EE4152
Digital Communications

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Content (Location: NTULearn > Content)

- Digital Communication Principles
- Information Theory
- Error Correcting Coding
- Optimum Signal Processing

Relevant Information

- This course builds upon material covered in EE3002/IM3002 (Communication Principles)
- Mathematical tools used in this subject include Fourier series, Fourier transform, probability theory, convolution and correlation.

Textbook & References

- **B P Lathi and Z Ding**,
- *Modern Digital and Analog Systems*, 4/Ed, Oxford University Press, 2010
- **S Haykin and M Moher**, *Communication Systems*, 5/Ed, John Wiley, 2010.
- **J G Proakis and M Salehi**, *Communication Systems Engineering*, 2/Ed, Prentice-Hall, 2002

Assessment Components

- **CA** - Assignment (10%) <~wk 5>
- **CA** – Quiz #1 (10%) <wk 6>, covering lecture topics in wks 1-5
- **CA** – Project Report (10%) <~wks 7-8>
- **CA** – Quiz #2 (10%) <wk 11>, covering lecture topics in wks 6-10
- **Exam** (60%)

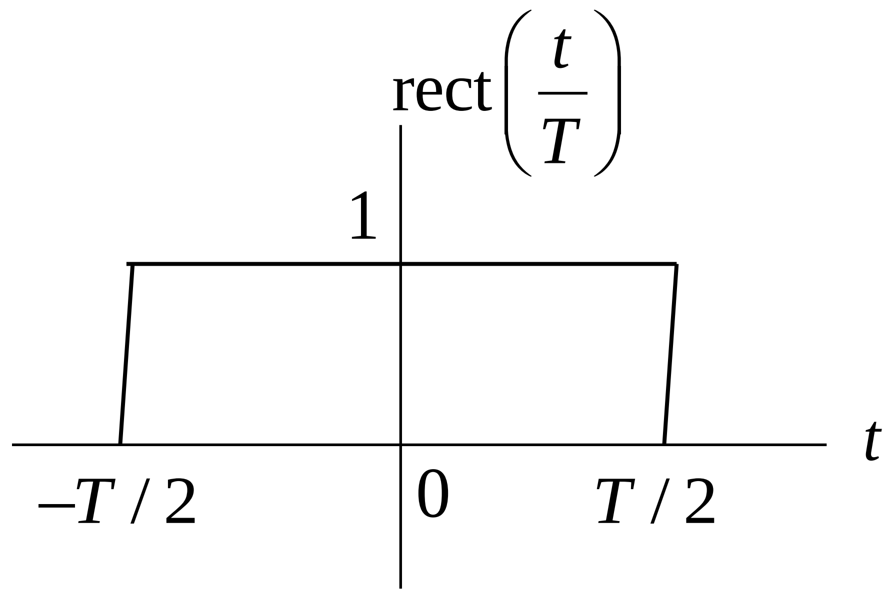
Digital Communication Principles

- Sampling Theorem & Aliasing
- Line Coding Schemes
- Autocorrelation and Power Spectral Density (PSD)
- Intersymbol Interference (ISI) and Pulse Shaping

Rectangular Function

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\leftrightarrow T\,\mathrm{sinc}(fT)$$

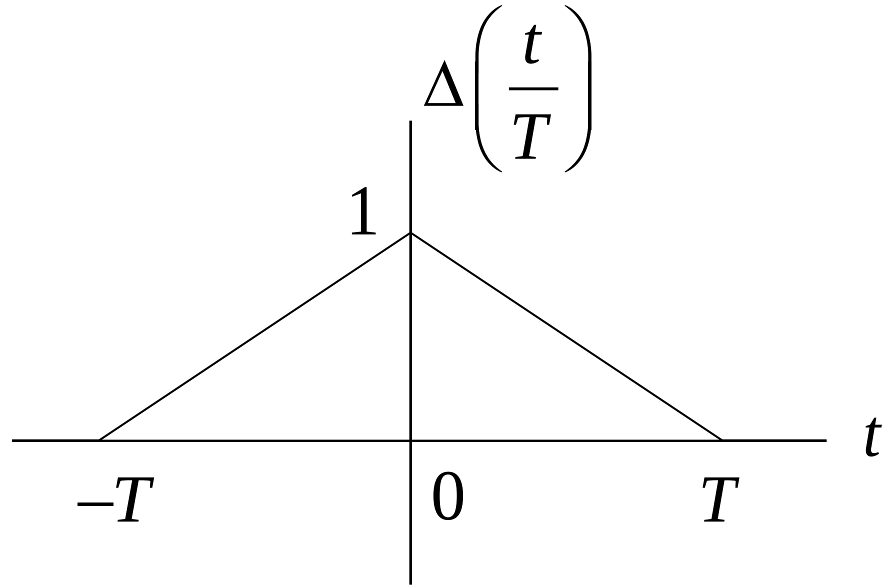


- $\Pi(t/T)$ ⇦ See pp. 70 (Lathi)

Triangular Function

$$\Delta\left(\frac{t}{T}\right) = \begin{cases} 1 - \dfrac{|t|}{T} & |t| \le T \\ 0 & \text{otherwise} \end{cases}$$

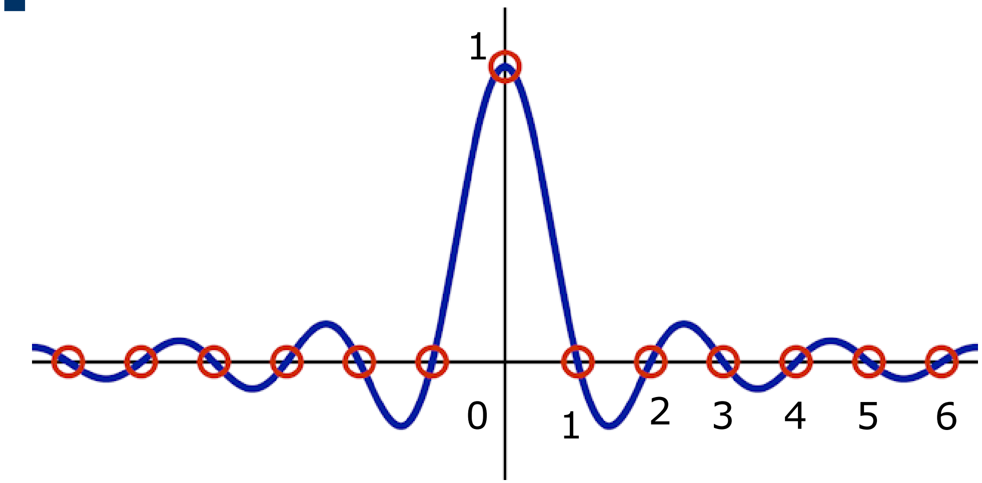
$$\leftrightarrow T\,\mathrm{sinc}^2(fT)$$



- $\Delta(t/2T)$ ⇦ See pp. 70 (Lathi)

Sinc Function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



where zero crossings are at $x = \pm 1, \pm 2, \pm 3, \cdots$

- sinc(x) = sin(x)/x ⇦ See pp. 70 (Lathi)

Fourier Series

If $g_p(t)$ is a periodic function with period T_0, then it can be represented as

$$g_p(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{j2\pi n f_0 t}$$

where $f_0 = 1/T_0$ and

$$C_n = \frac{1}{T_0} \int_{T_0} g_p(t) e^{-j2\pi n f_0 t} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Fourier Transform

Given that a time function $g(t)$ with finite energy

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

the Fourier transform pair is given by

$$G(f) = F[g(t)] = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt$$

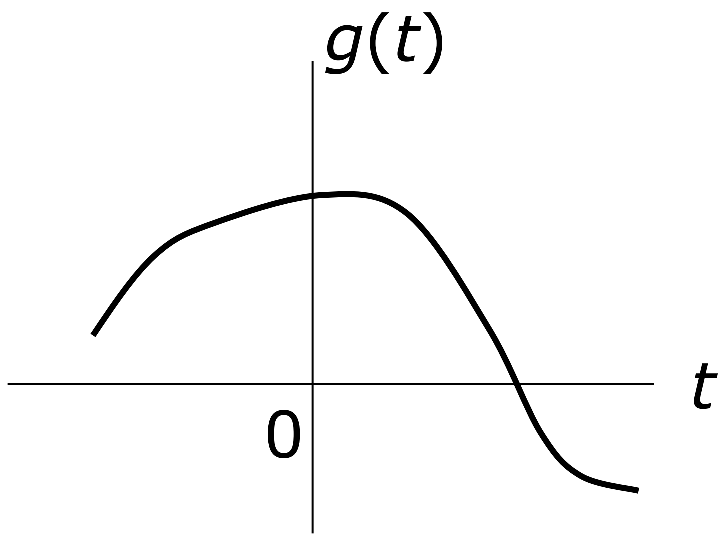
$$g(t) = F^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df$$

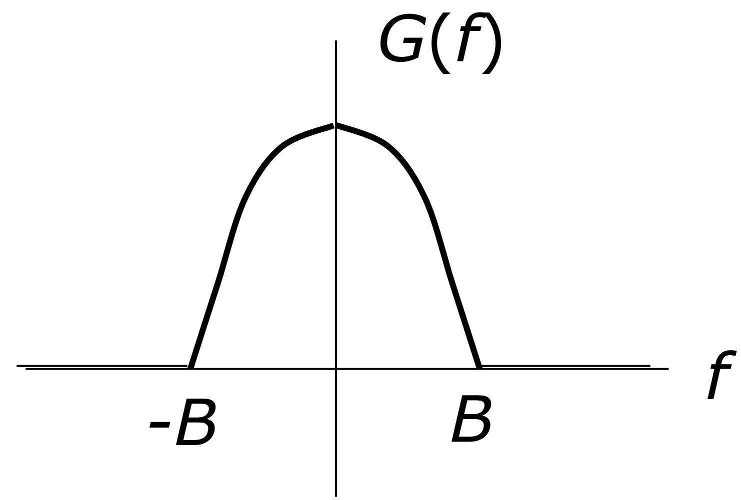
Sampling Theorem & Aliasing

Introduction

- Analog signals can be digitized through sampling and quantization.
- The sampling rate must be fast enough so that the analog signal can be reconstructed from the samples with acceptable accuracy.
- What is the appropriate sampling rate?

Band-Limited Signal



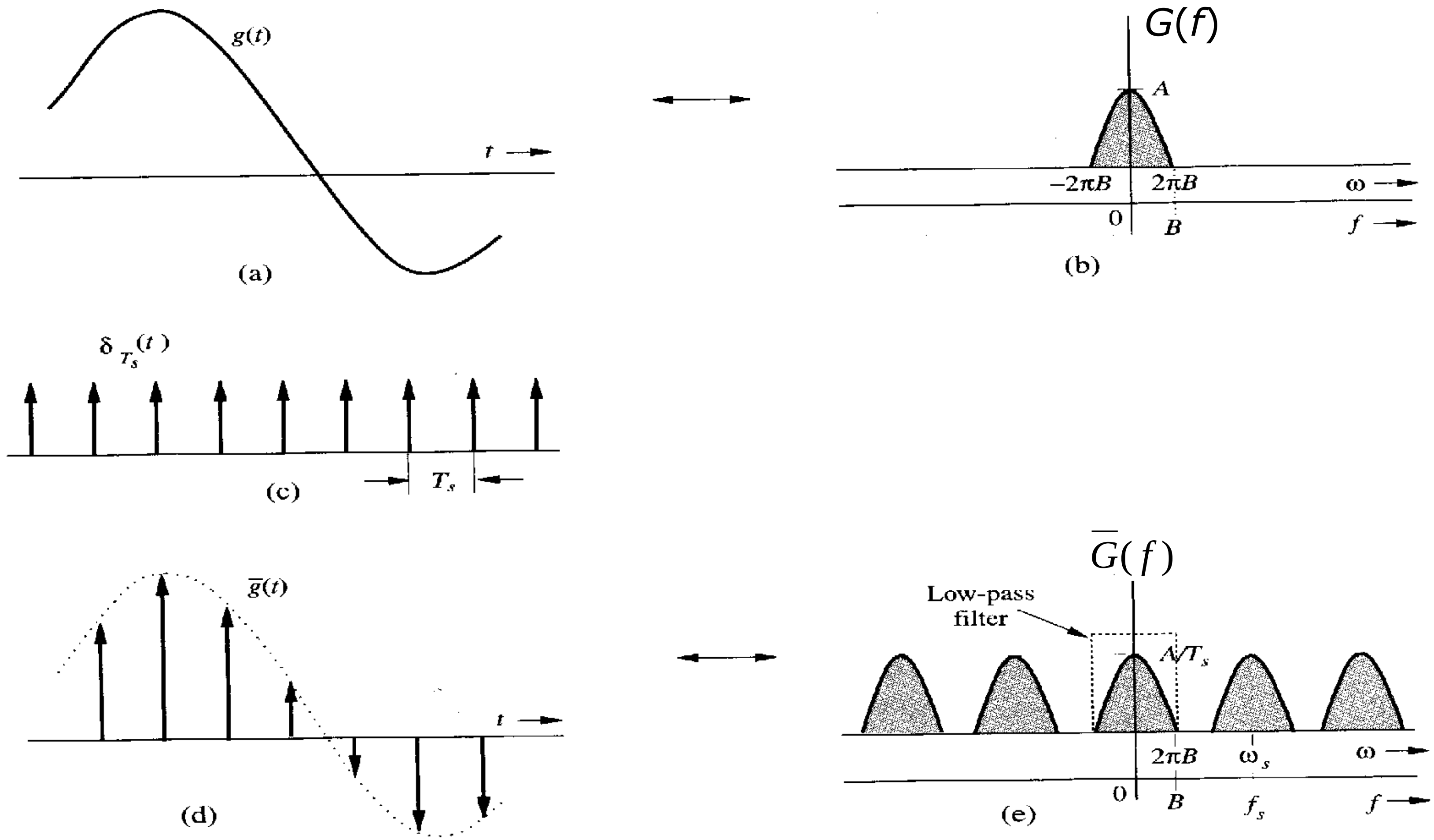


$G(f)$ is zero for $|f| > B$

Sampling Theorem

- A signal band-limited to B Hz can be reconstructed without any error from its samples taken uniformly at a sampling rate *greater than* or *equal to* $2B$ samples per second.
- The **sampling frequency** is $f_s \geq 2B$ Hz.
- The **sampling interval** is $T_s \leq 1/(2B)$ s.

Sampled Signal & its Fourier Transform



Proof:

- Consider a signal $g(t)$ band-limited to B Hz.
- Multiplying $g(t)$ by a unit impulse train yields the sampled signal

$$\bar{g}(t) = g(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Exercise

Find the Fourier series representation of the *periodic* unit impulse train

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Answer:

$$\delta_{T_s}(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}, \qquad f_s = 1/T_s$$

Proof: (...)

- Using the Fourier series for the impulse train, we have

$$\bar{g}(t) = g(t) \times \left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t} \right] = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} g(t) e^{j2\pi n f_s t}$$

- Taking the Fourier transform yields

Proof: (...)

