Name: ______________________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2016/17

MH2500– Probability and Introduction to Statistics

30 August 2016 Test 1 40 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed one side of an A4 size paper as cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(8 marks)**

(a) Suppose X is a geometric random variable with $p = 0.3$. Find $P(X = 5)$. Give your answer to three significant figures.

[Answer:]

$$P(X = 5) = (1 - p)^4 p = 0.7^4 \times 0.3 \approx 0.0720.$$

3 marks for up to $0.7^4 \times 0.3$ and 1 mark for the calculation.

(b) Let Y be the uniform random variable on $[3, 5]$. Give the cumulative distribution function of Y.

[Answer:]

For $y \in [3, 5]$,

$$F(y) = \int_3^y \frac{1}{5-3} dy = \left[\frac{y}{2}\right]_3^y = \frac{y-3}{2}.$$

Therefore,

$$F(y) = \begin{cases} 1, & y > 5; \\ \frac{y-3}{2}, & 3 \le y \le 5; \\ 0, & y < 3. \end{cases}$$

3 marks for $F(y) = \frac{y-3}{2}$ and 1 mark for the rest.

QUESTION 2. **(10 marks)**

(a) A deck of 52 cards is shuffled thoroughly and five cards are drawn. What is the probability that the five cards consist of two spades and three hearts and there are no kings?

5 marks each.

[Answer:]

Let A be the event that the five cards drawn consist of two spades and three hearts and no kings. There are 13 spades and 13 hearts in a deck of cards. Excluding the kings, there are 12 spades and 12 hearts. Thus

$$\begin{aligned} P(A) &= \frac{\binom{12}{3}\binom{12}{2}}{\binom{52}{5}} \\ &= \frac{121}{21658} \\ &\approx 0.00559 \end{aligned}$$

(b) There are n students in a room where n is an integer greater than 1. Write down in terms of n, the probability that at most one of them is born on 30 August. (For simplicity, assume there are 365 days in a year.)

[Answer:]

Let A be the event that at most one of the n students is born on 30 August, let B be the event that none of the students are born on 30 August, and C be the event that exactly one of the n students is born on 30 August. Then A is the disjoint union of B and C. Thus

$$\begin{aligned} P(A) &= P(B) + P(C) \\ &= \frac{364^n}{365^n} + \frac{\binom{n}{1}364^{n-1}}{365^n}. \end{aligned}$$

QUESTION 3. **(8 marks)**

A university conducted a survey on students' overall satisfaction level, and students are to answer whether they are "satisfied", "neutral", or "dissatisfied". Based on the survey results, 70% of the students who did the survey answered "satisfied", 10% of the students answered "neutral", and 20% of the students answered "dissatisfied". We are told that the proportion of Mathematics majors amongst those students who answered "satisfied", "neutral", and "dissatisfied" are 25%, 10%, and 5%, respectively.

(i) Find the probability that a randomly selected student is a Mathematics major.

(ii) Find the probability that a randomly selected student answered "satisfied" if we know that he is a Mathematics major.

Give your answer correct to three significant figures.

4 marks each

[Answer:]

(i) Let S, N, and D denote the event that a randomly selected student answered "satisfied", "neutral", and "dissatisfied", respectively. Let M denote that a randomly selected student is a Mathematics major. By the law of total probability,

$$\begin{aligned} P(M) &= P(M|S)P(S) + P(M|N)P(N) + P(M|D)P(D) \\ &= 0.25 \times 0.7 + 0.1 \times 0.1 + 0.2 \times 0.05 \\ &= 0.195. \end{aligned}$$

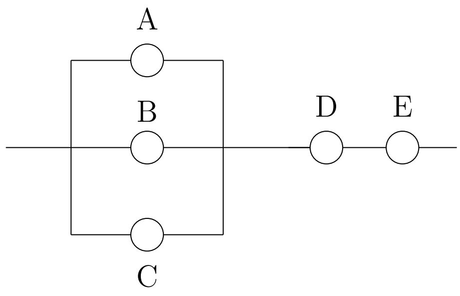
(ii)

$$P(S|M) = \frac{P(S \cap M)}{P(M)} = \frac{0.25 \times 0.7}{0.195} = \frac{35}{39} \approx 0.897.$$

QUESTION 4. **(8 marks)**

A system consists of five components, A, B, C, D, and E, connected in the way illustrated below. Assume units A, B, and C each fails with probability 0.1, and units D, E each fails with probability 0.2 and that all five units fail independently. What is the probability that the system works? Leave your answer correct to 3 significant figures.

