Project Hardness - Homework 7 (Chapter 10)
Due date: 2022/11/09 Wed 23:59:59

You are allowed to refer to any resources, but we encourage you to try by yourself as every problem is designed to be self-contained.

If your submission scored x points, $0.1x$ USD will be donated to the Armed Forces of Ukraine.

1. (40 points) You can assume that all problems are minimization problems.
   (a) (10 points) Assume a PTAS reduction from A to B. Prove that if $B \in$ PTAS, $A \in$ PTAS.
   (b) (10 points) Assume an APX reduction from A to B. Prove that if $B \in$ APX, $A \in$ APX.
   (c) (10 points) Assume $A \leq_L B$, Prove that there is an APX reduction from A to B.
   (d) (10 points) Assume $A \leq_L B, B \leq_L C$, Prove that $A \leq_L C$.

2. (40 points)
   (a) (10 points) Prove the following: MAX-3SAT-E3 $\leq_L$ INDEPENDENT SET-4
   (b) (10 points) Prove the following: For all $\Delta \geq 4$, INDEPENDENT SET-Δ is APX-Complete.
   (c) (10 points) Prove the following: INDEPENDENT SET-4 $\leq_L$ MAX-2SAT
   (d) (10 points) Prove the following: MAX-2SAT $\leq_L$ MAX-NAE-3SAT. Your resulting construction shall not have duplicated literals in clauses and 1-clauses.

3. (75.7 points) Given an undirected connected graph $G = (V, E)$, the *Minimum Degree Spanning Tree* problem asks to find a spanning tree with the smallest possible max degree. This problem is not in PTAS: Consider the reduction to the Hamiltonian Path. On the other hand, this problem is not known to be APX-hard, being a strong candidate for APX-Intermediate problem.

   Fürer and Raghavachari (1992) devised a combinatorial algorithm that computes the $OPT + 1$ solution, where OPT is the max degree of some optimal solution. Your task is to prove the main theorem of the aforementioned result.

   Theorem 5.1. Let T be a spanning tree of max degree exactly k of a graph G. Let OPT be the max degree of some optimal solution. Let S be the set of vertices of degree k in T. Let B be an arbitrary subset of vertices of degree $k-1$ in T. Let $S \cup B$ be removed from the graph, breaking the tree T into a forest F. Suppose G satisfies the condition that, there are no edges between different trees in F. Then $k \leq OPT + 1$. (*Hint:* Find the lower bound on the number of components in F.)