- In order to reconstruct $g(t)$, we have to recover $G(f)$ from $\bar{G}(f)$.
- This is possible if there is no overlap between successive cycles in $\bar{G}(f)$.
- The requirement is $f_s \geq 2B$.
- $G(f)$ can be recovered by passing $\bar{G}(f)$ through a LPF with bandwidth B Hz

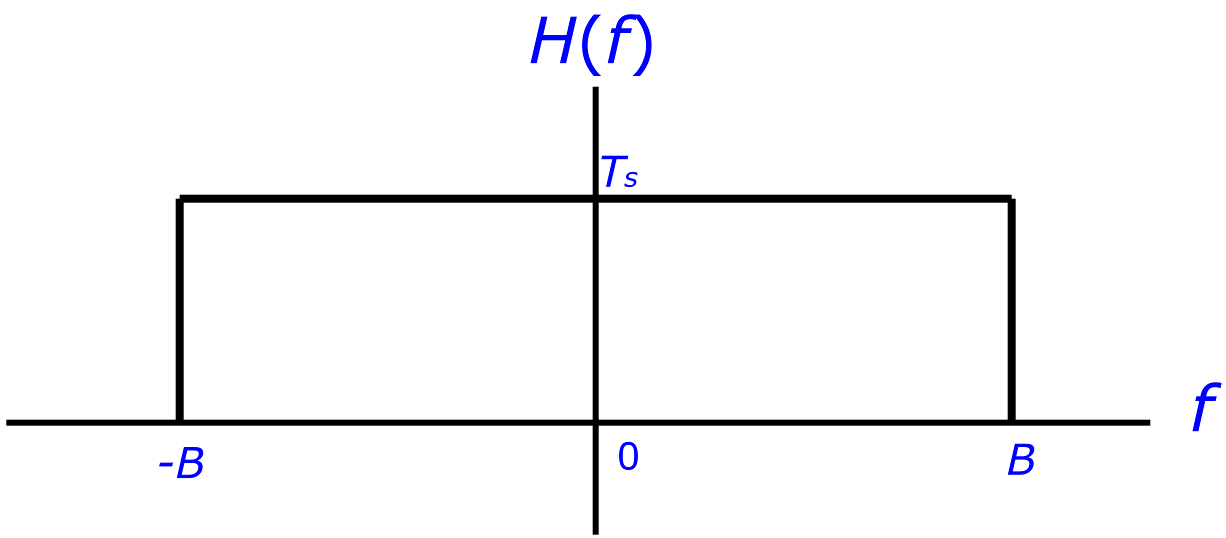
Nyquist Rate

- Since $f_s \geq 2B$, the minimum sampling rate is $2B$ samples per second. It is called the **Nyquist rate** for $g(t)$.
- Accordingly, the **Nyquist interval** for $g(t)$ is $1/2B$.

Reconstruction Filter

- The transfer function of the LPF is

$$H(f) = T_s \operatorname{rect}\left(\frac{f}{2B}\right)$$



Exercise

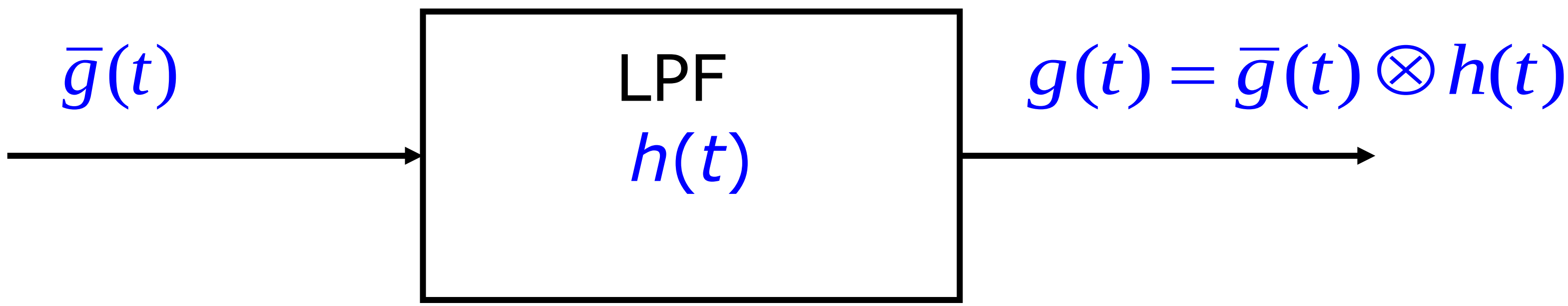
Find the impulse response of

$$H(f) = T_s \operatorname{rect}\left(\frac{f}{2B}\right)$$

Answer:

$$h(t) = F^{-1}\left[H(f)\right] = 2BT_s \operatorname{sinc}\left(2Bt\right)$$

Reconstruction of *g*(*t*)



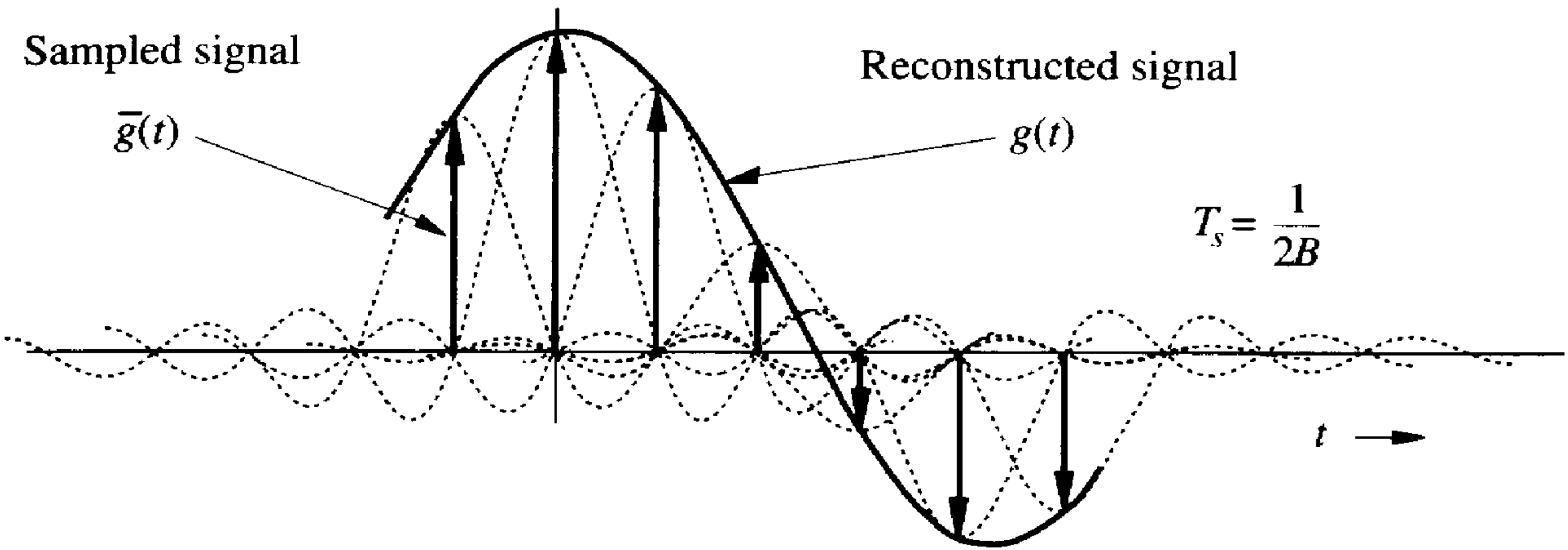
If $T_s = 1/2B$, then

$$h(t) = \text{sinc}\left(2Bt\right)$$

Interpolation Formula

$$
\begin{aligned}
g(t) &= \bar{g}(t) \otimes h(t) \\
&= \sum_k g\left(kT_s\right)\delta\left(t - kT_s\right) \otimes h(t) \\
&= \sum_k g\left(kT_s\right)\ h\left(t - kT_s\right) \\
&= \sum_k g\left(kT_s\right)\ \mathrm{sinc}\left[2B\left(t - kT_s\right)\right] \\
&= \sum_k g\left(kT_s\right)\ \mathrm{sinc}\left(2Bt - k\right)
\end{aligned}
$$

Weighted Sum of Sinc Functions



Example

Find a signal $g(t)$ that is band-limited to B Hz and whose samples are $g(0) = 1$ and
$g(\pm T_s) = g(\pm 2T_s) = g(\pm 3T_s) = \ldots = 0$
where the sampling interval T_s is the Nyquist interval for $g(t)$, i.e., $T_s = 1/2B$.

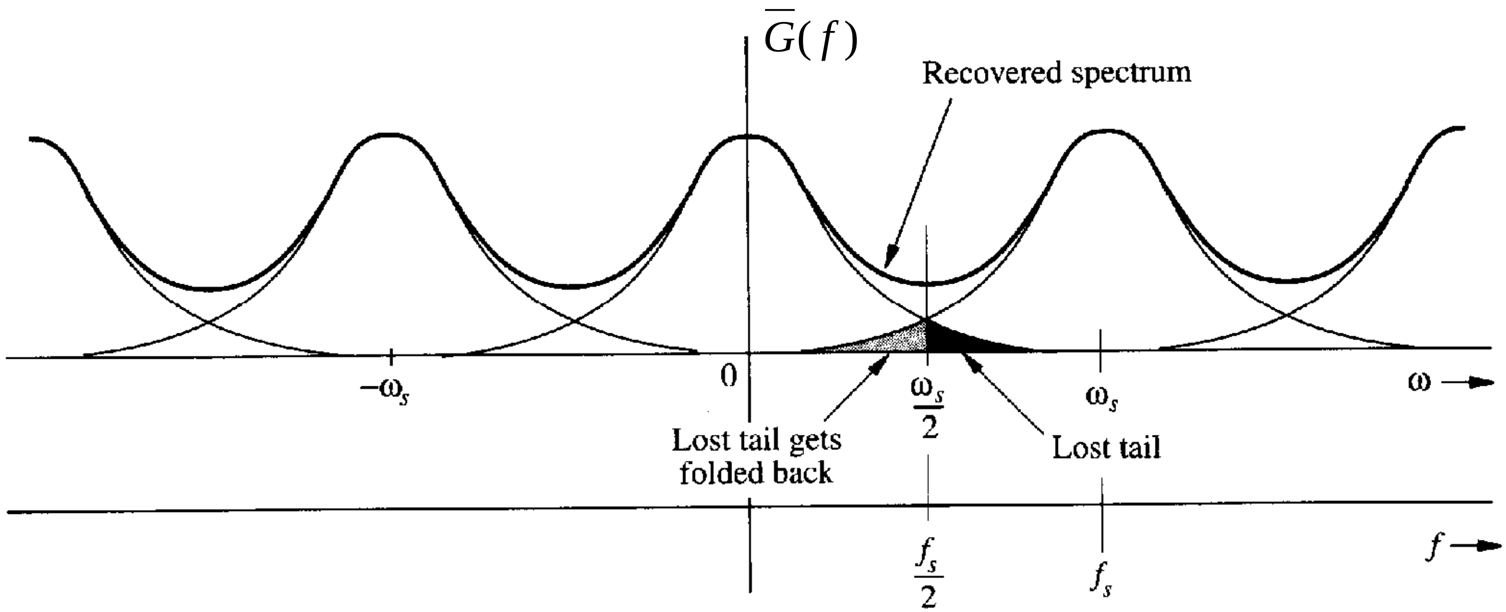
Substituting the sampled values into

$$g(t) = \sum_k g(kT_s)\ \operatorname{sinc}(2Bt - k)$$

we have $g(t) = \operatorname{sinc}(2Bt)$.

Undersampling

- If a signal is undersampled (sampled at a rate below the Nyquist rate), the spectrum of $\bar{G}(f)$ becomes



Undersampling (...)

- Because of the overlapping tails, it is not possible to exactly recover $g(t)$ from $\bar{g}(t)$.
- If the sampled signal $\bar{g}(t)$ is passed through a LPF with bw $f_s/2$, the output will be a distorted version of $g(t)$.

Distortion

There are two types of distortion:

(1) lost tail

(2) aliasing or tail inversion

Solution to Aliasing

- Aliasing can be eliminated by cutting the tail of $G(f)$ beyond $|f| > f_s/2$ **before the signal is sampled**.
- However, the distortion due to lost tail remains.
- In this way, the energy associated with distortion is reduced by half.

Exercise

Let $g(t) = 2a/(t^2 + a^2)$. Find the *essential bw* B such that the energy beyond which is equal to 1% of the total energy and also determine its Nyquist sampling rate.

Answer:

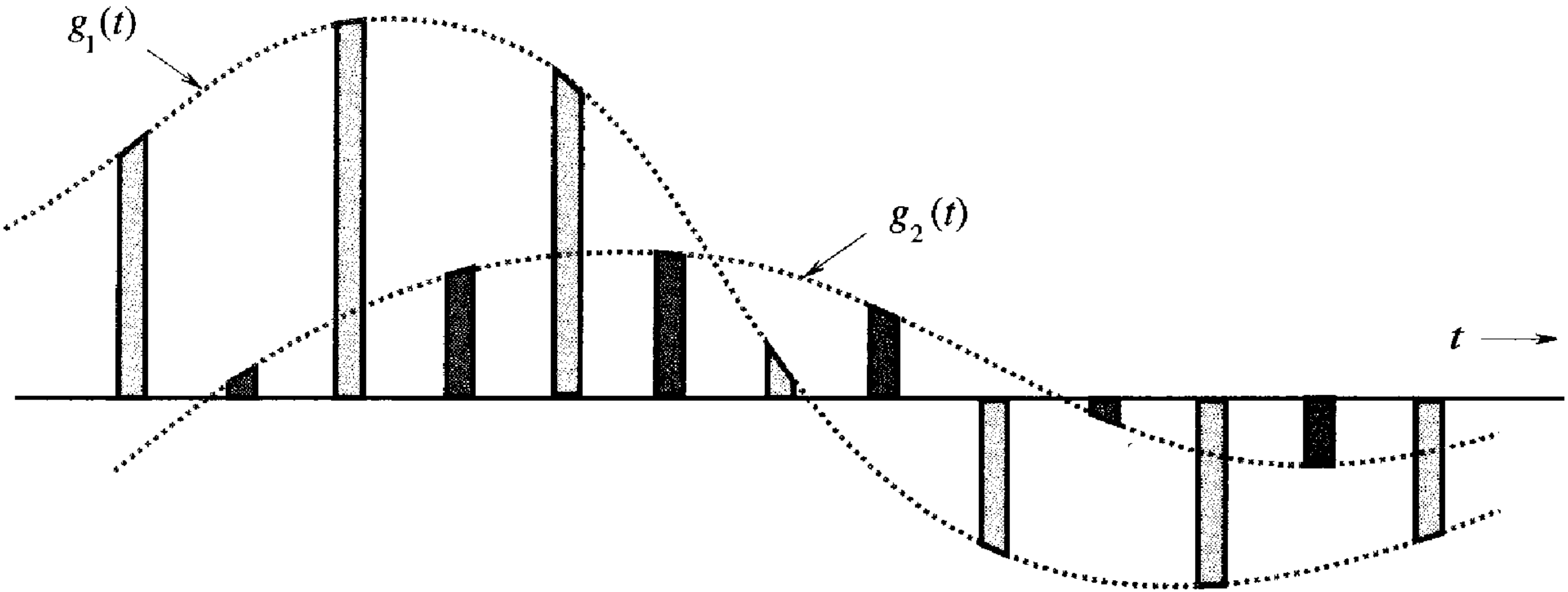
Practical/Natural Sampling

- Sampling a signal $g(t)$ by impulses is of theoretical interest only. In practice, sampling is accomplished by a train of pulses of finite width.
- Is it possible to recover $g(t)$ from the sampled signal?
- The answer is positive and it will be demonstrated in tutorial problems.

