Problem 1: (**50 points**) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A}$ the Hilbert matrix with elements

$$a_{ij} = \frac{1}{(i+j-1)}, \qquad i,j = 1, ..., n.$$

$\mathbf{b}$ is chosen in such a way that the exact solution is $\mathbf{x} = (1, 1, ..., 1)^T$.

For $n = 5, 9, 20, 100$:

(1) Find the condition number of $\mathbf{A}$. (**10 points**)
(2) Solve it with the direct method (using the MATLAB builtin function \, or Python builtin function numpy.linalg.solve). (**10 points**)
(3) Solve it with two iterative methods (1: Gauss-Seidel or PG; 2: PCG). For PCG, you can choose the preconditioner as the diagonal matrix made of the diagonal entries of the Hilbert matrix. (**20 points**)
(4) Compare the error for all three methods (direct + two iterative) and number of iterations for two iterative methods. (**10 points**)

Show your results like in the following table:

		\	PG		PCG	
n	$K(\mathrm{A}_n)$	Error	Error	Iter	Error	Iter
4	1.55e+04	7.72e-13	8.72e-03	995	1.12e-02	3
6	1.50e+07	7.61e-10	3.60e-03	1813	3.88e-03	4
8	1.53e+10	6.38e-07	6.30e-03	1089	7.53e-03	4
10	1.60e+13	5.24e-04	7.98e-03	875	2.21e-03	5
12	1.70e+16	6.27e-01	5.09e-03	1355	3.26e-03	5
14	6.06e+17	4.12e+01	3.91e-03	1379	4.32e-03	5

Problem 2: (**50 points**) Numerically solve the following 1D heat equation:

$$u_t(x,t) - u_{xx}(x,t) = -\sin(x)\sin(t) + \sin(x)\cos(t)\ , \quad x \in (0, \pi/2), t > 0\ .$$

subject to Dirichlet BCs:

$$u(0,t) = 0 \quad \text{and} \quad u(\pi/2, t) = \cos(t) \quad \text{for} \quad t > 0,$$

and the initial condition:

$$u(x,0) = \sin(x) \quad \text{for} \quad x \in [0, \pi/2].$$

Use one explicit and one implicit schemes for solving the resulting IVP until $T = \pi$.

(1) (**30 points**) Use some plots to demonstrate that your numerical solutions are accurate. The exact solution of this 1D heat equation is $u(x,t) = \sin(x)\cos(t)$.

(2) (**20 points**) Choose two different Δ_x and two different Δt, discuss how they affect the accuracy of your numerical solutions. To assist your discussion, you can evaluate the error as $\frac{\|\mathbf{u}(T)-\mathbf{u}^n\|^2}{\|\mathbf{u}(T)\|^2}$, and then compare the errors obtained with different Δ_x and Δt, where $\mathbf{u}(T)$ denotes the exact solution at $T = \pi$; $\mathbf{u}^n$ denotes your numerical solution at the last time step; $\|\|^2$ is square of the vector magnitude.