Supplementary material: Probing the structure of entanglement with entanglement moments

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In this supplement, we discuss entanglement moments when the measured system is an N dimensional Hilbert space, and we detail the Rabi model calculations presented in the main text. We described how to apply entanglement moments when the measured system is S^1 and the Bloch sphere, but the analysis is far more general. By utilizing the mathematics of $\mathbb{C}\mathrm{P}^{N-1}$, we can apply entanglement moments in N dimensions. For completeness, we also discuss our calculations in the Rabi model in depth. A wealth of information in the Rabi model is obtainable with exact calculations and simple numerics.

EXTENDING ANALYSIS TO ANY FINITE DIMENSIONAL HILBERT SPACE

We assume a bipartite system where the Hilbert space is the direct product of two other Hilbert spaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We can write states $|\Psi\rangle \in \mathcal{H}$ in terms of an orthonormal basis of $\mathcal{H}_B$, $\{|1\rangle, |2\rangle, \ldots, |N_B\rangle\}$, and (unnormalized) vectors $|\psi_i\rangle \in \mathcal{H}_A$,

$$|\Psi\rangle = \sum_i |\psi_i\rangle \otimes |i\rangle . \tag{1}$$

The vector $\langle\Psi|\Psi\rangle = 1$ while $\langle\psi_i|\psi_i\rangle \leq 1$ in general.

Let system $\mathcal{H}_A$ be an arbitrary N-dimensional Hilbert space. The space of normalized vectors is S^{2N-1}, but there is a U(1) gauge freedom in the distance measures given by

$$d^2_{(2n)}(i,j) = 1 - |\,\langle\tilde{\psi}_i|\tilde{\psi}_j\rangle\,|^2, \tag{2}$$

so the space is actually $\mathbb{C}\mathrm{P}^{N-1} = S^{2N-1}/\mathrm{U}(1)$. This is in fact a Hopf fibration from S^{2N-1} to $\mathbb{C}\mathrm{P}^{N-1}$ over the U(1) fiber. As with the other cases considered in the main text, we map $\langle\psi_i|\psi_i\rangle$ onto the function $\rho : \mathbb{C}\mathrm{P}^{N-1} \longrightarrow \mathbb{R}^+$.

The entanglement moments are then given by

$$C^2_{(n)} = \mathcal{N}_n \int_{\mathbb{C}\mathrm{P}^{N-1}} d\mu(z) \int_{\mathbb{C}\mathrm{P}^{N-1}} d\mu(w)\, \rho(z) d^2_{(2n)}(z,w)\rho(w). \tag{3}$$

As in the main text, the distance functions are known:

$$d^2_{(2n)}(z,w) = 1 - \Big|\sum_i z_i^* w_i\Big|^{2n}. \tag{4}$$

The distance function can be considered as a function of the on the *angle set*, so that we write

$$d^2_{(2n)}(z,w) = d^2_{(2n)}(2\Big|\sum_i z_i^* w_i\Big|^2 - 1). \tag{5}$$

Since Eq. (5) is an nth ordered polynomial in $2\left|\sum_i z_i^* w_i\right|^2 - 1$, we can expand it into Jacobi polynomials, $P_k^{(N-2,0)}(2\left|\sum_i z_i^* w_i\right|^2 - 1)$.

Now we need to use an addition formula for complex projective space as derived by [1, 2]. To develop the formula, we should write the space of functions, $L^2(\mathbb{C}\mathrm{P}^{N-1})$, as a direct sum of orthogonal subspaces in the following way. Dividing into the spaces of spherical harmonics, we have $L^2(S^{2N-1}) = H_1(2N) \oplus H_2(2N) \oplus \cdots$, where $H_m(2N)$ is the finite-dimensional vector space of harmonic polynomials homogeneous of degree m of $2N$ real variables that are restricted to S^{2N-1}. These should be further restricted to those that are just U(1) invariant since $\mathbb{C}\mathrm{P}^{N-1} = S^{2N-1}/\mathrm{U}(1)$. With this restriction, we follow the notation of [3] and write

$$L^2(\mathbb{C}\mathrm{P}^{N-1}) = H_{(0,0)}(N) \oplus H_{(1,1)}(N) \oplus H_{(2,2)}(N) \oplus \cdots , \tag{6}$$

where $H_{(m,m)}(N)$ are just the U(1) invariant parts of $H_m(2N)$.

Given this, we now state the addition theorem as written in [2]. Let $d_{k,N} = \dim H_{(k,k)}(N)$ and s_{kj} be an orthonormal basis in the space $H_{(k,k)}(N)$. Then the Jacobi polynomials become

$$P_k^{(N-2,0)}\left(2\Big|\sum_i z_i^* w_i\Big|^2 - 1\right) = \frac{1}{d_{k,N}}\binom{k+N-2}{k}\sum_{j=1}^{d_{k,N}} s^*_{kj}(z) s_{kj}(w). \tag{7}$$

Note that we can also calculate $d_{k,N}$ from formulae given in [2]. It is

$$d_{k,N} = \frac{2k+N-1}{N-1}\binom{k+N-2}{k}^2. \tag{8}$$

Just as before, we can expand our the distance function in terms of $P_k^{(N-2,0)}(2\left|\sum_i z_i^* w_i\right|^2 - 1)$, then expand that by the addition theorem and obtain

$$C^2_{(n)} = \mathcal{N}_n\left[1 - \sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{k+n+N-1}{n}\binom{k+N-1}{k}} \|\rho\|^2_{H_{(k,k)}(N)}\right], \tag{9}$$

where

$$\|\rho\|^2_{H_{(k,k)(N)}} = \sum_{j=1}^{d_{k,N}} \left|\int_{\mathbb{C}\mathrm{P}^{N-1}} d\mu(z)\, \rho(z) s^*_{kj}(z)\right|^2 \tag{10}$$

is the norm of the function in the finite subspace $H_{(k,k)}(N)$ – i.e., the norm in the kth harmonic. So the nth entanglement moment captures the information about the 1st through nth harmonic of the distribution.

