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Invest Wisely: a Mixed-integer Semi-closed-loop Dynamic Optimization Model

Investors are always eager to increase their returns and minimizing the risk by predicting the ups and downs of assets. Gold and bitcoin are two representative assets. In this paper, we build our model based on data in attachments to predict the possible best investment strategy. The performance of the model is evaluated by the investors' feelings when investing under the guidance of our strategy. Adjust the transaction costs to test its sensitivity to the cost, and finally write a memorandum including our strategy, model, and results to the trader.

In this paper, we introduce the **multi-stage fuzzy investment** method. By analyzing the characteristics of gold and bitcoin, investing by period is a good choice. The investment decision is mainly based on the price data before the trading day. Each investment change or maintain the current asset ratio [C,G,B], to achieve the purpose of reducing losses and increasing returns. Based on these demands, we construct the **mixed-integer semi-closed-loop dynamic optimization** model and solve it through **discrete approximate iterative algorithm**. The result is that the initial $1000 on 9/11/2016 will finally be the wealth equivalent to $62,211 on 9/10/2021.

To ensure that the model is convincing, regret value is used as reliable evaluation evidence. We set the risk level of the three assets (bitcoin: high-risk, gold: mid-risk and cash: risk-free). Then, plot the risk-return regret curve. The **regret value** of the adopted decision is compared with three other cases (all gold acquisition, all bitcoin acquisition, gold:bitcoin = 1:1). Here comes the conclusion: the regret level of the decisions made by the model is always at a low level (0.2~0.5). Regardless of the risk- return rate, the lower regret value is, the better the strategy is.

The adjustment of transaction costs is closely connected to changes in returns, which reflects the sensitivity of the strategy to costs. The cost in here is the commission (initially γ_{gold}=1%, $\gamma_{bitcoin}$=2%). Therefore, we adjust the commission percentage and reduce it to γ_{gold}=0.1%, $\gamma_{bitcoin}$= 0.2% and γ_{gold}=0.01%, $\gamma_{bitcoin}$=0.02%) respectively. It is obvious that the reduction in transaction costs leads to higher returns on the upfront decisions, an increase in the percentage of bitcoins invested, and a small increase in the final wealth.

Last but not least, we summarize the suggestions and write a memorandum to the trader to share our strategy, model, and results, hoping to bring a satisfying trading strategy.

Key Words: trapezoidal fuzzy number, discrete approximate iterative algorithm, regret value, dynamic optimization

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1. Introduction

Nowadays, market investment is one of the most important means to realize wealth appreciation. Through the purchase or sale of various assets in the market, investors can obtain different returns. In order to maximize the total return to investors, we can optimize the investment strategy and decide the amount and proportion of assets to be bought or sold based on market conditions.

Different assets have different risk levels, rewards and trading rules. Generally speaking, gold is less risky and less profitable; while bitcoin has higher risks and higher returns. The commission cost per transaction costs γ % of the amount traded. Usually, γ_{gold} is smaller than $\gamma_{bitcoin}$. For convenience, all "proportion of asset" in this paper refer to the proportion of the value of assets.

2. Assumptions and Justifications

1. **There is no limit to the maximum and minimum transaction amount.**
   To simplify the model, we assume that the minimum amount per transaction is infinitely close to 0, with no upper limit. And no debit or credit occurs. This applies to transactions of both gold and bitcoin.
2. **Investors make their decisions to purchase or sell based only on data from attachment, which are independent from people and circumstances around them.**
   Many factors exert influence on investors' decisions, such as the evaluation from people around them, the current situation of life, and the news on the Internet. In this question, when making a trading decision, only the attached data matters with no other factors.
3. **No major crises events affecting the world economy occurred during the investment period.**
   For example, uncontrollable events such as world wars and major natural disasters will have a huge impact on the economy. These impacts do not fully comply with market rules, so they cannot be correctly predicted.

3. Notations

Table 1 shows the notations commonly used in this paper and their description.

Table 1. Notations

Notations	Description
X_0	Initial assets owned by investors
t	An investment period
X_t	Total wealth at the end of period t
γ_i	The transaction cost of asset i
R_{it}	Fuzzy yield of risky assets i in period t
$\overline{svar}(X_t)$	The semi-variance of X_t
u_{it}	Investment amount to asset i on period t
u_{ft}	Investment amount to risk-free asset i on period t
l_{it}	The upper bound constraint of risk asset i on period t
p_{it}	The lower bound constraint of risk asset i on period t

4. Model Construction and Resolution

4.1 Problem Analysis

First of all, we note that this is a personal investment and the amount of investment is small, so the strategy should be in line with the market mechanism to reduce risks and increase returns.

As for gold trading, its price is affected by both short-term and long-term factors. Short-term factors, which are uncertain and contingent, let gold price fall more than its recent increase, so it is not recommended to invest in short-term timing. The medium and long-term timing model is influenced by the Federal Reserve's monetary policy and the characteristics of economic cycle. These economic characteristics can be made out obviously after their occurrence, but are hard to predict in advance.

Therefore, under the influence of complex and unpredictable driving factors, it is recommended to allocate gold assets for a long time. In this way, no matter how time flies or market influencing factors changes, we can finally obtain long-term benefits. The cost of such strategy is only to give up some band benefits that are difficult to grasp in the short term. It is not recommended that individual investors take risks for this kind of return. Therefore, for gold trading, we recommend maintaining long-term allocation of small positions, i.e. 5 % ~ 10 % of the position.

For bitcoin transactions, we prefer the automatic investment plan. In the stock market, automatic investment plan has been proved to be a suitable strategy for most people. By buying a certain amount of stock regularly, you can get the same returns with the market level, even higher. Since bitcoin's price change is also volatile, the automatic investment plan applies equally to bitcoin. By timing purchase, it effectively smooth out the price volatility, reduce holding costs and halve risk.

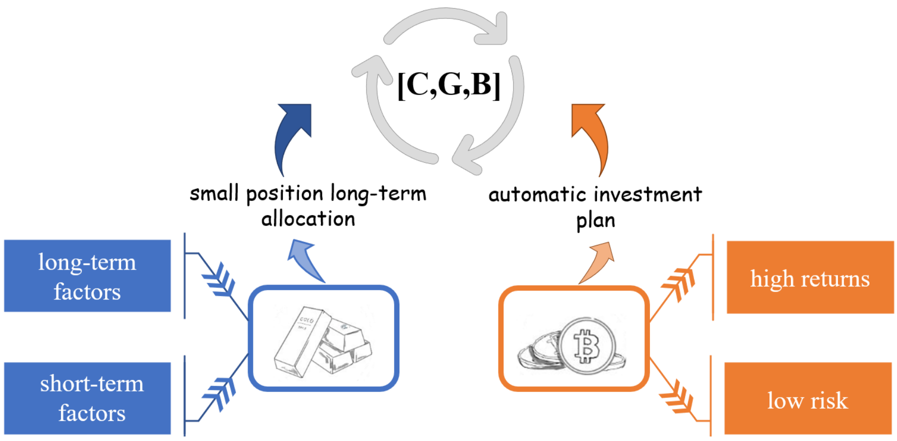


Figure 1. Gold and bitcoin

Securities returns, which can only be estimated according to limited information, can be considered as fuzzy variables rather than just random variables, which has been proved by many studies. Since the trapezoidal fuzzy number[1] has been shown to be an appropriate measure of security returns, we decide to predict the price of gold and bitcoin with fuzzy number. Based on this, establish a multi-stage decision-making model, which helps us to regularly change the proportion of holding assets and reduce trading in other time to reduce risks and improve returns.

