NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 7

For the tutorial on 6 October, let us discuss

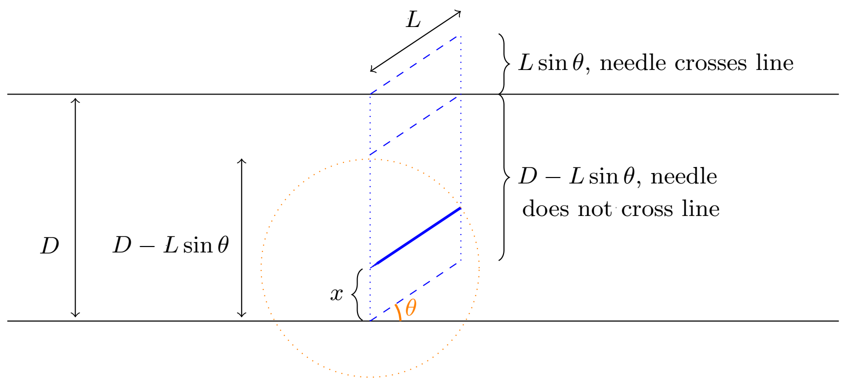
- Ex. 3.8.5, 6, 11, 18, 19, 30

Ex. 3.8.5. (Buffon's Needle Problem) A needle of length L is dropped randomly on a plane ruled with parallel lines that are distance D apart, where $D \geq L$. Show that the probability that the needle comes to rest crossing a line is $2L/(\pi D)$. Explain how this gives a mechanical means of estimating the value of π.

[Solution:]
It suffices to consider the two parallel lines which the needle head lies in. Let x denote the distance from the needle head to the lower line, and let θ be the angle of the needle relative to the lower line. Then (x, θ) where $0 \leq x < D$ and $0 \leq \theta < 2\pi$, determines the position of the needle on the vertical line segment.

We assume that the needle is equally likely to land anywhere and at any angle. Then

$$f_{X\Theta}(x, \theta) = \frac{1}{2\pi D} \qquad (\text{for } 0 \leq x < D \text{ and } 0 \leq \theta < 2\pi).$$



At a given angle $0 < \theta < \pi$, the needle does not cross the line if $0 < x < D - L\sin\theta$ and crosses the line if $D - L\sin\theta < x < D$. For angle $\pi < \theta < 2\pi$, the needle crosses the line if $0 < x < L|\sin\theta|$ and does not cross the line if $L|\sin\theta| < x < D$. By symmetry, we may calculate the probability by computing the probability for $0 < \theta < \pi/2$ and multiplying by 4. Thus the probability is

$$\begin{aligned} 4\int_0^{\pi/2} \int_{D-L\sin\theta}^{D} \frac{1}{2\pi D} dx d\theta &= \frac{2}{\pi D} \int_0^{\pi/2} L\sin\theta \; d\theta \\ &= \frac{2}{\pi D} [-L\cos\theta]_0^{\pi/2} \\ &= \frac{2L}{\pi D}. \end{aligned}$$

To find an estimate for π, we may drop a large number of pins on the plane and compute the proportion, p, of needles that crosses the parallel lines. Then p is an estimate of $2L/(\pi D)$ and so an estimate of π is given by $2L/(pD)$.

Ex. 3.8.6. A point is chosen randomly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the marginal densities of the x and y coordinates of the point.

[Solution:] First, we calculate the area of the ellipse. This can be computed as four times the area of the ellipse in the first quadrant. The area is

$$\begin{aligned}
4\int_0^b \int_0^{a\sqrt{1-y^2/b^2}} dx dy &= 4ab \int_0^{\pi/2} \sqrt{1-\sin^2(t)}\ \cos(t)dt \qquad \text{(substituting } y = b\sin(t))\\
&= 4ab \int_0^{\pi/2} \cos^2(t)dt \\
&= 2ab \int_0^{\pi/2} \cos(2t) + 1dt \\
&= ab[\sin(2t) + 2t]_0^{\pi/2} \\
&= ab\pi.
\end{aligned}$$

Thus, the joint density function is $f_{XY}(x, y) = \frac{1}{ab\pi}$. The marginal densities are

$$\begin{aligned}
f_X(x) &= \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \frac{1}{ab\pi} dy \\
&= \frac{2}{a\pi}\sqrt{1-\frac{x^2}{a^2}}, \qquad (-a \le x \le a),
\end{aligned}$$