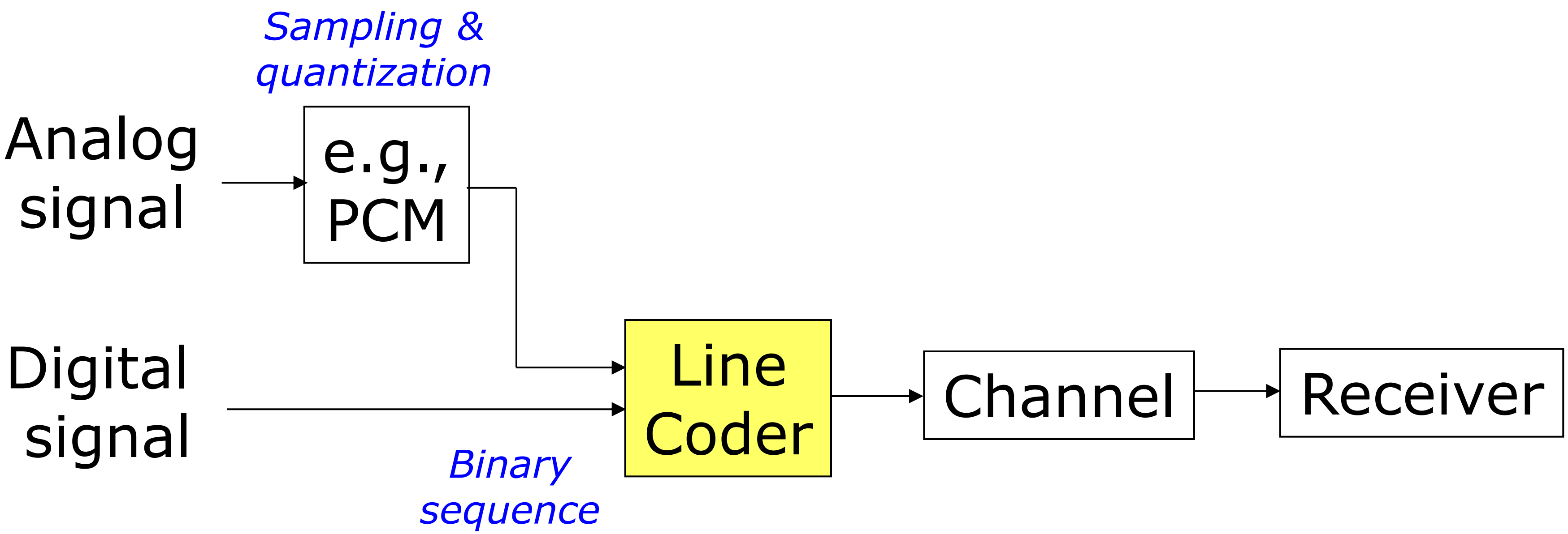
Time-Division Multiplexing (TDM)

- Because a pulse-modulated signal occupies only a small part of the channel time, several pulse-modulated signals can share the same channel on a time basis.

TDM of Two Signals

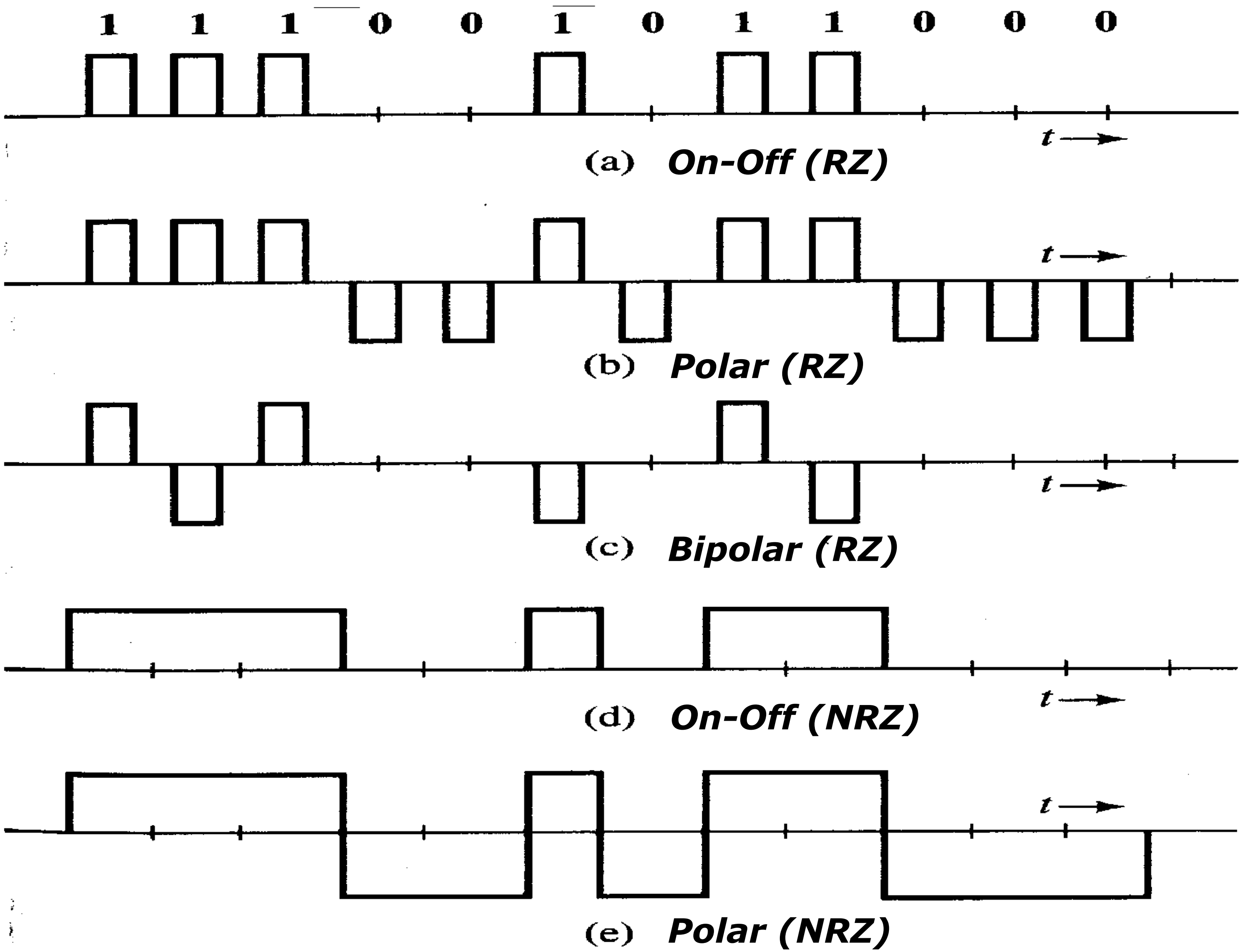


Line Coding


Sampling &
quantization
Analog
signal
e.g.,
PCM
Digital
signal
Binary
sequence
Line
Coder
Channel
Receiver

Introduction

- A digital signal is needed to map into electrical pulses or waveforms before transmitting over a channel. This process is called **line coding**.
- There are many ways of assigning waveforms to digital data.
- We shall restrict our discussion to the case of **binary data**.


1 1 1 0 0 1 0 1 1 0 0 0
t
(a) On-Off (RZ)
t
(b) Polar (RZ)
t
(c) Bipolar (RZ)
t
(d) On-Off (NRZ)
t
(e) Polar (NRZ)

On-Off Signaling

- '1' is transmitted by a pulse $p(t)$.
- '0' is transmitted by no pulse.
- Pulses can be **return-to-zero** (**RZ**), as shown in Fig. (a), or **nonreturn-to-zero** (**NRZ**), as shown in Fig. (d).

Polar Signaling

- '1' is transmitted by a pulse $p(t)$.
- '0' is transmitted by a pulse $-p(t)$.
- Pulses can be RZ (see Fig. (b)) or NRZ (see Fig. (e)).

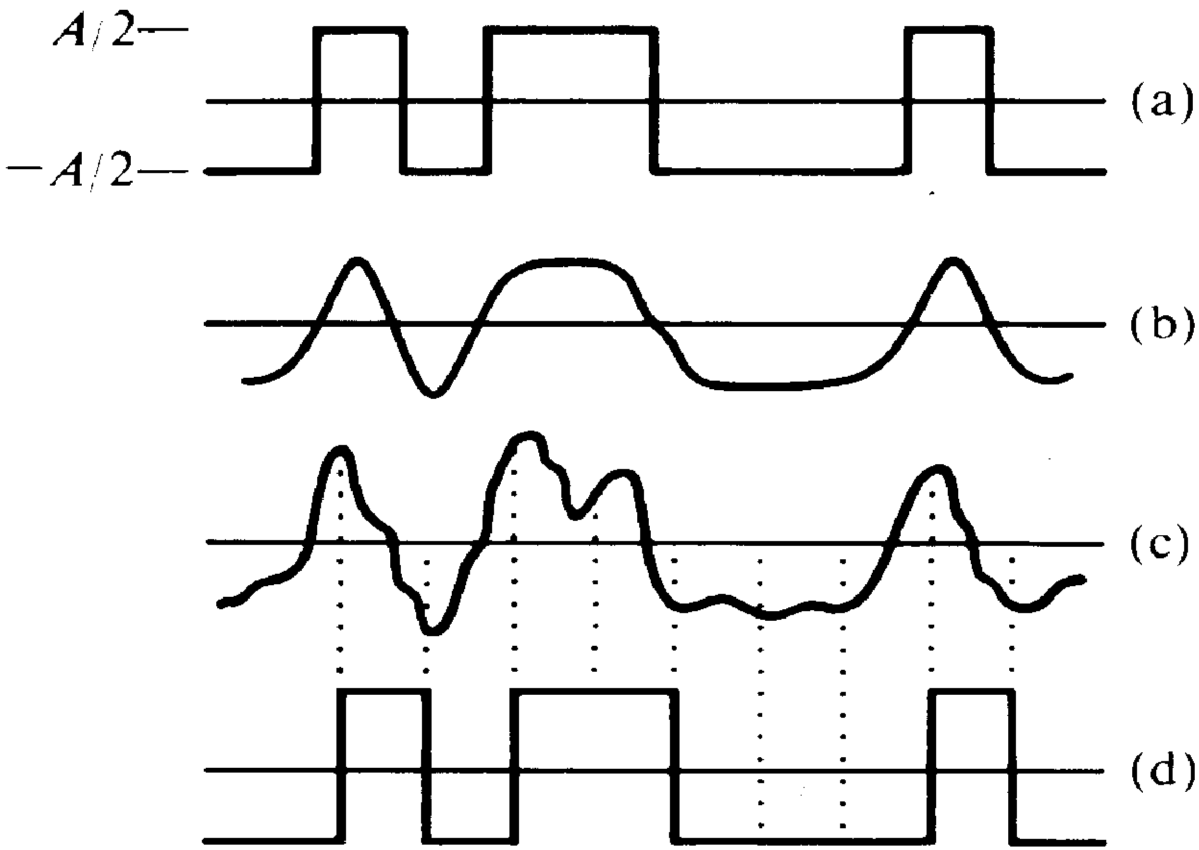
Bipolar Signaling

- '0' is encoded by no pulse.
- '1' is encoded by a pulse $p(t)$ or $-p(t)$, depending on whether the previous '1' is encoded by $-p(t)$ or $p(t)$.
- Pulses representing consecutive 1's are alternate in sign, as shown in Fig. (c).

Regenerative Repeaters

- The repeaters are used at regularly spaced intervals along a digital transmission path to detect the incoming signal and regenerate new clean pulses for further transmission along the path.
- This process periodically eliminates the accumulation of noise and signal distortion along the path.

Example



(a) Transmitted signal

(b) Received distorted signal (w/o noise)

(c) Received distorted signal (with noise)

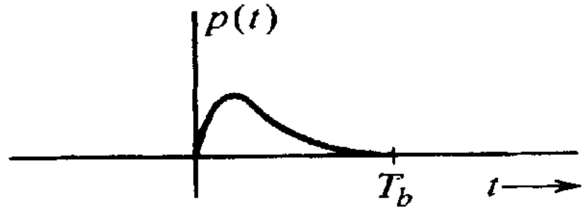
(d) Regenerated signal (delayed)

Desirable Properties of Line codes

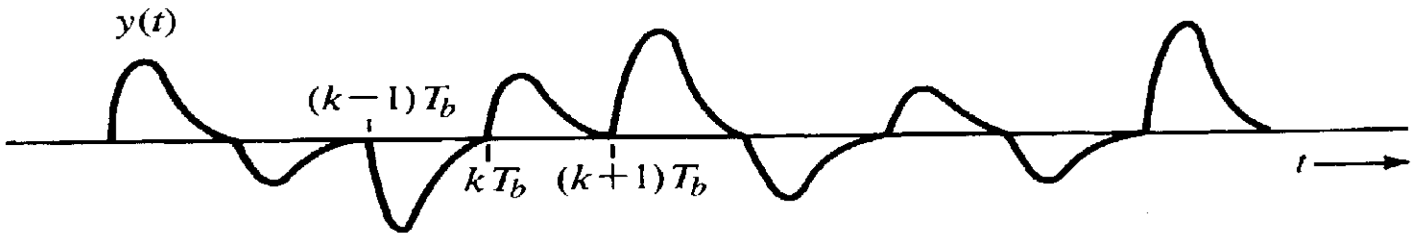
- **Transmission bandwidth (bw)** should be as small as possible.
- The **transmitted power** should be as small as possible for a given bw and a specific detection error probability.
- It is nice that the line code can **detect** or even **correct errors**.

Desirable Properties of Line codes (...)

