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Part IX
RV: Realized Volatility

This lecture note accompanies Chapter 12-13 of Taylor (2005) on realized volatility. Realized volatility plays a major role in modern financial econometrics and we shall give an introduction to the basic issues of the currently expanding area. We refer to McAleer and Medeiros (2008) and Andersen and Benzoni (2009) for a short introduction to the topic. A recent textbook dedicated to realized volatility (and related topics) is Aït-Sahalia and Jacod (2014).

The realized volatility is simple to compute as can be briefly exemplified (details below): The realized volatility for a given trading day $m \in \{1, ..., M\}$ based on N log-returns (that is, log-returns $r(m, i)$, $i = 1, ..., N$, measured over small time intervals, 10 seconds for example) is defined as:

$$V(m, N) = \sum_{j=1}^{N} r(m, j)^2.$$

What complicates matters is that *(i)* data are expensive and difficult to get (yet), and moreover difficult to 'handle' and *(ii)* underlying the simple computation is a theory for continuous time stochastic processes which motivates and explains the computation. One further thing must be stressed: The computation of $V(\cdot, \cdot)$ is model-free in the sense that the same quantity is computed for various data series independently of underlying econometric models. For each case one tries to make sure that for a rich enough class of underlying (continuous time) models the quantity can be interpreted as (integrated) 'volatility' (sometimes confusingly referred to as realized integrated variance also).

IX.1 A quick introduction to continuous time price processes (and quadratic variation).

Note first that the realized volatility for a given asset is based on the assumption that for any market time (say, within a given trading day) $t \in [0, T]$

there exists a corresponding price, denoted $P(t)$. As usual we will analyze the log-transform of this price, and hence define $p(t) = \log(P(t))$, or simply p_t. Naturally, this continuous time price is only observed whenever there is a transaction in the market, but in liquid markets, as for example the EUR-USD market, transactions are typically only separated by a few (milli-)seconds.

We continue by a quick overview of some key results for continuous time processes. Taylor (2005, chapter 13) provides a more wide-ranging survey, while for example Karatzas and Shreve (1991) gives a thorough treatment of continuous time processes.

Example IX.1 *(The Brownian motion) A key ingredient within continuous time processes is the* ***Brownian motion****. A stochastic process $(W(t) : t \geq 0)$ is called a Brownian motion if:*

1. $W(0) = 0$.
2. *W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then*
$$W(u) - W(t) \quad \textit{and} \quad W(s) - W(r)$$
are independent.
3. *The increments are normally distributed, i.e.*
$$W(t) - W(s) \overset{d}{=} N(0, t - s)$$
for all $0 \leq s \leq t$.
4. *W has continuous trajectories, i.e. it is continuous in t.*

Example IX.2 *A basic continuous time stochastic differential equation (SDE) is given by*
$$dp\left(t\right) = \sigma dW\left(t\right),$$
where $W\left(t\right)$ is a Brownian motion on the time interval $t \in [0, T]$, and σ is constant. This differential equation is identical to, or has solution,
$$p\left(t\right) = p\left(0\right) + \sigma W\left(t\right).$$
Thus, at any time point $p\left(t\right)$ is $N\left(p\left(0\right), \sigma^2 t\right)$ distributed. Moreover, by the definition of a Brownian motion, it has independent and Gaussian distributed increments,
$$p\left(t\right) - p\left(s\right) = \sigma\left(W\left(t\right) - W\left(s\right)\right) \overset{d}{=} N\left(0, \sigma^2\left(t - s\right)\right),$$

where $t > s \geq 0$. In particular,

$$\text{Var}\,(p\,(t)) = \text{Var}\,(p\,(t) - p\,(0)) = \sigma^2 t = \int_0^t \sigma^2 du \qquad \textit{and}$$

$$\text{Var}\,(p\,(t) - p\,(s)) = \sigma^2\,(t-s) = \int_s^t \sigma^2 du.$$

In fact, for any $t > s > u$, we have for example,

$$(p\,(t) - p\,(s)\,, p\,(s) - p\,(u))' \overset{d}{=} N\,(0, \Omega_t)\,, \qquad \textit{where}$$

$$\Omega_t = \sigma^2 \begin{pmatrix} t-s & 0 \\ 0 & s-u \end{pmatrix}.$$

That is, $p\,(t)$ is a (continuous time) random walk or a Brownian motion initiated in $p\,(0)$.