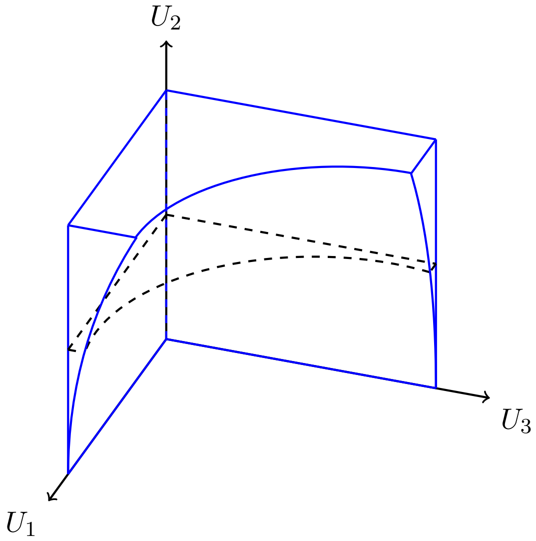
and

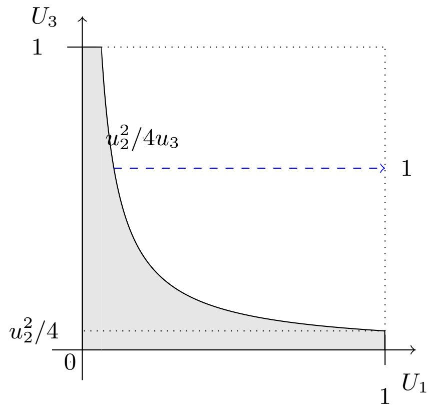
$$\begin{aligned}
f_Y(y) &= \int_{-a\sqrt{1-y^2/b^2}}^{a\sqrt{1-y^2/b^2}} \frac{1}{ab\pi} dx \\
&= \frac{2}{b\pi}\sqrt{1-\frac{y^2}{b^2}}, \qquad (-b \le y \le b).
\end{aligned}$$

Ex. 3.8.11. Let U_1, U_2, and U_3 be independent random variables uniform on [0,1]. Find the probability that the roots of the quadratic $U_1x^2 + U_2x + U_3$ are real.

[Solution:] The roots of a quadratic equation are real if and only if the discrimination is nonnegative. In this case, we need $D = U_2^2 - 4U_1U_3 \ge 0$.

The required probability is equal to the volume of the solid satisfying $u_2^2 > 4u_1u_3$ where $0 \le u_1, u_2, u_3 \le 1$.





For each value of u_2, the cross-sectional area is

$$\begin{aligned}1-\int_{u_2^2/4}^{1}\int_{u_2^2/(4u_3)}^{1} du_1du_3 &= 1-\int_{u_2^2/4}^{1} 1-\frac{u_2^2}{4u_3}du_3\\ &= 1-\left[u_3-\frac{u_2^2}{4}\ln u_3\right]_{u_2^2/4}^{1}\\ &= 1-\left[1-\frac{u_2^2}{4}+\frac{u_2^2}{2}\ln\frac{u_2}{2}\right]\\ &= \frac{u_2^2}{4}-\frac{u_2^2}{2}\ln\frac{u_2}{2}.\end{aligned}$$

Thus the probability is

$$\begin{aligned}\int_0^1 \frac{u_2^2}{4}-\frac{u_2^2}{2}\ln\frac{u_2}{2}\; du_2 &= \left[\frac{u_2^3}{12}\right]_0^1-\left[\frac{u_2^3}{6}\ln\frac{u_2}{2}\right]_0^1+\int_0^1\frac{u_2^2}{6}\; du_2\\ &= \frac{1}{12}+\frac{1}{6}\ln 2+\frac{1}{18}\\ &= \frac{5}{36}+\frac{1}{6}\ln 2\approx 0.254.\end{aligned}$$

Ex. 3.8.18. Let X and Y have the joint density function

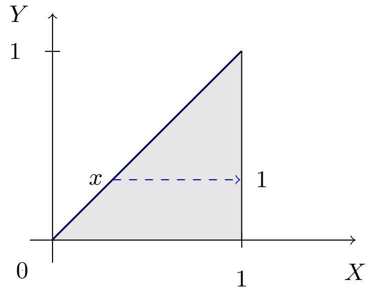
$$f(x,y)=k(x-y),\qquad 0\le y\le x\le 1,$$

and 0 elsewhere.

a. Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.
b. Find k.
c. Find the marginal densities of X and Y.
d. Find the conditional densities of Y given X and X given Y.