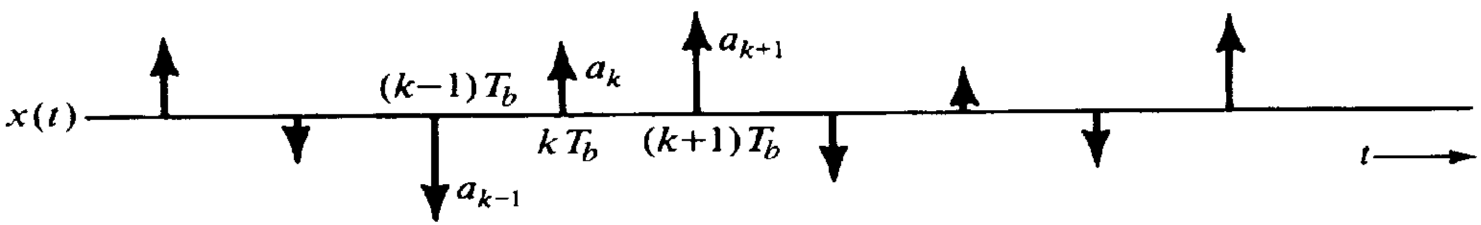
- It is desirable to have zero PSD at $f = 0$ (**dc component**) because ac coupling and transformers are used at the repeaters.
- It should be possible to **extract timing information** from the signal.
- It should be possible to transmit a digital signal correctly regardless of the data pattern (i.e., **transparent**).



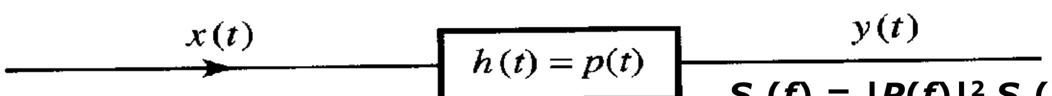
(a) ***Basic pulse***



(b) ***Transmitted pulse train***



(c) ***PAM signal***



$\boldsymbol{S_y(f) = |P(f)|^2 S_x(f)}$

(d) ***Relationship among x(t), y(t) and p(t)***

PSD of Line Codes

- The pulse train $y(t)$ is constructed from a basic pulse $p(t)$ every T_b sec with relative strength a_k for the pulse starting at $t = kT_b$.
- The k-th pulse of $y(t)$ is $a_k\, p(t - kT_b)$, where the elements of the data sequence $\{a_k\}$ are arbitrary and random.

PSD of Line Codes (...)

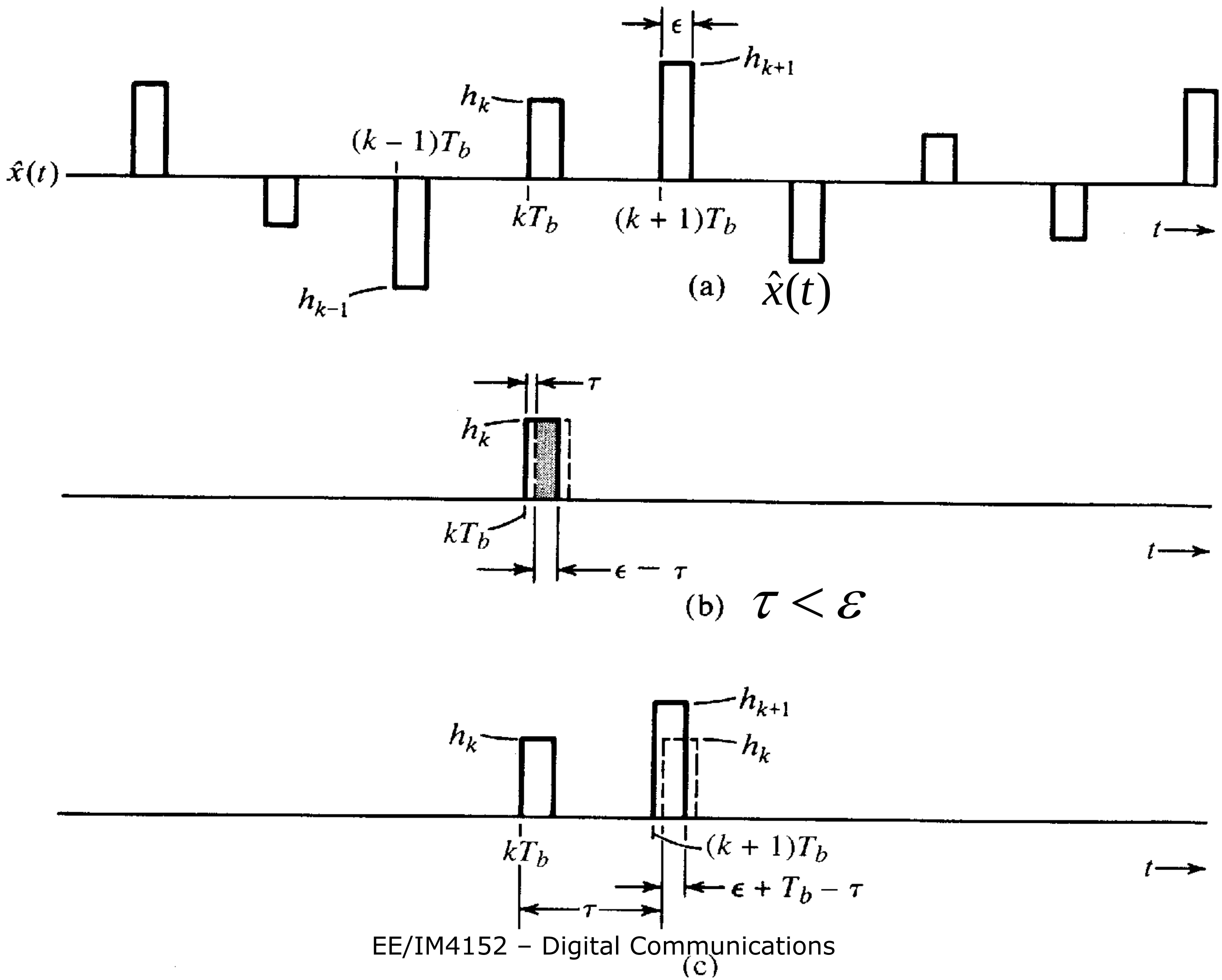
- We can find the PSD of $y(t)$ using the time-autocorrelation approach.

$$\Re_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, y(t-\tau)\, dt \leftrightarrow S_y(f)$$

- However, if the pulse $p(t)$ changes, we have to derive the PSD all over again.

PSD of Line Codes (...)

- The difficulty can be avoided by considering an impulse signal $x(t)$ that uses a unit impulse for the basic pulse $p(t)$, where the strength of the k-th impulse is a_k.
- If $x(t)$ is applied to the input of a filter with impulse response $h(t) = p(t)$, then the output will be $y(t)$.
- The desired PSD is $S_y(f)=|P(f)|^2 S_x(f)$.



ϵ
h_{k+1}
h_k
$(k-1)T_b$
$\hat{x}(t)$
kT_b
$(k+1)T_b$
$t \longrightarrow$
h_{k-1}
(a) $\hat{x}(t)$
τ
h_k
kT_b
$\epsilon - \tau$
$t \longrightarrow$
(b) $\tau < \varepsilon$
h_{k+1}
h_k
h_k
kT_b
$(k+1)T_b$
$t \longrightarrow$
$\epsilon + T_b - \tau$
τ
(c)

$\Re_x(\tau)$ and $S_x(f)$

- $\Re_x(\tau)$ is the time-autocorrelation function of the impulse train $x(t)$.
- $\Re_x(\tau) \leftrightarrow S_x(f)$
- To obtain $\Re_x(\tau)$, we replace the impulses of $x(t)$ by the rectangular pulses with width ε. The height of the k-th pulse is h_k.

Deriving $\mathscr{R}_x(\tau)$

- The k-th impulse should have the strength or area a_k, i.e.,

$$\varepsilon h_k = a_k$$

- If we denote the rectangular pulse train by $\hat{x}(t)$, then

$$\Re_{\hat{x}}(\tau) = \Re_{\hat{x}}(-\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \hat{x}(t)\,\hat{x}(t-\tau)\,dt$$

- $\mathscr{R}_x(\tau)$ can be obtained by letting $\varepsilon \to 0$.

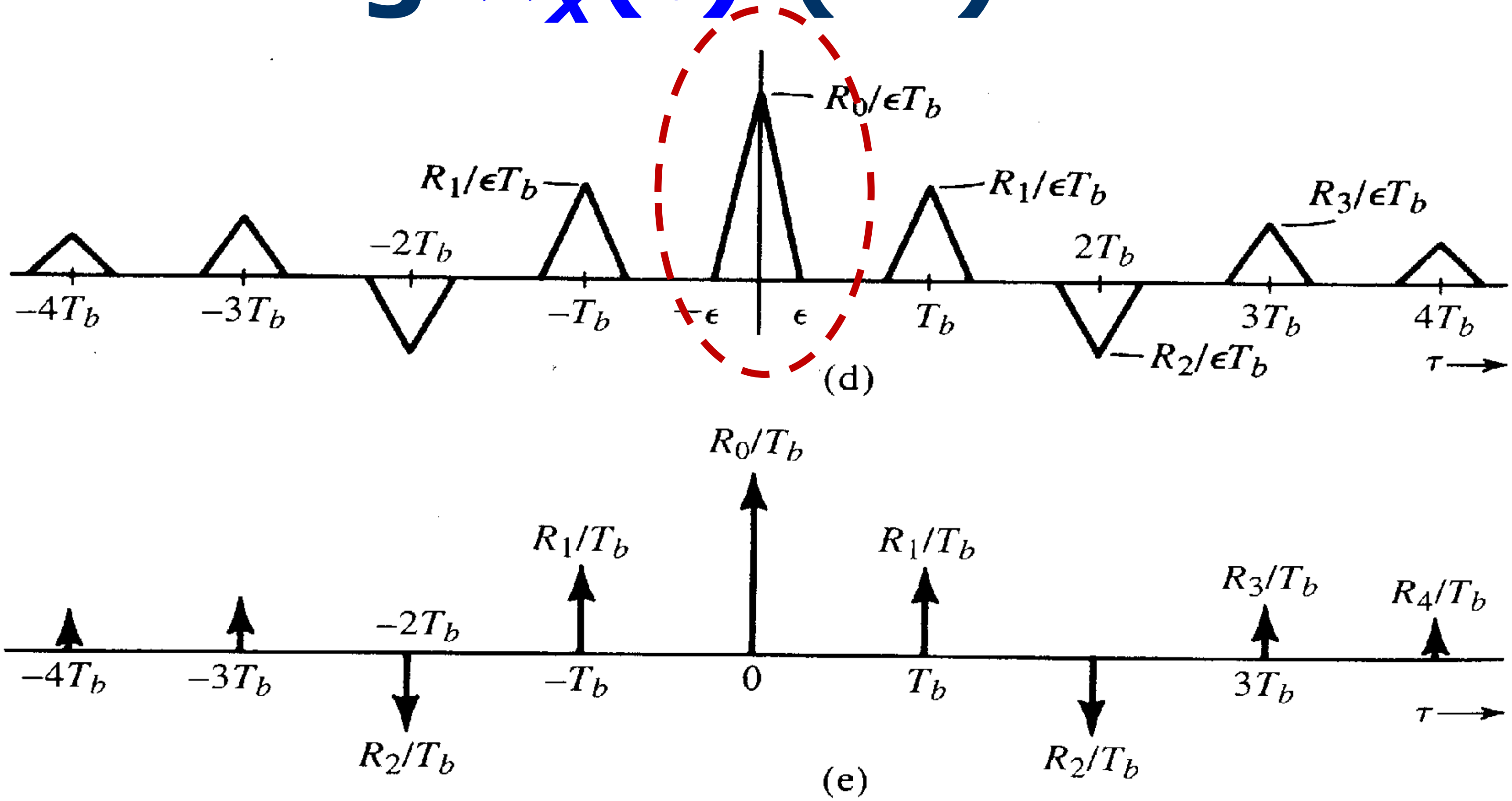
Deriving $\mathcal{R}_x(\tau)$ (...)

- Because $\Re_{\hat{x}}(\tau)$ is an even function of τ, we need to consider only positive τ.
- When $\tau < \varepsilon$ (see Fig. (b)),

$$\Re_{\hat{x}}(\tau) = \lim_{T\to\infty} \frac{1}{T} \sum_k h_k^2 \left(\varepsilon - \tau\right) = \lim_{T\to\infty} \frac{1}{T} \sum_k a_k^2 \left(\frac{\varepsilon - \tau}{\varepsilon^2}\right)$$

$$= \frac{1}{\varepsilon T_b} \left(\underbrace{\lim_{T\to\infty} \frac{T_b}{T} \sum_k a_k^2}_{R_0}\right)\left(1 - \frac{\tau}{\varepsilon}\right) = \underbrace{\frac{R_0}{\varepsilon T_b}\left(1 - \frac{|\tau|}{\varepsilon}\right)}_{\textit{See Fig. (d)}}$$

Deriving $\mathfrak{R}_x(\tau)$ (…)



Digression

- Suppose the time interval T contains N pulses, i.e., $N = T/T_b$. Then

$$R_0 = \lim_{T\to\infty} \frac{T_b}{T} \sum_k a_k^2$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_k a_k^2 = \left\langle a_k^2 \right\rangle$$

where $\langle x \rangle$ denotes the **time average** or **mean** of x.

Deriving $\mathcal{R}_x(\tau)$ (...)

- As we increase the shift τ further, the k-th pulse of $\hat{x}(t-\tau)$ will start overlapping the $(k+1)$-th pulse of $\hat{x}(t)$ as τ approaches T_b.
- Repeating the earlier steps, we obtain another triangle located at $\tau = T_b$ (see Fig. (d)).

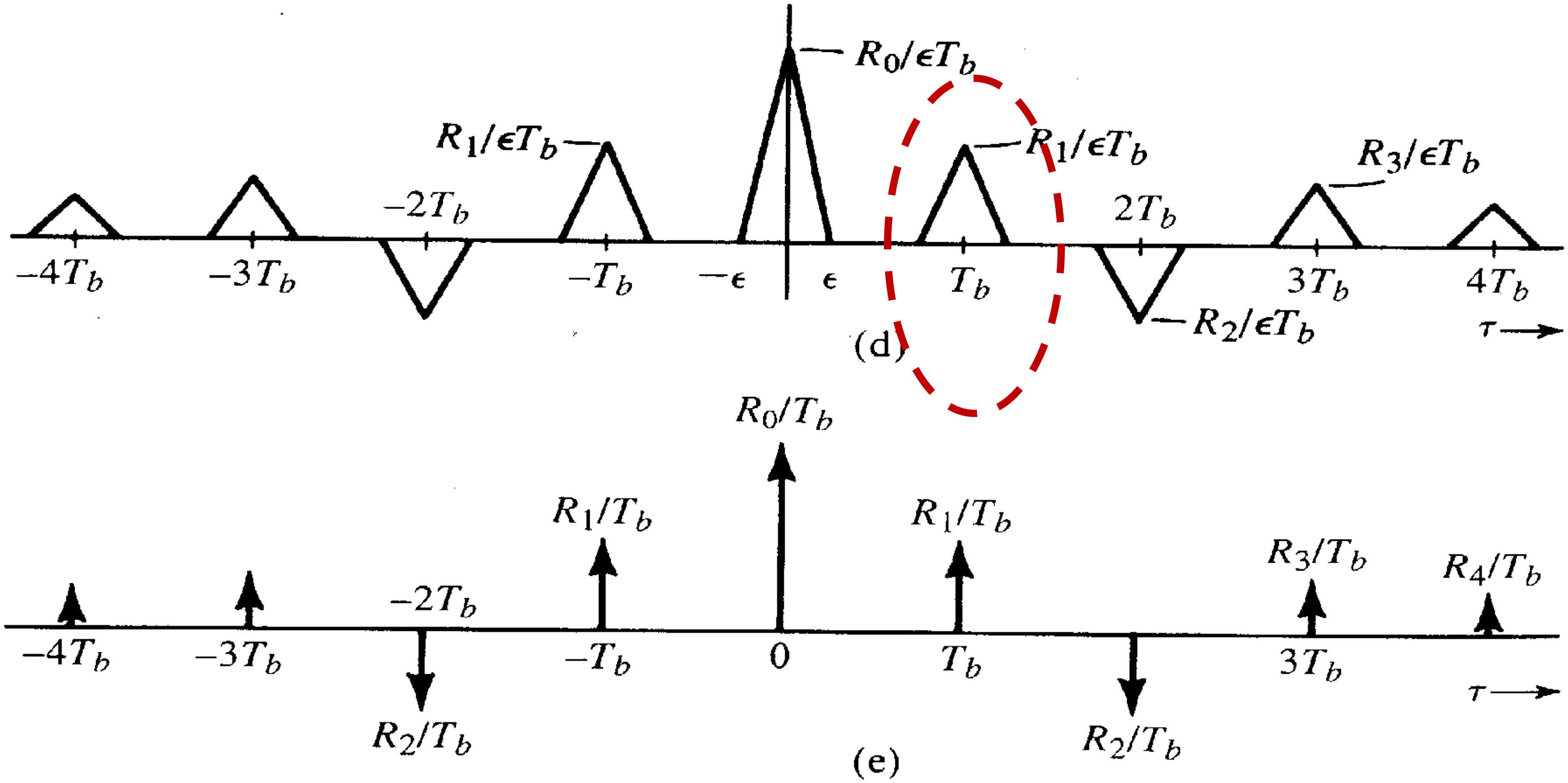
Deriving $\Re_x(\tau)$ (...)

