# Mathematical Statistics and Data Analysis
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# Answers

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## 1 Probability

**Ordering key, Real book v E-book**. $2 \leftrightarrow 6$, $4 \leftrightarrow 8$, $10 \rightarrow 24$, $12 \leftrightarrow 18$, $14 \leftrightarrow 62$, $16 \rightarrow 10$, $18 \leftrightarrow 26$, $20 \rightarrow 26$, $22 \leftrightarrow 32$, $24 \rightarrow 20$, $36 \leftrightarrow 38$, $40 \leftrightarrow 44$, $52 \leftrightarrow 56$, $72 \leftrightarrow 76$

1. (a) $\Omega = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$.
   (b)
   $$\begin{aligned} A &= \text{At least two heads} = \{hhh, hht, hth, thh\} \\ B &= \text{Heads in the first two flips} = \{hhh, hht\} \\ C &= \text{Tails in the last flip} = \{hht, htt, tht, ttt\} \end{aligned}$$
   (c)
   $$\begin{aligned} A^c &= \text{At most one heads} = \{htt, tht, tth, ttt\} \\ A \cap B &= \{hhh, hht\} \\ A \cup C &= \{hhh, hth, thh, hht, htt, tht, ttt\} \end{aligned}$$
2. (a) Argument. Venn-diagram.
   (b) Repeated use of the Addition Law.
3. $\Omega = \left\{ \begin{array}{l} rrr, rrg, rgr, grr, rgg, grg, ggr, rrw, rwr, wrr, ggw, gwg, wgg, rgw, rwg, grw, \\ gwr, wrg, wgr \end{array} \right\}$
4. Make use of the Addition Law and mathematical induction.
5. $C = (A \cap B^c) \cap (A \cup B) = (A \cap B)^c \cap (A \cup B)$.

6. (a) $\Omega = \{11, 12, 13, \ldots, 65, 66\}$

   i. $A = \left\{ \begin{array}{l} 14, 15, 16, 23, 24, 25, 26, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, \\ 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66 \end{array} \right\}$

   ii. $B = \{21, 31, 32, 41, 42, 43, 51, 52, 53, 54, 61, 62, 63, 64, 65\}$

   iii. $C = \{41, 42, 43, 44, 45, 46\}$

   i. $A \cap B = \{32, 41, 42, 43, 51, 52, 53, 54, 61, 62, 63, 64, 65\}$

   ii. $B \cup C = \{21, 31, 32, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 61, 62, 63, 64, 65\}$

   iii. $A \cap (B \cup C) = \{32, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 61, 62, 63, 64, 65\}$

7. Make use of the Addition Law and the fact that $\Pr(A \cup B) \leq 1$.

8. Venn-diagram

9. Make use of the Addition Law and the fact that $\Pr(A \cap B) \leq \min(\Pr(A), \Pr(B))$.

10. $\binom{n}{j} \left(\frac{1}{k}\right)^j \left(1 - \frac{1}{k}\right)^{n-j}, \quad j = 0, 1, \ldots, n.$

11. $P_4^7 / 10^4$.

12. $P_{26}^{256} = 256!/(256 - 26)!$.

13. (a) $10 \cdot (4^5 - 4) / \binom{52}{5}$.

    (b) $\binom{13}{1}\binom{4}{4}\binom{48}{1} / \binom{52}{5}$.

    (c) $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}\binom{44}{0} / \binom{52}{5}$.

14. Argue.

15. $4 \cdot 6 \cdot 3 = 72$

16. *Simpson's paradox (or Yule-Simpson effect).*

17.

$$p_A(k) = \frac{\binom{k}{0}\binom{100-k}{4}}{\binom{100}{4}} = \frac{\binom{100-k}{4}}{\binom{100}{4}}, \quad k = 0, 1, \ldots, 100$$

18. $6^4 = 1\,296$.

19. (a) $\binom{5}{1}\binom{2}{1}\binom{3}{1}\binom{2}{1} / \binom{12}{4}$

    (b) $10/33$.

20. $13!/(3! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \cdot 1! \cdot 1!)$.

21. $1/4$.

22. $1/24$.

23. $P_2^n = n(n-1)$.

24. $1/5525$.

25. $3! = 6$.

26. Approximation. $m = 1/(\lg n - \lg(n-k))$.

27. $P_5^{26}/26^5$.

28. $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}\binom{32}{5}$.

29. $\binom{10}{2}/\binom{47}{2}$.

30. 0.052, 0.301, 0.06.

31. $6!^2$.

32. $1 - \binom{40}{n}/\binom{52}{n}$. $n = 3$.

33. $7!/[(7-5)! \cdot 7^5]$.

34. See Tutorial 1.

35. Combinatorial identities.

36. 20 and 1 680.

37. $\binom{7}{2\ 2\ 3} = \frac{7!}{2!2!3!} = 210$.

38. $\binom{7}{3}$.

39. (a) $21!/26!$.
    (b) $n \geq -1/\lg\left(1 - \frac{21!}{26!}\right)$.

40. $\binom{12}{4}\binom{8}{4}\binom{4}{4}/3!$.

41. (a) $\left[\binom{7}{2} + \binom{8}{2} + \binom{9}{2}\right]/\binom{24}{2}$.
    (b) $\binom{7}{2}/\binom{24}{2}$.

42. $\binom{11}{4}\binom{7}{3}\binom{4}{3}\binom{1}{1}$.

43. $\binom{10}{4}\binom{6}{3}\binom{3}{3}$.

44. $8!$ (depending on interpretation).

45. Make use of the Multiplication Law and mathematical induction.

46. (a) $31/70$, (b) $21/31$.

47. (a) 11/45, (b) 6/11.

48. (a) 2/5, (b) 1/2.

49. (a) 4/7, (b) 3/7.

50. 1/5.

51. 2/5.

52. (a) 1/2.
    (b) 1/3.

53. 0.348.

54. (a) $\alpha p + (1-\beta)(1-p)$.
    (b) $\alpha(\alpha p + (1-\beta)(1-p)) + (1-\beta)(1-(\alpha p + (1-\beta)(1-p)))$.
    (c) $(1-\beta)/[2-(\alpha+\beta)]$.

55. (a) 0.484, 0.696, (b) 0.064, 0.614, 0.322.

56. $\left[\binom{52}{5} - \left(\binom{48}{5} + \binom{4}{1}\binom{48}{4}\right)\right] / \left[\binom{52}{5} - \binom{48}{5}\right] \approx 0.122.$

57. 2/3.

58. Use Bayes Theorem. It makes no difference.

59. (a) 2/3.
    (b) 5/6.
    (c) 4/5.

60. Check the Kolmogorov axioms.

61. 0.863.

62. $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$.

63. 1/3.

64. 5/8.

65. Make use of the complement rule, the Addition Law, and De Morgan's rule.

66. Use the definition.

67. Apply the definition of indepedence to the Addition Law.

68. False. Find a counterexample.

69. Yes, but only if one of the events i $\emptyset$.

70. Yes, but only if $\Pr(A) = 0$.

71. Use the definition of mutually independent events.

72.

$$\begin{aligned} p_2(0) &= p^2 q(1-q) + p(1-p)(1-q)^2 \\ p_2(1) &= pq(1-q) + (pq + (1-p)(1-q))^2 + p(1-p)q(1-q) \\ p_2(2) &= 2 \cdot (pq + (1-p)(1-q))(1-p)q \\ p_2(3) &= (1-p)^2 q^2 \end{aligned}$$

73. $\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}, \quad k = 0, 1, \ldots, n.$

74. $p^3 (2-p)^2$.

75. $p = 0.596\,97$.

76. $((1-p)(1+p))^n$.

77. $n = 14$.

78. (a) $\Pr(AA) = \Pr(Aa) = 1/2$.
    (b) Second generation. $\Pr(AA) = (p+q)^2$, $\Pr(aa) = (r+q)^2$, and $\Pr(Aa) = 1 - \left((p+q)^2 + (r+q)^2\right)$.

