Hardness HW 9

TAMREF
tamref.yun@snu.ac.kr

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Rules

- You can just link your former post instead of the solution. Otherwise, it is recommended to write the proof in your language.
- You are qualified if you **read** all the problems, and answered **at least 2** of them.

Question 1. Counting Bipartite Matching is Hard

Imitating the proof of $\sharp\text{BPM} \leq_{p.o.} \sharp\text{BMM}$, show that counting the number of bipartite matching is $\sharp\mathbb{P}-\text{complete}$.

Question 2. Another proof for hardness of $\sharp\text{CLIQUE}$

Construct a parsimonious reduction $\text{TH} - \text{POS} - 2\text{SAT} \to \text{CLIQUE}$. This gives an alternative proof about $\sharp\mathbb{P}-\text{complete}$ness of $\sharp\text{CLIQUE}$.

Question 3. UP-hardness of Permanent modulo k

In this problem, we investigate how Valiant(1979) could separate the complexity of $\text{per}(M) \mod 2^d$ and other $\text{per}(M) \mod k$ for $k \neq 2^d$.

> **Definition.** A decision problem $A = \{x : \exists^p y \text{ s.t. } (x, y) \in B\} \in \mathbb{NP}$ is called $\mathbb{UP}$ if $x \in A$ implies that there is **only one** y such that $(x, y) \in B$.

Given the lemmas below, define the reduction for $\mathbb{UP}$ and deduce that computing $\text{per}(M) \mod k$ is $\mathbb{UP}$-hard. According to complexityzoo, it is not known whether $\mathbb{UP}$ possess a complete problem.

(a) **(Valiant 1979, Lemma 3.1)** Given a CNF formula F, there is a function $f \in \mathbb{FP}$ such that $f(F)$ is a $\{-1, 0, 1, 2, 3\}$-matrix such that

$$\text{per}(f(F)) = 4^{t(F)} \cdot \sharp\text{SAT}(F)$$

- $t(F)$: twice the number of occurrences of literals $-$ the number of clauses in F. The only important fact is that $t(F)$ is a positive integer.

(b) **(Valiant 1979, Lemma 3.2)** For any $A \in \mathbb{NP}$, there is a parsimonious reduction $A \leq_p 3\text{SAT}$.

Question 4. Computing volume of a polytope is $\sharp P$-hard

Here, we define a **polytope** as an intersection of half-planes in n-dimension.

> **Definition.** Given an n-element POSET $(P, <)$, **order polytope** $\mathcal{O}_P$ of P is given by the sub-polytope of $C = [0,1]^n$, defined by
>
> $$\mathcal{O}_P := C \cap \bigcap_{p<q} \{(x_1, \cdots, x_n) : x_p < x_q\}$$

It is known that computing the number of **linear extensions (or, topological sorts)** of **POSETs (or, DAGs)** is $\sharp P-\mathrm{complete}$. (Brightwell & Winkler, 1991). Assuming the previous fact, deduce that computing volume of the polytope is $\sharp P$-hard, even for order polytopes.