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MATH 175 - H4L

Exercise 4.1: Gaussian Elimination

1 Introduction to the Problem

In this exercise, we consider the matrix equation $Ax = b$ given by

$$\begin{bmatrix} 1 & -1 & 3 & 4 \\ -4 & -6 & 15 & 20 \\ 4 & -3 & 3 & 7 \\ 1 & 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 26 \\ 19 \\ 17 \end{bmatrix}$$

and solve the system using variants of Gaussian Elimination. Particularly we apply Gaussian Elimination (a) without pivoting, (b) with partial pivoting, and (c) with scaled pivoting to the given matrix and compare their solutions.
To perform each of these Gaussian Elimination methods, we reconstruct the given problem into its augmented matrix form given by

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ -4 & -6 & 15 & 20 & 26 \\ 4 & -3 & 3 & 7 & 19 \\ 1 & 1 & -2 & 4 & 17 \end{array}\right]$$

2 Without Pivoting

Solving the problem using Gaussian Elimination without pivoting, we have

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ -4 & -6 & 15 & 20 & 26 \\ 4 & -3 & 3 & 7 & 19 \\ 1 & 1 & -2 & 4 & 17 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 + 4R_1, R_3 \to R_3 - 4R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ 0 & -10 & 27 & 36 & 54 \\ 0 & 1 & -9 & -9 & -9 \\ 0 & 2 & -5 & 0 & 10 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - \frac{1}{5}R_2]{R_3 \to R_3 + \frac{1}{10}R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ 0 & -10 & 27 & 36 & 54 \\ 0 & 0 & -6.3 & -5.4 & -3.6 \\ 0 & 0 & 0.4 & 7.2 & 20.8 \end{array}\right] \xrightarrow{R_4 \to R_4 + \frac{0.4}{6.3}R_2}$$

$$\begin{bmatrix} 1 & -1 & 3 & 4 & 7 \\ 0 & -10 & 27 & 36 & 54 \\ 0 & 0 & -6.3 & -5.4 & -3.6 \\ 0 & 0 & 0 & 6.857142857 & 20.57142857 \end{bmatrix}$$

Performing back substitution on the final augmented matrix, we have

$$x_4 = \frac{20.57142857}{6.857142857} = 3 \qquad x_2 = \frac{54 - 36(3) - 27(-2)}{-10} = 0$$

$$x_3 = \frac{-3.6 + 5.4(3)}{-6.3} = -2 \qquad x_1 = \frac{7 - 4(3) - 3(-2) + 1(0)}{1} = 1$$

Equivalently, the solution to the given matrix is $x = [x_1, x_2, x_3, x_4]^T = [1, 0, -2, 3]^T$.

3 With Partial Pivoting

We start again with the augmented matrix given by

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ -4 & -6 & 15 & 20 & 26 \\ 4 & -3 & 3 & 7 & 19 \\ 1 & 1 & -2 & 4 & 17 \end{array}\right]$$

- 1^{st} pass: Checking the entries in the first column, we see that the largest magnitude is a_{21} or a_{31} which is 4. For this, we arbitrarily choose row 2 as our pivot and interchange rows 1 and 2. After this, we perform elementary row operations to obtain the updated system

$$\left[\begin{array}{cccc|c} -4 & -6 & 15 & 20 & 26 \\ 0 & -2.5 & 6.75 & 9 & 13.5 \\ 0 & -9 & 18 & 27 & 45 \\ 0 & -0.5 & 1.75 & 9 & 23.5 \end{array}\right]$$

- 2^{nd} pass: Now, we look at the entries of the second column from row 2 to 4. We observe that the largest magnitude is 9 which is obtained from a_{32}. Thus, we interchange rows 2 and 3 and perform elementary row operations that will make the new (after interchange) a_{32} and a_{42} zero. After completing the elimination pass, we have

$$\left[\begin{array}{cccc|c} -4 & -6 & 15 & 20 & 26 \\ 0 & -9 & 18 & 27 & 45 \\ 0 & 0 & 1.75 & 1.5 & 1 \\ 0 & 0 & 0.75 & 7.5 & 21 \end{array}\right]$$