4.2 Problem Analysis and Modeling Ideas

Based on the analysis above, with the '**time inconsistency**' of investment strategy brought into model construction, a **multi-stage mean semi-variance fuzzy portfolio** model is proposed. It is based on transaction cost, loan constraint, threshold constraint, revenue demand and cardinality constraint, and the optimal investment strategy with time consistency is studied. Due to transaction cost, revenue demand and cardinality constraints, the model provides a **mixed-integer semi-closed-loop dynamic optimization model with path-dependence** to solve this problem. As for details, we'll use the discrete approximate iterative algorithm is to obtain the optimal time-consistent investment strategy in the following paper. Figure 2 is the flowchart of this process.

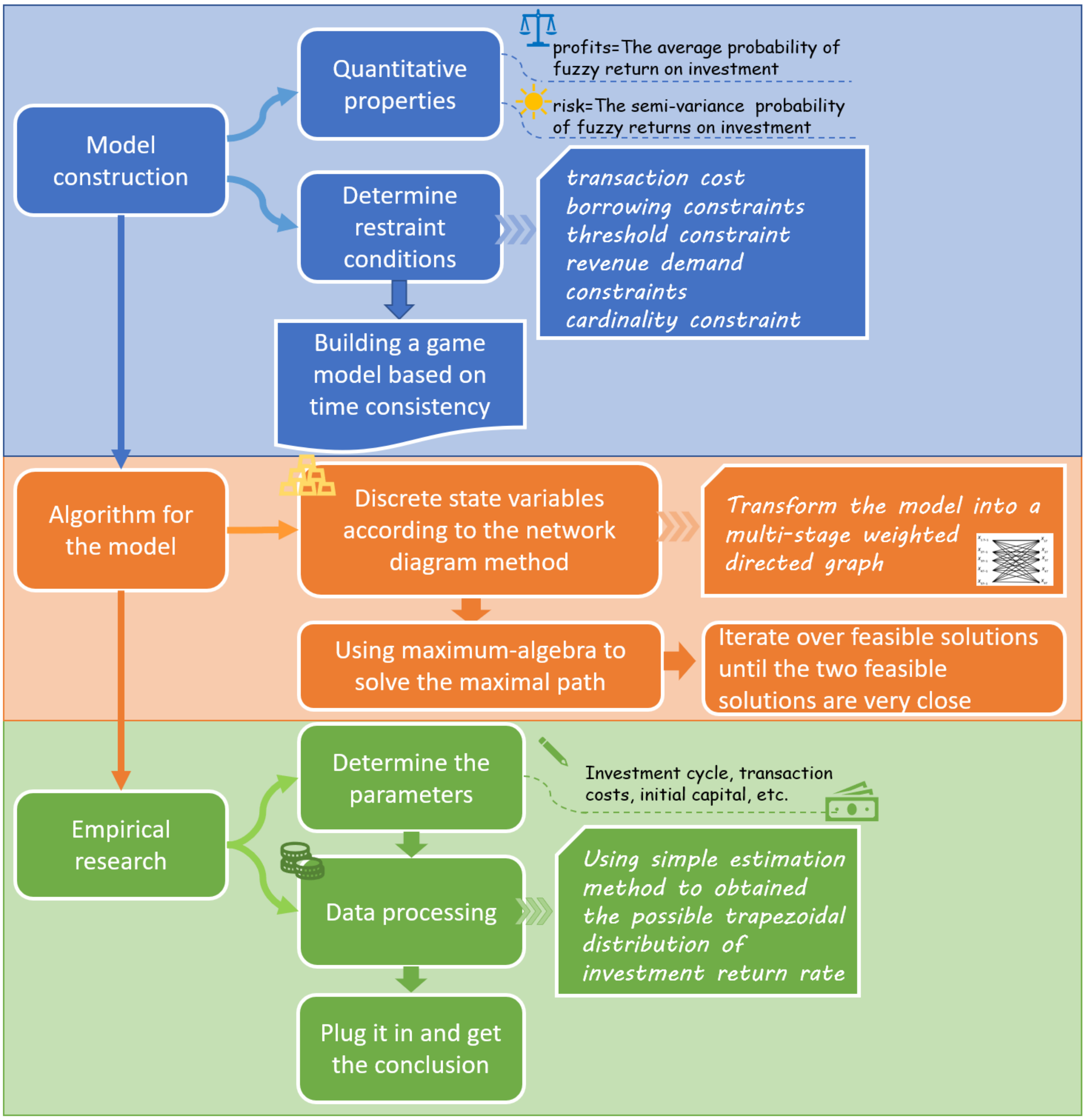


Figure 2. Model construction

4.3 Data Preprocessor

Before data analysis and decision making, it is a must to ensure the quality of data, which determines the prediction and generalization ability of the model. In order to ensure the accuracy, integrity and credibility of the data, we preprocess the data given by the attachments: *BCHAIN-MKPRU.csv* and *LBMA-GOLD.csv*.

Method *pandas.isnull.sum()* in python is used to process these two files. Some data is found in need of supplement. Consider a line as one item. Items without price data are included in the total but marked as incomplete. We find that there are 10 incomplete items out of a total of 1265 items in LBMA-GOLD.csv. In order to make the data smoother, we choose the mean interpolation method to supplement the data. For each incomplete item, the mean value of price on the day before and after the missing data day is supplemented into it. Table 2 shows the price data inserted into *LBMA-GOLD.csv*.

Table 2. Data inserted into file

Date	12/23/16	12/30/16	12/22/17	12/29/17	12/24/18	12/31/18	12/24/19	12/31/19	12/24/20	12/31/20
Price	1132.98	1148.45	1272	1301.5	1263.1	1281	1469.8	1521	1874.65	1915.4

4.4 The Necessity of Portfolio Investment

Focusing on gold and bitcoin, we mainly analyze their trading risks and correlation of price trends in this part.

With the development of market investment, people are more inclined to trade rationally and reduce risks while obtaining greater expected returns. Portfolio investment is a common way to avoid risks in investment management. By selecting different investment portfolios, we can maximize returns and minimize risks.