- The width of the triangle is 2ε and the height is $R_1/\varepsilon T_b$ at $\tau = T_b$, where

$$R_1 = \lim_{T\to\infty} \frac{T_b}{T} \sum_k a_k\, a_{k+1}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_k a_k\, a_{k+1} = \langle a_k\, a_{k+1} \rangle$$

Deriving $\mathcal{R}_x(\tau)$ (…)



Deriving $\mathfrak{R}_x(\tau)$ (...)

- Similarly, we can obtain a sequence of triangular pulses located at $\tau = 2T_b$, $3T_b$, $\ldots$. For the n-th pulse centered at nT_b, the height is $R_n/\varepsilon T_b$ and the area of the triangular pulse is R_n/T_b.
- As $\varepsilon \to 0$, the n-th pulse becomes an impulse with strength (or area) R_n/T_b (see Fig. (e)).

Deriving $S_x(f)$

- Hence, $\Re_x(\tau) = \sum_{n=-\infty}^{\infty} \frac{R_n}{T_b} \times \delta\left(t - nT_b\right)$

  The PSD is $S_x(f) = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} R_n\, e^{-j2\pi nfT_b}$

- Recognizing that $R_n = R_{-n}$, we have

$$S_x(f) = \frac{1}{T_b}\left(R_0 + 2\sum_{n=1}^{\infty} R_n \cos 2\pi nfT_b \right)$$

Deriving $S_y(f)$

$$S_y(f) = |P(f)|^2 S_x(f)$$

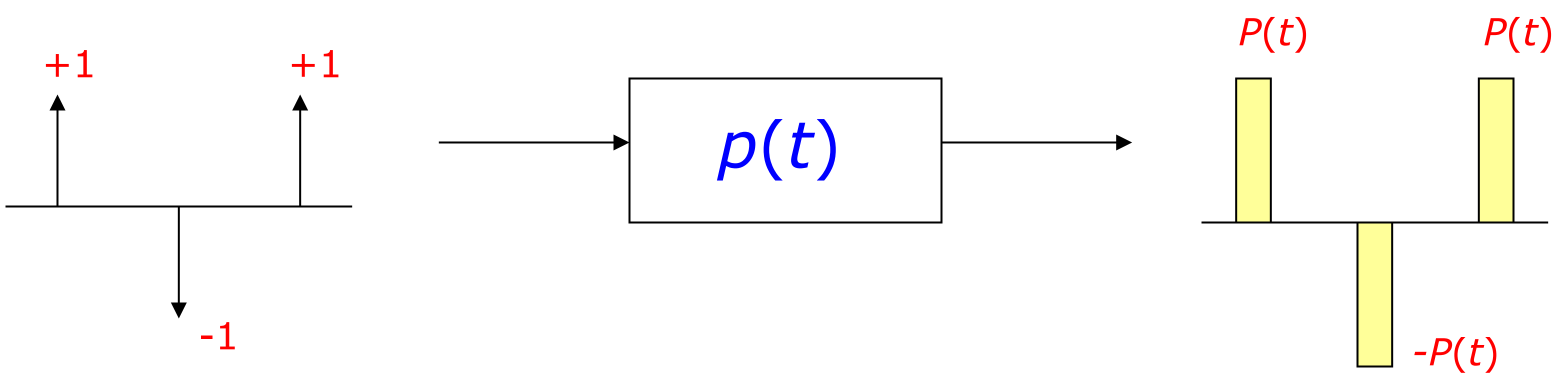
$$= \frac{|P(f)|^2}{T_b}\left(\sum_{n=-\infty}^{\infty} R_n\, e^{-j2\pi nfT_b}\right)$$

$$= \frac{|P(f)|^2}{T_b}\left(R_0 + 2\sum_{n=1}^{\infty} R_n \cos 2\pi nfT_b\right)$$

Using this result, we are able to find the PSDs of various line codes.

Example - Polar Signaling

- '1' → pulse $p(t)$
  '0' → pulse $-p(t)$



- a_k is equally likely to be 1 or -1
- $(a_k)^2 = 1$

Find R_0

$$R_0 = \lim_{N \to \infty} \frac{1}{N} \sum_k a_k^2$$

- Since there are N pulses in the interval T,

$$\sum_k a_k^2 = N$$

and

$$R_0 = \lim_{N \to \infty} \frac{1}{N} \left(N \right) = 1$$

Find R_1

a_k	a_{k+1}	$a_k a_{k+1}$
+1	+1	+1
+1	-1	-1
-1	+1	-1
-1	-1	+1

$$R_1 = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k \, a_{k+1}$$

- On average, out of N terms,

$$a_k \, a_{k+1} = 1 \qquad \text{for } N/2 \text{ terms}$$

$$a_k \, a_{k+1} = -1 \qquad \text{for } N/2 \text{ terms}$$

Hence,

$$R_1 = \lim_{N\to\infty} \frac{1}{N} \left[\frac{N}{2}(1) + \frac{N}{2}(-1) \right] = 0$$

Find R_n and $S_y(f)$

- Similarly, for $n > 1$,

$$R_n = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{2}(1) + \frac{N}{2}(-1)\right] = 0$$

Hence,

$$S_y(f) = \frac{|P(f)|^2}{T_b}\left(R_0 + 2\sum_{n=1}^{\infty} R_n \cos 2\pi n f T_b\right)$$

$$= \frac{|P(f)|^2}{T_b}$$

Special Case

- If $p(t)$ is a half-width rectangular pulse,

$$p(t) = \mathrm{rect}\left(\frac{t}{T_b/2}\right) = \mathrm{rect}\left(\frac{2t}{T_b}\right)$$

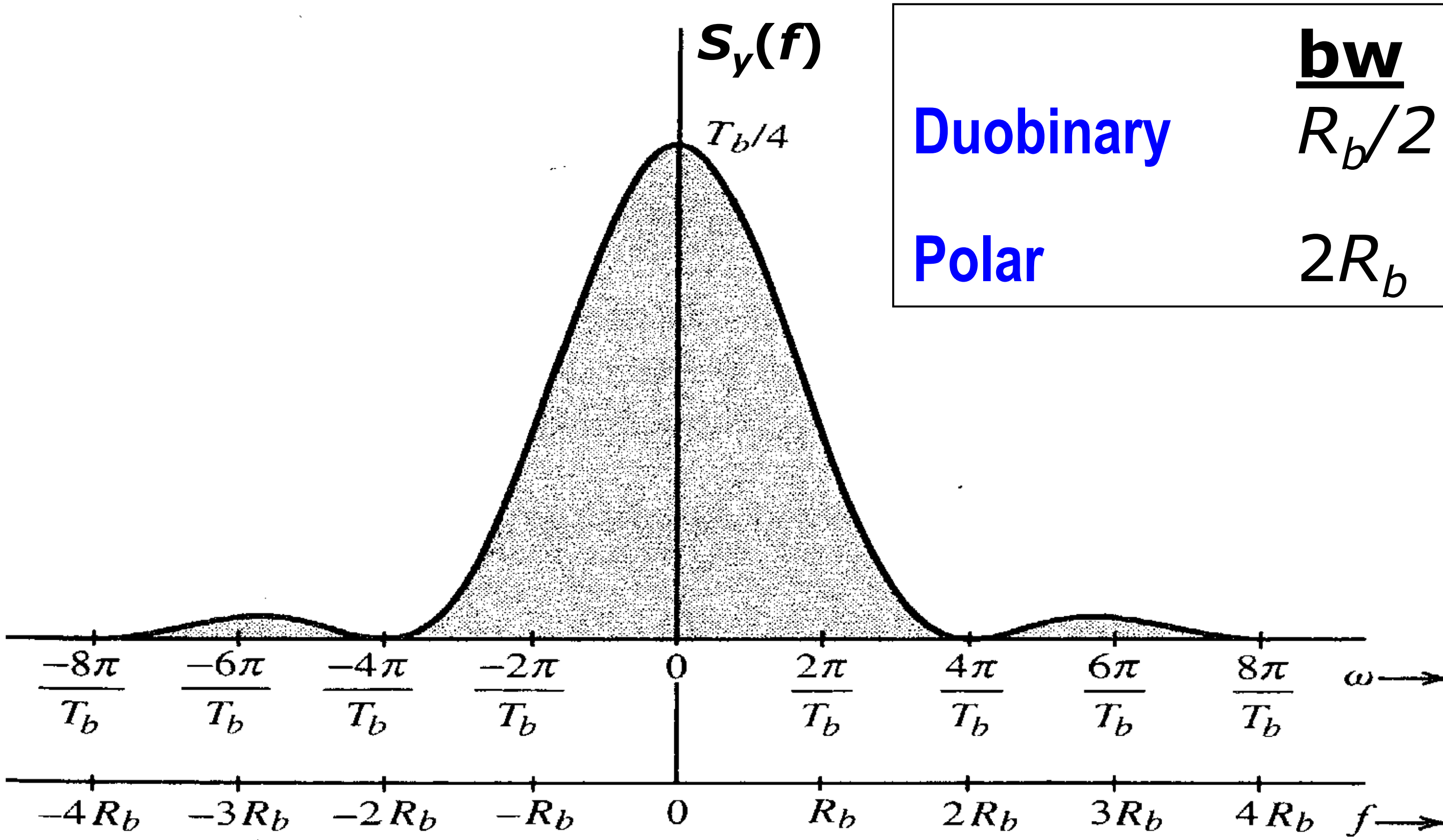
and

$$P(f) = \frac{T_b}{2}\,\mathrm{sinc}\left(\frac{fT_b}{2}\right)$$

Therefore,

$$S_y(f) = \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{fT_b}{2}\right)$$

PSD of a Polar Signal



	bw	**rate**
Duobinary	$R_b/2$	R_b
Polar	$2R_b$	R_b

|<- bw ->|

Disadvantages

- The bw is $2R_b$ Hz for the information rate R_b pulses per second. (Note that for an information rate of R_b pulses per second, the min. bw is $R_b/2$ Hz)
- Even though we use full-width pulses, the required bw is R_b Hz.
- Not suitable for an AC coupling environment
- No error detection and correction capability

Advantages

- **Most efficient**:
  For a given power level, the detection-error probability for polar signaling is the smallest.
- **Transparent**:
  Pulses are always present regardless of the bit sequence.
- **Timing extraction**:
  Rectification of a polar signal yields a periodic signal for timing extraction.

Achieving a DC Null by Pulse Shaping

- Since $S_y(f) = |P(f)|^2 S_x(f)$, we can force the PSD to have a DC null by selecting $P(f)$ properly.
- Because

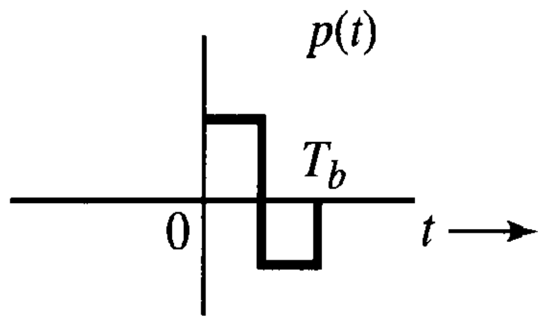
$$P(f) = \int_{-\infty}^{\infty} p(t)\, e^{-j2\pi ft}\, dt$$

we find the condition:

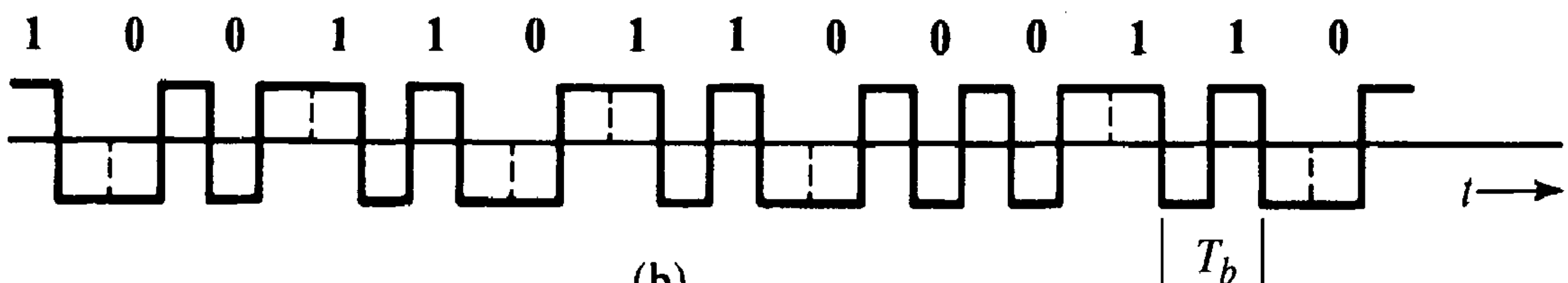
$$P(0) = \int_{-\infty}^{\infty} p(t)\, dt$$

