Microeconomics III, Ex. Class 7: Problem Set 3[a]

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[a] slides created by Thor Donsby Noe, adapted for autumn 2022 semester

Outline

PS3, Ex. 1 (A): Dominance and best response

PS3, Ex. 2 (A): Equilibrium selection

PS3, Ex. 3 (A): NE proof using IEWDS

PS3, Ex. 4 (A): Mixed strategy price competition

PS3, Ex. 5: Luxembourg as a rogue state (static game)

PS3, Ex. 6: Cournot Oligopoly with three firms

PS3, Ex. 7: Mixed Strategy Nash Equilibria - (p,q)-diagrams

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

PS3, Ex. 1 (A): Dominance and best response

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		Player 2	
		L	R
Player 1	U	5, 5	1, 6
	D	6, 1	2, 2

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Player 2

		L	C	R
Player 1	U	1, 0	1, 2	0, 1
	D	0, 3	0, 1	2, 0

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How do we know that the PSNE are the unique equilibria?
Iterated Elimination of Strictly Dominated Strategies (IESDS)! As the equlibria can be found by IESDS, these have to be the unique equilibria. (Game 1: eliminate U then L, Game 2: eliminate R, then D, then L).

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		Player 2		
		a	b	c
Player 1	A	2, 2	0, 0	-1, 2
	B	0, 0	0, 0	0, 0
	C	2, -1	0, 0	1, 1

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For **risk neutral** players (A, a) is the most reasonable as it maximizes payoff for both players.

For **risk averse** players avoiding A and a eliminates the risk of a negative payoff. (C, c) is more reasonable than (B, b) as the payoffs are positive.

PS3, Ex. 3 (A): NE proof using IEWDS

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3. (A) We have seen in the lectures that IESDS never eliminates a Nash Equilibrium. However, we saw in Problem Set 2 that this is not true if we do iterated elimination of weakly dominated strategies (IEWDS.) Go through the proof in the slides from lecture 2 and identify the step that is no longer true if we replace IESDS by IEWDS. That is, explain why the proof is no longer true when we replace ‘strict domination’ by ‘weak domination’.

PS3, Ex. 3 (A): NE proof using IEWDS

Informal proof: For the intuition, look at this example for now. At home, you can compare the two different formal proofs.

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		L	R
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NE is any strategy where no player can be strictly better off by deviating:

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IESDS: If a strategy was strongly dominated there would be a clear incentive to deviate from it, thus, by definition the strategy could not be a NE.

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IEWDS: In a NE where a player 1 is indifferent between the NE-payoff and her payoff from deviating, the NE-strategy can be weakly dominated if player 1's' alternative strategy would give a higher payoff in the case where player 2 deviates from his NE strategy as well.

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E.g. Player 1 is indifferent between $u_1(U, L)$ and $u_1(D, L)$, however, $u_1(D, R) > u_1(U, R)$, i.e. D weakly dominates U and U can be eliminated. I.e. eliminating the NE (U, L), leaving behind the reduced form game:

		Player 2	
		L	R
Player 1	D	3, 0	1, 2

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$$\forall s_2 \in S_2^n : \; u_1(s_1^*, s_2) < u_1(s_1^{'}, s_2) \quad (1)$$

where S_2^n is the set of player-2 strategies that have not been eliminated in rounds $1, ..., n-1$.

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- Contradiction! We can do the same for player 2. It follows that s_i^* survives IESDS for $i = 1, 2$.

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- **No** contradiction!

Conclusion: for a NE (s_1^*, s_2^*) IEWDS can eliminate s_1^* if $s_1^{'}, s_2^{'}$ exist such that:

$$\text{for } s_1^{'} \in S_1^n : \; u_1(s_1^*, s_2^*) = u_1(s_1^{'}, s_2^*)$$

and

$$\text{for } s_2^{'} \in S_2^n : \; u_1(s_1^*, s_2^{'}) < u_1(s_1^{'}, s_2^{'})$$

PS3, Ex. 4 (A): Mixed strategy price competition

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4. (A). Consider price competition between two firms when some consumers are informed about prices and others are not. Firms have zero marginal cost and they set price simultaneously; for the sake of this example, assume each price can only take one of the following values: 80, 54, 38. The market consists of two consumers. The uninformed consumer will visit a firm at random (probabilities $\frac{1}{2}, \frac{1}{2}$) and buy from it, regardless of the price. The informed consumer will visit the firm with the lowest price and buy from it. If both firms set the same price, assume that the informed consumer picks a firm at random (probabilities $\frac{1}{2}, \frac{1}{2}$).

(a) Argue that this game can be represented by the following bimatrix.

	p_2=80	p_2=54	p_2=38
p_1=80	80, 80	40, 81	40, 57
p_1=54	81, 40	54, 54	27, 57
p_1=38	57, 40	57, 27	38, 38

(b) Show that there is no Nash equilibrium in pure strategies.

(c) Confirm that the following strategy profile is a Nash equilibrium: each firm plays price 80 with probability 0.232, price 54 with probability 0.361, and price 38 with probability 0.407.

(d) Why do you think the equilibrium is so different from the standard Bertrand pricing game (i.e. where competition drives price down to marginal cost)?

