NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

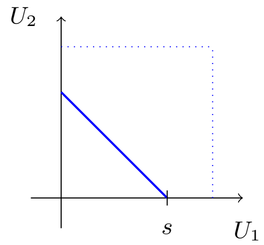
2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 8

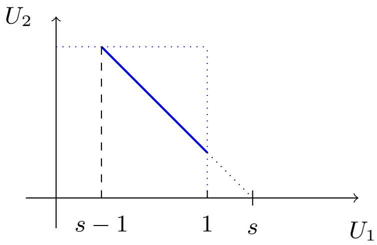
For the tutorial on 13 October, let us discuss

- Ex. 3.8.43, 48, 51, 67, 70, 74

Ex. 3.8.43. Let U_1 and U_2 be independent and uniform on [0,1]. Find and sketch the density function of $S = U_1 + U_2$.

[Solution:]





The joint density function is given by

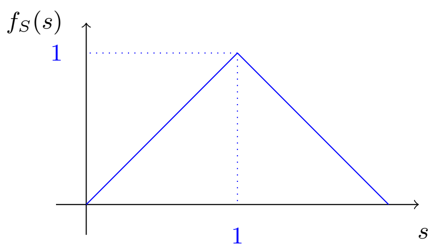
$$f_{U_1U_2}(x,y) = \begin{cases} 1, & 0 \le x, y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

From lecture, the density function is

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f(x, s-x)dx \\ &= \int_0^s dx = s, \qquad (0 \le s < 1), \end{aligned}$$

and

$$\begin{aligned} f_S(s) &= \int_{s-1}^{1} f(x, s-x)dx \\ &= 2 - s, \qquad (1 \le s \le 2), \end{aligned}$$



Ex. 3.8.48. Let T_1 and T_2 be independent exponentials with parameters λ_1 and λ_2. Find the density function of $T_1 + T_2$.

[Solution:] Let $Z = T_1 + T_2$. From lecture,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f(x, z-x)dx \\
&= \int_0^z \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(z-x)} dx \\
&= \lambda_1\lambda_2 \int_0^z e^{-\lambda_1 x - \lambda_2(z-x)} dx \\
&= \lambda_1\lambda_2 \left[\frac{1}{-\lambda_1 + \lambda_2} e^{-\lambda_1 x - \lambda_2(z-x)} \right]_0^z \\
&= \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 z} - e^{-\lambda_2 z}), \qquad z > 0.
\end{aligned}$$

Ex. 3.8.51. Let X and Y have the joint density function $f(x, y)$ and let $Z = XY$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|} dy.$$

[Solution:]

$$F_Z(z) = \int_{-\infty}^{0} \int_{z/x}^{\infty} f(x, y)\, dydx + \int_0^{\infty} \int_{-\infty}^{z/x} f(x, y)\, dydx.$$

Make a change of variable $y = v/x$, we find that

$$\begin{aligned}
F_Z(z) &= \int_{-\infty}^{0} \int_{z}^{-\infty} \frac{1}{x} f\left(x, \frac{v}{x}\right) dvdx + \int_0^{\infty} \int_{-\infty}^{z} \frac{1}{x} f\left(x, \frac{v}{x}\right) dvdx \\
&= \int_{-\infty}^{0} \int_{-\infty}^{z} \frac{1}{|x|} f\left(x, \frac{v}{x}\right) dvdx + \int_0^{\infty} \frac{1}{x} \int_{-\infty}^{z} f\left(x, \frac{v}{x}\right) dvdx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \frac{1}{|x|} f\left(x, \frac{v}{x}\right) dvdx \\
&= \int_{-\infty}^{z} \int_{-\infty}^{\infty} \frac{1}{|x|} f\left(x, \frac{v}{x}\right) dxdv.
\end{aligned}$$

Hence

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f\left(x, \frac{z}{x}\right) dx.$$

Ex. 3.8.67. A card contains n chips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential with parameter λ, find the density function of the card's lifetime.

[Solution:] First, let f denote the density function of an exponential function with parameter λ. Then

$$f(t) = \lambda e^{-\lambda t}, \qquad (t \geq 0),$$

and

$$F(t) = 1 - e^{-\lambda t}, \qquad (t \geq 0).$$

Let $X_1, X_2, \ldots X_n$ be the lifetime of the n chips. then the lifetime of the card is the lifetime of $X_{(2)}$. Let $U = X_{(2)}$ and then we require

- $X_{(1)} \leq u$,
- $U \in [u, u + du]$, and
- $X_{(3)}, \ldots, X_{(n)} \geq u$.

