MH2500 Probability and Introduction to Statistics

Handout 10 - Distributions Derived from the Normal Distribution

Synopsis

We discuss three probability distributions derived from the normal distribution $N(\mu, \sigma^2)$. These three distributions occur in many statistical problems.

- χ^2 distribution
- t distribution
- F distribution

χ^2 Distribution

Definition

If Z is a standard normal random variable, the distribution of $U = Z^2$ is called the chi-square distribution with 1 degree of freedom.

Definition

If $U_1, U_2, ..., U_n$ are independent chi-square random variables with 1 degree of freedom, the distribution of $V = U_1 + U_2 + \cdots + U_n$ is called the *chi-square distribution with n degrees of freedom* and is denoted by χ_n^2.

χ^2 Distribution

- The chi-square distribution with 1 degree of freedom is a special case of the gamma distribution with parameters $1/2$ and $1/2$ (Section 2.3).
- The sum of independent gamma random variables that have the same value of λ follows a gamma distribution (Section 4.5, Handout 8 Slide 26).
- The chi-square distribution with n degrees of freedom is a gamma distribution with $\alpha = n/2$ and $\lambda = 1/2$.

Density function of χ^2 distribution

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{\frac{n}{2}-1} e^{-v/2}, \quad 0 \leq v.$$

t Distribution

Definition

Let $Z \sim N(0,1)$ and $U \sim \chi_n^2$ and Z and U are independent, then the distribution of $Z/\sqrt{U/n}$ is called the t **distribution** with n degrees of freedom.

Proposition A

The density function of the t distribution with n degrees of freedom is

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}.$$

F Distribution

Definition

Let U and V be independent chi-square random variables with m and n degrees of freedom, respectively. The distribution of

$$W = \frac{U/m}{V/n}$$

is called the F distribution with m and n degrees of freedom and is denoted by $F_{m,n}$.

Proposition B

The density function of W is

$$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n} w\right)^{-(m+n)/2}, \quad w \geq 0.$$

The sample mean and sample variance

Definition

Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables, sometimes we refer to them as a **sample** from a normal distribution. Define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

These are called the **sample mean** and **sample variance**, respectively.

The sample mean and sample variance

Corollary A

$\overline{X}$ and S^2 are independently distributed.

Theorem B

The distribution of $(n-1)S^2/\sigma^2$ is the chi-square distribution with $n-1$ degrees of freedom.

Corollary B

Given $\overline{X}$ and S^2, then

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}.$$