The above example of a stochastic process $X\,(t)$ with $t \in [0, T]$ is obviously not a realistic model for log-returns and many extensions of this exist of which we provide some few key examples (more can be found in Taylor, 2005).

The variance computation for $X\,(t) = p\,(t)$ above is closely related to the concept of quadratic variation of a stochastic process usually denoted $[X](t)$ (and hence also to the concept of realized volatility as we shall see below). With $X(t)$ univariate this can be defined as the following (in probability) limit: For some fixed $t > 0$, let $\Pi_N = \{t_0, t_1, ..., t_k, .., t_N\}$ be a partition of the interval $[0, t]$ with $0 = t_0 \leq t_1 \leq ... \leq t_k \leq ... \leq t_N = t$. A measure of how fine this partition is, is given by the *mesh* $||\Pi_N|| = \max_{k=1,2,...,N} |t_k - t_{k-1}|$ and

$$[X]\,(t) = \underset{||\Pi_N|| \to 0}{\text{plim}} \sum_{k=1}^{N} [X(t_k) - X(t_{k-1})]^2\,. \tag{IX.1}$$

Likewise, the d-th power variation is defined as above, but with $|X(t_k) - X(t_{k-1})|^d$ replacing $[X(t_k) - X(t_{k-1})]^2$.

Working with continuous time processes can be difficult, and one of the difficulties is the concept of differentiation with respect to time t as for example the Brownian motion $W\,(t)$, while continuous, it is nowhere differentiable, while $W\,(t)^2$ is. A key tool working with these processes is the Ito-lemma (or Ito's rule). For a quite general class of univariate stochastic processes $X(t)$ defined by some stochastic differential equation (SDE; see Karatzas and Shreve, 1991), the Ito rule explains how functions of X_t evolve, such as for example $f\,(X_t) = \log X(t)$ (with $X(t) > 0$) and $f\,(X_t) = X_t^2$.

More precisely, with $f(t, X(t))$ some function of time t and of $X(t)$ (which is twice differentiable in $X(t)$), the famous Ito's rule states that,

$$df(t, X(t)) = \dot{f}_t(t, X(t))\, dt + \dot{f}_x(t, X(t))\, dX_t + \frac{1}{2}\ddot{f}_{xx}(t, X(t))\, d[X](t), \quad \text{(IX.2)}$$

where

$$\begin{aligned} \dot{f}_t(t, X(t)) &= \tfrac{\partial}{\partial t} f(t, X(t)), \qquad \dot{f}_x(t, X(t)) = \tfrac{\partial}{\partial X} f(t, X(t)), \\ \ddot{f}_{xx}(t, X(t)) &= \tfrac{\partial^2}{\partial X^2} f(t, X(t)) \end{aligned}$$

Example IX.3 *In terms of the previous example of the* ***Brownian motion****, it holds that the quadratic variation is* $[p](t) = \sigma^2[W](t)$*, where*

$$[W](t) = t = \int_0^t ds.$$

Moreover, with $f(t, W(t)) = W^2(t)$ *we have* $X(t) = W(t)$*,* $\dot{f}_t = 0$*,* $\dot{f}_x = 2W_t$ *and* $\ddot{f}_{xx} = 2$*, such that*