79. Similar to the previous one.
    (a) $\Pr(aa) = \Pr(AA) = 1/4$, $\Pr(Aa) = 1/2$.
    (b) $2/3$.
    (c) $\Pr(aa) = p/6$, $\Pr(AA) = (2-p)/3$, $\Pr(Aa) = (2+p)/6$.
    (d) $(4-p)/(6-p)$.

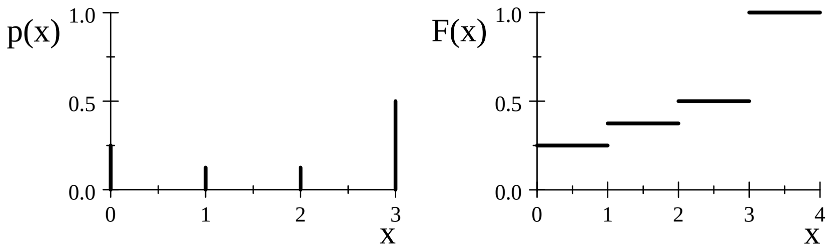
80. $\Pr(A \cap B \cap C) \neq \Pr(A)\Pr(B)\Pr(C)$.

# 2 Random Variables

**Ordering key, Real book v E-book**. $2 \leftrightarrow 6$, $8 \leftrightarrow 12$, $10 \leftrightarrow 14$, $20 \rightarrow 24$, $22 \rightarrow 20$, $24 \rightarrow 22$, $30 \leftrightarrow 32$, $46 \leftrightarrow 48$, $54 \leftrightarrow 56$, $58 \leftrightarrow 60$, $62 \leftrightarrow 64$, $68 \leftrightarrow 70$.

1.

| $x$ | $p(x)$ | $F(x)$ |
|---|---|---|
| 0 | 0.25 | 0.25 |
| 1 | 0.125 | 0.375 |
| 2 | 0.125 | 0.5 |
| 3 | 0.5 | 1.0 |




2. Argue.

3.

| $k$ | $F(k)$ | $p(k)$ |
|---|---|---|
| 0 | 0.0 | 0.0 |
| 1 | 0.1 | 0.1 |
| 2 | 0.3 | 0.2 |
| 3 | 0.7 | 0.4 |
| 4 | 0.8 | 0.1 |
| 5 | 1.0 | 0.2 |

4. Use the definitions.

5. Use the definition.

6. (a) $p(0) = 1/2$, $p(1) = 1/4$, $p(2) = 1/8$, $p(3) = 1/16$, $p(4) = 1/16$.
   (b) $p(0) = 5/16$, $p(1) = 3/8$, $p(2) = 1/4$, $p(3) = 1/16$.
   (c) $p(-4) = 1/16$, $p(-2) = 1/4$, $p(0) = 3/8$, $p(2) = 1/4$, $p(4) = 1/16$.
   (d) $p(0) = 1/8$, $p(3) = 1/2$, $p(4) = 3/8$.

7. $F(x) = 1 - p, 0 \leq x \leq 1$.

8. 0.0098 vs 0.00018.

9. $\frac{1}{2} \leq p \leq 1$.

10. (a)

$$p(k) = \begin{cases} (1-p_1)^n (1-p_2)^n p_1, & k = 2n+1 \\ (1-p_1)^{n+1} (1-p_2)^n p_2, & k = 2n+2 \end{cases}$$

(b) $p_1 / (p_1 + p_2 - p_1 p_2)$.

11. $\lfloor (n+1) p \rfloor$. *Remark.* If $(n+1) p \in \mathbb{Z}^+$, the binomial is bimodal with modes $(n+1) p$ and $(n+1) p - 1$.

12. Use the Binomial Theorem.

13. (a) $\Pr(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k} \approx 0.013$

    (b) $\Pr(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{2}\right)^{20} \approx 0.252$

14. 0.986 compared to 0.815.

15. Five games. $\Pr(X \geq 3) = \sum_{k=3}^{5} \binom{5}{k} \cdot 0.4^k \cdot 0.6^{5-k} \approx 0.3174$.
    Seven games. $\Pr(X \geq 4) = \sum_{k=4}^{7} \binom{7}{k} \cdot 0.4^k \cdot 0.6^{7-k} \approx 0.2898$.

16. Messy combinatorics.

17. $p(k) = (1-p)^k p, \quad k = 0, 1, 2, \ldots$

18. $p(k) = \binom{k+r-1}{r-1} (1-p)^k p^r, \quad k = 0, 1, 2, \ldots$

19. $F(k) = 1 - (1-p)^k, \quad k = 1, 2, \ldots$.

20. $\binom{2n-k}{n} \frac{1}{2^{2n-k}}, \quad k = 0, 1, \ldots, n.$

21. Se Tutorial 2.

22. $k = 2/\lg 2 = 6.64$, i.e. $k = 7$ since $k \in \mathbb{Z}^+$.

23. $\Pr(X = r + k) = \binom{r+k-1}{r} (1-p)^k p^r$.

24. $(3/4)^3$.

25. (a) $1 - 1.3 \times 10^{-8104000} \approx 0.99865$.

    (b) $\binom{104000}{2} (1.3 \times 10^{-8})^2 (1 - 1.3 \times 10^{-8})^{103998} \approx 9.1271 \times 10^{-7}$.

26. 0.77, 0.20, 0.026.

27. $X \sim Bi(100\,000, 0.001)$ but also approximately $Po(100)$.

28. Follow the instructions.

29. Follow the instructions.

30. $k = \lfloor \lambda \rfloor$.

31. (a) 0.283, (b) 20.79 minutes.

32. (a) $p(k) = 4^k e^{-4}/k!, \quad k = 0, 1, 2, \ldots$. $k = 4$.
    (b) 0.00274.

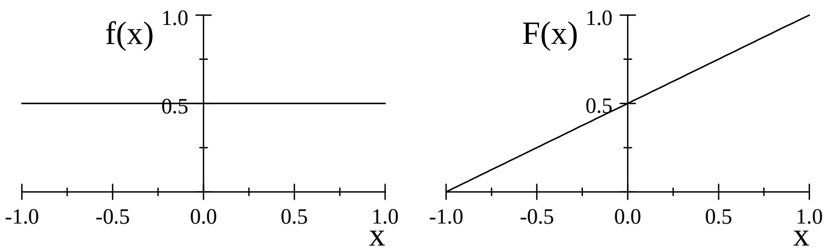
33. $f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, \quad x \geq 0$.

34.

$$F(x) = \frac{x+1}{2} + \frac{\alpha(x^2-1)}{4}, \quad -1 \leq x \leq 1$$

$$\eta = \begin{cases} -\frac{1}{\alpha} - \sqrt{\frac{1}{\alpha^2}+1}, & -1 \leq \alpha < 0 \\ -\frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2}+1}, & 0 < \alpha \leq 1 \end{cases}$$

35.




36. $p(k) = 1/n$, $k = 0, 1, 2, \ldots, n-1$. $X$ is said to be *discrete* uniform and we write $X \sim U[0, n-1]$.

37. 2/3.

38. Use basic properties of pdfs.

39. (a) Use the fact that $\tan^{-1}(x)$ is a strictly increasing function.
    (b) $f(x) = [\pi(1+x^2)]^{-1}, \quad -\infty < x < \infty$.
    (c) $\tan(0.4\pi) \approx 3.08$

40. (a) $c = 3$, (b) $F(x) = x^3$, $0 \leq x \leq 1$, (c) 0.124.

41. $q_1 = (\ln 4 - \ln 3)/\lambda$ and $q_3 = \ln 4/\lambda$.

42. $f_R(r) = 2\lambda\pi r e^{-\lambda\pi r^2}$, $r > 0$.

43. $f_R(r) = 4\lambda\pi r^2 e^{-4\lambda\pi r^3/3}$, $r > 0$.

44. $p_X(k) = e^{-\lambda k}\left(1 - e^{-\lambda}\right), \quad k = 0, 1, 2, \ldots$.

