NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 6

For the tutorial on 22 September, let us discuss

- Ex. 3.8.3, 7, 9, 13, 22, 23

Ex. 3.8.3. Three players play 10 independent rounds of a game, and each player has probability $\frac{1}{3}$ of winning each round. Find the joint distribution of the number of games won by each of the three players.

[Solution:] Let N_1, N_2, and N_3 denote the number of games won by each of the three players. Then the joint distribution of N_1, N_2, and N_3 follows a multinomial distribution with $n = 10$ and $p_1 = p_2 = p_3 = \frac{1}{3}$.

$$p(n_1, n_2, n_3) = \begin{cases} \binom{10}{n_1, n_2, n_3} \left(\frac{1}{3}\right)^{10}, & \text{if } n_1, n_2, n_3 \geq 0 \text{ and } n_1 + n_2 + n_3 = 10; \\ 0, & \text{otherwise.} \end{cases}$$

Ex. 3.8.7. Find the joint and marginal densities corresponding to the cdf

$$F(x, y) = (1 - e^{-\alpha x})(1 - e^{-\beta y}), \qquad x \geq 0, \qquad y \geq 0, \qquad \alpha > 0, \qquad \beta > 0.$$

[Solution:]

$$F_X(x) = \lim_{y\to\infty} F(x, y) = \lim_{y\to\infty} (1 - e^{-\alpha x})(1 - e^{-\alpha y}) = 1 - e^{-\alpha x}.$$

Hence

$$f_X(x) = \frac{d}{dx}(1 - e^{-\alpha x}) = \alpha e^{-\alpha x} \qquad (x \geq 0),$$

and $f_X(x) = 0$ otherwise. Similarly, $f_Y(y) = \beta e^{-\beta y}$ for $y \geq 0$ and $f_Y = 0$ otherwise.

The joint density is given by

$$\begin{aligned} f(x, y) &= \frac{\partial^2}{\partial x \partial y}(1 - e^{-\alpha x})(1 - e^{-\beta y}) \\ &= \frac{\partial}{\partial x}(1 - e^{-\alpha x})\beta e^{-\beta y} \\ &= \alpha\beta e^{-\alpha x - \beta y}, \qquad (x, y \geq 0), \end{aligned}$$

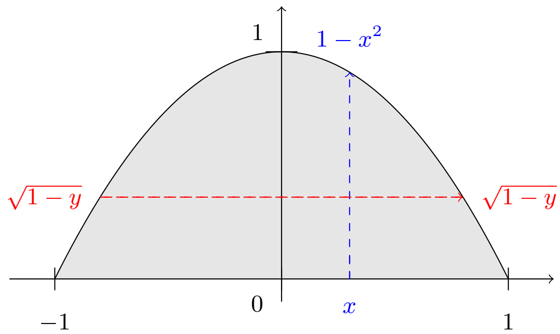
and $f(x, y) = 0$ otherwise.

Ex. 3.8.9. Suppose that (X, Y) is uniformly distributed over the region defined by $0 \leq y \leq 1 - x^2$ and $-1 \leq x \leq 1$.

a. Find the marginal densities of X and Y.

b. Find the two conditional densities.

[Solution:]



a. First, we calculate the area in which $f_{XY}(x,y)$ takes positive value.

$$\text{Area} = \int_{-1}^{1}\int_{0}^{1-x^2} 1 dy dx = \int_{-1}^{1} 1 - x^2 dx = \left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3}.$$

Hence the joint density is given by

$$f(x,y) = \begin{cases} \frac{3}{4}, & \text{if } (x,y) \text{ is in the shaded region;} \\ 0, & \text{otherwise.} \end{cases}$$

For $-1 \le x \le 1$,

$$f_X(x) = \int_0^{1-x^2} \frac{3}{4} dy = \frac{3}{4}\left[y\right]_0^{1-x^2} = \frac{3}{4}(1-x^2).$$

Similarly, for $0 \le y \le 1$,

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4} dx = \frac{3}{4}\left[x\right]_{-\sqrt{1-y}}^{\sqrt{1-y}} = \frac{3}{2}\sqrt{1-y}.$$

The above are valid for x and y satisfying $0 \le y \le 1 - x^2$ and $-1 \le x \le 1$.

b. The conditional densities are given by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{4}}{\frac{3}{2}\sqrt{1-y}} = \frac{1}{2\sqrt{1-y}}, \qquad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{4}}{\frac{3}{4}(1-x^2)} = \frac{1}{1-x^2}.$$

The above are valid for x and y satisfying $0 \le y \le 1 - x^2$ and $-1 \le x \le 1$.