$$dW^2(t) = 2W(t)dW(t) + dt.$$

Thus we see that while indeed $W^2(t)$ *is differentiable,* $dW^2(t)/dt \neq 2W(t)$ *as one might expect. Another implication is that* $W^2(t) = 2\int_0^t W(s)dW(s) + t$*, or the stochastic integral known from e.g. unit-root analysis in time series is given by,*

$$\int_0^t W(s)dW(s) = \frac{1}{2}\left(W^2(t) - t\right).$$

Example IX.4 *A large classic class of continuous time stochastic processes is given by the* ***Ito processes****, with* $\mu(t)$ *and* $\sigma(t)$ *(stochastic) functions of time* t*,*

$$dX(t) = \mu(t)\, dt + \sigma(t)dW(t),$$

or, equivalently,

$$X(t) = X(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)dW(s).$$

Here $\mu(t)$ *and* $\sigma(t)$ *are assumed to be adapted processes, i.e.* $\mu(t)$ *and* $\sigma(t)$ *are known at time* t*.*
It is beyond the scope of this note to discuss all aspects of the class of Ito processes. Importantly, the quadratic variation is

$$[X](s) = \int_0^t \sigma^2(s)ds.$$

Moreover, we emphasize the following important properties about $\int_0^t \sigma(s)dW(s)$*: It holds that*

$$E\left[\int_0^t \sigma(s)dW(s)\right] = 0$$

and

$$E\left[\left(\int_0^t \sigma(s)dW(s)\right)^2\right] = \int_0^t E[\sigma^2(s)]ds.$$

Lastly, if $\sigma(t)$ *is a deterministic function of time,*

$$\int_0^t \sigma(s)dW(s) \stackrel{d}{=} N\left(0, \int_0^t \sigma^2(s)ds\right).$$

Example IX.5 *The* ***Geometric Brownian Motion*** $X(t)$ *known from the Black-Scholes formula for option prices is given as the solution to,*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \qquad \text{with } [X](t) = \int_0^t \sigma^2 X^2(s)ds.$$

Applying Ito's rule to $f(t, X(t)) = \log X(t)$*, gives*

$$d\log X(t) = \frac{1}{X(t)}dX(t) - \frac{1}{2X(t)^2}d[X](t) = \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt,$$

that is,

$$d\log X(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t). \qquad \text{with } [\log X](t) = \int_0^t \sigma^2 ds = \sigma^2 t.$$

Note that this motivates why most of the recent literature on realized volatility often starts with the assumption that log*-prices,* $\log X(t)$*, have a general formulation of the form,*

$$d\log X(t) = \tilde{\mu}(t)dt + \tilde{\sigma}(t)dW_t,$$

with time-varying drift term $\tilde{\mu}(t)$ *and volatility* $\tilde{\sigma}(t)$*. In fact, most often* $\tilde{\sigma}(t)$ *is stochastic as in the discrete time SV models.*

We can write the solution for the Geometric Brownian motion as,

$$\log X(t) = \log X(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t),$$

from which we can find the explicit solution,

$$X(t) = X(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}.$$

Hence for $X(0)$ fixed, $X(t)$ is log-normal distributed with,

$E[X(t)] = X(0)\exp(\mu t)$ *and* $V[X(t)] = X(0)^2 \exp(2\mu t)\left(\exp\left(\sigma^2 t\right) - 1\right)$

Moreover, with $s < t$,

$$\log X(t) = \log X(s) + \left(\mu - \frac{1}{2}\sigma^2\right)(t-s) + \sigma\left(W(t) - X(s)\right).$$

Hence, the increments $\log X(t) - \log X(s)$ are Gaussian, and $X(t)$ conditional on $X(s)$ is log-normal.

