NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 3

For the tutorial on 1 September, let us discuss

- Ex. 2.5.6, 11, 13, 16, 22, 26.

Ex. 2.5.6. Let A and B be events, and let I_A and I_B be the associated indicator random variables. Show that

$$I_{A\cap B} = I_A I_B = \min\{I_A, I_B\}$$

and

$$I_{A\cup B} = \max\{I_A, I_B\}.$$

[Solution:] Let $\omega \in \Omega$.

Suppose $I_{A\cap B}(\omega) = 1$. Then $\omega \in A\cap B$ and so $\omega \in A$ and $\omega \in B$. Hence $I_A(\omega) = I_B(\omega) = 1$.

Suppose $I_{A\cap B}(\omega) = 0$. Then $\omega \notin A \cap B$, and so $\omega \notin A$ or $\omega \notin B$ (by De Morgan's law). Hence either $I_A(\omega) = 0$ or $I_B(\omega) = 0$, and in both cases, $I_A I_B(\omega) = \min\{I_A, I_B\}(\omega) = 0$.

Suppose $I_{A\cup B}(\omega) = 1$. Then $\omega \in A \cup B$ and so $\omega \in A$ or $\omega \in B$. Hence $I_A(\omega) = 1$ or $I_B(\omega) = 1$. In both cases, $\max\{I_A, I_B\}(\omega) = 1$.

Suppose $I_{A\cup B}(\omega) = 0$. Then $\omega \notin A \cup B$, and so $\omega \notin A$ and $\omega \notin B$ (by De Morgan's law). Hence $I_A(\omega) = I_B(\omega) = 0$, and so $\max\{I_A, I_B\}(\omega) = 0$.

Ex. 2.5.11. Consider the binomial distribution with n trials and probability p of success on each trial. For what value of k is $P(X = k)$ maximized? This value is called the **mode** of the distribution. (Hint: Consider the ratio of successive terms.)

[Solution:] As suggested by the hint, we work out the ratio

$$\frac{P(X = k)}{P(X = k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)}.$$

Then $\frac{(n-k+1)p}{k(1-p)} \geq 1$ if and only if $np - kp + p \geq k - pk$, which is equivalent to $k \leq np + p$.

Thus

$$P(X = k) \geq P(X = k-1) \qquad \text{when } k \leq (n+1)p$$

and

$$P(X = k) < P(X = k-1) \qquad \text{when } k > (n+1)p.$$

Thus $P(X = k)$ is maximized when $k = \lfloor (n+1)p \rfloor$. (If $(n+1)p$ is an integer, then $P(X = (n+1)p) = P(X = (n+1)p - 1)$ and are both maximized.)

Ex. 2.5.13. A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 12 items or more correct.

(a) What is the probability that the student passes?
(b) Answer the question in part (a) again, assuming that the student can eliminate two of the choices on each question.

[Solution:]

(a) Let X denote the number of questions that are answered correctly. Then X has a binomial distribution with $n = 20$ and $p = 1/3$. Then the required probability is

$$P(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k} \approx 0.0130$$

(b) In this case, X has a binomial distribution with $n = 20$ and $p = 1/2$. Thus, the required probability is

$$P(X \geq 12) = \sum_{k=12}^{20} \binom{20}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{20-k} \approx 0.252$$

Ex. 2.5.16. Show that if n approaches ∞ and r/n approaches p and m is fixed, the hypergeometric frequency function tends to the binomial frequency function with parameters m and p. Give a heuristic argument for why this is true.

[Solution:]

Recall that for a hypergeometric random variable X with parameters r, n, and m,

$$P(X = k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} = \frac{\frac{r!}{k!(r-k)!}\frac{(n-r)!}{(m-k)!(n-r-m+k)!}}{\frac{n!}{m!(n-m)!}}.$$

Since r/n approaches p, it means that as n tends to ∞, r also tends ∞ proportionately. Thus we try to regroup the products so that the r's and n's can be compared. We see that

$$\begin{aligned} P(X = k) &= \frac{m!}{k!(m-k)!} \frac{\frac{r!}{(r-k)!}\frac{(n-r)!}{(n-r-m+k)!}}{\frac{n!}{(n-m)!}} \\ &= \frac{m!}{k!(m-k)!} \frac{r(r-1)\cdots(r-k+1) \times (n-r)(n-r-1)\cdots(n-r-m+k+1)}{n(n-1)\cdots(n-m+1)} \\ &= \frac{m!}{k!(m-k)!} \frac{\frac{r}{n}(\frac{r}{n}-\frac{1}{n})\cdots(\frac{r}{n}-\frac{k-1}{n}) \times (1-\frac{r}{n})(1-\frac{r+1}{n})\cdots(1-\frac{r+m-k-1}{n})}{1(1-\frac{1}{n})\cdots(1-\frac{m-1}{n})}, \end{aligned}$$

where we divided by n^m in the numerator and in the denominator. From this expression, it is easy to see that as $n \to \infty$,

$$P(X = k) \to \binom{m}{k} p^k (1-p)^{m-k},$$

which is the binomial frequency function.

Heuristic argument: The probability $P(X = k)$ counts the probability of k successes in m draws without replacement, from a population of size n, of which r are successes. As $n \to \infty$, the ratio $r/n \to p$ means that for a very large n, the proportion of them that are "successes" is approximately p. Thus even without replacement, the probability of success for any draw remains as approximately p. Thus the distribution is approximately binomial.

Ex. 2.5.22. Three identical fair coins are thrown simultaneously until all three show the same face. What is the probability that they are thrown more than three times?

[Solution:] Let X denote the number of times the three coins are thrown until all three coins show the same face. The probability of all three coins showing the same face is $\frac{2}{8} = \frac{1}{4}$. Thus, X has a geometric distribution with $p = 1/4$.

$$P(X > 3) = \sum_{n=4}^{\infty} p(1-p)^{n-1} = \frac{1}{4}\frac{\left(\frac{3}{4}\right)^3}{1-\frac{3}{4}} = \frac{27}{64}.$$

Ex. 2.5.26. The university administration assures a mathematician that he has only 1 chance in 10,000 of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years, and always rides the elevator up to his office when h first arrives, what is the probability that he will never be trapped? That he will be trapped once? Twice? Assume that the outcomes on all the days are mutually independent (a dubious assumption in practice).

[Solution:] Let X denotes the number of times the mathematician is trapped in the 10 years. Then X has a binomial distribution with $n = 2600$ and $p = 0.0001$. Then the required probabilities are

$$\begin{aligned}
P(X=0) &= \binom{2600}{0}(0.0001)^0(0.9999)^{2600} = (0.9999)^{2600} = 0.771 \\
P(X=1) &= \binom{2600}{1}(0.0001)^1(0.9999)^{2599} = 0.200 \\
P(X=2) &= \binom{2600}{2}(0.0001)^2(0.9999)^{2598} = 0.0261.
\end{aligned}$$

Alternatively, since n is reasonably big, we could approximate this binomial distribution using a Poisson distribution with $\lambda = 2600(0.0001) = 0.26$. Then

$$\begin{aligned}
P(X=0) &= \frac{e^{-0.26}0.26^0}{0!} = 0.771 \\
P(X=1) &= \frac{e^{-0.26}0.26^1}{1!} = 0.200 \\
P(X=2) &= \frac{e^{-0.26}0.26^2}{2!} = 0.261.
\end{aligned}$$