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MATH 175 - H4L

Exercise 3.2: More on Root-finding

I. Introduction to the Problem

In this exercise, we consider the function

$$f(x) = 1 + \ln x - x \tag{3.2.1}$$

and initially determine the multiplicity of $x = 1$ as a root of f. After which, we implement the Newton-Raphson Method on f using an initial guess of $x_0 = 2$ with a jump-based stopping criterion with error tolerance $\epsilon = 10^{-8}$ and maximum iteration $N_{\max} = 100$. In the iteration process, we also calculate for the rate of convergence q of the sequences of estimates for the root r of f as well as the asymptotic error constant λ. Finally, we apply the Steffensen's method on f using the same initial guess $x_0 = 2$ and jump-based stopping criterion. The results of the two methods applied will then be compared and contrasted.

II. Determining Multiplicity

We note from the discussion that if f is sufficiently differentiable, then we say that r is a root of f with multiplicity m if and only if

$$f(r) = f'(r) = f''(r) = \cdots = f^{(m-1)}(r) = 0, f^{(m)}(r) \neq 0$$

Hence, to determine the multiplicity of $x = 1$ as a root of the function f in Equation 3.2.1, we differentiate f and stop after m times if $f^{(m)}(r) \neq 0$. Solving this we have

$$\begin{aligned}
f(x) &= 1 + \ln x - x & &\Longrightarrow f(1) = 1 + \ln 1 - 1 = 0 \\
f'(x) &= \frac{1}{x} - 1 & &\Longrightarrow f'(1) = \frac{1}{1} - 1 = 0 \\
f''(x) &= -\frac{1}{x^2} & &\Longrightarrow f''(1) = -\frac{1}{1^2} = -1 \neq 0
\end{aligned}$$

Since $f''(1) \neq 0$, then $x = 1$ is a root of f of multiplicity $m = 2$.

III. Newton-Raphson Implementation

We now apply the Newton-Raphson Method on f using the initial guess $x_0 = 2$ and with a jump-based stopping criterion with error tolerance $\epsilon = 10^{-8}$ and maximum iteration $N_{\max} = 100$. Note that $(n+1)^{\text{th}}$ iteration is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for } n \geq 0$$

We also compute for an estimate of the asymptotic error constant λ. Since we know from **I.** that $x = 1$ is a root of f, then we let $r = 1$ and solve λ by

$$\lim_{n \to \infty} \frac{|r - x_{n+1}|}{|r - x_n|^q} = \lambda \tag{3.2.2}$$

If the estimates for λ converges, we define q as the rate of convergence of our estimates for the root. Implementing these in MATLAB, we have the result in the next page.

n	x_n	rel jump	lambda (q=1)	lambda (q=2)
1	1.386294	3.068528E-01	0.386294	0.386294
2	1.172192	1.544421E-01	0.445754	1.153923
3	1.081540	7.733526E-02	0.473543	2.750084
4	1.039705	3.868118E-02	0.486938	5.971742
5	1.019595	1.934224E-02	0.493511	12.429392
6	1.009734	9.671324E-03	0.496766	25.351768
7	1.004851	4.835687E-03	0.498386	51.200036
8	1.002422	2.417847E-03	0.499193	102.898324
9	1.001210	1.208924E-03	0.499597	206.295775
10	1.000605	6.044619E-04	0.499798	413.091113
11	1.000302	3.022310E-04	0.499899	826.682007
12	1.000151	1.511155E-04	0.499950	1653.863905
13	1.000076	7.555774E-05	0.499975	3308.227756
14	1.000038	3.777887E-05	0.499987	6616.955485
15	1.000019	1.888944E-05	0.499994	13234.410956
16	1.000009	9.444718E-06	0.499997	26469.321905
17	1.000005	4.722359E-06	0.499998	52939.143806
18	1.000002	2.361179E-06	0.499999	105878.787615
19	1.000001	1.180590E-06	0.500000	211758.075200
20	1.000001	5.902949E-07	0.500000	423516.650392
21	1.000000	2.951474E-07	0.500000	847033.801015
22	1.000000	1.475737E-07	0.500000	1694068.101625
23	1.000000	7.378686E-08	0.500000	3388136.702846
24	1.000000	3.689343E-08	0.500000	6776273.915483
25	1.000000	1.844671E-08	0.500000	13552548.320365
26	1.000000	9.223357E-09	0.500000	27105097.293263

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Root is 1.0000000092 after 26 iterations.
The function value at the estimate is -1.701406E-16.
The rate of convergence is q = 1 with asympototic error constant lambda = 0.500000
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Figure 3.2.1: Result of Implementing Newton-Raphson Method on f in MATLAB.