45. (a) $1 - e^{-1} \approx 0.632$

    (b) $e^{-1/2} - e^{-3/2} \approx 0.383$

    (c) $x_{0.99} 10 \ln 100 \approx 46.05$.

46. Use the definition.

47. $x = (\alpha - 1) / \lambda$.

48. $\lambda = -\ln 0.95 \approx 0.05$.

49. (a) Use the definition.

    (b) Use the definition. Follow the instructions.

    (c) Follow the instructions.

    (d) Follow the instructions.

50. Use the definition of the gamma function. Consider both $s = t^2$ and $s = e^t$ (i.e. $t = \ln s$).

51. Follow the instructions.

52. (a) 0.25, (b) It is a linear transformation so it is Normal. Find its parameters.

53. (a) 0.3085, (b) 0.8351, (c) 21.45.

54.

$$f_Y(y) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{y^2}{2\sigma^2}}, \quad y > 0$$

55. $c = 1.96\sigma$.

56. Standardize.

57. See lecture notes.

58.

$$f_Y(y) = \frac{1}{\sigma y \sqrt{2\pi}} \exp\left\{-\frac{1}{2} \cdot \frac{(\ln y - \mu)^2}{\sigma^2}\right\}, \quad y > 0$$

59. $f_X(x) = \frac{1}{2\sqrt{x}}, \quad 0 < x < 1$.

60. $f_X(x) = 2x, \quad 0 < x < 1$.

61. $Y = cX \sim Ga(\alpha, \lambda/c)$.

62. $E \sim Ga\left(\frac{1}{2}, \frac{1}{m\sigma^2}\right)$.

63. $f(x) = \left[\pi\left(1 + x^2\right)\right]^{-1}, \quad -\infty < x < \infty$.

64. Use the Transformation Method.

65. $X = \left(-1 + 2\sqrt{1/4 - \alpha\,(1/2 - \alpha/4 - U)}\right)/\alpha.$

66. $X = (1-U)^{-1/\alpha}.$

67. (a) $f(x) = \beta\,(x/\alpha)^{\beta-1}\,e^{-(x/\alpha)^{\beta}}, \quad x > 0.$

(b) $X \sim Exp\,(1).$

(c) $X = \alpha\,(-\ln(1-U))^{1/\beta}.$

68.

$$f_V(v) = \frac{v^{-1/\alpha-1}}{\alpha\,(b-a)}, \quad b^{-\alpha} < v < a^{-\alpha}$$

69.

$$f_X(x) = (\lambda/3)\,(3/4\pi)^{-1/3}\,x^{-2/3}\exp\left\{-\lambda\left(\frac{3x}{4\pi}\right)^{1/3}\right\}, \quad x > 0$$

70.

$$f_X(x) = \frac{\lambda}{2\sqrt{x\pi}}\exp\left\{-\lambda\sqrt{\frac{x}{\pi}}\right\}, \quad x > 0$$

71. Follow the instructions.

72. *Not solved.*

# 3 Joint Distributions

**Ordering key, Real book v E-book**. 10 $\leftrightarrow$ 14, 16 $\leftrightarrow$ 20, 24 $\rightarrow$ 32, 26 $\rightarrow$ 24, 32 $\rightarrow$ 26, 36 $\leftrightarrow$ 38, 44 $\rightarrow$ 50, 46 $\rightarrow$ 48, 48 $\rightarrow$ 44, 50 $\rightarrow$ 46, 68 $\leftrightarrow$ 74, 70 $\leftrightarrow$ 72.

1. The joint distribution of $X$ and $Y$ is

| | $x$ | | | | |
|---|---|---|---|---|---|
| $y$ | 1 | 2 | 3 | 4 | $p(y)$ |
| 1 | 0.10 | 0.05 | 0.02 | 0.02 | 0.19 |
| 2 | 0.05 | 0.20 | 0.05 | 0.02 | 0.32 |
| 3 | 0.02 | 0.05 | 0.20 | 0.04 | 0.31 |
| 4 | 0.02 | 0.02 | 0.04 | 0.10 | 0.18 |
| $p(x)$ | 0.19 | 0.32 | 0.31 | 0.18 | |

   (a) Look at the margins of the cross table.

   (b) $p(x \mid Y=1)$

| | $x$ | | | |
|---|---|---|---|---|
| $y$ | 1 | 2 | 3 | 4 |
| 1 | $\frac{10}{19}$ | $\frac{5}{19}$ | $\frac{2}{19}$ | $\frac{2}{19}$ |

   $p(y \mid X=1)$

| | $y$ | | | |
|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 |
| 1 | $\frac{10}{19}$ | $\frac{5}{19}$ | $\frac{2}{19}$ | $\frac{2}{19}$ |

2. A generalization of the hypergeometric distribution.

   (a)
$$p_{X,Y,Z}(x,y,z) = \frac{\binom{p}{x}\binom{q}{y}\binom{r}{z}}{\binom{N}{n}}, \quad 0 \le x,y,z \le n, \quad x+y+z=n$$

   (b)
$$p_{X,Y}(x,y) = \frac{\binom{p}{x}\binom{q}{y}\binom{N-p-q}{n-x-y}}{\binom{N}{n}}, \quad 0 \le x,y \le n, \quad x+y \le n$$

   (c)
$$p_X(x) = \frac{\binom{p}{x}\binom{N-p}{n-x}}{\binom{N}{n}}, \quad 0 \le x \le n$$

   i..e., $X \sim Hyp(n,p,N)$.

3.
$$p_{X,Y,Z}(x,y,z) = \frac{n!}{x!y!z!}\left(\frac{1}{3}\right)^n, \quad 0 \le x,y,z \le n, \quad x+y+z=n$$

4.
$$\left(\frac{w-2r}{w+d}\right)^2 \text{ and } \left(1-\left(\frac{w-2r}{w+d}\right)^2\right)^n \text{, respectively}$$

5. *Buffon's Needle Problem. Hint.* Let $R$ represent the distance between the needle's midpoint and line, and let $\Theta$ represent the needle's direction (or angle). Argue that $R$ and $\Theta$ are independent random variables where $R \sim U(0, D/2)$ and $\Theta \sim U(0, \pi/2)$. This means that the joint pdf is given by

$$f_{R,\Theta}(r,\theta) = \frac{4}{\pi D}, \quad 0 \le r \le \frac{D}{2}, \quad 0 \le \theta \le \frac{\pi}{2}$$

In order for the needle to cross a line we must have that

$$r + L/2 \sin\theta \ge D/2 \Longleftrightarrow r \ge D/2 - L/2 \sin\theta$$

Solve the resulting double integral and you have the answer. Drop the needle a large number of times and let $r_A$ represent the relative number of times the needle crosses a line. Then

$$\pi \approx \frac{2L}{r_A D}$$

6.
$$f_X(x) = \frac{2\sqrt{1-\frac{x^2}{a^2}}}{a\pi}, \quad -a \le x \le a$$
$$f_Y(y) = \frac{2\sqrt{1-\frac{x^2}{b^2}}}{b\pi}, \quad -b \le x \le b$$

7. $f(x,y) = \alpha\beta e^{-(\alpha x+\beta y)}, \quad x, y > 0, \quad \alpha, \beta > 0$. $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta)$.

8. (a) $(i)$ $\Pr(X > Y) = 1/2$, $(ii)$ $\Pr(X + Y \le 1) = 3/14$, $(iii)$ $\Pr\left(X \le \frac{1}{2}\right) = 2/7$.
   (b) $f_X$ and $f_Y$ are identical. $f(x) = (6x^2 + 6x + 2)/7$, $0 \le x \le 1$.
   (c) $f_{Y|X}$ and $f_{X|Y}$ are identical. $f_{Y|X}(y \mid x) = 3(x+y)^2 / [3x^2 + 3x + 1]$, $0 \le y \le 1$.