Proper normalization of our distribution gives us $\|\rho\|^2_{H_{(0,0)}(N)} = 1$, since $H_{(0,0)}(N)$ is the space of constant functions. We can read off the normalization as

$$\mathcal{N}_n[\mathbb{C}\mathrm{P}^{N-1}] = \frac{\binom{n+N-1}{n}}{\binom{n+N-1}{n} - 1}. \tag{11}$$

This entire analysis reduces to the case of a Bloch sphere for $N = 2$, and we reproduce the Bloch sphere formula from the main text exactly.

ENTANGLEMENT CALCULATIONS IN THE RABI MODEL

For the Rabi model, $\mathcal{H}_A$ is a two level system and $\mathcal{H}_B$ is a harmonic oscillator, and we have

$$H = \omega a^\dagger a + g(a + a^\dagger)\sigma_x + \Delta\sigma_z, \tag{12}$$

where $a(a^\dagger)$ is the annihilation (creation) operater, σ_x and σ_z are the x and z Pauli matrices respectively, and ω, g, and Δ are constants (frequency of the oscillator, coupling, and Zeeman splitting, respectively).

The Rabi model's eigenstates have a particular form since the operator $\sigma_z \otimes P$, where P is the parity operator on the harmonic oscillator, commutes with the Hamiltonian. The states are

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle \otimes |\phi_\pm\rangle \pm |-\rangle \otimes P\,|\phi_\pm\rangle\right] \tag{13}$$

where $|+\rangle$ and $|-\rangle$ are the eigenstates of σ_x with eigenvalues $\sigma_x\,|\pm\rangle = \pm\,|\pm\rangle$, and $|\phi_\pm\rangle$ are unknown vectors in the Hilbert space of the harmonic oscillator.

The concurrence [4] for this system can be easily calculated, and happens to be

$$C^2_{(1)} = 1 - |\,\langle\phi_\pm|P|\phi_\pm\rangle\,|^2, \tag{14}$$

so the entanglement of these states just depends on the expectation value of the parity operator with harmonic oscillator wave functions.

We can exactly calculate things if we let $\Delta \to 0$. In that case the eigenstates are just

$$|\Psi_\pm; \Delta \to 0\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle \otimes |n\rangle_L \pm |-\rangle \otimes P\,|n\rangle_L\right], \tag{15}$$

where $|n\rangle_L$ is the nth state of the harmonic oscillator centered at $x = -\sqrt{2}g$.

The resulting concurrence is then

$$C^2_{(1)} = 1 - L_n(g^2/2)e^{-g^2/4}, \tag{16}$$

where $L_n(g^2/2)$ are Lagueere polynomials. As the coupling g increases, we see oscillations in the entanglement due to the Laguerre polynomials.

Braak [5] solved for the eigenstates when $\Delta \neq 0$. The eigenvalues can be calculated from $E^\pm_n = \xi^\pm_n - g^2/\omega$ and

$$0 = G_\pm(\xi^\pm_n) = \sum_{m=0}^{\infty} K_m(\xi^\pm_m)\left[1 \mp \frac{\Delta}{\xi^\pm_m - m\omega}\right]\left(\frac{g}{\omega}\right)^2, \tag{17}$$

where the coefficients $K_m(\xi)$ satisfy

$$mK_m = f_{n-1}(\xi)K_{m-1} - K_{m-2}, \tag{18}$$

$$K_0 = 1, \quad K_1(\xi) = f_0(\xi), \tag{19}$$

$$f_m(\xi) = \frac{2g}{\omega} + \frac{1}{2g}\left(m\omega - \xi + \frac{\Delta^2}{\xi - m\omega}\right). \tag{20}$$

The unnormalized eigenstates, written in Bargmann space [6], are

$$\begin{aligned}\phi^\pm_n(z) &= e^{gz}\sum_{n=0}^{\infty} K_n(\xi^\pm_n)(-z+g)^n \\ &= \pm e^{gz}\sum_{n=0}^{\infty} K_n(\xi^\pm_n)\Delta\frac{(z+g)^n}{\xi^\pm_n - n}.\end{aligned} \tag{21}$$

Taking normalization into account, we can obtain

$$C^2_{(1)}[|\Psi^\pm_n\rangle] = 1 - \left(\frac{\sum_{n=0}^{\infty} n!K_n(\xi^\pm_n)^2\frac{\Delta}{\xi^\pm_n - n}}{\sum_{n=0}^{\infty} n!K_n(\xi^\pm_n)^2}\right)^2. \tag{22}$$

The higher moments, $C^2_{(n)}$, do not admit a closed form expression when we specify the measurement basis as the eigenstates of $\hat{x} = \frac{1}{\sqrt{2}}(a + a^\dagger)$. However, it is straightforward to numerically calculate them. The eigenvalues can be calculated from Eq. 17 using a relatively simple rootfinding algorithm. Then with the eigenfunctions $(\psi_\uparrow(x)\ \psi_\downarrow(x))^T$ from 21, the probability distribution on the Bloch sphere is given from

$$\rho(\theta) = \int \mathrm{d}z\left(|\psi_\uparrow|^2 + |\psi_\downarrow|^2\right)\delta\left(\theta - \pi - 2\arctan\left(\frac{\psi_\uparrow}{\psi_\downarrow}\right)\right), \tag{23}$$

The entanglement moments, $C^2_{(n)}(g)$, can be plotted by extracting the Fourier components of the surface and adding them together appropriately.

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