PS3, Ex. 4 (A): Mixed strategy price competition

(a) The game in normal form and bimatrix:

Players: *Firm 1, Firm 2*. Strategies: $p_i \in S_i = S = \{80, 54, 38\}$

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Payoffs consist of *payoff from the informed consumer* + *payoff from the uninformed*. I.e. payoffs for player $i \neq j$:

$$u_i(p_i, p_j) = \begin{cases} p_i + \frac{1}{2}p_i & \text{if} \quad p_i < p_j \\ \frac{1}{2}p_i + \frac{1}{2}p_i & \text{if} \quad p_i = p_j \\ 0 + \frac{1}{2}p_i & \text{if} \quad p_i > p_j \end{cases}$$

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Which can be represented as:

	p_j=80	p_j=54	p_j=38
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p_i=54	$\frac{3}{2}54=81$, -	54, -	$\frac{1}{2}54=27$, -
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Check that firm i is indifferent between all pure strategies when the opposing firm's strategy is given by the probability distribution $\widehat{p_j} = (0.232, 0.361)$:

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Check that firm i is indifferent between all pure strategies when the opposing firm's strategy is given by the probability distribution $\widehat{p_j} = (0.232, 0.361)$:

$$u_i(p_i = 80, \widehat{p_j}) = 0.232 \cdot 80 + 0.361 \cdot 40 + (1 - 0.232 - 0.361) \cdot 40 = 49.280 \approx 49.3$$

PS3, Ex. 4 (A): Mixed strategy price competition

(c) Confirm that the following strategy profile is a Nash equilibrium: each firm plays price 80 with probability 0.232, price 54 with probability 0.361, and price 38 with probability 0.407.

Remember: In an equilibrium in mixed strategies, a player is indifferent between all pure strategies that she is choosing with positive probability.

	p_2=80	p_2=54	p_2=38
p_1=80	80, 80	40, 81	40, 57
p_1=54	81, 40	54, 54	27, 57
p_1=38	57, 40	57, 27	38, 38

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$$u_i(p_i = 54, \widehat{p_j}) = 0.232 \cdot 81 + 0.361 \cdot 54 + (1 - 0.232 - 0.361) \cdot 27 = 49.275 \approx 49.3$$

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$$u_i(p_i = 38, \widehat{p_j}) = 0.232 \cdot 57 + 0.361 \cdot 57 + (1 - 0.232 - 0.361) \cdot 38 = 49.267 \approx 49.3$$

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$$u_i(p_i = 38, \widehat{p_j}) = 0.232 \cdot 57 + 0.361 \cdot 57 + (1 - 0.232 - 0.361) \cdot 38 = 49.267 \approx 49.3$$

There are rounding errors as the exact mixed strategy profile is $\widehat{p_j} = \left(\frac{193}{833}, \frac{8127}{22491}\right)$.

PS3, Ex. 4 (A): Mixed strategy price competition

(d) Why do you think the equilibrium is so different from the standard Bertrand pricing game (i.e. where competition drives price down to marginal cost)?

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In the standard Bertrand Oligopoly price competition would lead to the perfectly competitive outcome (price = marginal cost), here:

$$p_1^* = p_2^* = c = 0$$

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In the standard Bertrand Oligopoly price competition would lead to the perfectly competitive outcome (price = marginal cost), here:

$$p_1^* = p_2^* = c = 0$$

Introduction of an uninformed consumer dampens the effect of price competition as a firm i can expect a revenue of at least $\frac{1}{2}p_i$ no matter what price p_i it sets.

Price competition could be increased by lowering the probability that an uninformed customer randomly picks the firm, i.e. through:

1. A higher share of informed customers.
2. More competing firms (however, other effects affect the outcome as well).

PS3, Ex. 5: Luxembourg as a rogue state (static game)

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Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican (V) and Denmark (D) are preparing an attack on Luxembourg's nuclear research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_v s_D,$$

where $s_i \in [0,1]$ is the share of its military capacity that country i ($i \in \{V, D\}$) uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country i is

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost.

(a) Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium (NE) of this game.

(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

Please take 10 min to work on Ex. 5

PS3, Ex. 5.a: Luxembourg as a rogue state (static game)

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Write expected payoff for player $i \neq j$.

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Expected payoff for player $i \neq j$:

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Find the best-response function for i.

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Find the best-response function for i:

$$FOC: \ \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

What is the NE?

(Hint: is the game symmetric?)

PS3, Ex. 5.a: Luxembourg as a rogue state (static game)

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(a) Find the NE in the static game:

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Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

$$s_i^* = \frac{1}{3} \equiv s^{NE}$$

i.e. $NE = \left\{(s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3})\right\}$

PS3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

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Find the best-response function for i:

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(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

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Write expected payoff for player $i \neq j$.

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Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Find the social planner target function.

PS3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

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$$= 2\bar{s} - 2\bar{s}^2$$

The social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO).

PS3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

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$$FOC: \ \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

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i.e. $NE = \left\{(s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3})\right\}$

(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$FOC: \ \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{S} = 0$$

$$\bar{S} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

PS3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

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$$FOC: \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{S} = 0$$

$$\bar{S} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

The SO is higher than the NE as the positive externality is not rewarded, which leads to an incentive to free ride.