Zero area under $p(t)$

Manchester (Split Phase) Signaling



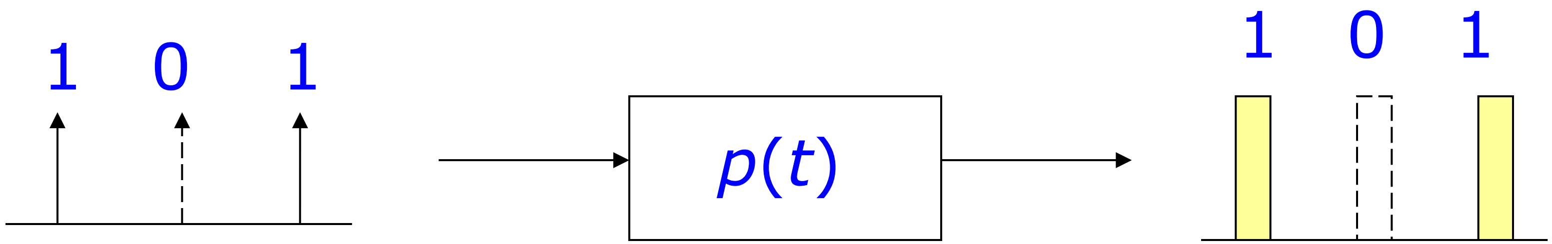
(a)



(b)

Example - on-Off Signaling

- '1' $\rightarrow$ pulse $p(t)$ and '0' $\rightarrow$ no pulse



- a_k is equally likely to be 1 or 0
- $(a_k)^2$ = 1 with probability 0.5

  = 0 with probability 0.5

Find R_0

$$R_0 = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k^2$$

- Since there are N pulses in the interval T,

$$\sum_k a_k^2 = N/2$$

and

$$R_0 = \lim_{N\to\infty} \frac{1}{N} \left(N/2\right) = 0.5$$

Find R_1

a_k	a_{k+1}	$a_k a_{k+1}$
1	1	1
1	0	0
0	1	0
0	0	0

$$R_1 = \lim_{N\to\infty} \frac{1}{N} \sum_k a_k\, a_{k+1}$$

- On average, out of N terms,

$$a_k\, a_{k+1} = 1 \qquad \text{for } N/4 \text{ terms}$$

$$a_k\, a_{k+1} = 0 \qquad \text{for } 3N/4 \text{ terms}$$

Hence,

$$R_1 = \lim_{N\to\infty} \frac{1}{N}\left[\frac{N}{4}\times 1 + \frac{3N}{4}\times 0\right] = \frac{1}{4}$$

Find R_n and $S_y(f)$

- Similarly, for $n > 1$, $R_n = 1/4$.
- Hence,

$$S_y(f) = \frac{|P(f)|^2}{T_b}\left(\sum_{n=-\infty}^{\infty} R_n \, e^{-j2\pi n f T_b}\right)$$

$$= \frac{|P(f)|^2}{T_b}\left(\frac{1}{2} + \frac{1}{4}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-j2\pi n f T_b}\right) = \frac{|P(f)|^2}{4T_b}\left(1 + \sum_{n=-\infty}^{\infty} e^{-j2\pi n f T_b}\right)$$

Exercise

Show that

$$\sum_{n=-\infty}^{\infty} e^{-j2\pi n f T_b} = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b} \right)$$

Special Case

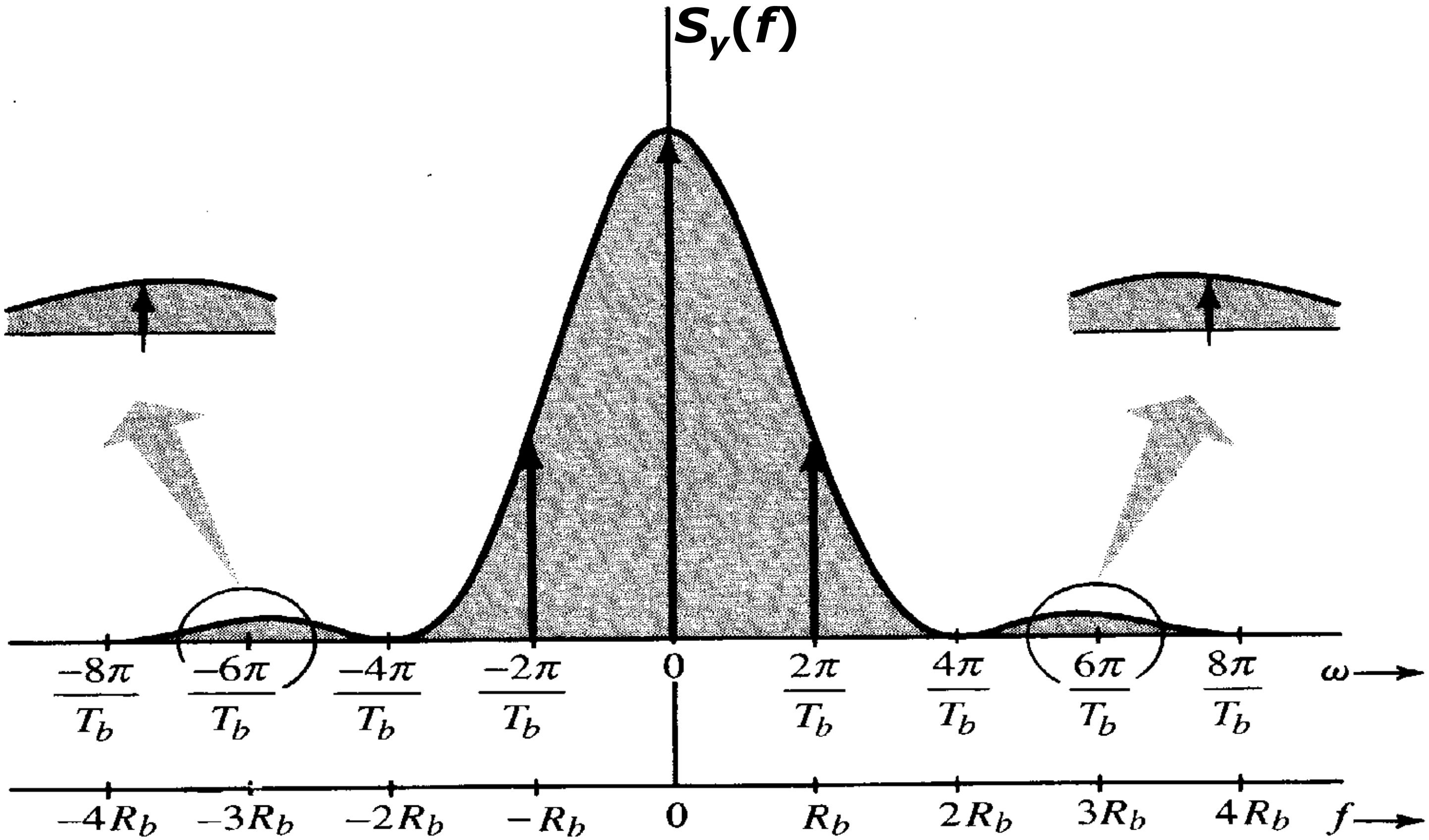
Using the result of the exercise, we have

$$S_y(f) = \frac{|P(f)|^2}{4T_b}\left[1 + \frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_b}\right)\right]$$

For the case of a half-width rectangular pulse,

$$S_y(f) = \frac{T_b}{16}\operatorname{sinc}^2\left(\frac{fT_b}{2}\right)\left[1 + \frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_b}\right)\right]$$

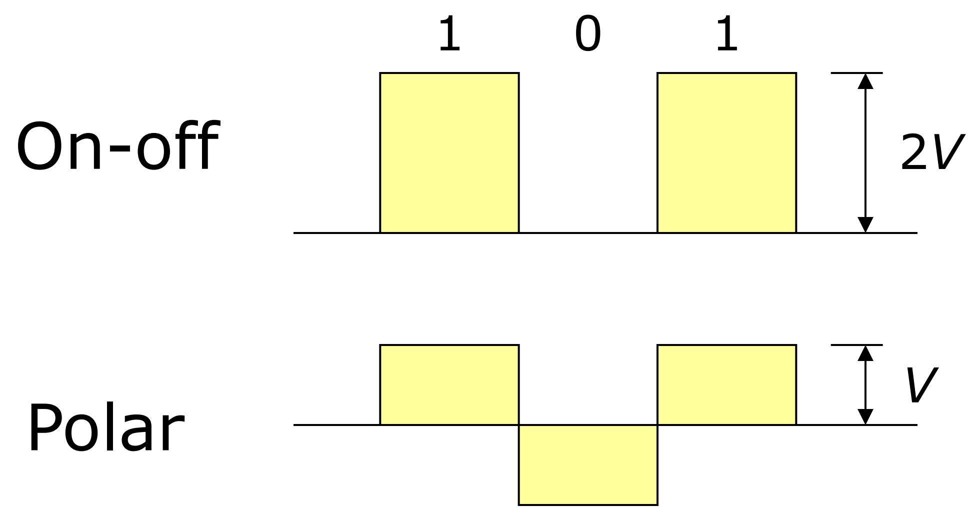
PSD of On-Off Signals



Exercise – Noise Immunity

The following two signals have the same noise immunity.

Show that the signal power of the on-off signal is twice that of the polar signal.



Alternatively, for a given signal power, the polar signal has a bigger noise immunity.

Disadvantages

- Excessive bw ($2R_b$)
- Not suitable for an AC coupling environment
- No error detection and correction capability
- Not transparent (if too many 0 bits)
- For a given value of transmitted power, it has less immunity to noise interference than the polar scheme

Exercise – PSD of Bipolar Signaling

- For the bipolar signaling scheme,
  '0' $\rightarrow$ no pulse
  '1' $\rightarrow$ pulse $p(t)$ or $-p(t)$ (alternating)
- Show that $R_0 = 1/2$, $R_1 = -1/4$ and $R_n = 0$ for $n > 1$ and, hence,

$$S_y(f) = \frac{|P(f)|^2}{2T_b}\left[1 - \cos 2\pi f T_b\right] = \frac{|P(f)|^2}{T_b}\sin^2\left(\pi f T_b\right)$$

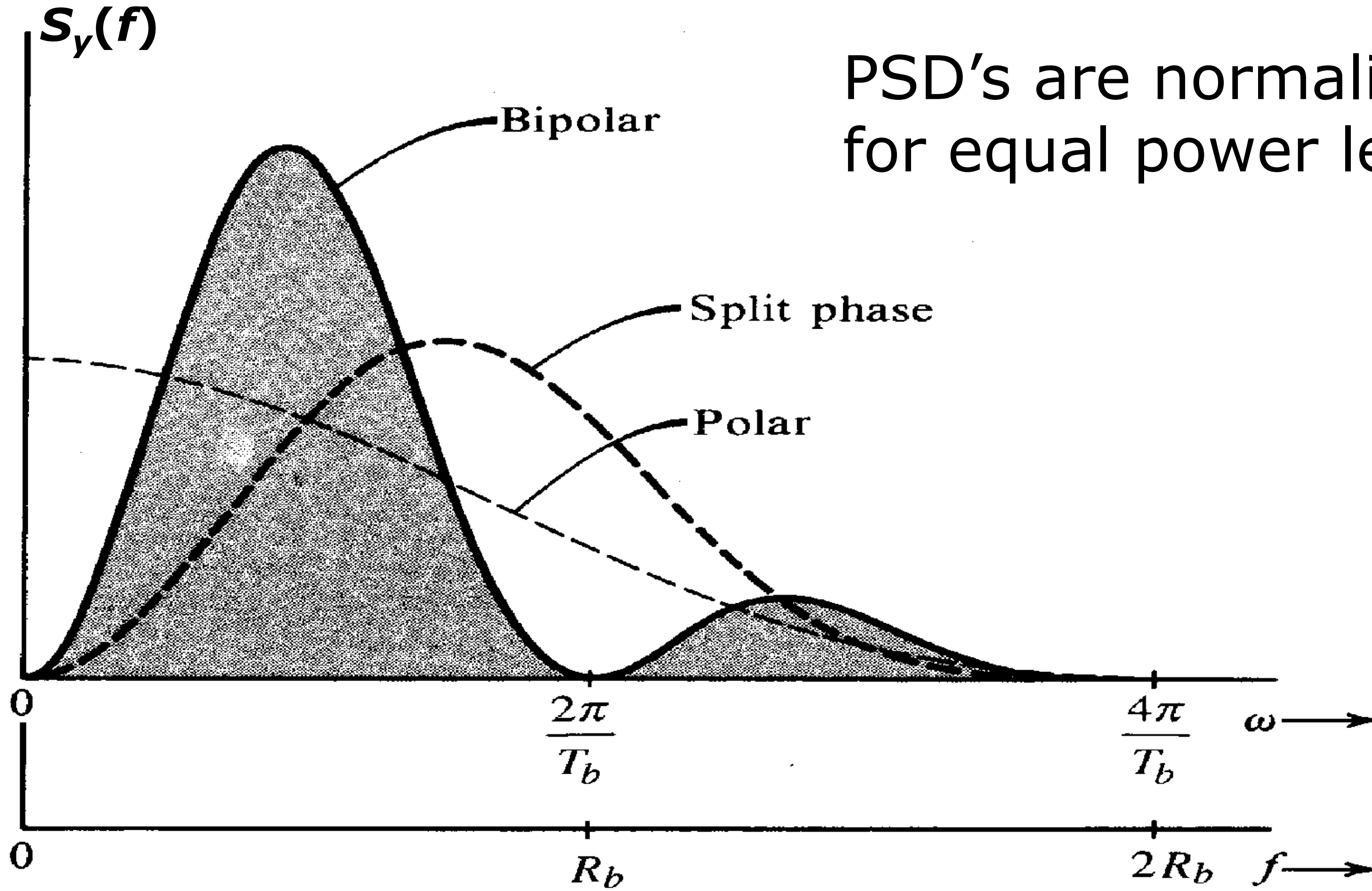
Exercise (…)

- For the case of half-width rectangular pulses,

$$S_y(f) = \frac{T_b}{4}\,\mathrm{sinc}^2\left(\frac{fT_b}{2}\right)\sin^2(\pi f T_b)$$

The essential bw of the signal is R_b, which is half that of polar or on-off signaling and twice the minimum requirement.

