) or Buiter (1981) analyze intergenerational welfare effects of capital account liberalization. Backus and Smith (1993) analyze international risk sharing with non-tradable goods.

Chapter 8

Real Frictions

We have so far abstracted from frictions that prevent the extension of credit (except for market incompleteness and borrowing constraints) or delay the adjustment of the capital stock and the labor force. In this chapter, we introduce such frictions and study their implications. We begin with capital adjustment costs before turning to frictional labor markets and credit frictions.

8.1 Capital Adjustment Costs

In the benchmark model, investment contributes one-for-one to the buildup of capital. We now relax this assumption and posit that the buildup of capital is subject to adjustment costs. As a consequence, firms solve dynamic optimization problems. Rather than renting capital on spot markets, they install and own it with a view on current and future adjustment costs.

8.1.1 Convex Adjustment Costs and Tobin's q

Suppose that a buildup of capital equal to I_t requires resources $I_t + A(I_t, K_t)$ where A denotes an adjustment cost function. For now, we abstract from depreciation ($\delta = 0$) and assume the following properties of A: Adjustment costs are weakly positive; equal to zero when there is no adjustment; as well as smooth and strictly convex. Formally, $A(I, K) \geq 0$; $A(0, K) = 0$; $A_I(0, K) = 0$; and $A_{II}(I, K) > 0$. An example of A that we will use is the function $A(I, K) = zI^2/(2K)$ for some constant $z > 0$. Note that with this adjustment cost function, a larger preexisting capital stock reduces the adjustment cost per unit of capital buildup.

Consider a firm operating a neoclassical production function, f, that faces wages, w_t, and a constant (for simplicity) gross interest rate, R, and that chooses investment, I_t, and labor demand, L_t, to maximize profits. There is no depreciation. The Lagrangian associated with the firm's program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} R^{-t}[f(K_t, L_t) - I_t - w_t L_t - A(I_t, K_t) - q_t(K_{t+1} - K_t - I_t)],$$

where the multiplier q_t is associated with the law of motion for installed capital. This multiplier—*Tobin's q*—represents the shadow value of installed capital relative to the price of "outside capital" or (investment) goods, which is normalized to unity. The first-order conditions with respect to L_t, K_{t+1}, and I_t, respectively, are given by

$$\begin{aligned} f_L(K_t, L_t) &= w_t, \\ f_K(K_{t+1}, L_{t+1}) &= A_K(I_{t+1}, K_{t+1}) - q_{t+1} + Rq_t, \\ q_t &= 1 + A_I(I_t, K_t). \end{aligned}$$

The first condition represents the usual labor demand relation: Conditional on installed capital the firm equalizes the marginal product of labor and the wage. The second condition can be written as an asset pricing relation,

$$q_t = \frac{f_K(K_{t+1}, L_{t+1}) - A_K(I_{t+1}, K_{t+1}) + q_{t+1}}{R}.$$

It states that the shadow price of installed capital equals the discounted shadow price in the subsequent period, plus the discounted dividend from installed capital; the dividend in turn equals the marginal product of capital (as usual), net of the reduction in future adjustment costs due to the higher capital stock (recall that $A_K < 0$). Iterating the equation forward yields (absent bubbles)

$$q_0 = \sum_{t=1}^{\infty} \frac{f_K(K_t, L_t) - A_K(I_t, K_t)}{R^t}.$$

Note that in the absence of adjustment costs, $q_t = 1$ and $f_K(K_t, L_t) = R - 1$, corresponding to our findings in the baseline model.

According to the condition $q_t = 1 + A_I(I_t, K_t)$, the shadow price of installed capital equals the replacement cost of capital plus the marginal adjustment cost. Due to the convexity of A, this condition yields a unique mapping from q_t to I_t (conditional on K_t)—for a given stock of installed capital, q is a sufficient statistic for investment.

The end-of-period value of the firm at date $t = 0$, V_0, is given by

$$V_0 = \sum_{t=1}^{\infty} \frac{f(K_t, L_t) - w_t L_t - A(I_t, K_t) - I_t}{R^t},$$

evaluated at the optimal investment and labor demand. If both f and A exhibit constant returns to scale, then the value of a marginal unit of installed capital, q_t, and the average value of installed capital, V_t / K_{t+1}, coincide ("marginal and average q coincide"). This follows from

$$\begin{aligned} K_{t+1} q_t &= \frac{f_K(K_{t+1}, L_{t+1})K_{t+1} - A_K(I_{t+1}, K_{t+1})K_{t+1} + q_{t+1}K_{t+1}}{R} \\ &= \frac{f(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - A_K(I_{t+1}, K_{t+1})K_{t+1} + (1 + A_I(I_{t+1}, K_{t+1}))(K_{t+2} - I_{t+1})}{R} \\ &= \frac{f(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - A(I_{t+1}, K_{t+1}) - I_{t+1} + q_{t+1}K_{t+2}}{R} \\ &= \sum_{s=t+1}^{\infty} \frac{f(K_s, L_s) - w_s L_s - A(I_s, K_s) - I_s}{R^{s-t}} = V_t. \end{aligned}$$

The fact that marginal and average q coincide implies that the gross rate of return on shares of the firm equals R.

Consider the quadratic adjustment cost function introduced above and assume for simplicity that labor demand is fixed at L. The optimality conditions then read

$$\begin{aligned} I_t z &= (q_t - 1)K_t, \\ Rq_t - q_{t+1} &= \\ &f_K(K_{t+1}, L) + \frac{z}{2}\left(\frac{I_{t+1}}{K_{t+1}}\right)^2 = f_K\left(K_t\left(1 + \frac{q_t - 1}{z}\right), L\right) + \frac{1}{2z}(q_{t+1} - 1)^2. \end{aligned}$$

The first equation states that the investment-capital ratio is proportional to $q_t - 1$. The second condition relates the marginal product of capital to q_t and q_{t+1}.

In steady state, the capital stock K satisfies $R = 1 + f_K(K, L)$ and investment and the shadow price are given by $I = 0$ and $q = 1$. A first-order Taylor expansion about the steady state thus yields the following linear dynamic system in the variables $dK_t \equiv K_t - K$ and $dq_t \equiv q_t - 1$:

$$\begin{aligned} dK_{t+1} &= dK_t + \frac{K}{z} dq_t, \\ dq_{t+1} &= \left(R - \frac{K f_{KK}(K, L)}{z}\right) dq_t - f_{KK}(K, L) dK_t. \end{aligned}$$

The matrix governing the system dynamics,

$$M \equiv \begin{bmatrix} 1 & K/z \\ -f_{KK}(K, L) & R - K f_{KK}(K, L)/z \end{bmatrix},$$

has one stable and one unstable eigenvalue. Since capital is predetermined while the shadow price is a "jump variable," the linear system is saddle path stable: For any initial level of installed capital, K_0, there exists a unique shadow price, q_0, such that starting from (K_0, q_0), the dynamic system prescribes a path that converges to the steady state.

Figure 8.1 illustrates the dynamics by means of a phase diagram in (K, q) space. The horizontal solid line depicts points at which the capital stock is constant, $K_{t+1} = K_t$. Below (above) that locus, $q_t < (>) 1$ and the capital stock falls (grows). The decreasing solid line depicts points with time invariant shadow prices, $q_{t+1} = q_t$. To the left (right) of that locus, the marginal product of capital is high (low) and the shadow price falls (rises). The dotted paths indicate adjustment paths starting from two initial capital stocks, K^{low} and K^{high}. All paths satisfy the two conditions governing system dynamics; only the blue paths follow the saddle path and converge to the steady state, $(K^{\text{ss}}, 1)$.

Using the phase diagram, we can analyze the effect of productivity and interest rates on investment. Both higher productivity and lower interest rates increase the discounted marginal products of capital; this is reflected in an outward shift of the steady-state shadow price relation (the decreasing solid line in figure 8.1). The saddle

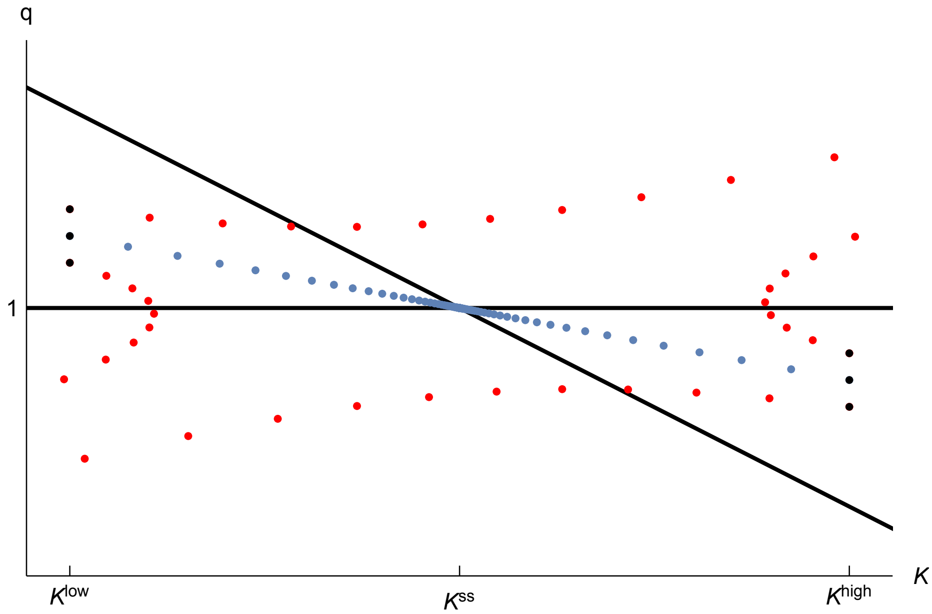

Figure 8.1: Dynamics of installed capital and its shadow price: Steady-state capital accumulation condition and shadow price relation (solid, in black) as well as dynamic adjustment paths off (red) and on (blue) the saddle path.

path therefore shifts out as well and for a given initial level of capital, the shadow price and firm value rise. Intuitively, the value of installed capital increases because the discounted marginal products are higher, and the marginal products are higher because the capital stock does not immediately adjust. Over time, the firm builds up capital until the shadow price reaches its steady-state value of one.

In general equilibrium, the capital stock and q_t interact with household consumption and saving. As usual, the household Euler equation relates the growth rate of consumption to the gross interest rate which now equals

$$R_{t+1} = \frac{f_K(K_{t+1}, L_{t+1}) - A_K(I_{t+1}, K_{t+1}) + q_{t+1}}{q_t}.$$

8.1.2 Non-Convex Adjustment Costs

If the adjustment cost function is not convex then q_t ceases to be a sufficient statistic for investment. We analyze this case in a two-period model and compare it to the situation without adjustment costs or with convex adjustment costs.

Suppose that the firm has an initial stock of installed capital, K_0; chooses investment or disinvestment, I_0; and maximizes firm value, $[f(K_1, L) + K_1]/R - A(I_0, K_0) - I_0$, subject to the law of motion $K_1 = K_0 + I_0$. Consider first the case without adjustment costs, $A(I_0, K_0) \equiv 0$. Let $K_1^\star$ denote the optimal capital stock at date $t = 1$ in this case, that is $f_K(K_1^\star, L) = r$. Clearly, in equilibrium, $I_0 = K_1^\star - K_0$ and $q_0 = 1$.

Consider next the convex adjustment costs discussed above (and let $q_1 = 1$ and $A_K(I_1, K_1) = 0$). Equilibrium then is characterized by the conditions

$$\begin{aligned} q_0 &= \frac{f_K(K_1, L) + 1}{R}, \\ q_0 &= 1 + A_I(I_0, K_0). \end{aligned}$$

Investment or disinvestment is smaller than in the frictionless case. Moreover, unless $K_1^\star = K_0$, $q_0 \neq 1$. The dotted and solid black lines in figure 8.2 illustrate the relation between K_0, q_0, and I_0 in the frictionless and the convex adjustment cost case, respectively.

As an example of non-convex adjustment costs, consider a fixed cost. Let $A(I_0, K_0) = A > 0$ iff $I \neq 0$ and zero otherwise. The optimal policy then consists of either not adjusting the capital stock at all, or fully adjusting it to the frictionless level. In the former case, q_0 differs from unity while in the latter, it does not. Either way, q_0 is not a sufficient statistic for investment. The blue lines in figure 8.2 illustrate the case with a fixed adjustment cost.

As another example of non-convex adjustment costs, consider proportional costs of adjustment, $A(I_0, K_0) = a|I_0|, a > 0$. Now, the firm faces a constant marginal cost of adjusting. If $K_1^\star - K_0$ is "small" in absolute value then the marginal gain of adjusting is smaller than a; consequently, $I_0 = 0$ and q_0 differs from unity. If the absolute value of $K_1^\star - K_0$ is "large," in contrast, then the firm adjusts to the point where the marginal gain of further adjustment equals the marginal cost. That is, $I_0 \neq 0$ in that range but the adjustment is incomplete and q_0 does not reach unity. The red lines in figure 8.2 illustrate this case.

8.1.3 Risk, Irreversibility, and the Option Value of Waiting

When the return on an investment project is risky and the investment cannot be reversed then the decision to invest destroys an *option*, namely the option to wait and to invest only if and when uncertainty has been resolved in a favorable way. This option has value and destroying it constitutes an opportunity cost of starting the project which is not accounted for in a naive present value calculation.

To see this, consider the problem of a risk neutral firm that chooses whether to invest a fixed amount 1 at date $t = 0$, not to invest, or to delay the decision. The investment is irreversible and once undertaken, pays a return ρ per period forever. The return can either be high or low, $\rho(h)$ or $\rho(l)$ respectively. Conditional on information available at date $t = 0$, the former occurs with probability $\eta(h)$ and the latter with probability $\eta(l) = 1 - \eta(h)$. At date $t = 1$, uncertainty is resolved and the firm learns ρ with certainty. The gross discount factor is given by $R = 1 + r > 1$.

Consider first the strategy of investing at date $t = 0$. The payoff of this strategy, V, equals

$$V = -1 + \frac{\eta(h)\rho(h) + \eta(l)\rho(l)}{r}.$$

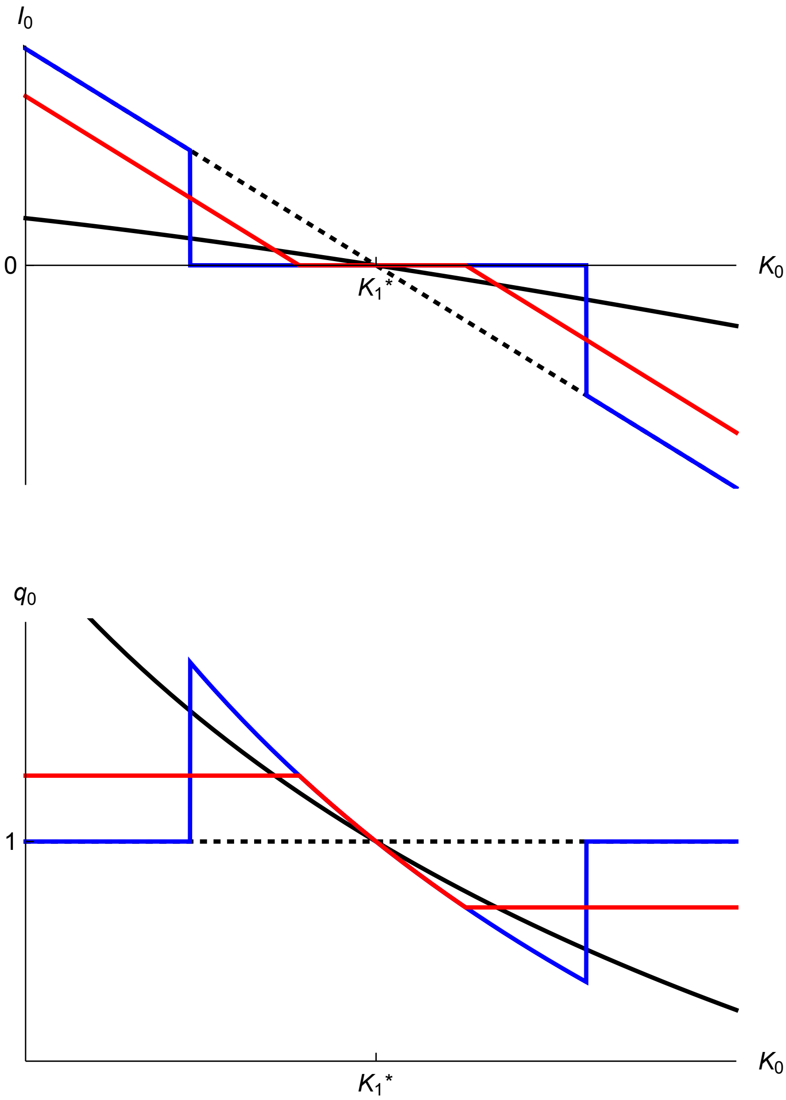

Figure 8.2: Convex and non-convex adjustment costs: Optimal investment (top) and q (bottom).

We assume that $V \geq 0$ (the risky project is profitable, $\eta(h)\rho(h) + \eta(l)\rho(l) \geq r$) but $-1 + \rho(l)/r < 0$ (the project is not profitable when the low return is realized).

Consider next the strategy of waiting and only investing at date $t = 1$ if this is profitable. The payoff of this second strategy, W, is given by

$$W = \frac{\eta(h)\max[0, -1 + \rho(h)/r] + \eta(l)\max[0, -1 + \rho(l)/r]}{R} = \frac{\eta(h)(-1 + \rho(h)/r)}{R},$$

where we use the fact that the project is profitable in the high but unprofitable in the low state.

If $\eta(h)$ is sufficiently close to one then the second strategy delays a project that is profitable with very high probability; accordingly $V > W$ because delaying is costly ($R > 1$). For smaller values of $\eta(h)$, in contrast, $V < W$. Waiting then is advantageous because the benefit of avoiding an unprofitable investment with probability $\eta(l)$, outweighs the cost of delaying a profitable investment with probability $\eta(h)$. While the option to invest now or later is "in the money" ($V \geq 0$) it is not optimal to exercise it already at date $t = 0$. For the opportunity cost of investing early—destroying the option to "wait and see"—exceeds the benefit ($W > V$).

If the investment were reversible the firm could fully recover its investment (net of the return in the first period) including interest at date $t = 1$. It would make use of this possibility in the low state and the payoff of investing early thus would equal

$$-1 + \eta(h)\frac{\rho(h)}{r} + \eta(l)\left(\rho(l) + \frac{R(1 - \rho(l))}{R}\right) = \eta(h)\left(-1 + \frac{\rho(h)}{r}\right) = RW,$$

which exceeds W when delay is costly. With reversibility the strategy to invest immediately therefore would dominate the strategy to "wait and see."

8.2 Labor Market Frictions

In the baseline model discussed in section 6.1, workers and firms meet on a competitive, frictionless labor market. The equilibrium wage is determined by the condition that labor demand equals supply. We now relax this assumption and introduce *matching frictions*. Specifically, we assume that job seekers and firms trying to fill vacancies must invest resources to search for each other, and that they meet randomly. This affects both job creation and wage determination. When a job seeker and a firm with a vacancy meet they form a *bilateral monopoly*. Rather than waiting for another match, the job seeker is willing to accept any wage higher than the marginal rate of substitution, and the firm is willing to accept any wage lower than labor's marginal product; there is thus space for negotiation. Accordingly, we introduce a new mechanism to determine wages.

8.2.1 Economy

The economy is inhabited by a representative firm and a representative household with a continuum of members that insure each other, as in the extensive margin model considered in subsection 6.1.1. At date t, a fraction x_t of household members consumes leisure; a fraction y_t supplies labor; and a fraction $z_t = 1 - x_t - y_t$ searches for a job. Workers employed by the firm perform two types of tasks: A share $1 - \nu_t$ contributes to production and the remaining share ν_t searches for new employees that is, the number of the firm's vacancies equals $y_t \nu_t$. The firm accumulates capital, k_t, which it uses jointly with productive labor, $y_t(1 - \nu_t)$, to produce output subject to the constant returns to scale production function f. Productivity, A_t, is exogenous and labor augmenting.

Vacancies are filled, and job seekers are employed, according to a constant returns to scale *matching function*, g. When the household has z_t job seekers and the firm tries to fill v_t vacancies then

$$g(z_t, v_t) = z_t \, g\left(1, \frac{v_t}{z_t}\right) \equiv z_t \, g(1, \theta_t)$$

vacancies are filled and job seekers get employed. Variable θ_t denotes *labor market tightness*, the ratio of vacancies to unemployed (job seekers). At date t, the share $\eta(\theta_t) \equiv g(1, \theta_t)/\theta_t = g(z_t, v_t)/v_t$ of vacancies is filled; the function η is decreasing. Correspondingly, the share $\theta_t \eta(\theta_t) = g(z_t, v_t)/z_t$ of job seekers (which increases in θ_t) gets employed. Both households and firms take labor market tightness as given. Employment is a state variable and employment relationships end with an exogenous probability, s.

For notational simplicity, we abstract from aggregate (productivity) risk and formulate the equilibrium conditions recursively. Let a_t denote household financial assets at date t; and q_{t+1} the price at date t of an asset that pays off one unit at the subsequent date.

8.2.2 Firms

The firm's value function, W, satisfies

$$\begin{aligned}
W(k_t, y_t, t) \quad &= \quad \max_{k_{t+1}, \nu_t} \; cf_t + q_{t+1} W(k_{t+1}, y_{t+1}, t+1) \\
\text{s.t.} \quad & \quad cf_t = f(A_t, k_t, y_t(1 - \nu_t)) + k_t(1 - \delta) - k_{t+1} - y_t w_t, \\
& \quad y_{t+1} = y_t(1 - s) + \eta(\theta_t) y_t \nu_t, \\
& \quad 0 \leq \nu_t \leq 1,
\end{aligned}$$

where cf_t denotes cash flow; $\delta \geq 0$ is the depreciation rate; and w_t denotes the wage. The first constraint defines cash flow as output net of gross investment and the wage bill; the second represents the law of motion for employment from the firm's perspective; and the third reflects the fact that the number of recruiters cannot exceed the number of employees.

The first-order conditions with respect to capital and recruiting, respectively, are

$$\begin{aligned}
1 \quad &= \quad q_{t+1} W_k(k_{t+1}, y_{t+1}, t+1), \\
f_y(A_t, k_t, y_t(1 - \nu_t)) \quad &= \quad \eta(\theta_t) q_{t+1} W_y(k_{t+1}, y_{t+1}, t+1).
\end{aligned}$$

The firm equates the resource cost of investment to the discounted marginal increase in firm value due to a higher capital stock; and the output loss due to more intense recruiting to the discounted marginal increase in firm value from higher employment, weighted by the probability of filling a vacancy.

Using the envelope conditions for capital and employment,

$$\begin{aligned} W_k(k_t, y_t, t) &= f_k(A_t, k_t, y_t(1-\nu_t)) + 1 - \delta, \\ W_y(k_t, y_t, t) &= f_y(A_t, k_t, y_t(1-\nu_t)) - w_t + (1-s)q_{t+1} W_y(k_{t+1}, y_{t+1}, t+1), \end{aligned}$$

we find

$$1 = q_{t+1}\left(f_k(A_{t+1}, k_{t+1}, y_{t+1}(1-\nu_{t+1})) + 1 - \delta\right), \tag{8.1}$$

$$\begin{aligned} f_y(A_t, k_t, y_t(1-\nu_t)) &= \eta(\theta_t) q_{t+1} \\ &\times \left(f_y(A_{t+1}, k_{t+1}, y_{t+1}(1-\nu_{t+1}))\left(1 + \frac{1-s}{\eta(\theta_{t+1})}\right) - w_{t+1}\right), \end{aligned} \tag{8.2}$$

$$W_y(k_t, y_t, t) = f_y(A_t, k_t, y_t(1-\nu_t))\left(1 + \frac{1-s}{\eta(\theta_t)}\right) - w_t. \tag{8.3}$$

The first equation is standard. The second equates the cost and benefit of recruiting. The last condition states that the marginal value of employment equals the marginal product of labor, net of the wage, plus the recruitment cost that the firm saves when employment is higher. (One employee at date t generates $1-s$ units of employment at date $t+1$; $1/\eta(\theta_t)$ recruiters at date t increase employment at date $t+1$ by one unit. An additional employee at date t thus saves $(1-s)/\eta(\theta_t)$ recruiters who can instead be employed in production.)

8.2.3 Households

The date t period utility of the representative household is given by $u(c_t) - \gamma(y_t + z_t)$, where c_t denotes consumption and $\gamma > 0$ measures the disutility of work or job search. We assume logarithmic preferences. The household's value function, V, thus satisfies

$$\begin{aligned} V(a_t, y_t, t) \;&=\; \max_{a_{t+1}, z_t} \ln(c_t) - \gamma(y_t + z_t) + \beta V(a_{t+1}, y_{t+1}, t+1) \\ \text{s.t.} \;&\quad c_t = a_t + w_t y_t - q_{t+1} a_{t+1}, \\ &\quad y_{t+1} = y_t(1-s) + \theta_t \eta(\theta_t) z_t, \\ &\quad 0 \le z_t \le 1 - y_t. \end{aligned}$$

The household also satisfies a natural borrowing limit. The first constraint is the budget constraint. The second constraint represents the law of motion for employment from the household's perspective, and the third constraint is the time use constraint.

The first-order condition with respect to assets,

$$\frac{1}{c_t} q_{t+1} = \beta V_a(a_{t+1}, y_{t+1}, t+1),$$

relates the intertemporal marginal rate of substitution to the price. The optimality condition with respect to labor market search,

$$\gamma = \beta\theta_t\eta(\theta_t)V_y(a_{t+1}, y_{t+1}, t+1),$$

equates the cost of job search (foregone utility from leisure) and the marginal benefit; the latter equals the discounted continuation value of employment, multiplied by the probability of a match.

Using the envelope conditions

$$\begin{aligned} V_a(a_t, y_t, t) &= \frac{1}{c_t}, \\ V_y(a_t, y_t, t) &= \frac{w_t}{c_t} - \gamma + \beta(1-s)V_y(a_{t+1}, y_{t+1}, t+1), \end{aligned}$$

we arrive at the equilibrium conditions

$$q_{t+1} = \beta\frac{c_t}{c_{t+1}}, \tag{8.4}$$

$$\gamma = \beta\theta_t\eta(\theta_t)\left(\frac{w_{t+1}}{c_{t+1}} - \gamma + \gamma\frac{1-s}{\theta_{t+1}\eta(\theta_{t+1})}\right), \tag{8.5}$$

$$V_y(a_t, y_t, t) = \frac{w_t}{c_t} - \gamma + \gamma\frac{1-s}{\theta_t\eta(\theta_t)}. \tag{8.6}$$

The first condition is the standard Euler equation. The second equates the cost and benefit of job search. The third condition states that higher employment generates two benefits: Marginal utility due to higher labor income, net of the disutility from work; and marginal utility from leisure because of reduced job search. (An additional unit of employment at date t generates $1-s$ units at date $t+1$, thus saving the household $(1-s)/(\theta_t\eta(\theta_t))$ job seekers at date t.)

8.2.4 Market Clearing and Wage Determination

Goods market clearing implies

$$c_t = f(A_t, k_t, y_t(1-\nu_t)) + k_t(1-\delta) - k_{t+1}. \tag{8.7}$$

Employment, y_t, is a state variable that changes in response to exogenous separations and endogenous hires. When a job seeker and a firm with a vacancy meet the two parties negotiate; if they can agree on a wage then they form a new employment relationship.

Let $\tilde{V}_y(a_t, y_t, t, \Delta w_t)$ and $\tilde{W}_y(k_t, y_t, t, \Delta w_t)$ denote the value to the household and the firm, respectively, of a marginal hire whose wage exceeds the market wage at date t by Δw_t, but in the future equals the market wage. From the two envelope conditions derived earlier,

$$\begin{aligned} \tilde{V}_y(a_t, y_t, t, \Delta w_t) &= \frac{\Delta w_t}{c_t} + V_y(a_t, y_t, t), \\ \tilde{W}_y(k_t, y_t, t, \Delta w_t) &= -\Delta w_t + W_y(k_t, y_t, t). \end{aligned}$$

Let $\mathcal{J}(\Delta w_t)$ denote the weighted average of the surpluses of the negotiating parties,

$$\mathcal{J}(\Delta w_t) \equiv \left(\tilde{V}_y(a_t, y_t, t, \Delta w_t)\right)^{\phi} \left(\tilde{W}_y(k_t, y_t, t, \Delta w_t)\right)^{1-\phi},$$

where ϕ and $1-\phi$ denote the bargaining weights of the job seeker and the firm, respectively.[1] With *Nash bargaining*, the equilibrium wage maximizes $\mathcal{J}(\Delta w_t)$ that is, the wage satisfies $\mathcal{J}'(0) = 0$ or equivalently,

$$\phi \frac{1}{c_t} \frac{1}{V_y(a_t, y_t, t)} = (1-\phi) \frac{1}{W_y(k_t, y_t, t)}.$$

Using conditions (8.3) and (8.6) and solving for the wage, we find

$$\begin{aligned} w_t &= \gamma c_t \left(1 - \frac{1-s}{\theta_t \eta(\theta_t)}\right) \\ &\quad + \phi \left\{ f_y(A_t, k_t, y_t(1-v_t)) \left(1 + \frac{1-s}{\eta(\theta_t)}\right) - \gamma c_t \left(1 - \frac{1-s}{\theta_t \eta(\theta_t)}\right) \right\}. \end{aligned} \tag{8.8}$$

The equilibrium wage has two components, represented by the two terms on the right-hand side of equation (8.8). First, the household's opportunity cost or *outside value*, namely the forgone utility from leisure net of the saved search cost (both in consumption terms). And second, the share ϕ of the *joint surplus*. The latter equals the sum of the household's and the firm's marginal values from a hire, net of the outside values of the two parties. The firm's marginal value equals $W_y(k_t, y_t, t)$ and its outside value equals zero while the household's marginal value net of the outside value (in consumption terms) equals $c_t V_y(a_t, y_t, t)$.

8.2.5 Equilibrium

In equilibrium, household assets represent the value of the representative firm; labor market tightness reflects optimal firm and household choices, $\theta_t \equiv y_t v_t / z_t$; and the laws of motion for employment as perceived by the firm and the household reduce to

$$y_{t+1} = y_t(1-s) + \eta\left(\frac{y_t v_t}{z_t}\right) y_t v_t. \tag{8.9}$$

Conditional on the state variables in the initial period an equilibrium is characterized by (8.1), (8.2), (8.4), (8.5), (8.7)–(8.9) as well as the definition of θ_t; the household budget constraint; and the borrowing limit.

Using (8.4) and (8.8), we can re-express conditions (8.2) and (8.5) as

$$\frac{f_y(A_t, k_t, y_t(1-v_t))}{c_t} = \beta \eta(\theta_t)(1-\phi)\Omega_{t+1}, \tag{8.10}$$

$$\gamma = \beta \theta_t \eta(\theta_t) \phi \Omega_{t+1}, \tag{8.11}$$

[1] We have assumed that agents take the wage as given when choosing a or k although these state variables may affect the bargaining outcome. To reconcile these features one could assume that counter parties do not observe individual assets or that the firm negotiates a uniform wage for all workers.

respectively, where we define (dropping arguments of the marginal product function for legibility)

$$\Omega_{t+1} \equiv \frac{f_y(t+1)}{c_{t+1}} - \gamma + \frac{1-s}{\eta(\theta_{t+1})}\left(\frac{f_y(t+1)}{c_{t+1}} + \frac{\gamma}{\theta_{t+1}}\right). \tag{8.12}$$

We will use these conditions below.

8.2.6 Constrained Pareto Optimality

Job seekers and firms with vacancies are *rationed*: They are willing to form employment relationships at the going wage but are unable to do so until being matched. When labor market tightness is high then firms are typically rationed for an extended period while job seekers quickly find a job; when tightness is low the situation is reversed. In either case, job seekers and firms exert negative *congestion externalities* and positive *thick market externalities*: Their search renders it harder for agents of the same type, but easier for agents of the other type to be matched.

A social planner cannot avoid the matching friction. But the planner internalizes the externalities, in contrast to firms and job seekers in the decentralized equilibrium. As a consequence, the (constrained optimal) social planner allocation typically differs from the equilibrium allocation. It is only when the bargaining weight of job seekers and firms assumes a particular value that the equilibrium allocation is constrained efficient.

To see this, consider the social planner's program. Letting P denote the value function of the social planner, we have

$$\begin{aligned} P(k_t, y_t, t) &= \max_{v_t, z_t} \ln(c_t) - \gamma(y_t + z_t) + \beta P(k_{t+1}, y_{t+1}, t+1) \\ \text{s.t.} \quad & (8.7), (8.9), \\ & 0 \leq z_t \leq 1 - y_t, \\ & 0 \leq v_t \leq 1. \end{aligned}$$

The first-order and envelope conditions reduce to a first-order condition for capital accumulation which corresponds to (8.1) and (8.4), as well as to

$$\frac{f_y(A_t, k_t, y_t(1-v_t))}{c_t} = -\gamma \frac{\eta'(\theta_t)\theta_t + \eta(\theta_t)}{\eta'(\theta_t)\theta_t^2}, \tag{8.13}$$

$$\gamma = -\beta \eta'(\theta_t)\theta_t^2 \Phi_{t+1}, \tag{8.14}$$

where we define (dropping again arguments)

$$\Phi_{t+1} \equiv \frac{f_y(t+1)}{c_{t+1}} - \gamma + \gamma \frac{1-s}{-\eta'(\theta_{t+1})\theta_{t+1}^2}.$$

Note the $\eta'(\theta_t)$ terms in the planner's optimality conditions; they reflect that the planner internalizes the search externalities. Note also that $g_z(z_t, v_t) = -\eta'(\theta_t)\theta_t^2$ and $g_v(z_t, v_t) = \eta'(\theta_t)\theta_t + \eta(\theta_t)$.

Condition (8.13) states that the planner equalizes the relative costs and benefits of recruiting and job search: Recruiting generates $g_v(z_t, v_t)$ matches and has opportunity cost $f_y(A_t, k_t, y_t(1-\nu_t))$ while job search generates $g_z(z_t, v_t)$ matches and has opportunity cost γc_t in consumption units. Condition (8.14) characterizes the efficient intensity of job search. It equalizes the utility cost of job search and the discounted, probability weighted benefit from a marginal match. This benefit reflects utility from consumption, net of the utility loss from working, plus the cost saving from reduced job search.

Let $\varphi(\theta_t)$ denote the elasticity of the matching function with respect to job search,

$$\varphi(\theta_t) \equiv \frac{g_z(z_t, v_t)}{g(z_t, v_t)} z_t = \frac{g_z(z_t, v_t)}{\eta(\theta_t)\theta_t}.$$

If the bargaining weight of the job seeker, ϕ, coincides with this elasticity that is, when the *Hosios* condition is satisfied, then the decentralized equilibrium allocation and the social planner allocation coincide and in particular, conditions (8.10)–(8.11) and (8.13)–(8.14) are identical.

