Problem 1. [20 pts] Momentum conservation

Consider a particle with electric charge q moving in the electrostatic field produced by each of the four charge configurations described below. What components of the particle linear momentum $\mathbf{p} = m\mathbf{v}$, and of the particle angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ will be conserved in each case?

(a) An infinite plane of charge, located on the plane $z = 0$.

(b) A semi-infinite homogeneous plane $z = 0$ and $y > 0$.

(c) An infinite homogeneous solid charged cylinder, with its axis along the y-axis.

(d) A finite homogeneous solid charged cylinder, with its axis along the y-axis, and its center at the origin.

(e) A homogeneous circular torus, with its axis along the z-axis.

Solution

Part (a)

The infinite plane of charge will mean that p_x, p_y, and L_z are conserved. The other rotations will rotate the plane and the z translations are broken, causing p_z to not be conserved.

Part (b)

In this case, of the above three, only one remains: p_x since that is the only symmetry left.

Part (c)

p_y and L_y are both conserved in this case. (other rotations and translations break the other four).

Part (d)

This breaks the result from (c) down to only having L_y conserved. No translation symmetry along y now.

Part (e)

Similar to (d) only the axis has changed so it only conserves L_z.

Problem 2. [20 pts] A system with one degree of freedom is described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{k}{x^2}. \tag{1}$$

Consider the transformation

$$x(t) \mapsto e^{-\epsilon/2}x(e^{\epsilon}t). \tag{2}$$

(a) Show that the infinitesimal version of this transformation is

$$\begin{aligned} \delta x(t) &= \left(t\dot{x}(t) - \frac{1}{2}x(t)\right)\epsilon \\ \delta \dot{x}(t) &= \left(t\ddot{x}(t) + \frac{1}{2}\dot{x}(t)\right)\epsilon \end{aligned} \tag{3}$$

(b) Show that this transformation is a symmetry of the Lagrangian and obtain the associated constant of motion Q.

(c) Check your result, i.e., show that $dQ/dt = 0$ when evaluated with the solutions of the equations of motion.

Solution

Part (a)

As we let ϵ become small, we get

$$\begin{aligned} x(t) &\mapsto (1-\epsilon/2)x((1+\epsilon)t) \\ &\mapsto x(t) + \left(t\dot{x}(t) - \frac{1}{2}x(t)\right)\epsilon \end{aligned} \tag{4}$$

This gives us

$$\delta x(t) = \left(t\dot{x}(t) - \frac{1}{2}x(t)\right)\epsilon. \tag{5}$$

The derivative is easily found too

$$\delta \dot{x}(t) = \left(t\ddot{x}(t) + \frac{1}{2}\dot{x}(t)\right)\epsilon. \tag{6}$$

Part (b)

There are two ways of showing this: One with the full transformation and the other with the infinitesimals. In this case, we show the infinitesimals

$$\begin{aligned} \delta L &= m\dot{x}\delta\dot{x} + \frac{2k}{x^3}\delta x \\ &= \left(mt\dot{x}\ddot{x} + \frac{1}{2}m\dot{x}(t)^2 + \frac{2kt\dot{x}}{x^3} - \frac{k}{x^2}\right)\epsilon \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 t - \frac{kt}{x^2}\right)\epsilon. \end{aligned} \tag{7}$$

The change in L being a full derivative implies this is a symmetry of the Lagrangian.

Using this, we can find Q using $Q = \frac{\partial L}{\partial \dot{x}}\frac{\partial \sigma}{\partial \epsilon} - \Lambda$, where $\delta L = \frac{\mathrm{d}\Lambda}{\mathrm{d}t}\epsilon$ and $\sigma(x(t),\epsilon) \approx x(t) + (t\dot{x} - \frac{1}{2}x)\epsilon + \cdots$

$$Q = m\dot{x}\left(t\dot{x} - \frac{1}{2}x\right) - \frac{1}{2}m\dot{x}^2 t + \frac{kt}{x^2} = \frac{1}{2}m\dot{x}^2 t - \frac{1}{2}m\dot{x}x + \frac{kt}{x^2}. \tag{8}$$

