Problem 1. [20 pts] Harmonic oscillator via Hamilton-Jacobi

Consider the Hamiltonian of a one-dimensional harmonic oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{1}$$

where ω is a constant. What is the the solution to the corresponding Hamiltonian-Jacobi equation $W(q,t)$ (you don't need to evaluate the integral over q)? And using W, find the dynamical trajectories $q(t)$ and $p(t)$.

Problem 2. [40 pts] Wave- to ray-optics with Hamilton-Jacobi

The setup: If light is going through a medium with a local index of refraction $n(y)$ and is partially moving in the y and z direction, it is governed by the following wave equation

$$\frac{n(y)^2}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial y^2} - \frac{\partial^2 E}{\partial z^2} = 0, \tag{2}$$

in this problem, we will derive ray optics which gives a particle description of these waves.

(a) To find how light moves in this medium when it is at frequency ω, substitute

$$E(y,z,t) = Ae^{-i(\omega t - \frac{\omega}{c}W(y,z))}$$

and keep terms only of order $(\omega/c)^2$ (high frequency approximation).

(b) Solve for $\frac{\partial W}{\partial z}$ putting the equation in the form of a Hamilton-Jacobi equation

$$\frac{\partial W}{\partial z} = -H\left(y, \frac{\partial W}{\partial y}\right). \tag{3}$$

What is the Hamiltonian $H(y,p)$? (*Hint*: Many times in this problem you will encounter square roots. The **positive** square roots will give you what you want but think about what the negative square roots mean physically; they are physical.)

(c) Compute the Lagrangian $L(y, \frac{\mathrm{d}y}{\mathrm{d}z}) = \frac{\mathrm{d}y}{\mathrm{d}z}p - H(y,p)$ eliminating (y,p) in favor of $(y, \frac{\mathrm{d}y}{\mathrm{d}z})$. *Note:* The spatial direction z is acting like "time" in this situation.

(d) Show that the "action" $S = \int L(y, \frac{\mathrm{d}y}{\mathrm{d}z})dz$ we are minimizing is related to the *least time* of a ray between two points (this is called "Fermat's principle")

$$T = \int_A^B \frac{n(y)}{c}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}\,dz = \int_A^B \frac{n(y)}{c}\sqrt{dy^2 + dz^2} \tag{4}$$

(e) Write a paragraph (at least 1/2-page) about what we have shown as a correspondence between classical waves and particles/rays. Compare this with the massive particles in classical mechanics. Is there an equation like Eq. (2) in classical mechanics? What about quantum mechanics?

(f) For $n(y) = n_0\frac{y}{y_0}$, solve for $y(z)$ assuming $y(0) = y_0$ and $y'(0) = 0$. Furthermore, using $W(y,z) = bz + g(y)$, solve for $W(y,z)$ and write out the full form of $E(y,z,t)$ in this approximation.