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Sufficiency

1. $f(p; x_1 \ldots x_n) = p^{x_1}(1-p)^{n-x_1} p^{x_2}(1-p)^{n-x_2} \cdots p^{x_n}(1-p)^{n-x_n}$

$= p^{\sum x_i}(1-p)^{n^2-\sum x_i}$

Let $h(x_1 \cdots x_n) = 1$, $v(T, p) = p^T(1-p)^{n^2-T}$

$\therefore f(p; x_1 \ldots x_n) = h(x_1 \cdots x_n) \cdot v(T, p)$

$\therefore T = \sum_{i=1}^{n} x_i$ is sufficient

2. $f(p; x_1 \ldots x_n) = \prod_{i=1}^{n} p(1-p)^{x_i}$

$= p^n (1-p)^{\sum x_i}$

Let $h(x_1 \ldots x_n) = 1$, $v(T, p) = p^n(1-p)^T$

$\therefore f(p; x_1 \ldots x_n) = h(x_1 \ldots x_n) \cdot v(T, p)$

$T = \sum_{i=1}^{n} x_i$ is sufficient

3. $f(p; x_1 \ldots x_n) = \prod_{i=1}^{n} C_{x_i+r-1}^{r-1} p^r (1-p)^{x_i}$

$= \underbrace{\left(\prod_{i=1}^{n} C_{x_i+r-1}^{r-1}\right)}_{h(x_1 \ldots x_n)} \underbrace{p^{nr}(1-p)^{\sum x_i}}_{}$ $\rightarrow$ Let $\sum x_i = T$, $v(T, p)$

$\therefore f(p; x_1 \ldots x_n) = h(x_1 \ldots x_n) \cdot v(T, p)$

$\therefore T = \sum_{i=1}^{n} x_i$ is sufficient

4. $f(\beta; x_1 \ldots x_n) = \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i} I_{x_i > 0}$

$$= \underbrace{\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)}_{h(x_1 \ldots x_n)} \underbrace{e^{-\beta \sum x_i}}_{v(T, \beta)} \qquad T = \sum x_i$$

$\therefore T = \sum_{i=1}^{n} x_i$ is sufficient

5. $f(\beta; x_1 \ldots x_n) = \underbrace{\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)}_{v(T, \alpha)} \underbrace{e^{-\beta \sum x_i}}_{h(x_1 \ldots x_n)} \qquad T = \prod_{i=1}^{n} x_i$

$\therefore T = \prod_{i=1}^{n} x_i$ is sufficient

MLE

1. (a) $\hat{p} = \frac{5}{43}$

(b) $\hat{p} = \frac{3}{38}$

2. $X_1 \ldots X_n \sim U(0, \theta)$

$L(\theta; x_1 \ldots x_n) = \prod_{i=1}^{n} \frac{1}{\theta} = \theta^{-n}$

$l(\theta; x_1 \ldots x_n) = -n \ln \theta$

$\frac{dl}{d\theta} = -\frac{n}{\theta}$

$\therefore L(\theta; x_1 \ldots x_n) = \theta^{-n}$ is a decreasing function for $\theta \geq X_{(n)}$

$\therefore \hat{\theta} = X_{(n)}$ is a consistent sequence

3. $X_1 - X_n \sim N(\mu, \sigma^2)$

$$L(\mu,\sigma; x_1 - x_n) = \prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}\right) = (2\pi\sigma^2)^{-\frac{n}{2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$l(\mu,\sigma; x_1 - x_n) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2}\ln\sigma^2 - \frac{n}{2}\ln 2\pi$$

$$\frac{dl}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0 \Longrightarrow \hat{\mu} = \frac{\Sigma x_i}{n} = \bar{x}$$

$$\frac{dl}{d\sigma^2} = \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2\sigma^2} = 0 \Longrightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

$$\therefore 95\%\ \text{MLE} = \hat{\mu} + 1.645\,\hat{\sigma}$$

$$= \bar{x} + 1.645\sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}}$$

4. MLE of $v = P(X>2)$

$$= P\left(Z > \frac{2-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{2-\mu}{\sigma}\right)$$

5. $f(x) = \dfrac{1}{\pi(1+(x-\theta)^2)}$, $-\infty < x < \infty$

$$L(\theta, x_1 - x_n) = \frac{1}{\pi^n\prod_{i=1}^{n}(1+(x-\theta)^2)} = \frac{1}{\pi^n}\cdot\frac{1}{(1-(x_i-\theta)^2)^n}$$

$$l(\theta, x_1 - x_n) = -n\ln\pi - \sum_{i=1}^{n}\ln(1+(x_i-\theta)^2)$$

$$\frac{dl}{d\theta} = \sum_{i=1}^{n}\frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = 0$$

$$\Longrightarrow \frac{2(x+\theta)}{1+(x+\theta)^2} - \frac{2(x-\theta)}{1+(x-\theta)^2} = 0$$

$$(x+\theta) + (x+\theta)(x-\theta)^2 - (x-\theta) - (x-\theta)(x+\theta)^2 = 0$$

$$2\theta + (x+\theta)(x-\theta)[x-\theta-(x+\theta)] = 0$$

$$2\theta - 2\theta(x+\theta)(x-\theta) = 0 \qquad n = 20$$

$$2\theta(1-x^2+\theta^2) = 0 \qquad \bar{x} = -1.1465$$

$$2\theta(\theta^2 + (1-x^2)) = 0$$

$$\therefore \hat{\theta} = \pm\sqrt{x^2-1}$$

$$\approx \pm 0.56$$

6. $\sum_{1}^{20} x = 6(20) = 120 \quad F(x) = 1 - e^{-\lambda x}$

$$P(X > 15) = e^{-\lambda 15}$$

$$L(\mu, X_{20}) = \frac{1}{\mu^{20}} \exp\left(-\frac{120}{\mu}\right) \exp\left(\frac{-15}{\mu}\right)$$

$$= \frac{1}{\mu^{20}} \exp\left(\frac{-135}{\mu}\right)$$

$$l(\mu, X_{20}) = 20 \ln \mu - 135\mu$$

$$\frac{dl}{d\mu} = \frac{20}{\mu} - 135 = 0 \Rightarrow \hat{\mu} = 6.75$$

7. $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$L(\lambda; x_1 - x_n) = e^{-\lambda n} \lambda^{\sum x_i} \left(\prod_n \frac{1}{x_i}\right)$$

$$l(\lambda; x_1 - x_n) = -n\lambda + \sum x_i \ln \lambda + \ln\left(\prod_n \frac{1}{x_i}\right)$$

$$\frac{dl}{d\lambda} = -n + \frac{\sum x_i}{\lambda} = 0 \Rightarrow \hat{\lambda} = \frac{\sum x_i}{n} = \bar{x}$$

$$\hat{sd} = \sqrt{\hat{\lambda}} = \sqrt{\bar{x}}$$

8. $f(x) = \lambda e^{-\lambda x}$

$$L(\lambda, x_1 - x_n) = \lambda^n e^{-\lambda \sum x_i}$$

$$l(\lambda, x_1 - x_n) = n \ln \lambda - \lambda \sum x_i$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \sum x = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

$$\text{MLE of Median} = \left(\hat{\lambda} - \ln 2, \hat{\lambda} + \frac{1}{3}\right) = \left(\frac{1}{\bar{x}} - \ln 2, \frac{1}{\bar{x}} + \frac{1}{3}\right)$$