Remark: The system works if all three conditions below are satisfied.
(1) at least one of A, B, C works,
(2) D works,
(3) E works.



[Answer:]

$$P(\text{at least one of A, B, C works}) = 1 - P(\text{all A, B, C fails}) = 1 - 0.1^3 = 0.999.$$

Thus

$$\begin{aligned} P(\text{system works}) &= P(\text{at least one of A, B, C works })P(\text{D works})P(\text{E works}) \\ &= (1 - 0.999)(1 - 0.2)(1 - 0.2) \\ &= 0.639. \end{aligned}$$

Name: ______________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2016/17

MH2500– Probability and Introduction to Statistics

20 September 2016 Test 2 40 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed ONE double-sided A4 size paper as cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(10 marks)**

(a) Suppose X and Y are discrete random variables each taking values 1, 2, and 3. Their joint frequency function is given as follows.

$$p(x,y) = \begin{cases} \frac{1}{8}, & \text{if } 1 \le x \le y \le 3; \\ \frac{1}{12}, & \text{if } 1 \le y < x \le 3. \end{cases}$$

(i) Find $p_Y(2)$. Leave your answer as a single fraction.

(ii) Find $p_{X|Y}(2|2)$. Leave your answer as a single fraction.

[Answer:]

(i)

$$p_Y(2) = \sum_{i=1}^{3} p(i,2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{1}{3}.$$

(ii)

$$p_{X|Y}(2|2) = \frac{p(2,2)}{p_Y(2)} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}.$$

(b) Suppose X and Y are continuous random variables with joint cdf

$$F(x,y) = \frac{1}{2}(x^2 y + xy^2), \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Find the marginal density of X.

[Answer:]

$$F_X(x) = \lim_{y\to\infty} F(x,y) = \lim_{y\to 1} \frac{1}{2}(x^2 y + xy^2) = \frac{1}{2}(x^2 + x).$$

Therefore, the marginal density is

$$f_X(x) = F'(x) = x + \frac{1}{2}.$$

– 5 marks each for parts (a) and (b).

QUESTION 2. **(8 marks)**

The number of traffic accidents at a road junction each day follows a Poisson distribution with parameter $\lambda = 0.5$.

(a) Find the probability that there are no accidents on a certain day.

(b) Suppose ten days are randomly selected. Find the probability that there is at least one day where no accidents occurred.

Give your answers correct to three significant figures.

[Answer:]

a. Let X denote the number of accidents in 1 day. Then X is Poisson with $\lambda = 0.5$. Then

$$P(X = 0) \;=\; \frac{0.5^0}{0!}e^{-0.5} \approx 0.607.$$

b. Let Y be the number of days out of the ten selected days where there are no accidents. Then Y is binomial with parameters $n = 10$ and $p = 0.607$. Therefore, the required probability is

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{10}{0}(0.607)^0(1 - 0.607)^{10} \approx 1.00$$

– 4 marks for each part.

QUESTION 3. **(8 marks)**

For some $a > 0$, let

$$f(x) = \begin{cases} \dfrac{a}{x^2}, & \text{if } x > a; \\ 0, & x \leq a. \end{cases}$$

(i) Show that f is a density.

(ii) Find the corresponding cdf and the median.

[Answer:]

(i)

$$\int_{=\infty}^{\infty} f(x)dx = \int_a^{\infty} \frac{a}{x^2}dx = \left[-\frac{a}{x}\right]_a^{\infty} = 1.$$

Hence f is a density.