Example IX.6 *The* ***Ornstein-Uhlenbeck*** *(OU) process is the continuous time equivalent of the discrete time AR(1) process. In its simplest form it is the solution to,*

$$dX(t) = -\kappa X(t)dt + \sigma dW(t).$$

To solve this, apply Ito's rule to $f(t, X(t)) = \exp(\kappa t)X(t)$, to get

$$\begin{aligned} d(\exp(\kappa t)X(t)) &= \kappa \exp(\kappa t)X(t)dt + \exp(\kappa t)\sigma dX(t) \\ &= \exp(\kappa t)\sigma dW(t) \end{aligned}$$

That is,

$$\exp(\kappa t)X(t) = X(0) + \int_0^t \exp(\kappa s)\sigma dW(s)$$

or

$$X(t) = \exp(-\kappa t)X(0) + \int_0^t \exp(-\kappa(t-s))\sigma dW(s)$$

and moreover,

$$[X](t) = \int_0^t \exp(-2\kappa(t-s))\sigma^2 ds = \frac{\sigma^2}{2\kappa}(1 - \exp(-2\kappa t)).$$

We also see that,

$$X(t) = \exp(-\kappa(t-s))X(s) + \int_s^t \exp(-\kappa(t-u))\sigma dW(s),$$

hence with observing at discrete time points with $t - s = 1$, the OU is an AR(1) process

$$X(t) = \phi X(t-1) + \varepsilon(t),$$

with autoregressive parameter $\phi = \exp(-\kappa)$ and i.i.d. Gaussian innovations $\varepsilon(t)$. The innovations $\varepsilon(t)$ are i.i.d. $N(0, \omega)$ with variance $\omega = \frac{1}{2\kappa}(1 - \exp(-2\kappa))$.

IX.2 Realized Volatility

Within a given trading day $m \in \{1, ..., M\}$ let $[m-1, m]$ denote the time interval over which a financial asset with price $p(\cdot)$ is traded. We focus on the case where the price of the asset is observed at $N+1$ equally-spaced discrete points in time, $\tau = t_0, t_1, ..., t_N$, with $t_i = m - 1 + i/N$, $i = 0, 1, ..., N$. Then we may define the ith log-return within trading day m as

$$\begin{aligned} r(m,i) &= p(t_i) - p(t_{i-1}), \quad i = 1, ..., N, \\ &= p(m-1+i/N) - p(m-1+(i-1)/N) \end{aligned}$$

Following McAleer and Medeiros (2008), we define the realized volatility is defined as follows.

Definition IX.1 *The realized volatility for a given day $m \in \{1, ..., M\}$ based on N equally-spaced log-returns is defined as*

$$V(m,N) = \sum_{j=1}^{N} r(m,j)^2.$$

Using the definition of return $r(m,j)$, the realized volatility can be rewritten as

$$\begin{aligned} V(m,N) &= \sum_{j=1}^{N} r(m,j)^2 \\ &= r(m,1)^2 + \cdots + r(m,N)^2 \\ &= [p(t_1) - p(t_0)]^2 + \cdots + [p(t_N) - p(t_{N-1})]^2 \\ &= [p(m-1+1/N) - p(m-1)]^2 + \cdots + [p(m) - p(m-1+(N-1)/N)]^2 \end{aligned}$$

Example IX.7 *The New York Stock Exchange (NYSE) is open Monday - Friday from 9:30 to 16:00 ET except on public holidays. Hence each trading day on the NYSE has 6.5 hours of trading. The table below states half hourly prices for the index from 10/2-09.*

Time	*log-price*	*price*
09:30	*6.7684*	*869.89*
10:00	*6.7506*	*854.60*
10:30	*6.7536*	*857.14*
11:00	*6.7549*	*858.22*
11:30	*6.7505*	*854.48*
12:00	*6.7396*	*845.25*
12:30	*6.7427*	*847.88*
13:00	*6.7342*	*840.71*
13:30	*6.7409*	*846.31*
14:00	*6.7325*	*839.25*
14:30	*6.7333*	*839.89*
15:00	*6.7445*	*849.39*
15:30	*6.7268*	*834.45*
16:00	*6.7180*	*827.16*

