Chapter 2. NP-Hardness via SAT and Planar SAT

#project-hardness

Hyunseo Jeong (Karuna)
jhwest@snu.ac.kr

Seoul National University

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Overview

What do we study in this Chapter?

- We will see some NP-complete graph problems and discuss NP-completeness of their planar graph versions.
- One way of proving NP-completeness of such planar graph problems is using **crossover gadget** that resolves the crossings of the graph, in result making a graph planar.
    - This requires making ad-hoc structures for each problem.
- One other approach we take is defining **Planar 3SAT** and directly reducing from it.
    - There are many variantions of Planar 3SAT with more requirements and still NP-complete, we'll talk about that.

Blowup

Before getting into the main topic, we define some terms that will be used for this & future chapters:

Blowup

- A reduction $f : A \leq_p B$ has a **linear blowup** if $|f(x)| \leq \mathcal{O}(|x|)$
- A reduction $f : A \leq_p B$ has a **quadratic blowup** if $|f(x)| \leq \mathcal{O}(|x|)$

First approach - crossover gadget

One good example demonstrating how crossover gadgets work is **Planar 3COL** problem.

3COL

- Instance : A graph G is given. For a XX-3COL problem, a graph has a property XX.
- Question : Does there exist a proper 3-coloring of G?

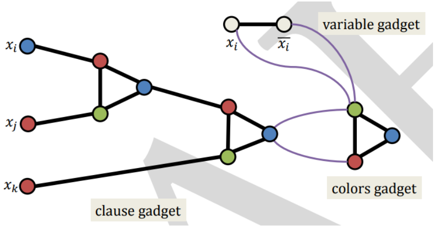
We first prove that 3COL is NP-complete. Then, we prove that 3COL $\leq_p$ Planar 3COL to show that Planar 3COL is NP-complete.

First approach - crossover gadget

Theorem

$3SAT \leq_p 3COL$, so 3COL is NP-complete.

Given any 3SAT instance φ, we construct a graph G so that $G \in 3COL$ iff $\varphi \in 3SAT$.



For each clause $x_i \vee x_j \vee x_k$, build a gadget described above. This forces at least one of x_i, x_j, x_k to be blue-colored, and the rest to be red-colored.

First approach - crossover gadget

Now we want to reduce Planar 3COL from 3COL.

- Given a graph G, we place the graph on the 2D plane in any way. There might be crossings between the edges.
- To make the graph planar, we replace each crossings with the following **crossover gadget**:

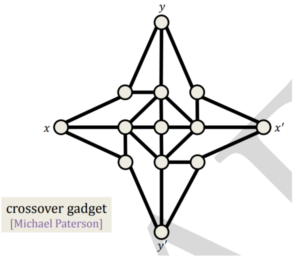


Figure: The crossover gadget for Planar 3COL

First approach - crossover gadget

It works like this:

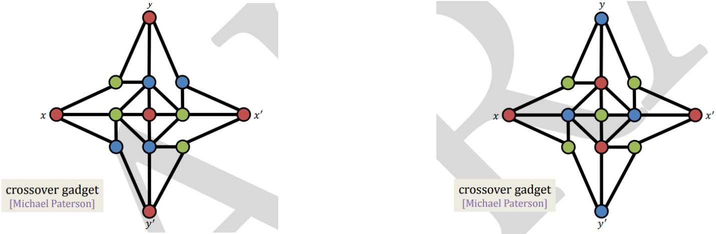


Figure: Two ways of 3-coloring the crossover gadget

First approach - crossover gadget

After replacing all crossings, we get a planar graph with the same 3-colorability with the original graph (I omitted details on the construction).
Since there may be $\mathcal{O}(N^2)$ crossings, this reduction has **quadratic blowup**.

Theorem (Garey et al. 1976)

3COL $\leq_p$ Planar 3COL, so Planar 3COL is NP-complete.

This is basically how methods involving crossover gadgets are done! All you need to do is to provide a crossover gadget and show that it works.

Limitations

But how to we even come up with the crossover gadget, for even complicated problems? Is it even true that crossover gadget always exist?
It turns out that the answer is negative.

Theorem (Gurhar et al. 2012)

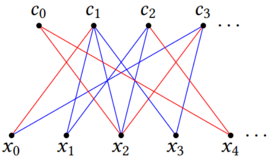
There is **NO** crossover gadget for the Planar Hamiltonian Cycle problem.

So we need another approach.

Second approach - Planar 3SAT

Our main idea is to define a **planar version of 3SAT**, and reduce Planar XX directly from Planar 3SAT.
We start from viewing an instance of a 3SAT as a bipartite graph.