(ii)

$$F(x) = \int_a^x \frac{a}{t^2}dt = \left[-\frac{a}{t}\right]_a^x = 1 - \frac{a}{x}, \qquad (x \geq a).$$

$$F(x) = \frac{1}{2} \qquad \Longleftrightarrow \qquad 1 - \frac{a}{x} = \frac{1}{2} \qquad \Longleftrightarrow \qquad \frac{a}{x} = \frac{1}{2}.$$

Hence $x = 2a$.

– 4 marks for each part.

QUESTION 4. **(8 marks)**

Let U be a uniform random variable on $[0, 1]$. Find the density of $V = e^{-3U}$.

[Answer]

$$\begin{aligned} P(V \leq v) &= P(e^{-3U} \leq v) \\ &= P(-3U \leq \ln v) \\ &= P\left(U \geq -\frac{1}{3}\ln v\right) \\ &= 1 + \frac{1}{3}\ln v. \end{aligned}$$

Thus, for $v \in [e^{-3}, 1]$,

$$\begin{aligned} f_V(v) &= \frac{d}{dv}\left(1 + \frac{1}{3}\ln v\right) \\ &= \frac{1}{3v}. \end{aligned}$$

Therefore,

$$f_V(v) = \begin{cases} \dfrac{1}{3v}, & v \in [e^{-3}, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Name: ______________________________ Tutorial group: T1

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2016/17

MH2500– Probability and Introduction to Statistics

18 October 2016 Test 3 40 minutes

<u>INSTRUCTIONS</u>

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.
5. You are allowed three sides of an A4 size paper as cheat sheet.

For graders only	Question	1	2	3	4	Bonus	Total
	Marks						

QUESTION 1. **(8 marks)**

If X and Y have the joint density function

$$f(x,y) = \begin{cases} x + \frac{5}{2}y, & 0 \le x \le y \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Z = Y - X$, find $P(Z \ge \frac{1}{2})$. Leave your answer as a fraction or to three significant figures.

[Answer:]

- Method 1

  We first find the region R which is the intersection of $0 \le x \le y \le 1$ and $y - x \ge \frac{1}{2}$. R is a triangle formed by $(0, \frac{1}{2})$, $(0, 1)$, and $(\frac{1}{2}, 1)$. The probability $P(Z \ge \frac{1}{2})$ is the integration of the density function $f(x, y)$ over region R,

$$\begin{aligned} P(Z \ge \frac{1}{2}) &= \iint_R f(x,y)dxdy = \int_{\frac{1}{2}}^{1} \int_0^{y-\frac{1}{2}} (x + \frac{5}{2}y)dxdy \\ &= \int_{\frac{1}{2}}^{1} \left(\frac{x^2}{2} + \frac{5}{2}xy \right) \Bigg|_{x=0}^{x=y-\frac{1}{2}} dy = \int_{\frac{1}{2}}^{1} \left(3y^2 - \frac{7}{4}y + \frac{1}{8} \right) dy \\ &= \left(y^3 - \frac{7}{8}y^2 + \frac{1}{8}y \right) \Bigg|_{y=\frac{1}{2}}^{y=1} = \frac{1}{4} - \frac{-1}{32} = \frac{9}{32} (= 0.28125) \end{aligned}$$

- Method 2

  Then density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, x+z)dx = \begin{cases} \int_0^{1-z} \left[x + \frac{5}{2}(x+z) \right] dx, & 0 \le z \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

  further

$$f_Z(z) = \begin{cases} -\frac{3}{4}z^2 - z + \frac{7}{4}, & 0 \le z \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

  So that

$$P(Z \ge \frac{1}{2}) = \int_{\frac{1}{2}}^{1} [-\frac{3}{4}z^2 - z + \frac{7}{4}]dz = \frac{9}{32} (= 0.28125).$$

QUESTION 2. **(10 marks)**

If X and Y have the joint density function

$$f(x,y) = \begin{cases} x+4y, & 0 \le y \le x \le 1; \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities of X and Y.

