Name: ______________________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2015/16

MH2500– Probability and Introduction to Statistics

22 September 2015 Test 2 40 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed two double-sided A4 size cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(6 marks)**

An urn contains 20 red balls and 10 white balls. Suppose 10 balls are drawn from the urn without replacement. Find the probability that amongst the 10 balls drawn, exactly 7 are red. Leave your answer as a fraction or correct to three significant figures.

[Answer:]

$$P(X=7) = \frac{\binom{20}{7}\binom{10}{3}}{\binom{30}{10}} = \frac{206720}{667667} \approx 0.310.$$

Final answer is $\frac{206720}{667667}$ or 0.310, including incomplete working .... 6 points.

Final answer is correct but wrong working .... case by case, maybe 3 points.

Working is correct but final answer is wrong (including wrong number of significance figures) .... 5 points.

Some parts are correct .... 3 points.

Nothing is correct .... 1 point.

Blank or as good as not attempted .... 0 points.

QUESTION 2. **(12 marks)**

Suppose an office receives telephone calls as a Poisson process with $\lambda = 0.3$ per min.

(a) Find the probability that there are 10 calls in a one hour interval. Give your answer to three significant figures.

(b) Let T denote the time taken (in minutes) between one phone call and two more phone calls, i.e., time between the first and the third phone calls. Find the probability density function of T.

[Answer:]

a. Let X denote the number of calls in one hour. Then X is Poisson with $\lambda = 60 \times 0.3 = 18$. Then

$$P(X = 10) \;=\; \frac{18^{10}}{10!}e^{-18} = 0.0150.$$

b. Note that the number of phone calls per t minutes is Poisson with parameter $0.3t$. Suppose at time 0, there is a phone call. Let T denote the time taken (in minutes) to receive two more phone calls. Then

$$\begin{aligned} P(T \le t) &= 1 - P(T > t) \\ &= 1 - P(\text{no phone calls in time } (0,t]) - P(\text{one phone call in time } (0,t]) \\ &= 1 - \frac{(0.3t)^0}{0!}e^{-0.3t} - \frac{(0.3t)^1}{1!}e^{-0.3t} \\ &= 1 - (1 + 0.3t)e^{-0.3t}. \end{aligned}$$

Hence the density is

$$f_T(t) = -0.3e^{-0.3t} + 0.3(1 + 0.3t)e^{-0.3t} = 0.09te^{-0.3t}.$$

Remark: This is Gamma density with $\lambda = 0.3$ and $\alpha = 2$. This is true in general. Suppose X is the number of events occurring in time and it follows a Poisson distribution with parameter λ. Let T be the length of time until the k-th event. Then T is Gamma with parameters $\alpha = k$ and λ.

Part (a) is worth 6 points, quite straight forward, most people get it right.

Part (b) is worth 6 points.

- 2 points for recognizing that number of phone calls per t minutes is Poisson with parameter $0.3t$.
- 3 points for some form of computation involving $1 - P(\text{no phone calls in } (0,t])$.

QUESTION 3. **(8 marks)**

Suppose U is a uniform random variable on [1,4) and let X be a random variable defined by $X = \dfrac{3}{4-U}$. Find the density function of X.

[Answer:]

$$
\begin{aligned}
P(X \le x) &= P\left(\frac{3}{4-U} \le x\right) \\
&= P\left(\frac{3}{x} \le 4-U\right) \\
&= P\left(U \le 4-\frac{3}{x}\right) && \text{(3 points up to here)} \\
&= \frac{4-\frac{3}{x}-1}{3} \\
&= 1-\frac{1}{x}, \qquad (1 \le x < \infty). && \text{(5 points up to here)}
\end{aligned}
$$

Hence

$$
f_X(x) = \frac{d}{dx}\left(1-\frac{1}{x}\right) = \frac{1}{x^2} \qquad (1 \le x < \infty). \qquad \text{(7 points up to here)}
$$

(1 point for stating the range where the formula is valid, i.e., $1 \le x < \infty$.)

QUESTION 4. **(8 marks)**

Three players, Alan, Bob, and Carl play 10 consecutive rounds of a game. Alan has probability $\frac{1}{2}$ of winning each round, while Bob and Carl each has probability $\frac{1}{4}$ of winning each round.

(a) Find the joint distribution of the number of games won by each of the three players.

(b) Find the marginal distribution of the number of games won by Bob.

[Answer]

(a) Let A, B, C be the number of games won by Alan, Bob, and Carl, respectively. Then

$$\begin{aligned} P(A=a, B=b, C=c) &= \binom{10}{a}\binom{10-a}{b}\binom{10-a-b}{c}\frac{1}{2^a}\frac{1}{4^b}\frac{1}{4^c} \\ &= \binom{10}{a, b, c}\frac{1}{4^{2a+b+c}}. \end{aligned}$$

(b)

$$P(B=b) = \binom{10}{b}\frac{1}{4^b}\left(\frac{3}{4}\right)^{10-b}.$$

4 points for each part.

2 or 3 points for each part answers that contains parts that are relevant and somewhat correct.

1 point for a good effort trial that is not correct.

0 for no attempt.