NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2015-2016

MH2500 – Probability and Introduction to Statistics

December 2015 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages including appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **FOUR** double-sided A4 size help sheets.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. The table with values of the standard normal distribution is included at the end of this examination paper.

QUESTION 1 **(10 marks)**

A professor, due to his inexperience, designed a confusing quiz question to determine whether students understood the lecture. For a student who understood the lecture, the probability of giving the correct answer to the quiz question is 0.75. On the other hand, for a student who did not understand the lecture, the probability of giving the correct answer is 0.1. It is estimated that 60% of the students understood the lecture and the remaining 40% did not.
Let U denote the event that a particular student understood the lecture. Let C denote the event that the student answered the question correctly. Find $P(U|C)$.

[Solution:] By Bayes Theorem,

$$\begin{aligned} P(U|C) &= \frac{P(C|U)P(U)}{P(C|U)P(U) + P(C|U^c)P(U^c)} \\ &= \frac{0.75 \times 0.6}{0.75 \times 0.6 + 0.1 \times 0.4} \\ &= \frac{45}{49} \approx 0.918. \end{aligned}$$

QUESTION 2 **(25 marks)**

Let $X_1, X_2, \ldots, X_4$ be four independent random variables each having the density function

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty; \\ 0, & x \le 0. \end{cases}$$

For $1 \le k \le 4$, let $X_{(k)}$ denote the k-th order statistic.

(a) Find the joint density function of $X_{(2)}$ and $X_{(3)}$.

(b) Let $Y = X_{(3)} - X_{(2)}$. Find the distribution of Y.

[Solution:] First, note that

$$F(x) = \int_0^x e^{-t} dt = [-e^{-t}]_0^x = 1 - e^{-x}.$$

(a) Let $U = X_{(3)}$ and $V = X_{(2)}$. Then

- $X_{(1)} \leq v$,
- $X_{(2)}$ lies in $[v, v + dv]$,
- $X_{(3)}$ lies in $[u, u + du]$,
- and $X_{(4)} \geq u$.

There are $4! = 24$ ways to choose the four variables. Hence

$$\begin{aligned} f(u,v) &= 24F(v)f(v)f(u)[1 - F(u)], \qquad 0 \leq v \leq u. \\ &= 24[1 - e^{-v}]e^{-v}e^{-u}e^{-u} \\ &= 24[1 - e^{-v}]e^{-v-2u}. \end{aligned}$$

Clearly $f(u,v) = 0$ otherwise.

(b) Either recall from lecture, or work out from definition, that for $Y = U - V$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(v + y, v) dv.$$

Hence

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} 12[1 - e^{-v}]e^{-3v-2y} dv \\ &= 24 \int_0^{\infty} e^{-3v-2y} - e^{-4v-2y} dv \\ &= 8e^{-2y} - 6e^{-2y} \\ &= 2e^{-2y}. \end{aligned}$$

Hence Y is exponential with parameter $\lambda = 2$.

QUESTION 3 **(25 marks)**

(a) Let X be a random variable with $0 \le X \le 1$ and $E(X) = \mu$. Show that:

(i) $0 \le \mu \le 1$;

(ii) $0 \le Var(X) \le \mu(1-\mu) \le \frac{1}{4}$. [Hint: Use $X^2 \le X$.]

(b) The result in Part (a) may be generalized as follows. Let X be a random variable with $a \le X \le b$ and $E(X) = \mu$. Show that:

(i) $a \le \mu \le b$;

(ii) $0 \le Var(X) \le (\mu - a)(b-\mu) \le \frac{1}{4}(b-a)^2$.