(b) Are X and Y independent?

(c) Find the conditional density of Y given X.

(d) Find the joint cumulative distribution function of X and Y.

[Answer:]

a.

$$f_X(x) = \int_0^x (x+4y)dy = (xy+y^2)\Big|_{y=0}^{y=x} = 3x^2, \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_y^1 (x+4y)dx = \left(\frac{1}{2}x^2 + 4xy\right)\Big|_{x=y}^{x=1} = -\frac{9}{2}y^2 + 4y + \frac{1}{2}, \qquad 0 \le y \le 1.$$

b. Since $f(x,y) = x+4y \neq (3x^2) \times (-\frac{9}{2}y^2 + 4y + \frac{1}{2}) = f_X(x) f_Y(y)$, X and Y are not independent.

c.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+4y}{3x^2}, \qquad 0 \le y \le x \le 1; \quad x \neq 0.$$

d. The joint CDF of X and Y is defined as $F(u,v) = P(x \le u, y \le v)$

- When $0 \le v \le u \le 1$,

$$F(u,v) = \int_0^v \int_y^u (x+4y)dxdy = \int_0^v \left(\frac{1}{2}x^2 + 4xy\right)\Big|_{x=y}^{x=u} dy$$

$$= \int_0^v \left(-\frac{9}{2}y^2 + 4uy + \frac{u^2}{2}\right) dy = \left[-\frac{3}{2}y^3 + 2uy^2 + \frac{u^2}{2}y\right]\Big|_{y=0}^{y=v}$$

$$= -\frac{3}{2}v^3 + 2uv^2 + \frac{u^2 v}{2}, \qquad 0 \le v \le u \le 1.$$

Changing the variables back to x and y, we have the CDF as

$$F(x,y) = -\frac{3}{2}y^3 + 2xy^2 + \frac{x^2 y}{2}, \qquad 0 \le y \le x \le 1.$$

- When $0 \le u \le 1$ and $v \ge u$,

$$F(u,v) = \int_0^u \int_y^u (x+4y)dxdy = \int_0^u \left(\frac{1}{2}x^2 + 4xy\right)\Big|_{x=y}^{x=1} dy$$

$$= \int_0^u \left(-\frac{9}{2}y^2 + 4uy + \frac{u^2}{2}\right) dy = \left[-\frac{3}{2}y^3 + 2uy^2 + \frac{u^2}{2}y\right]\Big|_{y=0}^{y=u}$$

$$= u^3, \qquad 0 \le u \le 1.$$

Changing the variables back to x and y, we have the CDF as

$$F(x,y) = x^3, \qquad 0 \le x \le 1; y \ge x.$$

- When $0 \le v \le 1$ and $u \ge 1$

$$F(u,v) = \int_0^v \int_y^1 (x+4y)dxdy = \int_0^v \left(\frac{1}{2}x^2 + 4xy\right)\Big|_{x=y}^{x=1} dy$$

$$= \int_0^v \left(-\frac{9}{2}y^2 + 4y + \frac{1}{2}\right) dy = \left[-\frac{3}{2}y^3 + 2y^2 + \frac{1}{2}y\right]\Big|_{y=0}^{y=v}$$

$$= -\frac{3}{2}v^3 + 2v^2 + \frac{v}{2}, \qquad 0 \le v \le 1.$$

Changing the variables back to x and y, we have the CDF as

$$F(x,y) = -\frac{3}{2}y^3 + 2y^2 + \frac{y}{2}, \qquad 0 \le y \le 1; x \ge 1.$$

- When $v \ge 1$ and $u \ge 1$, $F(u,v) = 1$, so

$$F(x,y) = 1.$$

- When $v \le 0$ or $u \le 0$, $F(u,v) = 1$, then

$$F(x,y) = 0.$$

QUESTION 3. **(8 marks)**

If X_1, X_2, and X_3 are independent random variables, each with the same exponential density function $f(x)$,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Find the joint density of $X_{(1)}$ and $X_{(3)}$. Note that $X_{(1)} \leq X_{(2)} \leq X_{(3)}$.

