MA677 HW2 Jiahao Xu

1.

y	$f(y)$
1	$\frac{1}{42}$
2	$\frac{2}{42}$
3	$\frac{6}{42}$
4	$\frac{4}{42}$
7	$\frac{7}{42}$
8	$\frac{16}{42}$
6	$\frac{6}{42}$

$$\therefore E(Y) = 1\times\frac{1}{42} + 2\times\frac{2}{42} + 3\times\frac{6}{42} + 4\times\frac{4}{42} + 6\times\frac{6}{42} + 7\times\frac{7}{42} + 8\times\frac{16}{42}$$
$$= 6$$

2. $f(x,y) = 12y^2$ for $0 \le y \le x \le 1$. find $E(XY)$

$$E(XY) = \int_0^1\int_0^x xy12y^2\,dy\,dx = \int_0^1\int_0^x 12xy^3\,dy\,dx$$
$$= \int_0^1 3xy^4\Big|_0^x\,dx$$
$$= \int_0^1 3x^5\,dx$$
$$= \frac{1}{2}$$

3. $X_1, X_2, X_3 \sim \text{Uniform}(0,1)$

$$E[(X_1 - 2X_2 + X_3)^2] = E(X_1^2 - 2X_1X_2 + X_1X_3 - 2X_1X_2 + 4X_2^2 - 2X_2X_3 + X_1X_3 - 2X_2X_3 + X_3^2)$$
$$= E(X_1^2 - 4X_1X_2 + 4X_2^2 + 2X_1X_3 + X_3^2 - 4X_2X_3)$$
$$= E(X_1^2) + 4E(X_2^2) + E(X_3^2) - 4E(X_1X_2) + 2E(X_1X_3) - 4E(X_2X_3)$$
$$= \frac{1}{3} + \frac{4}{3} + \frac{1}{3} - 4\times\frac{1}{2}\times\frac{1}{2} + 2\times\frac{1}{2}\times\frac{1}{2} - 4\times\frac{1}{2}\times\frac{1}{2}$$
$$= \frac{1}{2}$$

$E(X_1) = E(X_2) = E(X_3) = \frac{1}{2}$

$E(X_1^2) = E(X_2^2) = E(X_3^2) = \int_0^1 x_i^2\,dx = \frac{1}{3}$

4. $f(x) = e^{-x},\ x>0 \quad Y = e^{\frac{3x}{4}} \quad$ Find $E(Y)$

$\ln Y = \frac{3x}{4} \Rightarrow x = \frac{4}{3}\ln Y \quad f(y) = e^{-\frac{4}{3}\ln y}\left|\frac{4}{3y}\right| = \frac{4}{3}y^{-\frac{7}{3}}$

$\therefore E(Y) = \int_1^\infty y\frac{4}{3}y^{-\frac{7}{3}}dy = \int_1^\infty \frac{4}{3}y^{-\frac{4}{3}}dy = \frac{4}{3}\left[-3y^{-\frac{1}{3}}\right]_1^\infty = -4\left[y^{-\frac{1}{3}}\right]_1^\infty$

$= 4$

5. $Y = g(X) = 2X^2+1,\quad E(X^2) = 1^2\times\frac{1}{6} + 2^2\times\frac{1}{6} + 3^2\times\frac{1}{6} + 4^2\times\frac{1}{6} + 5^2\times\frac{1}{6} + 6^2\times\frac{1}{6} = \frac{91}{6}$

$$
\begin{aligned}
E(Y) &= E(2X^2+1) \\
&= 2E(X^2)+1 \\
&= 2\times\frac{91}{6}+1 \\
&= \frac{94}{3}
\end{aligned}
$$

6. $f(x) = 2(1-x),\ 0<x<1 \quad Y = (2X+1) \quad$ Find $E(Y^2)$

$Y^2 = (2X+1)^2 = 4X^2+4X+1$

$$
\begin{aligned}
E(Y^2) = E(4X^2+4X+1) &= 4E(X^2)+4E(X)+1 \\
&= 4\int_0^1 x^2\, 2(1-x)\,dx + 4\int_0^1 x\, 2(1-x)\,dx + 1 \\
&= 8\int_0^1 x^2-x^3\,dx + 8\int_0^1 x-x^2\,dx + 1 \\
&= 8\left[\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1 + 8\left[\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\right]_0^1 + 1 \\
&= 8\left(\tfrac{1}{3}-\tfrac{1}{4}\right) + 8\left(\tfrac{1}{2}-\tfrac{1}{3}\right) + 1 \\
&= 3
\end{aligned}
$$

7.
$$
\begin{aligned}
E[(aX+b)^n] &= E\left[\sum_{k=0}^{n}\binom{n}{k}(aX)^{n-k}\cdot b^k\right] \\
&= \sum_{k=0}^{n} E\left[\binom{n}{k}(aX)^{n-k}\cdot b^k\right] \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\cdot b^k\cdot E(X^{n-k})
\end{aligned}
$$

Therefore, $E[(aX+b)^n] = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i E(X^{n-i})$

8. $n=20$, $p=0.05$ $X+Y=20$, $E(X)=np=1$

$E(X-Y)=E(X-(20-X))$

$=E(2X-20)$

$=2E(X)-20$

$=-18$

It means for a random sample of 20 parts selected from the shipment, the expectation of defective parts less than good parts is 18.