PS3, Ex. 6: Cournot Oligopoly with three firms

PS3, Ex. 6: Cournot Oligopoly with three firms

There are three identical firms in an industry. Their production quantities are denoted q_1, q_2, and q_3. The inverse demand function is

$$p = 1 - Q, \text{ where } Q = q_1 + q_2 + q_3.$$

The marginal cost is zero.

(a) Compute the quantities in the Cournot equilibrium, i.e., the Nash Equilibrium of the game where the firms simultaneously choose quantities.

(b) What is the price in the Cournot-equilibrium?

(c) Show that if two of the three firms merge (transforming the industry into a duopoly), the profits of the merging firms decrease. Explain.

(d) What happens if all three firms merge?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1,2,3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

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Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

$$q_i^* = \frac{1 - 2q_i^*}{2}$$

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$$q_i^* = \frac{1}{4} \equiv q^{NE}$$

(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

PS3, Ex. 6: Cournot Oligopoly with three firms

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(c) Show that if two of the three firms merge (transforming the industry into a duopoly), the profits of the merging firms decrease. Explain.

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j)q_i$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

$$q_i^* = \frac{1 - 2q_i^*}{2}$$

$$q_i^* = \frac{1}{4} \equiv q^{NE}$$

(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j)q_i$$

BR function for firm i in the duopoly:

$$q_i = \frac{1 - q_j}{2}$$

$$q_i^* = \frac{1 - 2q_i^*}{2}, \quad q_i^* = q_j^*$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j)q_i$$

BR function for firm i in the duopoly:

$$q_i = \frac{1 - q_j}{2}$$
$$q_i^* = \frac{1 - 2q_i^*}{2}, \quad q_i^* = q_j^*$$
$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

By merging the rivalry is internalized by reducing joint output which increase market price and the profit margin:

$$p^* = 1 - q_m^* - q_3^* = \frac{1}{3} \Rightarrow \pi_m^* = \pi_3^* = \frac{1}{9}$$

Are Firm *1* and *2* better or worse off? Why?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j)q_i$$

BR function for firm i in the duopoly:

$$q_i = \frac{1 - q_j}{2}$$

$$q_i^* = \frac{1 - 2q_i^*}{2}, \quad q_i^* = q_j^*$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

By merging the rivalry is internalized by reducing joint output which increase market price and the profit margin:

$$p^* = 1 - q_m^* - q_3^* = \frac{1}{3} \Rightarrow \pi_m^* = \pi_3^* = \frac{1}{9}$$

However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

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Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

$$\pi_i = (1 - q_i - q_j)q_i$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

By merging the rivalry is internalized by reducing joint output which increase market price and the profit margin:

$$p^* = 1 - q_m^* - q_3^* = \frac{1}{3} \Rightarrow \pi_m^* = \pi_3^* = \frac{1}{9}$$

However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

(d) What happens if all three firms merge?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1,2,3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

(c) Firm *1* and *2* merge to firm m.

$$\pi_i = (1 - q_i - q_j)q_i$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

By merging the rivalry is internalized by reducing joint output which increase market price and the profit margin:

$$p^* = 1 - q_m^* - q_3^* = \frac{1}{3} \Rightarrow \pi_m^* = \pi_3^* = \frac{1}{9}$$

However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

(d) A full merger maximizes joint profits:

$$q^*_{\text{monopoly}} = p^*_{\text{monopoly}} = \frac{1}{2} \Rightarrow \pi^*_{\text{monopoly}} = \frac{1}{4} > \frac{2}{9}$$

PS3, Ex. 7: Mixed Strategy Nash Equilibria - (p,q)-diagrams

PS3, Ex. 7: Mixed Strategy Nash Equilibria - (p,q)-diagrams

Plot the mixed best responses of each player (in a "(p,q)-diagram" - see the textbook). And find all Nash equilibria (pure and mixed) in the games below

(a)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

(b)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

(c)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

(d)

		Player 2	
		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (p_2)	1, 2	4, 3
	s_3 (1-p_1-p_2)	0, 1	0, 3

PS3, Ex. 7: Mixed Strategy Nash Equilibria - (p,q)-diagrams

Plot the mixed best responses of each player (in a "(p,q)-diagram" - see the textbook). And find all Nash equilibria (pure and mixed) in the games below

(a)

Player 2

Player 1

	L (q)	R (1-q)
T (p)	0, 0	0, 0
B (1-p)	0, 0	1, 1

(b)

Player 2

Player 1

	L (q)	R (1-q)
T (p)	1, 3	1, 0
B (1-p)	1, 1	5, 5

(c)

Player 2

Player 1

	L (q)	R (1-q)
T (p)	3, 2	1, 2
B (1-p)	0, 1	1, 2

(d)

Player 2

Player 1

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

Hint: Find the probabilities q for which Player 1 is indifferent, e.g. $u_1(T, q) = u_1(B, q)$. and the probabilities p for which Player 2 is indifferent, e.g. $u_2(L, p) = u_2(R, p)$.

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1

	L (q)	R (1-q)
T (p)	0, 0	0, 0
B (1-p)	0, 0	1, 1

For which values of q is Player 1 indifferent?