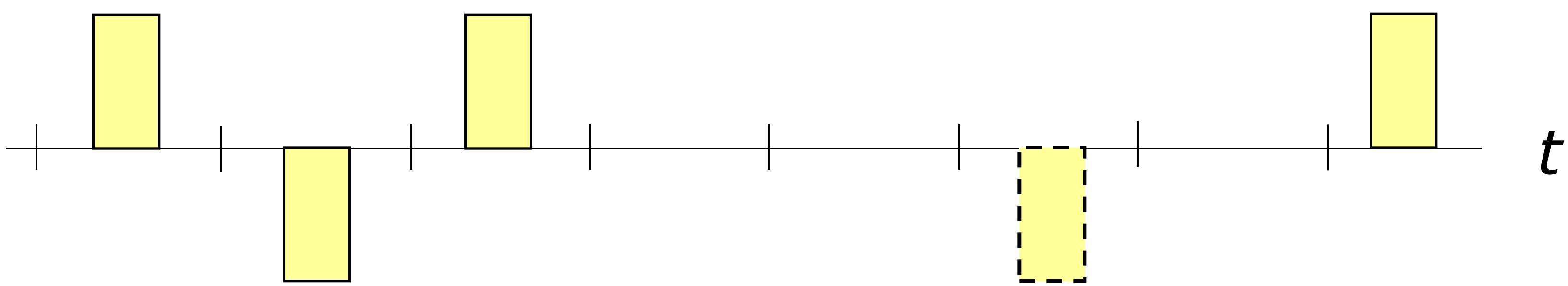
PSD of Various Signals



PSD's are normalized for equal power levels

Advantages

- DC null
- BW not excessive
- Single-error-detection capability
- Rectification gives a discrete component at the clock frequency



Pulse Shaping

Pulse Shaping

- The PSD of digital signals can be controlled by

$$S_y(f) = |P(f)|^2 S_x(f)$$

Pulse shape

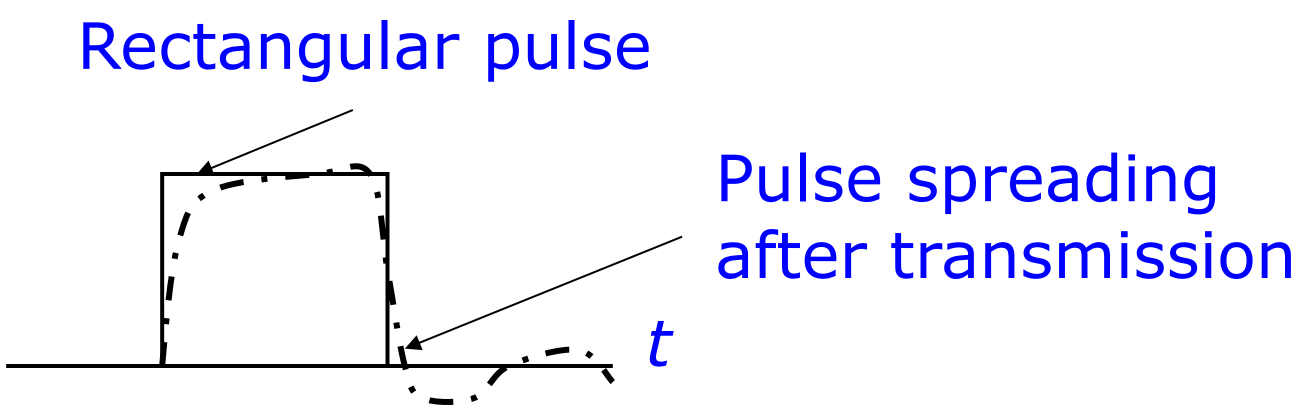
Line code

Rectangular Pulse

- In our previous study, we used a half-width rectangular pulse for illustration. Its bw is, strictly speaking, infinite.
- For example, the bw of a bipolar signal is infinite but the essential bw is R_b Hz. When it is sent over a channel of bw R_b Hz, a small portion of the spectrum is suppressed.

Pulse Spreading and ISI

- The consequence of this suppression is **pulse spreading**. Spreading of a pulse beyond its interval T_b will cause **intersymbol interference** (**ISI**).



Band-limited Signal ?

- We can try to resolve the problem by using band-limited pulses so that they can be transmitted intact over a band-limited channel.
- However, band-limited pulses **cannot** be time-limited.
- Spreading of a pulse beyond its interval T_b will interfere with neighboring pulses and cause ISI.

Solution

- Note that if there is no ISI at the **instants** of decision making, pulse amplitudes can be detected correctly despite pulse spreading.
- This is possible by choosing a properly shaped band-limited pulse.
- Need to study Nyquist’s criteria.

Nyquist's First Criterion

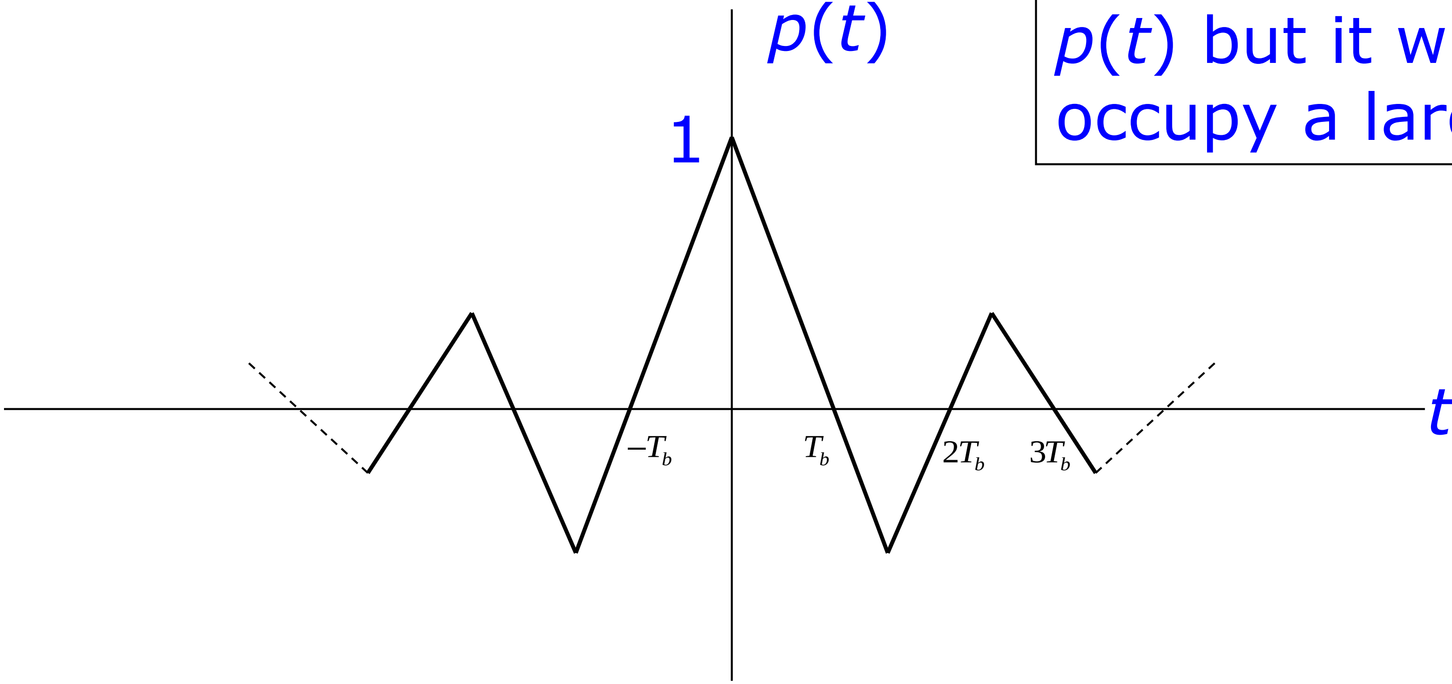
Nyquist achieves zero ISI by choosing the pulse shape as follows:

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm nT_b \end{cases}$$

where $T_b = 1/R_b$ is the separation between successive transmitted pulses and $n = 1, 2, 3, \ldots$

An Example

This is a possible $p(t)$ but it will occupy a large bw!



Note that $R_b = 1/T_\mathrm{b}$ is the pulse rate.

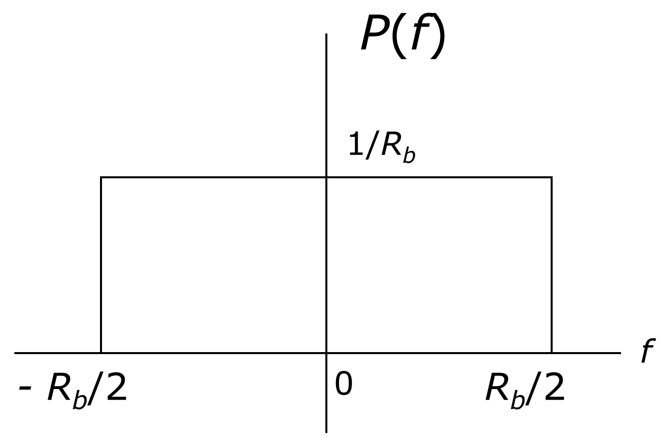
Exercise

Find the Fourier transform (FT) of

$$p(t) = \mathrm{sinc}\left(R_b t\right)$$

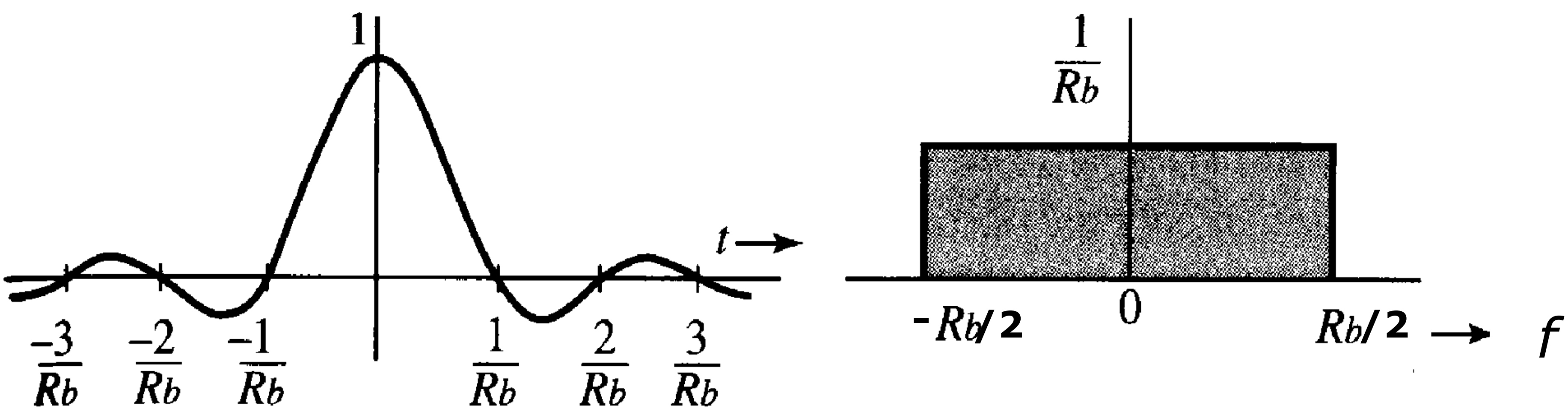
Answer:

$$P(f) = \frac{1}{R_b}\mathrm{rect}\left(\frac{f}{R_b}\right)$$



$p(t)$ with Minimum bw

If we restrict the pulse bw to be $R_b/2$, then only one pulse, $\text{sinc}(R_b t)$, satisfies the requirements as



Observations

- The bw of the pulse is $R_b/2$, which satisfies the minimum bw requirement for transmission of R_b bit/s.
- $p(t)$ can be generated as an impulse response of an ideal filter of bw $R_b/2$.
- Using this pulse, we can transmit at a rate R_b pulses per second without ISI.

Practical Problems

- The pulse starts at $t = -\infty$. (**Impractical**)
- The sinc pulse decays at a rate $1/t$. (**decays too slowly**)
- A small error in the transmission rate, sampling rate or sampling instants can add up to a very large value. Note that cumulative interference at any pulse center from all the remaining pulses is of the form $\Sigma(1/n)$. (**divergent series**)

Exercise

Show that the decay of

$$p(t) = \text{sinc}(R_b t)$$

is proportional to $1/t$.

Answer:

Control the decay

$$\text{sinc}(R_b t) = \frac{1}{\pi R_b}\left(\frac{1}{t}\right)\sin(\pi R_b t)$$

Better Solution

- To solve the decay problem, we need to relax the requirement of minimum bw and allow $R_b/2 < \text{bw} < R_b$. Hopefully, the pulse obtained will decay faster than $1/t$.
- Let us derive the requirement of $P(f)$ for $R_b/2 < \text{bw} < R_b$.

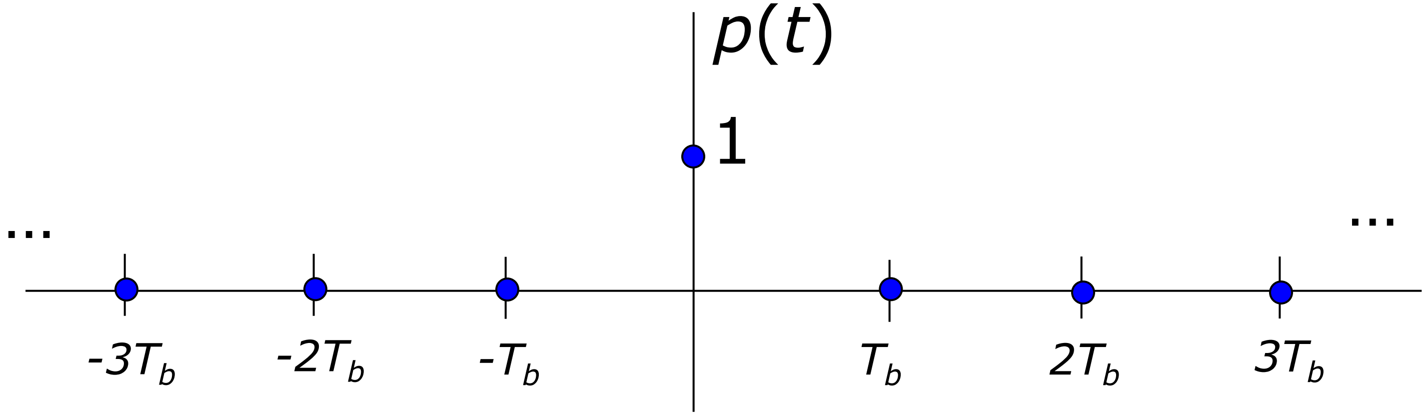
Derivation (1)

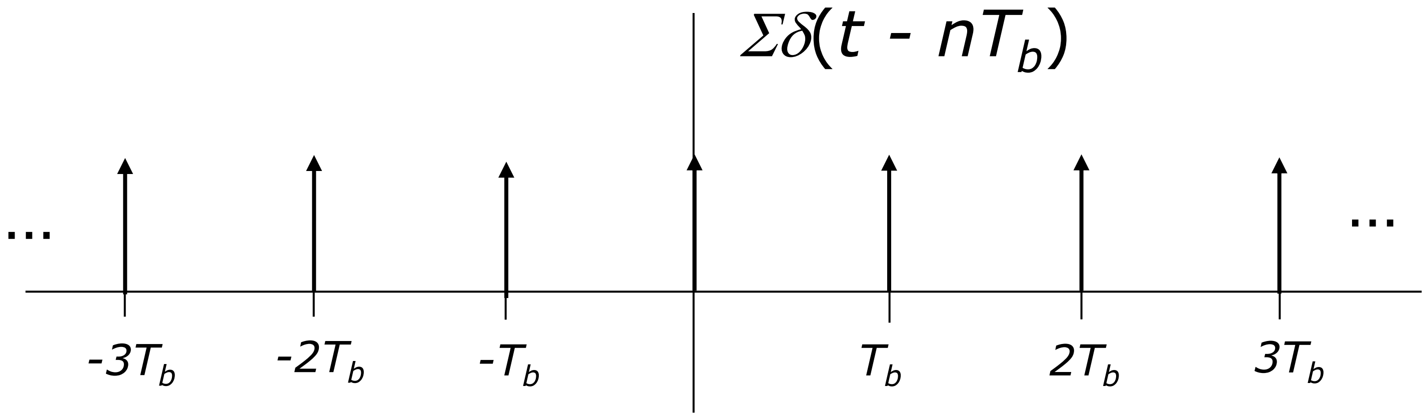
The desired pulse $p(t)$ satisfies

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm nT_b \end{cases}$$

If we sample $p(t)$ every T_b sec, then

$$p(t) \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT_b)}_{\frac{1}{T_b} \sum_{n=-\infty}^{\infty} e^{j2\pi nt/T_b}} = \delta(t)$$


p(t)
1
...
...
-3Tb
-2Tb
-Tb
Tb
2Tb
3Tb


Σδ(t - nTb)
...
...
-3Tb
-2Tb
-Tb
Tb
2Tb
3Tb

Derivation (2)