9. $f(x,y) = 3/4$, $-1 \le x \le 1$, $0 \le y \le 1 - x^2$.
   (a) $f_X(x) = 3(1-x^2)/4$, $-1 \le x \le 1$. $f_Y(y) = 3\sqrt{1-y}/2$, $0 \le y \le 1$.
   (b) $Y \mid X = x \sim U(0, 1-x^2)$ and $X \mid Y = y \sim U\left(-\sqrt{1-y}, \sqrt{1-y}\right)$.

10. (a) $X \sim Exp(1)$ and $f_Y(y) = 1/(y+1)^2$, $y \ge 0$. $X$ and $Y$ are dependent.
    (b) $Y \mid X = x \sim Exp(x)$ and $X \mid Y = y \sim Ga(2, y+1)$.

11. $(6\ln 2 + 5)/36 \approx 0.254$.

12. (a) $c = 1/8$.
    (b) $X \sim Ga\,(4,1)$ and $f_Y\,(y) = (1+|y|)\,e^{-|y|}/4$, $-\infty < y < \infty$.
    (c) $f_{Y|X}\,(y \mid x) = 3\,(x^2 - y^2)\,/\,(4x^3)\,,\ -x \le y < x$.
    $f_{X|Y}\,(x \mid y) = (x^2 - y^2)\,e^{-x}/\left[2\,(1+|y|)\,e^{-|y|}\right],\ x \ge |y|$.

13. $p\,(0) = 1/2$ and $p\,(1) = p\,(2) = 1/4$.

14. $f_{X,Y,Z}\,(x,y,z) = 3/\,(4\pi)\,,\ x^2 + y^2 + z^2 \le 1$.

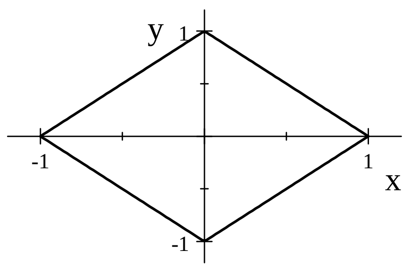
    (a) $f_X$, $f_Y$, and $f_Z$ are identical. Use polar coordinates. $f\,(x) = 3\,(1 - x^2)\,/4$, $-1 \le x \le 1$.
    (b) $f_{X,Y}$, $f_{X,Z}$, and $f_{Y,Z}$ are identical. $f\,(x,y) = 3\sqrt{1 - x^2 - y^2}/\,(2\pi)\,,\ x^2 + y^2 \le 1$.
    (c) $f_{X,Y|Z=0}\,(x,y) = 1/\pi$, $x^2 + y^2 \le 1$, i.e. $(X,Y) \mid Z = 0$ is uniformly distributed on the unit circle.

15. (a) $c = 3/\,(2\pi)$, (b) *Not solved*, (c) $1 - 1/\left(2\sqrt{2}\right) \approx 0.646$, (d) See Problem 14*a*, (e) $f_{Y|X}$ and $f_{X|Y}$ are identical.

$$f_{Y|X}\,(y \mid x) = 2\sqrt{1 - x^2 - y^2}/\left[\pi\left(1 - x^2\right)\right], y \le \sqrt{1 - x^2}.$$

16. $f\,(x_1, x_2) = 1/x_1$, $0 \le x_2 \le x_1 \le 1$, and $f_{X_2}\,(x_2) = -\ln x_2$, $0 \le x_2 \le 1$.

17. (a)

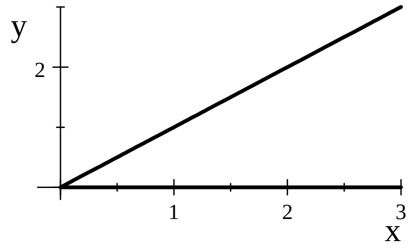



    (b) $f_X$ and $f_Y$ are identical. $f\,(x) = 1 - |x|\,,\ -1 \le x \le 1$.
    (c) $f_{Y|X}\,(y \mid x) = 1/\left[2\,(1 - |x|)\right],\ |x| - 1 \le y \le 1 - |x|$.

18. The joint pdf of $X$ and $Y$ is given by

$$f_{X,Y}(x,y) = k(x-y), \quad 0 \le y \le x \le 1$$

(a)




(b) $k = 6$.

(c) $f_X(x) = 3x^2$, $0 \le x \le 1$, and $f_Y(y) = 3(1-y)^2$, $0 \le y \le 1$.

(d) $f_{Y|X}(y \mid x) = 2(x-y)/x^2, 0 \le y \le x \le 1$, and $f_{X|Y}(x \mid y) = 2(x-y)/(1-y)^2$, $0 \le y \le x \le 1$.

19. (a) $\Pr(T_1 > T_2) = \beta/(\alpha+\beta)$, (b) $\Pr(T_1 > 2T_2) = \beta/(2\alpha+\beta)$.

20. $f_Z(z) = (T - |z|)/T^2$, $-T \le z \le T$.

21. Define indicator variables $I_x$ for which we have that $\Pr(I_x = 1) = R(x)$. Argue that $f_Y(y) = f_{X|I_y=1}(y)$. Use Bayes' Rule.

22. $N(t_0,t_1) \mid N(t_0,t_2) = n \sim Bi(n, (t_1-t_0)(/t_2-t_0))$.

23. $X \sim Bi(m, pr)$.

24. $\theta = x/n$.

25. Let $W$ be a random variable for which $\Pr(W=-1) = \Pr(W=1) = 1/2$. Now let $Y = XW$. Show that $f_{-X}(v) = f_X(-v)$. Use the Law of Total Probability.

26. $P \mid X = k \sim Beta(k+1, 2-k)$. Since there are just two cases, $P \mid X = 0 \sim Beta(1,2)$ and $P \mid X = 1 \sim Beta(2,1)$.

27. Use definitions.

28. See Section 3.3 pp78-79 for the definition of a copula. Use this.

29.

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \lambda\mu e^{-\lambda x} e^{-\mu y}\left[1 + \alpha\left(1 - 2e^{-\lambda x}\right)\left(1 - 2e^{-\mu y}\right)\right]$$

30.
$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = \begin{cases} u^{-\alpha}, & u^\alpha > v^\beta \\ v^{-b}, & u^\alpha < v^\beta \end{cases}$$

where it exists.

31. $X$ and $Y$ are dependent.

32. *Not yet complete.*

33. $\Theta \mid N = n \sim Beta\,(2, n)$.

34. $\Theta \mid X = x \sim Beta\,(x+3, n-x+3)$.

35. Use the same approach as in the example but with $Beta\,(14, 8)$ instead of $U\,(0,1)$ as prior distribution. Use the same technique as in Problem 34.

36. *Not solved.*

37. *Not solved.*

38. *Not solved.*

39. *Not solved.*

40. *Not solved.*

41. *Not solved.*

42. (a) Use the Law of Total Probability.
    (b) Use, e.g,. $c = e^2/\sqrt{2\pi}$. Look at the regions $|x| > 2$ and $-2 \le x \le 2$ separately.

43. $f_S\,(s) = s$, $0 < s < 1$ and $f_S\,(s) = 2 - s$, $1 \le s < 2$.

44. Let $S = X + Y$. Then
$$p\,(s) = \begin{cases} 1/9, & s = 2 \\ 2/9, & s = 3 \\ 3/9, & s = 4 \\ 2/9, & s = 5 \\ 1/9, & s = 6 \end{cases}$$

45. Let $N \sim Po\,(\lambda)$. Then $X \mid N = n \sim Bi\,(n, p_A)$. Use the fact that $p_X\,(k) = \sum_{n=0}^{\infty} p_{X,N}\,(k,n) = \sum_{n=0}^{\infty} p_{X|N=n}\,(k)\,p_N\,(n)$ to show that $X \sim Po\,(\lambda p_A)$.

46. Let $S = T_1 + T_2$ . $f_S\,(s) = \lambda_1 \lambda_2 \left(e^{-\lambda_2 s} - e^{-\lambda_1 s}\right) / \left(\lambda_1 - \lambda_2\right)$, $s > 0$.