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

Write up Player 1's best-response (BR) function, $p^*(q)$.

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

Player 1's BR function, $p^*(q)$:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0, 1] & \text{if} \quad q = 1 \end{cases}$$

Plot Player 1's BR function, $p^*(q)$, in a (p,q)-diagram.

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
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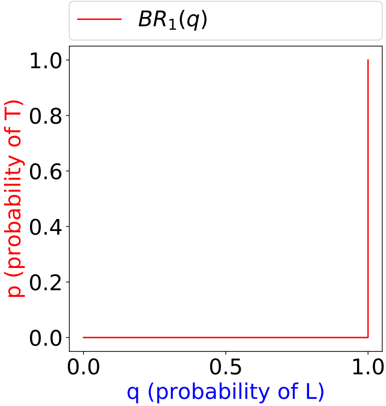


For which values of p is Player 2 indifferent?

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

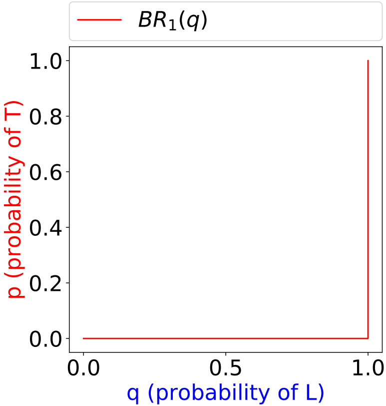
Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0, 1] & \text{if} \quad q = 1 \end{cases}$$

Player 2:

- Indifferent if $p = 1 \Rightarrow q \in [0, 1]$
- Prefers R if $p < 1 \Rightarrow q = 0$.



Write up Player 2's best-response (BR) function, $q^*(p)$

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1

	L (q)	R (1-q)
T (p)	0, 0	0, 0
B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

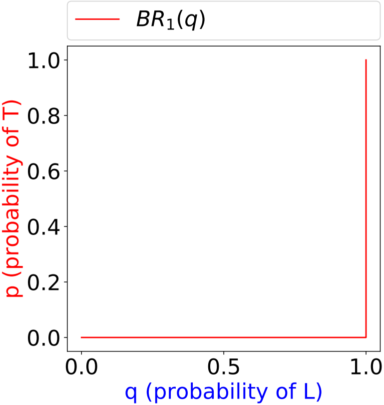
$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Player 2:

- Indifferent if $p = 1 \Rightarrow q \in [0, 1]$
- Prefers R if $p < 1 \Rightarrow q = 0$.

Player 2's BR function, $q^*(p)$:

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0,1] & \text{if} \quad p = 1 \end{cases}$$



Plot Player 2's BR function, $q^*(p)$

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 1		Player 2: L (q)	Player 2: R (1-q)
	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

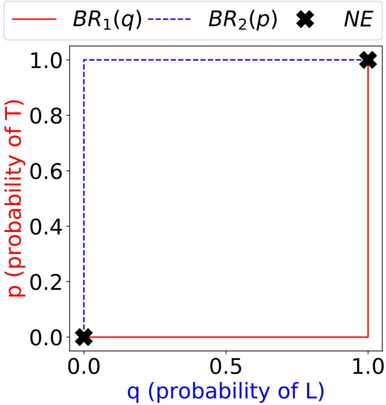
- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0, 1] & \text{if} \quad q = 1 \end{cases}$$

Player 2:

- Indifferent if $p = 1 \Rightarrow q \in [0, 1]$
- Prefers R if $p < 1 \Rightarrow q = 0$.

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0, 1] & \text{if} \quad p = 1 \end{cases}$$



Write up all NE (pure and mixed).

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1

	L (q)	R (1-q)
T (p)	0, 0	0, 0
B (1-p)	0, 0	1, 1

Player 1:

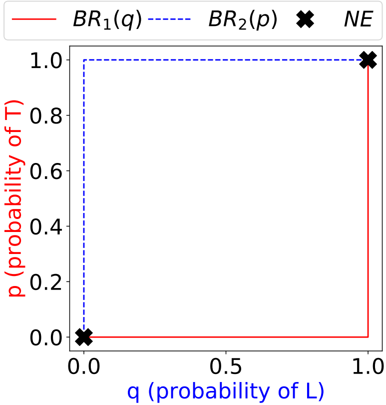
- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0, 1] & \text{if} \quad q = 1 \end{cases}$$

Player 2:

- Indifferent if $p = 1 \Rightarrow q \in [0, 1]$
- Prefers R if $p < 1 \Rightarrow q = 0$.

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0, 1] & \text{if} \quad p = 1 \end{cases}$$



Two Pure Strategy Nash Equilibria:

$$PSNE = \{(T, L), (B, R)\}$$

We find two Mixed Strategy NE (MSNE). Both coincide with the PSNE:

$$(p^*, q^*) = \{(1, 1), (0, 0)\}$$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

For which values of q is Player 1 indifferent?

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$

$$5q = 4 \Rightarrow q = 1$$

Write up Player 1's best-response (BR) function, $p^*(q)$*.*

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
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Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$

$$5q = 4 \Rightarrow q = 1$$

Player 1's BR function, $p^*(q)$:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Plot Player 1's BR function, $p^*(q)$, in a (p,q)-diagram.