This can be shown as follows: Using (8.11), the right-hand side of (8.10) can be expressed as $\gamma(1-\phi)/(\phi\theta_t)$ and thus (using the Hosios condition and the constant returns to scale property of the matching function), $\gamma g_v(z_t, v_t)/g_z(z_t, v_t)$. Moreover, using the modified condition (8.10), the last term of Ω_{t+1} in (8.12) can be expressed as $(1-s)\gamma/(\phi\eta(\theta_{t+1})\theta_{t+1})$. The Hosios condition and the above definition of the elasticity then implies $\Omega_{t+1} = \Phi_{t+1}$ and the result follows.

Intuitively, the Hosios condition guarantees that the private gains from job seeking or posting vacancies (which households or firms internalize) equal the social contributions of these activities. When the elasticity $\varphi(\theta_t)$ is high then job search strongly increases the odds that a vacancy is filled but it only has a minor negative effect on the probability that a job seeker is matched. In contrast, recruiting strongly reduces the odds that a vacancy is filled in this case while it only has a minor positive effect on the job finding probability of a job seeker. When $\varphi(\theta_t)$ is high it is therefore efficient to give strong incentives for job seeking that is, pay high wages.

8.2.7 The Case without Capital

Returning to the decentralized equilibrium consider a variant of the model without capital that is, let $f(A_t, k_t, y_t(1-\nu_t)) = A_t y_t(1-\nu_t)$. The first-order conditions for ν_t and z_t, the wage condition, the resource constraint, and the budget constraint then reduce to

$$\begin{aligned}
\frac{A_t}{c_t} &= \beta\eta(\theta_t)\left(\frac{A_{t+1}}{c_{t+1}}\left(1+\frac{1-s}{\eta(\theta_{t+1})}\right) - \frac{w_{t+1}}{c_{t+1}}\right), \\
\gamma &= \beta\theta_t\eta(\theta_t)\left(\frac{w_{t+1}}{c_{t+1}} - \gamma + \gamma\frac{1-s}{\theta_{t+1}\eta(\theta_{t+1})}\right), \\
\frac{w_t}{c_t} &= \gamma\left(1-\frac{1-s}{\theta_t\eta(\theta_t)}\right) + \phi\left\{\frac{A_t}{c_t}\left(1+\frac{1-s}{\eta(\theta_t)}\right) - \gamma\left(1-\frac{1-s}{\theta_t\eta(\theta_t)}\right)\right\}, \\
\frac{c_t}{A_t} &= y_t(1-\nu_t),
\end{aligned}$$

respectively.

Note that productivity, consumption, and the wage only appear in ratios. Productivity thus is fully reflected in consumption and the wage, and different productivity sequences are associated with the same equilibrium sequences for employment, recruiting, job search, and labor market tightness.

Intuitively, holding wages and consumption (and thus, interest rates) constant, a contemporaneous productivity increase renders recruiting more expensive relative to production, thereby inducing firms to recruit less and produce more (see the first condition). But higher production raises consumption, and less recruiting lowers future production and consumption. In equilibrium, this requires a fall in the interest rate (from the Euler equation), raising the incentive to invest in vacancies. With logarithmic preferences, the direct, negative effect on the incentive to recruit and the indirect, positive effect cancel. With capital, this result only would hold in steady state since capital cannot instantaneously adjust in line with productivity.

Exogenously imposed *wage stickiness* renders labor market outcomes more responsive to productivity. When firms anticipate elevated productivity but unchanged wages then recruiting becomes more profitable (see the optimality condition for ν_t above). Employment therefore rises more strongly than in an environment with Nash bargaining where wages increase with productivity. Wage stickiness is individually rational in the sense that it does not give rise to inefficiencies from the joint perspective of a household and a firm as long as the wage remains within the bargaining set delimited by the pair's reservation wages.

8.2.8 The Case without Capital and Leisure

To further simplify the analysis we abstract from leisure. Households thus are either employed or unemployed (searching for a job), $z_t = 1 - y_t$. We also assume that workers are risk neutral and that each firm has one job. The cost for a firm of posting a vacancy is b and a filled job generates output A. A worker consumes the wage income when employed and the return on home production, γ, when unemployed.

Let V, V^0, W, and W^0 denote the steady-state values of an employed and unemployed worker and a firm with a worker and a vacancy, respectively. For a given wage, w, and labor market tightness, $\theta \equiv v/z$, these values follow from the Bellman equations

$$\begin{aligned}
W &= A - w + \beta\left((1-s)W + sW^0\right), \\
W^0 &= -b + \beta\left(\eta(\theta)W + (1-\eta(\theta))W^0\right), \\
V &= w + \beta\left((1-s)V + sV^0\right), \\
V^0 &= \gamma + \beta\left(\theta\eta(\theta)V + (1-\theta\eta(\theta))V^0\right).
\end{aligned}$$

Due to *free entry*, a firm entering the labor market and posting a vacancy just breaks even,

$$W^0 = 0,$$

implying

$$\begin{aligned} V - V^0 + W &= \frac{A - \gamma + b\theta}{1 - \beta(1-s) + \beta\theta\eta(\theta)}, \\ w &= A - b\frac{1 - \beta(1-s)}{\beta\eta(\theta)}, \\ W &= \frac{b}{\beta\eta(\theta)}. \end{aligned}$$

Intuitively, the joint surplus reflects output net of home production as well as "saved" posting costs (we assume that the joint surplus is positive). Moreover, a higher cost of posting a vacancy or a lower probability of filling it (due to higher labor market tightness) must be compensated by a lower wage and a higher value W. Finally, the wage increases in output.

To close the model, we assume Nash bargaining. Maximizing $(V - V^0)^\phi (W - W^0)^{1-\phi}$ with respect to w (taking continuation values in the Bellman equations as given) implies that the surplus appropriated by the firm equals the joint surplus times the firm's bargaining weight,

$$W = (1 - \phi)(V - V^0 + W).$$

An equilibrium is a collection $(W, W^0, V, V^0, w, \theta)$ that solves the above equations.

The vertical line in figure 8.3 represents the equilibrium level of labor market tightness. When the bargaining power of workers, ϕ, rises or their outside options, γ, improve, or when the cost of posting a vacancy, b, increases then the incentive for firms to post vacancies can only be maintained by a corresponding increase in the probability of filling vacancies. This requires lower labor market tightness that is, a leftward shift of the schedule.

The decreasing schedule in figure 8.3 depicts the *Beveridge curve* relationship between steady-state unemployment and labor market tightness. To derive the curve we use the fact that with constant employment (and abstracting from leisure), the law of motion (8.9) implies $g(z, v) = s(1 - z)$ or

$$z = \frac{s}{s + \theta\eta(\theta)}.$$

The Beveridge curve is decreasing because both unemployment (job search) and vacancies increase the number of matches. In steady state, net inflows into employment equal zero. Additional vacancies thus must be accompanied by fewer job seekers, both to reduce the number of matches and to increase the number of outflows from employment, $s(1 - z)$. Since $\theta\eta(\theta)$ is concave the Beveridge curve is convex. (The same holds true when the curve is plotted in (v, z) space.)

Steady-state unemployment and vacancies are determined by the intersection of the Beveridge curve and the vertical schedule representing the equilibrium conditions.

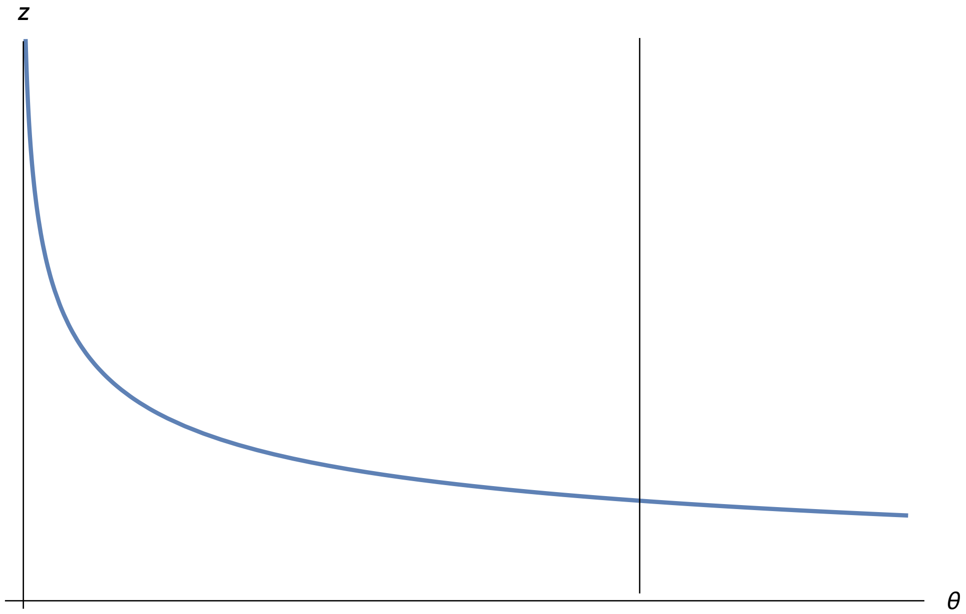

Figure 8.3: Steady-state labor market tightness and the Beveridge curve.

Note that *structural unemployment* is strictly positive in equilibrium; if it equalled zero firms would not recruit. When the matching technology improves or the separation rate, s, falls then the Beveridge curve shifts down and the vertical schedule shifts outward; structural unemployment falls.

8.3 Credit Frictions

8.4 Bibliographic Notes

Jorgensen (1963) presents the neoclassical theory of investment; Tobin (1969) discusses the relative price of installed capital; and Hayashi (1982) analyzes the relation between marginal and average q. Lucas and Prescott (1971), Baldwin and Meyer (1979), McDonald and Siegel (1986), and Dixit (1989, 1–2) study risky investment decisions and the option value of waiting; see Dixit and Pindyck (1994) for a textbook treatment.

Diamond (1982), Mortensen (1982), and Pissarides (1985) develop the search and matching model of the labor market. Pissarides (1990; 2000) describes the baseline model and reviews the literature. The presentation of the model with capital in the text follows Shimer (2010) who builds on Merz (1995). Hosios (1990) analyzes constrained efficiency and Hall (2005) proposes the model of individually-rational sticky wages.

Beyond the material covered in the chapter, McCall (1970) analyzes job search in partial equilibrium. Lucas and Prescott (1974) study a general equilibrium model of spatially separated labor markets each of which is competitive and subject to persistent productivity shocks; workers in markets with low productivity optimally move to other markets with higher expected demand, at the cost of temporary (frictional) unemployment.

Chapter 9

Money

An object used as *money* performs three functions. First, it serves as *unit of account* or *numeraire*. Second, it serves as *medium of exchange* or *means of payment* and helps mitigate the *double coincidence of wants* problem.[1] Finally, money serves as *store of value*, at least temporarily; if money did not maintain a stable value it could not usefully serve as a medium of exchange.

Money issued by governments typically takes the form of intrinsically useless *fiat money*. In earlier times, money used to be convertible into valuable objects like gold or silver, or it was a *commodity money* which had value in itself in addition to serving as money. Money constitutes *outside money* when it is a net asset of the private sector, for example because it is issued by the government or foreigners. In contrast, bank deposits (or cryptocurrency credits, when they perform the functions of money) constitute *inside money* because they are both a private sector asset (for deposit holders) and liability (for banks).

We introduce intrinsically useless money as unit of account in the household's consumption-saving problem and analyze equilibrium prices, returns, and exchange rates. Subsequently, we add frictions that create a role for such money as store of value and medium of exchange, even if it is dominated in return by other assets.

9.1 Unit of Account

Consider a generalization of the saving problem under risk studied in subsection 4.1.3. In addition to investments in "real" assets, $\{a^l_{t+1}(\epsilon^t)\}_l$, whose prices and returns are quoted in terms of the consumption good the infinitely-lived household may invest in "nominal" assets, $\{B^j_{t+1}(\epsilon^t)\}_j$, whose prices and returns are quoted in terms of money. The household may also invest in money, $M_{t+1}(\epsilon^t)$, which does not pay interest. Let $P_t(\epsilon^t)$ denote the *price level*—the price of the consumption good in terms of money;

[1] When an agent willing to exchange A for B meets a potential trading partner seeking to exchange C for A then agent 1 can sell A in exchange for money to agent 2 and money in exchange for B to a third party, and agent 2 can act similarly; with barter, all parties involved would have to meet at the same time (or extend credit).

money has value if the price level is finite.

The household's dynamic budget constraint at date t, history ϵ^t reads

$$\begin{aligned}\sum_l a^l_{t+1}(\epsilon^t) + \frac{\sum_j B^j_{t+1}(\epsilon^t) + M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} &= \\ w_t(\epsilon^t) + \frac{W_t(\epsilon^t)}{P_t(\epsilon^t)} - c_t(\epsilon^t) + \sum_l a^l_t(\epsilon^{t-1})R^l_t(\epsilon^t) &+ \frac{\sum_j B^j_t(\epsilon^{t-1})I^j_t(\epsilon^t) + M_t(\epsilon^{t-1})}{P_t(\epsilon^t)},\end{aligned} \tag{9.1}$$

where $w_t(\epsilon^t)$, $W_t(\epsilon^t)$, and $c_t(\epsilon^t)$ denote real and nominal exogenous incomes as well as consumption, respectively. The gross real rate of return on asset l is denoted by $R^l_t(\epsilon^t)$, and the gross nominal rate of return on asset j by $I^j_t(\epsilon^t) \equiv 1 + i^j_t(\epsilon^t)$. To convert nominal incomes, returns, or asset purchases into real terms they are divided by the price level.

The household maximizes discounted expected utility flows from consumption, $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[u(c_t(\epsilon^t))]$, subject to (9.1), a no-Ponzi-game condition, and the condition that money cannot be shorted, $M_{t+1} \geq 0$. Differentiating with respect to $a^l_{t+1}(\epsilon^t)$, $B^j_{t+1}(\epsilon^t)$, and $c_t(\epsilon^t)$, and letting $m_{t+1}(\epsilon^{t+1})$ denote the household's marginal rate of substitution, we arrive at the stochastic Euler equation for real assets (see section 5.1) and a parallel condition for nominal assets, with nominal returns adjusted for inflation,

$$1 = \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})R^l_{t+1}(\epsilon^{t+1})\right] = \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})\frac{I^j_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})}\right], \tag{9.2}$$

where $\Pi_{t+1}(\epsilon^{t+1}) \equiv P_{t+1}(\epsilon^{t+1})/P_t(\epsilon^t)$. In equilibrium, condition (9.2) holds for all non-monetary assets in the household's portfolio.

For now, we disregard the question whether the portfolio also includes money. In section 9.2, we take up that question and study the first-order condition with respect to $M_{t+1}(\epsilon^t)$.

9.1.1 Fisher Equation

Consider a real bond with risk-free gross real rate of return $R_{t+1}(\epsilon^t)$, and a nominal bond with risk-free gross nominal rate of return $I_{t+1}(\epsilon^t)$. If the household invests in both bonds, condition (9.2) implies

$$\begin{aligned} I^{-1}_{t+1}(\epsilon^t) &= \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})\Pi^{-1}_{t+1}(\epsilon^{t+1})\right] \\ &= R^{-1}_{t+1}(\epsilon^t)\mathbb{E}_t\left[\Pi^{-1}_{t+1}(\epsilon^{t+1})\right] + \mathbb{C}\mathrm{ov}_t\left[m_{t+1}(\epsilon^{t+1}), \Pi^{-1}_{t+1}(\epsilon^{t+1})\right]. \end{aligned}$$

This is a stochastic version of the *Fisher equation* which links inflation, real, and nominal interest rates.

In the absence of inflation risk, the Fisher equation reduces to

$$I_{t+1} = R_{t+1}\Pi_{t+1},$$

that is, the gross nominal interest rate equals the gross real rate times gross inflation. With risk, in contrast, the nominal interest rate reflects both the real rate, average (inverse) inflation, and a (positive or negative) inflation risk premium. Suppose for example that the covariance term is positive that is, inverse inflation covaries positively with the marginal rate of substitution, or inflation covaries positively with next period consumption. The inflation adjusted return on the nominal bond then covaries negatively with future consumption—the nominal bond is a good hedge (see section 5.2)—and the premium on the nominal bond is negative. Combining the Fisher equation with the term structure of real interest rates (see section 5.4), we may also derive the *term structure of nominal interest rates*.

9.1.2 Interest Parity and Nominal Exchange Rate

Condition (9.2) also relates returns on assets that are denominated in different currencies. Let $E_t(\epsilon^t)$ denote the *nominal exchange rate*, that is the price of one unit of foreign currency expressed in terms of domestic currency, and let a "star" denote foreign variables. The gross real rate of return on a foreign currency investment then equals

$$\frac{P_t(\epsilon^t)}{E_t(\epsilon^t)} I^{\star}_{t+1}(\epsilon^{t+1}) \frac{E_{t+1}(\epsilon^{t+1})}{P_{t+1}(\epsilon^{t+1})} = \frac{I^{\star}_{t+1}(\epsilon^{t+1})}{E_t(\epsilon^t)} \frac{E_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})},$$

because one unit of the good buys $P_t(\epsilon^t)$ units of domestic currency and thus, $P_t(\epsilon^t)/E_t(\epsilon^t)$ units of foreign currency; and next period, one unit of foreign currency purchases $E_{t+1}(\epsilon^{t+1})/P_{t+1}(\epsilon^{t+1})$ units of the good. We conclude that, when the household invests in the foreign-currency denominated bond, the equilibrium condition

$$1 = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{I^{\star}_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})} \frac{E_{t+1}(\epsilon^{t+1})}{E_t(\epsilon^t)} \right]$$

holds.

Consider a domestic- and a foreign-currency denominated bond with risk-free gross nominal rates of return $I_{t+1}(\epsilon^t)$ and $I^{\star}_{t+1}(\epsilon^t)$, respectively. When the household invests in both bonds, condition (9.2) and the discussion above imply the indifference condition

$$\mathbb{E}_t \left[\frac{m_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})} \right] I_{t+1}(\epsilon^t) = \mathbb{E}_t \left[\frac{m_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})} E_{t+1}(\epsilon^{t+1}) \right] \frac{I^{\star}_{t+1}(\epsilon^t)}{E_t(\epsilon^t)}.$$

Suppose first that it is known at date t that the exchange rate at date $t+1$ will equal $E_{t+1}(\epsilon^t)$, for example because the household engages in a forward contract. The indifference condition then reduces to the *covered interest parity* condition,

$$E_t(\epsilon^t) = E_{t+1}(\epsilon^t) \frac{I^{\star}_{t+1}(\epsilon^t)}{I_{t+1}(\epsilon^t)},$$

which states that the foreign-currency interest rate exceeds the domestic-currency yield by the rate at which the foreign currency depreciates between dates t and $t+1$. Intuitively, when investments in the two currencies expose the household to identical risks then the returns on the two strategies, expressed in domestic-currency terms, must be identical; otherwise the household would not hold both bonds.

Suppose next that the exchange rate at date $t+1$ is not known at date t. The indifference condition can then be expressed as

$$E_t(\epsilon^t) = \left(\mathbb{E}_t\left[E_{t+1}(\epsilon^{t+1})\right] + \mathbb{C}\mathrm{ov}_t\right)\frac{I^\star_{t+1}(\epsilon^t)}{I_{t+1}(\epsilon^t)},$$

where $\mathbb{C}\mathrm{ov}_t$ denotes a covariance term. When we abstract from this term, we arrive at the *uncovered interest parity* condition. It states that a foreign-currency interest rate premium is associated with an *expected* depreciation of the foreign currency.

According to the *law of one price* identical goods have the same price internationally as long as transportation costs and other impediments to trade are negligible. When all goods are tradeable and the law of one price applies then the real exchange rate equals one (see section 7.2) and (absolute) *purchasing power parity* holds that is, the nominal exchange rate reflects differences in international price levels:

$$E_t(\epsilon^t) = P_t(\epsilon^t)/P^\star_t(\epsilon^t).$$

When prices only respond sluggishly to macroeconomic shocks (for example because of price rigidities, see chapter 10) then purchasing power parity only holds in the long run, after price levels have adjusted. In contrast, interest parity always holds as it reflects asset market equilibrium conditions. The interplay between the two parities can give rise to non-monotone and volatile exchange rate dynamics.

Consider a domestic monetary expansion that temporarily lowers the interest rate and raises the domestic price level in the long run. Interest parity implies an expected exchange rate appreciation—a falling exchange rate—during the period of lower interest rates. At the same time, purchasing power parity implies that in the long run, the increased price level is accompanied by a higher exchange rate. These two requirements can only be jointly satisfied if the exchange rate *overshoots* in the short run to an even higher level than in the long run. Stated differently, with inflexible prices, a monetary expansion gives rise to a strong depreciation of the domestic currency which is partly undone during the transition to the new, long-run exchange rate.

9.2 Store of Value

So far, we have only analyzed the household's demand for non-monetary assets. Consider now the first-order condition with respect to $M_{t+1}(\epsilon^t)$ in the savings problem laid out in section 9.1. Since money pays no interest and cannot be shorted this is given by (recall that $m_{t+1}(\epsilon^{t+1})$ denotes the marginal rate of substitution)

$$1 \geq \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})\frac{1}{\Pi_{t+1}(\epsilon^{t+1})}\right],$$

with equality if the household chooses to hold money. Comparing this condition with condition (9.2) and abstracting for simplicity from risk, we conclude that the household holds intrinsically useless money only if $I_{t+1} = 1$ (the nominal interest rate equals zero) and $R_{t+1}^{-1} = \Pi_{t+1}$. The latter equality implies that money is a *bubble* whose price, $1/P_t$, grows at the real rate of interest. For such a bubble to be sustained in a rational expectations equilibrium the economy's growth rate must exceed the real interest rate (unless the return on the bubble is taxed) as is the case in an inefficient overlapping generations economy (see subsection 5.3.2). We turn to this case next.

9.2.1 Overlapping Generations

Consider an overlapping generations endowment economy. Absent money, the steady-state life-cycle consumption profile of each households equals its endowment profile, $(c_1, c_2) = (w_1, w_2)$, and the gross real interest rate equals the marginal rate of substitution,

$$R = \frac{u'(w_1)}{\beta u'(w_2)}.$$

Suppose that the gross population growth rate, ν, strictly exceeds R that is, the autarky equilibrium is inefficient.

Then there exists a Pareto superior equilibrium with valued, bubbly money that supports the efficient allocation with intergenerational transfers analyzed in subsection 3.2.7. In this superior equilibrium the old cohort creates M units of intrinsically useless money per capita of the young at date $t = 0$, which they sell to the young at price $1/P_0 < \infty$ in exchange for goods. At date $t = 1$, the then old households sell the money at price $1/P_1 = \nu/P_0$ to the young of cohort 1. Since there are ν young households per old household the money market clears, and since the price level has fallen by the factor ν, the gross real rate of return on money equals $P_0/P_1 = \nu$. The same pattern repeats in all subsequent periods.

The old at date $t = 0$ clearly benefit from this arrangement as they receive goods in exchange for the money they create. To see that the young benefit as well let $z \equiv M/P_0$ denote real balances at date $t = 0$. Under the monetary arrangement, welfare of a young household equals $u(w_1 - z) + \beta u(w_2 + z\nu)$; this exceeds $u(w_1) + \beta u(w_2)$ to the first order because

$$\begin{aligned}
&[u(w_1 - z) + \beta u(w_2 + z\nu)] - [u(w_1) + \beta u(w_2)] \\
&\qquad \approx -zu'(w_1) + z\beta\nu u'(w_2) > -zu'(w_1) + z\beta R u'(w_2) = 0.
\end{aligned}$$

Intuitively, money enables intergenerational transfers which help improve the allocation of consumption across cohorts and thus, over the life cycle (see subsection 3.2.7).

Note that there exists another equilibrium in which money is not valued. If the young at date t anticipate $P_{t+1} = \infty$ then money does not serve them as a store of value and accordingly, they do not give up resources in exchange for it, $P_t = \infty$. In an efficient economy, only the equilibrium without valued money exists.

9.2.2 Borrowing Constrained, Infinitely Lived Households

Consider next two groups of infinitely-lived representative households. Households in the first or "even" group receive a high endowment, $\bar{w}$, at even dates, starting at date $t = 0$, and a low endowment, $\underline{w}$, at odd dates; that is, their endowment stream is given by $\{\bar{w}, \underline{w}, \bar{w}, \ldots\}$. Households in the second or "odd" group, which has the same size, receive the same endowment stream shifted by one period, $\{\underline{w}, \bar{w}, \underline{w}, \ldots\}$. All households have the same time separable, strictly increasing and concave preferences. Let superscripts e and o denote variables of a typical household in the even or odd group, respectively.

Suppose first that markets are complete such that households of the two groups can trade with each other. In a competitive equilibrium a household in the even group solves

$$\max_{\{c_t^e, a_{t+1}^e\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^e) \text{ s.t. } c_t^e = w_t^e + a_t^e R_t - a_{t+1}^e, \;\; a_0^e = 0,$$

and a no-Ponzi-game condition. Here, c_t^e and a_t^e denote consumption and assets; R_t is the gross real interest rate; and w_t^e is high (low) in even (odd) periods. Households in the odd group solve a parallel program. The resource constraint is given by

$$c_t^e + c_t^o = w_t^e + w_t^o = \underline{w} + \bar{w}.$$

The households' Euler equations imply

$$\frac{u'(c_{t+1}^e)}{u'(c_t^e)} = \frac{u'(c_{t+1}^o)}{u'(c_t^o)}.$$

Since aggregate resources are constant over time this implies that households in the even (odd) group consume the constant amount c^e (c^o). Market clearing then requires $R_t = \beta^{-1}$. Note that households in the even (odd) group carry assets (liabilities) from even to odd periods, $a_1^e = -a_1^o = a_3^e = -a_3^o = \ldots > 0$. In contrast, members of both groups carry zero assets or liabilities from odd to even periods. At each date, $a_{t+1}^e + a_{t+1}^o = 0$. The complete markets equilibrium is Pareto optimal.

Suppose next that households face borrowing constraints that prevent strictly negative asset positions. A household in the even group then solves

$$\max_{\{c_t^e, a_{t+1}^e\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^e) \text{ s.t. } c_t^e = w_t^e + a_t^e R_t - a_{t+1}^e, \;\; a_0^e = 0, \;\; a_{t+1}^e \geq 0,$$

and its first-order conditions read

$$u'(c_t^e) = \beta R_{t+1} u'(c_{t+1}^e) + \mu_t^e, \;\; \mu_t^e a_{t+1}^e = 0,$$

where μ_t^e denotes the non-negative multiplier associated with the borrowing constraint. Households in the odd group solve a parallel program.

One stationary equilibrium in the incomplete markets environment is the autarkic equilibrium satisfying $c_t^e = w_t^e$, $c_t^o = w_t^o$, and $a_{t+1}^e = a_{t+1}^o = 0$ in all periods. The interest rate in this autarkic equilibrium is constant and satisfies

$$u'(\bar{w}) = \beta R u'(\underline{w}).$$

Another stationary equilibrium, which we focus on, supports a bubble. As in the autarkic equilibrium there is no lending and borrowing between members of the two groups. But unlike in the autarkic equilibrium, aggregate asset holdings are strictly positive, $a_{t+1}^e + a_{t+1}^o > 0$, and invested in a bubble. Households in the even (odd) group buy the bubble in even (odd) periods from members of the odd (even) group and sell it back in the subsequent period. When they buy the bubble, households satisfy the Euler equation subject to $\mu_t = 0$; and when they sell it they are borrowing constrained ($\mu_t > 0$). Letting a denote saving invested in the bubble the equilibrium satisfies

$$\begin{aligned} u'(\bar{w} - a) &= \beta R u'(\underline{w} + aR), \\ u'(\underline{w} + aR) &\geq \beta R u'(\bar{w} - a), \ \ a \geq 0. \end{aligned}$$

Goods market clearing requires aggregate consumption, $(\bar{w} - a) + (\underline{w} + aR)$, to equal the aggregate endowment, $\bar{w} + \underline{w}$, implying $R = 1$. (The autarkic equilibrium satisfies these equilibrium conditions with $a = 0$.)

Note that the bubbly equilibrium conditions require $1 \geq u'(\bar{c})/u'(\underline{c})$ where $\bar{c} \equiv \bar{w} - a$ and $\underline{c} \equiv \underline{w} + aR$ denote consumption of a household with high and low endowment, respectively. Conditional on the interest rate this restriction imposes an upper bound on the bubble; if the bubble exceeded the bound then households could smooth consumption and the constraint would no longer be binding. But this would require $R = \beta^{-1}$ which is ruled out by market clearing.

To motivate the borrowing constraints, note that the equilibrium conditions in the bubbly equilibrium exactly mimic the conditions in the overlapping generations model analyzed in subsection 9.2.1 (for $w_1 = \bar{w}$, $w_2 = \underline{w}$, $z = a$, and $\nu = 1$). Although households in the even and odd group are infinitely lived the binding borrowing constraints divide their budget sets into two-period components that resemble the budget sets of short-lived cohorts. Another motivation relates the borrowing constraints to spatially separated trading posts. Envision a turnpike that extends to infinity in both directions. Households in the even (odd) group travel in eastward (westward) direction along the turnpike and stop in each period at subsequent trading posts. A household therefore never meets another household more than once during its infinite lifetime, and no pair of households shares a trading partner in the past or the future. While this rules out any form of private lending and borrowing on a centralized market, it admits trading of a bubble.

In parallel to the overlapping generations economy the bubble in the *turnpike model* can be interpreted as real money balances. The gross interest rate on real balances in the turnpike model equals unity that is, the price level is constant over time, reflecting the constant population size.

9.3 Medium of Exchange

When intrinsically useless money is not a bubble that is, when its price grows at a rate smaller than the rate of interest, then it will only be held if it generates benefits in addition to serving as a store of value with low return. The key additional benefit is that money serves as a medium of exchange—it lubricates trade. We analyze this function of money in three models; the first emphasizes micro foundations, the other two tractability.

9.3.1 Matching Frictions

Consider an economy with a continuum of infinitely-lived households of measure one. Since households are anonymous, private lending and borrowing is ruled out and all trade is *quid pro quo*.

Households are risk neutral, and discount the future at factor β. They produce consumption goods at no cost and derive utility from the consumption of goods produced by a share $\gamma \in (0,1)$ of the other households. Conditional on meeting a potential trading partner, which occurs with probability $\theta \in (0,1)$, the probability of a double coincidence of wants therefore equals γ^2. Goods can be stored at no cost and are indivisible. Accepting a good in trade generates a small cost, ε, and consuming the preferred sort of good generates utility, u. Upon consuming, a household immediately produces a new good. Households thus always hold one good or, in a monetary equilibrium, alternatively one unit of money.

We conjecture that an equilibrium with valued money and constant price level, $P_t = 1$, exists. In such an equilibrium, a share $M \in (0,1)$ of households hold money, other households weakly prefer holding money over holding goods, and potential trading partners that hold a good therefore accept money with positive probability, $\hat{\mu} > 0$. Note that the medium of exchange role of money is coupled with a store of value role since money does not immediately change hands. Since accepting a good in trade has a small cost, no household with money buys a good except for own consumption.

Three types of transactions may occur: A household might exchange a good against another good or sell it against money, or a household with money may buy a good. Let μ denote the household's optimal probability of accepting money in exchange for goods—the household's best response to $\hat{\mu}$. The value of holding a good, V, and of holding money, W, then satisfy the Bellman equations

$$\begin{aligned} V &= \theta(1-M)\gamma^2(u-\varepsilon+\beta V) + \theta M\gamma \max_{\mu} \beta[\mu W + (1-\mu)V] \\ &\qquad +(1-\theta(1-M)\gamma^2 - \theta M\gamma)\beta V \\ &= \beta V + \theta\gamma\left((1-M)\gamma(u-\varepsilon) + M\beta \max_{\mu}[\mu(W-V)]\right), \\ W &= \theta(1-M)\gamma\hat{\mu}(u-\varepsilon+\beta V) + (1-\theta(1-M)\gamma\hat{\mu})\beta W \\ &= \beta W + \theta(1-M)\gamma\hat{\mu}(u-\varepsilon+\beta(V-W)). \end{aligned}$$

Intuitively, the value of holding the good derives from three sources: The possibility to exchange the good against the preferred consumption good; the possibility to exchange it against money if this improves the continuation value; and the continuation value if no trading partner is met. The value of holding money derives from the possibility that other households accept it in exchange for the preferred consumption good, and from the continuation value.

The first Bellman equation implies that the household's optimal response only depends on the difference between V and W: If $W > V$ then $\mu = 1$; if $W < V$ then $\mu = 0$; and if $W = V$ then $\mu \in [0,1]$. From the second Bellman equation, $\hat{\mu} = 0$ implies $W = 0$: When other households do not accept money then holding money has no value. Finally, combining the two Bellman equations yields the result that $\hat{\mu} < \gamma$ implies $W < V$ and thus, $\mu = 0$; $\hat{\mu} > \gamma$ implies $W > V$ and thus, $\mu = 1$; and $\hat{\mu} = \gamma$ implies $W = V$ and thus, $\mu \in [0,1]$. We conclude that conditional on $M \in (0,1)$ monetary equilibria exist: They require $\hat{\mu} = \mu \geq \gamma$.

The optimal monetary equilibrium under the veil of ignorance maximizes $(1-M)V + MW$ subject to $\hat{\mu} = \mu \geq \gamma$. Suppose that initially, a third party offers to exchange at most $\bar{M}$ units of money against the same number of goods. When households anticipate $\hat{\mu} > \gamma$ such that $W > V$ then all households that have the possibility to do so accept this offer. In the aggregate, this does not only lubricate trade, which has a positive effect on welfare, but it also reduces the number of goods available for consumption, which has a negative effect. This puts an upper bound on the optimal $\bar{M}$.

When households anticipate $\hat{\mu} = \gamma$ (such that $W = V$), in contrast, then the economy is better off in a non-monetary equilibrium because money does not lubricate trade beyond what the exchange of goods achieves.

9.3.2 Money in the Utility Function

Suppose next that both real balances, denoted by $z_t(\epsilon^t) \equiv M_{t+1}(\epsilon^t)/P_t(\epsilon^t)$, and "shopping time" provide transactions services and that the two can partly be substituted against each other. Specifically, we posit that purchasing $c_t(\epsilon^t)$ units of consumption goods requires $\ell(c_t(\epsilon^t), z_t(\epsilon^t))$ units of time, where function ℓ strictly increases in the transactions volume and decreases in real balances. Household utility increases in consumption and leisure, $\tilde{u}(c_t(\epsilon^t), x_t(\epsilon^t))$. Substituting the time constraint, $1 = x_t(\epsilon^t) + \ell(c_t(\epsilon^t), z_t(\epsilon^t))$, into the utility function $\tilde{u}$, we arrive at the reduced form *money-in-the-utility-function* specification

$$u(c_t(\epsilon^t), z_t(\epsilon^t)).$$

We assume that u is strictly increasing in consumption and, up to some point, in real balances (reflecting restrictions on the functions ℓ and $\tilde{u}$). We also assume that u is strictly concave with positive cross partial derivatives.

