Name: ______________________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2016/17

MH2500– Probability and Introduction to Statistics

20 September 2016 Test 2 40 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed ONE double-sided A4 size paper as cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(10 marks)**

(a) Suppose X and Y are discrete random variables each taking values 1, 2, and 3. Their joint frequency function is given as follows.

$$p(x,y) = \begin{cases} \frac{1}{8}, & \text{if } 1 \le x \le y \le 3; \\ \frac{1}{12}, & \text{if } 1 \le y < x \le 3. \end{cases}$$

(i) Find $p_Y(2)$. Leave your answer as a single fraction.

(ii) Find $p_{X|Y}(2|2)$. Leave your answer as a single fraction.

[Answer:]

(i)

$$p_Y(2) = \sum_{i=1}^{3} p(i,2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{1}{3}.$$

(ii)

$$p_{X|Y}(2|2) = \frac{p(2,2)}{p_Y(2)} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}.$$

(b) Suppose X and Y are continuous random variables with joint cdf

$$F(x,y) = \frac{1}{2}(x^2y + xy^2), \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Find the marginal density of X.

[Answer:]

$$F_X(x) = \lim_{y\to\infty} F(x,y) = \lim_{y\to 1} \frac{1}{2}(x^2y + xy^2) = \frac{1}{2}(x^2 + x).$$

Therefore, the marginal density is

$$f_X(x) = F'(x) = x + \frac{1}{2}.$$

– 5 marks each for parts (a) and (b).

QUESTION 2. **(8 marks)**

The number of traffic accidents at a road junction each day follows a Poisson distribution with parameter $\lambda = 0.5$.

(a) Find the probability that there are no accidents on a certain day.

(b) Suppose ten days are randomly selected. Find the probability that there is at least one day where no accidents occurred.

Give your answers correct to three significant figures.

[Answer:]

a. Let X denote the number of accidents in 1 day. Then X is Poisson with $\lambda = 0.5$. Then

$$P(X = 0) \;=\; \frac{0.5^0}{0!}e^{-0.5} \approx 0.607.$$

b. Let Y be the number of days out of the ten selected days where there are no accidents. Then Y is binomial with parameters $n = 10$ and $p = 0.607$. Therefore, the required probability is

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{10}{0}(0.607)^0(1 - 0.607)^{10} \approx 1.00$$

– 4 marks for each part.

QUESTION 3. **(8 marks)**

For some $a > 0$, let

$$f(x) = \begin{cases} \dfrac{a}{x^2}, & \text{if } x > a; \\ 0, & x \leq a. \end{cases}$$

(i) Show that f is a density.

(ii) Find the corresponding cdf and the median.

[Answer:]

(i)

$$\int_{=\infty}^{\infty} f(x)dx = \int_a^{\infty} \frac{a}{x^2}dx = \left[-\frac{a}{x}\right]_a^{\infty} = 1.$$

Hence f is a density.

(ii)

$$F(x) = \int_a^x \frac{a}{t^2}dt = \left[-\frac{a}{t}\right]_a^x = 1 - \frac{a}{x}, \qquad (x \geq a).$$

$$F(x) = \frac{1}{2} \qquad \Longleftrightarrow \qquad 1 - \frac{a}{x} = \frac{1}{2} \qquad \Longleftrightarrow \qquad \frac{a}{x} = \frac{1}{2}.$$

Hence $x = 2a$.

– 4 marks for each part.

QUESTION 4. **(8 marks)**

Let U be a uniform random variable on $[0,1]$. Find the density of $V = e^{-3U}$.

[Answer]

$$
\begin{aligned}
P(V \le v) &= P(e^{-3U} \le v) \\
&= P(-3U \le \ln v) \\
&= P\left(U \ge -\frac{1}{3}\ln v\right) \\
&= 1 + \frac{1}{3}\ln v.
\end{aligned}
$$

Thus, for $v \in [e^{-3}, 1]$,

$$
\begin{aligned}
f_V(v) &= \frac{d}{dv}\left(1 + \frac{1}{3}\ln v\right) \\
&= \frac{1}{3v}.
\end{aligned}
$$

Therefore,

$$
f_V(v) = \begin{cases} \dfrac{1}{3v}, & v \in [e^{-3}, 1]; \\ 0, & \text{otherwise.} \end{cases}
$$