47. Use convolution formula to show that $Z = X + Y \sim N\,(0,2)$, i.e. that
$$f_Z\,(z) = \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-z^2/4}, \quad -\infty < z < \infty$$

48. Use the convolution formula.

49. Let $S = X + Y$. $f_S(s) = \lambda e^{-\lambda s}\left(e^{\lambda z/2} - 1\right),\ s > 0$.

50. $f_Z(z) = \int_{-\infty}^{\infty} f_X(v+y) f_Y(y)\, dy$.

51. It follows that

$$F_Z(z) = \Pr(Z \le z) = \int_{-\infty}^{0}\int_{z/x}^{0} f(x,y)\, dydx + \int_{0}^{\infty}\int_{0}^{z/x} f(x,y)\, dydx$$

Make the change of variable $v = xy$.

52. *Remark.* We assume here that $X \sim U(0,1)$ and $Y \sim U(0,1)$. Let $Z = X/Y$. Then $f_Z(z) = 1/\left(2z^2\right),\ z \ge 1$.

53. 5/9.

54. The joint pdf of $\Theta, \Phi$, and $R$ is given by

$$f_{\Theta,\Phi,R}(\theta,\phi,r) = \frac{r^2 \sin\phi}{\sigma^3 (2\pi)^{3/2}} \exp\left\{-\frac{r^2}{2\sigma^2}\right\}, \quad r > 0, \quad 0 \le \phi \le \pi, \quad 0 \le \theta < 2\pi$$

Also, $f_\Theta(\theta) = 1/(2\pi),\ 0 \le \theta < 2\pi$, and $f_\Phi(\phi) = \sin\phi/2$, $0 \le \phi \le \pi$. Finally,

$$f_R(r) = \frac{\sqrt{2} r^2 \exp\left\{-\frac{r^2}{2\sigma^2}\right\}}{\sigma^3\sqrt{\pi}}, \quad r > 0$$

55. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.

(a) $f_{X,Y}(x,y) = 1/\left(2\pi\sqrt{x^2+y^2}\right)$, $x^2 + y^2 \le 1$, $(x,y) \ne (0,0)$.

(b) $f_X$ and $f_Y$ are identical.

$$f_X(x) = \frac{1}{\pi}\ln\frac{1+\sqrt{1-x^2}}{|x|}, \quad -1 \le x \le 1, \quad x \ne 0$$

(c) $R$ should not be $U(0,1)$. Instead consider $f_R(r) = 2r$.

56. $f_{R,\Theta}(r,\theta) = \lambda^2 r \exp\{-\lambda r(\cos\theta + \sin\theta)\},\ r > 0,\ 0 \le \theta < \frac{\pi}{2}$. $R$ and $\Theta$ are dependent.

57. Let $\mathbf{X} = \mathbf{BY}$ where

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

58. A linear transformation. Use the transformation theorem and compare the result to the pdf of a bivariate normal.

59. A linear transformation. Use the transformation theorem and compare the result to the pdf of a bivariate normal. *Remark.* It is better (less messy) to solve the problem using matrices.

60. *Not solved.*

61.
$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u-a}{b}, \frac{v-c}{d}\right) \cdot \left|\frac{1}{bd}\right|$$

62. $1 - e^{-1/2} \approx 0.3935$.

63. (a)
$$f_{U,V}(u,v) = \frac{1}{2} f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

(b)
$$f_{U,V}(u,v) = \frac{1}{2v} f_{X,Y}\left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right)$$

(c)
$$f_{U,V}(u,v) = \frac{1}{2} f_X\left(\frac{u+v}{2}\right) f_Y\left(\frac{u-v}{2}\right), \quad f_{U,V}(u,v) = \frac{1}{2r} f_X\left(\sqrt{\frac{u}{v}}\right) f_Y\left(\sqrt{uv}\right)$$

64.
$$f_{U,V}(u,v) = \frac{\lambda^2 v}{(u+1)^2} e^{-\lambda v}, \quad 0 < u, v < \infty$$

65. We are looking for the pdf of $V = X_{(1)}$. See Example A in Section 3.7.

66. $F_U(u) = (1 - e^{-2u})^3$, $u > 0$ and $f_U(u) = 6e^{-2u}(1 - e^{-2u})^2$, $u > 0$.

67. $f(x) = n(n-1)\lambda e^{-\lambda(n-1)x}\left(1 - e^{-\lambda x}\right)$, $x > 0$.

68. (a) $f_{U_{(1)},U_{(2)},U_{(3)}}(u_1, u_2, u_3) = 6$, $0 \le u_1 < u_2 < u_3 \le 1$.
(b) $1/27$.

69. $f_V(v) = n\beta\alpha^{-\beta} v^{\beta-1} e^{-n(v/\alpha)^\beta}$, $v > 0$.

70. Use the fact that $\Pr(V > v, U \le u) = (F(u) - F(v))^n$, $u < v$, in combination with the trivial fact that $\Pr(U \le u) = \Pr(V \le v, U \le u) + \Pr(V > v, U \le u)$.

71. $1 - \nu^n$, $0 < \nu < 1$.

72. $1/32$.

73. Difficult. Consider the mapping $(X_1, X_2, \ldots, X_n) \longrightarrow (X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ which is a permutation of the original variables. However, this is not a bijection, why we start by dividing $\mathbb{R}^n$ into $n!$ identical pieces.

74. Let $W_k$ represent the wating time until service of the $k$th job in the queue. Then, according to Example A in Section 3.7, $W_1 = X_{(1)} \sim Exp\,(n\lambda)$. More generally, $W_k \sim Ga\,(k, \lambda n)$.

75. See the differential method in Section 3.7.

76. A little bit messy, but pretty straightforward.

77. $U_{(k)} - U_{(k-1)} \sim \beta\,(1, n)\,,\ k = 2, 3, \ldots, n.$

78. Make the change of variable $x = yu$.

79. $R \sim Exp\,(\lambda)$.

80. (a) $n/\,(n+1)$.
    (b) $(n-1)\,/\,(n+1)$.

81. Same answer as in Problem 80.

# 4 Expected Values

**Ordering key, Real book v E-book**. $2 \leftrightarrow 4$, $12 \leftrightarrow 16$, $20 \rightarrow 30$, $26 \rightarrow 20$, $30 \rightarrow 26$, $34 \leftrightarrow 36$, $46 \leftrightarrow 48$, $62 \leftrightarrow 64$, $72 \leftrightarrow 74$, $88 \leftrightarrow 90$.

1. $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx \leq \int_{-\infty}^{\infty} Mf(x)\,dx = M\int_{-\infty}^{\infty} f(x)\,dx = M < \infty$.

2. (a) $E(X) = \alpha/(\alpha - 1)$, $\alpha > 1$.
   (b) $Var(X) = \alpha/\left[(\alpha-1)^2(\alpha-2)\right]$, $\alpha > 2$.

3. $E(X) = 3.1$, $Var(X) = 1.49$, $\alpha > 1$.

4. $E(X) = (n+1)/2$, $Var(X) = (n^2-1)/12$.

5. $E(X) = \alpha/3$, $Var(X) = (3-\alpha^2)/9$.

6. (a) $E(X) = 2/3$, (b) $Y \sim U(0,1)$ so $E(X) = 1/2$, (c) $E(X^2) = 1/2$, (d) Use both techniques.

7. (a) $E(X) = 5/8$, (b) $E(Y) = 7/8$. Use both techniques, (c) $E(X^2) = 7/8$, (d) $Var(X) = 31/64$. Use both techniques.

8. Count the number of times the term $p(k) = \Pr(X = k)$ appears. If $X \sim Ge(p)$ then $\Pr(X \geq k) = (1-p)^{k-1}$.

9. The largest $n$ such that $F_X(n-1) \leq 1 - c/s$.

10. $(n+1)/2$.