The household maximizes $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[u(c_t(\epsilon^t), z_t(\epsilon^t))]$ subject to (9.1), a no-Ponzi-game condition, and the constraint that money cannot be shorted. The first-order conditions are given by (9.2) for all assets other than money, and the additional condition

$$u_z(c_t(\epsilon^t), z_t(\epsilon^t)) = u_c(c_t(\epsilon^t), z_t(\epsilon^t)) - \beta \mathbb{E}_t \left[\frac{u_c(c_{t+1}(\epsilon^{t+1}), z_{t+1}(\epsilon^{t+1}))}{\Pi_{t+1}(\epsilon^{t+1})} \right]. \tag{9.3}$$

Intuitively, the marginal benefit of holding real balances for transactions services (on the left-hand side of the equation) is equalized with the marginal cost (on the right-hand side), namely marginal utility of consumption times the financial loss,

$$1 - \mathbb{E}_t[m_{t+1}(\epsilon^{t+1}) \Pi_{t+1}^{-1}(\epsilon^{t+1})],$$

that the household bears because money does not pay interest.

Combining conditions (9.2) and (9.3), we arrive at

$$\frac{u_z(c_t(\epsilon^t), z_t(\epsilon^t))}{u_c(c_t(\epsilon^t), z_t(\epsilon^t))} = 1 - \mathbb{E}_t[m_{t+1}(\epsilon^{t+1}) \Pi_{t+1}^{-1}(\epsilon^{t+1})] = \frac{i_{t+1}(\epsilon^t)}{I_{t+1}(\epsilon^t)}. \tag{9.4}$$

This represents a *money demand* function. It relates the demand for real balances, positively to the transactions volume (consumption), and negatively to the opportunity cost of holding money (the nominal interest rate). Stated differently, the condition relates the *velocity of money* (measured by the ratio $c_t(\epsilon^t)/z_t(\epsilon^t)$) positively to the nominal interest rate. The left equality in condition (9.4) also can be expressed as an asset pricing equation: Dividing condition (9.3) by $P_t(\epsilon^t)$ and iterating forward yields (absent a bubble)

$$\frac{u_c(c_t(\epsilon^t), z_t(\epsilon^t))}{P_t(\epsilon^t)} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \frac{u_z(c_{t+s}(\epsilon^{t+s}), z_{t+s}(\epsilon^{t+s}))}{P_{t+s}(\epsilon^{t+s})} \right].$$

The condition states that the market price of a unit of nominal balances, $1/P_t(\epsilon^t)$, equals the present discounted value of a dividend stream from nominal balances, normalized by the marginal utility of consumption. The dividend, in turn, is given by the transactions services provided by a unit of real balances, divided by the price level. Note that absent transaction services from money, the price of money equals zero. Unlike in the models studied in section 9.2 and subsection 9.3.1, however, money can have strictly positive value even if it is not valued in the infinite future.

In steady state, all real variables, including real balances are constant; nominal balances and the price level grow at a constant gross rate, μ say. Condition (9.2) thus implies $R = \beta^{-1}, I = \mu\beta^{-1}$: The money growth rate determines the inflation rate and affects the nominal interest rate but it has no effect on the real interest rate.

To derive the effect of μ on steady-state capital, income, and consumption we embed the household's problem in the representative agent general equilibrium environment studied in section 3.1. The condition $R = \beta^{-1}$ then determines the capital-labor ratio. Suppose first that the representative household's labor supply is fixed at unity (that is, the time endowment available for leisure and shopping time is measured net of working time). This implies that the representative household's capital and labor income are independent of μ. The budget constraint (9.1) thus reduces to

$c = f(k,1) - \delta k - (M_{t+1} - M_t)/P_t = f(k,1) - \delta k - (1-\mu^{-1})z$. We conclude that money is *neutral*: The steady-state allocation is not affected by a level change of nominal balances which is reflected in an equiproportional change of the price level. The growth rate of money, however, does affect the allocation (it reduces consumption).

Suppose, in addition, that the steady-state income loss that the household experiences from holding nominal balances, $(1-\mu^{-1})z$, is compensated by an equal-sized transfer income from the newly injected money that is, there is a "*helicopter drop*" of money. Or suppose alternatively that the household's real balances are negligible in size that is, the economy is in the "*cashless limit*". In this case, steady-state consumption is independent not only of the level of nominal balances but also of their growth rate, μ—money is *superneutral*: The growth rate of nominal balances does not affect the steady-state allocation except for real balances which fall if μ and thus, inflation and the nominal interest rate increase.

Outside of steady state, monetary neutrality prevails in the cashless limit provided that utility is separable between consumption and real balances such that the first-order conditions characterizing the household's "real" choices are independent of nominal balances and prices. If utility is non-separable, money generally is not neutral because real balances affect the marginal utility of consumption.

Suppose next that labor supply is endogenous. When working time enters preferences the interaction between the marginal disutility from working and the marginal utility from holding real balances creates a link between μ and steady-state labor supply and thus, consumption, even in the cashless limit. In general, money therefore is neither neutral nor superneutral.

9.3.3 Cash-in-Advance Constraint

Consider finally an environment in which only real balances provide transactions services. At the beginning of date t, asset markets open and the household rebalances its portfolio and replenishes money holdings, $M_{t+1}(\epsilon^t)$. Subsequently, it purchases consumption goods against money subject to the "*cash-in-advance constraint*"

$$\frac{M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} \geq c_t(\epsilon^t),$$

which stipulates that consumption purchases cannot exceed the real balances the household has acquired at the beginning of the period.

Nominal income in the form of cash, $W_{t-1}(\epsilon^{t-1})$, cannot be spent in the period when it is earned; the cash must rather be carried into the following period. The dynamic budget constraint (9.1) thus is replaced by the condition

$$\begin{aligned}\sum_l a^l_{t+1}(\epsilon^t) + \frac{\sum_j B^j_{t+1}(\epsilon^t) + M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} &= w_t(\epsilon^t) + \sum_l a^l_t(\epsilon^{t-1})R^l_t(\epsilon^t) \qquad (9.5)\\ &+\frac{\sum_j B^j_t(\epsilon^{t-1})I^j_t(\epsilon^t) + W_{t-1}(\epsilon^{t-1}) + M_t(\epsilon^{t-1}) - c_{t-1}(\epsilon^{t-1})P_{t-1}(\epsilon^{t-1})}{P_t(\epsilon^t)}.\end{aligned}$$

The three right-most terms in the constraint denote the real value of cash carried into the period which originates from two sources: Cash income generated at date $t-1$, and real balances acquired on the asset market at the beginning of date $t-1$ that were not spent on consumption.

The household maximizes $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[u(c_t(\epsilon^t))]$ subject to the cash-in-advance constraint, the budget constraint (9.5), and a no-Ponzi-game condition. The first-order conditions for this problem are given by (9.2) for all non-monetary assets, and the additional condition

$$\zeta_t(\epsilon^t) = u'(c_t(\epsilon^t)) - \beta \mathbb{E}_t \left[\frac{u'(c_{t+1}(\epsilon^{t+1}))}{\Pi_{t+1}(\epsilon^{t+1})} \right] = u'(c_t(\epsilon^t)) \frac{i_{t+1}(\epsilon^{t+1})}{I_{t+1}(\epsilon^{t+1})}, \tag{9.6}$$

where $\zeta_t(\epsilon^t)$ denotes the normalized multiplier associated with the cash-in-advance constraint.

Note the close parallels between the first-order conditions in the models with money in the utility function and with a cash-in-advance constraint, conditions (9.3) and (9.6). In both cases, the transactions-services benefit provided by money, which is given by the multiplier in the case with a cash-in-advance constraint, is equalized with the financial loss that the household bears when holding money. Note also that a strictly positive nominal interest rate implies that the multiplier $\zeta_t(\epsilon^t)$ is strictly positive as well such that real balances equal consumption that is, the velocity of money equals one. With a strictly positive nominal interest rate the budget constraint (9.5) thus reduces to

$$\begin{aligned}
&\sum_l a^l_{t+1}(\epsilon^t) + \frac{\sum_j B^j_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} = \\
&\qquad w_t(\epsilon^t) - c_t(\epsilon^t) + \sum_l a^l_t(\epsilon^{t-1}) R^l_t(\epsilon^t) + \frac{\sum_j B^j_t(\epsilon^{t-1}) I^j_t(\epsilon^t) + W_{t-1}(\epsilon^{t-1})}{P_t(\epsilon^t)}.
\end{aligned}$$

Naturally, the close parallels between conditions (9.3) and (9.6) extend to the asset pricing representation. Dividing the first equality in (9.6) by $P_t(\epsilon^t)$ and iterating forward yields (absent a bubble)

$$\frac{u'(c_t(\epsilon^t))}{P_t(\epsilon^t)} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \frac{\zeta_{t+s}(\epsilon^{t+s})}{P_{t+s}(\epsilon^{t+s})} \right].$$

The condition states that the price of a unit of nominal balances is strictly positive as long as the cash-in-advance constraint binds in the present or the future.

With a different timing convention the velocity of money may differ from unity. Suppose that the asset market opens at the end of a period, after uncertainty has been revealed and consumption goods have been purchased. Households then choose money holdings before they know the state of nature in the subsequent period when they spend the money, and depending on the state of nature, the cash-in-advance constraint may be slack.

Formally, with the modified timing convention the cash-in-advance constraint reads

$$\frac{M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} \geq c_t(\epsilon^t)$$

and the budget constraint is given by (9.1). The cash-in-advance constraint now drives a wedge between the marginal utility of consumption and the marginal utility of wealth, $\lambda_t(\epsilon^t)$, because holding money to purchase consumption goods in the subsequent period implies that interest income is foregone: $u'(c_t(\epsilon^t)) = \lambda_t(\epsilon^t) + \xi_t(\epsilon^t)$. As a consequence, condition (9.2) only holds in modified form, with the marginal utility of consumption replaced by the shadow value of income, $\lambda_t(\epsilon^t)$. Also, from the first-order condition for money holdings, condition (9.6) is replaced by

$$\frac{\beta \mathbb{E}_t \left[\xi_{t+1}(\epsilon^{t+1})\Pi_{t+1}^{-1}(\epsilon^{t+1})\right]}{\lambda_t(\epsilon^t)} = \frac{i_{t+1}(\epsilon^{t+1})}{I_{t+1}(\epsilon^{t+1})},$$

reflecting the fact that the benefit of investing in money—relaxing the cash-in-advance constraint in the subsequent period—is stochastic under the modified timing convention.

9.4 Bibliographic Notes

Since Aristotle (350 B.C.E.), scholars ponder the functions of money and its nature as a social convention. Fisher (1896) discusses the effect of inflation on interest rates. Keynes (1923) and Cassel (1918) discuss the interest parity and purchasing parity conditions, respectively. The overshooting model is due to Dornbusch (1976).

Samuelson (1958), Shell (1971), Gale (1973), and Wallace (1980) analyze money in the overlapping generations model; see also Kehoe and Levine (1985) on indeterminacy and Grandmont (1985) on endogenous cycles in that model. The turnpike model is due to Townsend (1980), see also Woodford (1990).

The model in section 9.3.1 follows Kiyotaki and Wright (1993). Sidrauski (1967) analyzes the effects of money in the utility function. Baumol (1952) and Tobin (1956) study the demand for money when the latter serves transaction purposes and portfolio rebalancing is costly. Saving (1971) relates the money-in-the-utility-function specification to a shopping time specification, see also Feenstra (1986) and Croushore (1993).

The cash-in-advance constraint is due to Clower (1967); Grandmont and Younes (1972), Lucas (1980; 1982), and Svensson (1985) analyze models with cash-in-advance constraints. Lucas and Stokey (1987) study a model with "*cash goods*" and "*credit goods*," where only purchases of the former are subject to a cash-in-advance constraint; since households substitute between the two categories of goods the velocity of money is interest elastic.

Walsh (2017) provides an overview over the literature.

Beyond the material covered in the chapter, Bewley (1980) analyzes money demand in a stochastic incomplete markets environment where households hold money to self insure. Kiyotaki and Wright (1989) study the emergence of specific commodities as commodity monies. Lagos and Wright (2005) develop a tractable matching model of monetary exchange. Rocheteau and Nosal (2017) review the "new monetarist" approach to monetary economics and finance in the tradition of Kiyotaki and Wright (1989; 1993) and Lagos and Wright (2005).

Chapter 10

Price Setting and Price Rigidity

Firms and other economic agents often *set* prices rather than taking them as given. To account for this, we replace the assumption of "small" firms that behave competitively by the alternative assumption of monopolistic competition between firms that produce less than perfect substitutes. Once we model price setting, we can also take the additional step of introducing infrequent price adjustment and thus, *price rigidity*.

We analyze the distortions and aggregate demand externalities associated with price setting as well as the output effects of nominal disturbances when prices are sticky.

10.1 Price Setting

To model price setting we introduce *monopolistic competition*. We assume that some firms have pricing power because there are no perfect substitutes for the differentiated goods they produce. Facing a *finite* price elasticity of demand these firms charge a price above marginal cost.

Formally, we distinguish between a continuum of monopolistically competitive *intermediate good* producers and competitive *final good* producers. Intermediate good producer j employs a constant returns to scale technology to produce intermediate good variety j, $y_t(j,\epsilon^t)$, which it sells at price $P_t(j,\epsilon^t)$ to final good producers. The latter combine the varieties to produce the final good, $y_t(\epsilon^t)$. Their production function is given by

$$y_t(\epsilon^t) = \left(\int_0^1 y_t(j,\epsilon^t)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{10.1}$$

where $\varepsilon > 1$, reflecting limited substitutability between varieties (see the discussion in subsection 2.2.3).

A final good producer takes the intermediate good prices, $\{P_t(j,\epsilon^t)\}_{i\in[0,1]}$, as given and maximizes final good production subject to a given level of input costs, Z say. The

Lagrangian associated with this program reads

$$\mathcal{L} = \left(\int_0^1 y_t(j, \epsilon^t)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda \left(\int_0^1 y_t(j, \epsilon^t) P_t(j, \epsilon^t) dj - Z \right).$$

Note that the multiplier, λ, equals the inverse of the final good price, $P_t(\epsilon^t)$: Marginally raising Z increases the optimal $y_t(\epsilon^t)$ by λ; since $Z = y_t(\epsilon^t)P_t(\epsilon^t)$, this implies $\lambda^{-1} = P_t(\epsilon^t)$.

The first-order condition with respect to $y_t(j, \epsilon^t)$,

$$y_t(j, \epsilon^t)^{-\frac{1}{\varepsilon}} y_t(\epsilon^t)^{\frac{1}{\varepsilon}} = \lambda P_t(j, \epsilon^t),$$

thus can be expressed as

$$y_t(j, \epsilon^t) = \left(\frac{P_t(j, \epsilon^t)}{P_t(\epsilon^t)} \right)^{-\varepsilon} y_t(\epsilon^t). \tag{10.2}$$

Condition (10.2) determines the relative demand for varieties; higher substitutability (higher ε) renders the relative demand more price elastic. Substituting condition (10.2) into the spending constraint yields

$$P_t(\epsilon^t) y_t(\epsilon^t) = Z = \int_0^1 y_t(j, \epsilon^t) P_t(j, \epsilon^t) dj = y_t(\epsilon^t) P_t(\epsilon^t)^{\varepsilon} \int_0^1 P_t(j, \epsilon^t)^{1-\varepsilon} dj,$$

which reduces to

$$P_t(\epsilon^t) = \left(\int_0^1 P_t(j, \epsilon^t)^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}. \tag{10.3}$$

Condition (10.3) defines the final good price index.

Turning to the intermediate good firms, a producer of variety j maximizes profits subject to the demand curve represented by (10.2). Since the intermediate goods technology exhibits constant returns to scale, the producer's marginal and average cost is independent of the quantity produced and thus, the same for all intermediate good producers. Letting $\psi_t(\epsilon^t)$ denote marginal cost expressed in units of the final good, an intermediate good producer maximizes

$$\left(\frac{P_t(j, \epsilon^t)}{P_t(\epsilon^t)} - \psi_t(\epsilon^t) \right) y_t(j, \epsilon^t) \text{ s.t. } (10.2),$$

taking $P_t(\epsilon^t)$ and $y_t(\epsilon^t)$ as given. The first-order condition of this program with respect to $P_t(j, \epsilon^t)$ reduces to

$$P_t(j, \epsilon^t) = \psi_t(\epsilon^t) P_t(\epsilon^t) \frac{\varepsilon}{\varepsilon - 1},$$

that is, an intermediate good producer sets its price as a *markup* over nominal marginal cost. Higher substitutability (higher ε) reduces pricing power and implies a lower markup. In the limit where $\varepsilon \to \infty$, the markup disappears and price equals nominal marginal cost. In an equilibrium where all producers adjust their prices every period, $P_t(j, \epsilon^t) = P_t(\epsilon^t)$ and $1 = \psi_t(\epsilon^t)\varepsilon/(\varepsilon - 1)$.

Monopolistic competition depresses output below the first-best level. If the intermediate and final good producers were to jointly optimize they would maximize final good output net of input costs for intermediate good production,

$$\left(\int_0^1 y_t(j, \epsilon^t)^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 y_t(j, \epsilon^t)\psi_t(\epsilon^t) dj.$$

Accordingly, they would satisfy the first-order condition

$$y_t(j, \epsilon^t)^{-\frac{1}{\varepsilon}} y_t(\epsilon^t)^{\frac{1}{\varepsilon}} = \psi_t(\epsilon^t).$$

Efficiency requires that the marginal product of an intermediate good in final good production (represented by the term on the left-hand side of the equation) equals that good's marginal cost (on the right-hand side). In equilibrium, this efficiency condition is violated because the marginal product exceeds marginal cost by the markup (this follows from condition (10.2) and optimal price setting). The *wedge* arises in equilibrium because intermediate good producers maximize profits without internalizing that these profits constitute outlays for final good producers.

Since marginal costs are constant across varieties, efficiency also requires that the same quantity of each variety is produced. In the equilibrium with flexible prices that we have studied so far, this condition is satisfied. When intermediate good firms charge different prices and thus, supply different quantities, however, then the second efficiency condition is (also) violated in equilibrium. We will return to this point in section 10.2.

Suppose that intermediate good producers are subject to a small *nominal rigidity*, for instance because they incur a small "*menu cost*" when publishing an updated price list after a price adjustment. Consider a small shock which implies that the actual price charged by a producer deviates from the profit maximizing one. The envelope condition implies that the profit loss from a small price deviation is of second order. How large a deviation the producer accepts before adjusting its price and incurring the menu cost depends on the extent of *real rigidity*: Smaller costs of satisfying demand at unchanged prices—a more elastic supply—make the producer accept larger deviations. Small nominal rigidities therefore can induce a producer to adjust its price only infrequently as long as the degree of real rigidity is sufficiently large.

The second-order loss for an individual producer from not adjusting its price contrasts with first order, potentially large gains for the economy as a whole. Consider again a small shock which implies that the actual prices charged by intermediate good producers lie slightly below the profit maximizing ones (conditional on $P_t(\epsilon^t)$ and $y_t(\epsilon^t)$). When all intermediate good producers refrain from raising their prices, for example because of small nominal (and large real) rigidities, then the final good becomes cheaper and the quantity of final good output rises. This has *first-order* benefits for all producers because equilibrium output was inefficiently low to start with.

The difference between the second-order private losses from not raising one's price and the associated first-order social gains reflects an *aggregate demand externality*. Since the prices set by individual producers enter the final good price index they have an indirect effect on aggregate demand and thus, on output. Moreover, since the firm sector produces at an inefficient level (due to the markups) a positive indirect effect on output has a first-order welfare effect—the demand externality—which individual producers do not internalize.

10.2 Price Rigidity

When it is costly to reset prices then the benefit from price adjustment must be sufficiently large for a producer to opt for a reset. A *state dependent* price setting strategy conditions the timing and extent of price adjustment on the producer's state, in parallel to some of the investment strategies studied in section 8.1. With a *time dependent* strategy, prices are reset after specific time intervals. We focus on a tractable specification of time dependent pricing, without modeling the costs of price adjustment or other determinants of the time intervals.

Suppose that after exogenously determined random durations, the intermediate good producers analyzed in section 10.1 reset their prices optimally. In each period, a fraction $\theta \in (0, 1)$ of producers must keep their prices unchanged while a fraction $1 - \theta$ is "allowed" to reset prices optimally. The probability that a producer resets its price for the first time after h periods thus equals $\theta^{h-1}(1 - \theta)$, and the expected duration of a price equals $(1 - \theta)^{-1}$.

Producers are committed to supply whatever quantity is demanded at the given price. Since a producer may not be able to reset its price in the following period the firm's program is a dynamic one, in contrast to the program studied in section 10.1. Let $m_{t,t+h}(\epsilon^{t+h})$ denote the marginal rate of substitution between date t, history ϵ^t and date $t + h$, history ϵ^{t+h} which the producer uses for discounting. Taking the sequences $\{y_{t+h}(\epsilon^{t+h}), \psi_{t+h}(\epsilon^{t+h}), P_{t+h}(\epsilon^{t+h}), m_{t,t+h}(\epsilon^{t+h})\}_{h=0}^{\infty}$ as given, the producer chooses $P_t(j, \epsilon^t)$ to maximize (using condition (10.2))

$$\mathbb{E}_t \sum_{h=0}^{\infty} \theta^h m_{t,t+h}(\epsilon^{t+h}) \left(\frac{P_t(j, \epsilon^t)}{P_{t+h}(\epsilon^{t+h})} - \psi_{t+h}(\epsilon^{t+h}) \right) \left(\frac{P_t(j, \epsilon^t)}{P_{t+h}(\epsilon^{t+h})} \right)^{-\varepsilon} y_{t+h}(\epsilon^{t+h}).$$

Letting $\rho_t(j, \epsilon^t) \equiv P_t(j, \epsilon^t)/P_t(\epsilon^t)$ denote the relative price chosen by producer j when a reset is possible, the program can equivalently be expressed as

$$\mathbb{E}_t \sum_{h=0}^{\infty} \theta^h m_{t,t+h}(\epsilon^{t+h}) \left(\frac{\rho_t(j, \epsilon^t) P_t(\epsilon^t)}{P_{t+h}(\epsilon^{t+h})} - \psi_{t+h}(\epsilon^{t+h}) \right) \left(\frac{\rho_t(j, \epsilon^t) P_t(\epsilon^t)}{P_{t+h}(\epsilon^{t+h})} \right)^{-\varepsilon} y_{t+h}(\epsilon^{t+h}).$$

Differentiating with respect to $\rho_t(j, \epsilon^t)$ yields the first-order condition

$$\mathbb{E}_t \sum_{h=0}^{\infty} \theta^h m_{t,t+h}(\epsilon^{t+h}) y_{t+h}(\epsilon^{t+h}) \left(\frac{P_t(\epsilon^t)}{P_{t+h}(\epsilon^{t+h})} \right)^{1-\varepsilon} \times \\ \left(\rho_t(j, \epsilon^t) - \psi_{t+h}(\epsilon^{t+h}) \frac{\varepsilon}{1-\varepsilon} \frac{P_{t+h}(\epsilon^{t+h})}{P_t(\epsilon^t)} \right) = 0. \quad (10.4)$$

Intuitively, the producer aims at achieving the desired markup of $\varepsilon/(\varepsilon - 1)$ "on average" over the duration of the set price. Charging a different markup is costly as it forces to satisfy demand at a price that is not profit maximizing. When prices can be reset every period, $\theta \to 0$, we recover the solution of the static program, $\rho_t(j, \epsilon^t) = \psi_t(\epsilon^t)\varepsilon/(\varepsilon - 1)$.

As mentioned in section 10.1, asynchronized price adjustments induce *price dispersion* and the associated quantity differences across intermediate good varieties imply a waste of resources. To see this, recall from condition (10.2) that

$$\int_0^1 y_t(j, \epsilon^t) dj = y_t(\epsilon^t) \int_0^1 \left(\frac{P_t(j, \epsilon^t)}{P_t(\epsilon^t)} \right)^{-\varepsilon} dj \equiv y_t(\epsilon^t) \Delta_t(\epsilon^t),$$

where $\Delta_t(\epsilon^t)$ is a measure of price dispersion. When all producers set the same price then $\Delta_t(\epsilon^t) = 1$ and $\int_0^1 y_t(j, \epsilon^t) dj = y_t(\epsilon^t)$. Otherwise, $\Delta_t(\epsilon^t) > 1$ and $\int_0^1 y_t(j, \epsilon^t) dj > y_t(\epsilon^t)$. Intuitively, price differences imply that final good producers demand—and intermediate good producers supply—high (low) quantities of relatively cheap (expensive) varieties whose marginal contribution to final good output is relatively small (large).

When producers adjust their prices infrequently the *rigidity* or *stickiness* of prices at the firm level is reflected in aggregate price level rigidity: the distribution of prices in the previous period is a state variable. This distribution is non-degenerate since producers reset at different times. However, since all firms that do reset their price in a given period reset it to the same level, $\rho_t(j, \epsilon^t) = \rho_t(\epsilon^t)$, the aggregate price level only depends on this common choice as well as on the aggregate price level in the previous period. From condition (10.3),

$$P_t(\epsilon^t)^{1-\varepsilon} = \left(\int_0^1 P_t(j, \epsilon^t)^{1-\varepsilon} dj \right) = \left(\theta \int_0^1 P_{t-1}(j, \epsilon^{t-1})^{1-\varepsilon} dj + (1-\theta)(\rho_t(\epsilon^t) P_t(\epsilon^t))^{1-\varepsilon} \right).$$

Dividing by $P_t(\epsilon^t)^{1-\varepsilon}$, this yields a relationship between gross inflation and the optimally chosen relative price,

$$1 = \theta \Pi_t(\epsilon^t)^{\varepsilon-1} + (1-\theta)\rho_t(\epsilon^t)^{1-\varepsilon}. \quad (10.5)$$

10.3 Price Rigidity in General Equilibrium

To analyze the implications of price setting and price rigidity in general equilibrium we augment the real business cycle model studied in section 6.3.1 by a monopolistically competitive intermediate goods sector.

10.3.1 Firms

Following the analysis in sections 10.1 and 10.2 competitive final good producers purchase varieties from monopolistically competitive intermediate good firms which reset prices after exogenously determined, random durations. The intermediate good producers rent capital and labor from households and employ the constant returns to scale technology f. Producer j's output is given by

$$y_t(j,\epsilon^t) = A_t(\epsilon^t) f(k_t(j,\epsilon^t), L_t(j,\epsilon^t)),$$

where $A_t(\epsilon^t)$, $k_t(j,\epsilon^t)$, and $L_t(j,\epsilon^t)$ denote productivity, capital, and labor input.

Intermediate good producers choose the minimum-cost capital-labor ratio. Due to constant returns to scale this ratio is the same for all producers. The Lagrangian associated with the cost minimization program is given by

$$\mathcal{L} = r_t(\epsilon^t)k_t(j,\epsilon^t) + w_t(\epsilon^t)L_t(j,\epsilon^t) - \lambda\left(A_t(\epsilon^t)f(k_t(j,\epsilon^t), L_t(j,\epsilon^t)) - y_t(j,\epsilon^t)\right),$$

where $r_t(\epsilon^t)$ and $w_t(\epsilon^t)$ denote the rental rate and the wage, respectively. Differentiating with respect to the factor inputs and positing a Cobb-Douglas production function with capital share α, we find

$$\psi_t(\epsilon^t) = \frac{1}{A_t(\epsilon^t)}\left(\frac{r_t(\epsilon^t)}{\alpha}\right)^{\alpha}\left(\frac{w_t(\epsilon^t)}{1-\alpha}\right)^{1-\alpha}, \tag{10.6}$$

$$\frac{k_t(j,\epsilon^t)}{L_t(j,\epsilon^t)} = \frac{\alpha}{1-\alpha}\frac{w_t(\epsilon^t)}{r_t(\epsilon^t)}. \tag{10.7}$$

Condition (10.6) gives the marginal costs of a producer that employs the minimum-cost capital-labor ratio given in condition (10.7).

While intermediate goods producers always choose the minimum-cost factor input combination they do not necessarily supply the profit maximizing quantity of output. This is a consequence of the fact that firms adjust prices only infrequently and supply whatever quantity is demanded at those prices. Conditional on this constraint, however, firms do optimally set prices. The corresponding equilibrium conditions are given by conditions (10.4) and (10.5).

10.3.2 Households

The representative household maximizes $\sum_{t=0}^{\infty}\beta^t \mathbb{E}_0[u(c_t(\epsilon^t), x_t(\epsilon^t))]$ where u is strictly increasing and concave, and $c_t(\epsilon^t)$ and $x_t(\epsilon^t)$ denote consumption and leisure. The household also demands real balances; money demand is decreasing in the nominal interest rate. For simplicity, we assume that money holdings are negligible as far as their effect on the household budget constraint or marginal utility of consumption and leisure is concerned, for example because the economy is in the "cashless limit" (see subsection 9.3.2).

The household works, consumes, and saves. Its portfolio includes capital, $k_{t+1}(\epsilon^t)$, nominal bonds, $B_{t+1}(\epsilon^t)$, and negligible real balances. The gross real rate of return on capital equals $R_{t+1}(\epsilon^{t+1}) \equiv 1-\delta+r_{t+1}(\epsilon^{t+1})$ where δ denotes the depreciation rate; and the nominally riskfree gross rate of return on bonds equals $I_{t+1}(\epsilon^t)$. The household's dynamic budget constraint is given by condition (9.1), with just one type of nominal bond, no real bonds, negligible real balances, and firms profits as an additional source of household income.

Maximizing with respect to $k_{t+1}(\epsilon^t)$, $B_{t+1}(\epsilon^t)$, $c_t(\epsilon^t)$, and $x_t(\epsilon^t)$ and simplifying yields the first-order conditions (see section 6.3.1 and condition (9.2))

$$\begin{aligned}
u_c(c_t(\epsilon^t),x_t(\epsilon^t)) &= \beta\mathbb{E}_t\left[u_c(c_{t+1}(\epsilon^{t+1}),x_{t+1}(\epsilon^{t+1}))R_{t+1}(\epsilon^{t+1})\right], && (10.8)\\
u_c(c_t(\epsilon^t),x_t(\epsilon^t)) &= \beta\mathbb{E}_t\left[u_c(c_{t+1}(\epsilon^{t+1}),x_{t+1}(\epsilon^{t+1}))\frac{I_{t+1}(\epsilon^t)}{\Pi_{t+1}(\epsilon^{t+1})}\right], && (10.9)\\
u_x(c_t(\epsilon^t),x_t(\epsilon^t)) &= w_t(\epsilon^t)u_c(c_t(\epsilon^t),x_t(\epsilon^t)). && (10.10)
\end{aligned}$$

10.3.3 Market Clearing

Market clearing requires the aggregate supplies and demands for labor and capital to be equalized:

$$\begin{aligned}
1-x_t(\epsilon^t) &= \int_0^1 L_t(j,\epsilon^t)dj,\\
k_t(\epsilon^{t-1}) &= \int_0^1 k_t(j,\epsilon^t)dj.
\end{aligned}$$

(Note that the aggregate supply of capital is predetermined, in contrast to the aggregate demand.) Firm optimality and labor market clearing implies that the capital-labor ratio of each intermediate good producer equals $k_t(\epsilon^{t-1})/L_t(\epsilon^t)$ with $L_t(\epsilon^t) \equiv 1-x_t(\epsilon^t)$. Due to constant returns to scale this implies

$$\int_0^1 y_t(j,\epsilon^t)dj = A_t(\epsilon^t)\int_0^1 f(k_t(j,\epsilon^t),L_t(j,\epsilon^t))dj = A_t(\epsilon^t)f(k_t(\epsilon^{t-1}),L_t(\epsilon^t)).$$

Accordingly, the economy's resource constraint satisfies

$$c_t(\epsilon^t)+k_{t+1}(\epsilon^t)-k_t(\epsilon^{t-1})(1-\delta) = y_t(\epsilon^t) = \frac{1}{\Delta_t(\epsilon^t)}A_t(\epsilon^t)f(k_t(\epsilon^{t-1}),L_t(\epsilon^t)), \quad (10.11)$$

where $\Delta_t(\epsilon^t)$ denotes the price dispersion measure defined in section 10.2. Price dispersion has the same effect on the resource constraint as lower productivity.

The nominal bond is in zero net supply, $B_t(\epsilon^t)=0$.

10.3.4 General Equilibrium

To simplify the analysis, we linearize the equilibrium conditions about the steady state associated with constant productivity and zero inflation, $A=\Pi=1$. Other, non-stationary equilibria may exist but we disregard them for now. Let a circumflex denote infinitesimal relative deviations from the steady state.

Since in steady state, $m_{t,t+h}(\epsilon^{t+h}) = \beta^h$, $\psi\varepsilon/(1-\varepsilon) = 1$, and $\rho(j) = 1$, the price setting condition (10.4) implies

$$\sum_{h=0}^{\infty} \theta^h \beta^h \mathbb{E}_t \left[\hat{\rho}_t(\epsilon^t) - \hat{\psi}_{t+h}(\epsilon^{t+h}) - \hat{P}_{t+h}(\epsilon^{t+h}) + \hat{P}_t(\epsilon^t) \right] = 0$$

or

$$\hat{\rho}_t(\epsilon^t) + \hat{P}_t(\epsilon^t) = (1-\theta\beta) \sum_{h=0}^{\infty} \theta^h \beta^h \mathbb{E}_t \left[\hat{\psi}_{t+h}(\epsilon^{t+h}) + \hat{P}_{t+h}(\epsilon^{t+h}) \right].$$

That is, the steady-state deviation of producer j's newly set price (on the left-hand side) and the discounted sum of deviations of nominal marginal costs (on the right-hand side) are equalized. We may re-express the previous condition as a first-order difference equation,

$$\hat{\rho}_t(\epsilon^t) + \hat{P}_t(\epsilon^t) = (1-\theta\beta) \left(\hat{\psi}_t(\epsilon^t) + \hat{P}_t(\epsilon^t) \right) + \theta\beta \mathbb{E}_t \left[\hat{\rho}_{t+1}(\epsilon^{t+1}) + \hat{P}_{t+1}(\epsilon^{t+1}) \right],$$

for which we seek a stationary solution. Since the inflation rate, $\pi_{t+1}(\epsilon^{t+1})$, equals $\hat{P}_{t+1}(\epsilon^{t+1}) - \hat{P}_t(\epsilon^t)$, this equation can equivalently be written as

$$\hat{\rho}_t(\epsilon^t) = (1-\theta\beta)\hat{\psi}_t(\epsilon^t) + \theta\beta \mathbb{E}_t \left[\hat{\rho}_{t+1}(\epsilon^{t+1}) + \pi_{t+1}(\epsilon^{t+1}) \right].$$

From condition (10.5), the linearized law of motion for inflation is given by $(1-\theta)\hat{\rho}_t(\epsilon^t) = \theta\hat{\Pi}_t(\epsilon^t) = \theta\pi_t(\epsilon^t)$. Combining this condition with the difference equation derived above yields the *New Keynesian Phillips curve*,

$$\pi_t(\epsilon^t) = \frac{(1-\theta)(1-\theta\beta)}{\theta} \hat{\psi}_t(\epsilon^t) + \beta \mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})]. \tag{10.12}$$

The Phillips curve relates inflation to marginal costs because firms target a desired markup, and to future inflation because firms take into account that they might not be able to reset their price quickly. With more price rigidity or patience (a higher θ or β) inflation responds less strongly to marginal costs.