By introducing $\boldsymbol{DCC}$ coefficient[2], we can study the dynamic correlation between gold and bitcoin in the market. DCC-GARCH model is widely used in this field. Its general form is:

$$
\begin{aligned}
& r_t|F_{t-1} \sim N(0, H_t) \\
& H_t = D_t R_t D_t \\
& Q_t = (1-\alpha-\beta)\overline{Q} + \alpha\mu_{t-1}\mu_{t-1}^T + \beta Q_{t-1} \\
& D_t = diag\left(\sqrt{h_{11,t}}, \sqrt{h_{22,t}}, \dots \sqrt{h_{kk,t}}\right) \\
& R_t = \left(diag(Q_t)\right)^{-\frac{1}{2}} Q_t \left(diag(Q_t)\right)^{-\frac{1}{2}}
\end{aligned} \tag{1}
$$

where r_t is the conditional revenue rate of k kinds of assets, F_{t-1} is the set of collected information, H_t is the conditional covariance matrix, R_t is the dynamic conditional correlation index matrix, Q_t is the covariance matrix and $\overline{Q}$ is the unconditional covariance matrix of standardized residual. α and β are positive parameters which meet $\alpha+\beta < 1$.

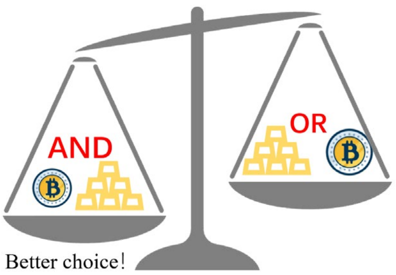


Figure 3. Investment choice

Through this model, we establish a single variable GARCH (1,1) model for the return ratio series of bitcoin and gold, and draw the conclusion that the dynamic correlation between bitcoin and gold is characterized by remarkable persistence. The details are not important to the main problem we want to solve, so we omitted them in this paper. In short, it is supposed to invest both gold and bitcoin to gain more return while lower the risks.

As for volatility, it describes the uncertainty of asset returns, which is used to reflect the risk level of financial assets. According to how volatility changes, assets can be devided into assets with different risks. Common types include high-risk assets, mid-risk assets, and risk-free assets.

Unlike gold and bitcoin, the U.S. dollars is affected by many factors and is largely controlled, so its value holds for a long time, and the volatility is very low. It can be considered as a risk-free asset. As for gold, its price and volatility changes more violent than U.S. dollar but less violent than bitcoin. Thus, gold can be considered as a mid-risk asset and bitcoin can be considered as a high-risk asset, as Figure 4 shows.

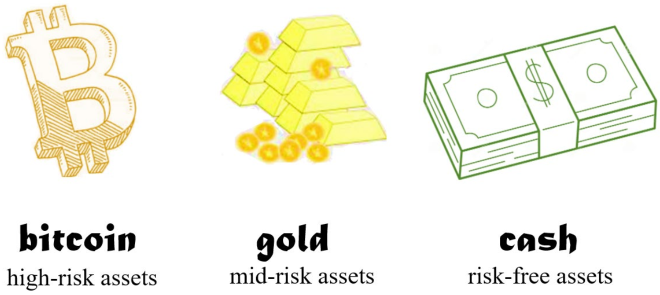


Figure 4. Risks of assets

4.5 Definition of Parameters

Assume that there are n kinds of risk assets and 1 kind of risk-free asset for investors to choose. The initial wealth is X_0. Let t stand for investment period, with T stages. At the first stage of investment stage, the investor can only invest with X_0. The asset portfolios changes at the beginning of eachstage and investment last for T stages.

Let R_{it} stand for the fuzzy yield of risky assets i in period t, and $R_t = (R_{1t}, R_{2t}, \dots R_{nt})'$. Set X_t as the total wealth at the end of period t. In period t, set $\overline{svar}(X_t)$ as the semi-variance of X_t, u_{it} as the investment in asset i, u_{ft} as the investment in risk-free asset i, l_{it} and p_{it} as the upper and lower bound constraint of risk asset i on period t. Let $u\_t = (u_{1t}, u_{2t}, \dots, u_{nt}, u_{ft})'$.

4.6 Parameter Quantization

The securities market is a complex system with dynamic changes. It is difficult for people to obtain the overall information of the random distribution of securities returns. The return can only be estimated according to the historical information of securities. According to most of the research[3] in this field, securities returns can be considered as fuzzy variables rather than random variables. In addition, considering the uncertainty of market environment, trapezoidal fuzzy number is often used to measure the yield of securities. Many scholars have discussed the fuzzy portfolio optimization problem deeply, such as Carlsson[4], Anne Trefethen[5], Zhang Peng and Zhang Weiguo[6], etc.

The return and risk of asset portfolios are measured by the possibilistic mean value and the possibilistic standard semi-variance of asset fuzzy return, additionally. Clearly, the whole process is self-financing, since the investor did not add additional funds during this period.

Record the return rate of risk assets as R_{it}, where $R_{it} = (a_{it}, b_{it}, \alpha_{it}, \beta_{it})$.

In period t, the possibilistic mean value r_{pt} of the assets $u_t = (u_{1t}, u_{2t}, \dots u_{nt}, u_{ft})$ can be calculated as:

$$r_{pt} = \sum_{i=1}^{n} \bar{M}(R_{it})u_{it} + r_{ft}(X_{t-1} - \sum_{i=1}^{n} u_{it}) - X_{t-1}$$
$$= (r_{fi} - 1)X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft}\right) u_{it}, t = 1,2, \dots, T \quad (2)$$

Assuming the transaction cost is $\gamma\%$ in this paper. Obviously, the transaction cost of asset i in period t is $u_{it} \times \gamma\%$, and the total cost of the asset portfolio $u_t = (u_{1t}, u_{2t}, \dots, u_{nt}, u_{ft})$ is:

$$C_t = \sum_{i=1}^{n} u_{it} \times \gamma, \quad t = 1,2,\dots,T \tag{3}$$

Then, the net return ratio of asset portfolio X_t by the end of period t is:

$$r_{Nt} = (r_{ft} - 1)X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} - \sum_{i=1}^{n} \gamma_{it} |u_{it}| \tag{4}$$

and the total wealth held by investor by the end of period t is:

$$X_t = r_{Nt} + X_{t-1} = r_{ft}X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} - \sum_{i=1}^{n} \gamma_{it} |u_{it}| \tag{5}$$

where $t = 1,2,\dots,T$.

4.7 Determination of Constraint Conditions

The threshold constraint of multi-stage portfolio is:

$$l_{it} \le u_{it} \le p_{it} \tag{6}$$

where l_{it} and p_{it} as the upper and lower bound constraint of u_{it}.

