NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2016/17 Semester 1 MH2500 Probability and Introduction to Statistics Tutorial 2

As 25 August is union day, tutorials for groups T1–T8 are cancelled. Please work on these problems on your own and refer to the solution file that will be uploaded on NTULearn. For tutorial groups T9–T11, classes will be as usual.

- Ex. 1.8.52, 56, 60, 64, 68, 72.

Ex. 1.8.52. A couple has two children. What is the probability that both are girls given that the oldest is a girl? What is the probability that both are girls given that one of them is a girl?

[Solution:] Let A denote the event that both are children are girls, B denote the event that the oldest is a girl, and C denote the event that at least one of them is a girl.

Then

$$P(A) = \frac{1}{4}, \qquad P(B) = \frac{1}{2}, \qquad P(C) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Note that $A \cap B = A$ and so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

Similarly, $A \cap C = A$ and so

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Ex. 1.8.56. (modified) Suppose that 5 cards are dealt from a 52-card deck.

(i) What is the probability of at least two kings given the first card one is a king?
(ii) What is the probability of at least two kings given there is at least one king?

[Solution:]

(i) After the first card is dealt, there are 51 cards remaining with 3 kings in it. Let A be the event that there is at least one king in the 2nd to 5-th card. Then

$$P(A) = 1 - P(A^c) = 1 - \frac{\binom{48}{4}}{\binom{51}{4}} = \frac{922}{4165} \approx 0.221.$$

(ii) Let B be the event that there are at least 2 kings and let C be the event that there is at least 1 king.

$$P(C) = 1 - P(C^c) = 1 - \frac{\binom{48}{5}}{\binom{52}{5}} = \frac{18472}{54145} \approx 0.341,$$
$$P(B) = 1 - P(B^c) = 1 - \frac{\binom{48}{5} + \binom{4}{1}\binom{48}{4}}{\binom{52}{5}} = \frac{2257}{54145} \approx 0.0417,$$
$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{P(B)}{P(C)} = \frac{2257}{18472} \approx 0.122.$$

Ex. 1.8.60. A factory runs three shifts. In a given day, 1% of the items produced by the first shift are defective, 2% of the second shift's items are defective, and 5% of the third shift's items are defective.

(i) If the shifts all have the same productivity, what percentage of the items produced in a day are defective?
(ii) If an item is defective, what is the probability that it was produced by the third shift?

[Solution:]

(i) Percentage of items produced that are defective $= \dfrac{1+2+5}{3}\% \; = \; 2.67\%$
(ii) Probability the third shift produced a defective item given that a defective item is produced $= \dfrac{0.05}{0.01+0.02+0.05} = \dfrac{5}{8} = 0.625.$

Alternative presentation: Let A denote the event that an item is defective and let B denote the event that the third shift produced the item. then

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{5\% \times \frac{1}{3}}{2.67\%} \qquad \text{(one out of three shifts)} \\ &= 0.625. \end{aligned}$$

Ex. 1.8.64. If B is an event with $P(B) > 0$, show that the set function $Q(A) = P(A|B)$ satisfies the axioms for a probability measure. Thus, for example,

$$P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B).$$

[Solution:] Clearly $Q(\Omega) = P(\Omega \cap B)/P(B) = 1$ and for any $A \subseteq \Omega$, clearly $Q(A) = P(A \cap B)/P(B) \geq 0$. Hence the first two axioms are satisfied.

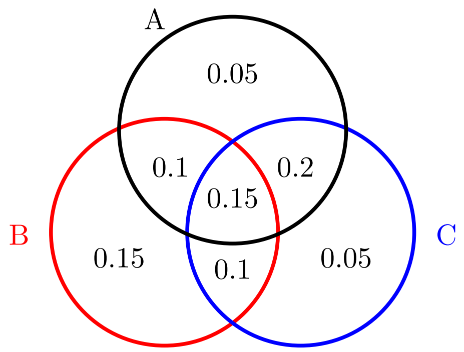
Supposed $A_1, A_2, \ldots, A_n, \ldots$ are mutually disjoint, then

$$\begin{aligned} Q\left(\bigcup_{i=1}^{\infty} A_i\right) &= \frac{1}{P(B)} P\left(B \cap \bigcup_{i=1}^{\infty} A_i\right) \\ &= \frac{1}{P(B)} P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right) \\ &= \frac{1}{P(B)} \sum_{i=1}^{\infty} P(A_i \cap B) \\ &= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} \\ &= \sum_{i=1}^{\infty} Q(A_i), \end{aligned}$$

and so the third axiom is also satisfied.

Ex. 1.8.68. If A is independent of B and B is independent of C, then A is independent of C. Prove this statement or give a counterexample if it is false.

[Solution:] False. Here's a counterexample. Let events A, B, and C have probabilities given in the vein diagram.



Then $P(A) = P(B) = P(C) = 0.5$, and

$$P(A \cap B) = 0.25 = P(A)P(B) \qquad P(B \cap C) = 0.25 = P(B)P(C),$$

which implies that A is independent of B and B is independent of C. However, $P(A \cap C) = 0.35 \neq P(A)P(C)$.

Another example. We flip a coin twice. Then the sample space is

$$\Omega = \{hh, ht, th, tt\},$$

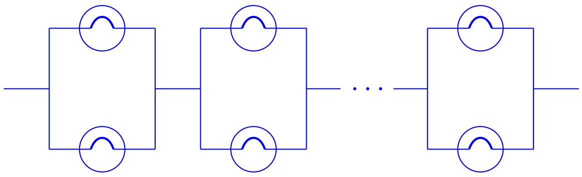
where h denotes head and t denotes tail. Let both A and C denote the same event that the first flip is a head, let B be the event that the second flip is a head. Then

$$A = C = \{ht, hh\} \qquad \text{and} \qquad B = \{th, hh\}.$$

Then $P(A \cap B) = P(B \cap C) = \frac{1}{4} = P(A)P(B)$ but $P(A \cap C) = P(A) = \frac{1}{2} \neq \frac{1}{2} \times \frac{1}{2} = P(A)P(C)$.

Ex. 1.8.72. Suppose that n components are connected in series. For each unit, there is a backup unit and the system fails if and only if both a unit and its backup fail. Assuming that all the units are independent and fail with probability p, what is the probability that the system works?

[Solution:]
A simple way to understand this system is to think of each unit and backup unit as a pair of light bulbs that where each bulb fails with probability p, and the pairs are connected in series as illustrated. Then the system "works" if for each pair of light bulbs, at least one of them lights up.



Let A be the event that the system works. The probability that a unit and its backup both fail is p^2, and so the probability that either a unit or its backup works is $1 - p^2$. Thus

$$P(A) = (1 - p^2)^n.$$