11. *Not finished.*

12. Expected values for linear transformations.

13. Use the fact that $x = \int_0^x dt$. Change the order of integration in the resulting double integral.

14. Identical to Problem 6.

15. Expected gain is $2a/n$ for both options.

16. Hints. $f(\xi - x) = f(\xi + x)$. $E(X) = \int_{-\infty}^{\xi} xf(x)\,dx + \int_{\xi}^{\infty} xf(x)\,dx$. In the second integral, make the change of variable $y = 2\xi - x$. Then $f(2\xi - y) = f(y)$ and $\int_{-\infty}^{\xi} f(y)\,dy = 1/2$.

17. $X_{(k)} \sim Beta(k, n-k+1)$. Therefore $E\left(X_{(k)}\right) = k/(n+1)$ and $Var\left(X_{(k)}\right) = [k(n-k+1)]/\left[(n+1)^2(n+2)\right]$.

18. $(n-1)/(n+1)$.

19. $1/(n+1)$.

20. $\left(1-e^{-\lambda}\right)/\lambda$.

21. $1/3$.

22. $1/4$.

23. $2/\lambda^2$ and $1/\lambda^2$, respectively.

24. Use the fact that the $E$ operator represents a double sum.

25. $2\alpha(\alpha+1)/\lambda^2$.

26. Doesn't exist.

27. 1. Also the variance is 1.

28. $n\left(1-\left(1-\frac{p}{n}\right)^m\right)$.

29. See lecture notes.

30. $n\sum_{k=n-r+1}^{n}\frac{1}{k}$.

31. $E(1/X)=\ln 2\neq 2/3=1/E(X)$.

32. $E(1/X)=\lambda/(\alpha-1)$, $\alpha>1$.

33. See lecture notes for ideas.

34. *Not solved.*

35. $r/p$.

36. $2/3$.

37. $p<(1/k)^{1/k}$

38. *Not solved.*

39. (a) 4 606.
    (b) 9 991.7, i.e. approximately 10 000.

40. $997/1296\approx 0.77$.

41. $998/216\approx 4.62$. $\Pr(X>99)\leq 0.047$.

42. See lecture notes (for $k=2$).

43. Use Theorem A of Section 4.3.

44. $Var(X) - Var(Y)$.

45. $-np_ip_j$.

46. $E(Z) = \left(\alpha + \sqrt{1-\alpha^2}\right)\mu$ and $\rho_{U,Z} = \alpha$.

47. $Cov(X,Z) = -Var(X)$ and $\rho_{X,Z} = -\sqrt{Var(X)/(Var(X)+Var(Y))}$.

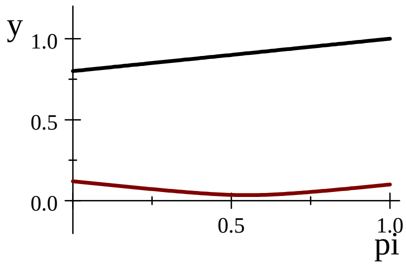
48. Use Theorem A of Section 4.3.

49. (a) Use Theorem A of Section 4.1.2 and use the fact that $X$ and $Y$ are independent.
    (b) $\alpha = \sigma_Y^2/(\sigma_X^2 + \sigma_Y^2)$.
    (c) $1/3 < \sigma_X^2/\sigma_Y^2 < 3$.

50. Hint. Due to independence, $Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i)$.

51. $\pi_i = 1/n$.

52. Denote the two securities $R_1$ and $R_2$. Let $\pi = (p, 1-p)$ be a strategy.
    (a) $R_1$ since $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$. Hence, $\pi = (1, 0)$.
    (b) For $\pi = (0.5, 0.5)$ it follows that $(\mu(\pi), \sigma(\pi)) = (0.9, 0.036)$.
    (c) For $\pi = (0.8, 0.2)$ it follows that $(\mu(\pi), \sigma(\pi)) = (0.96, 0.062)$.
    (d) For $\pi = (p, 1-p)$ it follows that
    $(\mu(\pi), \sigma(\pi)) = \left(0.2p + 0.8, \sqrt{0.043\,6p^2 - 0.048\,p + 0.014\,4}\right)$.




53. Different methods. Use, e.g., *Cauchy-Schwarz inequality.*

54. $Cov(U,V) = \sigma_Z^2$ and $\rho_{U,V} = \sigma_Z^2/\sqrt{(\sigma_X^2+\sigma_Z^2)(\sigma_Y^2+\sigma_Z^2)}$.

55. $E(T) = n(n+1)\mu/2$ and $Var(T) = n(n+1)(2n+1)\sigma^2/6$.

56. $Cov(S,T) = n(n+1)\sigma^2/2$ and $\rho_{S,T} = \sqrt{(3n+3)/(4n+2)}$.

57. $Var(XY) = \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2$.

58. (a) $E(Z) = (f(x+h) - f(x))/h$ and $Var(Z) = 2\sigma^2/h^2$. So $E(Z) \approx f'(x)$ for small values of $h$.
    (b) *Not solved.*
    (c) *Not solved.*

59. The domain is not rectangular so $X$ and $Y$ cannot be independent. By symmetry, $E(XY) = 0$ and $E(X) = E(Y) = 0$. Formal calculations are a bit messy.

60. $E(X) = E(SY) = E(S)E(Y) = 0$ and $E(XY) = E(SY^2) = E(S)E(Y^2) = 0$ so $Cov(X,Y) = 0$. In order for $X$ and $Y$ to be independent, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for *every* $(x,y)$. However, for given value of $y$ we have positive density only for $x = \pm y$.

61. (a) $Cov(X,Y) = 1/36$.
    (b) $E(X \mid Y = y) = y/2$ and $E(Y \mid X = x) = (1+x)/2$.
    (c) $X \mid Y \sim U(0,Y)$ and $Y \mid X \sim U(1,X)$.
    (d) $\widehat{Y} = E(Y \mid X = x) = (1+x)/2$ and $MSE_{\widehat{Y}} = 1/24$.
    (e) The same as in $d$.

62. Standardize, and use rules for how to calculate the covariance when dealing with linear transformations.

63. See Problem 3.8.
    (a) $Cov(X,Y) = -5/588$ and $\rho = -25/199$.
    (b) $E(Y \mid X = x) = (6x^2 + 8x + 3) / [4(3x^2 + 3x + 1)]$.

64. See Problem 3.1.
    (a) $Cov(X,Y) = 0.5096$ and $\rho = 0.515$.
    (b) $E(Y \mid X = 1) = \frac{34}{19}$, $E(Y \mid X = 2) = \frac{17}{8}$, $E(Y \mid X = 3) = \frac{88}{31}$, $E(Y \mid X = 4) = \frac{29}{9}$. The pmf of $W = E(Y \mid X)$ is given by

| $w$ | $\frac{34}{19}$ | $\frac{17}{8}$ | $\frac{88}{31}$ | $\frac{29}{9}$ |
|---|---|---|---|---|
| $p(w)$ | 0.19 | 0.32 | 0.31 | 0.18 |

65. It is necessary for $E(S_n \mid N = n) = E(S_n)$ to be true.

66. 5/3.

67. 3/2 and 1/6.

68. Use the law of total variance and use the fact that $Var(E(Y \mid X)) \geq 0$.

69. *Not solved.*

70. Use the definitions of conditional distrbutions and independence.

71. $Y \mid X = x \sim Hyp\,(x, m, n)$, and so $E\,(Y \mid X = x) = mx/n$.

72. $\mu^2$ and $\sigma^2\,(\mu + \mu^2)$.

73. $3n/4$ (or $np\,(1+p)$ in the general case).

74. $n \cdot (1-p) + p\,(n+1)\,/2$.

75. $E\,(U) = 1/\,(2\lambda)$ and $Var\,(U) = 5/\left(12\lambda^2\right)$.

76. $Y \mid X = x \sim U\left(0, \sqrt{1-x^2}\right)$ so $E\,(Y \mid X = x) = \sqrt{1-x^2}/2$.
$X \mid Y = y \sim U\left(-\sqrt{1-y^2}, \sqrt{1-y^2}\right)$ so $E\,(X \mid Y = y) = 0$.