Set the lower bound constrain of proportion of investment ratio on risk-free assets to be $u_{ft}^{b} (u_{ft}^{b} \le 0)$, then the borrowing constraints of risk-free assets in period t is:

$$u_{ft} = X_{t-1} - \sum_{i=1}^{n} u_{it} \ge u_{ft}^{b} \tag{7}$$

According to formula 1, the semi-variance of asset portfolio u_{it} is:

$$\overline{svar}_t(u_t) = u_t' H_t^- u_t \tag{8}$$

where its standard semi-variance is $\sqrt{\overline{svar}_t(u_t)}$. Assume that H_t^- is a semi-positive definite matrix.

$$H_t^- = \left(Cov_t^-(r_{it}, r_{jt}) \right)_{n \times n} \tag{9}$$

$$\begin{aligned} Cov_t^-(r_{it}, r_{jt}) = & \frac{(b_{it} - a_{it})(\beta_{jt} + \alpha_{jt}) + (b_{jt} - a_{jt})(\beta_{it} + \alpha_{it})}{12} \\ & + \left[\frac{(\beta_{it} + \alpha_{it})(\beta_{jt} + \alpha_{jt})}{36} \right] + \frac{(\beta_{it} - \alpha_{it})(\beta_{jt} - \alpha_{jt})}{4} + \frac{\beta_{it}\beta_{jt}}{18} \end{aligned} \tag{10}$$

The cardinality constraint is:

$$\sum_{i=1}^{n} z_{it} \le K, z_{it} \in \{0,1\} \tag{11}$$

where K is the maximum number of risk-free assets, and

$$z_{it} = \begin{cases} 1 & choose\ asset\ i \\ 0 & not\ choose\ asset\ i \end{cases} \tag{12}$$

4.8 Game Model Construction

The traditional multi-stage mean-standard semi-variance portfolio optimization model only considers the expected value of end-of-period wealth and standard semi-variance. In the real world, however, investors care not only about the expected value and standard semi-variance of end-of-period wealth, but also about which during the investment period. In other words, the expectation and the semi-variance of the portfolio are different in the t period. Therefore, this paper uses standard semi-variance to measure risk. Set the weight coefficient $w_t > 0$ and risk preference coefficient $\eta_t > 0, t = 1,2,..,T$.

Time-consistent strategy means when $t_1 < t_2$, the optimal policies based on these two stages are the same. Few scholars have studied the time consistency of the multi-stage mean-standard semi-variance portfolio model with transaction costs, borrowing constraints, threshold constraints, income demand and cardinality constraints. In this paper, we restate this problem as a game problem. In this part, we will study the optimal strategy of generalized multi-stage mean-quasi-semi-variance fuzzy portfolio model with time consistency under multiple realistic constraints.

According to Bjork and Murguci's research, a definition based on non-cooperative game was given:

Definition 1. Consider a fixed control law $u^{TC}(k-1)$. Set $u(k-1) = (u_{k-1}, u_k^{TC}, \dots, u_{T-1}^{TC})$. u_{k-1} can be any control variable. For any $k(k = 1,2,\dots,T)$, if $u^{TC}(k-1)$ is the same time-consistent strategy, then it comes the following result:

$$Max_{u_{k-1}} u_{k-1}\big(X_{k-1}, u(k-1)\big) = u_{k-1}(X_{k-1}, u^{TC}(k-1))$$

Such time-consistent law $u^{TC}(k-1)$ is called the **perfect Nash equilibrium strategy** of sub-game. Under a non-cooperative game frame, assume that a gamer starts at position$(k-1, X_{k-1})$. According to Definition 1, since he follows strategy $(u_k^{TC}, \dots, u_{T-1}^{TC})$, he can only choose strategy u_{k-1} to maximize $u_{k-1}\big(X_{k-1}, u(k-1)\big)$.

The following formula 13 shows a common equivalent transformation in time- consistent strategy:

$$\begin{aligned}
& w_T\left[\overline{M}_T(X_T) - \eta_T\sqrt{\overline{svar}_{T-1}(X_T)}\right] \\
&= w_T\left[\overline{M}_T\left(r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n} C_{iT}|u_{iT} - u_{i(T-1)}|\right)\right] \\
&\quad -w_T[-\eta_T\sqrt{\overline{svar}_{T-1}\left(r_{fT}X_{T-1} + \sum_{i=1}^{n}(r_{iT} - r_{fT})u_{iT} - \sum_{i=1}^{n} C_{iT}|u_{iT} - u_{i(T-1)}|\right)}] \\
&= w_T\left[\sum_{i=1}^{n}\left(\frac{a_{iT}+b_{iT}}{2} + \frac{\beta_{iT}-\alpha_{iT}}{6} - r_{fT}\right)u_{iT} + \eta_T\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ijT}^{-}u_{iT}u_{jT}}\right] \\
&\quad -w_T\left[r_{fT}X_{T-1} + \sum_{i=1}^{n} C_{iT}|u_{iT} - u_{(i(T-1))}|\right]
\end{aligned} \tag{13}$$

In actual investment, we can easily conclude the following conditions:

- Expected return should be maximized;
- Investment to risk-free assets is a must;

- The number of assets is no more than K;
- The end-of-period wealth are supposed to be larger than X_{0T}

Thus, base on formula 6, formula 8, formula 10, formula 11, formula 12, formula 13 and the conclusions above, we can get the following strategy model:

$$Max \sum_{t=1}^{T} w_t \left[\begin{array}{l} r_{ft}\, X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} \\ - \sum_{i=1}^{n} c_{iT} \left| u_{it} - u_{i(t-1)} \right| - \eta_t \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ijT}^{-} u_{it} u_{jt}} \end{array} \right] \tag{14}$$

$$s.t. \begin{cases} X_t = r_{ft} X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} - \sum_{i=1}^{n} c_{it} \left| u_{it} - u_{i(t-1)} \right| \\ X_T \ \geq X_{0T} \\ X_{t-1} - \sum_{i=1}^{n} u_{it} \ \geq u_{ft}^{b} \\ \sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0,1\} \\ l_{it} z_{it} \leq u_{it} \leq p_{it} z_{it}, i = 1, \cdots, n, t = 1, \cdots, T \end{cases} \tag{15}$$

This is the mixed-integer semi-closed-loop dynamic optimization model with path-dependence. It takes both returns and risks into consideration to help investors gain more while avoiding risks. The following parts will solve it and put it into practical application.

4.9 Model Resolution

In this part, we will use discrete approximate iterative algorithm[6] to solve this problem. First, discretize the state variables based on network graph methods. Through this way, we can turn the model into a multi-stage weighted directed graph. Next, solve the maximum path by the maximum algebraic method. Here we can get the feasible solution. Finally, based on the feasible solution, keep iterating until the last 2 solutions are very close.

The details are as follows:

Step 1. Disperse the interval $[X_t^{min}, X_t^{max}]$ into four equations. Therefore, discrete state variable x_{it} at stage t can be obtained. ($i = 1,2,\ldots,5, t = 1,2..,T$).

Step 2. Let $F_t(j,k)$ stand for the multi-stage weighted directed graph. Solve the edges of $F_t(j,k)$.Build the graph as Figure 5 shows.

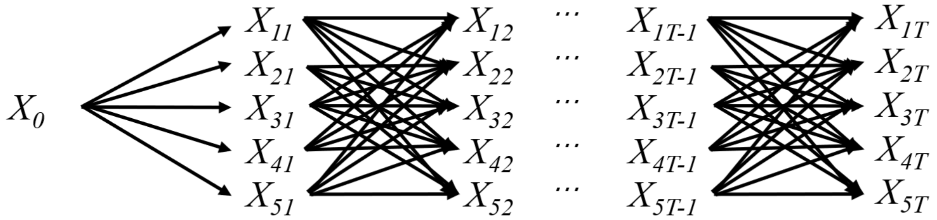


Figure 5.Multi-stage weighted directed graph

Step 3. Based on discrete approximate iterative algorithm, get the longest path $F^{(1)}$ of $F_t(j,k)$ after the first iteration:

$$F^{(1)} = F_1^{(1)} \otimes F_2^{(1)} \otimes \ \dots \ \otimes F_T^{(1)} \tag{16}$$

where $F_1^{(1)} = \left(F_1^{(1)}(1,j)\right)_{1\times 5}, F_2^{(1)} = \left(F_2^{(1)}(I,j)\right)_{5\times 5}, \dots, F_T^{(1)} = \left(F_T^{(1)}(I,j)\right)_{5\times 5}$

Step 4. Continue iteration. The $(k+1)$-th iteration can be described as follows:

Let the longest path of the k-th iteration $F^{(k)}$ be $X_0 \to X_{i_1 1}^{(k)} \to X_{i_2 2}^{(k)} \to \cdots \to X_{i_T T}^{(k)}$.