Part (c)

Lastly, we can use the equations of motion to see if this is conserved

$$m\ddot{x} = \frac{2k}{x^3}. \tag{9}$$

Doing the math

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{1}{2}m\dot{x}^2 + m\dot{x}\ddot{x}t - \frac{1}{2}m\ddot{x}x - \frac{1}{2}m\dot{x}^2 + \frac{k}{x^2} - \frac{2kt}{x^3} \\ &= m\dot{x}\ddot{x}t - \frac{1}{2}m\ddot{x}x + \frac{k}{x^2} - \frac{2kt\dot{x}}{x^3} \\ &= \frac{2k\dot{x}t}{x^3} - \frac{k}{x^2} + \frac{k}{x^2} - \frac{2kt\dot{x}}{x^3} \\ &= 0.\end{aligned} \tag{10}$$

Problem 3. [20 pts] Particle in electromagnetic field

Consider the Lagrangian of a non-relativistic particle of mass m and electric charge q in an electromagnetic field

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}, \tag{11}$$

where $\phi(t,\mathbf{r})$ and $\mathbf{A}(t,\mathbf{r})$ are the electromagnetic potentials, in terms of which the components of the electric and magnetic fields can be written as

$$E_i = -\partial_i\phi - \frac{1}{c}\partial_t A_i, \quad B_i = \epsilon_{ijk}\partial_j A_k, \tag{12}$$

where $\partial_i \equiv \partial/\partial x_i$ and ϵ_{ijk} is the totally antisymmetric symbol (Levi-Civita symbol).

(a) Write the Euler-Lagrange equations and show that they reproduce the Lorentz force

$$m\ddot{\mathbf{r}} = q\mathbf{E} + \frac{q}{c}\dot{\mathbf{r}}\times\mathbf{B}, \tag{13}$$

Hint: Use the identity $\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, where δ_{ij} is the Kronecker delta. You will need to use your ability to manipulate indices in this problem.

(b) Solve the equations of motion for the case

$$\phi = 0, \quad \mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B}, \tag{14}$$

with $\mathbf{B} = (0,0,B)$ in Cartesian coordinates and B is a constant.

(c) Show that the rotations around the z-axis are a symmetry of the Lagrangian, and obtain the associated conserved quantity. Use again $\mathbf{B} = (0,0,B)$.

Solution

Part (a)

We can write out the Lagrangian with indices

$$L = \frac{1}{2} m \dot{r}_i \dot{r}_i - q\phi + \frac{q}{c} \dot{r}_i A_i. \tag{15}$$

Then we can compute the equations of motion

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{r}_i} &= \frac{\partial L}{\partial r_i} \\
\frac{\mathrm{d}}{\mathrm{d}t} \left(m\dot{r}_i + \frac{q}{c} A_i \right) &= -q \frac{\partial \phi}{\partial r} + \frac{q}{c} \dot{r}_j \frac{\partial A_j}{\partial r_i} \\
m\ddot{r}_i + \frac{q}{c} \frac{\partial A_i}{\partial r_j} \dot{r}_j + \frac{q}{c} \frac{\partial A_i}{\partial t} &= -q \frac{\partial \phi}{\partial r_i} + \frac{q}{c} \dot{r}_j \frac{\partial A_j}{\partial r_i} \\
m\ddot{r}_i &= q \left(-\frac{\partial \phi}{\partial r_i} - \frac{1}{c} \frac{\partial A_i}{\partial t} \right) + \frac{q}{c} \left(\dot{r}_j \frac{\partial A_j}{\partial r_i} - \dot{r}_j \frac{\partial A_i}{\partial r_j} \right) \\
m\ddot{r}_i &= q \left(-\partial_i \phi - \frac{1}{c} \partial_t A_i \right) + \frac{q}{c} (\dot{r}_j \partial_i A_j - \dot{r}_j \partial_j A_i)
\end{aligned} \tag{16}$$