Since there are no first-order effects on the price dispersion index, $\hat{\Delta}_t(\epsilon^t) = 0$, the resource constraint (10.11) can be expressed as

$$\begin{aligned} \hat{c}_t(\epsilon^t)\frac{c}{y} + \left(\hat{k}_{t+1}(\epsilon^t) - (1-\delta)\hat{k}_t(\epsilon^{t-1}) \right) \frac{k}{y} &= \\ \hat{y}_t(\epsilon^t) &= \hat{A}_t(\epsilon^t) + \alpha\hat{k}_t(\epsilon^{t-1}) + (1-\alpha)\hat{L}_t(\epsilon^t). \end{aligned} \tag{10.13}$$

From condition (10.7),

$$r_t(\epsilon^t) - r = r \left(\hat{w}_t(\epsilon^t) + \hat{L}_t(\epsilon^t) - \hat{k}_t(\epsilon^{t-1}) \right). \tag{10.14}$$

Combining conditions (10.6) and (10.7) and using the resource constraint (10.13) to simplify yields

$$\hat{\psi}_t(\epsilon^t) = \hat{w}_t(\epsilon^t) + \hat{L}_t(\epsilon^t) - \hat{y}_t(\epsilon^t). \tag{10.15}$$

Finally, turning to the household's first-order conditions, equations (10.8)–(10.10), we assume that preferences are additively separable in consumption and leisure. Defining the positive inverse elasticities $\sigma \equiv -u_{cc}(c,x)c/u_c(c,x)$ and $\eta \equiv -u_{xx}(c,x)(1-x)/u_x(c,x)$, we find

$$\begin{aligned} -\sigma\hat{c}_t(\epsilon^t) &= -\sigma\mathbb{E}_t\left[\hat{c}_{t+1}(\epsilon^{t+1})\right] + \mathbb{E}_t[r_{t+1}(\epsilon^{t+1}) - r], && (10.16)\\ -\sigma\hat{c}_t(\epsilon^t) &= -\sigma\mathbb{E}_t\left[\hat{c}_{t+1}(\epsilon^{t+1})\right] + \mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - (r-\delta)\right], && (10.17)\\ \eta\hat{L}_t(\epsilon^t) &= \hat{w}_t(\epsilon^t) - \sigma\hat{c}_t(\epsilon^t). && (10.18) \end{aligned}$$

(To derive condition (10.16), we approximate $(1-\delta+r_{t+1}(\epsilon^{t+1}))/(1-\delta+r)-1$ by $(1+r_{t+1}(\epsilon^{t+1})-r)-1$. A parallel approximation implies condition (10.17).)

Equation (10.17)—the *dynamic IS-curve*—relates contemporaneous activity (consumption) to the nominal interest rate, similarly to a Keynesian consumption function. Unlike in the standard Keynesian model, however, the dynamic IS-curve also contains expectation terms.

Conditions (10.12)–(10.18) constitute a system of eight equations in eight endogenous variables $(\pi_t(\epsilon^t), \hat{\psi}_t(\epsilon^t), \hat{c}_t(\epsilon^t), \hat{k}_{t+1}(\epsilon^t), \hat{y}_t(\epsilon^t), \hat{L}_t(\epsilon^t), r_t(\epsilon^t), \hat{w}_t(\epsilon^t))$ and two exogenous variables $(\hat{A}_t(\epsilon^t), i_{t+1}(\epsilon^t))$.

10.3.5 Analysis

To build intuition for the workings of the model consider first the case with perfectly flexible prices, $\theta = 0$. As discussed earlier, relative prices and real marginal costs are constant in this case,

$$\rho_t(j,\epsilon^t) = 1 = \psi_t(\epsilon^t)\varepsilon/(\varepsilon-1),$$

and the Phillips curve (10.12) does not constitute an equilibrium condition. Since $\hat{\psi}_t(\epsilon^t) = 0$ it follows from conditions (10.14) and (10.15) that $\hat{y}_t(\epsilon^t) = \hat{w}_t(\epsilon^t) + \hat{L}_t(\epsilon^t) = \hat{k}_t(\epsilon^{t-1}) + \hat{r}_t(\epsilon^t)$. The resource constraint (10.13) is unchanged.

The modified linearized equilibrium conditions thus parallel the conditions in the real business cycle model (see subsection 6.3.1), with two differences. First, the modified conditions characterize deviations from a distorted (due to monopolistic competition) steady state, rather than from the non-distorted steady state in the real business cycle model. Second, the modified conditions include one equation, condition (10.17), that is not present in the real business cycle model. This additional equation combined with condition (10.16) yields the Fisher equation,

$$\mathbb{E}_t\left[r_{t+1}(\epsilon^{t+1})\right] - \delta = i_{t+1}(\epsilon^t) - \mathbb{E}_t\left[\pi_{t+1}(\epsilon^{t+1})\right],$$

which equates the expected real interest rate (on the left-hand side) and the inflation adjusted nominal rate (on the right-hand side). Conditional on the rental rates on capital associated with the equilibrium allocation as well as on a nominal interest rate the condition pins down expected (not realized) inflation. In conclusion, the model with flexible prices resembles the real business cycle model; the allocation in the model is independent of inflation; the nominal interest rate pins down expected inflation; and with risk, actual *inflation* and thus, the *price level* are *indeterminate*.

With price rigidity, $\theta > 0$, this *classical dichotomy* between the real allocation and nominal variables breaks down. On the one hand, the Phillips curve (10.12) links inflation to marginal costs and thus, the allocation. On the other, the nominal interest rate affects the allocation. Because the Phillips curve constrains inflation, a change in $i_{t+1}(\epsilon^t)$ translates into a change of the inflation adjusted nominal interest rate. From the Euler equations this induces consumption and investment responses which in turn feed into wages, rental rates, and marginal costs. In the long run, consumption and capital return to their steady-state values and the classical dichotomy holds.

Independently of the degree of price rigidity, the nominal interest rate is linked to a money market equilibrium condition which is left implicit because we have not spelled out the household's money demand.[1] As long as this demand is interest elastic a (possibly tiny) adjustment in the supply of real balances triggers a change of the market clearing nominal interest rate. The no-arbitrage relations underlying the term structure of interest rates also hold independently of the degree of price stickiness. Anticipated future changes of nominal interest rates affect the nominal term structure and, over horizons over which the classical dichotomy fails to hold, also the real term structure. The Fisher equation links current and expected future nominal and real rates as well as inflation.

When there is no capital, $\alpha = k_t(\epsilon^{t-1}) = 0$, the model simplifies substantially. Conditions (10.14) and (10.16) are superfluous in this case, and the resource constraint, intratemporal first-order condition, and marginal cost condition, respectively, reduce to

$$\begin{aligned}
\hat{c}_t(\epsilon^t) &= \hat{y}_t(\epsilon^t) = \hat{A}_t(\epsilon^t) + \hat{L}_t(\epsilon^t), \\
\hat{w}_t(\epsilon^t) &= (\eta+\sigma)\hat{y}_t(\epsilon^t) - \eta\hat{A}_t(\epsilon^t), \\
\hat{\psi}_t(\epsilon^t) &= (\eta+\sigma)\hat{y}_t(\epsilon^t) - (\eta+1)\hat{A}_t(\epsilon^t).
\end{aligned}$$

The two core equations, the Phillips and dynamic IS-curve respectively, are given by

$$\begin{aligned}
\pi_t(\epsilon^t) &= \frac{(1-\theta)(1-\theta\beta)}{\theta}\left((\eta+\sigma)\hat{y}_t(\epsilon^t) - (\eta+1)\hat{A}_t(\epsilon^t)\right) + \beta\mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})], \\
\hat{y}_t(\epsilon^t) &= \mathbb{E}_t\left[\hat{y}_{t+1}(\epsilon^{t+1})\right] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) + \ln(\beta)\right],
\end{aligned}$$

where $\ln(\beta)$ equals the negative of the steady-state real interest rate.

The output deviations in the conditions above represent deviations from steady state. We can alternatively express these conditions in terms of *output gaps* that is, deviations of output from the level under flexible prices, or *natural level of output*, $y_t^n(\epsilon^t)$. Letting $\chi_t(\epsilon^t)$ denote the output gap and using the fact that the steady-state output

[1] Recall that we have assumed that real balances are negligible as far as their effect on the budget constraint or the marginal utility of consumption is concerned.

levels y^n and y are identical, we have

$$\chi_t(\epsilon^t) \equiv \hat{y}_t(\epsilon^t) - \hat{y}_t^n(\epsilon^t) = \frac{y_t(\epsilon^t) - y_t^n(\epsilon^t)}{y}.$$

Recall that with flexible prices, real marginal costs are constant such that $(\eta + \sigma)\hat{y}_t^n(\epsilon^t) = (\eta + 1)\hat{A}_t(\epsilon^t)$. It follows that we can rewrite the two core equations as

$$\begin{aligned} \pi_t(\epsilon^t) &= \kappa\chi_t(\epsilon^t) + \beta\mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})], && (10.19) \\ \chi_t(\epsilon^t) &= \mathbb{E}_t\left[\chi_{t+1}(\epsilon^{t+1})\right] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - r_{t+1}^n(\epsilon^{t+1})\right], && (10.20) \end{aligned}$$

where $\kappa \equiv (1-\theta)(1-\theta\beta)(\eta+\sigma)/\theta$ and

$$r_{t+1}^n(\epsilon^{t+1}) \equiv -\ln(\beta) + \sigma(\mathbb{E}_t[\hat{y}_{t+1}^n(\epsilon^{t+1})] - \hat{y}_t^n(\epsilon^t))$$

denotes the real interest rate in the flexible price economy—the *natural rate of interest*.[2] Conditional on bounded sequences $\{i_{t+1}(\epsilon^t), r_{t+1}^n(\epsilon^{t+1})\}_{t=0}^{\infty}$, bounded sequences $\{\pi_t(\epsilon^t), \chi_t(\epsilon^t)\}_{t=0}^{\infty}$ that satisfy conditions (10.19) and (10.20) constitute equilibria in the sticky price economy without capital.

In subsequent chapters we analyze this model further.

10.4 Bibliographic Notes

The model with monopolistically competitive intermediate good producers builds on Dixit and Stiglitz (1977). Akerlof and Yellen (1985), Mankiw (1985), and Blanchard and Kiyotaki (1987) analyze monetary models with distortions due to price setting; they show that second-order private benefits of adjustment contrast with first-order social benefits, see also Cooper and John (1988). Farhi and Werning (2016) and Korinek and Simsek (2016) analyze aggregate demand externalities and contrast them with pecuniary externalities.

The stochastic price adjustment assumption adopted in section 10.2 is due to Calvo (1983). Rotemberg (1982) derives the New Keynesian Phillips curve under the assumption that firms face quadratic costs of price adjustment. Fischer (1977) and Taylor (1979) consider deterministic, staggered price adjustment mechanisms, see also Chari, Kehoe and McGrattan (2000). Dotsey, King and Wolman (1999), Gertler and Leahy (2008), Golosov and Lucas (2007) study the implications of state-dependent price setting. Mankiw and Reis (2002) show that slow information diffusion ("*sticky information*") in a model with continuous price adjustments generates inflation rather than price stickiness.

Yun (1996) and King and Wolman (1993) formulate and analyze early versions of the New Keynesian business cycle model. Erceg, Henderson and Levin (2000) introduce

[2] Note that $r_{t+1}^n(\epsilon^{t+1})$ denotes a real interest rate, not a rental rate of capital. The relationship between flexible price output (or consumption) growth and the natural rate of interest corresponds to the Euler equation or dynamic IS-curve in the flexible price economy.

wage in addition to price stickiness in that model. Obstfeld and Rogoff (1995) and Kollmann (2001) analyze price stickiness in open economy models.

Woodford (2003), Galí (2008), and Walsh (2017) provide an overview over the literature and contain detailed discussions.

Chapter 11

The Government

Fiscal and monetary policies affect budget constraints, change incentives, and absorb resources. To analyze these effects, we introduce a government sector. Our equilibrium concept remains unchanged—households and firms optimize, and markets clear—but requires refinements: We assume that agents in the private sector take the government's policy as given; we require that the government also satisfies a budget constraint; and we account for the government's resource use when specifying market clearing conditions. When a specific policy implements an equilibrium so defined then we say that the policy is *feasible*.

We analyze the macroeconomic effects of fiscal policy including taxation, government consumption, debt and social security; identify conditions under which policy changes are neutral; and study how monetary policy jointly with fiscal policy affects inflation and output.

11.1 Taxation and Government Consumption

Consider the representative agent economy analyzed in section 3.1, augmented by a government sector that levies taxes on labor income, at rate τ_t^w, and on the return on saving, at rate τ_t^k, to finance government spending.[1] The budget constraint of a household reads

$$a_{t+1} = a_t R_t (1 - \tau_t^k) + w_t (1 - \tau_t^w) - c_t$$

and household optimization thus implies the Euler equation

$$u'(c_t) = \beta R_{t+1} (1 - \tau_{t+1}^k) u'(c_{t+1}).$$

The tax on capital income reduces the net return on household saving (or borrowing). A higher τ_{t+1}^k therefore induces the same type of income and substitution effects as

[1] To simplify notation, we assume in this chapter that the latter tax is levied on the gross return that is, the principal is taxed as well. If, more realistically, the tax only were levied on income, at rate θ_t say, then the budget constraint would read $a_{t+1} = a_t(1 + (r_t - \delta)(1 - \theta_t)) + w_t(1 - \tau_t^w) - c_t$. Clearly, $\theta_t = \tau_t^k + \tau_t^k / (r_t - \delta)$.

a lower R_{t+1} (see subsection 2.1.1), and it discourages saving. In contrast, the labor income tax does not induce a substitution effect since labor is supplied inelastically; it only reduces household wealth.

Taxes finance government consumption, g_t, which may or may not be valued by households. We assume that it is not valued, or that preferences are separable between private and government consumption such that the household's first-order conditions are unaffected by g_t. For now, we abstract from government deficits or surpluses. This implies the government budget constraint

$$g_t = a_t R_t \tau_t^k + w_t \tau_t^w.$$

Substituting into the household's constraint yields

$$a_{t+1} = a_t R_t + w_t - c_t - g_t,$$

indicating that the tax financed government consumption reduces disposable income.

Combining the budget constraints of households, firms, and the government and imposing the market clearing and equilibrium conditions discussed in section 3.1 yields the core equilibrium conditions

$$\begin{aligned} k_{t+1} &= k_t(1-\delta) + f(k_t, 1) - c_t - g_t, \\ u'(c_t) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)(1 - \tau_{t+1}^k) u'(c_{t+1}). \end{aligned}$$

Compared to the core conditions in the model without government, (3.8) and (3.9), the resource constraint (or GDP identity) now accounts for government consumption, and the tax rate on capital income enters the Euler equation.

Consider a steady state which is characterized by the conditions

$$\begin{aligned} c &= f(k, 1) - \delta k - g, \\ 1 &= \beta(1 + f_K(k, 1) - \delta)(1 - \tau^k). \end{aligned}$$

The resource constraint states that replacement investment, δk, and total consumption, $c + g$, equal output. Conditional on the capital stock, private consumption therefore falls when government consumption rises. The Euler equation states that the after-tax return on saving equals β^{-1}; a tax on capital income—but not on labor income—therefore reduces the capital stock. Accordingly, steady-state private consumption is maximal (conditional on g) if the government only levies labor income taxes.

Off steady state, capital income taxation generates inferior outcomes too. To see this, note that the equilibrium conditions with labor but no capital income taxation are identical to the conditions when households pay a *lump-sum tax* equal to g_t, that is a tax whose proceeds households cannot affect. In turn, the conditions with a lump-sum tax correspond to the optimality conditions in a Robinson Crusoe economy where only the resource constraint binds and the "government" consumes g_t. This implies that the equilibrium allocation subject to labor income taxes is Pareto optimal, in contrast to the allocation subject to capital income taxes.

While in equilibrium labor and capital income taxes generate the same revenue, the two instruments have different incentive effects since only capital income taxes can be avoided. From the perspective of an individual household that takes tax rates as given and internalizes the private cost of paying taxes, a capital income tax discourages saving. The benefit of paying taxes—a lower equilibrium tax rate for everybody, given that total tax revenue must equal g_t—is not internalized. This drives a *wedge*, $1 - \tau^k_{t+1}$, between the social marginal rate of transformation, R_{t+1}, and the private marginal rate of transformation, $R_{t+1}(1 - \tau^k_{t+1})$, which the household equalizes with the private marginal rate of substitution.

These findings generalize. Not only does government consumption reduce household wealth, but its financing by means of taxes that induce substitution effects gives rise to welfare reducing *tax distortions*. Abstracting from distributive implications (which are absent in this environment with homogeneous households), taxes that induce substitution effects (here, capital income taxes) therefore generate Pareto inferior outcomes than taxes that do not induce such effects (here, labor income taxes).

In endogenous growth models of the type considered in subsection 6.2.2, a tax induced reduction in the after-tax interest rate lowers the economy's equilibrium *growth rate*, with potentially large welfare consequences.

11.2 Government Debt and Social Security

Next, we introduce government debt as a source of funding. We restrict the analysis to non-distorting taxation of labor income at rate τ_t and allow for population growth at gross rate ν. The resource constraint is given by

$$\nu k_{t+1} = f(k_t, 1) + k_t(1 - \delta) - c_t - g_t,$$

where capital as well as private and government consumption are expressed in per-worker terms.

Government debt allows to intertemporally decouple tax collections and government spending. The dynamic budget constraint of the government reads

$$\nu b_{t+1} = b_t R_t + g_t - \tau_t w_t,$$

where b_t denotes the stock of government debt per worker. The constraint states that a *primary deficit*, $g_t - \tau_t w_t > 0$, must be financed by debt issuance in excess of debt service (repayment of principal plus interest). Equivalently, a *deficit*, $b_t(R_t - 1) + g_t - \tau_t w_t > 0$, increases the government's indebtedness. Note that we do not distinguish between the interest rates on government debt and capital. Market clearing requires that households are indifferent between the two assets and thus, absent risk, that the interest rates coincide.

Let $q_0 \equiv 1$ and $q_t \equiv (R_1 \cdots R_t)^{-1}$. If the no-Ponzi-game condition $\lim_{t\to\infty} q_t \nu^{t+1} b_{t+1} \leq 0$ is imposed with equality, then the dynamic budget constraints imply the intertemporal government budget constraint

$$0 = b_0 R_0 + \sum_{t=0}^{\infty} q_t \nu^t (g_t - \tau_t w_t).$$

11.2.1 Government Debt with a Representative Agent

Consider first the representative agent model. The representative family maximizes $\sum_{t=0}^{\infty} \beta^t \nu^t u(c_t)$ subject to the dynamic budget constraint and a no-Ponzi-game condition. The Euler equation, dynamic and intertemporal budget constraints are given by

$$\begin{aligned} u'(c_t) &= \beta R_{t+1} u'(c_{t+1}), \\ \nu a_{t+1} &= a_t R_t + w_t(1-\tau_t) - c_t, \\ 0 &= a_0 R_0 + \sum_{t=0}^{\infty} q_t \nu^t (w_t(1-\tau_t) - c_t). \end{aligned}$$

Suppose the government initially balances its budget in each period such that $b_t = 0$ and $\tau_t w_t = g_t$, implying $k_t = a_t$. Consider a change of policy that alters the timing of tax collections while holding its present value (at the initial interest rates) constant. For example, suppose that the government reduces tax collections at date t by Δ per capita and increases taxes at date $t+1$ by $\Delta R_{t+1}/\nu$ per capita. The government also issues debt Δ per capita at date t and fully services it in the subsequent period out of the additional tax revenue.

Capital accumulation, consumption, interest rates and wages are not affected by this policy change. To establish this result, we conjecture that wages and interest rates indeed remain unchanged and we verify that the initial capital and consumption sequences then continue to satisfy all equilibrium conditions. Under the conjecture, the representative family's budget set is unaffected by the policy change because the tax cut at date t and the present value of the tax hike in the subsequent period are equal in absolute value,

$$-\Delta + \frac{\Delta R_{t+1}}{\nu} \frac{\nu}{R_{t+1}} = 0.$$

The initial consumption sequence therefore remains optimal for the household. From the household's dynamic budget constraint, this implies that the family increases saving when the government runs a deficit, by exactly the same amount. Since $a_{t+1} = b_{t+1} + k_{t+1}$, capital accumulation and thus, wages and interest rates therefore remain unchanged. Since the government's budget constraints are satisfied as well the conjecture is verified. We conclude that the equilibrium allocation (except for government debt) remains unchanged.

This result is an instance of the *Ricardian equivalence* proposition. The proposition states that for a given government consumption sequence (and thus, present discounted value of taxes) the timing of tax collections does not affect the equilibrium allocation. Note that the proposition makes a statement about changes in government financing, not government consumption. The proposition holds under three key assumptions. First, households and the government save or borrow at the same interest rates. Second, the policy change does not shift the tax burden from one group to another. (With a representative family this is satisfied by assumption.) And third, taxes are not distorting. These assumptions guarantee that a change of government financing does not alter budget sets in the private sector.

11.2.2 Government Debt with Overlapping Generations

Consider next the two-period lived overlapping generations model analyzed in section 3.2. In general equilibrium, we have

$$\begin{aligned} u'(c_{1,t}) &= \beta R_{t+1} u'(c_{2,t+1}), \\ \nu(b_{t+1}+k_{t+1}) &= w_t(1-\tau_t) - c_{1,t}, \\ c_{2,t+1} &= \nu(b_{t+1}+k_{t+1}) R_{t+1}, \end{aligned}$$

where $c_{1,t}$ and $c_{2,t}$ denote consumption of a young and old household at date t, respectively. Note that we have imposed the market clearing condition $a_{t+1} = b_{t+1} + k_{t+1}$.

In this economy, Ricardian equivalence does not hold. Reducing taxes at date t and increasing them at date $t+1$ shifts the tax burden from workers in cohort t to those in the subsequent cohort. The debt issued to finance the government deficit and acquired by workers at date t constitutes net wealth for this group in the sense that they do not have to contribute future resources to service it. Because of the tax cut's positive wealth effect on cohort t, the workers increase their saving by less than the amount of the tax cut, raising consumption.

Capital accumulation therefore slows down: Government debt *crowds out* capital. As a consequence, interest rates rise and wages fall in the subsequent period. Cohort t does not only benefit from a lighter tax burden but also from a higher return on saving while cohort $t+1$ bears a heavier tax burden and receives lower wages.

Along a balanced growth path with constant per-capita values, the government budget constraint reads

$$\nu b = bR + g - \tau w.$$

Absent population growth, taxes equal government consumption and interest payments on debt; with population growth, taxes fall short of these spending items because new debt is issued in each period. If the economy is inefficient, $R < \nu$, then the revenue raised from additional debt issuance exceeds interest payments and the government may purchase goods without ever collecting taxes, simply by holding the debt-to-worker ratio constant ($g > 0, \tau = 0, b > 0$; note that this violates the no-Ponzi-game condition). Debt is welfare increasing under these circumstances because it reduces capital over accumulation. Moreover, debt is a *bubble*: While it never generates dividends (tax revenues) it is rolled over forever at a positive price.

11.2.3 Pay-As-You-Go Social Security with Overlapping Generations

Maintaining the overlapping generations structure, consider finally a *pay-as-you-go social security system*. In contrast to a *fully funded* system where households contribute resources into accounts and consume the return after retirement (and where in equilibrium, the contributions fund capital accumulation) the contributions in a pay-as-you-go system finance benefits to the contemporaneous retirees. That is, the pay-as-you-go system is a transfer system. Each old household at date t receives a transfer, $T_t \nu$, that is fully financed by labor income taxes levied at rate τ_t^s, such that $T_t = w_t \tau_t^s$. Abstracting from debt and government consumption equilibrium is characterized by the conditions

$$\begin{aligned} u'(c_{1,t}) &= \beta R_{t+1} u'(c_{2,t+1}), \\ \nu k_{t+1} &= w_t(1-\tau_t^s) - c_{1,t}, \\ c_{2,t+1} &= \nu k_{t+1} R_{t+1} + T_{t+1} \nu. \end{aligned}$$

For any feasible social security policy $\{\tau_t^s, T_t\}_{t \geq 0}$, there exists an equivalent tax-and-debt policy $\{\tau_t, b_{t+1}\}_{t \geq 0}$ of the type considered in subsection 11.2.2 that implements the same equilibrium allocation. To see this, note first that under the social security policy, the intertemporal budget constraint of a household in cohort t is given by

$$c_{1,t} + \frac{c_{2,t+1}}{R_{t+1}} = w_t(1-\tau_t^s) + \frac{T_{t+1}\nu}{R_{t+1}},$$

while under the tax-and-debt policy, the constraint reads

$$c_{1,t} + \frac{c_{2,t+1}}{R_{t+1}} = w_t(1-\tau_t).$$

Given the equilibrium prices supported by the social security policy, the budget sets characterized by the two constraints are identical if the present value of taxes net of transfers under the social security policy, $w_t \tau_t^s - T_{t+1}\nu / R_{t+1}$, equals taxes under the tax-and-debt policy, $w_t \tau_t$. Since $T_{t+1} = w_{t+1} \tau_{t+1}^s$, this implies the equivalence condition

$$\tau_t = \tau_t^s - \frac{w_{t+1} \tau_{t+1}^s \nu}{w_t R_{t+1}},$$

which maps the sequence of social security tax rates into a sequence of tax rates.

A second equivalence condition follows from the requirement that the dynamic budget constraints of the government (or households) be satisfied. The requirement that two policies pay the same amount of funds to the old at date t,

$$\nu b_t R_t = \nu w_t \tau_t^s,$$

relates the sequence of social security tax rates to a debt sequence. Since neither policy affects the resource constraint or the factor price conditions we conclude that the two equivalence conditions map any feasible social security policy into a tax-and-debt policy that implements the same equilibrium allocation and prices. Absent restrictions on the available tax and transfer instruments, similar mappings can be derived in environments where social security taxes are distorting, households long-lived and heterogenous within a cohort, or outcomes stochastic.

Intuitively, under the social security policy, households save little because they receive transfers in old age. Under the equivalent tax-and-debt policy, they save more because they pay lower taxes when young but do not receive transfers when old. The difference in saving exactly corresponds to the debt the government issues under the tax-and-debt policy. In light of this equivalence, one refers to the present discounted value of the already committed future social security benefits as the *implicit debt* of the pay-as-you-go financed social security system.

Since the implicit debt associated with a social security policy and the explicit debt associated with the equivalent tax-and-debt policy entail the same financial commitments, focusing on the latter and disregarding the former can be misleading. For example, explicit debt does not comprehensively measure the fiscal burden a policy imposes on future generations as these generations also have to contribute resources to service the implicit debt (unless the government defaults on the latter).

Generational accounts provide a more comprehensive measure. The *generational account* of a group of households is the present discounted value of that group's remaining lifetime net taxes. From the government's intertemporal budget constraint, the sum of all generational accounts equals the present discounted value of current and future government consumption plus the outstanding government debt. Generational accounts thus account for commitments both due to explicit and implicit liabilities.

Suppose that at date $t = 0$ a pay-as-you-go social security system $\{\tau_t^s, T_t\}_{t\geq 0}$ is introduced. The effect on the budget set of an old household at date $t = 0$ and on the budget set of a member of cohort $t \geq 0$, respectively, are given by

$$w_0\tau_0^s\nu \quad \text{and} \quad -w_t\tau_t^s + \frac{w_{t+1}\tau_{t+1}^s\nu}{R_{t+1}}.$$

Using the relations derived earlier, we can represent this in terms of an equivalent policy that finances a transfer to the old at date $t = 0$ out of taxes and debt which subsequent cohorts service over time, see table 11.1. The first generation receiving social security benefits clearly is made better off. Whether subsequent generations benefit or loose depends on whether the equilibrium is efficient or not. Along an inefficient balanced growth path ($R < \nu$) subsequent generations also benefit because

$$-w\tau^s + \frac{w\tau^s\nu}{R} > 0.$$

Along an efficient path, in contrast, they are harmed.

In stochastic environments, a history-contingent social security policy (or equivalent tax-and-debt policy with history-contingent returns on government debt) can contribute to inter generational risk sharing. This may be valuable because, absent such policies, overlapping generations cannot implement all ex-ante beneficial insurance arrangements (see section 4.4). A social security policy that provides annuities may also contribute to intra generational risk sharing by insuring longevity risk.

	Pay-as-you-go		Explicit debt
Effect on household budget at date t			
lifetime net taxes:	τ_t	$=$	τ_t
$+$ taxes on young households	τ_t^s	$>$	τ_t
$-$ discounted old age benefits	$\frac{T_{t+1}\nu_{t+1}}{R_{t+1}}$	$>$	0
Effect on government budget at date t			
cash flow, $t=0$:	0	$=$	0
$+$ total cash inflow, $t=0$	$N_0\tau_0^s$	$=$	$N_0\tau_0^s$
$+$ taxes on young households, $t=0$	$N_0\tau_0^s$	$>$	$N_0\tau_0$
$+$ debt issued, $t=0$	0	$<$	$N_0 b_1 \nu_1$
$-$ total cash outflow, $t=0$	$N_0 T_0$	$=$	$N_0\theta_0$
$-$ transfer to old households, $t=0$	$N_0 T_0$	$=$	$N_0\theta_0$
cash flow, $t>0$:	0	$=$	0
$+$ total cash inflow, $t>0$	$N_t\tau_t^s$	$=$	$N_t\tau_t^s$
$+$ taxes on young households, $t>0$	$N_t\tau_t^s$	$>$	$N_t\tau_t$
$+$ debt issued, $t>0$	0	$<$	$N_t b_{t+1}\nu_{t+1}$
$-$ total cash outflow, $t>0$	$N_t T_t$	$=$	$N_t b_t R_t$
$-$ transfer to old households, $t>0$	$N_t T_t$	$>$	0
$-$ debt service, $t>0$	0	$<$	$N_t b_t R_t$

Note: N_t denotes the size of cohort t and $\nu_{t+1} \equiv N_{t+1}/N_t$ its possibly time-varying growth rate. Wages are normalized to one. Equivalence then requires $\nu_{t+1}b_{t+1} = \tau_t^s - \tau_t$ and $T_{t+1} = b_{t+1}R_{t+1}$. In the economy with government debt, the transfer to the initial old, θ_0, corresponds with the transfer paid under the pay-as-you-go system.

Table 11.1: Equivalence of pay-as-you-go social security and explicit government debt.

11.3 Equivalence of Policies

The Ricardian equivalence proposition discussed in subsection 11.2.1 describes *equivalence classes* of fiscal policies whose members implement the same equilibrium allocation that is, the same sequences for consumption, capital, wages, and interest rates but not necessarily for financial assets like government debt. Our discussion of equivalent pay-as-you-go social security and tax-and-debt policies in subsection 11.2.3 identified another type of equivalence classes. We now unify these discussions and present additional applications.

11.3.1 General Equivalence Result

Let μ denote the state at the initial date and let φ denote a policy. Equivalence classes relate pairs of policies and states. A pair (μ, φ) and another pair $(\hat{\mu}, \hat{\varphi})$ belong to the same equivalence class if and only if both pairs implement the same equilibrium allocation.[2]

A direct approach to establishing that (μ, φ) and $(\hat{\mu}, \hat{\varphi})$ belong to the same equivalence class relies on characterizing the equilibrium allocations implemented by each pair (if they exist) and showing that they are identical. An indirect approach relies on establishing that the choice sets of households and firms are not affected by the change of policy. Suppose a pair (μ, φ) implements an equilibrium and suppose that another pair, $(\hat{\mu}, \hat{\varphi})$, satisfies the following conditions:

i. μ and $\hat{\mu}$ encode identical production possibilities, and restrictions on inputs and/or outputs of firms are identical across policies;

ii. households' choice sets are identical if evaluated at the equilibrium prices;

iii. at the equilibrium allocation and prices, $(\hat{\mu}, \hat{\varphi})$ satisfies the government's dynamic budget constraints.

The two pairs then belong to the same equivalence class.

This can be seen as follows: Conjecture that equilibrium prices under (μ, φ) and (μ', φ') are the same. With household choice sets unchanged, household demand functions are unaltered since preferences do not depend on policy. With constraints on production unaffected, firm net supply functions are unaltered. The original household and firm choices (except possibly for financial assets) thus remain optimal and clear markets. Private sector choices and the government's new policy also satisfy the relevant budget constraints. Given that the equilibrium allocation under (μ, φ) and (μ', φ') is the same, the conjecture is verified.

11.3.2 Applications

Note that the reasoning supporting the general equivalence result parallels the arguments establishing Ricardian equivalence as well as equivalence of pay-as-you-go social security and tax-and-debt policies. There, the choice set of a household is the set of affordable consumption allocations over the household's lifetime, and condition i. is trivially satisfied because the initial capital stock which corresponds to μ is held constant. But the result applies much more broadly as the following examples show.

Heterogeneity Suppose that taxes are non distorting but households are heterogeneous within cohorts. An equivalence class (conditional on some initial state) then consists of policies that satisfy the government budget constraints and impose on each household a given household specific present discounted value of taxes.

[2]For simplicity, we disregard issues related to multiplicity of equilibria.

Tax Distortions Suppose that households value consumption and leisure such that labor income taxes are distorting. The choice set of a household then is given by the set of affordable consumption and leisure combinations. An equivalence class (conditional on the initial state) consists of policies that satisfy the government budget constraints and impose on each household a given household specific lifetime *tax function* which specifies the present discounted value of taxes as a function of the household's choices. For example, one tax policy in such an equivalence class might tax labor income at date t at rate τ_t^w, while another policy in the same class might tax labor income at date t at rate $\tau_t^w R_{t+1}$ but collect the tax only in the subsequent period. Since both policies have the same effect on the household's choice set an equilibrium allocation implemented by the former policy also constitutes an equilibrium allocation under the latter. As with "standard" Ricardian equivalence, however, the two policies are associated with different levels of government debt.

