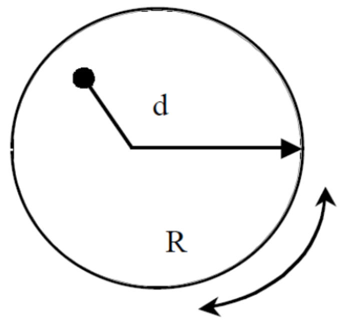
Problem 1. [20 pts] Moment of Inertia

Three point masses of identical mass m are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. Find the moment of inertia tensor around the origin, the principal moments of inertia, and a set of principal axes.

Problem 2. [20 pts] Pendulum disk



Consider a pendulum formed by suspending a uniform disk of radius R at a point a distance d from its center. The disk is free to swing only in the plan of the picture.

(a) Using the parallel axis theorem, or calculating it directly, find the moment of inertia I for the pendulum about an axis a distance d $(0 \leq d < R)$ from the center of the disk.

(b) Find the gravitational torque on the pendulum when displaced by an angle ϕ.

(c) Find the equation of motion for small oscillations and give the frequency ω. Further, find the value of d corresponding to the maximum frequency, for fixed R and m.

Problem 3. [20 pts] Angular moment in different frames

A bar of negligible weight and length ℓ has two identical masses m on each end. The bar is forced to rotate about an axis passing through its center and making an angle θ with it. The rotation is such that the angular velocity (pseudo-)vector $\boldsymbol{\omega}$ does not change in time.

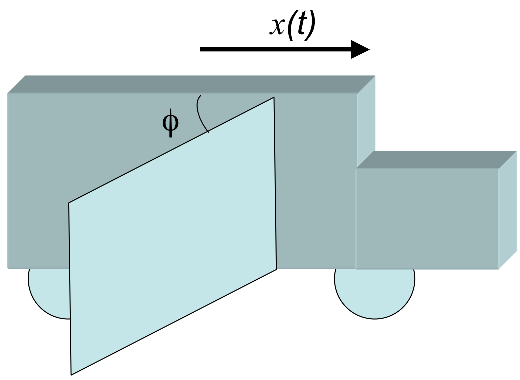
(a) Write the angular momentum $\mathbf{L}$ of the bar in *body* coordinates, in terms of the three components of $\boldsymbol{\omega}$. Compute the time derivative of $\mathbf{L}$ in the body frame, and from Euler's equations, find the components of the torque that are driving bar (along the principal axes of inertia).

(b) Consider now an *inertial* frame of reference with origin at the center of the bar and third axis in the direction of $\boldsymbol{\omega}$. Compute the bar's angular momentum in this frame by adding the angular momentum of each mass, $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ where $L_i = m\mathbf{r}_i \times \mathbf{v}_i$, and compute the time derivative of $\mathbf{L}$ in the inertial frame.

(c) Find now the transformation that relates the inertial and body frame, and use it to show that the two expressions you derived for $\mathbf{L}$ agree with each other. Check also that the time derivatives of $\mathbf{L}$ in the two frames are indeed related by

$$\dot{\mathbf{L}}|_{\text{inertial}} = \boldsymbol{\omega} \times \mathbf{L} + \dot{\mathbf{L}}|_{\text{body}} \tag{1}$$

Problem 4. [20 pts] A car door



A car door begins moving on a horizontal road, with the door accidentally left open with an initial angle ϕ_0 ($\phi = 0$ being when the door is full closed). The motion of the car is described by a function $X(t)$. The door has mass M, width W, height H, and a negligble thickness. Assume this is a primitive car, where the hinges allow a full rotation of the door.

(a) Write the Lagrangian of the door, considering it as a rotating rigid body.

(b) Find the differential equation for the angle of the door with the car, in terms of the (assumed known) position $X(t)$ of the car

(c) Describe qualitatively the door's motion when the car moves

- with uniform velocity.
- with uniform positive acceleration.
- with uniform negative acceleration.

(d) Under what conditions can the door oscillate harmonically about an equilibrium position? What is the frequency of the oscillation?