This corresponds to $N = 13$*. Hence the realized volatility based on half hourly returns for the 10/2-09 can be computed as*

$$V(\text{"10/2-09"}, 13) = (6.7506 - 6.7684)^2 + \cdots + (6.7180 - 6.7268)^2 = 0.0011798$$

If one instead wanted the realized volatility based on hourly returns (counting from 9:30, corresponding to $N = 6$*) it would be computed as*

$$\begin{aligned} V(\text{"10/2-09"}, 6) &= (6.7536 - 6.7684)^2 + (6.7505 - 6.7536)^2 \\ &\quad + \cdots + (6.7268 - 6.7333)^2 = 0.00039274 \end{aligned}$$

Remark IX.1 *The computation of* $V(m, N)$ *is based on the log-price observed at the points* $t_i = m - 1 + i/N$, $i = 0, 1, ..., N$*. Here the point* $t_N = m$ *should be understood as the last point where the price is observed during trading day* m*, and not the first observation at trading day* $m + 1$*. These two prices may differ quite a lot due to so-called overnight effects. Likewise, when computing* $V(m+1, N)$ *the point* $t_0 = m$ *corresponds to the first point where the price is observed during trading day* $m + 1$*. In order to deal with this issue, as clarified in the section below on market time, one could consider* $[m - 1, m[$ *as the time interval of trading at trading day* m*. This would lead to introducing some additional notation and conventions which we do not seek to deal with in this note.*

IX.2.1 Quadratic Variation and Realized Volatility

The realized volatility is computed as

$$V(m,N)=\sum_{j=1}^{N}r(m,j)^2=\sum_{j=1}^{N}\left(p(t_j)-p(t_{j-1})\right)^2.$$

A main result is that for a quite general class of SDE's for $X(t)=\log p\,(t)$, on the form

$$dX(t)=d\log p(t)=\mu dt+\sigma(t)dW(t),\qquad [X](t)=\int_0^t\sigma^2(s)ds,$$

(with σ_t possibly stochastic as in the much applied SV models), we have ,

$$V(m,N)\xrightarrow{p}\int_{m-1}^{m}\sigma(s)^2ds=[X](m)-[X](m-1)$$

as $N\to\infty$, i.e. as we increase the sample frequency (the number of sampling points per trading day).

That is, the realized volatility $V\,(m,N)$ converges in probability as $N\to\infty$ to the intra-day quadratic variation, which again is the integrated variance[1] *of p_t.*

To give an understanding why this is the case we demonstrate it for two specific examples.

Example IX.8 *Assume $dp(t)=\sigma dW(t)$ such that $[p](t)=\sigma^2 t$, and*

$$p(t)-p(s)=\sigma\left(W(t)-W(s)\right)\overset{d}{=}N\left(0,\sigma^2\left(t-s\right)\right).$$

Hence with $t=m-1+j/N$ and $s=m-1+(j-1)/N$, $t-s=1/N$ and therefore,

$$\begin{aligned}V(m,N)&=\sum_{j=1}^{N}r(m,j)^2=\sum_{j=1}^{N}\left(p(m-1+j/N)-p(m-1+(j-1)/N)\right)^2\\&=\sigma^2\frac{1}{N}\sum_{j=1}^{N}z_j^2,\qquad z_j\ \sim i.i.d.N\left(0,1\right).\end{aligned}$$

[1] Therefore sometimes the term "realized volatility" can be misleading as it is really the "realized integrated volatility".

Now, as $N \to \infty$, the LLN for i.i.d. processes gives directly,

$$V(m,N) = \sigma^2 \frac{1}{N} \sum_{j=1}^{N} z_j^2 \xrightarrow{p} \sigma^2 E z_j^2 = \sigma^2 = [p](m) - [p](m-1).$$

That is, the realized volatility $V(m,N)$ converges in probability to the intraday integrated volatility, σ^2.