To show an equivalence with the force equation, we can substitute

$$\begin{aligned}
m\ddot{r}_i &= qE_i + \frac{q}{c} \epsilon_{ijk} \dot{r}_j B_k \\
&= q \left(-\partial_i \phi - \frac{1}{c} \partial_t A_i \right) + \frac{q}{c} \epsilon_{ijk} \epsilon_{kmn} \dot{r}_j \partial_m A_n \\
&= q \left(-\partial_i \phi - \frac{1}{c} \partial_t A_i \right) + \frac{q}{c} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \dot{r}_j \partial_m A_n \\
&= q \left(-\partial_i \phi - \frac{1}{c} \partial_t A_i \right) + \frac{q}{c} (\dot{r}_j \partial_i A_j - \dot{r}_j \partial_j A_i).
\end{aligned} \tag{17}$$

These two equations match, proving the Lagrangian reproduces the Lorentz force.

Part (b)

In cartesian coordinates $A_x = -\frac{1}{2} yB$, $A_y = \frac{1}{2} xB$, and $A_z = 0$.

$$\begin{aligned}
m\ddot{x} &= \frac{qB}{c} \dot{y}, \\
m\ddot{y} &= -\frac{qB}{c} \dot{x}, \\
m\ddot{z} &= 0.
\end{aligned} \tag{18}$$

Defining $\omega_c = \frac{qB}{mc}$, there are numerous ways of solving the equations, we will take a derivative so that $\dddot{x} = -\omega_c^2 \dot{x}$ (and similarly for y). This implies that

$$\begin{aligned}
\dot{x} &= A' \cos \omega_c t + B' \sin \omega_c t, \\
\dot{y} &= C' \cos \omega_c t + D' \sin \omega_c t.
\end{aligned} \tag{19}$$

Or integrating this (and redefining constants)

$$x(t) = A\cos\omega_c t + E\sin\omega_c t + x_0,$$
$$y(t) = C\cos\omega_c t + D\sin\omega_c t + y_0. \tag{20}$$

In the above x_0 and y_0 represent just the point we are rotating around.

$$-\omega_c^2(A\cos\omega_c t + E\sin\omega_c t) = \omega_c^2(-C\sin\omega_c t + D\cos\omega_c t), \tag{21}$$

which sets $A = -D$ and $E = C$. (One can at this point assume many things about the coordinates without loss of generality—for instance that the circle is centered at the origin and it begins on the x-axis.) At $t = 0$, we assume $x(0) = x_1$ and $y(0) = y_1$ while $\dot{x}(0) = v_x$ and $\dot{y}(0) = v_y$. This will give us

$$\begin{aligned} A + x_0 &= x_1, \\ C\omega_c &= v_x, \\ C + y_0 &= y_1, \\ -A\omega_c &= v_y. \end{aligned} \tag{22}$$

This gives us $A = -v_y/\omega_c$ and $C = v_x/\omega_c$ while $x_0 = x_1 + v_y/\omega_c$ and $y_0 = y_1 - v_x/\omega_c$. Taken together

$$\begin{aligned} x(t) &= (-v_y\cos\omega_c t + v_x\sin\omega_c t)/\omega_c + x_1 + v_y/\omega_c, \\ y(t) &= (v_x\cos\omega_c t + v_y\sin\omega_c t)/\omega_c + y_1 - v_x/\omega_c. \end{aligned} \tag{23}$$

(Variations on this also work.)

Part (c)

The Lagrangian takes the form

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - \frac{qB}{2c}y\dot{x} + \frac{qB}{2c}x\dot{y}. \tag{24}$$

In cylindrical coordinates $x(t) = r\cos\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\,\sin\theta$, $y(t) = r\sin\theta$, and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\,\cos\theta$, which gives

$$-y\dot{x} + x\dot{y} = r^2\dot{\theta}, \tag{25}$$

and the full Lagrangian becomes

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\omega_c r^2\dot{\theta}. \tag{26}$$

This now makes it clear that $\theta \to \theta + \theta_0$ is a symmetry of the Lagrangian.

The conserved quantity for rotations about z is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2(\dot{\theta} + \tfrac{1}{2}\omega_c). \tag{27}$$