77. (a) $Cov\,(X, Y) = 1$ and $\rho = 1/\sqrt{2}$.
    (b) $E\,(X \mid Y = y) = y/2$ and $E\,(Y \mid X = x) = 1 + x$.
    (c) $E\,(X \mid Y) \sim Ga\,(2, 2)$ and $f_{Y|X}\,(w) = e^{-(w-1)}$, $w \geq 1$.

78. Use the fact that $f\,(x) = f\,(-x)$ to show $E\,(X^3) = 0$. In Problem 16 it was shown that $E\,(X) = 0$, and so it follows that

$$E\left(\frac{X - E\,(X)}{\sqrt{Var\,(X)}}\right)^3 = (Var\,(X))^{-3/2}\,E\left(X^3\right) = 0$$

79. $M\,(t) = 1/2 + 3e^t/8 + e^{2t}/8$. $M'\,(0) = 5/8 = E\,(X)$ and $M''\,(0) = 7/8 = E\,(X^2)$.

80. $M\,(t) = 2e^t/t - 2\,(e^t - 1)\,/t^2$. $M'\,(0) = 2/3 = E\,(X)$ and $M''\,(0) = 1/2 = E\,(X^2)$.

81. $M\,(t) = pe^t + 1 - p$. $M^{(k)}\,(0) = p = E\left(X^k\right)$.

82. $M_Y\,(t) = (M_X\,(t))^n = (pe^t + 1 - p)^n$.

83. $Y = \sum_{i=1}^n X_i$. $M_Y\,(t) = (pe^t + 1 - p)^{\sum_{i=1}^n n_i}$, i.e. $Y \sim Bi\,(\sum_{i=1}^n n_i, p)$.

84. $Y = \sum_{i=1}^n X_i$. $M_Y\,(t) = \prod_{i=1}^n\,(p_i e^t + 1 - p_i)^{n_i}$.

85. $M\,(t) = pe^t/\left[1 - (1-p)\,e^t\right]$.

86. $M\,(t) = pe^t/\left[1 - (1-p)\,e^t\right]^r$.

87. $p_i = p$ for all $i$.

88. Let $Z = X/\sigma$. Then $Z \sim N(0,1)$ and $X^n = \sigma^n Z^n$ and so $E(X^n) = \sigma^n E(Z^n)$. Taylor expansion of $M_Z(t)$ yields

$$M_Z(t) = 1 + \frac{1}{2}t^2 + \frac{1}{2!2^2}t^4 + \frac{1}{3!2^3}t^6 + \frac{1}{4!2^4}t^8 + \cdots$$

If we differentiate $M_Z(t)$ an *odd* number of times, the smallest term will contain $t$ and so $E(Z^n) = M_Z^{(n)}(0) = 0$. If we differentiate $M_Z(t)$ an *even* number of times, then $E(Z^n) = M_Z^{(n)}(0) = (n-1)(n-3)\cdot \cdots \cdot 3 \cdot 1 = (n-1)!!$.

89. $M_Y(t) = \exp\left\{t\sum_{i=1}^n \alpha_i\mu_i + \frac{t^2}{2}\sum_{i=1}^n \alpha_i^2\sigma_i^2\right\}$.
$Y = \sum_{i=1}^n \alpha_i X_i \sim N\left(\sum_{i=1}^n \alpha_i\mu_i, \sum_{i=1}^n \alpha_i^2\sigma_i^2\right)$.

90. $M_Z(t) = M_X(\alpha t)\, M_Y(\beta t)$.

91. $M_{cX}(t) = 1 - t/(\lambda/c)$, i.e. $cX \sim Exp(\lambda/c)$.

92. Let $\Theta \sim Ga(\alpha,\lambda)$, $\alpha \in \mathbb{Z}^+$ and $X \mid \Theta = \theta \sim Po(\theta)$. The unconditional mgf of $X$ is given by $M_X(t) = [\lambda/(\lambda - e^t + 1)]^\alpha$. The mgf of $Y = \alpha + X$ is given by

$$M_Y(t) = \left(\frac{\lambda e^t}{\lambda - e^t + 1}\right)^\alpha = \left(\frac{\frac{\lambda}{1+\lambda}e^t}{1 - \frac{1}{1+\lambda}e^t}\right)^\alpha$$

that is, $Y \sim NegBin\left(\alpha, \frac{\lambda}{1+\lambda}\right)$.

93. Let $X_1, X_2, \ldots$ be a sequence such that $X_i \sim Exp(\lambda_i)$, and let $N \sim Ge(p)$ independent of the $X$'s. The mgf of $S = X_1 + X_2 + \cdots + X_N$ is given by $M_S(t) = \lambda p/(\lambda p - t)$, i.e. $S \sim Exp(\lambda p)$.

94. (a) First confirm that $G^{(k)}(s) = \sum_{i=k}^\infty i(i-1)\cdots(i-k+1)\, s^{i-k} p_i$. Then, using the fact that $0^0 = 1$, conclude that $G^{(k)}(0) = k! \cdot p_k$.
   (b) *Remark.* We have to be careful since it is not clear that the function is differentiable in $t = 1$. However, by letting $t \nearrow 1$ it works. Using $a$, we confirm that $G^{(k)}(1) = \sum_{i=k}^\infty i(i-1)\cdots(i-k+1)\, p_i = E[X(X-1)\cdots(X-k+1)]$.
   (c) $M(\ln t) = E\left(e^{X\ln t}\right) = E\left(\left(e^{\ln t}\right)^X\right) = E\left(t^X\right) = G(t)$.
   (d) Låt $X \sim Po(\lambda)$. $G(s) = e^{\lambda(s-1)}$.

95. $M(s,t) = E\left(e^{sX+tY}\right) = E\left(e^{sX}e^{tY}\right) = E\left(e^{sX}\right)E\left(e^{tY}\right) = M_X(s)\,M_Y(t)$

96.
$$\frac{\partial^2 M(s,t)}{\partial s \partial t}\Big|_{s=t=0} = \sum_x \sum_y xyp(x,y) = E(XY)$$

97. $M_W(t) = M_{aX+bY}(t) = M_X(at)\,M_Y(bt)$. Determine $M_W'(0)$ and $M_W''(0)$.

98. $E(S) = \lambda\mu$ and $Var(S) = \mu\lambda(\lambda+1)$.

99. (a) $g(x) = \sqrt{x}$.

$$\mu_Y \approx \sqrt{\mu_X} - \frac{\sigma_X^2}{8\mu_X^{3/2}}, \qquad \sigma_Y^2 \approx \frac{\sigma_X^2}{4\mu_X}$$

(b) $g(x) = \ln x$.

$$\mu_Y \approx \ln \mu_X - \frac{\sigma_X^2}{2\mu_X^2}, \qquad \sigma_Y^2 \approx \frac{\sigma_X^2}{\mu_X^2}$$

(c) $g(x) = \sin^{-1} x = \arcsin x$.

$$\mu_Y \approx \sin^{-1} \mu_X + \frac{\mu_X \sigma_X^2}{2(1-\mu_X^2)^{3/2}}, \qquad \sigma_Y^2 \approx \frac{\sigma_X^2}{1-\mu_X^2}$$

100. *Exact.* $\mu_Y = \ln 2/10 = 0.069315$ and $\sigma_Y^2 = \left(1 - 2\cdot(\ln 2)^2\right)/200 = 0.00019547$. *Approximate.* $\mu_Y \approx 0.06\,913\,6$ and $\sigma_Y^2 \approx 0.00016461$.

101. $\mu_Y \approx \sqrt{\lambda} - 1/\left(8\sqrt{\lambda}\right)$ and $\sigma_Y^2 \approx 1/4$.

102. $\mu_\Theta \approx \arctan(y_0/x_0)$ and $\sigma_\Theta^2 \approx \sigma^2/(x_0^2+y_0^2)$.

103. $\sigma_V \approx 0.02\pi = 0.062832$.