Multiple Tax Instruments Suppose that the government taxes consumption expenditures at rate τ_t^c, capital income at rate τ_t^k, and labor income at rate τ_t^w. The household's dynamic budget constraint reads

$$a_{t+1} = a_t R_t (1 - \tau_t^k) + w_t (1 - x_t)(1 - \tau_t^w) - c_t (1 + \tau_t^c),$$

where x_t denotes leisure. Integrating the dynamic budget constraints and imposing a no-Ponzi-game condition yields the intertemporal budget constraint

$$a_0 R_0 (1 - \tau_0^k) + \sum_{t=0}^{\infty} q_t \kappa_t \left(w_t (1 - x_t)(1 - \tau_t^w) - c_t (1 + \tau_t^c) \right) = 0,$$

where we define $\kappa_0 \equiv 1$ and $\kappa_t \equiv [(1 - \tau_1^k) \cdots (1 - \tau_t^k)]^{-1}, t > 0$. Letting $\xi_t \equiv (1 + \tau_t^c)/(1 + \tau_0^c)$, we can rewrite this as

$$\frac{a_0 R_0 (1 - \tau_0^k)}{1 + \tau_0^c} + \sum_{t=0}^{\infty} q_t \kappa_t \xi_t \left(w_t (1 - x_t) \frac{1 - \tau_t^w}{1 + \tau_t^c} - c_t \right) = 0.$$

Note that from the household's perspective, the price at date t of leisure relative to consumption equals $w_t(1 - \tau_t^w)/(1 + \tau_t^c)$ and the price of consumption at date $t + 1$ relative to consumption at date t equals $q_{t+1}\kappa_{t+1}\xi_{t+1}/(q_t \kappa_t \xi_t) = (1 + \tau_{t+1}^c)/(R_{t+1}(1 - \tau_{t+1}^k)(1 + \tau_t^c))$. That is, the *tax wedges*

$$\frac{1 - \tau_t^w}{1 + \tau_t^c} \text{ and } \frac{1 + \tau_{t+1}^c}{(1 - \tau_{t+1}^k)(1 + \tau_t^c)}$$

distort the consumption-leisure and consumption-saving choices, respectively.

If $a_0 R_0 = 0$ then only the two tax wedges, not the three tax rates individually, affect the household's budget set. Feasible tax sequences generating the same wedge sequences therefore constitute an equivalence class in this case. For example, a feasible tax policy employing all three tax instruments is equivalent to another policy that only relies on a specific combination of capital and labor income taxes.

If $a_0 R_0 \neq 0$ then the budget set also depends on $(1-\tau_0^k)/(1+\tau_0^c)$. Note that a capital income tax levied on date $t=0$ financial wealth can be replicated by consumption taxes. To see this, suppose for simplicity that the initial policy imposes no taxes except for capital income taxes at date $t=0$, $\tau_0^k > 0$. This is equivalent to a policy with no capital income taxes but a positive consumption tax at date $t=0$, $\tau_0^c = (1-\tau_0^k)^{-1} - 1$; positive consumption taxes in all other periods (to keep the intertemporal wedges unchanged), $\tau_t^c = \tau_0^c$; and labor income subsidies in all periods (to keep the relative prices between consumption and leisure unchanged), $\tau_t^w = -\tau_t^c$.

11.4 Fiscal-Monetary Policy Interaction

Governments issue nominal debt and central bank money. This implies that the government budget constraint links fiscal and monetary policy instruments as well as the price level. The fact that money carries low or no interest also gives rise to a novel revenue source, seignorage.

11.4.1 Consolidated Government Budget Constraint

The government collects taxes net of government consumption, $\tau_t(\epsilon^t) - g_t(\epsilon^t)$ (the primary surplus, expressed in real terms); redeems maturing real and nominal debt, $b_t(\epsilon^{t-1})$ and $B_t(\epsilon^{t-1})$ respectively; and issues new debt as well as additional money balances, $b_{t+1}(\epsilon^t)$, $B_{t+1}(\epsilon^t)$, and $M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})$. Liabilities issued at date t and maturing at date $t+1$ are indexed by the history ϵ^t. Real or *inflation indexed debt* pays the potentially history-contingent gross rate of return $R_{t+1}(\epsilon^{t+1})$, expressed in real terms; nominal debt pays the gross rate of return $I_{t+1}(\epsilon^{t+1})$, expressed in nominal terms, which translates into the real rate of return $I_{t+1}(\epsilon^{t+1})\Pi_{t+1}^{-1}(\epsilon^{t+1})$, where $\Pi_{t+1}(\epsilon^{t+1})$ denotes gross inflation. Throughout, debt positions should be interpreted as net debt positions of the government.

Let $\{m_{t+1}(\epsilon^{t+1})\}_{t\geq 0}$ denote the asset pricing kernel. Standard asset pricing (see equation (5.1) in section 5.3 and equation (9.2) in section 9.1) implies that the equilibrium price of real and nominal debt equals unity and $P_t^{-1}(\epsilon^t)$ (the inverse of the price level), respectively. To see the latter, rewrite the Euler equation for nominal debt as

$$1 = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{I_{t+1}(\epsilon^{t+1})}{\Pi_{t+1}(\epsilon^{t+1})} \right] = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{I_{t+1}(\epsilon^{t+1})/P_{t+1}(\epsilon^{t+1})}{1/P_t(\epsilon^t)} \right]$$

and note that $I_{t+1}(\epsilon^{t+1})/P_{t+1}(\epsilon^{t+1})$ is the real payoff of nominal debt. It follows that the equilibrium price of nominal debt equals $1/P_t(\epsilon^t)$.

Also, when issuing one unit of money, the government receives $1/P_t(\epsilon^t)$ units of the good in exchange. The government's *consolidated dynamic budget constraint* therefore reads

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} = \tag{11.1}$$
$$\tau_t(\epsilon^t) - g_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)}.$$

Condition (11.1) states that the real value of government debt including interest (on the left-hand side) equals the primary surplus plus revenue from new debt and money issuance (on the right-hand side). Note that the real value of government debt may be history-contingent for two reasons: Because of contingent interest rates or, with nominal debt, a stochastic price level.

Solving the dynamic budget constraint forward (using the Euler equations) and imposing a no-Ponzi-game condition yields

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} = \tag{11.2}$$
$$\sum_{j=0}^{\infty} \mathbb{E}_t \left[(m_{t+1}(\epsilon^{t+1}) \cdots m_{t+j}(\epsilon^{t+j})) \times \right.$$
$$\left. \left(\tau_{t+j}(\epsilon^{t+j}) - g_{t+j}(\epsilon^{t+j}) + \frac{M_{t+1+j}(\epsilon^{t+j}) - M_{t+j}(\epsilon^{t+j-1})}{P_{t+j}(\epsilon^{t+j})} \right) \right].$$

Equation (11.2) states that the market value of outstanding debt equals the present discounted value of current and future primary surpluses including *seignorage* revenues, where seignorage is defined as the resources the government collects in exchange for the money it issues.

If we define government liabilities more broadly to include debt *and* outstanding money balances then the dynamic budget constraint rewritten as

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} =$$
$$\tau_t(\epsilon^t) - g_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)}$$

can be solved forward to yield

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} =$$
$$\sum_{j=0}^{\infty} \mathbb{E}_t \left[(m_{t+1}(\epsilon^{t+1}) \cdots m_{t+j}(\epsilon^{t+j})) \times \right.$$
$$\left. \left(\tau_{t+j}(\epsilon^{t+j}) - g_{t+j}(\epsilon^{t+j}) + \frac{m_{t+1+j}(\epsilon^{t+1+j})\ M_{t+1+j}(\epsilon^{t+j})\ i_{t+1+j}(\epsilon^{t+1+j})}{P_{t+1+j}(\epsilon^{t+1+j})} \right) \right].$$

The last term on the right-hand side represents an alternative measure of seignorage, namely the cost reduction for the government due to the fact that money does not pay interest, in contrast to debt. This cost reduction enters the budget constraint in parallel to a tax revenue. If money paid interest, for example because *reserves*—the deposits of commercial banks at the central bank—paid *interest on reserves* then no such seignorage term would be present.

11.4.2 Seignorage Needs as Driver of Inflation

Consider a deterministic economy without debt where the government fixes taxes and government consumption at some exogenous values, $g - \tau > 0$. For an equilibrium to exist, seignorage revenue then must be sufficient to balance the budget and equation (11.1) simplifies to

$$g - \tau = \frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1} - M_t}{M_t} \frac{M_t}{P_{t-1}} \frac{P_{t-1}}{P_t}.$$

The right-hand side of the equation indicates that seignorage is proportional to the private sector's money demand, M_t / P_{t-1}. Accordingly, the government is constrained in its ability to raise seignorage revenue.

Assume that M_t grows at the constant gross rate γ_M and the private sector's demand for real balances depends negatively on the nominal interest rate and thus (from the Fisher equation), expected inflation. Expected and actual inflation are equal to each other and to the money growth rate, reflecting a quantity theory relationship. The budget constraint can then be expressed as

$$g - \tau = \frac{\gamma_M - 1}{\gamma_M} \cdot \text{money demand}(\gamma_M).$$

The right-hand side of this equation is the product of an increasing and a decreasing function of γ_M: On the one hand, higher money growth increases the *inflation tax* that households pay when acquiring additional money in order to keep real balances constant; but on the other hand, they reduce real balances and thus, the tax base in response to higher inflation and nominal interest rates. The two counteracting effects imply a *seignorage Laffer curve*—a hump shaped relationship between the "tax rate," γ_M, and the seignorage revenue. Except for too high levels of seignorage revenue, there exist two money growth rates that generate that revenue, a low and a high one.

The turnpike model analyzed in subsection 9.2.2 provides micro foundations for an inflation elastic money demand function. Consider an equilibrium in which non-constrained households buy the bubble $a = M_{t+1}/P_t$ and sell it at value M_{t+1}/P_{t+1} in the subsequent period; the gross return on the bubble thus equals the inverse gross inflation rate, Π^{-1}. The difference between bubble purchases (by non-constrained households) and sales (by constrained households) in a period, $a(1 - \Pi^{-1})$, corresponds to the new bubble sales by the government. To satisfy the government budget constraint, these sales have to equal the primary deficit,

$$g - \tau = a(1 - \Pi^{-1}) = a\pi/\Pi,$$

where π denotes the net inflation rate.

Let $\bar{c} \equiv \bar{w} - a$ and $\underline{c} \equiv \underline{w} + a\Pi^{-1}$ denote equilibrium consumption of a household with high and low endowment, respectively. The resource constraint is given by $\bar{c} + \underline{c} = \bar{w} + \underline{w} - g$; the Euler equation of a household investing in the bubble reads

$$u'(\bar{c}) = \beta\Pi^{-1}u'(\underline{c});$$

and the borrowing constraint implies the condition $1 \geq u'(\bar{c})/u'(\underline{c})$ which is necessarily satisfied when $a \geq 0$ and $g - \tau > 0$.

For a given inflation rate the Euler equation pins down the demand for real balances, a, and thus seignorage revenue, $a\pi/\Pi$. Note that inflation affects seignorage revenue twofold. On the one hand, it gives rise to income and substitution effects on the demand for the bubble (see subsection 2.1.2); with logarithmic utility and $\underline{w} = 0$, these effects cancel. On the other hand, higher inflation increases the tax on bubble holdings as well as new bubble sales. When the bubble demand is sufficiently elastic seigniorage is a hump shaped function of the inflation rate.

11.4.3 A Simple Cash-in-Advance Economy

As a laboratory for our subsequent analyses, consider a simple endowment economy inhabited by an infinitely lived representative household that owns a history-contingent endowment sequence, $\{w_t(\epsilon^t)\}_{t\geq 0}$, and a government with history-contingent resource requirement $\{g_t(\epsilon^t)\}_{t\geq 0}$. Households may not consume their own endowments, and consumption goods can only be sold to, and bought from, other households against cash (see subsection 9.3.3). The government must also use cash to purchase goods. The economy's resource constraint reads $w_t(\epsilon^t) = c_t(\epsilon^t) + g_t(\epsilon^t)$. The households' and government's cash-in-advance constraints (which bind because of positive interest rates) are given by $c_t(\epsilon^t) = M^h_{t+1}(\epsilon^t)/P_t(\epsilon^t)$ and $g_t(\epsilon^t) = M^g_{t+1}(\epsilon^t)/P_t(\epsilon^t)$, respectively. Let $M_{t+1}(\epsilon^t) \equiv M^h_{t+1}(\epsilon^t) + M^g_{t+1}(\epsilon^t)$. Securities are traded and money holdings chosen after the state of nature is realized, before cash transactions take place.

The household's dynamic budget constraint reads

$$\begin{aligned}\tau_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M^h_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} = \\ b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M^h_t(\epsilon^{t-1})}{P_t(\epsilon^t)} + \frac{w_{t-1}(\epsilon^{t-1}) - c_{t-1}(\epsilon^{t-1})}{\Pi_t(\epsilon^t)},\end{aligned}$$

where the last term on the right-hand side represents the real value of cash inflows from endowment sales, net of cash outflows for consumption purchases in the previous period. Since the cash-in-advance constraint binds, this collapses to

$$\begin{aligned}\tau_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + c_t(\epsilon^t) = \\ b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + w_{t-1}(\epsilon^{t-1})\Pi_t^{-1}(\epsilon^t).\end{aligned}$$

Because the revenue from endowment sales accrues in cash that must be carried into the next period, inflation or even low deflation ($\Pi_t^{-1}(\epsilon^t) < R_t(\epsilon^t)$) acts as a (non-distorting) *tax* on sales.

The initial state in the economy is $\mu = (M_0^h, M_0^g, b_0 R_0, B_0 I_0)$ and a policy is given by

$$\varphi = \{\tau_t(\epsilon^t), g_t(\epsilon^t), b_{t+1}(\epsilon^t), R_{t+1}(\epsilon^{t+1}), B_{t+1}(\epsilon^t), I_{t+1}(\epsilon^{t+1}), M_{t+1}(\epsilon^t)\}_{t\geq 0}.$$

The endogenous variables are $\{c_t(\epsilon^t), P_t(\epsilon^t), m_{t+1}(\epsilon^{t+1}), M_{t+1}^h(\epsilon^t), M_{t+1}^g(\epsilon^t)\}_{t\geq 0}$. Equilibrium requires

$$\begin{aligned}
c_t(\epsilon^t) &= w_t(\epsilon^t) - g_t(\epsilon^t), \\
m_{t+1}(\epsilon^{t+1}) &= \beta \frac{u'(w_{t+1}(\epsilon^{t+1}) - g_{t+1}(\epsilon^{t+1}))}{u'(w_t(\epsilon^t) - g_t(\epsilon^t))}, \\
\frac{M_{t+1}^h(\epsilon^t)}{M_{t+1}^g(\epsilon^t)} &= \frac{w_t(\epsilon^t) - g_t(\epsilon^t)}{g_t(\epsilon^t)}, \\
1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) R_{t+1}(\epsilon^{t+1}) \right], \\
1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) I_{t+1}(\epsilon^{t+1}) \Pi_{t+1}^{-1}(\epsilon^{t+1}) \right], \\
P_t(\epsilon^t) &= \frac{M_{t+1}(\epsilon^t)}{w_t(\epsilon^t)},
\end{aligned}$$

as well as the government budget constraints, conditions (11.1) and (11.2). Walras' Law implies that the household budget constraints then are satisfied as well.

Since policies in the same equivalence class implement the same allocation they must include the same government consumption sequence; and since all other elements of a policy do not affect the equilibrium allocation, all feasible policies belong to the same equivalence class. Feasible policies may only differ from each other with respect to taxes, debt instruments, money balances, and nominal interest rates. Changes of these instruments may alter the equilibrium price level sequence, as we will show now.

11.4.4 Inflation Effects of Government Financing

Irrelevance of Debt Composition Note first a neutrality result: A feasible change of the composition of government debt between real and nominal debt accompanied by no change of taxes or money supply does not alter inflation. Such a policy change neither affects the level of total indebtedness in real terms nor total debt issuance in real terms. For example, a feasible policy φ with positive real and nominal debt implements the same equilibrium inflation as a modified policy $\hat{\varphi}$ with zero nominal debt and

$$\begin{aligned}
\hat{b}_t(\epsilon^{t-1}) \hat{R}_t(\epsilon^t) &= b_t(\epsilon^{t-1}) R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1}) I_t(\epsilon^t)}{P_t(\epsilon^t)},\ t \geq 1, \\
1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \hat{R}_{t+1}(\epsilon^{t+1}) \right].
\end{aligned}$$

Throughout the subsection, we therefore abstract from nominal debt, without loss of generality.

Policy Mixes We consider feasible policy changes affecting taxes, seignorage and debt subject to (11.1), (11.2), the asset pricing condition, and the cash-in-advance constraint. To satisfy the equilibrium conditions, the policies before and after the change, φ and $\hat{\varphi}$ respectively, and the associated price level sequences, $\{P_t\}_{t\geq 0}$ and $\{\hat{P}_t\}_{t\geq 0}$, must be related as follows:

$$\begin{aligned}
\hat{\tau}_t(\epsilon^t) + \frac{\hat{M}_{t+1}(\epsilon^t) - \hat{M}_t(\epsilon^{t-1})}{\hat{P}_t(\epsilon^t)} + \hat{b}_{t+1}(\epsilon^t) - \hat{b}_t(\epsilon^{t-1})\hat{R}_t(\epsilon^t) & \\
= \tau_t(\epsilon^t) + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} + b_{t+1}(\epsilon^t) - b_t(\epsilon^{t-1})R_t(\epsilon^t), & \\
\hat{P}_t(\epsilon^t) = \hat{M}_{t+1}(\epsilon^t)/w_t(\epsilon^t), & \\
0 = \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})(\hat{R}_{t+1}(\epsilon^{t+1}) - R_{t+1}(\epsilon^{t+1}))\right]. &
\end{aligned}$$

The new policy must also satisfy condition (11.2). We consider several special cases of such policy changes.

Current vs. Future Taxes Delaying taxation and financing the temporary revenue shortfall by issuing government debt leaves the money supply unchanged. The equilibrium price level sequence then is unchanged as well. For example, altering taxes and debt issuance according to

$$\begin{aligned}
\hat{\tau}_0(\epsilon^0) &= \tau_0(\epsilon^0) - \Delta, \\
\hat{b}_1(\epsilon^0) &= b_1(\epsilon^0) + \Delta, \\
\hat{\tau}_1(\epsilon^1) &= \tau_1(\epsilon^1) + R_1(\epsilon^1)\Delta
\end{aligned}$$

(where $\Delta > 0$) has no effect on price levels.

Seignorage vs. Future Taxes A one-time change of the composition of financing between seignorage and debt coupled with a subsequent change of taxes has a permanent effect on the price level. Formally, let

$$\begin{aligned}
\hat{\tau}_0 &= \tau_0, \\
\hat{b}_1(\epsilon^0) + \frac{\hat{M}_1(\epsilon^0) - M_0}{\hat{P}_0(\epsilon^0)} &= b_1(\epsilon^0) + \frac{M_1(\epsilon^0) - M_0}{P_0(\epsilon^0)}, \\
\hat{P}_0(\epsilon^0) &= \hat{M}_1(\epsilon^0)/w_0(\epsilon^0),
\end{aligned}$$

where the two seignorage terms differ. Since the policy $\hat{\varphi}$ does not involve further changes in seignorage the effect on the price level is permanent.[3] The change of debt

[3]To see this, consider for simplicity a deterministic setting. The cash-in-advance constraints imply $\{\hat{M}_{t+1}/\hat{P}_t\}_{t\geq 0} = \{M_{t+1}/P_t\}_{t\geq 0}$. $(\hat{M}_1 - M_0)/\hat{P}_0 \neq (M_1 - M_0)/P_0$ implies $\hat{P}_0 \neq P_0$ and $\hat{M}_1 \neq M_1$. Unchanged seignorage revenues at date $t = 1$, $(\hat{M}_2 - \hat{M}_1)/\hat{P}_1 = (M_2 - M_1)/P_1$, implies $\hat{M}_1/\hat{P}_1 = M_1/P_1$ and thus, $\hat{P}_1 \neq P_1$ and $\hat{M}_2 \neq M_2$. The argument extends to subsequent periods.

issuance at date $t = 0$ implies $\hat{b}_1(\epsilon^0)\hat{R}_1(\epsilon^1) \neq b_1(\epsilon^0)R_1(\epsilon^1)$ for some history ϵ^1. Long-term budget balance therefore requires an appropriate adjustment of taxes subsequent to ϵ^1. With this adjustment, all equilibrium conditions are met.

An *open market operation* where the government sector purchases government debt from households against cash constitutes an example of such a policy change.

Current vs. Future Seignorage A debt financed reduction of seignorage that is compensated by a subsequent increase of seignorage permanently alters the price level. In fact, such a monetary contraction coupled with a subsequent expansion implies a higher price level in the long run.

To understand this result, consider a deterministic environment with constant endowment, w, and gross interest rate, $R > 1$. Feasible policy φ involves no seignorage revenues, $M_t = M$, such that $P_t = P = M/w$. Under the modified policy, $\hat{\varphi}$, money balances are reduced at date $t = 0$ and kept constant until date $t = T - 1$ when they are increased again. That is,

$$\begin{aligned} \hat{M}_{t+1} &= M - \Delta_1,\ t = 0, \ldots, T-2, \\ \hat{M}_{t+1} &= M - \Delta_1 + \Delta_T,\ t = T-1, T, \ldots. \end{aligned}$$

The cash-in-advance constraints imply

$$\begin{aligned} \hat{P}_t &= (M - \Delta_1)/w,\ t = 0, \ldots, T-2, \\ \hat{P}_t &= (M - \Delta_1 + \Delta_T)/w,\ t = T-1, T, \ldots. \end{aligned}$$

Under $\hat{\varphi}$, seignorage revenues at date $t = 0$ and $t = T - 1$, respectively, equal $-\Delta_1 w/(M - \Delta_1)$ and $\Delta_T w/(M - \Delta_1 + \Delta_T)$. To satisfy the budget constraint (11.2), the present value of these revenues must equal zero, implying

$$\frac{\Delta_T w}{M - \Delta_1 + \Delta_T} = R^{T-1}\frac{\Delta_1 w}{M - \Delta_1} \ \Rightarrow\ \frac{\Delta_T}{\Delta_1} = R^{T-1}\frac{M - \Delta_1 + \Delta_T}{M - \Delta_1}.$$

Note that $\Delta_T > \Delta_1$, both since $R > 1$ and $\hat{P}_{T-1} > \hat{P}_0$. That is, following a monetary contraction at date $t = 0$, money balances and thus, the price level *increase* in the long run. This so-called "*unpleasant monetarist arithmetic*" does not contradict the existence of a close link between money growth and inflation. It rather emphasizes that monetary policy has fiscal implications (the monetary contraction generates a revenue shortfall) and that fiscal needs may force a monetary expansion that generates inflation. Note also that a postponement of the expansionary policy increases the long-run price level (P_T increases in T).

11.4.5 Game Of Chicken

When monetary and fiscal policy are controlled by separate authorities—a central bank on the one hand and a fiscal authority on the other—then the institutional structure governing their interplay can have important macroeconomic implications.

Suppose the fiscal authority "moves first" in the sense of committing to history-contingent tax and government consumption sequences before the monetary authority chooses the money supply. By moving first, the fiscal authority shifts responsibility for implementing an equilibrium to the monetary authority; the latter must generate sufficient seignorage to satisfy the intertemporal budget constraint. In this *game of chicken* the central bank's choice set thus is restricted by the actions of the fiscal authority. Although the central bank may wish to conduct a monetary policy aimed at stabilizing the price level say, its second mover status can render that impossible.

Threats to price stability of this kind can be countered by instituting a different arrangement that guarantees *central bank independence* and assigns the first mover advantage to the monetary authority. An independent central bank is relieved of the responsibility for intertemporally balancing the budget; that responsibility lies with the fiscal authority.

11.4.6 Fiscal Theory of the Price Level

When nominal debt is outstanding at date $t = 0$, the government's intertemporal budget constraint (11.2) may not only be balanced by appropriate choices of government consumption, taxes, or seignorage revenues, but also by a revaluation of nominal debt through a change of the price level. The *fiscal theory of the price level* emphasizes this possibility. It views the intertemporal budget constraint (11.2) not as a constraint on government actions but as an equilibrium condition that determines the price level.

The fiscal theory of the price level can be motivated in a one-period model. Suppose that nominal liabilities $B_0 I_0$ are outstanding at date $t = 0$ which is the last period; both money balances and seignorage are negligible; and there is no real debt. Suppose further that the choice of fiscal policy is *non-Ricardian*: Rather than balancing the government's budget by setting $\tau_0 - g_0 = B_0 I_0 / P_0$ for whatever equilibrium price level is realized (as would be the case in the *Ricardian* case), the government sets $\tau_0 - g_0$ independently of P_0. An equilibrium may still exist as long as $P_0 = B_0 I_0 / (\tau_0 - g_0)$. In this case, the price level is fiscally determined.[4]

For a more careful analysis, consider the model introduced in subsection 11.4.3. We abstract from real debt and assume that the nominal interest rate is risk-free such that $B_{t+1}(\epsilon^t) I_{t+1}(\epsilon^t)$ is constant across all histories ϵ^{t+1} subsequent to history ϵ^t. Accordingly, the dynamic budget constraint of the government reads

$$\frac{B_t(\epsilon^{t-1}) I_t(\epsilon^{t-1})}{P_t(\epsilon^t)} = \tau_t(\epsilon^t) - g_t(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)},$$

[4]There is an even simpler mechanism without debt to fiscally determine the price level. It relies on the government setting government consumption in nominal terms and tax revenue in real terms, or vice versa. In either case, the budget balance requirement pins down a price level.

and the intertemporal budget constraint at date $t = 0$ is given by

$$\frac{B_0(\epsilon^{-1})I_0(\epsilon^{-1}) + M_0(\epsilon^{-1})}{P_0(\epsilon^0)} = \sum_{j=0}^{\infty} \mathbb{E}_0 \left[(m_1(\epsilon^1) \cdots m_j(\epsilon^j)) \left(\tau_j(\epsilon^j) - g_j(\epsilon^j) + \frac{m_{j+1}(\epsilon^{j+1})\, M_{j+1}(\epsilon^j)\, i_{j+1}(\epsilon^{j+1})}{P_{j+1}(\epsilon^{j+1})} \right) \right].$$

The remaining equilibrium conditions are

$$\begin{aligned} 1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) I_{t+1}(\epsilon^t) \Pi_{t+1}^{-1}(\epsilon^{t+1}) \right], \\ P_t(\epsilon^t) &= \frac{M_{t+1}(\epsilon^t)}{w_t(\epsilon^t)}, \end{aligned}$$

where $m_{t+1}(\epsilon^{t+1})$ is pinned down by the resource constraint.

A *policy regime* is a mapping from a set $\mathcal{P}$ of strictly positive price level sequences into a set whose elements are sets of policies. With each price level sequence $\{P_t(\epsilon^t)\}_{t\geq 0}$ in $\mathcal{P}$, the policy regime associates a set of policies, $\Phi(\{P_t\}_{t\geq 0})$, such that the price level sequence and any policy in the set satisfy the above equilibrium conditions, except possibly the intertemporal budget constraint. A policy regime is Ricardian if the price level sequence and policies also satisfy the intertemporal budget constraint. Otherwise, the policy regime is non-Ricardian: there exist some price level sequences in $\mathcal{P}$ and associated policies that do not satisfy the intertemporal budget constraint. The non-Ricardian regime rules these price level sequences out as equilibrium price level sequences.

Consider for example a deterministic environment with constant endowments, $w_t = w$, and government consumption, $g_t = g$, implying $m_{t+1} = \beta$. A (Ricardian or non-Ricardian) policy regime then imposes the following restrictions on policy:

$$\begin{aligned} M_{t+1} &= P_t w, \\ I_{t+1} &= \Pi_{t+1}/\beta, \\ B_{t+1} &= B_t I_t - P_t \left(\tau_t - g + w - \Pi_t^{-1} w \right). \end{aligned}$$

If the regime is Ricardian then policy also satisfies the intertemporal budget constraint for any strictly positive P_0 that is, policy satisfies

$$\frac{B_0 I_0 + M_0}{P_0} = \sum_{j=0}^{\infty} \beta^j \left(\tau_j - g + \frac{w\, i_{j+1}}{I_{j+1}} \right),$$

where $i_t = I_t - 1$. If the regime is non-Ricardian, in contrast, then policy does not necessarily satisfy the latter restriction. Since the intertemporal budget constraint must hold in equilibrium a non-Ricardian regime may rule out certain price level sequences that are not ruled out under a Ricardian regime. In this sense, the non-Ricardian policy regime determines the price level (sequence).

Suppose that $B_0 I_0 + M_0 \neq 0$ and monetary policy fixes the nominal interest rate at value I. Equilibrium inflation then is constant at value $\Pi = \beta I$ and money supply grows at the gross rate Π, implying $P_t = P_0 \Pi^t$ and $M_{t+1} = P_0 \Pi^t w$. We consider fiscal policy regimes that relate positive price level sequences that grow at a constant rate, to fiscal policies $\{\tau_t, g, B_{t+1}\}_{t \geq 0}$. In any fiscal policy regime, fiscal policy satisfies

$$B_{t+1} = B_t I - P_0 \Pi^t \left(\tau_t - g + w - \Pi^{-1} w \right).$$

If the fiscal policy regime is Ricardian then, for any $P_0 > 0$, policy also satisfies

$$\frac{B_0 I_0 + M_0}{P_0} = \sum_{j=0}^{\infty} \beta^j \left(\tau_j - g + w \frac{I-1}{I} \right),$$

that is, P_0 imposes a constraint on fiscal policy. Since there is no other condition to determine the initial price level, P_0 is *indeterminate* (see section 11.5). Under a non-Ricardian fiscal policy regime, in contrast, fiscal policy is not constrained by the latter condition; as a consequence, it may determine the price level.

Suppose alternatively that monetary policy fixes the money supply at date t at value M_{t+1} such that the equilibrium price level equals $P_t = M_{t+1}/w$. Under a non-Ricardian fiscal policy regime the price level now is over determined; except for knife edge cases, only a Ricardian fiscal policy regime is consistent with equilibrium.

That a non-Ricardian policy regime may determine the "initial" price level and thereby revalue "initially" outstanding nominal debt does not mean that the government can choose primary surpluses and seignorage revenues arbitrarily. Standard asset pricing and rational expectations imply that, when nominal debt is issued for the first time (before date $t = 0$) the government cannot raise more resources in present discounted value terms than it repays in the future. Accordingly, the intertemporal budget constraint binds at the time of debt issuance. This can also be seen by noting that, at the "truly initial" date $t = -1$ say when $B_{-1} I_{-1} + M_{-1} = 0$ the price level P_{-1} cannot revalue outstanding liabilities. A non-Ricardian policy regime therefore does not allow the government to escape long-run budget balance. Similarly, a non-Ricardian policy regime does not provide a *nominal anchor*—it does not determine the price level—before nominal liabilities have been issued for the first time; it may only contribute, in a stochastic environment, to determining history-contingent inflation rates.

11.4.7 Stability under Policy Rules

Mechanically, a non-Ricardian policy regime determines the equilibrium price level because only a specific price level prevents explosive debt dynamics: The equilibrium conditions without the intertemporal budget constraint determine the path of government debt in real terms, conditional on its starting value, and this path satisfies the government's no-Ponzi-game condition (or the household's transversality condition) only for a specific starting value and thus, a specific initial price level.

The same mechanism may be at work when a policy regime prescribes certain ad-hoc policy rules, for example rules specifying how the interest rate and taxes are set in response to inflation and the stock of outstanding debt. To understand the argument, it suffices to consider a deterministic setting. Suppose the policy regime prescribes that the nominal interest rate responds to inflation, and taxes net of government consumption respond to the stock of real debt at the end of the previous period,

$$\begin{aligned} I_{t+1} &= \alpha \Pi_t, \\ \tau_t - g_t &= \gamma \frac{B_t}{P_{t-1}}, \end{aligned}$$

where α and γ are fixed parameters.[5] Suppose also, as before, that the equilibrium gross real interest rate equals β^{-1}. The Fisher equation, $I_{t+1} = \Pi_{t+1}/\beta$, and the interest rate rule then imply

$$\Pi_{t+1} = \alpha\beta\Pi_t.$$

We allow for a cash-in-advance constraint or a money demand function that depends on the interest rate; in either case, $M_{t+1}/P_t = w_t \zeta(I_{t+1})$ for some function ζ. The dynamic budget constraint thus can be expressed as

$$\begin{aligned} \frac{B_{t+1}}{P_t} &= \frac{B_t}{P_{t-1}} \left(\frac{I_t}{\Pi_t} - \gamma \right) - \frac{M_{t+1} - M_t}{P_t} \\ &= \frac{B_t}{P_{t-1}} (\beta^{-1} - \gamma) - \chi(w_t, w_{t-1}, \Pi_t) \end{aligned}$$

for some function χ, where the second equality uses the Fisher equation, the money demand function, the interest rate rule, and the equilibrium condition $\Pi_{t+1} = \alpha\beta\Pi_t$.

Linearizing the two dynamic equations yields a linear difference equation system in two variables, the deviation of Π_t from its steady-state value—which is not predetermined at date t—and the deviation of B_t/P_{t-1} from its steady-state value—which is predetermined at date t. The matrix determining the stability of the system is given by

$$M \equiv \begin{bmatrix} \alpha\beta & 0 \\ \xi & \beta^{-1} - \gamma \end{bmatrix}$$

for some constant ξ; its eigenvalues equal $\alpha\beta$ and $\beta^{-1} - \gamma$.

Since we study a linear approximation about the system's steady state we restrict attention to bounded solutions. (This is a more stringent stability requirement than the no-Ponzi-game condition in the fiscal theory of the price level.) Three cases may be distinguished, see appendix B.5. First, if both eigenvalues of M are unstable then no bounded solution exists. Second, if both eigenvalues are stable then any initial inflation rate together with the predetermined real debt value gives rise to a bounded solution. As a consequence, *sunspot shocks* may buffet the system. Finally, if exactly one eigenvalue is stable then the system is saddle-path stable and the two policy rules pin down a unique inflation rate conditional on the predetermined real debt level.

[5] We disregard additive constant terms since they are irrelevant for the argument.

Specifically, if $|\alpha\beta| > 1$ (*"active monetary policy"*), inflation in the initial period must equal a specific value to guarantee stable inflation dynamics (the difference equation for inflation is solved forward to yield a bounded solution). But in this case, debt dynamics only are bounded if $|\beta^{-1} - \gamma| < 1$ (*"passive fiscal policy"*). The situation parallels the one in the fiscal theory of the price level when the policy regime is Ricardian.