[Answer:] Let $g(v, u)$ be the joint density function of $X_{(1)}$ and $X_{(3)}$. Using the differential argument, we can have

$$g(v, u)dvdu = \binom{3}{1} f(v)dv \binom{2}{1} f(u)du \binom{1}{1} [F(u) - F(v)],$$

and therefore $g(v, u) = 6f(v)f(u)[F(u) - F(v)]$. Note that $F(x)$ is the cumulative distribution function of the exponential random variables X_i's. $F(x)$ takes the following form,

$$F(x) = \int_0^x f(\tau)d\tau = \int_0^x \lambda e^{-\lambda\tau} d\tau = [-e^{-\lambda\tau}]|_{\tau=0}^{\tau=x} = 1 - e^{-\lambda x}, \quad x \geq 0.$$

Explicitly, $g(v, u)$ can be written as

$$g(v, u) = 6f(v)f(u)[F(u) - F(v)] = 6\lambda^2 e^{-\lambda(v+u)}[e^{-\lambda v} - e^{-\lambda u}], \quad 0 \leq v \leq u.$$

Equivalently,

$$g(v, u) = 6\lambda^2[e^{-\lambda(2v+u)} - e^{-\lambda(v+2u)}], \quad 0 \leq v \leq u.$$

QUESTION 4. **(8 marks)**

X is a binomial random variable with parameters n and p.

(a) Find E(X).

(b) Let $Y = X(X-1)$, find E(Y).

(c) Let $Z = X^2$, find E(Z). (Hint: Z=X+Y.)

(d) Find Var(X).

[Answer]

(a) The binomial distribution can be viewed as a linear combination of n independent Bernoulli distributions. The expected value of a Bernoulli random variable with parameter p is p. So $E(X) = np$.

We can also work with the definition of expectation. For a binomial random variable X with parameters n and p, $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$. Hence

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}, \end{aligned}$$

and let $j = k - 1$

$$\begin{aligned} E(X) &= \sum_{j=0}^{n-1} \frac{n!}{j!(n-1-j)!} p^{j+1} (1-p)^{n-1-j} \\ &= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j} \end{aligned}$$

$\frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j}$ is the probability of a binomal random variable with parameters $n-1$ and p and j successes. So, the sum from 0 to $n-1$ is 1. This gives $E(X) = np$.

(b) If we once again view a binomal distribution as a linear combination of n Bernoulli random variables, Var(X) should be easily obtained as Var(X) $= np(1-p)$, given the fact that the variance of a Bernoulli random variable is $p(1-p)$. Further, $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$, we have $E(X^2) = \mathrm{Var}(X) + [E(X)]^2 = np(1-p) + n^2p^2$. Therefore $E[X(X-1)] = E(X^2) - E(X) = np(1-p) + n^2p^2 - np = n(n-1)p^2$.

Alternatively, we can use the definition of expectation to find $E[X(X-1)]$ as follows,

$$
\begin{aligned}
E[X(X-1)] &= \sum_{k=0}^{n} k(k-1)\binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k},
\end{aligned}
$$

and let $j = k-2$

$$
\begin{aligned}
E[X(X-1)] &= \sum_{j=0}^{n-2} \frac{n!}{j!(n-2-j)!} p^{j+2} (1-p)^{n-2-j} \\
&= n(n-1)p^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} p^j (1-p)^{n-2-j}
\end{aligned}
$$

$\frac{(n-2)!}{j!(n-2-j)!} p^j (1-p)^{n-2-j}$ is the probability of a binomal random variable with parameters $n-2$ and p and j successes. So, the sum from 0 to $n-2$ is 1. This gives $E[X(X-1)] = n(n-1)p^2$.

(c) If the binomial distribution is considered as a linear combination of n independent Bernoulli distrubtions, we have already solved all the problems of Question 4. Otherwise, $E(X^2) = E[X(X-1)+X] = E[X(X-1)] + E(X) = n(n-1)p^2 + np$.