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
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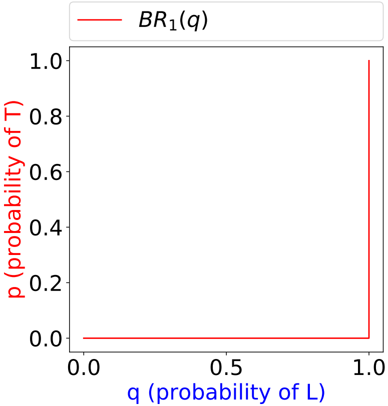


For which values of p is Player 2 indifferent?

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$

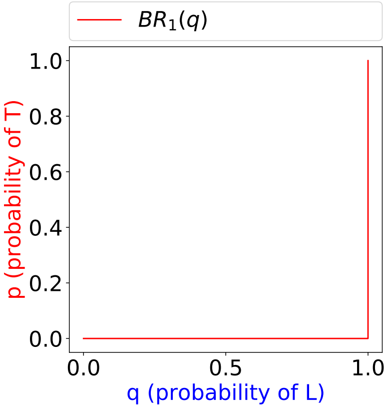
$$5q = 4 \Rightarrow q = 1$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Player 2 is indifferent if:

$$3p + 1(1-p) = 0p + 5(1-p)$$

$$7p = 4 \Rightarrow p = \frac{4}{7}$$



Write up Player 2's best-response (BR) function, $q^*(p)$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$

$$5q = 4 \Rightarrow q = 1$$

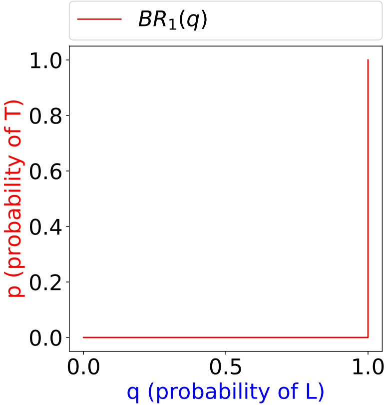
$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Player 2 is indifferent if:

$$3p + 1(1-p) = 0p + 5(1-p)$$

$$7p = 4 \Rightarrow p = \frac{4}{7}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 4/7 \\ q \in [0,1] & \text{if} \quad p = 4/7 \\ q = 1 & \text{if} \quad p > 4/7 \end{cases}$$



Plot Player 2's BR function, $q^*(p)$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1 - q)$$

$$5q = 4 \Rightarrow q = 1$$

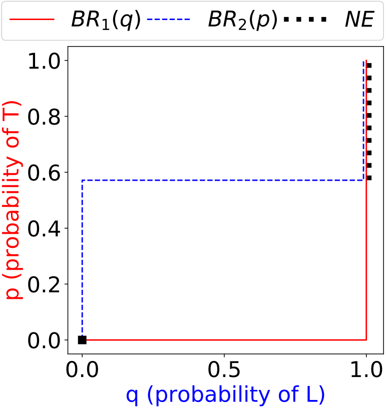
$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Player 2 is indifferent if:

$$3p + 1(1 - p) = 0p + 5(1 - p)$$

$$7p = 4 \Rightarrow p = \frac{4}{7}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 4/7 \\ q \in [0,1] & \text{if} \quad p = 4/7 \\ q = 1 & \text{if} \quad p > 4/7 \end{cases}$$



Write up all NE (pure and mixed).

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$

$$5q = 4 \Leftrightarrow q = 1$$

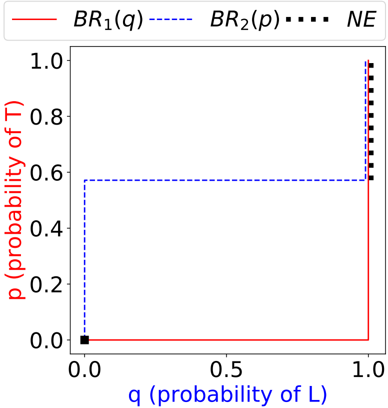
$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$

Player 2 is indifferent if:

$$3p + 1(1-p) = 0p + 5(1-p)$$

$$7p = 4 \Leftrightarrow p = \frac{4}{7}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 4/7 \\ q \in [0,1] & \text{if} \quad p = 4/7 \\ q = 1 & \text{if} \quad p > 4/7 \end{cases}$$



The pure and mixed strategy NE are:

$$(p^*, q^*) = \left\{ (0,0); (1,1); \left(p \in \left[\frac{4}{7}, 1\right), q = 1 \right) \right\}$$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

For which values of q is Player 1 indifferent?

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

Write up Player 1's best-response (BR) function, $p^*(q)$.

Player 2

		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$

$$q = 0$$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1-q) = (1-q)$$

$$q = 0$$

Player 1's BR function, $p^*(q)$:

$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Plot Player 1's BR function, $p^*(q)$, in a (p,q)-diagram.

