Parametrized Complexity
Part 2

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Correction

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DEFINITION

Let A and B be parameterized problems. A **parameterized reduction** of A onto B maps an instance (x, k) of A to an instance (x', k') of B such that

1. x' depends only on x.
2. k' depends only on k.
3. The function that maps x to x' can be computed in $f(k) \cdot poly(|x|)$ time.
4. k' is bounded by a computable function of k.
5. $(x, k) \in A$ if and only if $(x', k') \in B$.

Introduction

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- As with polynomial-time-reduction, parameterized reduction also yields its own hierarchical complexity class structure, which will be the main topic of today's lecture.

The Complexity Class W[1]

[PROBLEM] Nondeterministic-Turing-Machine-Acceptance(NTMA$_k$)

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- It is believed, though not as strong as P$\neq$NP, that $\mathrm{NTMA}_k \notin$FPT.
- We can now define a new complexity class under the assumption that $\mathrm{NTMA}_k \notin$FPT.

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Polynomial Time Reduction	Parameterized Reduction

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Problem A is **NP-complete** if A is NP and NP-hard	Problem A is **W[1]-complete** if A is W[1] and W[1]-hard

The Complexity Class W[1]

THEOREM

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- As any independent set of size k can be found, if any, within path length of $O(k^2)$, we conclude that Independent-Set$_k$ $\in$W[1].

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- The vertices of the graph will be grouped into k^2 cells labelled $\{1, 2, \cdots, k\} \times \{1, 2, \cdots, k\}$, where vertices in each cell form a clique.

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- The vertices of the graph will be grouped into k^2 cells labelled $\{1, 2, \cdots, k\} \times \{1, 2, \cdots, k\}$, where vertices in each cell form a clique.
- By adding some more edges, we want this graph to have an independent set of size $k \times k$ if and only if the machine has an accepting path of length k.

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  2. For all $1 \le i < k$ and j and $(p, a) \in \Sigma \times Q$, put edges between (p, a) at cell (i, j) and every vertices at the row $i + 1$ which is incompatible with (p, a) according to the transition function of the machine.

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- Clearly, this graph has a clique of size k^2 if and only if the initial machine has an accepting path of length k. Therefore, Independent-Set$_k$ is W[1]-complete.

■

The Complexity Class W[1]

COROLLARY

Clique$_k$ is W[1]-complete.

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- For each vertex v of G, create $\Delta - \deg(v)$ new vertices $v'_1, \cdots, v'_{\Delta-\deg(v)}$, and connect each v_i and v'_j.
- Since for each $l \geq 3$, all clique of size l of the resulting graph is contained in one of G_is, it has a clique of size k if and only if the original graph has a clique of size k.

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- We'll later define the class W[2], where Dominating-Set$_k$ has been proven to be complete on.

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    5. Connect w_e with every vertex of $V_i - \{u\}$ and $V_j - \{v\}$.
- As there's no edge between x_i and y_i, every dominating set of size k in the resulting graph must contain exactly one vertex from each V_i.
- Now it is clear from the construction that the original graph has an independent set of size k if and only if the resulting graph has a dominating set of size k.

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Instance: A boolean circuit C and an integer $k \geq 0$.

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[PROBLEM] Circuit-SAT$_k$

Instance: A boolean circuit C and an integer $k \geq 0$.
Question: Is there a valid assignment of the input where exactly k variables are set to True?

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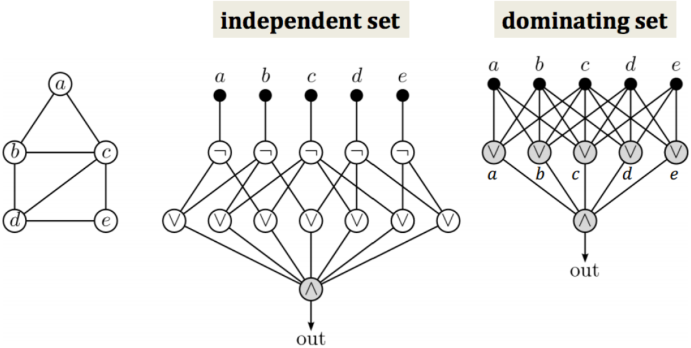