Then the density function is

$$\begin{aligned} f_U(u) &= \frac{n!}{1!1!(n-2)!} F(u) f(u) [1 - F(u)]^{n-2} \\ &= n(n-1)(1 - e^{-\lambda u}) \lambda e^{-\lambda u} [1 - (1 - e^{-\lambda u}]^{n-2} \\ &= n(n-1) \lambda e^{-(n-1)\lambda u} (1 - e^{-\lambda u}) \end{aligned}$$

Ex. 3.8.70. If five numbers are chosen at random in the interval $[0, 1]$, what is the probability that they all lie in the middle half of the interval?

[Solution:] Method 1: Using ordered statistics. Let the five random numbers be $X_1, X_2, \ldots, X_5$ and let $U = X_{(5)}, V = X_{(1)}$. Then all five numbers are in the middle have interval if and only if

$$\frac{1}{4} \leq X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq X_{(4)} \leq X_{(5)} \leq \frac{3}{4}.$$

First, we find the joint density function for U and V. By the differential argument, we require

- $V \in [v, v + dv]$.
- $X_{(2)}, X_{(3)}, X_{(4)}$ in $[v, u]$, and
- $U \in [u, u + du]$.

Let $f(x)$ be the uniform density function on $[0, 1]$. Thus

$$f_{U,V}(u, v) = \frac{5!}{1!3!1!} f(v)(u - v)^3 f(u) = 20(u - v)^3 \qquad (0 \leq v \leq u \leq 1).$$

The required probability is

$$
\begin{aligned}
20\int_{1/4}^{3/4}\int_{1/4}^{u}(u-v)^3 dv du &= 20\int_{1/4}^{3/4}\left[-\frac{1}{4}(u-v)^4\right]_{1/4}^{u} du \\
&= 5\int_{1/4}^{3/4}\left(u-\frac{1}{4}\right)^4 du \\
&= 5\left[\frac{1}{5}\left(u-\frac{1}{4}\right)^5\right]_{1/4}^{3/4} \\
&= \frac{1}{32}.
\end{aligned}
$$

Method 2: We want each random number to lie in the interval $[1/4, 3/4]$, and since the random numbers are chosen at random, they have the uniform distribution on $[0, 1]$. Thus the probability of any one random number lying in the interval $[1/4, 3/4]$ is $\frac{1}{2}$. Since the random numbers are independent of one another, the probability is

$$
\left(\frac{1}{2}\right)^5 = \frac{1}{32}.
$$

Ex. 3.8.74. Let U_1, U_2, and U_3 be independent uniform random variables.

a. Find the joint density of $U_{(1)}, U_{(2)}$, and $U_{(3)}$.
b. The locations of three gas stations are independently and randomly placed along a mile of highway. What is the probability that no two gas stations are less than $\frac{1}{3}$ mile apart?

[Solution:]

a. Let $X = U_{(1)}$, $Y = U_{(2)}$, and $Z = U_{(3)}$. Then in order to apply the differential argument, we require
   - $X \in [x, x + dx]$,
   - $Y \in [y, y + dy]$, and
   - $Z \in [z, z + dz]$.

   Let f be the uniform density function on $[0, 1]$. Then

$$
f(x, y, z) = \frac{3!}{1!1!1!} f(x)f(y)f(z) = 6 \qquad (0 \le x \le y \le z).
$$

b. The probability is

$$\begin{aligned}
\int_{2/3}^{1}\int_{1/3}^{z-1/3}\int_{0}^{y-1/3} 6\ dxdydz &= \int_{2/3}^{1}\int_{1/3}^{z-1/3} [6x]_0^{y-1/3}\ dydz \\
&= \int_{2/3}^{1}\int_{1/3}^{z-1/3} [6y-2]\ dydz \\
&= \int_{2/3}^{1} \left[3y^2-2y\right]_{1/3}^{z-1/3} dz \\
&= \int_{2/3}^{1} \left[3z^2-4z+\frac{4}{3}\right] dz \\
&= \left[z^3-2z^2+\frac{4}{3}z\right]_{2/3}^{1} \\
&= 1-2+\frac{4}{3}-\frac{8}{27}+\frac{8}{9}-\frac{8}{9} \\
&= \frac{1}{27}.
\end{aligned}$$