The best solution to the longest path in Figure 5 is also the feasible solution to the multi-stage mean semi-variance fuzzy portfolio model. Based on this, disperse the variables from stage 1 to stage T into four equations, as the following steps shows.

1) Disperse X_2^{min} and $X_{i_2 2}^{(k)}$, $X_{i_2 2}^{(k)}$ and X_2^{max} into two internal structure which are the same. The 5 disperse points of X_2, which are $X_2^{min}, X_{22}^{(k+1)}, X_{i_1 2}^{(k+1)}, X_{32}^{(k+1)}$ and X_2^{max} can be solved.
2) Based on $\left(X_{i_3 3}^{(k)}, \dots, X_{i_{T+1} T}^{(k)}\right)$, disperse the variables from stage 3 to stage T into 5 disperse points in the same way. The weight of stage t can also be easily solved.
3) The longest path of $(k+1)$-th iteration $F^{(k+1)}$ and another feasible solution can be calculated :

$$F^{(k+1)} = F_1^{(k+1)} \otimes F_2^{(k+1)} \otimes \dots \otimes F_T^{(k+1)} \tag{17}$$

where $F_1^{(k+1)} = \left(F_1^{(k+1)}(1,j)\right)_{1\times 5}, F_2^{(k+1)} = \left(F_2^{(k+1)}(I,j)\right)_{5\times 5} .. F_T^{(k+1)} = \left(F_T^{(k+1)}(I,j)\right)_{5\times 5}$.

If $\left|F^{(k+1)} - F^{(k)}\right| < 10^{-6}$, the longest path $F^{(k+1)}$ is the approximate solution to our model. Else, keep iterating until it does.

4.10 Application in Practical Investment

In this part, based on the data in the attachments and the model above, we figure out how much the initial investment worth on 9/10/2021.

The first thing needs doing is clearing parameters:

- According to the historical data on closing prices of gold and bitcoin from 09/11/2016 to 09/10/2021, consider a month as a period. Thus, $T = 60$.
- Initial wealth $X_0 = \$1000$.
- To ensure that investment is profitable, the ultimate wealth should be greater than the initial wealth. So $X_T > X_0 = 1000$.
- There are 2 kinds of risk assets (gold, bitcoin) and 1 kind of risk-free asset(cash), thus, $n = 2, K = 2$.
- Set $i = 1$ stand for gold and $i = 2$ stand for bitcoin. The transaction costs of gold and bitcoin are $\gamma_1 = 0.01, \gamma_2 = 0.02$.
- Since there will be no additional funds in the investment process, that is, no borrowings or arrears, the lower bound of the risk-free asset investment ratio is 0. Also, the amount of transactions in the period t is less than or equal to the total wealth in period $t-1$. Thus, $u_{ft}^b = 0$, $X_{t-1} - \sum_{i=1}^{n} u_{it} \geq 0$.

- Due to threshold constraints, the position of gold should be kept between 5 % and 20 %. Thus, $0.05X_t z_{1t} \le u_{1t} \le 0.2X_t z_{1t}, t = 1, \cdots, T$.

The model described by formula 18 can be embodied as follows:

$$Max \sum_{t=1}^{T} w_t \begin{bmatrix} r_{ft} X_{t-1} + \sum_{i=1}^{n} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} \\ - \sum_{i=1}^{n} c_{iT} \left| u_{it} - u_{i(t-1)} \right| - \eta_t \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{\bar{ijT}} u_{it} u_{jt}} \end{bmatrix} \quad (18)$$

$$s.t. \begin{cases} X_t = r_{ft} X_{t-1} + \sum_{i=1}^{2} \left(\frac{a_{it} + b_{it}}{2} + \frac{\beta_{it} - \alpha_{it}}{6} - r_{ft} \right) u_{it} - \sum_{i=1}^{2} c_{it} \left| u_{it} - u_{i(t-1)} \right| & (a) \\ X_T \ge 1000 & (b) \\ X_{t-1} - \sum_{i=1}^{2} u_{it} \ge 0 & (c) \\ \sum_{i=1}^{2} z_{it} \le 2, z_{it} \in \{0,1\} & (d) \\ 0.05X_t z_{1t} \le u_{1t} \le 0.2X_t z_{1t}, t = 1, \cdots, T. & (e) \\ \sum_{t=1}^{60} z_{1t} \ge 60 \times 60\% & (f) \end{cases} \quad (19)$$

Formula 19 is the wealth we hope to achieve, formula 19(a) corresponds to the iterative process, formula 19(b) corresponds to profit demand, formula 19(c) corresponds to the self-financing process, formula 19(d) corresponds to the product category and formula 19(e) corresponds to the threshold constraint of gold. In order to ensure that gold assets are in a low position for a long time, its low-position periods are not less than 60% of the total period, as formula 19(f) says.

Solve it according to the method in Section 4.7 and get the trapezoidal distribution of the probability of return on assets in each period.

Figure 6. Daily price of gold and bitcoin

To better understand the transaction in each period, the price charts of gold and bitcoin are shown above as Figure 6 shows. Visualize the results of the model as the following Figure 7, 8 and 9 show.