- There is one vertex for each variables and clauses, and an edge is added if a clause contains a variable.
- Each edge has a color representing whether the literal is negated or not.



$$
\begin{aligned}
c_0 &= (\neg x_2 \wedge \neg x_4) \\
c_1 &= (\neg x_0 \wedge x_1 \wedge x_2 \wedge x_3) \\
c_2 &= (x_1 \wedge x_2 \wedge \neg x_4) \\
c_3 &= (x_0 \wedge \neg x_2 \wedge x_3) \\
&\cdots
\end{aligned}
$$

Second approach - Planar 3SAT

Planar 3SAT is defined in an obvious way.

Planar 3SAT

- Instance : A CNF formula φ is given, with the guarantee that its corresponding graph is planar.
- Question : Is φ satisfiable?

Note. Planarity test and finding a planar embedding of a planar graph can be both done in **linear** time, so we may just assume that a planar graph (associated to the formula) is given as an instance.

Second approach - Planar 3SAT

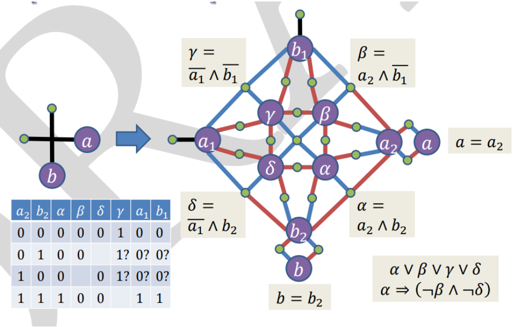
Planar 3SAT is a hard problem.

Theorem

3SAT $\leq_p$ Planar 3SAT, so Planar 3SAT is NP-complete.

Since we know only one way of proving hardness of a planar problem, we make a good use of it.

Second approach - Planar 3SAT



a_2	b_2	α	β	δ	γ	a_1	b_1
0	0	0	0	0	1	0	0
0	1	0	0		1?	0?	0?
1	0	0		0	1?	0?	0?
1	1	1	0	0		1	1

Figure: Crossover gadget for Planar 3SAT

Some additional (and easy) work should be done, since some clauses have 2 or 4 variables.

Second approach - Planar 3SAT

Now we can prove hardness of a planar graph problem by a reduction from Planar 3SAT. This is mainly done in this way:

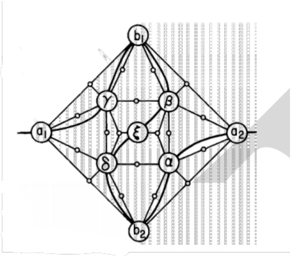
- Let G be any instance (which is a bipartite graph) of Planar 3SAT.
- We find a polynomial-time reduction from G to H **which preserves planarity**, such that $H \in A$ iff $G \in$ Planar 3SAT.

To find a reduction, or to prove a reduction preserves planarity, we often need stronger arguments than the original Planar 3SAT.

Second approach - Planar 3SAT

Theorem

Assume that in the given graph, all the variable vertices are connected in a cycle. Even so, the problem remains NP-complete.



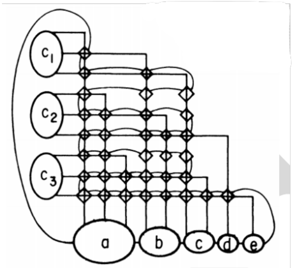


Figure: How a cycle is formed

Why do we care? The example we will see next will answer this.

Second approach - Planar 3SAT

As an example of reduction from Planar 3SAT, we prove that **Planar Directed Hamiltonian Cycle** is NP-complete. Note that this can't be proved by reduction from Hamiltonian Cycle using crossover gadgets.

Hamiltonian Cycle

- Instance : A graph *G* is given. For XX Ham Cycle, the graph has a property XX.
- Question : Is there a cycle that visits every vertex exactly once?

Theorem (Lichtenstein, 1982)

Planar 3SAT $\leq_p$ Planar Directed Ham Cycle, so Planar Directed Ham Cycle is NP-complete.

Second approach - Planar 3SAT

Given any Planar 3SAT instance φ, we have to construct a directed planar graph G such that $G \in$ Planar Directed Ham Cycle iff $\varphi \in$ Planar 3SAT.

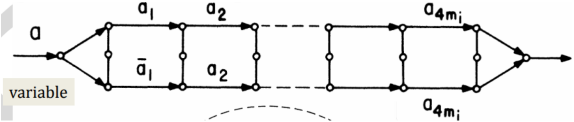


Figure: A variable gadget

We replace each variable vertex with a gadget shown above. A Hamiltonian cycle must alternate between two sides of a gadget, so there are two possibilities. We think of one being that $x_i = \texttt{TRUE}$, one being $x_i = \text{FALSE}$.

Second approach - Planar 3SAT

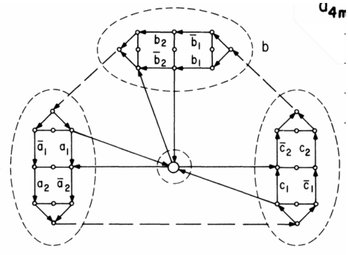


Figure: Clause connected to variables

A clause is replaced with a single vertex. For each literal in a clause, we add two directed edges as shown in the figure. Finally we make a cycle that goes through all variable gadgets, we have shown this is possible in previous slides.

