Bonus Problem (Moment generating functions)

Problem. For the random variable X, the moments can be expressed as

$$E\left(X^n\right) = \frac{2^n}{n+1}, \quad n = 1, 2, 3, \ldots$$

Find some (in fact, the unique) distribution of X having these moments.
Hint: Study the moment generating function of X and use the fact that

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}$$

Solution. Following the hint, it follows that

$$M_X(t) = E\left(e^{tX}\right) = E\left(\sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left(X^n\right)$$

Next we use the fact that the moments of X are known.

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{2^n}{n+1} = \sum_{n=0}^{\infty} \frac{(2t)^n}{(n+1)!}$$

We notice that the summand resembles the power series expansion of e^{2t}. We must, however, first make some adjustments. First we let $m = n+1$, i.e.

$$M_X(t) = \sum_{m=1}^{\infty} \frac{(2t)^{m-1}}{m!} = \frac{1}{2t} \sum_{m=1}^{\infty} \frac{(2t)^m}{m!}$$

Now the summand is correct, but because of the change of variable the summation starts at $m = 1$ and not at $m = 0$ as it should. We notice that the term for $m = 0$ is $(2t)^0 / 0! = 1$, and therefore

$$\sum_{m=1}^{\infty} \frac{(2t)^m}{m!} = \sum_{m=0}^{\infty} \frac{(2t)^m}{m!} - 1$$

And so, finally,

$$M_X(t) = \frac{1}{2t}\left(\sum_{m=0}^{\infty} \frac{(2t)^m}{m!} - 1\right) = \frac{e^{2t} - 1}{2t}$$

which we recognize as the mgf of $U(0, 2)$.