Alternatively, if $|\beta^{-1} - \gamma| > 1$ (*"active fiscal policy"*), inflation (and thus, the price level) in the initial period must adjust to guarantee stable debt dynamics. Stable inflation dynamics then require $|\alpha\beta < 1|$ (*"passive monetary policy"*). The situation is akin to a non-Ricardian policy regime in the fiscal theory of the price level when the government fixes the nominal interest rate.

11.5 Determinate Inflation and Output

In this chapter, we have encountered several instances of price level indeterminacy. In the model with flexible prices discussed in subsection 10.3.5, risk renders inflation indeterminate when the government sets the nominal interest rate. In the fiscal theory of the price level analyzed in subsection 11.4.6, a Ricardian policy regime may render the price level and the real value of government debt indeterminate. And in the model with ad-hoc policy rules considered in subsection 11.4.7, sufficiently passive rules also rendered inflation and real debt indeterminate.

We now study the source of price level indeterminacy in more detail and analyze the role of monetary policy in the determination of the price level. We first consider a flexible price environment before turning to a model with rigid prices. Throughout the analysis we assume that fiscal policy is Ricardian. Since we analyze linearized equilibrium conditions we are looking for bounded solutions.

11.5.1 Flexible Prices

Consider the model with flexible prices analyzed in subsection 10.3.5. Recall that in this environment the classical dichotomy holds: The nominal interest rate or the money supply do not affect the real allocation; the real interest rate equals the natural interest rate, $r_t^n(\epsilon^t)$, (which follows a stationary process); and inflation is determined by the Fisher equation which reads, in linearized form,

$$\mathbb{E}_t[r_{t+1}^n(\epsilon^{t+1})] = i_{t+1}(\epsilon^t) - \mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})].$$

Suppose first that the government determines a history contingent sequence of nominal interest rates to be implemented independently of the values of other, endogenous model variables. Such an *interest rate peg* only pins down expected inflation in this environment, not actual inflation (see subsection 10.3.5). That is, *inflation* and thus, the *price level* (and, for a given money demand function, nominal balances) are *indeterminate*. The source of the indeterminacy is that the nominal and natural interest rate sequences only constrain the price level's expected change in each period. The consequence is that inflation might respond to *sunspot shocks*. Appendix B.5 contains a formal discussion.

Suppose next that rather than pegging the interest rate the government follows an *interest rate rule* which specifies the interest rate as a function of inflation. In particular, let $i_{t+1}(\epsilon^t) = \bar{\imath}_{t+1}(\epsilon^t) + \phi\pi_t(\epsilon^t)$ where $\phi > 0, \phi \neq 1$ and $\{\bar{\imath}_{t+1}(\epsilon^t)\}_{t\geq 0}$ is a stationary sequence (for example, a constant sequence); both this sequence and the parameter ϕ are chosen by the government. In line with our assumption that the steady-state inflation rate equals zero we impose that in steady state, $\bar{\imath} = r^n$. Substituting the rule into the Fisher equation yields

$$\mathbb{E}_t[r^n_{t+1}(\epsilon^{t+1})] = \bar{\imath}_{t+1}(\epsilon^t) + \phi\pi_t(\epsilon^t) - \mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})].$$

When $\phi < 1$, $\lim_{T\to\infty} \mathbb{E}_t[\pi_{t+T}(\epsilon^{t+T})]$ is bounded irrespective of the value of $\pi_t(\epsilon^t)$. Conditional on $\{r^n_{t+1}(\epsilon^{t+1}), \bar{\imath}_{t+1}(\epsilon^t)\}_{t\geq 0}$, the difference equation and the boundedness requirement therefore do not pin down $\pi_t(\epsilon^t)$, similarly to the case when ϕ equals zero. That is, *inflation* is *indeterminate* and as a consequence, it might respond to *sunspot shocks*, see appendix B.5.

When $\phi > 1$, in contrast, *inflation* is *determinate*. Conditional on the information at date t, the only inflation level that satisfies the difference equation and the boundedness requirement is given by (see appendix B.5)

$$\pi^\star_t(\epsilon^t) = \sum_{s=1}^{\infty} \phi^{-s}\mathbb{E}_t[r^n_{t+s}(\epsilon^{t+s}) - \bar{\imath}_{t+s}(\epsilon^{t+s-1})].$$

Any other level of inflation would imply that expected future inflation diverges, violating the boundedness requirement. Intuitively, when the government raises the nominal interest rate by more than one to one in response to higher inflation ($\phi > 1$) it introduces an unstable root in the law of motion for equilibrium inflation and effectively threatens to let inflation diverge unless $\pi_t(\epsilon^t) = \pi^\star_t(\epsilon^t)$. This property of the rule—the *Taylor principle*—in combination with the boundedness requirement pins down $\pi_t(\epsilon^t)$.

A potential problem with this argument concerns the assumption that the policy rule is linear. When savers have access to cash, nominal interest rates cannot be lowered (substantially) below zero because this would trigger portfolio shifts out of interest bearing assets into cash. The nominal interest rate therefore is constrained by a *zero lower bound* or more realistically, an *effective lower bound* that falls short of zero because cash holdings generate transaction and storage costs. In the presence of such a lower bound the interest rate rule cannot be globally linear.

Suppose, then, that the interest rate rule is given by

$$i_{t+1}(\epsilon^t) = \max[\underline{i}, \bar{\imath}_{t+1}(\epsilon^t) + \phi\pi_t(\epsilon^t)],$$

where $\underline{i}$ denotes the effective lower bound and $\phi > 1$. Substituting this modified rule into the Fisher equation and assuming that $\underline{i} < r^n$, we find that the model has two steady-state inflation rates. The first, unstable steady state corresponds to the solution derived previously, $\pi^\star = (r^n - \bar{\imath})/(\phi - 1) = 0$. The second, stable steady state, $\underline{\pi}$, satisfies $\underline{\pi} = \underline{i} - r^n < 0$. Around this second, deflationary steady state the dynamics are stable. Conditional on information at date t, any inflation rate $\pi_t(\epsilon^t) \in [\underline{\pi}, \pi^\star]$ thus constitutes an equilibrium outcome.

Suppose finally that the government pegs the money supply. Let $p_t(\epsilon^t) \equiv \ln(P_t(\epsilon^t))$ denote the logarithm of the price level such that approximately, $\pi_{t+1}(\epsilon^{t+1}) = p_{t+1}(\epsilon^{t+1}) - p_t(\epsilon^t)$. Moreover, approximate the money demand function (9.4) derived in subsection 9.3.2 by the relationship

$$\ln(M_{t+1}(\epsilon^t)) - p_t(\epsilon^t) = -\eta i_{t+1}(\epsilon^t),$$

where $\eta > 0$.[6] Substituting this relationship into the Fisher equation and defining $z_{t+s}(\epsilon^{t+s}) \equiv (\eta r^n_{t+s}(\epsilon^{t+s}) + \ln(M_{t+s}(\epsilon^{t+s-1})))/(1+\eta)$ yields

$$p_t(\epsilon^t) = \mathbb{E}_t[z_{t+1}(\epsilon^{t+1})] + \frac{\eta}{1+\eta}\mathbb{E}_t[p_{t+1}(\epsilon^{t+1})].$$

Iterating forward, we arrive at

$$p_t(\epsilon^t) = \sum_{s=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^s \mathbb{E}_t[z_{t+1+s}(\epsilon^{t+1+s})],$$

where we rule out hyperinflation. Since the government pegs the money supply and the natural interest rate is exogenous we conclude that the *price level* is *determinate*. This conclusion is not necessarily robust to relaxing the assumption of a linear money demand function or to allowing for hyperinflation.

11.5.2 Rigid Prices

Suppose now that prices are rigid and output is demand determined. The New Keynesian Phillips curve and dynamic IS-curve, respectively, are given by

$$\begin{aligned}
\pi_t(\epsilon^t) &= \kappa \chi_t(\epsilon^t) + \beta \mathbb{E}_t[\pi_{t+1}(\epsilon^{t+1})], \\
\chi_t(\epsilon^t) &= \mathbb{E}_t\left[\chi_{t+1}(\epsilon^{t+1})\right] - \frac{1}{\sigma}\mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1}) - r^n_{t+1}(\epsilon^{t+1})\right],
\end{aligned}$$

where $\chi_t(\epsilon^t)$ denotes the output gap, and the parameters κ, β, σ are positive (and $\beta < 1$) (see equations (10.19) and (10.20) in subsection 10.3.5).

We consider linear stochastic policy rules that map inflation, the output gap, and a stationary *monetary policy shock*, $\zeta_t(\epsilon^t)$, into the nominal interest rate,

$$i_{t+1}(\epsilon^t) = \bar{\imath}_{t+1}(\epsilon^t) + \phi_\pi \pi_t(\epsilon^t) + \phi_\chi \chi_t(\epsilon^t) + \zeta_t(\epsilon^t),$$

where the coefficients ϕ_π and ϕ_χ are strictly positive. Substituting the rule into the dynamic IS-curve, we arrive at a difference equation system in two non-predetermined

[6]Since we consider the case where the real allocation is independent of monetary policy we suppress the possible dependence of money demand on the transactions volume (e.g., consumption).

endogenous variables, inflation and the output gap, as well as three exogenous variables, the intercept of the policy rule, the natural rate of interest, and the monetary policy shock.

The system is saddle path stable if and only if the condition

$$\phi_\pi + \frac{1-\beta}{\kappa}\phi_\chi > 1$$

is satisfied. To gain intuition for this restriction consider a deterministic environment with constant intercept and natural rate, $\bar{\imath} = r^n$. We know that in this case, one equilibrium is given by zero inflation and a zero output gap at all times. Suppose there exists another equilibrium with a constant, non-zero inflation rate, $\bar{\pi}$ say. From the Phillips curve, this requires that the output gap is constant as well, at level $\bar{\chi} = (1-\beta)\bar{\pi}/\kappa$; and from the dynamic IS-curve, this implies the restriction $\phi_\pi\bar{\pi} + \phi_\chi\bar{\chi} = \bar{\pi}$ which corresponds to the above condition with the inequality sign replaced by an equality.

We conclude that when the above condition is "just" violated, then there exists a continuum of equilibria with constant, non-zero inflation and output gaps. When the condition is satisfied, in contrast, then it satisfies the *Taylor principle*—a rise in inflation would lead to a rise in the nominal interest rate by more than one-to-one; since this would imply explosive dynamics, *inflation* and *output* are *determinate*.

11.6 Real Effects of Monetary Policy

Even with perfectly flexible prices a monetary policy shock may affect the allocation when frictions other than price rigidity prevent an immediate response by the price level that could neutralize the effects of the shock. We discuss two such frictions, imperfect information and market segmentation. Thereafter, we return to a setting with price rigidity where a change in the nominal interest rate affects the allocation (see subsection 10.3.5) and we analyze the underlying transmission mechanism in more detail. Throughout the section, we assume that policy is Ricardian.

11.6.1 Flexible Prices

Consider first the effect of *imperfect information*. Envision an overlapping generations economy without capital or storage; as in the model analyzed in subsection 9.2.1, young households only save by holding money. Unlike in that model, however, young households produce output. How much they optimally produce depends on the expected return on saving and thus, on the current and expected future price level. The price level in turn is affected by two shocks. First, a monetary policy shock; absent information frictions, this shock would be neutral. Second, a real shock with asymmetric effects on different subgroups among the young; groups favored by the shock optimally increase their production.

Crucially, households do not directly observe the two shocks; they can only, imperfectly infer them from the observed equilibrium price level (see the discussion of rational expectations equilibrium in section 1.3). The solution to this *signal extraction problem* implies that when the price level increases, households rationally attach positive probability both to a favorable real shock and an expansionary monetary shock. Accordingly, all households expand production although, under perfect information, the monetary shock would have been neutral.

Consider next the consequences of *market segmentation* due to restrictions on specific types of financial transactions. When these restrictions give rise to distinct valuations of money in different markets or for different groups of agents then a monetary policy shock may not be neutralized by a price level adjustment which would imply neutrality in the absence of segmentation.

Suppose for example that firms and households are subject to a cash-in-advance constraint (see subsection 9.3.3). At the beginning of a period, before observing the monetary policy shock, households deposit money at banks; banks, in turn, lend the money to firms which need to fund their purchases of inputs in advance. Once households have chosen their portfolio the government transfers new money to banks. When this transfer is larger than expected the equilibrium price level rises by more than anticipated and households' real balances do not suffice to pay for the planned consumption. Banks, on the other hand, hold more cash than planned and lend it at cheaper rates to firms—the monetary expansion triggers a *liquidity effect* and this stimulates production. The real allocation therefore varies with the monetary policy shock. If the money injection symmetrically benefited banks and households, or if banks and households could trade money after the monetary policy shock, neutrality would prevail.

11.6.2 Rigid Prices

Consider the framework discussed in subsection 11.5.2 and assume that the condition for determinacy is satisfied ($\phi_\pi \kappa + (1-\beta)\phi_\chi > \kappa$). Since inflation and the output gap are not predetermined and the intercept of the policy rule and the natural rate of interest are constant, $\pi_t(\epsilon^t)$ and $\chi_t(\epsilon^t)$ are functions of $\zeta_t(\epsilon^t)$ only (see equation (B.3) in appendix B.5). Specifically, when the policy shock $\zeta_t(\epsilon^t)$ follows a first-order autoregressive process with coefficient $\rho \in [0, 1)$,

$$\begin{aligned}
\pi_t(\epsilon^t) &= -\kappa\omega\zeta_t(\epsilon^t), \\
\chi_t(\epsilon^t) &= -(1-\beta\rho)\omega\zeta_t(\epsilon^t),
\end{aligned}$$

where $\omega \equiv ((1-\beta\rho)(\sigma(1-\rho)+\phi_\chi)+\kappa(\phi_\pi-\rho))^{-1}$ is strictly positive. Substituting these expressions into the dynamic IS-curve and the interest rate rule yields the real and nominal interest rates, respectively, as functions of the policy shock:

$$\begin{aligned}
\mathbb{E}_t\left[i_{t+1}(\epsilon^t) - \pi_{t+1}(\epsilon^{t+1})\right] &= r^n + \sigma(1-\rho)(1-\beta\rho)\omega\zeta_t(\epsilon^t), \\
i_{t+1}(\epsilon^t) &= \bar{\imath} + (\sigma(1-\rho)(1-\beta\rho) - \kappa\rho)\omega\zeta_t(\epsilon^t).
\end{aligned}$$

Following a positive monetary policy shock, $\zeta(\eta^t) > 0$, the nominal interest rate exceeds the level implied by the systematic part of the interest rate rule. This leads to lower inflation, a lower output gap, and (since the natural level of output is unaffected by the nominal interest rate) lower output. When $\rho > 0$, these effects are persistent, otherwise they are temporary.

Associated with the output contraction is an increase in the real interest rate. Paradoxically, however, the nominal interest rate need not rise. For sufficiently high values of ρ, the response of $i_{t+1}(\epsilon^t)$ to $\pi_t(\epsilon^t)$ and $\chi_t(\epsilon^t)$ through the systematic part of the policy rule may more than offset the direct positive effect of the policy shock. Intuitively, when ρ is sufficiently high, the policy shock generates expectations of negative inflation in subsequent periods and this increases the nominal relative to the real interest rate.

Recall that κ, the coefficient on the output gap in the Phillips curve, depends on the frequency of price adjustments by firms, $1-\theta$. As $\theta \to 0$ (perfectly flexible prices), $\kappa \to \infty$ and the effect of the policy shock on the output gap equals zero. Inflation marginally responds by $-(\phi_\pi - \rho)^{-1}$ and the nominal interest rate as well as expected inflation in the subsequent period by $-\rho(\phi_\pi - \rho)^{-1}$; accordingly, the real interest rate is not affected. Consistent with the results established in subsection 11.5.1, we thus find that expected inflation and the nominal interest rate move in tandem; and the impact effect of the monetary policy shock on inflation is determined by the restriction that the expected inflation sequence under the policy rule be bounded.

For $\theta \to 1$ (completely rigid prices), in contrast, $\kappa \to 0$ and inflation is unaffected by the shock. The output gap marginally responds by $-(\phi_\chi + \sigma(1-\rho))^{-1}$ because, with zero inflation, it is determined by the dynamic system (see the equations in subsection 11.5.2)

$$\chi_t(\epsilon^t) = \mathbb{E}_t\left[\chi_{t+1}(\epsilon^{t+1})\right] - \frac{1}{\sigma}\mathbb{E}_t\left[\phi_\chi \chi_t(\epsilon^t) + \zeta_t(\epsilon^t)\right]$$

and the requirement that future expected output gaps remain bounded. The marginal effect of the policy shock on both the nominal and the real interest rate equals $\sigma(1-\rho)(\phi_\chi + \sigma(1-\rho))^{-1}$. A higher persistence of the shock implies a stronger output effect but potentially a weaker interest rate response. When the persistence is very high ($\rho \to 1$) interest rates do not respond at all.

11.7 Bibliographic Notes

Baxter and King (1993) and Barro (1990) analyze tax financed government consumption or investment in the neoclassical growth model and the Ak model, respectively.

The modern formulation of the Ricardian equivalence proposition is due to Barro (1974). Diamond (1965) studies debt in the OLG model. Auerbach, Gokhale and Kotlikoff (1994) discuss generational accounting, Breyer (1989) and Rangel (1997) analyze equivalent social security reforms, and Ball and Mankiw (2007) analyze risk sharing properties of social security systems.

Gonzalez-Eiras and Niepelt (2015) state a general neutrality result. Bassetto and Kocherlakota (2004) derive the neutrality result for policy changes that involve distorting taxes.

Cagan (1956) analyzes need-for-seignorage driven (hyper)inflation. The "unpleasant monetarist arithmetic" is due to Sargent and Wallace (1981). Subsection 11.4.4 follows Sargent (1987, 5.4). Leeper (1991) analyzes "active" and "passive" policy rules, and the fiscal theory of the price level is due to Sims (1994) and Woodford (1995), see also Aiyagari and Gertler (1985) and Kocherlakota and Phelan (1999). For critiques, see Bassetto (2002), Buiter (2002), and Niepelt (2004*a*).

Sargent and Wallace (1975) establish price level indeterminacy under an interest rate rule. Interest rate rules often are referred to as "*Taylor rules*", afterTaylor (1993). Taylor (1999) and Woodford (2001) discuss the Taylor principle and Bullard and Mitra (2002) analyze the conditions for price level determinacy; see also Atkeson, Chari and Kehoe (2010). Benhabib, Schmitt-Grohé and Uribe (2002) analyze determinacy under interest rate rules in the presence of an effective lower bound. Brock (1974) and Obstfeld and Rogoff (1983) analyze determinacy when the government pegs the money supply. Poole (1970) assesses how interest and money supply targets stabilize output and prices in an IS-LM framework with shocks to the IS and LM schedules.

Following Friedman (1968) and Phelps (1970), Lucas (1972) analyzes a model with imperfect information and real effects of monetary policy shocks. Grossman and Weiss (1983), Rotemberg (1984), and Lucas (1990) analyze models with segmented markets; see also Alvarez, Atkeson and Kehoe (2002) on endogenous segmentation.

Walsh (2017, 5) provides an overview over the literature on the effects of monetary policy under imperfect information and with segmented markets. Woodford (2003), Galí (2008, 3), and Walsh (2017, 8) discuss the stability properties of the New Keynesian model and its transmission mechanism and provide an overview over the related literature.

Beyond the material covered in the chapter, Wallace (1981) and Chamley and Polemarchakis (1984) derive neutrality results in economies with money as a store of value.

Chapter 12

Optimal Policy

When policy affects the equilibrium allocation then some policies are better than others. The "*Ramsey program*" is to choose the best (according to some criterion) admissible and feasible policy that is, the best policy using instruments at the government's disposable and implementing an equilibrium. The solution to the Ramsey program—the *Ramsey policy*—implements the *Ramsey allocation*. Both admissibility and feasibility constraints typically bind: A social planner that chooses among feasible allocations always can do at least as good as a Ramsey government that chooses among feasible allocations which can be implemented as equilibrium outcomes given the set of admissible policy instruments.

We start by reviewing fundamental results on Ramsey tax policies from public finance before turning to macroeconomic applications.

12.1 The Primal and Dual Approach to Optimal Taxation

Consider a complete markets economy with N goods and H households. Firms operate a constant returns to scale technology whose production frontier is characterized by the function f, $y_1 = f(y_2, \ldots, y_N)$, where y denotes the vector of net outputs. Taking the vector of producer prices, p, as given firms maximize profits. Price taking and constant returns to scale imply that profits equal zero. (Alternatively, firms are price takers and all profits are fully taxed away.)

Taking the vector of consumer prices, q, as given household h chooses the vector of net supplies and demands (e.g., supply of labor, demand for consumption), x^h, to maximize utility, u^h, subject to its budget constraint. The demand function of household h for good j is denoted $x_j^h(q)$, and for all goods $x^h(q)$. The household's indirect utility function, v^h, is defined as

$$v^h(q) \equiv \max_{x^h} u^h(x^h) \text{ s.t. } q \cdot x^h = 0.$$

Let $x \equiv \sum_h x^h$ and $x(q) \equiv \sum_h x^h(q)$.

The government maximizes a social welfare function, v, which aggregates $(u^1, \ldots, u^H)$. It has access to a technology whose production frontier is characterized by the function

g, $z_1 = g(z_2, \ldots, z_N)$, where z denotes the vector of government net outputs. The government also has access to good-specific, proportional excise taxes; consumer prices therefore may differ from producer prices, $q \neq p$.

Market clearing requires $x = y + z$. When firms and households satisfy their budget constraints and all markets clear, Walras' Law implies that the government satisfies its budget constraint as well.

12.1.1 Primal Approach

Abstract from government production and assume that the government has an exogenous resource requirement for good 1, $z_1 \leq 0$; all other components of z equal zero. In a competitive equilibrium, the first-order conditions of households and firms, the budget constraints of households, the market clearing condition, and the resource constraint are satisfied:

$$
\begin{aligned}
\frac{u_j^h(x^h)}{u_1^h(x^h)} &= \frac{q_j}{q_1}, \\
-f_j(y_2, \ldots, y_n) &= p_j/p_1, \ j > 1, \\
q \cdot x^h &= 0, \\
x &= y + z, \\
y_1 &= f(y_2, \ldots, y_N).
\end{aligned}
$$

Note that we can normalize $p_1 = q_1 = 1$ because the household does not have exogenous income.

From the first-order conditions of firms, the government can select any point on the production frontier by choosing p. This choice does not affect household demand because the government may choose q independently of p and there are no profits. Accordingly, the first-order conditions of firms do not constrain the government; they only determine the price vector p that supports a feasible y. We may therefore drop the first-order conditions of firms, and thus p, from the set of equilibrium conditions that constrain the Ramsey government.

Further simplifying these conditions, we eliminate consumer prices, q, by substituting the household first-order conditions in the budget constraints, arriving at

$$
\begin{aligned}
\frac{u_j^1(x^1)}{u_1^1(x^1)} &= \frac{u_j^h(x^h)}{u_1^h(x^h)}, \\
\sum_{j=1}^{N} u_j^h(x^h) x_j^h &= 0, \\
x_1 - z_1 &= f(x_2, \ldots, x_N).
\end{aligned}
$$

The first two conditions, the *implementability constraints*, reflect equilibrium in the private sector: Marginal rates of substitution are equal across households because all households face the same consumer prices; and each household satisfies its budget constraint with prices expressed in terms of marginal rates of substitution. The last condition imposes market clearing subject to the resource constraint.

Note that the equations above do not involve prices. The Ramsey program thus can be specified as a constrained choice of allocation; this is referred to as the *primal approach*. Note also that the constraint set is smaller than the constraint set of a social planner; the latter does not have to satisfy the implementability constraints. Finally, note that restrictions on the admissible tax instruments, for example that certain tax rates have to be equal to each other, would give rise to additional implementability constraints.

Suppose that $H = 1$ such that there is just one implementability constraint. The Ramsey program then reads

$$\max_x u^1(x) \text{ s.t. } \sum_{j=1}^{N} u_j^1(x) x_j = 0, \;\; x_1 - z_1 = f(x_2, \ldots, x_N).$$

Letting λ and μ denote the multipliers associated with the implementability and resource constraint, respectively, the first-order conditions read

$$\begin{aligned} u_1^1(x)[1 + \lambda(1 - E^1)] &= \mu, \\ u_j^1(x)[1 + \lambda(1 - E^j)] &= -\mu f_j(x_2, \ldots, x_N), \;\; j > 1, \end{aligned}$$

where we define

$$E^j \equiv -\sum_{i=1}^{N} \frac{u_{ji}^1(x) x_i}{u_j^1(x)}$$

as the sum of the elasticities of marginal utility, $u_j^1(x)$, with respect to each of the goods. A social planner not bound by the implementability constraint ($\lambda = 0$) optimally sets all marginal rates of substitution equal to the corresponding marginal rates of transformation. The Ramsey government cannot do that (unless $z_1 = 0$ such that the implementability constraint is slack) because equilibrium and specifically, government budget balance requires distorting taxes.

Since $u_j^1(x) = \alpha^1 q_j$ where α^1 denotes the marginal utility of income, the first-order conditions can be written as

$$\alpha^1 q_j [1 + \lambda(1 - E^j)] = \mu p_j.$$

Using $p_1 = q_1 = 1$ to solve for λ and substituting yields

$$\frac{q_j - p_j}{q_j} = 1 - \frac{\alpha^1}{\mu}[1 + \lambda(1 - E^j)] = \frac{\mu - \alpha^1}{\mu} \frac{E^j - E^1}{1 - E^1}, \; j > 1.$$

If $-E^1 \to \infty$ (completely inelastic demand for good 1) then optimal tax rates are uniform across goods 2 to N. If $-E^1 \to 0$ (completely elastic demand for good 1) then there are no income effects on $x_j, j > 1$, and the optimal tax rates are proportional to E^j. If, moreover, the cross-partials of the utility function equal zero (independent demands) then E^j reduces to the inverse of the own-price elasticity and the standard partial equilibrium result follows: Optimal tax rates are inversely proportional to the price elasticity.[1]

Suppose that $u(x) \equiv u(w(x_1, \ldots, x_n), x_{n+1}, \ldots, x_N)$ where the function w is homothetic. This implies that E^j is the same for all $j \leq n$ and thus, that the first-order conditions combine to

$$\frac{u_j^1(x)}{u_1^1(x)} = -f_j(x_2, \ldots, x_N),\ 1 < j \leq n.$$

Equivalently, $q_j/p_j = q_1/p_1$, that is the optimal ad valorem tax rates on goods 1 to n are *uniform* (and equal to zero since we normalized the tax rate on good 1 to be zero).

12.1.2 Dual Approach

The *dual approach* specifies the Ramsey program as a choice of prices and thus taxes, rather than allocation. We allow for government production and multiple households and let $v(q) \equiv v(v^1(q), \ldots, v^H(q))$. Normalizing $p_1 = q_1 = 1$, the program reads

$$\max_{q_2,\ldots,q_N,z_2,\ldots,z_N} v(q) \text{ s.t. } x_1(q) - g(z_2, \ldots, z_N) = f(x_2(q) - z_2, \ldots, x_N(q) - z_N)$$

and the first-order conditions are given by

$$\begin{aligned} 0 &= \lambda \left(f_j(x_2(q) - z_2, \ldots, x_N(q) - z_N) - g_j(z_2, \ldots, z_N) \right),\ j > 1, \\ \frac{\partial v(q)}{\partial q_j} &= \lambda \left(\frac{\partial x_1(q)}{\partial q_j} - \sum_{i=2}^{N} f_i \frac{\partial x_i(q)}{\partial q_j} \right),\ j > 1. \end{aligned}$$

The first condition represents optimality of z_j. It implies that as long as resources are scarce ($\lambda \neq 0$) *aggregate production efficiency* is optimal: The marginal rates of transformation should be equalized across private and public and more generally, across all productive sectors. Production efficiency rules out sector specific taxes on production factors, e.g., sector specific investment subsidies; taxes on goods used as intermediate inputs in certain sectors; sector specific payroll taxes; or tariffs that drive a wedge between domestic and world market producer prices. (Note that the production efficiency result also implies the uniform commodity taxation result.)

The second condition represents optimality of q_j. Using the equilibrium expression for producer prices and letting $t_j \equiv q_j - p_j$, it can be written as

$$\frac{\partial v(q)}{\partial q_j} = \lambda \sum_{i=1}^{N} p_i \frac{\partial x_i(q)}{\partial q_j} = -\lambda \frac{\partial}{\partial t_j} \sum_{i=1}^{N} t_i x_i(q).$$

[1] From $u_j^1(x) = q_j \alpha^1$, we have $u_{jj}^1(x) x_j'(q_j) = \alpha^1$ and thus, $E^j = -u_{jj}^1(x) x_j / u_j^1(x) = -x_j / [x_j'(q_j) q_j]$.

(The second equality uses the budget constraint, $\sum_i p_i x_i(q) = \sum_i (q_i - t_i) x_i(q) = -\sum_i t_i x_i(q)$.) At the optimum, a price change thus affects the social welfare function proportionally to the resource cost of meeting the induced change in demand, or proportionally to the induced change in tax revenue. The latter implication would also have followed if we had maximized the social welfare function subject to the government budget constraint (for given producer prices) rather than the resource constraint.

Roy's Identity states that $\partial v^h(q)/\partial q_j = -\alpha^h x_j^h(q)$ where α^h denotes household h's marginal utility of income. Letting $\beta^h \equiv \partial v(q)/\partial v^h \alpha^h$ denote household h's *social marginal utility of income*, the first-order condition for prices can be expressed as

$$\sum_{h=1}^{H} \beta^h x_j^h(q) = \lambda \sum_{h=1}^{H} \left(x_j^h(q) + \sum_{i=1}^{N} t_i s_{ij}^h(q) - x_j^h(q) \sum_{i=1}^{N} t_i \frac{\partial x_i^h(q)}{\partial I} \right).$$

Here, s_{ij}^h denotes the derivative of household h's compensated demand for good i with respect to q_j, and $\partial x_i^h(q)/\partial I$ denotes the income effect. (The decomposition uses the Slutsky equation.) Finally, letting $\gamma^h \equiv \beta^h/\lambda + \sum_{i=1}^{N} t_i \partial x_i^h(q)/\partial I$ denote the *social marginal valuation of income*—the marginal valuation of h's income given that this income also generates tax revenue—the condition simplifies to the *many-person Ramsey rule*,

$$\frac{\sum_{h=1}^{H} (\gamma^h - 1) x_j^h(q)}{x_j(q)} = \frac{\sum_{h=1}^{H} \sum_{i=1}^{N} t_i s_{ij}^h(q)}{x_j(q)}.$$

The right-hand side of the condition constitutes a measure of the tax induced substitution effects on $x_j(q)$ (using the symmetry of the Slutzky matrix). If producer prices were constant, consumers compensated, and the derivatives of compensated demand constant, then it would equal the relative change in demand for $x_j(q)$. The left-hand side accounts for differences in the social marginal valuation of income and the expenditures shares. If γ^h is the same for all households or $x_j^h(q)/x_j(q)$ the same across goods, then the left-hand side reduces to a constant and the tax induced substitution effects are constant across goods. Otherwise, they are not.

Suppose the government can also levy a poll tax. (An excise tax on labor supply combined with a poll tax is equivalent to a linear labor income tax.) Optimality then requires

$$\sum_{h=1}^{H} \beta^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{N} p_i \frac{\partial x_i^h(q)}{\partial I}.$$

Since

$$\sum_{h=1}^{H} \sum_{i=1}^{N} p_i \frac{\partial x_i^h(q)}{\partial I} = \sum_{h=1}^{H} \sum_{i=1}^{N} (q_i - t_i) \frac{\partial x_i^h(q)}{\partial I} = \sum_{h=1}^{H} \left(1 - \sum_{i=1}^{N} t_i \frac{\partial x_i^h(q)}{\partial I} \right),$$

we conclude that with a poll tax, the average γ^h equals unity. Accordingly, the left-hand side of the many-person Ramsey rule is the covariance between γ^h and $x_j^h(q)/x_j(q)$. Under the Ramsey policy, the relative change in demand for good j is proportional to this covariance: Consumption of goods in strong demand by households with high social marginal valuation of income should be discouraged less.

With just one consumer, $v = v^1$ and the many-person Ramsey rule reduces to the condition

$$\gamma^1 - 1 = \frac{\sum_{i=1}^N t_i s_{ij}^1(q)}{x_j(q)},$$

implying that the tax induced substitution effects should be constant across goods. If a poll tax is available such that $\gamma^1 = 1$ no commodity taxes are levied.

12.2 Tax Smoothing

In the macroeconomic context, the public finance question of *which* goods to tax corresponds to the problem of *when* to tax in order to minimize tax distortions or negative distributive implications. From an efficiency viewpoint optimal tax rates typically fluctuate little over time and across states that is, government spending and taxation are decoupled. The key instrument to sustain such decoupling is government indebtedness and insurance.

12.2.1 Complete Markets

Consider a representative household economy without capital. The household is endowed with one unit of time per period. At date t, history ϵ^t time can be transformed into $w_t(\epsilon^t)$ units of the good. Household preferences over consumption, c, and leisure, x, are represented by the standard utility function $\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u(c_t(\epsilon^t), x_t(\epsilon^t))]$ where u is strictly concave and β denotes the discount factor.

To finance a given stream of government consumption, $\{g_t(\epsilon^t)\}_{t \geq 0}$, the government taxes labor income at rates $\{\tau_t(\epsilon^t)\}_{t \geq 0}$ and issues Arrow securities of arbitrary maturity; markets are complete. Without loss of generality, taxes on consumption are normalized to zero (see the discussion in subsection 11.3.2 and note that absent a technology to transform resources intertemporally, there are no intertemporal producer prices).

Variable ${}_t b_s(\epsilon^{t-1}, \epsilon^s), s \geq t$, denotes claims vis-a-vis the government held at date t, given that history ϵ^{t-1} occurred; one claim entitles to one unit of the consumption good at date s after history ϵ^s conditional on history ϵ^{t-1}. The marginal distribution of ϵ^t is denoted by $H_t(\epsilon^t)$ and its density by $h_t(\epsilon^t)$.