[Solution:]

(a) (i) Since $0 \le X \le 1$,

$$0 \;=\; \int_0^1 0dx \;\le\; \int_0^1 xf(x)dx \;=\; E(X) \;\le\; \int_0^1 f(x)dx \;=\; 1.$$

(ii) Since $0 \le X^2 \le X \le 1$, we see that $E(X^2) \le E(X) \le \mu$. Hence

$$Var(X) = E(X^2) - [E(X)]^2 \le \mu - \mu^2 = \mu(1-\mu).$$

By using calculus (or by completing the square), the maximum of a function $f(x) = x(1-x)$ occurs at $x = 1/2$. Hence $\mu(1-\mu) \le \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$.

(b) (i) Set $Y = \frac{X-a}{b-a}$. Then $0 \le Y \le 1$ and

$$\mu_Y = \frac{\mu_X - a}{b-a},$$

which implies that

$$\mu_X = a + (b-a)\mu_Y.$$

Hence $a \le \mu_X \le b$.

(ii)

$$Var(Y) = Var\left(\frac{X-a}{b-a}\right) = \frac{Var(X)}{(b-a)^2}.$$

By Part (a)(ii), this means

$$0 \le \frac{Var(X)}{(b-a)^2} \le \mu_Y(1-\mu_Y) \le \frac{1}{4}.$$

Hence,

$$0 \leq Var(X) \leq (b-a)^2 \mu_Y (1-\mu_Y) \leq \frac{1}{4}(b-a)^2.$$

It remains to see that

$$(\mu_X - a)(b - \mu_X) = (a + (b-a)\mu_Y - a)(b - a - (b-a)\mu_Y) = (b-a)^2 \mu_Y (1-\mu_Y)$$

and this completes Part (b)(ii).

QUESTION 4 **(25 marks)**

Suppose (X, Y) is uniform on the unit disk (centered at origin with radius 1).

(a) Find f_X and f_Y, the marginal density functions of X and Y, respectively.

(b) Show that X and Y are not independent.

(c) Prove that $Cov(X, Y) = 0$.

[Solution:]

(a) Clearly the joint density function is

$$f_{XY}(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}, \qquad (-1 \leq y \leq 1).$$

Similarly, $f_X(x) = \frac{2\sqrt{1-x^2}}{\pi}, \qquad (-1 \leq x \leq 1).$

(b) Clearly

$$f_X(x) \cdot f_Y(y) \neq f_{XY} = \frac{1}{\pi}.$$

Hence they are not independent.

(c)

$$E[X] = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{\pi} dy dx = \int_{-1}^{1} x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy dx = \int_{-1}^{1} \frac{2x\sqrt{1-x^2}}{\pi} dx = 0$$

Similarly, $E[Y] = 0$.

Next,

$$E[XY] = \int_{=1}^{1} x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y}{\pi} dy dx = \int_{=1}^{1} \frac{x}{\pi} 0 dx = 0.$$

Hence

$$Cov[X,Y] = E[XY] - E[X]E[Y] = 0.$$

QUESTION 5 **(15 marks)**

Suppose $X_1, X_2, \ldots, X_{30}$ are independent Poisson random variables with $\lambda = 2$. Use the central limit theorem to approximate

$$P\left(\sum_{i=1}^{30} X_i > 50\right).$$

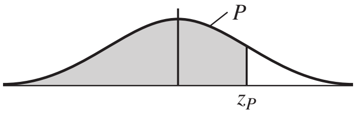
[Solution:] Since each X_i is Poisson with $\lambda = 2$, both the mean and variance are 2. Therefore,

$$Var\left(\sum_{i=1}^{30} X_i\right) = E\left(\sum_{i=1}^{30} X_i\right) = 60$$

By the CTL,

$$\begin{aligned} P\left(\sum_{i=1}^{30} X_i > 50\right) &\approx P\left(Z > \frac{60-50}{\sqrt{60}}\right) \qquad \text{where } Z \sim \text{N(0,1)} \\ &= P(Z > 1.291) \\ &= \Phi(1.29) \approx 0.9015. \end{aligned}$$

Cumulative Normal Distribution — Values of P corresponding to z_p for the Standard Normal Curve.



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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