From the table, we see that a good estimate is obtained after 26 iterations. The estimated root is $r^* = 1.0000000092$ with $f(r^*)$ having an order of magnitude of 10^{-16}. This verifies the goodness of r^* as a root of f. Moreover, we see that for $q = 1$ used on Equation 3.2.2, the value converges to $\lambda = 0.5$. Therefore, we conclude that the rate of convergence is $q = 1$ and the asymptotic error constant is $\lambda = 0.5$. That is, the Newton-Raphson Method resulted with a *linear* rate of convergence.

III. Steffensen's Method Implementation

We now implement the Aitken-I Method on f using the same initial conditions and stopping criterion from **II.** This method is also known as the *Steffensen's Method*. In this method, we follow the same pattern from Newton-Raphson except that for every iteration of multiple of 3, i.e., $3, 6, 9, \ldots$, we use the formula

$$x_n = x_{n-1} + \frac{\lambda}{1-\lambda}(x_{n-1} - x_{n-2}) \tag{3.2.3}$$

to find the n^{th} estimate for the root where $\lambda = 0.5$ as obtained from **II.** Moreover, we compute for the asymptotic error constant λ given from Equation 3.2.2 and use the values 1 and 2 for q. Implementing this in Excel, we have the results given in the next page.

			NR	Aitken	NR	Aiten
n	x_n	rel_jump	lambda	lambda	lambda	lambda
0	2.00000000000		q = 1	q=1	q=2	q=2
1	1.38629436112	0.306853	0.3862944		0.386294	
2	1.17219218899	0.154442	0.4457538		1.153923	
3	0.95809001686	0.182651	0.2433907	0.2433907	1.413483	1.413483
4	0.97874597423	0.02156	0.5071352		12.10058	
5	0.98929688773	0.01078	0.5035805		23.69342	
6	0.99984780124	0.010665	0.01422	0.01422	1.32859	1.32859
7	0.99992389676	7.61E-05	0.5000254		3285.344	
8	0.99996194741	3.81E-05	0.5000127		6570.189	
9	0.99999999807	3.81E-05	5.0765E-05	5.0765E-05	1.334064	1.334064
10	0.99999999807	0	1		5.18E+08	

Figure 3.2.2: Result of Implementing Steffensen's Method on f in Excel.

We note that the method halted after 10 iterations. An estimate $r^* = 0.99999999807$ is obtained with $f(r^*)$ having an order of magnitude of 10^{-18}, hence a quite good estimate and better than the estimate obtained using the Newton-Raphson Method. It can also be observed that the asymptotic error constant λ for $q = 1$ implemented with Aitken-I decreases and converges to 0. A value of $\lambda = 0$ obtained with $q = 1$ implies that the rate of convergence for the method is greater than 1. That is, the rate of convergence is *superlinear*.

From these results, we conclude that the Newton-Raphson Method did not work smoothly for the given function since it incurred a rate of convergence of $q = 1$ instead the expected $q = 2$. This is due to the result we obtained from **I.** in which $x = 1$ is a root of f of multiplicity 2. However, the result from **I.** verified that since the Newton-Raphson Method converged to a root of multiplicity 2, in this case the root is $x = 1$, then the asymptotic error constant is

$$\lambda = \frac{m-1}{m} = \frac{2-1}{2} = 0.5$$

for this root.

The result from **II.** then showed that the rate of convergence for Newton-Raphson is $q = 1$ or linear. Hence, we can apply the Aitken method. Implementing the Aitken-I (or Steffensen's) Method as shown in **III.**, showed that the rate of convergence is accelerated from linear to superlinear. Therefore, it is concluded that the Steffensen's Method improved the sequence of estimates for the root $x = 1$.

Remark: *The program used to solve this problem is made in MATLAB and Excel and is made solely by the author of this paper.*