Example IX.9 *Assume next that $dp(t) = \sigma dW(t)$ where $\{\sigma(t)\}_{t\in[0,T]}$ is independent of $\{W(t)\}_{t\in[0,T]}$. Conditional on $\{\sigma(t)\}_{t\in[0,T]}$ we have*

$$[p](t)|\,\{\sigma(t)\}_{t\in[0,T]} = \int_0^t \sigma^2(u)du,$$

and

$$p(t) - p(s)|\,\{\sigma(t)\}_{t\in[0,T]} = \int_s^t \sigma(u)dW_u \bigg|\, \{\sigma(t)\}_{t\in[0,T]} \overset{d}{=} N\left(0, \int_s^t \sigma^2(u)du\right).$$

We therefore have, conditional on $\{\sigma(t)\}_{t\in[0,T]}$ and with $z_j \sim i.i.d. N(0,1)$,

$$\begin{aligned}
V(m,N) &= \sum_{j=1}^{N} r(m,j)^2 = \sum_{j=1}^{N} \left(p(m-1+j/N) - p(m-1+(j-1)/N)\right)^2 \\
&= \sum_{j=1}^{N} \left(\int_{m-1+(j-1)/N}^{m-1+j/N} \sigma^2(u)du\right) z_j^2 \\
&= \sum_{j=1}^{N} \left(\int_{m-1+(j-1)/N}^{m-1+j/N} \sigma^2(u)du\right) + \sum_{j=1}^{N} \left(\int_{m-1+(j-1)/N}^{m-1+j/N} \sigma^2(u)du\right)\left(z_j^2 - 1\right) \\
&= \int_{m-1}^{m} \sigma^2(u)du + \sum_{j=1}^{N} \left(\int_{m-1+(j-1)/N}^{m-1+j/N} \sigma^2(u)du\right)\left(z_j^2 - 1\right)
\end{aligned}$$

For the second term, we can use Chebychev's inequality[2], *to see that (condi-*

[2]For any constant $\eta > 0$, $P(|X| > \eta) \leq E[X^2]/\eta^2$.

tional on $\{\sigma(t)\}_{t\in[0,T]}$*)*

$$\begin{aligned}
&P\left(\left|\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)\left(z_j^2-1\right)\right|>\eta\right)\\
&\leq \frac{E\left[\left|\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)\left(z_j^2-1\right)\right|^2\right]}{\eta^2}\\
&= \frac{E\left[\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)^2\left(z_j^2-1\right)^2\right]}{\eta^2}\\
&= \frac{2}{\eta^2}\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)^2
\end{aligned}$$

since the $z_j's$ *are independent,* $E[z_j^2-1]=0$*, and* $E[(z_j^2-1)^2]=2$*. Next, assuming that* $\max_{u\in[0,T]}\sigma^2(u)<\infty$*,*

$$\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\leq\left(\max_u\sigma^2(u)\right)\frac{1}{N},$$

and hence

$$\begin{aligned}
\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)^2 &\leq \left(\max_u\sigma^2(u)\right)^2\frac{1}{N^2}\sum_{j=1}^{N}\\
&= \left(\max_u\sigma^2(u)\right)^2\frac{1}{N}\to 0
\end{aligned}$$

as $N\to\infty$*. Hence,*

$$P\left(\left|\sum_{j=1}^{N}\left(\int_{m-1+(j-1)/N}^{m-1+j/N}\sigma^2(u)du\right)\left(z_j^2-1\right)\right|>\eta\right)\to 0,$$

meaning that the second term tends to zero in probability. We conclude that

$$V(m,N)\xrightarrow{p}\int_{m-1}^{m}\sigma^2(u)du=[p](m)-[p](m-1)\qquad \text{as } N\to\infty.$$

Example IX.10 *(Market Microstructure Noise) The previous examples show that the RV is a consistent estimator for QV as the sampling frequency* $N\to\infty$*. However, due to so-called market microstructure noise (MMN), the result is hardly applicable in practice; see e.g. Andersen and Benzoni*