Ex. 3.8.13. A fair coin is thrown once; if it lands heads up, it is thrown a second time. Find the frequency function of the total number of heads.

[Solution:] The possible outcomes are T, HT, and HH. Let X denote the number of heads. Then

$$p(X=0) = P(\text{coin lands head in the first throw}) = \frac{1}{2},$$

$$p(X=1) = P(\text{coin lands tail in the first throw and head in the second throw}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

$$p(X=2) = P(\text{coin lands heads in the two throws}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Ex. 3.8.22. Consider a Poisson process on the real line and denote by $N(t_1, t_2)$ the number of events in the interval (t_1, t_2). If $t_0 < t_1 < t_2$, find the conditional distribution of $N(t_0, t_1)$

given that $N(t_0, t_2) = n$. (Hint: Use the fact that the numbers of events in disjoint subsets are independent.)

[Solution:] By definition of conditional probability,

$$\begin{aligned} p_{N(t_0,t_1)|N(t_0,t_2)}(m|n) &= \frac{P(N(t_0,t_1) = m \text{ and } N(t_0,t_2) = n)}{P(N(t_0,t_2) = n)} \qquad (0 \le m \le n) \\ &= \frac{P(N(t_0,t_1) = m) \text{ and } P(N(t_1,t_2) = n-m)}{P(N(t_0,t_2) = n}, \end{aligned}$$

since for a Poison process, the number of events in disjoint subsets are independent.

Let the Poisson process have parameter λ per unit interval. Then $N(t_0, t_1)$, $N(t_1, t_2)$, and $N(t_0, t_2)$ are Poisson with parameters $\lambda(t_1 - t_0)$, $\lambda(t_2 - t_1)$, and $\lambda(t_2 - t_0)$, respectively. Therefore,

$$\begin{aligned} p_{N(t_0,t_1)|N(t_0,t_2)}(m|n) &= \frac{P(N(t_0,t_1) = m) \text{ and } P(N(t_1,t_2) = n-m)}{P(N(t_0,t_2) = n)} \\ &= \frac{\frac{[(t_1-t_0)\lambda]^m}{m!} e^{-\lambda(t_1-t_0)} \frac{[(t_2-t_1)\lambda]^{n-m}}{(n-m)!} e^{-\lambda(t_2-t_1)}}{\frac{[(t_2-t_0)\lambda]^n}{n!} e^{-\lambda(t_2-t_0)}} \\ &= \binom{n}{m} \left(\frac{t_1-t_0}{t_2-t_0}\right)^m \left(\frac{t_2-t_1}{t_2-t_0}\right)^{n-m}. \end{aligned}$$

Thus the conditional distribution of $N(t_0, t_1)$ given that $N(t_0, t_2) = n$ is Binomial with parameters n and $\frac{t_1-t_0}{t_2-t_0}$.

Ex. 3.8.23. Suppose that, conditional on N, X has a binomial distribution with N trials and probability p of success, and that N is a binomial random variable with m trials and probability r of success. Find the unconditional distribution of X.

[Solution:] The (unconditional) frequency function for X is

$$\begin{aligned} p_X(k) &= \sum_{n=k}^{m} p_{X|N}(x|n) p_N(n) \\ &= \sum_{n=k}^{m} \binom{n}{k} p^k (1-p)^{n-k} \binom{m}{n} r^n (1-r)^{m-n} \\ &= \frac{m!}{(m-k)!k!} p^k r^k \sum_{n=k}^{m} \frac{(m-k)!}{(n-k)!(m-n)!} (1-p)^{n-k} r^{n-k} (1-r)^{m-n} \\ &= \binom{m}{k} (pr)^k \sum_{j=0}^{m-k} \binom{m-k}{j} [(1-p)r]^j (1-r)^{m-k-j} \qquad (j = n-k) \\ &= \binom{m}{k} (pr)^k \left\{[(1-p)r] + (1-r)\right\}^{m-k} \\ &= \binom{m}{k} (pr)^k (1-rp)^{m-k}. \end{aligned}$$

Hence X follows a binomial distribution with parameters m and probability pr.