NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 4

For the tutorial on 8 September, let us discuss

- Ex. 2.5.32, 34, 42, 44, 46, 48, 49, 51.

Ex.2.5.32. Suppose that in a city, the number of suicides can be approximated by a Poisson process with $\lambda = 0.33$ per month.

a. Find the probability of k suicides in a year for $k = 0, 1, 2, \ldots$. What is the most probable number of suicides?

b. What is the probability of two suicides in one week?

[Solution:]

a. Let X be the number of suicides in a year. Then X is Poisson with parameter $\omega = 12\lambda = 3.96$.
$$P(X = k) = \frac{3.96^k}{k!}e^{-3.96}.$$
Note that the ratio $P(X = k)/P(X = k-1) = 3.96/k$ and so for $k = 1, 2, 3$, we see that $P(X = k) > P(X = k-1)$. For $k \geq 4$, we note that $3.96/k < 1$ and so $P(X = k) < P(X = k-1)$. Hence the most probable number of suicides is $\lfloor 3.96 \rfloor = 3$.

b. Let Y be the number of suicides in one week. Then Y is Poisson with parameter $\mu = 3.96/52 \approx 0.076154$. Thus
$$P(Y = 2) = \frac{(0.76154)^2}{2!}e^{-0.76154} \approx 0.00269.$$

Ex.2.5.34. Let $f(x) = (1+\alpha x)/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$ otherwise, where $-1 \leq \alpha \leq 1$. Show that f is a density, and find the corresponding cdf. Find the quartiles and the median of the distribution in terms of α.

[Solution:] For $-1 \leq x \leq 1$ and $-1 \leq \alpha \leq 1$, it is clear that $f(x) \geq 0$ and $f(x)$ is continuous. Hence, to show that f is a density, it suffices to show that $\int_{-\infty}^{\infty} f(x)dx = 1$.
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} \frac{1+\alpha x}{2}dx = \left[\frac{1}{2}x + \frac{\alpha}{4}x^2\right]_{-1}^{1} = 1.$$
Next,
$$\int_{-1}^{x} \frac{1+\alpha u}{2}du = \left[\frac{1}{2}u + \frac{\alpha}{4}u^2\right]_{-1}^{x} = \frac{1}{2}(x+1) + \frac{\alpha}{4}(x^2-1).$$
Thus, the cdf is
$$P(X \leq x) = \begin{cases} 0, & \text{if } x \leq -1; \\ \frac{1}{2}(x+1) + \frac{\alpha}{4}(x^2-1), & \text{if } -1 < x < 1 \ ; \\ 1, & \text{if } x \geq 1. \end{cases}$$

Finally, we solve for x, the following equation.

$$\frac{1}{2}(x+1)+\frac{\alpha}{4}(x^2-1)=\frac{j}{4}, \qquad j=1,2,3.$$

Case 1: $\alpha=0$. It suffices to solve

$$\frac{1}{2}(x+1)=\frac{j}{4}, \qquad j=1,2,3,$$

and the solutions are $x=-\frac{1}{2},0,\frac{1}{2}$.

Case 2: $\alpha\neq 0$. Completing the square gives

$$\frac{\alpha}{4}(x+\frac{1}{\alpha})^2-\frac{1}{4\alpha}+\frac{2-\alpha-j}{4}=0.$$

Solving this in terms of α gives

$$x_j=-\frac{1}{\alpha}\pm\frac{1}{\alpha}\sqrt{1-2\alpha+\alpha^2+j\alpha}.$$

Substituting $j=1,2,3$ gives the lower quartile, median, and upper quartile, respectively.

Ex.2.5.42. Find the probability density for the distance from an event to its nearest neighbor for a Poisson process in the plane.

[Solution:] It is given that events occur as "a Poisson process in the plane". This means we may assume that events occur in the Cartesian plane following a Poisson distribution with parameter λ for some λ. In other words, we assume that the number of events observed in any unit square of the plane is Poisson and on average, there are λ events per unit square.

Let E denote an event and we wish to find the probability density function for the distance from E to the nearest neighbour F. Let X denote the distance between E and F. Then $X\le r$ means that F lies in the circular disk centered at E with radius r (boundary included), while $X>r$ means E is the only event in the circular disk centered at E with radius r.