(2009, Section 6) and the references therein for a discussion of potential sources of MMN. To fix ideas, suppose, as in Example IX.8, that the true (efficient) returns are given by

$$r^*(m,j) = (p(m-1+j/N) - p(m-1+(j-1)/N)), \quad j = 1,\dots,N,$$

with $dp_t = \sigma dW_t$. Due to MMN, we observe $r^*(m,j)$ subject to noise, that is, we observe

$$r(m,j) = r^*(m,j) + \varepsilon_j,$$

where $\{\varepsilon_j\}_{j=1}^N$ is an i.i.d. process with $E[\varepsilon_t] = 0$, $\mathrm{Var}[\varepsilon_t] = \sigma_\varepsilon^2$ and $E[\varepsilon_t^4] < \infty$, and assume that the processes $\{\varepsilon_j\}_{j=1}^N$ and $\{p(t)\}_{t\in[m-1,m]}$ are independent. Note that the moments of ε_j do not depend on the frequency N. It holds that, with $V(m,N) = \sum_{j=1}^N r(m,j)^2$,

$$E\left[V(m,N)|\{p(t)\}_{t\in[m-1,m]}\right] = \sum_{j=1}^N r^*(m,j)^2 + N\sigma_\varepsilon^2$$

and

$$\mathrm{Var}\left[V(m,N)|\{p(t)\}_{t\in[m-1,m]}\right] = 4\sigma_\varepsilon^2 \sum_{j=1}^N r^*(m,j)^2 + N\,\mathrm{Var}\left(\varepsilon_t^2\right).$$

This suggests that the observed RV, $V(m,N)$, is an unreliable estimator for the RV of the efficient return variation, $\sum_{j=1}^N r^*(m,j)^2$. In particular, when the data are sample at very high-frequency, that is, when N is large, $V(m,N)$ may have little to do with the efficient returns. See, e.g., Zhang et al. (2005) for a rigorous discussion, and Hansen and Lunde (2006) for a setting where $r^*(m,j)$ and ε_j are dependent.

Example IX.11 *(Estimation of the conditional mean) Suppose that*

$$dp(t) = \mu dt + \sigma dW(t),$$

or equivalently,

$$p(t) = p(0) + \mu t + \sigma W(t),$$

with some initial fixed value $p(0)$ and $\mu \in \mathbb{R}$ and $\sigma > 0$. Suppose that we have N returns for a period $[0,T]$, say, one day, given by

$$r(i) = p(t_i) - p(t_{i-1}), \quad \text{with } t_i = \frac{i}{N}T, \quad i = 1,\dots,N.$$

Hence,

$$r(i) = \mu \frac{T}{N} + \sigma(W(t_i) - W(t_{i-1})),$$

and, using the properties of a Brownian motion, we have that $\{r(i)\}_{i=1,\dots,N}$ *is an i.i.d. process with*

$$r(i) \stackrel{d}{=} N\left(\mu \frac{T}{N}, \sigma^2 \frac{T}{N}\right).$$

For estimating μ *and* σ^2*, we have that the log-likelihood function is given by*

$$L_N(\mu, \sigma^2) = -\sum_{i=1}^{N} \log(\sigma^2) + \frac{\left(r(i) - \mu \frac{T}{N}\right)^2}{\sigma^2 \frac{T}{N}},$$

and it follows that the maximum likelihood estimator for μ *is given by*

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^{N} r(i) = \frac{p(T) - p(0)}{T} = \mu + \frac{\sigma W(T)}{T}.$$

