NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 1

For the tutorial on 18 August, let us discuss

- Ex. 1.8.4, 6, 9, 17, 24, 29, 35, 38.

Ex. 1.8.4. Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

[Solution:]

There are many ways to present this. One way is to use induction on n. Here's another way. Recall that for any two sets A and B, we may express $A \cup B$ as a disjoint union $A \cup (A^c \cap B)$ and so

$$P(A \cup B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B) \leq P(A) + P(B),$$

where in the last inequality, we used the fact that $A^c \cap B \subseteq B$.

We apply this idea on a union of n sets. We express $\bigcup_{i=1}^{n} A_i$ as a disjoint union

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \cup \cdots \cup (A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \cap A_n).$$

Note that each set $A_1^c \cap A_2^c \cap \cdots \cap A_{k-1}^c \cap A_k \subseteq A_k$ for $k = 1, 2, \ldots, n$, and so

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \cap A_n) \\ &\leq P(A_1) + P(A_2) + \cdots + P(A_n). \end{aligned}$$

Ex. 1.8.6. modified

Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

a. List the sample space.

b. List the elements that make up the following events: (1) $A =$ the sum of the two values is at least five but less than 8, (2) $B =$ the value of the first die is higher than the value of the second, (3) $C =$ the first value is 4.

c. List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.

[Solution:]

a. The sample space is

$$\Omega = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), \ldots, (6,1), (6,2), \ldots, (6,6)\},$$

b.

$A = \{(1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (3,3), (3,4), (4,1), (4,2), (4,3), (5,1), (5,2), (6,1)\}$,

$B = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$,

$C = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$.

c. List the elements of the following events:

$$\begin{aligned}
A \cap C &= \{(4,1),(4,2),(4,3)\},\\
B \cup C &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),\\
&\qquad (5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5)\},\\
A \cap (B \cup C) &= \{(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(6,1)\}.
\end{aligned}$$

Ex. 1.8.9. The weather forecaster says that the probability of rain on Saturday is 25% and that the probability of rain on Sunday is 25%. Is the probability of rain during the weekend 50%? Why or why not?

[Solution:]

Let A be the event that it rains on Saturday and B be the event that it rains on Sunday. Then the event that it rains during the weekend can be denoted by $A \cup B$ and

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.25 - P(A \cap B) = 0.5 - P(A \cap B).$$

So the probability of rain during the weekend is not 50%, (unless $P(A \cap B) = 0$, i.e., the probability of rain on both Saturday and Sunday is 0, which is unlikely the case.)

Ex. 1.8.17. In acceptance sampling, a purchaser samples 4 items from a lot of 100 and rejects the lot if 1 or more are defective. Graph the probability that the lot is accepted as a function of the percentage of defective items in the lot. (A sketch will do.)

[Solution:]

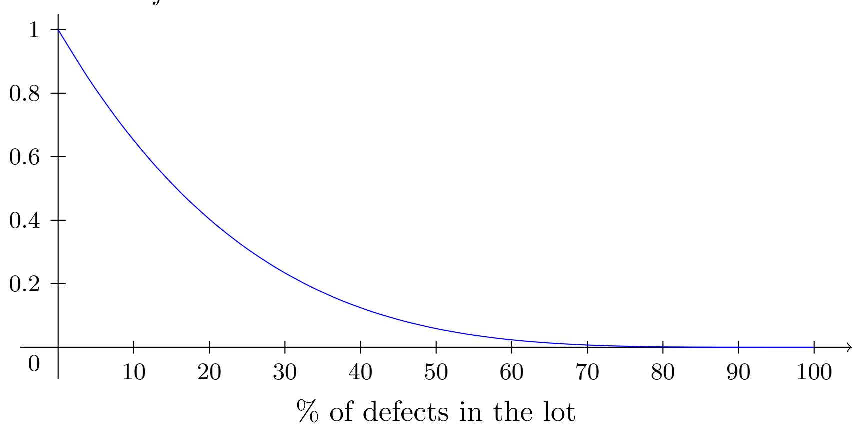
Let $f : [0, 100] \to [0, 1]$ be the desired function, i.e.,

$$f(x) = \text{probability that a lot with } x\% \text{ defective items is accepted.}$$

Given a lot with x defective items, if the lot is accepted, it means that all 4 items sampled are not defective, and the probability that all 4 items sampled are not defective is

$$\begin{aligned}
f(x) &= \frac{\binom{100-x}{4}}{\binom{100}{4}} = \frac{(100-x)(99-x)(98-x)(97-x)}{100 \cdot 99 \cdot 98 \cdot 97}\\
&= \frac{1}{100 \cdot 99 \cdot 98 \cdot 97}(x-100)(x-99)(x-98)(x-97).
\end{aligned}$$

We recognize that this is a degree 4 polynomial with positive leading term. The following is a sketch of the function f.



Ex. 1.8.24. A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

[Solution:] Let A be the event that the four aces are all next to each other. To count the number of ways A could occur, we may first pretend the four aces as one card and then count the number of rearrangement of 49 cards, and for each such rearrangement, we consider the rearrangement amongst the four aces. Thus

$$\begin{aligned} P(A) &= \frac{\text{number of ways of } A \text{ could occur}}{\text{total number of rearrangements}} \\ &= \frac{49! \times 4!}{52!} \\ &= \frac{24}{52 \times 51 \times 50} = \frac{1}{5525} \approx 0.000181. \end{aligned}$$

Ex. 1.8.29. A poker player is dealt three spades and two hearts. He discards the two hearts and draws two more cards. What is the probability that he draws two more spades?

[Solution:]
The poker player is drawing two cards from the remaining deck of 47 cards, of which 10 are spades. Thus the probability is

$$\frac{\binom{10}{2}}{\binom{47}{2}} = \frac{45}{1081} \approx 0.0416.$$

Ex. 1.8.35.b Prove the following identity both algebraically and by interpreting their meaning combinatorially.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

[Solution:]
Algebraically:

$$\begin{aligned} \binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{(n-1)!}{r!(n-r)!}(r+n-r) \\ &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r}. \end{aligned}$$

Combinatorially: We can count the number of ways of choosing r objects from n objects without replacement by fixing on one particular object and then considering whether it is chosen or not.

Case 1, the object is chosen, and so it remains to choose another $r-1$ objects from the remaining $n-1$ objects. The number of ways this can occur is $\binom{1}{1}\binom{n-1}{r-1} = \binom{n-1}{r-1}$.

Case 2, the object is not chosen, and so we shall choose r objects from the remaining $n-1$ objects. The number of ways this can occur is $\binom{1}{0}\binom{n-1}{r} = \binom{n-1}{r}$.

The sum of number of ways in both cases is equal to the total number of ways in choosing r objects from n objects, in other words,

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}.$$

Ex. 1.8.38. What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$?

[Solution:]

We can think of the expansion of $(x+y)^7$ as choosing either an x or a y from each of the seven factors $(x+y)$ and multiplying them. E.g. $x \times x \times y \times y \times y \times y \times x$

$$(x+y)\ (x+y)\ (x+y)\ (x+y)\ (x+y)\ (x+y)\ (x+y)$$

Note that we can only choose either x or y in the factor $(x+y)$ but not both. Thus the coefficient of x^3y^4 can be counted as the number of ways of choosing 3 x's from the seven factors of $(x+y)$, multiplied by choosing 4 y's from the remaining 4 factors of $(x+y)$. Thus the coefficient of x^3y^4 is

$$\binom{7}{3}\binom{4}{4} = 35.$$