MH2500 Probability and Introduction to Statistics

Handout 5 - Expected Values - I

Synopsis

We study the expected value of a random variable, and in particular, we evaluate the expected value of many distributions. We also prove some formulas for computing expected values of functions of random variables and apply them in examples.

- Expected value of a Random Variable.
- Expectations of Functions of Random Variables.
- Expectations of Functions of Linear combinations of Random Variables.

Definition

The expected value of a random variable is the same as its weighted average, i.e, the sum of each value multiplied by its "weight" (which is its probability).

Definition

If X is a discrete random variable with frequency function $p(x)$, the expected value of X, denoted by $E(X)$, is

$$E(X) = \sum_i x_i p(x_i).$$

provided that $\sum_i |x_i| p(x_i) < \infty$. If the sum diverges, the expectation is undefined.

- $E(X)$ is also referred to as the **mean** of X and is often denoted as μ or μ_X (We used that in the normal distribution).
- $E(X)$ can also be viewed as the center of mass of the frequency function.

Example A

Roulette

A roulette wheel has the numbers 1 through 36 as well as 0 and 00. If you bet \$1 that an odd number comes up, you win or lose \$1 according to whether that event occurs. Find your expected gain (or loss).

. . . .

Let X denote your net gain. Then X takes values 1 or -1 and

$$P(X = 1) = \frac{18}{38}$$

and

$$P(X = -1) = \frac{20}{38}$$

Hence

$$E(X) = 1 \times \frac{18}{38} + (-1) \times \frac{20}{38} = -\frac{1}{19} \quad .$$

Example B

Expectation of a Geometric Random Variable

Suppose that items produced in a plant are independently defective with probability p. Items are inspected one by one until a defective item is found. On average, how many items must be inspected?

. . . .

Let X be the number of items inspected. Then X is a geometric random variable with

$$P(X = k) = q^{k-1}p$$

where $q = 1 - p$.

Example B con't

Therefore,

$$
\begin{aligned}
E(X) &= \\
&= \\
&= \\
&= \\
&= \\
&= \qquad .
\end{aligned}
$$

Example C - Poisson

Roulette

Find the expected value of a Poisson random variable.

. . . .

Let X be a Poisson random variable with parameter λ. Then

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \\ &= \\ &= \quad . \end{aligned}$$

Example D - St Petersburg Paradox

A gambler has the following strategy for playing a sequence of games. He starts of betting \$1: If he loses, he doubles his bet; and he continues to double his bet until he finally wins.

To analyze this scheme, suppose that the game is fair and that he wins or loses the amount he bets. At trial 0, he bets \$1; if he loses, he bets \$2 at trial 1; and if he has not won by the k-th trial, he bets $\$2^k$. When he finally wins, he will be \$1 ahead, which can be checked by going through the scheme for the first few values of k.

Calculate the expected return. This seems like a foolproof way to win \$1, but is anything wrong with this scheme? Explain.

Example D - St Petersburg Paradox

Solution:
Let X denote the amount of money bet on the very last game (that he wins). The probability that he loses k-times followed by one win is

$$P(X = 2^k) = \frac{1}{2^{k+1}}$$

and

$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} nP(X = n) \\ &= \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} \\ &= \infty. \end{aligned}$$

Formally, $E(X)$ is not defined. Practically, this scheme is flawed because it does not take into account the enormous amount of capital required.

Definition

Definition

If X is a continuous random variable with density $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

provided that $\int |x|f(x)dx < \infty$. If the integral diverges, the expectation is undefined.

Again, $E(X)$ can be regarded as the center of mass of the density.

Example E - Gamma

Suppose X follows a gamma density with parameters α and λ. Find the expected value.

. . . .

By definition,

$$E(X) = \int_0^\infty \qquad\qquad\qquad dx.$$

Note that $\lambda^{\alpha+1} x^\alpha e^{-\lambda x}/\Gamma(\alpha+1)$ is a gamma density, and so

$$\int_0^\infty \frac{\lambda^{\alpha+1} x^\alpha e^{-\lambda x}}{\Gamma(\alpha+1)} dx = 1.$$

Therefore,

$$E(X) = \qquad\qquad \int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x} dx =$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

For $\alpha = 1$, we see that the mean of the exponential density is $E(X) = 1/\lambda$. In contrast, the median of the exponential density is $\log 2/\lambda$

Example F - Normal

Roulette

Suppose $X \sim N(\mu, \sigma^2)$. Find the expected value of X.