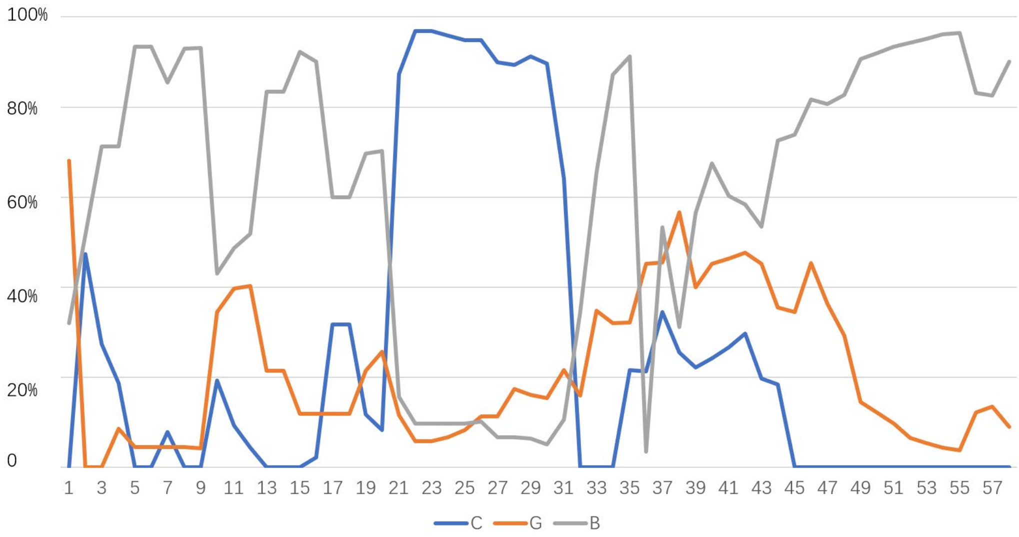


Figure 7. Proportion of assets in each period

Figure 7 shows the proportion of assets in total wealth at the end of each period. Blue, orange and gray lines represent cash, gold and bitcoin, respectively. We find that in periods 1 to 5 and 30 to 35, bitcoin is bought rapidly and massively. This is because the model judges that the price of bitcoin will continue to increase during this period and in the next several period, and the risk of decline in the short term is small, so a large number of purchases are made. In period 20 to 22, although we suffer some losses, the rapid sell-off of large amounts of gold and bitcoin helps us avoid the continuing decline. In period 34 and 35, most bitcoins are promptly and decisively sold, and gold, whose return is more stable, was invested. This further ensures the income when avoiding risks. In period 36 to 60, the slow purchase of bitcoin is due to its relatively low valuation caused by the rapid decline before. At this time, increasing the purchase is conducive to grasping the subsequent upward trend. Also, it is worth attention that although it is less likely to continue to fall after falling, that is, the risk is less, the model still suggests us to buy slowly to share the risk.

The reasons and details of other transactions are similar to the above analysis, so they are not described too much here. Table 3 shows the total wealth and the ratio among the three assets in some periods of our investment. Full data can be found in Appendix A in this paper.

Table 3. Part of the investment

Period	[C,G,B]			X_t
	Cash	Gold	Bitcoin	
1	0	680	320	959
2	478	0	522	1044
3	277	0	723	1059
4	190	86	724	1089
		…		
31	533	179	286	11025
32	350	126	522	12368
		…		
58	58	0	99	901
59	59	0	79	921
60	60	0	64	936

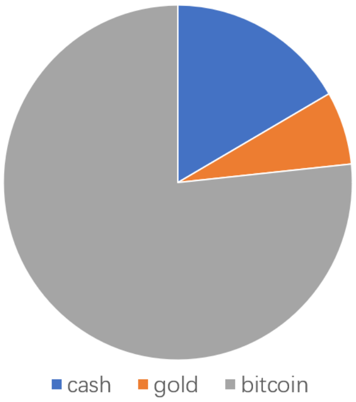


Figure 8. "Champion times" ratio

In order to more intuitively feel the key investment assets in different periods, count the assets with the highest proportion in each period, and call this asset the "champion" of this period. Count the "champion" times of the three assets. Bitcoin wins "champions" for 46 times, gold wins 10 times and cash wins 4 times. Their "champion times" ratio is shown in Figure 8.

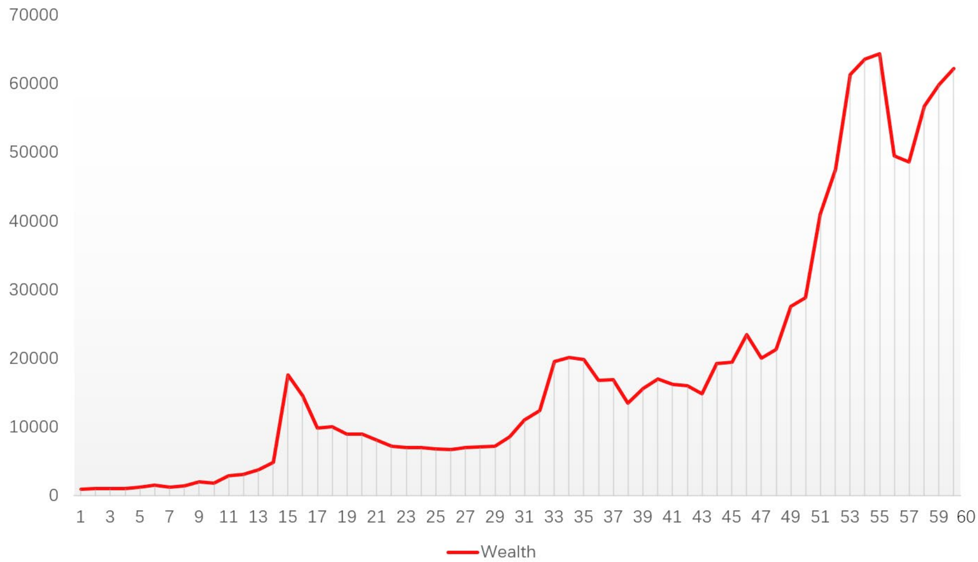


Figure 9. Total wealth

Figure 9 shows us our wealth intuitively. It can be found that the strategy based on the model helps us invest better, which means getting higher returns while avoiding risks. For example, in period 15 to 20 and 53 to 57, the market environment is quite unfavorable. Although there are losses, they are kept to a level that is far less than the market decline. And in period 46 to 55, we also keep up with the pace of market profits, obtained quite reliable income.

At the same time, it should be noted that compared with the losses in period 15 to 20 and 53 to 57, although the decline of the market in the latter is much larger than that in the former, the decline caused by investment based on the model is not much different. This shows that with the increase of historical data, the model's resistance to sudden market deterioration continues to increase. If more historical data are available, the stronger our ability to predict and bear risks, the more radical strategies can be adopted to achieve higher returns.

In a nutshell, by 09/10/2021, the initial $1000 finally reaches assets equivalent to **$62211**. The rate of return has reached an impressive 100%

5. Model Analysis

5.1 Model Demonstration

To prove the superiority of our model, we introduce specific measures of traders' regret psychology to prove that our model is the best model. When other conditions are the same, the lower the degree of traders' regret is, the more satisfying the decision is and the more successful the model is.

In actual financial activities, considering that investors are "bounded rational", investors' psychological factors will affect their investment behavior. In investment decision-making, any portfolio in the market is an alternative. Investors will compare the expected returns and risks of different portfolios, when investors have an unlimited number of alternative portfolios.

This part compares the investor's portfolio with the portfolio that is likely to receive the maximum return, to get the investor's "**regret value**" when the return is not optimal. Similarly, compare the investor's portfolio with the portfolio that is likely to have the least risk, to get the "regret value" when the portfolio does not meet the minimum risk. Because of the psychology of regret aversion, it is difficult for investors to maximize wealth and minimize risk from the perspective of absolute rationality when making investment decisions. Thus, add the psychology of regret aversion into investment decisions, hoping that the investment results will not be too regretful to be accepted.

We introduce regret psychology into the demonstration of the rationality of the model. It shows that the decision made by our model makes the degree of regret of investors lower than other choices. In other words, it makes investors more satisfied, which means our model is more successful. It works for different types of investors.