Taking FT on both sides, we have

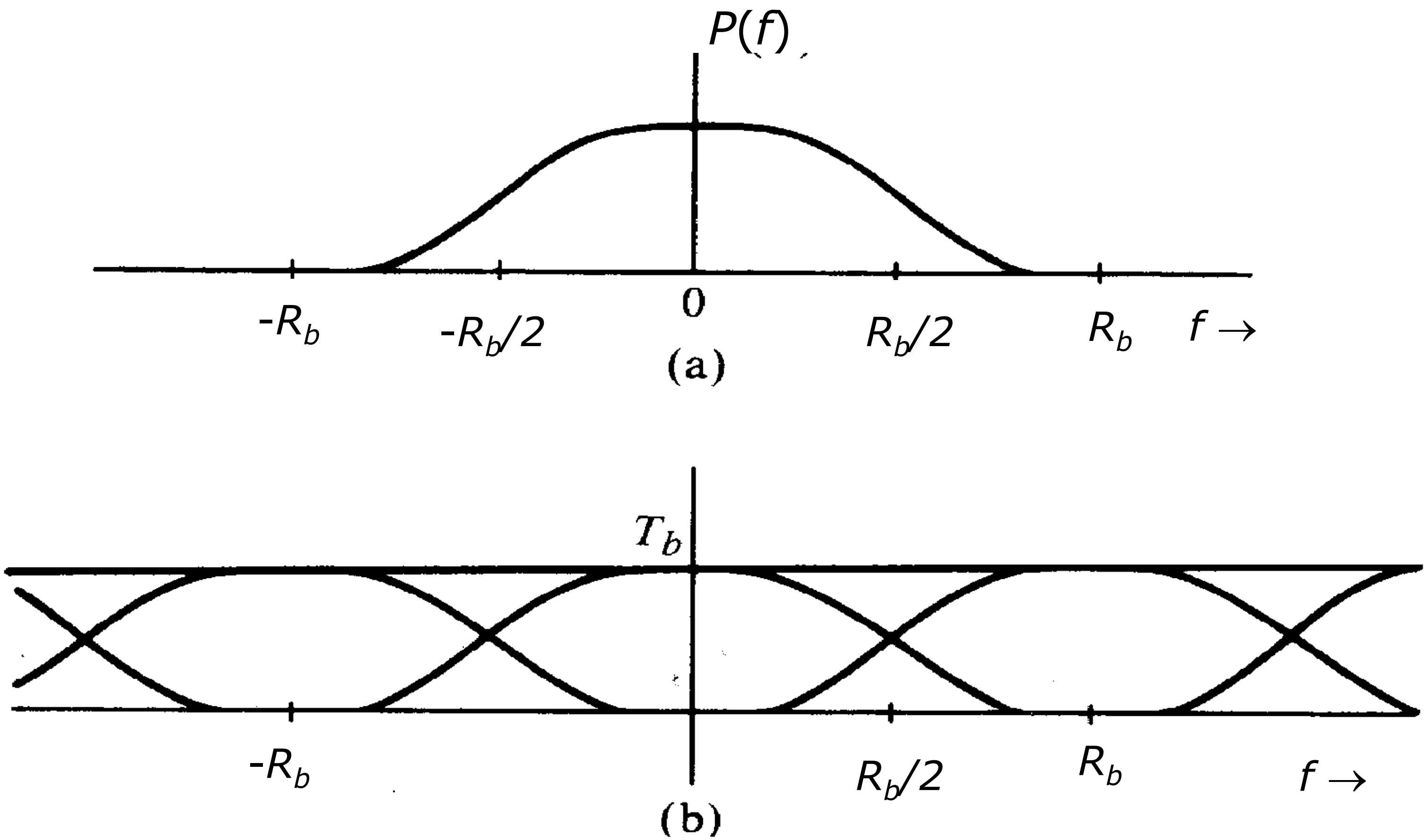
$$\frac{1}{T_b}\sum_{n=-\infty}^{\infty} P\left(f - \frac{n}{T_b} \right) = 1$$

or

$$\sum_{n=-\infty}^{\infty} P\left(f - nR_b \right) = T_b$$

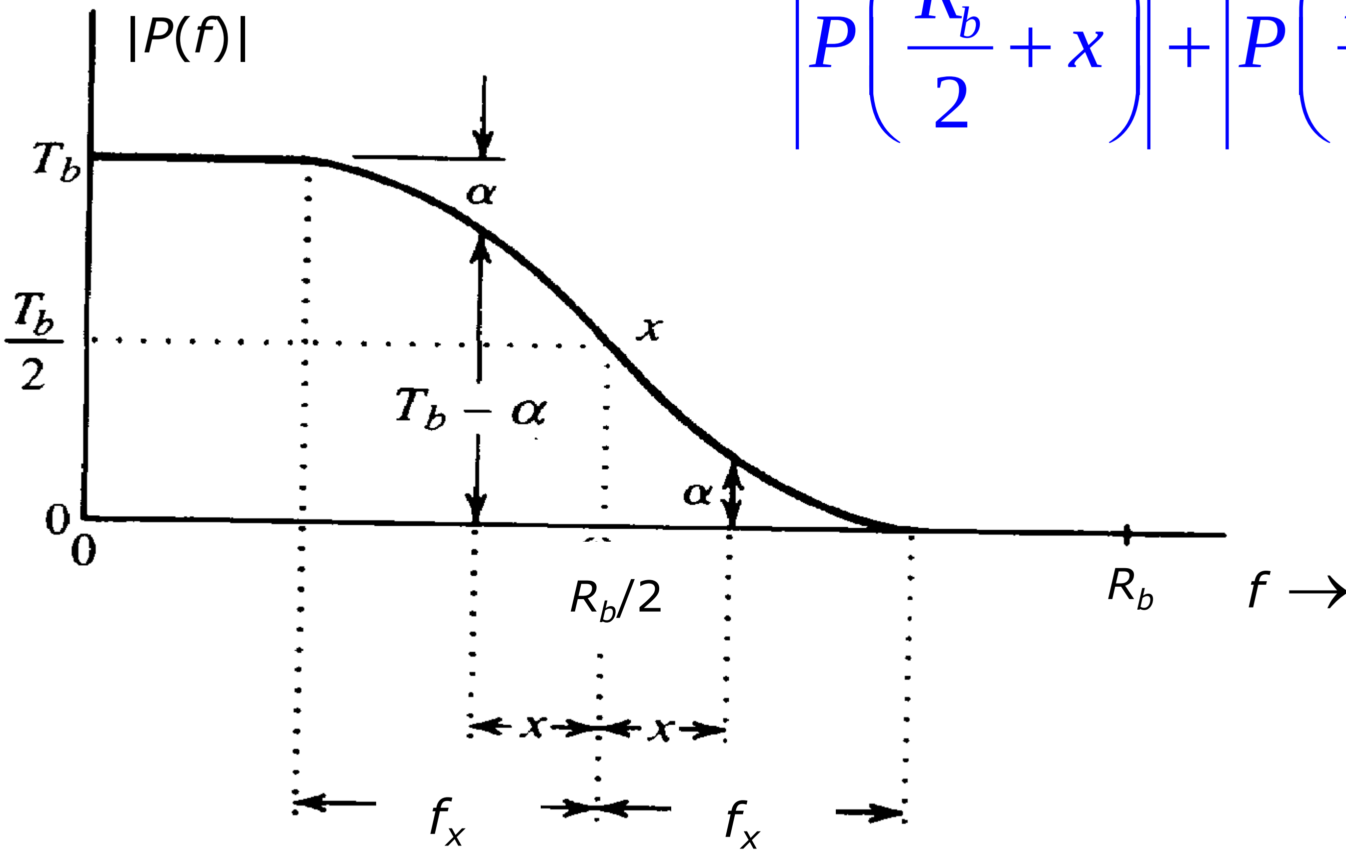
Thus, the sum of the spectra formed by repeating $P(f)$ every R_b is a constant.

Derivation (3)



Vestigial Spectrum

$$\left| P\left(\frac{R_b}{2}+x\right)\right| + \left| P\left(\frac{R_b}{2}-x\right)\right| = T_b$$



Roll-Off Factor *r*

- Let r be the ratio of the excess bw f_x to the theoretical min. bw $R_b/2$,

Roll-off factor $$r = \frac{f_x}{R_b/2} = \frac{2f_x}{R_b}$$

- Then the bw B_T of $P(f)$ in Hz is

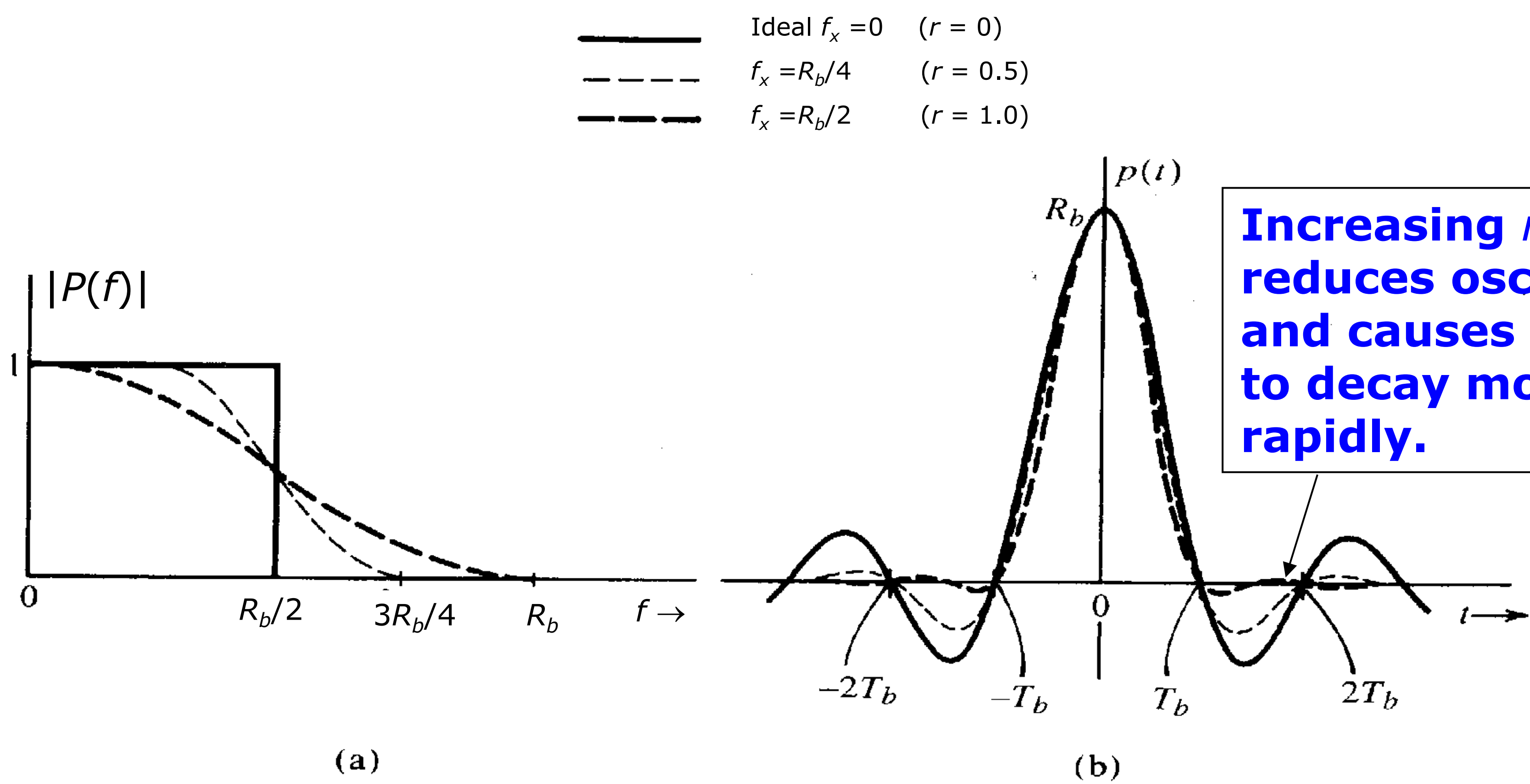
$$B_T = \frac{R_b}{2} + r\frac{R_b}{2} = (1+r)\frac{R_b}{2}$$

Family of Pulses

$$P(f) = \begin{cases} \dfrac{1}{2}\left\{1 - \sin\left(\dfrac{\pi[f-(R_b/2)]}{2f_x}\right)\right\} & \left|f - \dfrac{R_b}{2}\right| < f_x \\ 0 & |f| > \dfrac{R_b}{2} + f_x \\ 1 & |f| < \dfrac{R_b}{2} - f_x \end{cases}$$

satisfies the Nyquist 1st criterion.

Satisfying Nyquist Criterion



Increasing r reduces oscillation and causes it to decay more rapidly.

Exercise - Raised-Cosine

When $r = 1$,

$$P(f) = \frac{1}{2}\left(1 + \cos\frac{\pi f}{R_b}\right)\mathrm{rect}\left(\frac{f}{2R_b}\right)$$

$$= \cos^2\left(\frac{\pi f}{2R_b}\right)\mathrm{rect}\left(\frac{f}{2R_b}\right)$$

Find the time-domain pulse $p(t)$.

Answer:

$$p(t) = R_b \frac{\cos \pi R_b t}{1 - 4R_b^2 t^2}\,\mathrm{sinc}(R_b t)$$

Observations of Raised-Cosine ($r = 1$)

- bw = R_b Hz
- $p(0) = R_b$
- Zero crossings at T_b, $2T_b$, $3T_b$, …
  and at points midways between all the signaling instants
- Decays rapidly, as $1/t^3$
- Relatively insensitive to timing error

Example

For a scheme of pulse transmission using the Nyquist 1st criterion, determine the pulse transmission rate R_b in terms of bw B_T and the roll-off factor r.

Solution:

Since $B_T = (1 + r)R_b/2$, we have $R_b = 2B_T/(1+r)$. Because $0 \leq r \leq 1$, the pulse transmission rate varies from $2B_T$ *to* B_T, depending on the choice of r.