(d) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$.

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2015-2016

MH2500 – Probability and Introduction to Statistics

December 2015 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages including appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **FOUR** double-sided A4 size help sheets.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. The table with values of the standard normal distribution is included at the end of this examination paper.

QUESTION 1 **(10 marks)**

A professor, due to his inexperience, designed a confusing quiz question to determine whether students understood the lecture. For a student who understood the lecture, the probability of giving the correct answer to the quiz question is 0.75. On the other hand, for a student who did not understand the lecture, the probability of giving the correct answer is 0.1. It is estimated that 60% of the students understood the lecture and the remaining 40% did not.
Let U denote the event that a particular student understood the lecture. Let C denote the event that the student answered the question correctly. Find $P(U|C)$.

[Solution:] By Bayes Theorem,

$$\begin{aligned} P(U|C) &= \frac{P(C|U)P(U)}{P(C|U)P(U) + P(C|U^c)P(U^c)} \\ &= \frac{0.75 \times 0.6}{0.75 \times 0.6 + 0.1 \times 0.4} \\ &= \frac{45}{49} \approx 0.918. \end{aligned}$$

QUESTION 2 **(25 marks)**

Let $X_1, X_2, \ldots, X_4$ be four independent random variables each having the density function

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty; \\ 0, & x \leq 0. \end{cases}$$

For $1 \leq k \leq 4$, let $X_{(k)}$ denote the k-th order statistic.

(a) Find the joint density function of $X_{(2)}$ and $X_{(3)}$.

(b) Let $Y = X_{(3)} - X_{(2)}$. Find the distribution of Y.

[Solution:] First, note that

$$F(x) = \int_0^x e^{-t} dt = [-e^{-t}]_0^x = 1 - e^{-x}.$$

(a) Let $U = X_{(3)}$ and $V = X_{(2)}$. Then

- $X_{(1)} \leq v$,
- $X_{(2)}$ lies in $[v, v + dv]$,
- $X_{(3)}$ lies in $[u, u + du]$,
- and $X_{(4)} \geq u$.

There are $4! = 24$ ways to choose the four variables. Hence

$$\begin{aligned} f(u,v) &= 24F(v)f(v)f(u)[1 - F(u)], \qquad 0 \leq v \leq u. \\ &= 24[1 - e^{-v}]e^{-v}e^{-u}e^{-u} \\ &= 24[1 - e^{-v}]e^{-v-2u}. \end{aligned}$$

Clearly $f(u,v) = 0$ otherwise.

(b) Either recall from lecture, or work out from definition, that for $Y = U - V$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(v + y, v)dv.$$

Hence

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} 12[1 - e^{-v}]e^{-3v-2y}dv \\ &= 24\int_0^{\infty} e^{-3v-2y} - e^{-4v-2y}dv \\ &= 8e^{-2y} - 6e^{-2y} \\ &= 2e^{-2y}. \end{aligned}$$

Hence Y is exponential with parameter $\lambda = 2$.

QUESTION 3 **(25 marks)**

(a) Let X be a random variable with $0 \leq X \leq 1$ and $E(X) = \mu$. Show that:

(i) $0 \leq \mu \leq 1$;

(ii) $0 \leq Var(X) \leq \mu(1-\mu) \leq \frac{1}{4}$. [Hint: Use $X^2 \leq X$.]

(b) The result in Part (a) may be generalized as follows. Let X be a random variable with $a \leq X \leq b$ and $E(X) = \mu$. Show that:

(i) $a \leq \mu \leq b$;

(ii) $0 \leq Var(X) \leq (\mu - a)(b-\mu) \leq \frac{1}{4}(b-a)^2$.

