Problem 1. [20 pts] Momentum conservation

Consider a particle with electric charge q moving in the electrostatic field produced by each of the four charge configurations described below. What components of the particle linear momentum $\mathbf{p} = m\mathbf{v}$, and of the particle angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ will be conserved in each case?

(a) An infinite plane of charge, located on the plane $z = 0$.

(b) A semi-infinite homogeneous plane $z = 0$ and $y > 0$.

(c) An infinite homogeneous solid charged cylinder, with its axis along the y-axis.

(d) A finite homogeneous solid charged cylinder, with its axis along the y-axis, and its center at the origin.

(e) A homogeneous circular torus, with its axis along the z-axis.

Problem 2. [20 pts] A system with one degree of freedom is described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{k}{x^2}. \tag{1}$$

Consider the transformation

$$x(t) \mapsto e^{-\epsilon/2}x(e^{\epsilon}t). \tag{2}$$

In otherwords, $\sigma_x(x(t), \epsilon) = e^{-\epsilon/2}x(e^{\epsilon}t)$ in the language of class.

(a) Show that the infinitesimal version of this transformation is

$$\begin{aligned} \delta x(t) &= \left(t\dot{x}(t) - \frac{1}{2}x(t)\right)\epsilon \\ \delta \dot{x}(t) &= \left(t\ddot{x}(t) + \frac{1}{2}x(t)\right)\epsilon \end{aligned} \tag{3}$$

(b) Show that this transformation is a symmetry of the Lagrangian and obtain the associated constant of motion Q.

(c) Check your result, i.e., show that $dQ/dt = 0$ when evaluated with the solutions of the equations of motion.

Problem 3. [20 pts] Particle in electromagnetic field

Consider the Lagrangian of a non-relativistic particle of mass m and electric charge q in an electromagnetic field

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}, \tag{4}$$

where $\phi(t, \mathbf{r})$ and $\mathbf{A}(t, \mathbf{r})$ are the electromagnetic potentials, in terms of which the components of the electric and magnetic fields can be written as

$$E_i = -\partial_i \phi - \frac{1}{c}\partial_t A_i, \quad B_i = \epsilon_{ijk}\partial_j A_k, \tag{5}$$

where $\partial_i \equiv \partial/\partial x_i$ and ϵ_{ijk} is the totally antisymmetric symbol (Levi-Civita symbol).

(a) Write the Euler-Lagrange equations and show that they reproduce the Lorentz force

$$m\ddot{\mathbf{r}} = q\mathbf{E} + \frac{q}{c}\dot{\mathbf{r}} \times \mathbf{B}, \tag{6}$$

Hint: Use the identity $\epsilon_{ijk}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, where δ_{ij} is the Kronecker delta. You will need to use your ability to manipulate indices in this problem.

(b) Solve the equations of motion for the case

$$\phi = 0, \quad \mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}, \tag{7}$$

with $\mathbf{B} = (0, 0, B)$ in Cartesian coordinates and B is a constant.

(c) Show that the rotations around the z-axis are a symmetry of the Lagrangian, and obtain the associated conserved quantity. Use again $\mathbf{B} = (0, 0, B)$.