Due to the irrational behavior of investors in the process of investment, such as chasing up and down, overconfidence and regret aversion, many traditional portfolio models cannot explain the anomalies in financial markets well. Therefore, a large number of scholars began to develop and improve portfolio theory from the perspective of investor behavior. Chorus C.G proposed a **Generalized Random Regret Minimization** model[7] in 2014. Based on Chorus's model, this part describes the investor's regret psychology from the perspective of return and risk.

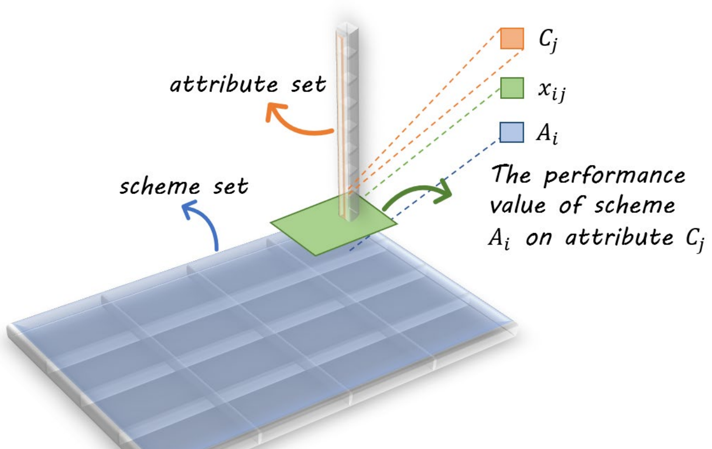


Figure 10. Performance value

Assume that there are n feasible schemes. $A = \{A_1, A_2, \dots, A_n\}$ is the set of schemes. $C = \{C_1, C_2, \dots, C_m\}$ is the set of attributes. x_{ij} is the performance value of scheme A_i on attribute C_j. Performance values of different schemes can be compared under the same attribute.

In the financial market, investors use the initial wealth to invest. If the final wealth is less than the maximum wealth available, investors may regret. This kind of regret value can be calculated as follows:

$$R_1 = l\,n\{\gamma + e^{\beta_1(r_{max} - r)}\} \tag{20}$$

where r_{max} is max available yield rate, r is the actual yield, γ is the investor's degree of regret and β_1 is the investor's sensitivity to yield rate.

In the process of investment, investors hope to reduce the risk by diversifying investment.

If the portfolio risk is greater than the minimum risk that investors can bear, investors will regret to do so. This kind of regret value can be calculated as follows:

$$R_2 = ln\{\gamma + e^{\beta_2(\delta - \delta_{min})}\} \tag{21}$$

where δ_{min} is the lowest acceptable risk, δ is the actual risk, γ is the investor's degree of regret and β_2 is the investor's sensitivity to risks.

Usually, the higher the expected return of an investor, the greater the risk he is willing to take. Thus, investor's sensitivity to yield rate and risks can be considered as nearly negative correlation. Let

$$\beta_1 + \beta_2 = 1 \tag{22}$$

If $\beta_1 > 0.5$, then the investor is more sensitive to yield. If $\beta_2 > 0.5$, then the investor is more sensitive to risks. Otherwise, the investor is equally sensitive to yield and risks. The total regret value is:

$$R = R_1 + R_2 \tag{23}$$

In this paper, cash is considered as risk-free asset, gold is considered as mid-risk asset and bitcoin is considered as high-risk asset, according to their fluctuation. Investors are divided into conservative investors, normal investors and radical investors, according their investment style. As for conservative investors, $\beta_1 > 0.5$. As for normal investors, $\beta_1 = 0.5$. As for racial investors, $\beta_1 < 0.5$.

According to the model, we can get an investment proposal. Next, we will explore the regret value R of investors with different styles when investing based on this proposal. The score is used to measure the success of the model, and the results are compared with the results of *equal proportional investment*[1].

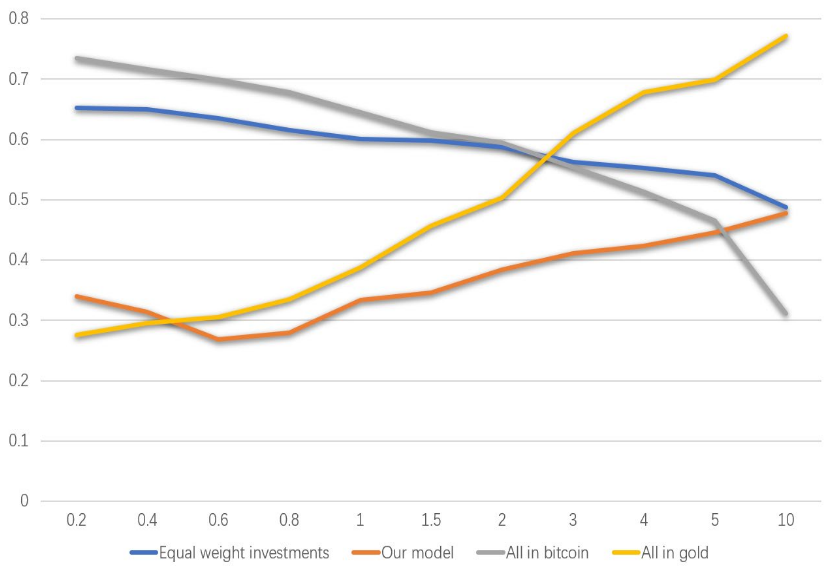


Figure 11. Regret value of different kinds of investments

Figure 11 shows the investors' regret value under different investment strategies as the sensitivity to income changes. In which, the horizontal axis is β_1/β_2 and the vertical axis is the regret value. As β_2 can actually approach to 0, this figure is actually a partial representation of the real situation. This is because most investors have a relatively balanced consideration of risks and benefits, and some investors can only accept minimal risks, but few investors are willing to ignore risks. Therefore, it can be considered that β_2 is usually not less than 0.1. So this figure shows most of the cases. The blue line represents the strategy of

[1] equal proportional investment means purchasing at $[C, G, B] = [333,333,333]$ on 09/11/2016.

investing $1,000 in cash, gold and bitcoin on average at the initial moment, the orange line represents the strategy made according to our model, the gray line represents the strategy of investing only bitcoin, and the yellow line represents the strategy of investing only gold. It can be found that in most of the cases, our model brings less regret to investors. In other words, our model is the best in most of the cases.

5.2 Sensitive Analysis

To verify the sensitivity of the model to transaction costs, we let the transaction cost changes in a certain range. The sensitivity of the model is captured by the change in the transaction amount at each stage and the final return.

When transaction costs fluctuate, the investment decisions given by the model will also be affected. Trades will reduce or even stop in some stages if transaction costs rise, even if the investment phase remains unchanged. And it will rise in some stages with the falling of transaction costs.

Therefore, we can change transaction costs and then observe the changes in investment suggestions given by the model. In this way, the sensitivity of the model to transaction costs is explored. We set three pairs of transaction costs of gold and bitcoin: $\gamma_1 = 0.01$ and $\gamma_2 = 0.02$, $\gamma_1 = 0.001$ and $\gamma_2 = 0.002, \gamma_1 = 0.0001$ and $\gamma_2 = 0.0002$. Here are the results of our model.

When $\gamma_1 = 0.01$ and $\gamma_2 = 0.02$, the result is clearly displayed in 4.8.