A circular disk of radius r has area πr^2 and so is Poisson with parameter $\mu=\pi r^2\lambda$. The same is true if we remove the point E from the disk, which is also known as a punctured disk. Let U_r denote the number of events in a punctured disk of radius r. Then U_r is Poisson with parameter $\pi r^2\lambda$. Thus, for a given r,

$$\begin{aligned}F_X(r)=P(X\le r)&=1-P(X>r)\\&=1-P(U_r=0)\\&=1-e^{-\pi r^2\lambda}.\end{aligned}$$

Hence the required density function is

$$f_X(r)=\frac{d}{dr}(1-e^{\pi r^2\lambda})=-2r\pi\lambda e^{-\pi r^2\lambda}.$$

Ex.2.5.44. Let T be an exponential random with parameter λ. Let X be a discrete random variable defined as $X=k$ if $k\le T<k+1$, $k=0,1,\ldots$. Find the frequency function of X.

[Solution:] Recall that T being an exponential random variable, has density function

$$f_T(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0;\\ 0, & 0.\end{cases}$$

For $k = 0, 1, 2, \ldots$,

$$\begin{aligned} P(X = k) &= P(k \le T < k+1) \\ &= \int_k^{k+1} \lambda e^{-\lambda x} dx \\ &= \left[-e^{-\lambda x}\right]_k^{k+1} \\ &= e^{-\lambda k}(1 - e^{-\lambda}). \end{aligned}$$

Ex.2.5.46. Recall the gamma function is defined as

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, \qquad x > 0$$

and the gamma density function is given by

$$g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, \qquad t > 0,$$

where α and λ are two positive parameters. Show that the gamma density integrates to 1.

[Solution:] We use the substitution $u = \lambda t$. Then $du = \lambda dt$, and we see that

$$\begin{aligned} \int_0^\infty g(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda t^\alpha e^{-\lambda t} dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du \qquad \text{(via substitution } u = \lambda t\text{)} \\ &= \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) = 1. \end{aligned}$$

Ex.2.5.48. T is an exponential random variable, and $P(T < 1) = 0.05$. What is λ?

[Solution:] Recall that

$$P(T < 1) = 1 - e^{-\lambda} = 0.05.$$

Hence $e^{-\lambda} = 0.95$, which solving for λ gives

$$\lambda = -\log(0.95) \approx 0.0513.$$

Ex. 2.5.51. Show that the normal density integrates to 1. (Hint: First make a change of variables to reduce the integral to that for the standard normal. The problem is then to show that $\int_{-\infty}^{\infty} \exp -x^2/2dx = \sqrt{2\pi}$. Square both sides and reexpress the problem as that of showing

$$\left(\int_{-\infty}^{\infty} \exp(-x^2/2)dx\right)\left(\int_{-\infty}^{\infty} \exp(-y^2/2)dy\right) = 2\pi.$$

Write the product of integrals as a double integral and change to polar coordinates. You might not have learnt these yet, so just assume the following are true.

$$\begin{aligned}\left(\int_{-\infty}^{\infty} \exp(-x^2/2)dx\right)\left(\int_{-\infty}^{\infty} \exp(-y^2/2)dy\right) &= \iint\limits_{R^2} \exp(-(x^2+y^2)/2)dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r d\theta dr \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r \; d\theta dr.\end{aligned}$$

Integrate to show that it is equal to 2π.)

[Solution:]

Suppose $U \sim N(\mu, \sigma^2)$. Then by the change of variable $x = (u-\mu)/\sigma$, we see that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-\mu)^2/2\sigma^2} du = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2} \sigma dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Thus, it suffices to show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$, which is equivalent to showing

$$\left(\int_{-\infty}^{\infty} e^{-x^2/2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right) = 2\pi.$$

Continuing from the hint, the left side equals

$$\begin{aligned}\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} r \; d\theta dr &= \int_0^{\infty} \left[\theta e^{-r^2/2} r\right]_0^{2\pi} dr \\ &= 2\pi \int_0^{\infty} r e^{-r^2/2} dr \\ &= 2\pi \left[-e^{-r^2/2}\right]_0^{\infty} \\ &= 2\pi.\end{aligned}$$