- 3^{rd} pass: Finally, we look at the third column rows $3-4$. We see here that the largest magnitude is obtained from a_{33} which is 1.75. So, we don't interchange any rows and proceed to perform elementary row operations on row 4 that will force $a_{34}=0$. The resulting augmented matrix after the elimination pass is

$$\left[\begin{array}{cccc|c} -4 & -6 & 15 & 20 & 26 \\ 0 & -9 & 18 & 27 & 45 \\ 0 & 0 & 1.75 & 1.5 & 1 \\ 0 & 0 & 0 & 6.8571... & 20.5714... \end{array}\right]$$

Performing back substition to the final augmented matrix, we have the following:

$$x_4 = \frac{20.5714...}{6.85714...} = 3 \qquad x_2 = \frac{45 - 27(3) - 18(-2)}{-9} = 0$$

$$x_3 = \frac{1 - 1.5(3)}{1.75} = -2 \qquad x_1 = \frac{26 - 20(3) - 15(-2) + 6(0)}{-4} = 1$$

Equivalently, the solution to the given matrix is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

4 With Scaled Pivoting

We start again with the augmented matrix given by

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 7 \\ -4 & -6 & 15 & 20 & 26 \\ 4 & -3 & 3 & 7 & 19 \\ 1 & 1 & -2 & 4 & 17 \end{array}\right]$$

For this variation, we initialize a row tracker matrix $r = [1, 2, 3, 4]^T$ and compute for the scale vector. Doing so, we have $s = [4, 20, 7, 4]^T$.

- 1^{st} pass: We now compute for the scaled magnitudes of the entries in column1, rows 1 to 4. The results are tabulated below.

j	1	2	3	4
r_k	1	2	3	4
M_k	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{4}{7}$	$\frac{1}{4}$

  This implies that $M = M_3$ and so we'll pivot on Row 3. Updating the row tracker we have $r = [3, 2, 1, 4]^T$. The elimination pass will then yield

$$\left[\begin{array}{cccc|c} 0 & -0.25 & 2.25 & 2.25 & 2.25 \\ 0 & -9 & 18 & 27 & 45 \\ 4 & -3 & 3 & 7 & 19 \\ 0 & 1.75 & -2.75 & 2.25 & 12.25 \end{array}\right]$$

- 2^{nd} pass: Similar with the first pass, we shall pick the pivot element by comparing the scaled magnitudes for column 2, rows 2 to 4. We have

j	2	3	4
r_k	2	1	4
M_k	$\frac{9}{20}$	$\frac{1}{28}$	$\frac{7}{16}$

  This implies that $M = M_2$ and so we'll pivot on Row 2. Updating the row tracker we have $r = [3, 2, 1, 4]^T$. The elimination pass will then yield

$$\left[\begin{array}{cccc|c} 0 & 0 & 1.75 & 1.5 & 1 \\ 0 & -9 & 18 & 27 & 45 \\ 4 & -3 & 3 & 7 & 19 \\ 0 & 0 & 13 & 18 & 28 \end{array}\right]$$

- 3^{rd} pass: We pick again the pivot element using the scaled magnitudes, this time from column 3, rows 3 to 4. We have

j	3	4
r_k	1	4
M_k	$\frac{7}{16}$	$\frac{3}{16}$

  This implies that $M = M_1$ and so we'll pivot on Row 1. Updating the row tracker we have $r = [3, 2, 1, 4]^T$. The elimination pass will then yield

$$\left[\begin{array}{cccc|c} 0 & 0 & 1.75 & 1.5 & 1 \\ 0 & -9 & 18 & 27 & 45 \\ 4 & -3 & 3 & 7 & 19 \\ 0 & 0 & 0 & 6.85714... & 20.5714... \end{array}\right]$$

Performing back substition to the final augmented matrix, we have the following:

$$x_4 = \frac{20.5714...}{6.85714...} = 3 \qquad x_2 = \frac{45 - 27(3) - 18(-2)}{-9} = 0$$

$$x_3 = \frac{1 - 1.5(3)}{1.75} = -2 \qquad x_1 = \frac{19 - 7(3) - 3(-2) + 3(0)}{4} = 1$$

Equivalently, the solution to the given matrix is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$