We adopt the primal approach. The benevolent government maximizes household welfare subject to the resource constraints

$$c_t(\epsilon^t) + g_t(\epsilon^t) = w_t(\epsilon^t)(1 - x_t(\epsilon^t))$$

and the implementability constraint

$$\sum_{t=0}^{\infty} \int \beta^t [u_c(c_t(\epsilon^t), x_t(\epsilon^t))(c_t(\epsilon^t) - {}_0 b_t(\epsilon^{-1}, \epsilon^t)) - u_x(c_t(\epsilon^t), x_t(\epsilon^t))(1 - x_t(\epsilon^t))] dH_t(\epsilon^t) = 0.$$

The latter integrates the household's (complete markets) intertemporal budget constraint,

$$\sum_{t=0}^{\infty} \int q_t(\epsilon^t)[c_t(\epsilon^t) - (1-\tau_t(\epsilon^t))w_t(\epsilon^t)(1-x_t(\epsilon^t)) - {}_0b_t(\epsilon^{-1},\epsilon^t)]d\epsilon^t = 0,$$

as well as the household first-order conditions,

$$\begin{aligned} u_c(c_0, x_0)q_t(\epsilon^t) &= \beta^t h_t(\epsilon^t)u_c(c_t(\epsilon^t), x_t(\epsilon^t)), \\ u_c(c_t(\epsilon^t), x_t(\epsilon^t))w_t(\epsilon^t)(1-\tau_t(\epsilon^t)) &= u_x(c_t(\epsilon^t), x_t(\epsilon^t)), \end{aligned}$$

where $q_t(\epsilon^t)$ denotes the price at date $t = 0$ of a unit of the good at date t after history ϵ^t. (The resource and budget constraints imply the government's intertemporal budget constraint.)

The government's choice variables are consumption and leisure. Let ν and $-\beta^t\mu_t(\epsilon^t)h_t(\epsilon^t)$ denote the multipliers associated with the implementability and resource constraints, respectively. Suppressing histories to improve legibility, the first-order conditions are given by

$$\begin{aligned} (1+\nu)u_c(c_t, x_t) + \nu(u_{cc}(c_t, x_t)(c_t - {}_0b_t) - u_{xc}(c_t, x_t)(1-x_t)) &= \mu_t, \\ (1+\nu)u_x(c_t, x_t) + \nu(u_{cx}(c_t, x_t)(c_t - {}_0b_t) - u_{xx}(c_t, x_t)(1-x_t)) &= \mu_t w_t. \end{aligned}$$

The conditions state that the government accounts for three types of effects when increasing c_t or x_t. First, the direct effects on the objective function. Second, the resource costs, represented by the terms multiplying μ_t. And third, the marginal effects on the implementability constraint, represented by the terms multiplying ν. The latter effects reflect both higher outlays for consumption or leisure and changes of the marginal rates of substitution—corresponding to changed inter- and intratemporal prices.

Together with the implementability and resource constraints the first-order conditions fully characterize the Ramsey allocation. From the household's first-order conditions, the latter implies a history contingent sequence of optimal tax rates. Finally, from the government's intertemporal budget constraint, the taxes imply a unique sequence of optimal government indebtedness since the value of outstanding debt equals the market value of future primary surpluses.

We make three key observations. First, in stochastic environments, the optimal indebtedness implied by the Ramsey tax sequence generally is stochastic. That is, the rate of return on government debt is not risk-free. Second and related, the *shadow value of public funds*—the government's valuation of public relative to private sector wealth, ν—is constant over time and across histories. And third, tax rates at date t only depend on $({}_0b_t, w_t, g_t)$.

To understand the first and second point, recall that the implementability constraint incorporates all competitive equilibrium conditions beyond the resource constraint. The multiplier associated with the implementability constraint therefore gives the shadow cost of the competitive equilibrium requirement; specifically, it represents the cost of the fact that taxes distorts the allocation. With complete markets households smooth the shadow value of income over time and across histories, and a parallel result holds for the government. As a consequence, complete markets imply that the difference between the shadow value of government and private sector funds, ν, is constant.

If, in contrast, the government did not face complete markets but, to take an extreme example, had to satisfy a balanced budget restriction at each date and history then the single implementability constraint would be replaced by a history contingent sequence of such constraints with an associated sequence of multipliers. In that case, the government's inability to decouple the timing of tax collections on the one hand and government spending on the other would imply that the shadow cost of public funds varies over time and across histories. We return to this point in subsection 12.2.2.

To understand the third point on the structure of optimal taxes, note that the two first-order conditions combine to an equation in $(\nu, {}_0b_t, w_t, c_t, x_t)$ while the resource constraint contains the variables (g_t, w_t, c_t, x_t). Since the structure of either equation is history independent the equilibrium allocation at a date and history is an invariant function of the exogenous state, $({}_0b_t, w_t, g_t)$, as well as of the constant multiplier, ν. As a consequence, the tax rates in two histories are identical as well as long as the contemporaneous state is the same in the two histories. In environments with additional state variables, e.g. capital, this complete markets result generalizes.

To characterize the Ramsey tax policy in more detail, we manipulate the optimality conditions to find two auxiliary conditions,

$$\begin{aligned}(1+\nu)[u_c(c_t, x_t)(c_t - {}_0b_t) - u_x(c_t, x_t)(1-x_t)] + \nu Q_t + (g_t + {}_0b_t)\mu_t &= 0, \\ \nu Q + \sum_{t=0}^{\infty} \int \beta^t (g_t + {}_0b_t)\mu_t dH_t &= 0,\end{aligned}$$

where $Q_t < 0, Q < 0$, and $\mu_t > 0$.[2] The first condition implies that, absent government spending in a history ($g_t = {}_0b_t = 0$), the tax rate nevertheless is strictly positive when public funds are scarce ($\nu > 0$). The second condition states that the shadow value of public funds equals zero if the market value of government consumption and predetermined debt service equals zero.

To understand the implications of these findings, suppose first that $g_t + {}_0b_t = 0$ in all histories or that $\sum_{t=0}^{\infty} \int \beta^t u_c(c_t, x_t)(g_t + {}_0b_t)dH_t = 0$. As we have just seen, $\nu = 0$ in this case. The first-order conditions then imply that the allocation is not distorted, $u_c(c_t, x_t)w_t = u_x(c_t, x_t)$, and tax rates therefore equal zero. Intuitively, there is no point in levying distorting taxes when the government's income from initial asset holdings (negative ${}_0b_t$) suffices to finance government consumption.

When $\nu > 0$, in contrast, then the government needs to raise taxes. Suppose next that $(w_t, g_t, {}_0b_t)$ is constant across histories. As discussed above, (c_t, x_t) and thus, tax rates then are constant as well. Accordingly, the government budget is balanced at all times.

[2]The first equation results from multiplying the government's first-order conditions by $c_t - {}_0b_t$ and $x_t - 1$, respectively, summing them, and using the resource constraint. The second equation follows from integrating the first condition, weighting by β^t, summing over time, and using the intertemporal budget constraint.

Third, consider a deterministic environment with constant productivity, w, and $g_t = 0$ at all dates except at date $t = T$ when $g_T > 0$. (From now on, we let ${}_0b_t = 0$ in all histories.) Our findings imply that tax rates at all dates $t \neq T$ are constant and strictly positive. Intuitively, since tax distortions are convex in the tax rate, optimal tax rates vary less than government spending—the optimal *tax smoothing* policy spreads tax collections over time to reduce average tax distortions. Accordingly, the government accumulates assets before date $t = T$ and services debt thereafter.

Next, consider the same scenario except that at date $t = T$, government consumption is stochastic and can take two values: $g_T > 0$ or $g_T = 0$. Our findings imply that tax rates are constant and strictly positive except at date $t = T$ when $g_T > 0$. Intuitively, the tax smoothing prescription applies both over time and across histories. Implementing an equilibrium with constant tax revenue before and after date $t = T$ requires the government's indebtedness at date $t = T + 1$ to be independent of g_T. Since the government budget at date $t = T$ is not balanced this requires that the government's indebtedness at date $t = T$ is contingent: When $g_T > 0$ then government debt is lower than when $g_T = 0$. That is, between $t = T - 1$ and $t = T$, the rate of return on government debt is contingent on the realization of g_T—the private sector (partially) insures the government against the high government consumption shock.

Finally, if (w_t, g_t) follows a deterministic cycle then (c_t, x_t) and tax rates follow a deterministic cycle as well and the government's budget is balanced over the cycle. Similarly, if (w_t, g_t) follows a stationary Markov process then (c_t, x_t) and tax rates inherit the stochastic properties of the state.

We have seen that with complete markets and stochastic (w_t, g_t), the tax smoothing Ramsey policy relies on contingent government indebtedness. One mechanism to deliver such contingency is to make the coupon payment contingent on the realization of the state. A more subtle mechanism, which works even when coupons are risk-free, relies on an appropriate choice of maturity structure. Since shocks to productivity and government consumption alter equilibrium consumption they also affect the term structure of interest rates which in turn affects the market value of outstanding long-term debt. For example, a rise in interest rates devalues outstanding long-term debt while it does not devalue maturing liabilities. Generically, the contingent government indebtedness under the complete markets Ramsey policy is spanned by the contingent term structure of interest rates associated with the Ramsey allocation. That is, even with risk-free coupons on government debt, markets are complete as long as the maturity structure of government debt is sufficiently rich.

12.2.2 Incomplete Markets

Short-Term, Risk-Free Debt Assume now that the government only issues one-period debt, with a risk-free coupon, implying that government indebtedness is non-contingent. The complete markets Ramsey allocation characterized in subsection 12.2.1 may no longer be implementable in this case and the properties of the Ramsey policy change.

Let $b_t(\epsilon^{t-1})$ denote debt with a safe return at date t that is, claims vis-a-vis the government that are due at date t in any history subsequent to the specific history ϵ^{t-1}

(the claims are measurable with respect to ϵ^{t-1} rather than ϵ^t as before). For convenience, we assume that productivity equals unity at all times. Using the resource constraint and adopting a shorthand notation, let $u_t(\epsilon^t) \equiv u(c_t(\epsilon^t), 1 - c_t(\epsilon^t) - g_t(\epsilon^t))$, $u_{c,t}(\epsilon^t) \equiv u_c(c_t(\epsilon^t), 1 - c_t(\epsilon^t) - g_t(\epsilon^t))$, and similarly for $u_{x,t}(\epsilon^t)$.

In competitive equilibrium, the household satisfies its intratemporal first-order condition and stochastic Euler equation (due to market incompleteness); the household or equivalently, the government satisfies its dynamic budget constraint; government debt or assets are bounded; and the resource constraint is met. From the intratemporal first-order condition and the resource constraint, we can express the government's primary surplus, $\tau_t(\epsilon^t)(1 - x_t(\epsilon^t)) - g_t(\epsilon^t)$, as

$$s_t(\epsilon^t) \equiv \left(1 - \frac{u_{x,t}(\epsilon^t)}{u_{c,t}(\epsilon^t)}\right)(c_t(\epsilon^t) + g_t(\epsilon^t)) - g_t(\epsilon^t).$$

Accordingly, the government's dynamic budget constraint incorporating the household optimality conditions and the resource constraint reads

$$b_t(\epsilon^{t-1}) \leq s_t(\epsilon^t) + \beta \mathbb{E}_t \left[\frac{u_{c,t+1}(\epsilon^{t+1})}{u_{c,t}(\epsilon^t)} b_{t+1}(\epsilon^t)\right],$$

where we assume that the government may pay lump-sum transfers, thus the inequality constraint.

Iterating this equation (with equality) forward, applying the law of iterated expectations, and assuming $\lim_{T\to\infty} \beta^T u_{c,T}(\epsilon^T) = 0$ almost surely, yields the implementability constraint

$$u_{c,0} b_0 = \sum_{t=0}^{\infty} \int \beta^t u_{c,t} s_t dH_t(\epsilon^t),$$

where we suppress histories to improve legibility. (We will go back and forth between suppressing histories or not.) There are two differences between this implementability constraint and the one in subsection 12.2.1. First, the constraint here incorporates the resource constraint. Second, it is derived from the government's rather than the private sector's intertemporal budget constraint. Neither difference is important.

However, the implementability constraint does not yet reflect the restriction that indebtedness be non-contingent (see also the discussion of equation (4.1) on page 47). To incorporate this restriction, we additionally impose the condition that debt at date t and thus, the present value of primary surpluses from date t onwards be the same for all ϵ^t conditional on ϵ^{t-1}. This *measurability constraint* can be stated as

$$u_{c,t} b_t = \sum_{s=t}^{\infty} \int \beta^{s-t} u_{c,s} s_s dH_s(\epsilon^s | \epsilon^t) \; \forall \epsilon^t | \epsilon^{t-1},\ t \geq 1,$$

where b_t on the left-hand side of the condition is measurable with respect to ϵ^{t-1}. Note that the measurability constraint at date $t \geq 1$ has the same form as the implementability constraint that holds at date $t = 0$. We also impose boundedness conditions, requiring that b_t and thus, the right-hand side of the measurability constraint normalized by $u_{c,t}$, lies between some bounds $\underline{M}$ and $\bar{M}$.

Let $\beta^t h_t(\epsilon^t)\gamma_t(\epsilon^t)$ denote the multiplier associated with the implementability constraint (for $t = 0$) and the measurability constraints (for $t \geq 1$) with $\gamma_0 \leq 0$; and let $\beta^t h_t(\epsilon^t)\xi_{1,t}(\epsilon^t)$ and $\beta^t h_t(\epsilon^t)\xi_{2,t}(\epsilon^t)$ denote the multipliers associated with the upper and lower bounds, respectively. The Lagrangian of the government's program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \int \beta^t \Big\{u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t}\bar{M} - \xi_{2,t}\underline{M}) \\ -(\gamma_t + \xi_{1,t} - \xi_{2,t}) \left(\sum_{s=t}^{\infty} \int \beta^{s-t} u_{c,s} s_s dH_s(\epsilon^s|\epsilon^t) \right) \Big\} dH_t(\epsilon^t).$$

This can be rewritten as

$$\mathcal{L} = \sum_{t=0}^{\infty} \int \beta^t \left\{ u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t}\bar{M} - \xi_{2,t}\underline{M}) - u_{c,t} s_t \sum_{s=0}^{t} (\gamma_s + \xi_{1,s} - \xi_{2,s}) \right\} dH_t(\epsilon^t)$$

or

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \int \beta^t \left\{ u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t}\bar{M} - \xi_{2,t}\underline{M}) - u_{c,t} s_t \nu_t \right\} dH_t(\epsilon^t) \\ \text{s.t.} \quad & \nu_t = \nu_{t-1} + \gamma_t + \xi_{1,t} - \xi_{2,t},\ \nu_{-1} = 0. \end{aligned}$$

Differentiating with respect to $c_t(\epsilon^t)$ and $b_{t+1}(\epsilon^t)$ yields the first-order conditions

$$\begin{aligned} u_{c,t} - u_{x,t} - \nu_t((u_{cc,t} - u_{cx,t})s_t + u_{c,t}s_{c,t}) + (u_{cc,t} - u_{cx,t})(\gamma_t b_t + \xi_{1,t}\bar{M} - \xi_{2,t}\underline{M}) &= 0, \\ \int \gamma_{t+1} u_{c,t+1} dH_{t+1}(\epsilon^{t+1}|\epsilon^t) &= 0, \end{aligned}$$

respectively. The first optimality condition relates the allocation in a period to the level of debt as well as to multipliers which may not only vary over time ($\gamma_t, \xi_{1,t}, \xi_{2,t}$) but also accumulate ($\nu_t$). The second condition states that the shadow cost of the measurability constraint in utility terms, $\gamma_{t+1} u_{c,t+1}$, should equal zero on average.

To understand these conditions, consider first the hypothetical complete-markets case where debt service at date t is measurable with respect to ϵ^t rather than ϵ^{t-1} (and $\xi_{1,t} = \xi_{2,t} = 0$). The second optimality condition then changes to $\gamma_{t+1} = 0$ and ν_t is constant across histories; the optimality conditions thus reduce to the equivalent of the conditions in subsection 12.2.1. Intuitively, with complete markets, there is no cost associated with having to satisfy the intertemporal budget constraint after the initial period, conditional on satisfying it in the initial period. Stated differently, the optimal choice of contingent indebtedness equalizes the shadow cost of the government budget constraint across histories.

With incomplete markets, in contrast, the government cannot equalize the shadow cost across histories. With a risk-free debt position it can only equalize this cost on average, over time. After a negative shock to the budget the intertemporal budget constraint tightens ($\gamma_t < 0$ and ν_t decreases) and going forward, the Ramsey allocation is more distorted than it would have been after a positive shock which leads to an increase of ν_t. The tightening or relaxation of the budget constraint is permanently reflected in the multiplier ν and thus, in the Ramsey allocation and tax policy. In contrast to the complete markets case, tax rates therefore do not only inherit the stochastic properties of the government consumption shock but also reflect its history.

It is instructive to consider the special case with linear utility, $u(c, x) = c + H(x)$ where H is increasing and concave. From the household's first-order conditions, the asset pricing kernel then equals β, and $\tau_t = 1 - H'(x)$. Tax revenue thus is a function of x_t,

$$\rho(x_t) = (1 - H'(x_t))(1 - x_t)$$

say. Under standard assumptions, it is a strictly concave function over the domain $[\underline{x}, \bar{x}]$ where $\underline{x}$ denotes the undistorted level of leisure ($\rho(\underline{x}) = 0$) and $\bar{x} < 1$ denotes the level where $\rho(\bar{x})$ attains the maximum of the Laffer curve. Inverting ρ yields leisure as a strictly convex function of tax revenue, $\chi(\rho_t)$ say. Function χ is defined over the domain $[0, \rho(\bar{x})]$.

Since utility at date t equals $1 - \chi(\rho_t) - g_t + H(\chi(\rho_t))$ we may formulate the Ramsey program with tax revenue and debt as the choice variables. The program reads

$$\begin{aligned}
\max_{\{\rho_t(\epsilon^t), b_{t+1}(\epsilon^t)\}_{t\geq 0}} \quad & -\sum_{t=0}^{\infty} \int \beta^t \mathcal{D}(\rho_t) dH_t(\epsilon^t) \\
\text{s.t.} \quad & b_t \leq \rho_t - g_t + \beta b_{t+1}, \\
& \underline{M} \leq b_{t+1} \leq \bar{M},
\end{aligned}$$

where we define the *deadweight loss* $\mathcal{D}(\rho_t) \equiv \chi(\rho_t) - H(\chi(\rho_t))$. Note that $\mathcal{D}$ is strictly convex over the domain $[0, \rho(\bar{x})]$ and reaches a minimum at $\rho_t = 0$.

This Ramsey program is isomorphic to a consumption-saving problem with the utility function $-\mathcal{D}(\rho_t)$, which has a bliss point at $\rho_t = 0$ (no deadweight loss); an interest rate equal to the inverse of the time discount factor; negative income shocks (government consumption); an asset with a risk-free return; and the natural borrowing limit. If the Markov process for government consumption has a nontrivial invariant distribution then "utility" in this program converges to the bliss point that is, the Ramsey tax rate converges to zero. The government accumulates a sufficiently large stock of assets to finance an infinite sequence of maximum government consumption; whenever the realization of government consumption is lower than its maximum value the government pays lump-sum transfers to the households.

This can also be seen from the optimality conditions derived earlier. Under the linear utility assumption these conditions are given by

$$\begin{aligned}
1 - H'(x_t) - \nu_t[1 - H'(x_t) + (1 - x_t)H''(x_t)] &= 0, \\
\nu_t = \nu_{t-1} + \gamma_t + \xi_{1,t} - \xi_{2,t}, \ \nu_{-1} &= 0, \\
\mathbb{E}_t[\gamma_{t+1}] &= 0,
\end{aligned}$$

where the first condition can be expressed as $-\mathcal{D}'(\rho_t) = \nu_t$. Whenever the debt limits do not bind ν_t and thus, the marginal deadweight loss follows a martingale. That is, the Ramsey policy keeps the expected marginal tax distortion constant over time.

Compare this to a complete markets environment where the Ramsey policy keeps the marginal tax distortion constant across all histories.

Since the government can pay lump-sum transfers, $\xi_{2,t} = 0$, ν_t is nonpositive, and $\mathbb{E}_t[\nu_{t+1}] = \nu_t + \mathbb{E}_t[\xi_{1,t+1}] \geq \nu_t$ that is, ν_t is a nonpositive submartingale. If the process for government consumption has an absorbing state then ν_t and taxes converge to a strictly negative and positive value, respectively. Otherwise, ν_t and taxes converge to zero as the government accumulates assets. With an ad-hoc restriction on the accumulation of government assets, $\xi_{2,t}$ may differ from zero; this undermines the convergence result.

A Broader Portfolio Returning to the case with general preferences, assume next that the government holds a broader portfolio of liabilities and assets, including physical capital. Markets are incomplete.

We assume that a Markov process governs government consumption and productivity, and we formulate the Ramsey program recursively. Output $f(k_\circ, 1 - x_\circ(\epsilon_\circ), \epsilon_\circ)$ depends on the predetermined capital stock, $k_\circ$, labor input, $1 - x_\circ(\epsilon_\circ)$, and a productivity shock, reflected by $\epsilon_\circ$. To simplify the notation we let $u_c(\epsilon_\circ) \equiv u_c(c_\circ(\epsilon_\circ), x_\circ(\epsilon_\circ))$, $f_K(\epsilon_\circ) \equiv f_K(k_\circ, 1 - x_\circ(\epsilon_\circ), \epsilon_\circ)$, etc.

The state at the beginning of a period, *before* the realization of the shock, includes the economy's capital stock, $k_\circ$; the government's net liabilities, $b_\circ$; the shock in the previous period, ϵ_-; and marginal utility in the previous period, θ_-. The choice variables in the government's program include the risk-free interest rate on government debt, $R_\circ$; government holdings of capital, $k_\circ^g$; exposures to arbitrary securities (in zero net supply), $\{e_\circ^i\}_i$, with exogenous gross returns $\{R^i(\epsilon_\circ)\}_i$; as well as variables which vary with the shock realization, namely consumption and leisure, $c_\circ(\epsilon_\circ)$ and $x_\circ(\epsilon_\circ)$; the capital stock at the beginning of the subsequent period, $k_+(\epsilon_\circ)$; government net liabilities at the beginning of the subsequent period, $b_+(\epsilon_\circ)$; and the labor income tax rate, $\tau_\circ(\epsilon_\circ)$. (In the initial period, the risk-free interest rate is given.)

The constraints of the government's program are given by

$$\begin{aligned}
\theta_- &= \beta \mathbb{E}[u_c(\epsilon_\circ) R_\circ | \epsilon_-], \\
\theta_- &= \beta \mathbb{E}[u_c(\epsilon_\circ)(1 + f_K(\epsilon_\circ) - \delta) | \epsilon_-], \\
\theta_- &= \beta \mathbb{E}[u_c(\epsilon_\circ) R^i(\epsilon_\circ) | \epsilon_-], \\
\tau_\circ(\epsilon_\circ) &= 1 - \frac{u_x(\epsilon_\circ)}{u_c(\epsilon_\circ) f_L(\epsilon_\circ)}, \\
R_\circ b_\circ - \omega_\circ(\epsilon_\circ) + g_\circ(\epsilon_\circ) &\leq \tau_\circ(\epsilon_\circ)(1 - x_\circ(\epsilon_\circ)) f_L(\epsilon_\circ) + b_+(\epsilon_\circ), \\
c_\circ(\epsilon_\circ) + g_\circ(\epsilon_\circ) + k_+(\epsilon_\circ) &= (1 - \delta) k_\circ + f(\epsilon_\circ), \\
\underline{M}(\cdot) \leq u_c(\epsilon_\circ) b_+(\epsilon_\circ) &\leq \bar{M}(\cdot),
\end{aligned}$$

where

$$\omega_\circ(\epsilon_\circ) \equiv \sum_i e_\circ^i (R^i(\epsilon_\circ) - R_\circ) + k_\circ^g (1 + f_K(\epsilon_\circ) - \delta - R_\circ)$$

denotes the return on the government's portfolio $(\{e_\circ^i\}_i, k_\circ^g)$. The first three constraints represent the household's Euler equations for risk-free government debt, capital, and the other assets. The fourth constraint relates the labor income tax rate to the household's marginal rate of substitution. The remaining constraints represent the (government) budget constraint, the resource constraint, and the debt limits.

Using the household's first-order conditions to substitute out $\tau_\circ$ and $R_\circ$ and letting $\tilde{b}_\circ \equiv b_\circ \theta_-$, we can express the budget constraint as

$$\left(\frac{\tilde{b}_\circ}{\beta \mathbb{E}[u_c(\epsilon_\circ)|\epsilon_-]} - \tilde{\omega}_\circ(\epsilon_\circ) + g_\circ(\epsilon_\circ)\right) u_c(\epsilon_\circ) \leq (u_c(\epsilon_\circ) f_L(\epsilon_\circ) - u_x(\epsilon_\circ))(1 - x_\circ(\epsilon_\circ)) + \tilde{b}_+(\epsilon_\circ),$$

where $\tilde{\omega}_\circ(\epsilon_\circ)$ differs from $\omega_\circ(\epsilon_\circ)$ in that $R_\circ$ is replaced by $\theta_- / (\beta \mathbb{E}[u_c(\epsilon_\circ)|\epsilon_-])$. The constraint set of the government is characterized by this modified budget constraint as well as the Euler equations, the resource constraint, and the debt limits. The Bellman equation reads

$$\begin{aligned} V(k_\circ, \tilde{b}_\circ, \theta_-, \epsilon_-) &= \max \mathbb{E}[u(\epsilon_\circ) + \beta V(k_+(\epsilon_\circ), \tilde{b}_+(\epsilon_\circ), u_c(\epsilon_\circ), \epsilon_\circ)|\epsilon_-] \\ \text{s.t.} & \quad \text{constraint set,} \end{aligned}$$

and the choice variables are $k_\circ^g, \{e_\circ^i\}_i, \{c_\circ(\epsilon_\circ), x_\circ(\epsilon_\circ), k_+(\epsilon_\circ), \tilde{b}_+(\epsilon_\circ)\}_{\epsilon_\circ}$. Note that in accordance with our definition of the state, the value function represents the unconditional value, "before" the realization of $\epsilon_\circ$.

Let $\nu_\circ(\epsilon_\circ) \cdot \text{prob}(\epsilon_\circ|\epsilon_-)$ denote the multiplier associated with the budget constraint (the shadow value of public funds) when $\epsilon_\circ$ is realized. The government's first-order conditions with respect to $\tilde{b}_+(\epsilon_\circ)$ (assuming debt limits do not bind), $e_\circ^i$, and $k_\circ^g$, respectively, are given by

$$\begin{aligned} &\nu_\circ(\epsilon_\circ) + \beta V_b(k_+(\epsilon_\circ), \tilde{b}_+(\epsilon_\circ), u_c(\epsilon_\circ), \epsilon_\circ) = 0, \\ &\mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ)(R^i(\epsilon_\circ) - R_\circ)|\epsilon_-] = 0, \\ &\mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ)(1 + f_K(\epsilon_\circ) - \delta - R_\circ)|\epsilon_-] = 0, \end{aligned}$$

and the envelope condition implies

$$V_b(k_\circ, \tilde{b}_\circ, \theta_-, \epsilon_-) = -\sum_{\epsilon_\circ} \nu_\circ(\epsilon_\circ) \text{prob}(\epsilon_\circ|\epsilon_-) \frac{u_c(\epsilon_\circ)}{\beta \mathbb{E}[u_c(\epsilon_\circ)|\epsilon_-]} = -\frac{R_\circ}{\theta_-} \mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ)|\epsilon_-].$$

Combined, these equations yield the optimality conditions

$$\begin{aligned} \nu_-(\epsilon_-)\theta_- &= \beta \mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ) R_\circ|\epsilon_-], \\ \nu_-(\epsilon_-)\theta_- &= \beta \mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ) R^i(\epsilon_\circ)|\epsilon_-], \\ \nu_-(\epsilon_-)\theta_- &= \beta \mathbb{E}[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ)(1 + f_K(\epsilon_\circ) - \delta)|\epsilon_-]. \end{aligned}$$

These conditions resemble the stochastic Euler equations characterizing a household's portfolio choice. They differ insofar as marginal utility is replaced by the product of marginal utility and the shadow value of public funds. Intuitively, the Ramsey policy equalizes the (average) valuation of public funds exactly as a household equalizes (average) marginal utility.

Note that the first condition coincides with the result derived earlier, namely that the change of the government budget multiplier, weighted by marginal utility, equals zero on average. This follows from

$$\mathbb{E}\left[\beta R_\circ \nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ) - \nu_-(\epsilon_-)\theta_- | \epsilon_-\right] = \beta R_\circ \mathbb{E}\left[(\nu_\circ(\epsilon_\circ) - \nu_-(\epsilon_-)) u_c(\epsilon_\circ) | \epsilon_-\right] = 0.$$

The second and third optimality condition generalize this result. For any asset or liability in the government's portfolio, the Ramsey policy satisfies

$$\beta \mathbb{E}\left[\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ) \left(R^i(\epsilon_\circ) - R_\circ\right) | \epsilon_-\right] = 0.$$

That is, a more diversified portfolio results in a smoother multiplier and thus, better insurance for the government. If the portfolio were sufficiently diversified for the government (and the household) to face complete markets then the multiplier would be constant across histories.

The optimality conditions can be used to derive an asset pricing kernel for government projects. Letting $\mu_\circ(\epsilon_\circ) \cdot \text{prob}(\epsilon_\circ|\epsilon_-)$ denote the multiplier associated with the resource constraint when shock $\epsilon_\circ$ is realized, this kernel is given by

$$\beta \frac{\nu_\circ(\epsilon_\circ) u_c(\epsilon_\circ) + \mu_\circ(\epsilon_\circ)}{\nu_-(\epsilon_-)\theta_- + \mu_-(\epsilon_-)}.$$

12.2.3 Capital Income Taxation

A Neutrality Result Consider the model with capital of subsection 12.2.2 and assume that the government may impose state-contingent tax rates on the return on capital, in addition to labor income. Capital income tax rates, $\tau^k_\circ(\epsilon_\circ)$, only enter the household's Euler equation for capital and the government's budget constraint:

$$\begin{aligned}
&\theta_- = \beta \mathbb{E}[u_c(\epsilon_\circ)(1 + (1 - \tau^k_\circ(\epsilon_\circ))(f_K(\epsilon_\circ) - \delta))|\epsilon_-], \\
&R_\circ b_\circ - \omega_\circ(\epsilon_\circ) + g_\circ(\epsilon_\circ) \leq \tau_\circ(\epsilon_\circ)(1 - x_\circ(\epsilon_\circ)) f_L(\epsilon_\circ) + \tau^k_\circ(\epsilon_\circ) k_\circ (f_K(\epsilon_\circ) - \delta) + b_+(\epsilon_\circ).
\end{aligned}$$

This implies a neutrality result: Optimal state-contingent capital income tax rates are indeterminate if the government faces complete markets, reflected in a portfolio with state-contingent returns, $\omega_\circ(\epsilon_\circ)$.

Intuitively, from the household's Euler equation, a redistribution of capital income taxes across states does not affect capital accumulation as long as the average tax wedge (suitably weighted) does not change. While such a redistribution alters the government's revenue in each of the affected states, this can be offset by adjusting the government portfolio as long as markets are complete. An equilibrium allocation thus can be implemented, for instance, either with state-contingent capital income taxes and bonds with risk-free coupons, or with risk-free capital income taxes and bonds with state-contingent coupons.

Zero Capital Income Taxation Consider a deterministic setting and suppose that the economy is inhabited by an infinitely lived, representative agent. The implementability constraint incorporates consumption and leisure sequences, $\{c_t, x_t\}_{t\geq 0}$, as well as the initial capital stock including interest, $k_0 R_0$, and tax rate, τ_0^k, (both predetermined). Write this constraint as

$$\iota_0(c_0, x_0, k_0 R_0, \tau_0^k) + \sum_{t=0}^{\infty} \beta^t \iota(c_t, x_t) = 0$$

for some functions ι and ι_0. The Lagrangian associated with the Ramsey program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{v(c_t, x_t, \nu) - \mu_t(c_t + g_t + k_{t+1} - (1-\delta)k_t - f(k_t, 1-x_t))\} + \nu \iota_0(c_0, x_0, k_0 R_0, \tau_0^k),$$

where we define $v(c_t, x_t, \nu) \equiv u(c_t, x_t) + \nu\, \iota(c_t, x_t)$; ν and $\beta^t \mu_t$ denote the multipliers associated with the implementability and resource constraints, respectively.

The first-order conditions with respect to $c_t, t \geq 1$, and k_{t+1}, respectively, are given by

$$\begin{aligned} v_c(c_t, x_t, \nu) &= \mu_t,\ t \geq 1, \\ \mu_t &= \beta \mu_{t+1}(1 - \delta + f_K(k_{t+1}, 1 - x_{t+1})). \end{aligned}$$

Combining these conditions yields the key equation of interest which we report together with the household's Euler equation:

$$\begin{aligned} v_c(c_t, x_t, \nu) &= \beta(1 - \delta + f_K(k_{t+1}, 1 - x_{t+1}))v_c(c_{t+1}, x_{t+1}, \nu),\ t \geq 1, \\ u_c(c_t, x_t) &= \beta(1 + (1 - \tau_{t+1}^k)(f_K(k_{t+1}, 1 - x_{t+1}) - \delta))u_c(c_{t+1}, x_{t+1}). \end{aligned}$$

The two equations imply that under the Ramsey policy capital income is not taxed for $t \geq 2$ whenever $v_c(c_t, x_t, \nu)$ and $u_c(c_t, x_t)$ grow at the same rate. Two alternative assumptions guarantee such equal growth and thus, optimality of *zero capital income taxation*. The first assumption relates to preferences. If preferences are separable between consumption and leisure, and homothetic, then $v_c(c_t, x_t, \nu)$ is proportional to $u_c(c_t, x_t)$. In this case, the zero capital taxation result is an instance of the uniform commodity taxation result discussed in subsection 12.1.1. Recall from subsection 11.3.2 that capital income taxation is equivalent to time varying taxation of consumption. When consumption taxes are normalized to zero and the structure of preferences calls for uniform taxation of consumption, capital income therefore must not be taxed. Note that standard CIES preferences satisfy the separability and homotheticity condition.

The second, alternative assumption is the steady-state assumption. If the Ramsey allocation converges to a steady state then both $v_c(c_t, x_t, \nu)$ and $u_c(c_t, x_t)$ are constant over time and the optimality of zero capital income taxation follows. In fact, it also follows in richer environments as long as in steady state, the derivative of the implementability constraint(s) with respect to capital equal(s) zero and the multiplier(s) of the constraint(s) are constant. (This holds true, for example, in the steady state of an economy with heterogenous households whose capital income—but not wage income—is taxed at a uniform rate.) Note that a strictly positive steady-state capital income tax rate would give rise to an ever increasing tax wedge between early and late consumption.

An upper bound on the tax rate, $\tau^k_{t+1} \leq \bar{\tau}^k$, which can be expressed as

$$1 - \left(\frac{u_c(c_t, x_t)}{\beta u_c(c_{t+1}, x_{t+1})} - 1 \right) / (f_K(k_{t+1}, 1 - x_{t+1}) - \delta) \leq \bar{\tau}^k,$$

introduces additional terms in the government's optimality conditions. As long as this constraint binds (forcing taxes to be spread over a longer horizon) the first key equation above contains additional terms, and this undermines the $\tau^k = 0$ implication. The constraint may bind forever.