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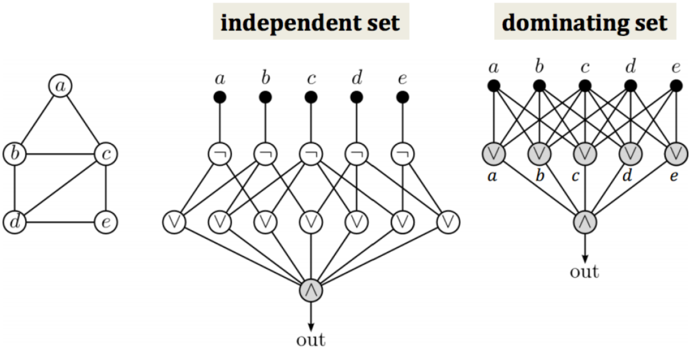
As we've noted earlier, locality of Clique_k / Independent-Set_k plays a part in the distinguishment of these two groups.
We formalize this concept to define other classes in W-Hierarchy.

W-Hierarchy



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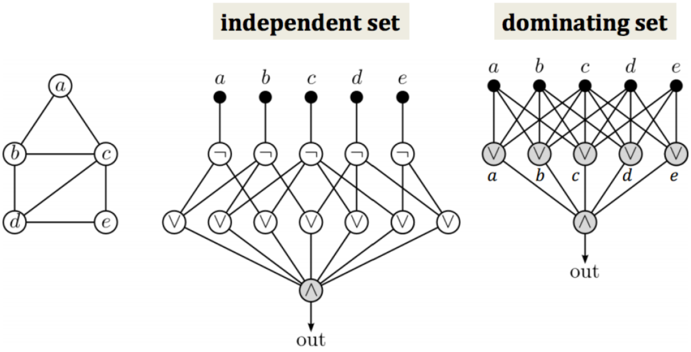

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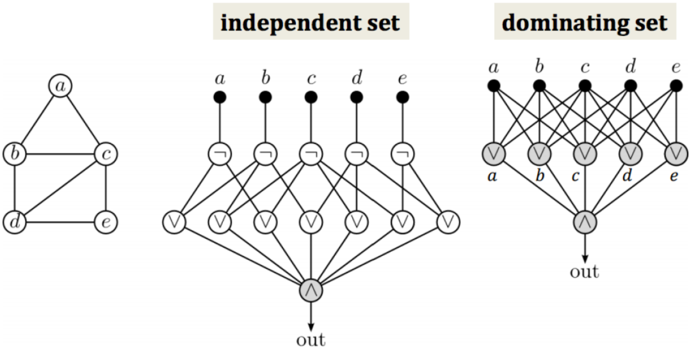

1. In the Independent-Set$_k$ circuit, all the input-output path pass through only one gate that has large number of inputs.
2. In the Dominating-Set$_k$ circuit, all the input-output path pass through two gates that has large number of inputs.

W-Hierarchy

DEFINITION

1. A **large gate** is a gate that has more than some fixed constant number. We let the constant be 2 for now.

W-Hierarchy

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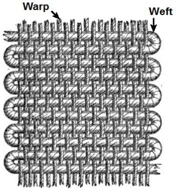

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6. $A_k \in$ XP if there exists computable functions f and g such that A_k can be solved in time $f(k) \cdot n^{g(k)}$.

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- FPT $\neq$ XP

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- Recall that assuming ETH, Vertex-Cover requires $2^{\Omega(n)}$ time to solve.
- If Vertex-Cover$_k$ could be solved in $2^{o(k)} \cdot n^c$ time, since $k \leq n$, this would yield a $2^{o(n)} \cdot n^c$ algorithm for Vertex-Cover, violating the ETH.

■

As Vertex-Cover has been shown to be solvable in time $O(k \cdot n + 1.28^k)$ by Chen et al., this bound is tight.

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- Recall that 3COL requires $2^{\Omega(n)}$ time assuming ETH.
- We give a reduction from 3COL to Clique$_k$.

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5. $G' = (V', E')$ is the final graph.

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- Now assume Clique$_k$ is solvable in $f(k) \cdot n^{o(k)}$ time. Then there exists a monotonously increasing unbounded function s such that Clique$_k$ is solvable in $f(k) \cdot n^{k/s(k)}$ time.

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- Set k to be the maximum value such that $f(k) \leq n$ and $k^{k/s(k)} \leq n$, and let g be the minimum value between the inverse of the two functions above.
- Now the runtime of our 3-coloring algorithm is
  $f(k) \cdot ((k \cdot 3^{n/k})^{k/s(k)}) \leq n \cdot k^{k/s(k)} \cdot 3^{n/s(k)} \leq n^2 \cdot 3^{n/s(g(n))} \leq 2^{o(n)}$
  which violates the ETH.

∎

The End