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1		L (q)	R (1-q)
	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

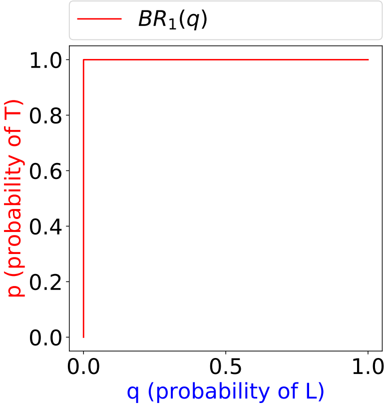
Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$

$$q = 0$$

Player 1's BR function, $p^*(q)$:

$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$



For which values of p is Player 2 indifferent?

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

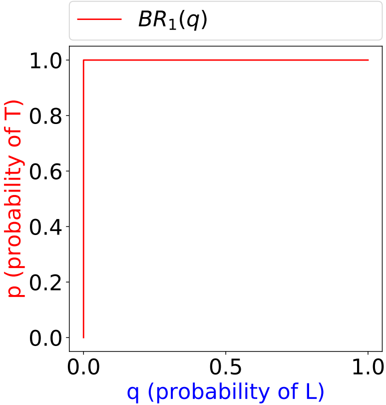
Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$
$$q = 0$$

$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Player 2 is indifferent if:

$$2p + (1 - p) = 2$$
$$p + 1 = 2 \Rightarrow p = 1$$



Write up Player 2's best-response (BR) function, $q^*(p)$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1-q) = (1-q)$$

$$q = 0$$

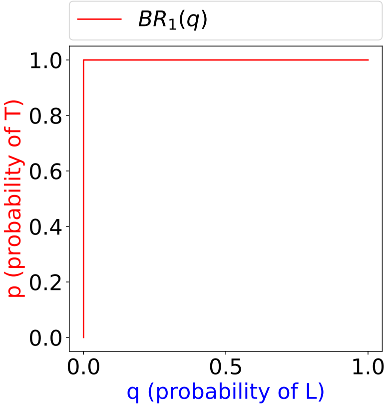
$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Player 2 is indifferent if:

$$2p + (1-p) = 2$$

$$p + 1 = 2 \Rightarrow p = 1$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0,1] & \text{if} \quad p = 1 \end{cases}$$



Plot Player 2's BR function, $q^*(p)$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$

$$q = 0$$

$$BR_1(q) = \begin{cases} p \in [0, 1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Player 2 is indifferent if:

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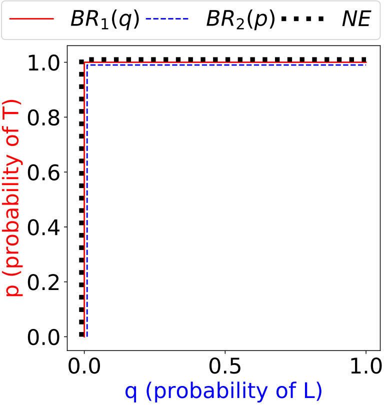
$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0, 1] & \text{if} \quad p = 1 \end{cases}$$



Write up all NE (pure and mixed).

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1-q) = (1-q)$$

$$q = 0$$

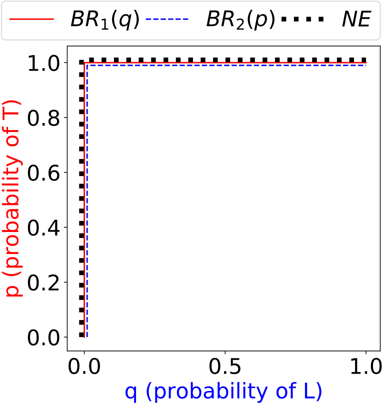
$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Player 2 is indifferent if:

$$2p + (1-p) = 2$$

$$p + 1 = 2 \Rightarrow p = 1$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0,1] & \text{if} \quad p = 1 \end{cases}$$



Three Pure Strategy NE (PSNE) exist:

$$PSNE = \{(T,L),(T,R),(B,R)\}$$

What about Mixed Strategy NE (MSNE), (p^*, q^*)*?*

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1-q) = (1-q)$$

$$q = 0$$

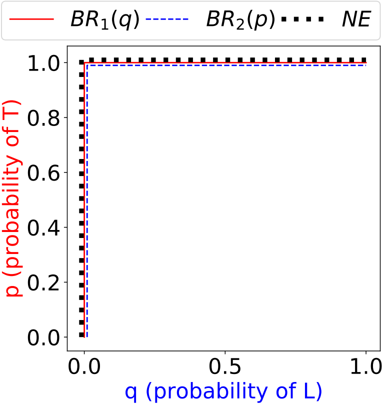
$$BR_1(q) = \begin{cases} p \in [0,1] & \text{if} \quad q = 0 \\ p = 1 & \text{if} \quad q > 0 \end{cases}$$

Player 2 is indifferent if:

$$2p + (1-p) = 2$$

$$p + 1 = 2 \Rightarrow p = 1$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1 \\ q \in [0,1] & \text{if} \quad p = 1 \end{cases}$$



$$PSNE = \{(T, L), (T, R), (B, R)\}$$

The three PSNE are contained in the two mixed strategy NE (MSNE), (p^*, q^*):

$$\{(p \in [0,1)\,, q = 0)\,; (p = 1, \in (0,1])\}$$

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

Can we reduce the bi-matrix?