Therefore Planar 3SAT $\leq_p$ Planar Directed Ham Cycle.

Variants of Planar SAT

We saw from previous slides that adding extra constraints to the instance makes the reduction relatively easier. So we are interested in how far we can go keeping the problem NP-complete.

First we can add this to the previous theorem.

Theorem

Given an instance of a Planar 3SAT, it can be drawn in a way such that

- A variable vertex is a horizontal segment on the x-axis.
- A clause vertex is a horizontal segment parallel not on the x-axis.
- Every edge between a clause vertex and the variable vertex is a vertical segment.

This kind of SAT has a name **Planar Rectilinear 3SAT**.

Variants of Planar SAT

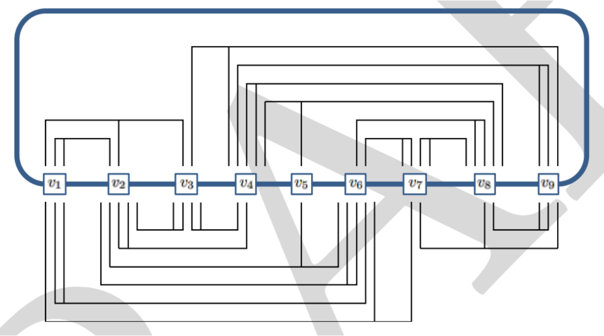


Figure: Planar Rectilinear 3SAT

Since we may assume that all the vertices are connected in a cycle, we can draw the graph as in the figure, such that the clauses form a nested tree structre on both sides of a cycle.

Variants of Planar SAT

This version of Planar SAT makes things a lot easier in many cases:

- Geometric arguments are now unnecessary because only axis-parallel segments are used.
- We may assume that the graph is drawn on a grid.

Variants of Planar SAT

Consider a monotone version of Planar Rectilinear 3SAT:

Planar Monotone Rectilinear 3SAT

- Instance : A Planar Rectilinear 3SAT instance φ is given, where all clauses are monotone, and all positive clauses are above the x-axis and negative clauses are below the x-axis.
- Question : Is φ satisfiable?

And this is also NP-complete.

Theorem

Planar Rectilinear 3SAT $\leq_p$ Planar Monotone Rectilinear 3SAT. Hence Planar Monotone Rectilinear 3SAT is NP-complete.

Variants of Planar SAT

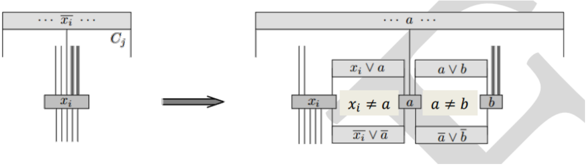


Figure: Reducing from PR-3SAT to PMR-3SAT

Variants of Planar SAT

Surprisingly, there are some variants that belong to **P**!

- Planar Rectilinear 3SAT where all clauses are above x-axis
- Planar not-all-equal 3SAT

Variants of Planar SAT

Planar MaxCut

- A planar graph $G = (V, E)$, and a parameter k is given.
- Find a partition of $V = V_1 \cup V_2$ such that edges between V_1 and V_2 is at least k.

Theorem (Hadlock, 1975)

Planar MaxCut is in P.

Theorem (Moret, 1988)

Planar not-all-equal 3SAT $\leq_p$ Planar MaxCut. Hence Planar not-all-equal 3SAT is in P.

Note that we have to reduce **from** a problem we want to show that is in P.

Variants of Planar SAT

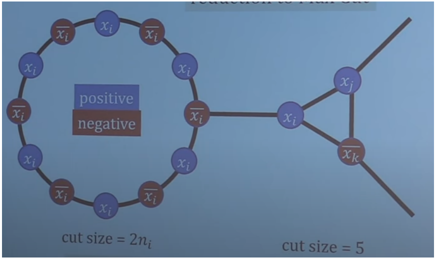


Figure: Reduction to Planar MaxCut

Replace each variable vertex with a cycle of even length. Replace each clause vertex with a triangle. Each clause vertex connects to distinct vertices of a cycle.

More problems

It is rather boring to explain all problems in the book, so I excluded most of them from the slide. These are some problems that are proved NP-complete:

- Planar Vertex Cover (with max degree = 3)
- Planar 3COL (with max degree = 4)
- Dominating Set
- Shakashaka
- Planar 1-in-3SAT
- Planar Positive Rectilinear 1-in-3SAT
    - Minimum Weight Triangulation
- Flattening Fixed-Angle Chains
- Corral
- Planar k-Means
- Multi-Robot Path Planning Problems on Planar Graphs
- 1-in-Degree Decomposition
- Tracking paths problem