By definition,

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx.$$

By a change of variables $z = x - \mu$, the equation becomes

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}\frac{z^2}{\sigma^2}} dz + \frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{z^2}{\sigma^2}} dz.$$

The first integral is 0 as the integrand is an odd function. The second integral is μ. Hence

$$E(X) = \mu.$$

Markov's Inequality

Theorem

If X is a random variable with $P(X \geq 0) = 1$ and for which $E(X)$ exists, then $P(X \geq t) \leq E(X)/t$.

Proof: We only prove the discrete case. The continuous case is similar.

$$\begin{aligned} E(X) &= \sum_x xp(x) \\ &= \sum_{x<t} xp(x) + \sum_{x \geq t} xp(x). \end{aligned}$$

Since $P(X \geq 0) = 1$, we see that X only take on nonnegative values. Hence

$$E(X) \geq \qquad\qquad \geq \qquad\qquad = tP(X \geq t).$$

Expectations of functions of random variables.

We could use methods from previous chapter, or we could use the following theorem.

Theorem A

Suppose $Y = g(X)$.

(a) If X is discrete with frequency function $p(x)$, then

$$E(Y) = \sum_x g(x)p(x).$$

provided that $\sum |g(x)|p(x) < \infty$.

(b) If X is continuous with density function $f(x)$, then

$$E(Y) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

provided that $\int |g(x)|f(x)dx < \infty$.

Proof:

We only prove the discrete case. The proof for the continuous case follows the same idea but is beyond the scope of this course.

By definition,

$$E(Y) = \sum_i y_i p_Y(y_i).$$

Let A_i denote the set of x's mapped to y_i by g, that is, $x \in A_i$ if $g(x) = y_i$. Then

$$\begin{aligned} E(Y) &= \sum_i y_i \sum_{x \in A_i} p(x) \\ &= \sum_i \sum_{x \in A_i} y_i p(x) \\ &= \sum_i \sum_{x \in A_i} g(x) p(x) \\ &= \sum_x g(x) p(x), \end{aligned}$$

where in the last equality, we used the fact that the A_i's are disjoint and every x belongs to some A_i.

Example

Suppose X takes values 1 and 2 with probability $1/2$ and $Y = 1/X$. Find $E(X)$ and $E(Y)$.

. . . .

$$E(X) = \qquad\qquad\qquad\qquad .$$

$$E(Y) = \qquad\qquad\qquad\qquad .$$

. . . .

Remark: $E(g(X)) \neq g(E(X))$.

Example

Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{4}x, & 1 \leq x \leq 3; \\ 0, & \text{otherwise.} \end{cases}$$

and let $Y = \frac{1}{2}X^2$. Find $E(Y)$.

. . . .

$$\begin{aligned} E(Y) &= \int_1^3 \\ &= \frac{1}{8}\int_1^3 x^3 dx \\ &= \frac{1}{32}[x^4]_1^3 \; = \; \frac{5}{2}. \end{aligned}$$

Theorem B

Theorem B

Suppose that $X_1, \ldots, Xn$ are jointly distributed random variables and $Y = g(X_1, \ldots, X_n)$.

a. If the X_i's are discrete with frequency function $p(X_1, \ldots, x_n)$, then

$$E(Y) = \sum_{x_1, \ldots, x_n} g(x_1, \ldots, x_n) p(x_1, \ldots, x_n)$$

provided that $\sum_{x_1, \ldots, x_n} |g(x_1, \ldots, x_n)| p(x_1, \ldots, x_n) < \infty$.

b. If the X_i's are continuous with joint density function $f(x_1, \ldots, x_n)$, then

$$E(Y) = \int \int \ldots \int g(x_1, \ldots, x_n) p(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

provided that the integral with $|g|$ in place of g converges.

Corollary

Corollary

If X and Y are independent random variables and g and h are fixed functions, then

$$E(g(X)h(Y)] = [E(g(X))][E(h(Y))]$$

provided that the expectations on the right hand side exist.