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (1-p_1)	1, 2	4, 3

For which values of q is Player 1 indifferent?

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (1-p_1)	1, 2	4, 3

Player 1 is indifferent if:

$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

For which values of p is Player 2 indifferent?

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

Player 1

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (1-p_1)	1, 2	4, 3

Player 1 is indifferent if:

$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

$$p_1 = 1 - p_1 \Rightarrow p_1 = \frac{1}{2}$$

Plot Player 1's BR function, $p^*(q)$, in a (p,q)-diagram.

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

Player 1	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (1-p_1)	1, 2	4, 3

Player 1 is indifferent if:

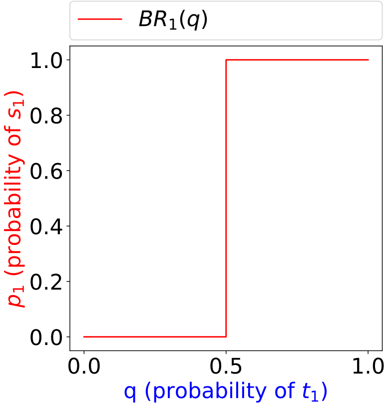
$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

$$p_1 = 1 - p_1 \Rightarrow p_1 = \frac{1}{2}$$



Plot Player 2's BR function, $q^*(p)$

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 $(1\text{-}q)$
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 $(1\text{-}p_1\text{-}p_2)$	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-$p_2 = 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 $(1\text{-}q)$
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 $(1\text{-}p_1)$	1, 2	4, 3

Player 1 is indifferent if:

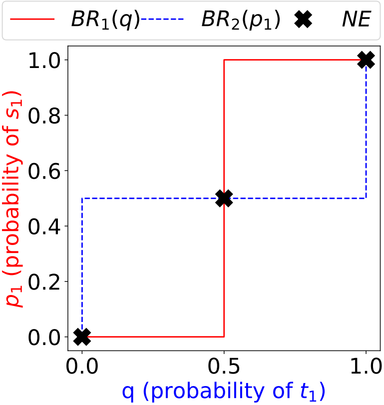
$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

$$p_1 = 1 - p_1 \Rightarrow p_1 = \frac{1}{2}$$



Write up all NE (pure and mixed), both in the reduced game and in the full game.

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria - (p,q)-diagrams

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (1-p_1)	1, 2	4, 3

Player 1 is indifferent if:

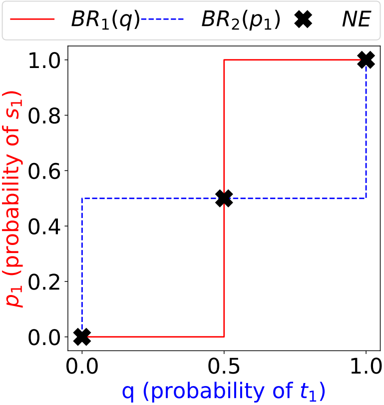
$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

$$p_1 = 1 - p_1 \Rightarrow p_1 = \frac{1}{2}$$



In the reduced game, three NE exist:

$$(p_1^*, q^*) = \{(0,0), (1/2, 1/2), (1,1)\}$$

And in the full game: $\left[(p_1^*, p_2^*), (q^*)\right] =$

$$\left\{[(0,1),(0)]\,;\left[\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2}\right)\right];[(1,0),(1)]\right\}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

Find all (pure and mixed) Nash equilibria in the following game:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

Find all (pure and mixed) Nash equilibria in the following game:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Hints:

1. Highlight the best responses in the matrix.
2. Find the relationship between q_1 and q_2 for which **Player 1 is indifferent**.
3. Write up the **best responses for Player 1**: $p^*(q_1, q_2)$, i.e. $BR_1(q_1, q_2)$.
4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.
5. Write up the **best responses for Player 2**:

$$BR_2(p) = (q_1^*(p), q_2^*(p)) = \begin{cases} \vdots & \vdots \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a Mixed Strategy Nash Equilibriumm (MSNE) both players must be indifferent between their respective pure strategies.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

1. Highlight the best responses in the matrix:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Which Pure Strategy Nash Equilibria (PSNE) exist?

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

1. Highlight the best responses in the matrix:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

No Pure Strategy Nash Equilibrium (PSNE) exist.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

No Pure Strategy Nash Equilibrium (PSNE) exist.

2. Find the relationship between q_1 and q_2 for which **Player 1 is indifferent**.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

2. Find the relationship between q_1 and q_2 for which **Player 1 is indifferent**:

Player 1 is indifferent if:

$$4q_1 + 2q_2 = 2q_1 + q_2 + 5(1 - q_1 - q_2)$$

$$7q_1 + 6q_2 = 5$$

$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1 is indifferent if:

$$4q_1 + 2q_2 = 2q_1 + q_2 + 5(1 - q_1 - q_2)$$

$$7q_1 + 6q_2 = 5$$

$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

3. Write up the **best responses for Player 1**: $p^*(q_1, q_2) =$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1 is indifferent if:

$$4q_1 + 2q_2 = 2q_1 + q_2 + 5(1 - q_1 - q_2)$$

$$7q_1 + 6q_2 = 5$$

$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

3. Write up the **best responses for Player 1**: $p^*(q_1, q_2) =$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.

Player 2 is indifferent between L and C if:

$$p + 3(1-p) = 3p + 2(1-p)$$

$$1 - p = 2p$$

$$p = \frac{1}{3}$$

If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.