[Solution:]

a. The required region is shaded in the figure below.



b.

$$
\begin{aligned}
1 &= \int_0^1 \int_0^x k(x-y)\ dydx \\
&= \int_0^1 \left[kxy - \frac{ky^2}{2}\right]_0^x\ dx \\
&= \int_0^1 \frac{kx^2}{2}\ dx \\
&= \left[\frac{kx^3}{6}\right]_0^1 = \frac{k}{6}.
\end{aligned}
$$

Hence $k = 6$.

c.

$$f_X(x) = \int_0^x 6(x-y)dy = [6xy - 3y^2]_0^x = 3x^2, \qquad (0 \leq x \leq 1)$$

and

$$f_Y(y) = \int_y^1 6(x-y)dx = [3x^2 - 6xy]_y^1 = 3 - 6y - 3y^2 + 6y^2 = 3(y-1)^2, \qquad (0 \leq y \leq 1).$$

d. The conditional densities of Y given X and X given Y are

$$f_{X|Y}(x|y) = \frac{6(x-y)}{3(y-1)^2} = \frac{2(x-y)}{(y-1)^2} \qquad (0 \leq y \leq x \leq 1)$$

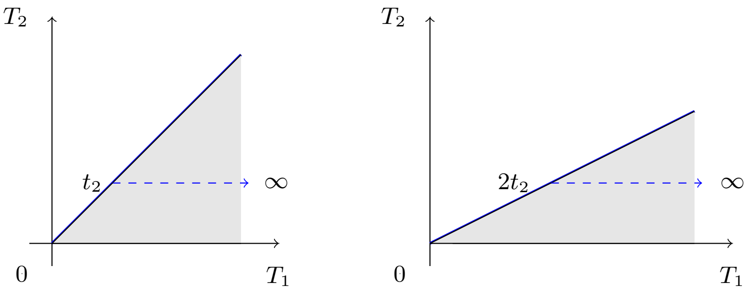
and

$$f_{Y|X}(y|x) = \frac{6(x-y)}{3x^2} = \frac{2(x-y)}{x^2} \qquad (0 \leq y \leq x \leq 1).$$

Ex. 3.8.19. Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β, respectively. Find (a) $P(T_1 > T_2)$ and (b) $P(T_1 > 2T_2)$.

[Solution:] Since the two random variables are independent, the joint density function is

$$f_{T_1,T_2}(t_1, t_2) = \alpha\beta e^{-\alpha t_1 - \beta t_2}.$$



a.

$$\begin{aligned}
P(T_1 > T_2) &= \int_0^\infty \int_{t_2}^\infty \alpha\beta e^{-\alpha t_1 - \beta t_2} \, dt_1 dt_2 \\
&= \int_0^\infty \left[-\beta e^{-\alpha t_1 - \beta t_2} \right]_{t_2}^\infty dt_2 \\
&= \int_0^\infty \left[\beta e^{-(\alpha+\beta)t_2} \right] dt_2 \\
&= \left[-\frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t_2} \right]_0^\infty \\
&= \frac{\beta}{\alpha+\beta}.
\end{aligned}$$

b.

$$\begin{aligned}
P(T_1 > 2T_2) &= \int_0^\infty \int_{2t_2}^\infty \alpha\beta e^{-\alpha t_1 - \beta t_2} \, dt_1 dt_2 \\
&= \int_0^\infty \left[-\beta e^{-\alpha t_1 - \beta t_2} \right]_{2t_2}^\infty dt_2 \\
&= \int_0^\infty \left[\beta e^{-(2\alpha+\beta)t_2} \right] dt_2 \\
&= \left[-\frac{\beta}{2\alpha+\beta} e^{-(\alpha/2+\beta)t_2} \right]_0^\infty \\
&= \frac{\beta}{2\alpha+\beta}.
\end{aligned}$$

Ex. 3.8.30. For $0 \le \alpha \le 1$ and $0 \le \beta \le 1$, show that $C(u,v) = \min(u^{1-\alpha}v, uv^{1-\beta})$ is a copula (the Marshall-Olkin copula) (valid for $0 \le u, v \le 1$. Somehow the author expects us to know that u and v cannot be negative or else $C(u,v)$ is not defined, and u, v cannot exceed 1, or else $C(u,v)$ may exceed 1.) What is the joint density?

[Solution:] Note that $C(u,v)$ is the joint cumulative distribution function. The marginal cdf of U is

$$F_U(u) = C(u,1) = \min(u^{1-\alpha}, u) = u.$$

Hence the marginal distribution of U is uniform. Similarly, the marginal cdf of V is

$$F_V(v) = C(1,v) = \min(v, v^{1-\beta}) = v.$$

Thus, the marginal distribution of V is also uniform, and so $C(u,v)$ is a copula.

The joint density is given by

$$\frac{\partial^2}{\partial u \partial v} C(u,v) = \begin{cases} \dfrac{\partial^2}{\partial u \partial v}(u^{1-\alpha} v), & \text{if } v^\beta \leq u^\alpha; \\ \dfrac{\partial^2}{\partial u \partial v}(u v^{1-\beta}), & \text{if } u^\alpha \leq v^\beta, \end{cases}$$

$$= \begin{cases} (1-\alpha) u^{-\alpha}, & \text{if } v^\beta \leq u^\alpha; \\ (1-\beta) v^{-\beta}, & \text{if } u^\alpha \leq v^\beta. \end{cases}$$