In particular, if X and Y are independent, then $E(XY) = E(X)E(Y)$.

. . . .

For proof, see tutorial 8.

Expectations of linear combinations of Random Variables

Theorem A

If $X_1, \ldots, X_n$ are jointly distributed random variables with expectations $E(X_i)$ and Y is a linear function of the X_i's where

$$Y = a + \sum_{i=1}^{n} b_i X_i,$$

then

$$E(Y) = a + \sum_{i=1}^{n} b_i E(X_i).$$

. . . .

We only prove the continuous case, and only for $n = 2$. The discrete case is similarly proved.

Proof

From Theorem B (slide 18), we have

$$
\begin{aligned}
E(Y) &= \int\int (a + b_1 x_1 + b_2 x_2) f(x_1, x_2) dx_1 dx_2 \\
&= a \int\int f(x_1, x_2) dx_1 dx_2 + b_1 \int\int x_1 f(x_1, x_2) dx_1 dx_2 \\
&\quad + b_2 \int\int x_2 f(x_1, x_2) dx_1 dx_2
\end{aligned}
$$

The first double integral is 1. The second double integral is

$$
\begin{aligned}
\int\int x_1 f(x_1, x_2) dx_1 dx_2 &= \int x_1 \int f(x_1, x_2) dx_2 dx_1 \\
&= \int x_1 f_{X_1}(x_1) dx_1 \\
&= E(X_1).
\end{aligned}
$$

Proof cont

Similarly, the third integral is $E(X_2)$. Hence the integral is

$$E(Y) = a + b_E(X_1) + b_2 E(X_2).$$

It remains to check that the expectation is well defined, i.e.,

$$\int\int (a + b_1 x_1 + b_2 x_2) f(x_1, x_2) dx_1 dx_2 < \infty.$$

This can be verified by noting that

$$|a + b_1 x_1 + b_2 x_2| \leq |a| + |b_1||x_1| + |b_2||x_2|$$

and the assumption that $E(X_i)$ exist.

Example B

Suppose you collect coupons and there are n distinct types of coupons, and that on each trial you are equally likely to get a coupon of any of the types.

Let X_1 be the number of trials up to and including the trial on which the first coupon is collected: $X_1 = 1$
Let X_2 be the number of trials from then on up to and including the trial on which the next coupon different from the first is obtained.
We define $X_3, \ldots, X_n$ similarly.

Then the total number of trials needed to collect all n coupons is, $X = X_1 + X_2 + \cdots + X_n$. Find $E(X)$.

....

First, we work out what is $E(X_r)$ for $1 \leq r \leq n$. At that point, $r-1$ coupons have been collected and so each trial has probability $(n - r + 1)/n)$ of success.

Example B

Therefore, X_r is a geometric random variable and

$$E(X_r) = \frac{n}{n-r+1} \qquad \text{(Slide 5)}.$$

Therefore,

$$\begin{aligned} E(X) &= \sum_{r=1}^{n} E(X_r) \\ &= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n \sum_{r=1}^{n} \frac{1}{r}. \end{aligned}$$

. . . .

Remark: We could further apply the approximation

$$\sum_{r=1}^{n} \frac{1}{r} = \log n + \gamma + \varepsilon_n.$$

Example E

An investor plans to apportion an amount of capital, C_0, between two investments placing a fraction π, $0 \le \pi \le 1$ in one investment and a fraction $1 - \pi$ in the other for a fixed period of time. Denoting the returns (final value divided by initial value) on the investments by R_1 and R_2.

Express her rate of returns and expected returns at the end of the period in terms of C_0, R_1, R_2, $E(R_1)$ and $E(R_2)$.

. . . .

Capital at the end of the period, $C_1 = \pi C_0 R_1 + (1-\pi)C_0 R_2$. Hence

$$R = \frac{C_1}{C_0} = \pi R_1 + (1-\pi)R_2.$$

Hence her expected return is

$$E(R) = \qquad\qquad\qquad\qquad .$$

. . . .

How should she choose π? It seems that if $E(R_1) > E(R_2)$ then she should choose $\pi = 1$, else $\pi = 0$. We shall see later that there's more to consider.