NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 9

For the tutorial on 20 October, let us discuss

- Ex. 4.7.10, 18, 20, 29, 32, 34.

Ex. 4.7.10. A list of n items is arranged in random order; to find a requested item, they are searched sequentially until the desired item is found. What is the expected number of items that must be searched through, assuming that each item is equally likely to be the one requested? (Questions of this nature arise in the design of computer algorithms.)

[Solution:]

Let X be the position of the desired object in the list of n objects. Since the order is random, the object is equally likely to be at any of the positions from 1 to n, and hence $P(X = k) = \frac{1}{n}$ for $k = 1, 2, \ldots, n$. Hence the expected number of items that must be searched through is

$$E(X) = \sum_{k=1}^{n} k \cdot \frac{1}{n} \;=\; \frac{1}{n}\sum_{k=1}^{n} k \;=\; \frac{1}{n}\frac{n(n+1)}{2} \;=\; \frac{n+1}{2}.$$

Ex. 4.7.18. If $U_1, \ldots, U_n$ are independent uniform random variables, find $E(U_{(n)} - U_{(1)})$.

[Solution:] Let $U = U_{(n)}$ and $V = U_{(1)}$ and let $R = U - V$. Then from lecture (Handout 4 slide 31/31), we learnt that the density of R is given by

$$f_R(r) = n(n-1)r^{n-2}(1-r), \qquad 0 \le r \le 1.$$

Hence

$$E(U_{(n)} - U_{(1)}) \;=\; \int_0^1 n(n-1)r^{n-1}(1-r)dr \;=\; \left[n(n-1)\left(\frac{1}{n}r^n - \frac{1}{n+1}r^{n+1}\right)\right]_0^1 \;=\; \frac{n-1}{n+1}.$$

(Alternatively, evaluate $E(U_{(n)})$ and $E(U_{(1)})$ separately and use the fact that $E(U_{(n)} - U_{(1)}) = E(U_{(n)}) - E(U_{(1)})$.)

Ex. 4.7.20 A stick of unit length is broken into two pieces. Find the expected ratio of the length of the longer piece to the length of the shorter piece.

[Solution:]

Let x be the length of the shorter piece, $(0 \le x \le \frac{1}{2})$. Since nothing was mentioned about how the stick is broken, we are going to assume that the stick is equally likely to be broken at any point between 0 to $\frac{1}{2}$. Hence the density of X is

$$f_X(x) = 2 \qquad \left(0 \le x \le \frac{1}{2}\right).$$

The length of the longer piece is $1-x$ and so the ratio is given by $r = g(x) = \frac{1-x}{x} = \frac{1}{x} - 1$, where $r \geq \frac{1}{2}$. Hence, by theorem from lecture,

$$E(R) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \;=\; \int_0^{1/2} \left(\frac{1}{x} - 1\right) 2 \; dx = 2 \left[\ln x - x\right]_0^{1/2} = \text{undefined}.$$

On the other hand, the expectation of the ratio of the shorter length to the longer length would have been defined and given by

$$\int_0^{1/2} \frac{x}{1-x} 2 \; dx \;=\; 2 \int_0^{1/2} \left(\frac{1}{1-x} - 1\right) dx \;=\; 2 \left[-\ln(1-x) - x\right]_0^{1/2} \;=\; 2\ln 2 - 1 \;\approx\; 0.386.$$

Ex. 4.7.29. Prove Corollary A of Section 4.1.1.

> If X and Y are independent random variables and g and h are fixed functions, then $E[g(X)h(Y)] = \{E[g(X)]E[h(Y)]\}$, provided that the expectations on the right-hand side exist.

[Solution:] We only prove the continuous case. The discrete case is similar.

Let $f_{X,Y}$ denote the joint density function of X and Y. Since X and Y are independent, $f_{X,Y} = f_X \cdot f_Y$ where f_X and f_Y are the marginal densities of X and Y, respectively. Then

$$\begin{aligned}
E[g(X)h(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) f_{X,Y}(x,y) \; dydx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) f_X(x) f_Y(y) \; dydx \\
&= \int_{-\infty}^{\infty} g(x) f_X(x) \int_{-\infty}^{\infty} h(y) f_Y(y) \; dydx \qquad (1) \\
&= \int_{-\infty}^{\infty} g(x) f_X(x) \; dx \int_{-\infty}^{\infty} h(y) f_Y(y) \; dy \qquad (2) \\
&= \{E[g(X)]E[h(Y)]\}.
\end{aligned}$$

We arrive at equation (1) after treating $g(x)f_X(x)$ as a constant with respect to y and we arrive at equation (2) by treating $\int_{-\infty}^{\infty} h(y) f_Y(y) \; dy$ as a constant with respect to x.

Ex. 4.7.32. Let X have a gamma distribution with parameters α and λ. For those values of α and λ for which it is defined, find $E(1/X)$.

[Solution:]

$$\begin{aligned}
E(1/X) &= \int_0^{\infty} \frac{1}{t} \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \; dt \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-2} e^{-\lambda t} \; dt.
\end{aligned}$$

Note that for a Gamma distribution with parameters $\alpha - 1$ and λ,

$$\frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^{\infty} t^{\alpha-2} e^{-\lambda t} \; dt = 1.$$

Hence we express

$$\begin{aligned} E(1/X) &= \frac{\lambda\Gamma(\alpha-1)}{\Gamma(\alpha)} \left\{ \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^\infty t^{\alpha-2} e^{-\lambda t}\, dt \right\} \\ &= \frac{\lambda\Gamma(\alpha-1)}{\Gamma(\alpha)} \\ &= \frac{\lambda}{\alpha-1}, \end{aligned}$$

since $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. The expression for $E(1/X)$ is valid provided $\alpha \neq 1$.

Ex. 4.7.34. Let X be uniform on $[0,1]$, and let $Y = \sqrt{X}$. Find $E(Y)$
(a) by finding the density of Y and then finding the expectation, and
(b) by using Theorem A of Section 4.1.1. (i.e., $E(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$.)

[Solution:]
a. For $0 \leq y \leq 1$,

$$P(Y \leq y) \;=\; P(\sqrt{X} \leq y) \;=\; P(X \leq y^2) \;=\; y^2.$$

Hence $f_Y(y) = 2y$. Therefore,

$$E(Y) \;=\; \int_0^1 2y^2\, dy \;=\; \frac{2}{3}.$$

b.

$$E(Y) \;=\; \int_0^1 \sqrt{x} dx \;=\; \left[\frac{2}{3} x^{3/2} \right]_0^1 \;=\; \frac{2}{3}.$$