104. (a) $\sigma_Y^2 \approx \sigma_R^2 \sin^2 \mu_\Theta + \sigma_\Theta^2 \mu_R^2 \cos^2 \mu_\Theta$.

(b) *Not solved.*

# 5 Limit Theorems

1. Use Chebyshev's inequality.

2. See previous problem. Combine the use of Chebyshev's inequality and the triangle inequality.

3. 0.0228 (Normal approximation). The exact poisson probability is 0.0233.

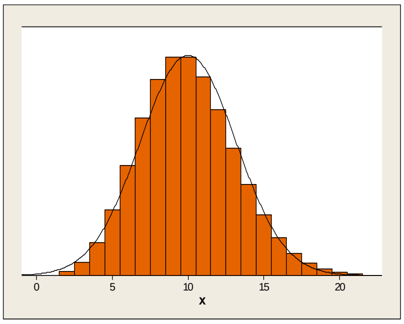
4. $\Delta = 16.45$

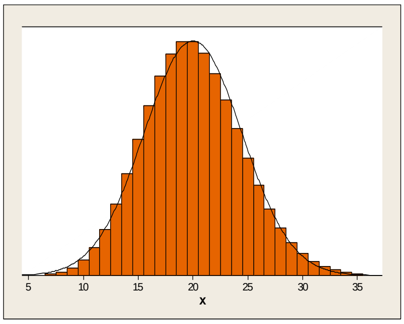
5. Rewrite the expression for $M_{X_n}(t)$ as $(1 + a_n/n)^n$ where $a_n \to \lambda(e^t - 1)$ as $n \to \infty$.

6. See lecture notes.

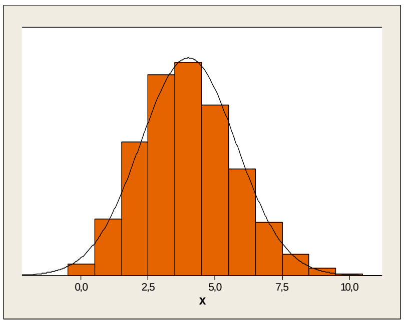
7. Use the definition of *continuity* together with the definition of *convergence in probability*.

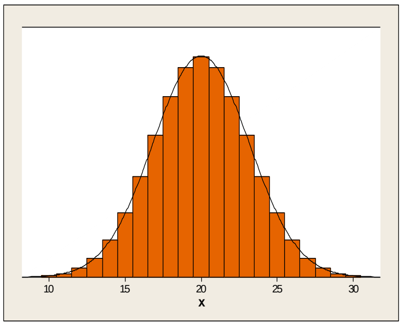
8. Use a statistical/mathematical package. Here we compare the pmfs of $Po(10)$ and $Po(20)$ with the pdfs of $N(10, 10)$ and $N(20, 20)$, respectively.






9. Use a statistical/mathematical package. Here we compare the pmfs of $Bi(20, 0.2)$ and $Bi(40, 0.5)$ with the pdfs of $N(4, 3.2)$ and $N(20, 10)$, respectively.






10. Without continuity correction: $\Pr(S < 300) \approx \Pr(Z < -2.93) = 0.0017$. With continuity correction: $\Pr(S < 300) \approx \Pr(Z < -2.96) = 0.0015$.

11. As $n$ grows larger, $\lambda$ becomes smaller which is not allowed so CLT doesn't apply in this situation. For different values of $n$, there are different collections of i.i.d. random variables that are summed.

12. (a) 0.363, (b) 0.245, (c) 0.042.

13. $N(0, 15)$. He is most likely to be close to where he started.

14. $N(10, 40/3)$.

15. Without continuity correction: $\Pr(S < -75) \approx \Pr(Z < -2.12) = 0.017$. With continuity correction: $\Pr(S < -75) \approx \Pr(Z < -2.14) = 0.016$.

16. $\Pr(S \leq 10) \approx \Pr(Z < -3.16) = 0.0008$.

17. $n = 97$.

18. 0.0228.

19. Use a statistical/mathematical package.

    (a) $\int_0^1 \cos(2\pi x)\, dx = 0$.

    (b) $\int_0^1 \cos(2\pi x^2)\, dx \approx 0.244$.

20. Using the calculation formula it follows that

$$Var\left(\widehat{I}(f)\right) = \frac{1}{n}\left[\int_0^1 (f(x))^2\, dx - \left(\int_0^1 f(x)\, dx\right)^2\right]$$

    In Problem 19$a$ this translates to $Var\left(\widehat{I}(f)\right) = 1/(2n)$.

21. A generalization of the Monte Carlo-method.

    (a) Use the definition of expectation.

    (b)

$$Var\left(\widehat{I}(f)\right) = \frac{1}{n}\left[\int_a^b \frac{(f(x))^2}{g(x)} dx - (I(f))^2\right]$$

    (c) In order to find an improvement, we have to find a pdf definied on $(0, 1)$ such that

$$Var\left(\widehat{I}(f)\right) - Var\left(\widehat{I}_{U(0,1)}(f)\right) = \frac{1}{n}\int_0^1 \frac{(f(x))^2 (1 - g(x))}{g(x)} dx < 0$$

22. $\Delta = 0.00141$

23. Let $X$ and $Y$ be i.i.d. $U(0,1)$. Furthermore, let

$$Z = \begin{cases} 0, & (X,Y) \notin A \\ 1, & (X,Y) \in A \end{cases}$$

Generate pseudrandom numbers $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. Then

$$\widehat{A} = \frac{1}{n} \sum_{i=1}^{n} Z_i = \overline{Z} \approx \int\int_A dxdy = A$$

24. $n = 10\,616$.

25. $S \overset{Approx}{\sim} N(0,30)$. $\overline{X}_n = S/50 \overset{Approx}{\sim} N(0,3/250)$. $S/\sqrt{50} \overset{Approx}{\sim} N(0,3/5)$.

26. $S \sim Bi(25, 0.3)$.

27. Prove first that $(1 + a/n)^n \to e^a$ as $n \to \infty$. Rewrite as

$$\left(1 + \frac{a}{n}\right)^n = \exp\left\{n \ln\left(1 + \frac{a}{n}\right)\right\}$$

and now use the fact that

$$\begin{aligned} \lim_{n\to\infty} n \ln\left(1 + \frac{a}{n}\right) &= a \lim_{n\to\infty} \frac{\ln\left(1 + \frac{a}{n}\right)}{a/n} \overset{h=a/n}{=} a \lim_{h\to 0} \frac{\ln(1+h)}{h} \overset{\ln 1 = 0}{=} \\ &= a \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = a \left(\frac{d}{dt} \ln t\right)_{t=1} = a \end{aligned}$$

In order to prove the more general result, we use the so called *Squeeze Theorem*. If $g(x) \leq f(x) \leq h(x)$ for all $x$, and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = a$ then $\lim_{x\to\infty} f(x) = a$.

28. Use the definitions of convergence. We have that $X_n \xrightarrow{d} \delta(0)$ where $\delta(0)$ is a degenerated, or one point, distribution with all of its probability mass in the point $x = 0$.

29. $U_{(n)} \xrightarrow{d} \delta(1)$, where $\delta(1)$ is a degenerated, or one point, distribution with all of its probability mass in the point $u = 1$. If we for example let $Z_n = n\left(U_{(n)} - 1\right)$, then

$$\lim_{n\to\infty} F_{Z_n}(z) = e^z, \quad z \leq 0$$

30. (a) $S_n \overset{Approx}{\sim} N\left(\frac{n}{2}, \frac{n}{12}\right)$. (b) $\overline{U} = S_n/n \overset{Approx}{\sim} N\left(\frac{1}{2}, \frac{1}{12n}\right)$. (c) $S_n - \frac{n}{2} \overset{Approx}{\sim} N\left(0, \frac{n}{12}\right)$. (d) $(S_n - n/2)/n \overset{Approx}{\sim} N\left(0, \frac{1}{12n}\right)$. (e) $(S_n - n/2)/\sqrt{n} \overset{Approx}{\sim} N\left(0, \frac{1}{12}\right)$.