12.2.4 Heterogeneous Households

When households are homogeneous the assumption that the government levies distorting taxes makes little sense: If everybody is the same, non-distorting lump-sum taxes clearly are preferable, and feasible. With heterogenous agents, in contrast, government policy is motivated by distributive concerns in addition to efficiency considerations. These concerns give rise to a trade-off between equity and efficiency and can rationalize taxation of an endogenous tax base.

Consider a variant of the economy analyzed in subsection 12.2.1 with two rather than one group of households, groups a and b with population shares η^a and $\eta^b = 1 - \eta^a$ respectively. Both groups have the same preferences but their labor productivities may differ, $w^a_t \neq w^b_t$. For simplicity, we abstract from risk. The resource constraint is given by

$$\eta^a c^a_t + \eta^b c^b_t + g_t = \eta^a w^a_t (1 - x^a_t) + \eta^b w^b_t (1 - x^b_t).$$

In each period, the government has two tax instruments at its disposal: A proportional labor income tax levied at rate τ_t, and a lump-sum tax, θ_t. The government's objective function is given by $\omega \eta^a U^a + \eta^b U^b$ where U^i denotes welfare of a member of group i and ω denotes some weight. The intertemporal budget constraint of a household in group $i = a, b$ reads

$$\sum_{t=0}^{\infty} q_t [c^i_t - (1 - \tau_t) w^i_t (1 - x^i_t) + \theta_t] = 0,$$

where q_t denotes the price at date $t = 0$ of a unit of the good at date t. The household first-order conditions

$$\begin{aligned} u_c(c^i_0, x^i_0) q_t &= \beta^t u_c(c^i_t, x^i_t), \\ u_c(c^i_t, x^i_t) w^i_t (1 - \tau_t) &= u_x(c^i_t, x^i_t) \end{aligned}$$

and the budget constraints yield the implementability constraints

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t [u_c(c_t^i, x_t^i)(c_t^i + \theta_t) - u_x(c_t^i, x_t^i)(1 - x_t^i)] = 0, \\
&u_c(c_t^a, x_t^a)/u_c(c_0^a, x_0^a) = u_c(c_t^b, x_t^b)/u_c(c_0^b, x_0^b), \\
&u_c(c_t^a, x_t^a) w_t^a / u_x(c_t^a, x_t^a) = u_c(c_t^b, x_t^b) w_t^b / u_x(c_t^b, x_t^b).
\end{aligned}$$

Let ν^a and ν^b denote the multipliers associated with the first two (intertemporal) implementability constraints. The optimal choice of lump-sum tax at date t satisfies

$$\nu^a u_c(c_t^a, x_t^a) + \nu^b u_c(c_t^b, x_t^b) = 0,$$

that is, the Ramsey policy sets the average multiplier equal to zero. To understand this result suppose first that group b did not exist such that only the first implementability constraint were present. The optimality condition for θ_t then would collapse to $\nu^a = 0$, indicating that the competitive equilibrium constraint is not binding for the Ramsey government. Intuitively, the Ramsey policy could implement the first best because the government could costlessly transfer resources from the private to the public sector.

With heterogeneous households, the lump-sum tax still allows the government to extract resources without distorting household choices. But since the lump-sum tax cannot be differentiated across groups the government generally does not reach first best (with respect to its objective function). Ideally, the government would costlessly transfer resources not only from the private to the public sector but also from the less to the more favored group. Since the latter is not possible, the choice of θ_t "at least" equalizes the average value of the multiplier with zero. If, by chance, the optimal lump-sum tax happens to implement the desired wealth distribution, then ν^a and ν^b individually equal zero as well. Otherwise, the Ramsey policy also employs labor income taxes, at the cost of generating tax distortions.

From the third implementability constraint, marginal utility grows at identical rates across groups. This implies that all but one lump-sum tax are redundant instruments (their first-order conditions are multiples of each other). With multiple lump-sum taxes, a Ricardian equivalence result applies: A change of timing of lump-sum taxes accompanied by suitable debt operations does not alter the equilibrium allocation.

To see how the timing of labor income taxes can affect the wealth distribution let $u(c, x) \equiv \ln(c) + \gamma \ln(x)$ and disregard lump-sum taxes. The implementability constraints then read

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t [1 - \gamma(1 - x_t^i)/x_t^i] = 0, \\
&\frac{c_0^a}{c_0^b} = \frac{c_t^a}{c_t^b} = \frac{w_t^a}{w_t^b} \frac{x_t^a}{x_t^b}
\end{aligned}$$

or

$$\sum_{t=0}^{\infty} \beta^t \left[1 - \gamma \left(\frac{1}{x_t^a} - 1\right)\right] = 0,$$
$$\sum_{t=0}^{\infty} \beta^t \left[1 - \gamma \left(\frac{c_0^a}{c_0^b} \frac{w_t^b}{w_t^a} \frac{1}{x_t^a} - 1\right)\right] = 0.$$

From the first equation, raising x_0^a requires lowering x_t^a at some other date. From the second equation, this translates into a change of (time invariant) relative consumption, c_0^a/c_0^b, if relative productivity varies over time, $w_0^a/w_0^b \neq w_t^a/w_t^b$. Specifically, an increase in x_0^a (corresponding to a tax hike at date $t = 0$) and corresponding decrease of x_t^a raises c_0^a/c_0^b if $w_0^a/w_0^b \leq w_t^a/w_t^b$. Wealth is redistributed for two reasons. First, collecting taxes in periods where one group is relatively more productive shifts the tax burden to that group. Second, tax induced changes in interest rates affect debtors and creditors asymmetrically.

12.3 Social Insurance and Saving Taxation

When households are exposed to idiosyncratic risk the government can provide insurance by redistributing among agents ex post. When outcomes depend on households' efforts, however, then insurance undermines incentives, giving rise to a trade-off between insurance and incentives.

To analyze the consequences of this trade-off for the taxation of saving, consider a two-period economy with a continuum of measure one of ex ante identical households, indexed by i. Household i receives an endowment w_0 in the first period which can be consumed or saved at the gross interest rate R_1. In the second period, the household works and consumes. Labor productivity, w_1^i, is random and i.i.d. across the population. Household i's preferences are given by

$$U^i \equiv u(c_0^i) + \beta \mathbb{E}_0[u(c_1^i(\epsilon^1)) + v(x_1^i(\epsilon^1))],$$

where u and v are strictly increasing and concave.

Labor productivity and labor supply are private information. An insurance scheme in the second period thus can only be conditioned on (c_0^i, c_1^i, y_1^i) where $y_1^i \equiv w_1^i(1 - x_1^i)$ denotes labor income. The government maximizes the social welfare function $\int_i U^i di$ subject to the constraint

$$\left(w_0 - \int_i c_0^i di\right) R_1 + \int_i w_1^i(1 - x_1^i) di = \int_i c_1^i di$$

as well as incentive compatibility constraints which map (c_0^i, w_1^i) and the insurance scheme into the household's optimal labor supply.

Rather than studying a particular insurance scheme we adopt a general approach. By the *revelation principle* without loss of generality we may restrict attention to direct mechanisms that induce truth telling. A mechanism maps household i's self-reported productivity level, ρ_1^i, together with the observed c_0^i into (c_1^i, y_1^i). It induces truth telling if households find it optimal to choose $\rho_1^i = w_1^i$ for all possible w_1^i.

Let $(c_0^\star, c_1^\star, x_1^\star)$ denote a consumption and leisure profile under the optimal mechanism and consider a marginal change of allocation that has no effect on households' incentives. Such a change results from a reduction of c_0 by $\Delta/u'(c_0^\star)$ (more saving) where $\Delta > 0$ is small, and an increase of $c_1^\star(w_1^i)$ by $\Delta/(\beta u'(c_1^\star(w_1^i)))$ for all realizations w_1^i. Note that the policy change does not affect utility along the equilibrium path, for any w_1^i.

If the initial allocation is optimal then the policy change must be resource neutral, otherwise the government could have done better. Optimality thus requires

$$\frac{\Delta}{u'(c_0^\star)} R_1 = \mathbb{E}_0 \left[\frac{\Delta}{\beta u'(c_1^\star(w_1^i))} \right],$$

that is, the gain in second period resources (represented on the left-hand side of the equation) equals the loss (represented on the right-hand side, using the fact that w_1^i is i.i.d.). Cancelling Δ we have established that the optimal allocation satisfies a *reciprocal Euler equation*.

Absent incentive problems the government could perfectly insure consumption in the second period; dropping the expectations operator and inverting the condition would then yield the standard Euler equation. If the insurance scheme needs to provide incentives, in contrast, then consumption in the second period must be random (dependent on observed labor income and thus, unobserved productivity) and Jensen's inequality implies

$$u'(c_0^\star) < \beta R_1 \mathbb{E}_0 \left[u'(c_1^\star(w_1^i)) \right].$$

Since households can save at the risk-free interest rate R_1 this implies that the optimal mechanism must impose a tax on the return on saving.

Intuitively, with moral hazard, saving imposes a social cost, in addition to the cost and benefit internalized by a household. Specifically, a larger asset position reduces the covariance between labor supply and consumption in the second period and thus, undermines incentives. The optimal policy addresses this adverse effect by discouraging saving.

12.4 Optimal Monetary Policy

12.4.1 Friedman Rule

12.4.2 Managing Price Rigidity

12.5 Bibliographic Notes

Ramsey (1927) introduces the primal approach. Atkinson and Stiglitz (1972) establish the results discussed in subsection 12.1.1. Diamond and Mirrlees (1971*a*; 1971*b*) introduce the dual approach and prove the production efficiency result. The many-person Ramsey rule is due to Diamond (1975). Atkinson and Stiglitz (1976) show that commodity taxes are superfluous, with a linear labor income tax if specific conditions are satisfied; or with a nonlinear labor income tax as discussed by Mirrlees (1971) if preferences are weakly separable between consumption and leisure.

The model presented in subsection 12.2.1 is due to Lucas and Stokey (1983). Angeletos (2002) shows that markets are complete even with non-contingent coupons as long as the maturity structure is sufficiently rich (see also Gale, 1990). Aiyagari, Marcet, Sargent and Seppälä (2002), Werning (2003), and Farhi (2010) analyze the model presented in subsection 12.2.2. The special case with linear utility provides micro foundations for Barro's (1979) and Bohn's (1990) tax-smoothing results. The indeterminacy result for optimal capital income taxes is due to Zhu (1992) and Chari, Christiano and Kehoe (1994). Chamley (1986) and Chari et al. (1994) derive the optimality of zero capital income taxes for CIES preferences. Chamley (1986) and Judd (1985) derive steady-state results and Straub and Werning (2014) clarify the conditions under which these results apply; see also Chari, Nicolini and Teles (2016). Farhi (2010) derives optimal capital income taxes in the model discussed in subsection 12.2.2. Subsection 12.2.4 follows Werning (2007) and Niepelt (2004*b*). Atkinson and Sandmo (1980) and King (1980) derive conditions for optimal capital income taxation in the OLG model.

Diamond and Mirrlees (1978), Rogerson (1985), and Golosov, Kocherlakota and Tsyvinski (2003) establish the optimality of an intertemporal wedge in the presence of moral hazard.

For an overview over the optimal taxation and Ramsey policy literature, see Atkinson and Stiglitz (1980, 12–14), Stiglitz (1987), Chari and Kehoe (1999), and Salanié (2003, 3–5). Woodford (2003), Galí (2008, 4, 5), and Walsh (2017, 8) discuss optimal policy in the New Keynesian model and provide an overview over the related literature.

Chapter 13

Equilibrium Policy

Chapter 14

Some Useful Models

Appendix A

Mathematical Tools

A.1 Constrained Optimization

Consider a maximization problem with one equality constraint and one inequality constraint:

$$\max_x f(x) \text{ s.t. } g(x) = 0, h(x) \leq 0.$$

Functions f, g, h are continuous and differentiable and $x \in \mathbb{R}^n$. Form the Lagrangian $\mathcal{L}(x, \lambda, \mu) \equiv f(x) - \lambda g(x) - \mu h(x)$.

Suppose that $x^\star \in \mathbb{R}^n$ is a local maximizer of f on the constraint set. Suppose furthermore that the Jacobian matrix at $x^\star$ of the binding constraints has full rank. Then, there exist multipliers $\lambda^\star$ and $\mu^\star$ such that $\partial\mathcal{L}(x^\star, \lambda^\star, \mu^\star)/\partial x_i = 0$, $i = 1, \ldots, n$; $\mu^\star \geq 0$; $g(x^\star) = 0$; $h(x^\star) \leq 0$; and the complementary slackness condition $\mu^\star h(x^\star) = 0$ is satisfied.

Differentiating the Lagrangian with respect to the choice variables thus yields necessary conditions for a local maximum. The result extends to the case with multiple inequality and/or equality constraints.

A.2 Infinite-Horizon Dynamic Programming

The *sequence problem* is defined by

$$V^\star(a_0) = \max_{\{a_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(a_t R + w - a_{t+1}) \text{ s.t. } a_0 \text{ given, } a_{t+1} \in A(a_t). \tag{SP}$$

Rather than imposing a no-Ponzi-game condition we require the choice variable to lie in the set described by the correspondence $A(a_t)$. A lower bound on this set implies that debt cannot be rolled over forever (if interest rates are strictly positive) and thus, rules out Ponzi games.

The *Bellman equation* associated with the sequence problem reads

$$V(a_t) = \max_{a_{t+1} \in A(a_t)} \{u(a_t R + w - a_{t+1}) + \beta V(a_{t+1})\} \text{ for all } a_t \in \mathcal{A}. \tag{BE}$$

Since the problem is time-autonomous, the time indices of the state and the control variable in (BE) do not carry significance. $\mathcal{A}$ denotes the state space.

A.2.1 Principle of Optimality

We assume that the set $A(a_t)$ is nonempty for all $a_t \in \mathcal{A}$. We also assume that for all sequences $\{a_{t+1}\}_{t\geq -1}$ that start with $a_0 \in \mathcal{A}$ and satisfy $a_{t+1} \in A(a_t)$, the infinite sequence in (SP) exists and is finite. Under these conditions, $V^\star(a_t) = V(a_t)$ for all $a_t \in \mathcal{A}$. Moreover, a plan $\{a_{t+1}\}_{t\geq 0}$ conditional on $a_0 \in \mathcal{A}$ that attains $V^\star(a_0)$ in (SP) also solves (BE), and the reverse statement holds as well; this is the *Principle of Optimality*.

The proofs of these results use the fact that if the infinite sum in (SP) exists and is finite, then it can be expressed as the sum of a contemporaneous payoff and a continuation payoff, similarly to the two terms on the right-hand side of (BE).

A.2.2 Uniqueness of V

If in addition, the set $\mathcal{A}$ is bounded and complete; $A(a_t)$ is nonempty, bounded, and complete for all $a_t \in \mathcal{A}$; and A and u are continuous, then a unique continuous and bounded function V satisfying (BE) as well as an optimal plan $\{a_{t+1}\}_{t\geq 0}$ solving (SP) or (BE) exist.

The uniqueness result follows from mathematical theorems on *contractions*. Note that the right-hand side of (BE) constitutes an operator on the value function, $T(V)$ say: For any value function V on the right-hand side of (BE) the operator returns a value function on the left-hand side. The solution to (BE) then satisfies $V = T(V)$ and the function V constitutes a fixed point of the operator T.

Under the maintained assumptions, the maximization problem on the right-hand side of (BE) has a solution such that T exists and in fact, is continuous. The operator T therefore maps a set of continuous functions into the same set. Moreover, it satisfies Blackwell's sufficient conditions for a contraction.[1] But if an operator constitutes a contraction, then it has a unique fixed point. Moreover, repeated application of the operator generates a sequence of functions that converges to the fixed point. For an arbitrary continuous function V_0, repeated application of the operator thus generates a sequence of functions, $V_0, T(V_0), T(T(V_0)), T(T(T(V_0))), \ldots$, that converges to the fixed point V.

A.2.3 Properties of V

If in addition, u is concave and A convex, then the value function V in (BE) is strictly concave and the optimal plan $\{a_{t+1}\}_{t\geq 0}$ solving (SP) or (BE) is unique.

[1] A metric space $(\mathcal{M}, d)$ is a set $\mathcal{M}$ whose elements can be added, multiplied by scalars, and for pairs of which a norm or distance d is defined. An operator T that maps a metric space into itself is a contraction if there exists a $\rho \in [0, 1)$ such that $d(T(m), T(n)) \leq \rho d(m, n)$ for all $m, n \in \mathcal{M}$.

If in addition, u is strictly increasing in the state and A is monotone, then the value function V in (BE) is strictly increasing.

If in addition, u is continuously differentiable on the interior of its domain, then the value function V in (BE) is differentiable.

A.3 Systems of Linear Difference Equations

Consider a system of two difference equations,

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \text{ or } z_{t+1} = Mz_t.$$

If M is diagonal (i.e., $b = c = 0$) then the two equations are uncoupled: we can solve them independently of each other, yielding $x_t = a^t x_0$ and $y_t = d^t y_0$ for arbitrary x_0, y_0. If M is not diagonal, we can use eigenvalues and -vectors to transform the system and render the equations uncoupled.

Suppose that M has two distinct and real eigenvalues, ρ_1 and ρ_2, with associated eigenvectors, v_1 and v_2, satisfying

$$M[v_1\ v_2] = [v_1\ v_2] \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \text{ or } MV = VP.$$

Pre-multiplying the original system $z_{t+1} = Mz_t$ by V^{-1} then yields a transformed, uncoupled system in the vector $Z \equiv V^{-1}z$ with diagonal matrix entries equal to the eigenvalues of M:

$$Z_{t+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} Z_t \text{ and therefore } Z_t = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t Z_0.$$

Using $z_t = VZ_t$, the latter equation can be transformed back to yield

$$z_t = V \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t Z_0 = V \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t V^{-1} z_0.$$

Note that M^t is given by VP^tV^{-1}. Letting $\varphi_0 \equiv V^{-1}z_0$ we conclude that

$$z_t = V \begin{bmatrix} \rho_1^t & 0 \\ 0 & \rho_2^t \end{bmatrix} \varphi_0 \text{ or } z_t = \varphi_{0[1]} \rho_1^t v_1 + \varphi_{0[2]} \rho_2^t v_2.$$

This first-order difference equation system in z has a family of solutions with two degrees of freedom, corresponding to the two elements of z_0 or φ_0. For a definite solution, we need two restrictions. An initial condition for an element of z_0 constitutes such a restriction. If an eigenvalue is unstable then the requirement that system dynamics be stable also implies a restriction; for example, $\rho_1 > 1$ and system stability imply $\varphi_{0[1]} = 0$.

If the eigenvalues of M are not distinct or if they are complex then the solution strategy must be adapted. The extension to the case with more than two variables is immediate.

A.4 Bibliographic Notes

Simon and Blume (1994, 18, 19), Mas-Colell et al. (1995, M.K), and Acemoglu (2009, A.11) review Lagrangian methods. Stokey and Lucas (1989, 3, 4) and Acemoglu (2009, 6) contain detailed proofs and discussions on dynamic programming. Acemoglu (2009, Example 6.5) covers the saving problem and discusses an approach to guarantee compactness of $\mathcal{A}$ in that program. Simon and Blume (1994, 23) cover linear difference equations, including the case of repeated or complex eigenvalues.

Beyond the material covered in the chapter, Stokey and Lucas (1989, 7–9) and Acemoglu (2009, 16) cover dynamic programming under risk.

Appendix B

Technical Discussions

B.1 Transversality Condition in Infinite-Horizon Saving Problem

The household's program is given by

$$\max_{\{a_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(a_t R_t + w_t - a_{t+1}) \text{ s.t. } a_0 \text{ given, } \lim_{T\to\infty} q_T a_{T+1} \geq 0.$$

Let $\hat{a} \equiv \{\hat{a}_{t+1}\}_{t\geq 0}$ denote a plan that satisfies the Euler equation at all times as well as $\lim_{T\to\infty} q_T \hat{a}_{T+1} = 0$. Let $\bar{a} \equiv \{\bar{a}_{t+1}\}_{t\geq 0}$ denote an alternative plan that satisfies $\lim_{T\to\infty} q_T \bar{a}_{T+1} \geq 0$. We want to show that the former dominates the latter.

For brevity, let $\hat{u}_t \equiv u(\hat{a}_t R_t + w_t - \hat{a}_{t+1})$ and $\hat{u}'_t \equiv u'(\hat{a}_t R_t + w_t - \hat{a}_{t+1})$ and similarly for $\bar{u}_t$ and $\bar{u}'_t$. Strict concavity of u and positive marginal utility imply $\hat{u}_t + R_t \hat{u}'_t(\bar{a}_t - \hat{a}_t) - \hat{u}'_t(\bar{a}_{t+1} - \hat{a}_{t+1}) > \bar{u}_t$ if $\hat{a}_t \neq \bar{a}_t$ or $\hat{a}_{t+1} \neq \bar{a}_{t+1}$. If $\hat{a}_{t+1} \neq \bar{a}_{t+1}$ for some $t \in \{0, \ldots, T\}$, we thus have

$$\sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t) < \sum_{t=0}^{T} \beta^t \{R_t \hat{u}'_t(\bar{a}_t - \hat{a}_t) - \hat{u}'_t(\bar{a}_{t+1} - \hat{a}_{t+1})\} = \beta^T \hat{u}'_T(\hat{a}_{T+1} - \bar{a}_{T+1}),$$

where we use $\hat{a}_0 = \bar{a}_0$ and $\hat{u}'_t = \beta R_{t+1} \hat{u}'_{t+1}$. From the Euler equation, $\beta^T \hat{u}'_T = \hat{u}'_0 q_T$. If $\hat{a}$ is not identical to $\bar{a}$, it follows that

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t) < \lim_{T\to\infty} \hat{u}'_0 q_T (\hat{a}_{T+1} - \bar{a}_{T+1}) = \lim_{T\to\infty} -\hat{u}'_0 q_T \bar{a}_{T+1} \leq 0$$

such that $\sum_{t=0}^{\infty} \beta^t \hat{u}_t > \sum_{t=0}^{\infty} \beta^t \bar{u}_t$. Satisfying $\lim_{T\to\infty} q_T \hat{a}_{T+1} = 0$ therefore is optimal.

B.2 Representative Household

B.3 Transversality Condition in Infinite-Horizon Planner Problem

Let $g(k_t) \equiv k_t(1-\delta) + f(k_t, 1)$ with f denoting the neoclassical production function. Note that the function g is strictly concave. The program is given by

$$\max_{\{k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(g(k_t) - k_{t+1}) \text{ s.t. } k_0 \text{ given}, \ \ k_{t+1} \geq 0.$$

Let $\hat{k} \equiv \{\hat{k}_{t+1}\}_{t\geq 0}$ denote a plan that satisfies the Euler equation and complementary slackness condition, $\hat{u}'_t = \beta g'(\hat{k}_{t+1})\hat{u}'_{t+1} + \hat{\mu}_t$ and $\hat{\mu}_t \hat{k}_{t+1} = 0$ respectively, at all times, as well as $\lim_{T\to\infty} \beta^T \hat{u}'_T \hat{k}_{T+1} = 0$. Let $\bar{k} \equiv \{\bar{k}_{t+1}\}_{t\geq 0}$ denote an alternative plan that satisfies $\lim_{T\to\infty} \beta^T \bar{u}'_T \bar{k}_{T+1} \geq 0$. Here, we let $\hat{u}_t \equiv u(g(\hat{k}_t) - \hat{k}_{t+1})$ and $\hat{u}'_t \equiv u'(g(\hat{k}_t) - \hat{k}_{t+1})$ and similarly for $\bar{u}_t$ and $\bar{u}'_t$. We want to show that $\hat{k}$ dominates $\bar{k}$.

Strict concavity of u and g and positive marginal utility imply $\hat{u}_t + \hat{u}'_t g'(\hat{k}_t)(\bar{k}_t - \hat{k}_t) - \hat{u}'_t(\bar{k}_{t+1} - \hat{k}_{t+1}) > \bar{u}_t$ if $\hat{k}_t \neq \bar{k}_t$ or $\hat{k}_{t+1} \neq \bar{k}_{t+1}$. If $\hat{k}_{t+1} \neq \bar{k}_{t+1}$ for some $t \in \{0, \dots, T\}$, we thus have

$$\begin{aligned}
\sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t) &< \sum_{t=0}^{T} \beta^t \{\hat{u}'_t g'(\hat{k}_t)(\bar{k}_t - \hat{k}_t) - \hat{u}'_t(\bar{k}_{t+1} - \hat{k}_{t+1})\} \\
&= \beta^T \hat{u}'_T(\hat{k}_{T+1} - \bar{k}_{T+1}) - \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1},
\end{aligned}$$

where we use $\hat{k}_0 = \bar{k}_0$, the Euler equation, and the complementary slackness condition. If $\hat{k}$ is not identical to $\bar{k}$, it follows that

$$\begin{aligned}
\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t) &< \lim_{T\to\infty} \left\{ \beta^T \hat{u}'_T(\hat{k}_{T+1} - \bar{k}_{T+1}) - \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1} \right\} \\
&= -\lim_{T\to\infty} \left\{ \beta^T \hat{u}'_T \bar{k}_{T+1} + \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1} \right\} \leq 0,
\end{aligned}$$

since marginal utility, the capital stock, and the multiplier all are weakly positive. We conclude that $\sum_{t=0}^{\infty} \beta^t \hat{u}_t > \sum_{t=0}^{\infty} \beta^t \bar{u}_t$. Satisfying $\lim_{T\to\infty} \beta^T \hat{u}'_T \hat{k}_{T+1} = 0$ therefore is optimal.

B.4 Non-Expected Utility

B.5 Linear Rational Expectations Models

B.5.1 Single Equation Model

Backward- and Forward-Solution of a Difference Equation

Consider the difference equation

$$x_{t+1} = \alpha x_t + s_t,$$

where the variable x_t is endogenous and predetermined and the variable s_t is exogenous. The sequence $\{s_t\}_{t\geq 0}$ is bounded. Iterating the equation backward yields

$$x_{t+1} = s_t + \alpha s_{t-1} + \alpha^2 s_{t-2} + \ldots + \alpha^t s_0 + \alpha^{t+1} x_0.$$

Iterating forward implies

$$x_{t+T} = \alpha^T x_t + \alpha^{T-1} s_t + \alpha^{T-2} s_{t+1} + \ldots + \alpha s_{t+T-2} + s_{t+T-1}.$$

When $|\alpha| < 1$, then $\lim_{T\to\infty} x_{t+T}$ is bounded irrespective of the value x_t. When $|\alpha| > 1$, in contrast, this does not hold true. To see this, define

$$x_t^\star \equiv -\lim_{T\to\infty} \sum_{j=1}^{T} \frac{s_{t+j-1}}{\alpha^j}.$$

Since $\{s_t\}_{t\geq 0}$ is bounded and $|\alpha| > 1$, $x_t^\star$ is well defined. For $x_t = x_t^\star$, $\lim_{T\to\infty} x_{t+T}$ equals zero. But for $x_t = x_t^\star + \Delta$ with $\Delta \neq 0$, x_{t+T} diverges as T increases. We conclude that for $|\alpha| > 1$ there exists a unique value for x_t, namely $x_t^\star$, such that $\{x_t\}_{t\geq 0}$ is bounded.

Rational Expectations Model

Consider next a stochastic (but still bounded) sequence $\{s_t\}_{t\geq 0}$.[1] The model is given by two conditions. First, the difference equation

$$\mathbb{E}_t[y_{t+1}] = \alpha y_t + s_t \quad \text{or} \quad y_{t+1} = \alpha y_t + s_t + \delta_{t+1},$$

where the variable y_t is endogenous and non-predetermined, and δ_{t+1} denotes a forecast error satisfying $\mathbb{E}_t[\delta_{t+1}] = 0$. And second, the restriction that $\lim_{T\to\infty} \mathbb{E}_t[y_{t+T}]$ be bounded.

Iterating forward and using the law of iterated expectations yields

$$\mathbb{E}_t[y_{t+T}] = \alpha^T y_t + \alpha^{T-1} s_t + \mathbb{E}_t[\alpha^{T-2} s_{t+1} + \ldots + \alpha s_{t+T-2} + s_{t+T-1}].$$

[1] To simplify the notation, we do not index variables by history.

When $|\alpha| < 1$, then $\lim_{T\to\infty} \mathbb{E}_t[y_{t+T}]$ is bounded irrespective of the value y_t. That is, the model determines the expectation $\mathbb{E}_{t-1}[y_t] = \alpha y_{t-1} + s_{t-1}$, but not the actual realization, y_t. The model thus leaves room for a *sunspot shock* to affect y_t.

When $|\alpha| > 1$, in contrast, boundedness of $\lim_{T\to\infty} \mathbb{E}_t[y_{t+T}]$ requires that y_t equals

$$y_t^\star \equiv -\lim_{T\to\infty} \mathbb{E}_t \sum_{j=1}^{T} \frac{s_{t+j-1}}{\alpha^j},$$

for parallel reasons as in the deterministic case. The model determines the expectation $\mathbb{E}_{t-1}[y_t] = \alpha y_{t-1} + s_{t-1}$, and it also determines the actual realization, $y_t = y_t^\star$. As in the case with $|\alpha| < 1$, the forecast error $\delta_t = y_t^\star - \mathbb{E}_{t-1}[y_t]$ is unpredictable. But unlike in that case, it only reflects the effect of new information about $\{s_{t+j}\}_{j\geq 0}$ on $y_t^\star$; the model leaves no room for a sunspot shock to affect y_t.

B.5.2 Multiple Equation Model

The model consists of the system of difference equations

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t[y_{t+1}] \end{bmatrix} = M \begin{bmatrix} x_t \\ y_t \end{bmatrix} + N s_t \tag{B.1}$$

or equivalently,

$$\begin{bmatrix} \underset{(n_x\times 1)}{x_{t+1}} \\ \underset{(n_y\times 1)}{y_{t+1}} \end{bmatrix} = \underset{(n\times n)}{M} \begin{bmatrix} \underset{(n_x\times 1)}{x_t} \\ \underset{(n_y\times 1)}{y_t} \end{bmatrix} + \underset{(n\times n_s)}{N} \underset{(n_s\times 1)}{s_t} + \begin{bmatrix} \underset{(n_x\times 1)}{0} \\ \underset{(n_y\times 1)}{\delta} \end{bmatrix}.$$

There are n_x predetermined variables (including for example the capital stock), denoted by x_t; n_y non-predetermined variables (e.g., consumption), denoted by y_t; and n_s exogenous bounded variables (e.g., productivity), denoted by s_t. The number of endogenous variables equals $n = n_x + n_y$, and δ_t denotes a vector of forecast errors. The model imposes the additional restriction that the endogenous variables be bounded.

Using the notation of appendix A.3, matrix M can be represented as the product of matrices that contain its eigenvectors and eigenvalues,

$$M = VPV^{-1} \equiv \underset{(n\times n)}{[v_1\ v_2 \cdots v_n]} \underset{(n\times n)}{\begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{bmatrix}} \underset{(n\times n)}{V^{-1}} \equiv V \begin{bmatrix} \underset{(n_<\times n_<)}{P_{<<}} & \underset{(n_<\times n_>)}{0} \\ \underset{(n_>\times n_<)}{0} & \underset{(n_>\times n_>)}{P_{>>}} \end{bmatrix} V^{-1},$$

where $n_<$ ($n_>$) denotes the number of eigenvalues whose absolute value is weakly smaller than (exceeds) unity, respectively. We assume that the eigenvalues are distinct and real and ordered in ascending absolute value, $|\rho_1| < |\rho_2| < \ldots < |\rho_n|$. Let

$$Z_t \equiv \begin{bmatrix} \underset{(n_<\times 1)}{Z_{<t}} \\ \underset{(n_>\times 1)}{Z_{>t}} \end{bmatrix} \equiv V^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underset{(n_<\times n_s)}{N_<} \\ \underset{(n_>\times n_s)}{N_>} \end{bmatrix} \equiv V^{-1} N. \tag{B.2}$$

Premultiplying equation (B.1) by V^{-1} yields $\mathbb{E}_t[Z_{t+1}] = PZ_t + V^{-1}Ns_t$. Iterating forward, we arrive at

$$\mathbb{E}_t[Z_{t+T}] = P^T Z_t + \sum_{j=0}^{T-1} P^{T-1-j} V^{-1} N \mathbb{E}_t[s_{t+j}].$$

Boundedness ($\lim_{T\to\infty} \mathbb{E}_t[Z_{t+T}] = 0$) thus requires

$$\lim_{T\to\infty} \left((P_{>>})^T Z_{>t} + \sum_{j=0}^{T-1} (P_{>>})^{T-1-j} N_{>} \mathbb{E}_t[s_{t+j}] \right) = 0$$

or

$$Z_{>t} = -\sum_{j=0}^{\infty} (P_{>>})^{-1-j} N_{>} \mathbb{E}_t[s_{t+j}].$$

That is, the requirement that system dynamics be stable imposes $n_>$ restrictions on Z_t. The initial conditions for the predetermined variables x_t impose n_x additional restrictions. We may distinguish three cases:

No Solution, $n_y < n_>$

When $n_y < n_>$ (and thus, $n_x > n_<$) then the stability requirement imposes more restrictions than there are non-predetermined variables that could adjust to satisfy them. In general, the model has no solution in this case.

Determinacy, $n_y = n_>$

When $n_y = n_>$ then the stability requirement imposes as many restrictions as there are non-predetermined variables. All endogenous variables thus are uniquely pinned down (as long as the relevant submatrix of V^{-1} is invertible such that a given $Z_{>t}$ implies a unique y_t in (B.2)). The model dynamics satisfy

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = V \begin{bmatrix} P_{<<} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + V \begin{bmatrix} N_< \\ 0 \end{bmatrix} s_t + V \begin{bmatrix} \underset{(n_< \times n_>)}{0} \\ \underset{(n_> \times n_>)}{I} \end{bmatrix} Z_{>t+1}. \quad \text{(B.3)}$$

Typically, $y_t \neq \mathbb{E}_{t-1}[y_t]$. The forecast error, δ_t, reflects the effect of new information about $\{s_{t+j}\}_{j\geq 0}$ on $Z_{>t}$.

Indeterminacy, $n_y > n_>$

When $n_y > n_>$ then the stability requirement does not pin down the non-predetermined variables; there are $n_y - n_>$ degrees of freedom. Suppose, for example, that $n_x = n_s = 0$ such that the model reduces to

$$y_{t+1} = My_t + \delta_{t+1}$$

and $Z_{>t} = 0$. In this case, $n_<$ elements of y_t can freely be chosen in each period without triggering explosive dynamics. Although $s_t = 0$ in all periods the forecast errors may be non-zero. In particular, y_t may respond to non-fundamental sunspot shocks.

B.6 Bibliographic Notes

Sargent and Wallace (1973) discuss the forward solution and Blanchard and Kahn (1980) propose the solution strategy for linear rational expectations models, following Vaughan (1970). Miao (2014, 2) reviews alternative solution strategies.

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