Problem 1: Approximate the integration of the function

$$f(x) = 4x^3 - (x+3)^{-2} - 2x + 5$$

in the interval $[-2, 1]$ using the following three quadrature rules:
(1) Gaussian quadrature; **(10 Points)**
(2) Midpoint sum; **(15 Points)**
(3) Simpson sum. **(15 Points)**
For (1), is your result close to the true value?
For (2) and (3), compute the error and examine how the error decays with increasing n (the number of data points that you use). Before the error is as small as comparable with rounding error, you should be able to see the error decaying with the theoretical order for each rule.

Problem 2: Consider the equations of motion of a system of two masses, springs, and dampers

$$\begin{aligned} m\ddot{x}_1(t) + c\dot{x}_1(t) + kx_1(t) &= c\dot{x}_2(t) + kx_2(t) \\ m\ddot{x}_2(t) + c\dot{x}_2(t) + kx_2(t) &= c\dot{x}_1(t) + kx_1(t)\,. \end{aligned}$$

The initial conditions for this system are $x_1(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ and $x_2(0) = 1$. Additionally we have $m = 1, k = 10$, and $c = 1$.

1. Solve for $x_1(t)$ and $x_2(t)$. You can do this manually via the Laplace transform or you can use the software (wolframalpha.com, Maple, Mathematica might be helpful) **(10 Points)**.

2. Transform the system into a first-order system by introducing the variables $y_1(t) = \dot{x}_1(t)$ and $y_2(t) = \dot{x}_2(t)$ so that you obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \\ \dot{x}_2(t) \\ \dot{y}_2(t) \end{bmatrix} = \boldsymbol{B} \begin{bmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{bmatrix}\,. \tag{1}$$

   Provide the specific form for $\boldsymbol{B}$ **(10 Points)**.

3. Solve Eq. (1) using different numerical methods until end time $T = 5$. Compute the numerical solutions for $x_1(t), x_2(t)$ with the time step sizes

$$\Delta t \in \{10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$$

   and compute the global errors of your numerical simulation for both x_1 and x_2.
   a) Use the explicit Euler method **(5 Points)**
   b) Use the explicit Heun's method **(5 Points)**
   c) Use the 4th-order Runge-Kutta method **(10 Points)**
   For each method, plot the global error vs. Δt in log-log scale, and check if the slope for each method matches the theoretical order of convergence.

4. Vary the spring constant to $k = 10^3$. The problem now becomes stiff. Solve Eq. (1) again using different numerical methods until end time $T = 1$.

a) Use the implicit Euler method. Plot the global error vs. Δt in log-log scale, and check if the slope matches the theoretical order of convergence. **(10 Points)**

b) Use the 4th-order Runge-Kutta method. Since RK method is an explicit method, it is only conditionally stable, meaning that you cannot obtain a stable solution if Δt is too large. Find the maximum Δt that you still can obtain a stable numerical solution. **(10 Points)**