Hence, for a fixed time interval, that is, with T *fixed, the estimator for* μ *does not depend on the sampling frequency* N*, and depends only on the time span of the data,* T*. This is in contrast to the realized volatility, where the infill asymptotics* ($N \to \infty$) *are used for showing convergence to the quadratic variation. It holds that*

$$E[\hat{\mu}] = \mu,$$

and

$$V(\hat{\mu}) = \frac{\sigma^2}{T},$$

so that for any $\epsilon > 0$

$$P(|\hat{\mu} - \mu| > \epsilon) \leq \frac{V(\hat{\mu})}{\epsilon^2} = \frac{\sigma^2}{T\epsilon^2}.$$

Hence, the estimator $\hat{\mu}$ *is unbiased but does only converge (in probability) to* μ *as* $T \to \infty$*, that is, as we increase the time span of the data.*

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Appendix: Calendar time and market time

Since the *realized volatility* measure is based on intra-day transactions the usual approach of simply denoting the (daily) returns 1 through T no longer suffices. For empirical applications, we typically need a way of timekeeping, which allows a specific time and date of a transactions as well as market specific conditions such as open-close hours and holidays. If we were to use simple calender time (e.g. 13:35 on the 22$^{\text{nd}}$ of February 2007) we would need to separately keep track of whether the market was open or closed at this time. As a solution we will use market time, which is defined below. For a further discussion see also Andersen, Bollerslev, Diebold & Labys (2001).

Definition IX.2 *Market time is defined specific to a given market. For any time point (where the market is open) it is defined as the fraction*

$$t = \frac{\text{"no. of hours of trading since initial date"}}{\text{"no. of hours of trading per day"}}.$$

By convention market time 0 corresponds to "opening of trade" (that is the time of day where the market opens) at the starting date. Hence all integer valued market times ($t = 0, 1, ..., T - 1$), correspond to opening time of the market on day $t+1$. When the considered period holds T trading days market time, t, belongs to the interval $[0, T[$.

To fix ideas consider the following two examples.

Example IX.12 *Consider the New York Stock Exchange (NYSE), which is open Monday - Friday from 9:30 to 16:00 ET except on public holidays. Assume we wish to analyze the following period: Friday the 6/2-09 at 9:30 to Tuesday the 10/2-09 at 16:00. What is the market time (t) of a transaction, which occurred on Tuesday at 11:54:30 and of one that occurred at 9:30 on the same day?*

1) Each trading day on the NYSE has 6.5 hours of trading. Hence from Friday the 6/2-09 at 9:30 to Tuesday at 11:54:30 there has been $6.5 + 6.5 + (11 + 54/60 + 30/60^2) - (9 + 30/60) = 15.40833$ *hours of trading. The market time of Tuesday at 11:54:30 is therefore* $t = 15.40833/6.5 = 2.370512$.

2) By convention 9:30 on Tuesday, which is the opening time of the NYSE on the third day of our period, has the market time $t = 2$.

Example IX.13 *Next consider the EUR-USD foreign exchange (FX) market, which trades 24 hours a day, 7 days a week. However, trading volume is much reduced during the weekends and we will therefore assume that the*

EUR-USD market is open from Monday at 0:00:00 to Friday at 23:59:59 GMT. Again we wish to analyze the period: Friday the 6/2-09 at 0:00:00 to Tuesday the 10/2-09 at 23:59:59. What is the market time (t) of a transaction, which occurred on Tuesday at 11:54:30?

Each trading day on the EUR-USD market has 24 hours of trading. Hence from Friday the 13/2-09 at 0:00 to Tuesday at 11:54:30 there has been $24 + 24 + (11 + 54/60 + 30/60^2) - (0 + 0/60) = 59.90833$ *hours of trading. The market time of Tuesday at 11:54:30 is therefore* $t = 59.90833/24 = 2.496180$.

From the examples and definition it follows that all transactions from trading day no. m, $m \in \{1, ..., T\}$ have market time in the interval $[m-1, m[$. Note that the point m is excluded from the interval, as this point corresponds to the beginning of the next trading day. This insight is important when manipulating databases of high frequency transactions