Player 2 is indifferent between L and C if:

$$p + 3(1-p) = 3p + 2(1-p)$$
$$1 - p = 2p$$
$$p = \frac{1}{3}$$

If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

Player 2 is indifferent between L and R if:

$$p + 3(1-p) = 4p$$
$$3 = 6p$$
$$p = \frac{1}{2}$$

If $p < 1/2$ prefer L; if $p > 1/2$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.

Player 2 is indifferent between L and C if:

$$\begin{aligned} p + 3(1-p) &= 3p + 2(1-p) \\ 1 - p &= 2p \\ p &= \frac{1}{3} \end{aligned}$$

If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

Player 2 is indifferent between L and R if:

$$\begin{aligned} p + 3(1-p) &= 4p \\ 3 &= 6p \\ p &= \frac{1}{2} \end{aligned}$$

If $p < 1/2$ prefer L; if $p > 1/2$ prefer R.

Player 2 is indifferent between C and R if:

$$3p + 2(1-p) = 4p \Leftrightarrow 2 = 3p \Leftrightarrow p = \frac{2}{3}$$

If $p < 2/3$ prefer C; if $p > 2/3$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

5. Write up the **best responses for Player 2, dependent on p**: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right)$

Case: $p < \frac{1}{3}$:
L $\succ_2 C, L \succ_2 R \Rightarrow play\ L$

Case: $p = \frac{1}{3}$:
L $\sim_2 C, L \succ_2 R, C \succ_2 R \Rightarrow \ L \sim_2 C$

Case: $p \in (\frac{1}{3}, \frac{1}{3})$:
C $\succ_2 L, R \succ_2 L, C \succ_2 R \Rightarrow play\ C$

Case: $p = \frac{1}{2}$:
C $\succ_2 L, R \sim_2 L, C \succ_2 R \Rightarrow play\ C$

Case: $p \in (\frac{1}{2}, \frac{2}{3})$:
C $\succ_2 L, R \succ_2 L, C \succ_2 R \Rightarrow play\ C$

Case: $p = \frac{2}{3}$:
C $\succ_2 L, R \sim_2 L, C \sim_2 R \Rightarrow C \sim_2 R$

Case: $p > \frac{2}{3}$:
C $\succ_2 L, R \sim_2 L, R \succ_2 C \Rightarrow play\ R$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

Player 2: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

Player 2: $BR_2(p) = \big(q_1^*(p), q_2^*(p)\big) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 1: $p < 1/3$:

$$BR_2\left(p < \frac{1}{3}\right) = \{(1,0)\} \Rightarrow$$

$$\underbrace{1}_{q_1} + \frac{6}{7}\underbrace{0}_{q_2} > \frac{5}{7}$$

- MSNE, Case 2: $p = 1/3$:

$$BR_2\left(p = \frac{1}{3}\right) = \{(x, 1-x) : x \in [0,1]\} \Rightarrow$$

$$\underbrace{x}_{q_1} + \frac{6}{7}\underbrace{1-x}_{q_2} > \frac{5}{7}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

Player 2: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 3: $p \in (\frac{1}{3}, \frac{2}{3})$:

$$BR_2\left(p \in (\frac{1}{3}, \frac{2}{3})\right) = \{(0,1)\} \Rightarrow$$

$$\underbrace{0}_{q_1} + \frac{6}{7}\underbrace{1}_{q_2} > \frac{5}{7}$$

- MSNE, Case 4: $p = \frac{2}{3}$:

$$BR_2\left(p = \frac{2}{3})\right) = \{(0,x)\} \Rightarrow$$

$$\underbrace{0}_{q_1} + \frac{6}{7}\underbrace{x}_{q_2} = \frac{5}{7}$$

$$\frac{6}{7}x = \frac{5}{7} \Rightarrow x = \frac{5}{6}$$

$$\Rightarrow BR_1\left(0, \frac{5}{6}\right) = [0,1] \ni \frac{2}{3}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

Player 2: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 5: $p > \frac{2}{3}$:

$$BR_2\left(p > \frac{2}{3}\right) = \{(0,0)\} \Rightarrow$$

$$\underbrace{0}_{q_1} + \frac{6}{7}\underbrace{0}_{q_2} < \frac{5}{7}$$

$\Rightarrow BR_2\left(\frac{2}{3}\right) = \left(0, \frac{5}{6}\right)$ is a unique MSNE:

$$[(p^*), (q_1^*, q_2^*)] = \left\{\left[\left(\frac{2}{3}\right), \left(0, \frac{5}{6}\right)\right]\right\}$$

Closing Remarks

- The first mandatory assignment is due on Sep. 30, 12 pm (noon).
- Individual hand-in!
- Hand-ins should contain your name and/or your KU-ID
- Hand in either via:
- Mail: malterattenborg@econ.ku.dk
- Direct message on Absalon
- Pigeon Box: Hall in building 26, right hand-side coming from the main entrance (PhDs section)