Name: ______________________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2015/16

MH2500– Probability and Introduction to Statistics

20 October 2015 Test 3 40 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed three double-sided A4 size cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(12 marks)**

Suppose the joint cumulative distribution function of continuous random variables X and Y is

$$F_{X,Y}(x,y) = \frac{1}{10}(3x^3y + xy^2), \qquad 0 \le x \le 1 \text{ and } 0 \le y \le 2.$$

(a) Find $f_{X,Y}\left(\frac{1}{3}, \frac{1}{2}\right)$, where $f_{X,Y}$ is the joint density of X and Y.

(b) Find $F_X(x)$ and $F_Y(y)$, the marginal cumulative distribution functions for X and Y, respectively.

(c) Find $f_{Y|X}(y|x)$, the conditional density of Y given X.

[Answer:]

(a)

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F_{X,Y}(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{10}(3x^3 + 2xy)\right) = \frac{1}{10}(9x^2 + 2y).$$

Therefore

$$f_{X,Y}\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{10}\left(9 \cdot \frac{1}{9} + 2 \cdot \frac{1}{2}\right) = \frac{1}{5}.$$

This is valid for $0 \le x \le 1$ and $0 \le y \le 2$. Elsewhere, $f(x,y) = 0$.

(b)

$$F_X(x) = F_{X,Y}(x,2) = \frac{1}{10}(6x^3 + 4x), \qquad 0 \le x \le 1.$$
$$F_Y(y) = F_{X,Y}(1,y) = \frac{1}{10}(3y + y^2), \qquad 0 \le y \le 2.$$

(c) First, the marginal density of X is

$$\frac{d}{dx}F_X(x) = \frac{1}{10}(18x^2 + 4) = \frac{1}{5}(9x^2 + 2).$$

Therefore, for $0 \le x \le 1$, the conditional density is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{10}(9x^2 + 2y)}{\frac{1}{5}(9x^2 + 2)} = \frac{9x^2 + 2y}{18x^2 + 4}$$
$$= \frac{1}{2} + \frac{y - 1}{9x^2 + 2}.$$

4 marks for each part.

Note that 1 mark is deducted if you did not say that the expression for $F_X(x)$ is valid only for $0 \le x \le 1$, $F_X(x) = 0$ if $x < 0$ and $F_X(x) = 1$ if $x > 1$, etc. The mark is typically deducted off part b.

QUESTION 2. **(8 marks)**

Let X and Y have the joint density function

$$f(x,y) = k(2x+y), \qquad 0 \le x \le 2 \text{ and } 0 \le y \le 2,$$

and 0 elsewhere.

(a) Find k. Leave your answer as a fraction.

(b) Find $P(Y < X+1)$. Leave your answer as a fraction or to three significant figures.

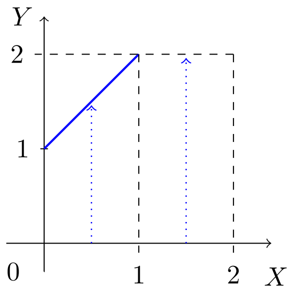
[Answer:]

a.

$$\begin{aligned}
\int_0^2 \int_0^2 k(2x+y)\; dydx &= k\int_0^2 \left[2xy + \frac{1}{2}y^2\right]_0^2 \; dx \\
&= k\int_0^2 4x+2\; dx \\
&= k\left[2x^2+2x\right]_0^2 \\
&= 12k.
\end{aligned}$$

Since the total probability is 1, we must have $k = \frac{1}{12}$.

b. We plot the line $Y = X+1$ and we wish to integrate over the area in the square of sides 2 that is below the line $Y = X+1$.



From the diagram, we could find the probability by splitting the integral into two. The probability is given by

$$\begin{aligned}
P(Y<X+1) &= \int_0^1 \int_0^{x+1} \frac{1}{6}x + \frac{1}{12}y \; dydx + \int_1^2 \int_0^2 \frac{1}{6}x + \frac{1}{12}y\; dydx \\
&= \int_0^1 \left[\frac{1}{6}xy + \frac{1}{24}y^2\right]_0^{x+1} \; dydx + \int_1^2 \left[\frac{1}{6}x^2 + \frac{1}{24}y^2\right]_0^2 \; dx \\
&= \int_0^1 \left[\frac{1}{6}x(x+1) + \frac{1}{24}(x+1)^2\right] dx + \int_1^2 \left[\frac{1}{3}x + \frac{1}{6}\right] dx \\
&= \left[\frac{1}{18}x^3 + \frac{1}{12}x^2 + \frac{1}{72}(x+1)^3\right]_0^1 + \left[\frac{1}{6}x^2 + \frac{1}{6}x\right]_1^2 \\
&= \frac{1}{18} + \frac{1}{12} + \frac{8-1}{72} + \frac{1}{6}(4-1)\frac{1}{6}(2-1) \\
&= \frac{65}{72}.
\end{aligned}$$

Alternatively, do $1 - \int_0^1 \int_{x+1}^2 kf(x,y)\, dydx$.

Part (a) is worth 4 marks, quite straight forward, most people get it right.

Part (b) is worth 4 marks.

- A correct set up of the integral(s) earns 3 marks or more.
- Most wrong setups, such as integrating over the wrong region, earns 2 marks.

QUESTION 3. **(6 marks)**

A number, N, is chosen randomly from the set $\{1, 2, 4\}$. A fair coin is then flipped N time. Let H denote the number heads obtained. Find the conditional distribution of N given $H = 2$.

[Answer:]

$$\begin{aligned}
P(H=2) &= P(H=2|N=1)P(N=1) + P(H=2|N=2)P(N=2) + P(H=2|N=4)P(N=4) \\
&= 0 \cdot \frac{1}{3} + \binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 \cdot \frac{1}{3} + \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \cdot \frac{1}{3} \\
&= 0 + \frac{1}{12} + \frac{2}{16} \\
&= \frac{10}{48}.
\end{aligned}$$

$$\begin{aligned}
P(N=1|H=2) &= \frac{P(N=1 \text{ and } H=2)}{P(H=2)} = \frac{0}{\frac{10}{48}} = 0 \\
P(N=2|H=2) &= \frac{P(N=2 \text{ and } H=2)}{P(H=2)} = \frac{\binom{2}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 \cdot \frac{1}{3}}{\frac{10}{48}} = \frac{2}{5} \\
P(N=4|H=2) &= \frac{P(N=4 \text{ and } H=2)}{P(H=2)} = \frac{\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \cdot \frac{1}{3}}{\frac{10}{48}} = \frac{3}{5}.
\end{aligned}$$

1 mark for correct formula for $P_{N|H}$ somewhere.

1 mark for $H|N$ has binomial distribution.

1 mark for N has uniform distribution.

5 marks if answer is mostly correct except for minor computation errors.

QUESTION 4. **(8 marks)**

Let X and Y have the joint density function $f(x, y)$ and let $Z = 2X + Y$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - 2x)\, dx$$

by completing the proof below.

Proof:

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-2x} f(x, y)\, dy dx.$$

Making a change of variable $y = \ldots$

[Answer]

Set $y = v - 2x$. Then $dy = dv$. . . . 2 marks

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z} f(x, v - 2x)\, dv dx \qquad (3 \text{ marks})$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(x, v - 2x)\, dx dv. \qquad (2 \text{ marks})$$

Differentiating with respect to z, we arrive at . . . 1 mark

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z - 2x)\, dx.$$