When $\gamma_1 = 0.001$ and $\gamma_2 = 0.002$, Table 4 shows the proportion of assets [C,G,B] and the total wealth X_t in different period.

Table 4. Total wealth and proportion of assets

Period	[C,G,B]			X_t
	Cash	Gold	Bitcoin	
1	0	680	320	998
2	317	0	683	1067
3	160	91	749	1069
4	0	43	957	1157
5	0	43	957	1157
		…		
60	0	46	954	63157

When $\gamma_1 = 0.0001$ and $\gamma_2 = 0.0002$, Table 5 shows the proportion of assets [C,G,B] and the total wealth X_t.

Table 5. Total wealth and proportion of assets

Period	[C,G,B]			X_t
	Cash	Gold	Bitcoin	
1	0	680	320	998
2	317	0	683	1067
3	160	91	749	1069
4	0	43	957	1157
5	0	43	957	1157
		…		
60	0	46	954	63157

We can clearly see that with the increase of transaction cost, the transaction amount and times in most of the stages are gradually decreasing. From this, we can draw the conclusion that our model is very sensitive to transaction cost.

6. Strengths and Weakness

Strengths:

- **Comprehensiveness**. What our model takes into account are not only returns, but also risks investors to take when investing. It takes as little risk as possible while making the investor's return as large as possible.
- **Innovativeness**. Introduce a specific measure of investor regret psychology to demonstrate the superiority of the model, and use it to measure the success of the model.
- **Applicability**. Theoretically, in the absence of major international events, the model can continuously update itself over long periods of time and is highly applicable.

Weakness:

- Because the risks taken investors are considered, the decisions proposed are on the conservative side, which may cause investors to miss out on certain gains.
- The decision is mainly influenced by the price data without taking other factors into account such as international situation, policy implications, etc. The factors are relatively homogeneous and may lead to results that differ from the actual optimal ones.

To: Mrs./Mr. Trader
From: Team # 2222182
Date: 2/21/2022

Investment in Gold and Bitcoin

Background

This is a memorandum of the model about small-amount personal investment.

- Assets: cash , gold , bitcoin
- Goal: low tisk, high return, low regret value

Descriptions

Strategy

Gold:
- Low position, i.e. 5%~20%
- Long-term holding

Bitcoin:
- Automatic investment plan
- Modify the investment amount according to the market

Reasons

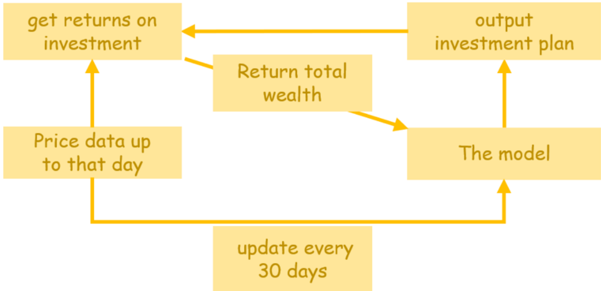
- The price of gold is influenced by many factors and hard to predict
- The price of bitcoin fluctuates violently. Its high return is born with high risks. Automatic investment plan shares the risk equally under the condition of obtaining no less than the average market income.

Descriptions

how to construct the model?

In a nutshell, predict first, plan next, make decisions last. Make sure it balances returns and risks.

how to use this model?



Strengths and weakness of the model

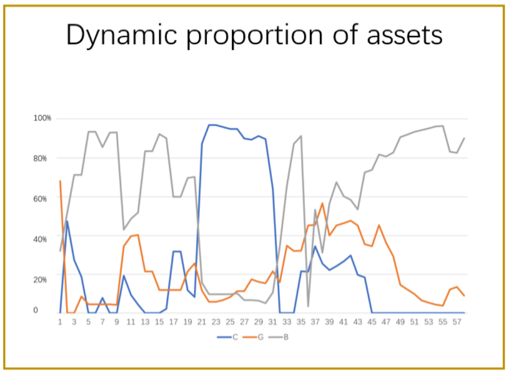
Strengths

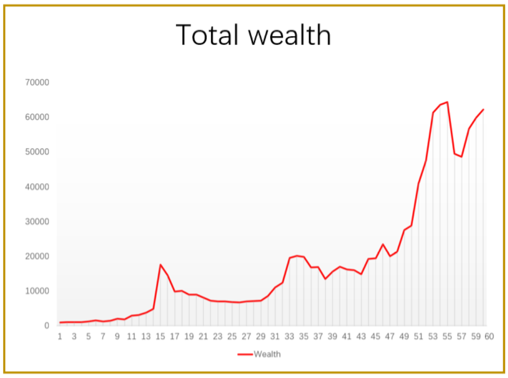
- High return with low risks
- Low regret value of investors
- Easy to handle

Weakness

- May give up some return to lower the risks

The result of the model





Questions you may have

Q1: How to prove the superiority of the model?

A: We introduce the specific measurement of the degree of investors' unsatisfaction. By comparing with other choices, we draw the conclusion that decisions made by our model make investors regret less, gain more and take less risks. That is how we prove its superiority.

Q2: It is sensitive to transaction costs?

A: Yes. The lower transaction costs are, the more we trade, and the more we earn.

Q3: Is the length of the period variable?

A: Yes. But we suggest you do not change it after you make your decision.

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Appendix A

Period	[C, G, B]			X_t	Period	[C, G, B]			X_t
	Cash	Gold	Bitcoin			Cash	Gold	Bitcoin	
1	0	680	320	959	31	533	179	286	11025
2	478	0	522	1044	32	350	126	522	12368
3	277	0	723	1059	33	0	128	872	19531
4	190	86	724	1089	34	0	259	741	20097
5	0	45	955	1243	35	0	259	741	19864
6	0	45	955	1512	36	179	376	445	16778
7	79	46	875	1263	37	177	374	449	16878
8	0	46	954	1439	38	345	466	209	13524
9	0	44	956	2074	39	282	326	252	15994
10	219	392	489	1856	40	163	335	502	17005
11	95	407	498	2892	41	184	355	461	16221
12	44	419	537	3110	42	202	359	439	15997
13	0	204	796	3765	43	231	352	417	14851
14	0	204	796	4869	44	154	277	568	19264
15	0	108	892	17597	45	156	274	570	19452
16	20	113	867	14556	46	0	356	644	23517
17	306	114	579	9877	47	0	344	656	20001
18	306	114	579	10013	48	0	261	739	21356
19	113	208	679	8992	49	0	137	862	27567
20	79	245	676	9015	50	0	117	883	28859
21	763	101	136	8113	51	0	94	906	41005
22	862	51	86	7235	52	0	65	935	47598
23	862	51	86	6997	53	0	54	946	61357
24	852	61	86	7004	54	0	38	962	63571
25	833	75	92	6841	55	0	35	965	64401
26	841	100	59	6755	56	0	127	873	49560

27	841	100	59	6998	57	0	139	861	48650
28	791	152	56	7112	58	0	99	901	56751
29	890	157	53	7199	59	0	79	921	59934
30	882	161	57	8549	60	0	64	936	62211