[Solution:]

(a) (i) Since $0 \leq X \leq 1$,

$$0 = \int_0^1 0dx \leq \int_0^1 xf(x)dx = E(X) \leq \int_0^1 f(x)dx = 1.$$

(ii) Since $0 \leq X^2 \leq X \leq 1$, we see that $E(X^2) \leq E(X) \leq \mu$. Hence

$$Var(X) = E(X^2) - [E(X)]^2 \leq \mu - \mu^2 = \mu(1-\mu).$$

By using calculus (or by completing the square), the maximum of a function $f(x) = x(1-x)$ occurs at $x = 1/2$. Hence $\mu(1-\mu) \leq \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$.

(b) (i) Set $Y = \frac{X-a}{b-a}$. Then $0 \leq Y \leq 1$ and

$$\mu_Y = \frac{\mu_X - a}{b-a},$$

which implies that

$$\mu_X = a + (b-a)\mu_Y.$$

Hence $a \leq \mu_X \leq b$.

(ii)

$$Var(Y) = Var\left(\frac{X-a}{b-a}\right) = \frac{Var(X)}{(b-a)^2}.$$

By Part (a)(ii), this means

$$0 \leq \frac{Var(X)}{(b-a)^2} \leq \mu_Y(1-\mu_Y) \leq \frac{1}{4}.$$

Hence,

$$0 \le Var(X) \le (b-a)^2\mu_Y(1-\mu_Y) \le \frac{1}{4}(b-a)^2.$$

It remains to see that

$$(\mu_X - a)(b-\mu_X) = (a+(b-a)\mu_Y - a)(b-a-(b-a)\mu_Y) = (b-a)^2\mu_Y(1-\mu_Y)$$

and this completes Part (b)(ii).

QUESTION 4 **(25 marks)**

Suppose (X, Y) is uniform on the unit disk (centered at origin with radius 1).

(a) Find f_X and f_Y, the marginal density functions of X and Y, respectively.

(b) Show that X and Y are not independent.

(c) Prove that $Cov(X, Y) = 0$.

[Solution:]

(a) Clearly the joint density function is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2+y^2 \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}, \qquad (-1 \le y \le 1).$$

Similarly, $f_X(x) = \frac{2\sqrt{1-x^2}}{\pi}, \qquad (-1 \le x \le 1).$

(b) Clearly

$$f_X(x) \cdot f_Y(y) \ne f_{XY} = \frac{1}{\pi}.$$

Hence they are not independent.

(c)

$$E[X] = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{x}{\pi} dy dx = \int_{-1}^{1} x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy dx = \int_{-1}^{1} \frac{2x\sqrt{1-x^2}}{\pi} dx = 0$$

Similarly, $E[Y] = 0$.

Next,

$$E[XY] = \int_{=1}^{1} x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y}{\pi} dydx = \int_{=1}^{1} \frac{x}{\pi} 0dx = 0.$$

Hence

$$Cov[X, Y] = E[XY] - E[X]E[Y] = 0.$$

QUESTION 5 **(15 marks)**

Suppose $X_1, X_2, \ldots, X_{30}$ are independent Poisson random variables with $\lambda = 2$. Use the central limit theorem to approximate

$$P\left(\sum_{i=1}^{30} X_i > 50\right).$$

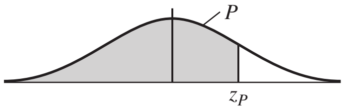
[Solution:] Since each X_i is Poisson with $\lambda = 2$, both the mean and variance are 2. Therefore,

$$Var\left(\sum_{i=1}^{30} X_i\right) = E\left(\sum_{i=1}^{30} X_i\right) = 60$$

By the CTL,

$$\begin{aligned} P\left(\sum_{i=1}^{30} X_i > 50\right) &\approx P\left(Z > \frac{60-50}{\sqrt{60}}\right) \qquad \text{where } Z \sim \text{N(0,1)} \\ &= P(Z > 1.291) \\ &= \Phi(1.29) \approx 0.9015. \end{aligned}$$

Cumulative Normal Distribution — Values of P corresponding to z_p for the Standard Normal Curve.



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for –1.62 equals $1 - .9474 = .0526$.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

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