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MATH 175 - H4L

Exercise 3: More Iterative Methods in Root-finding

In this exercise, we will use the Muller's and the Bairstow's Method to calculate all the roots of each of the two given polynomial functions f and g. For both processes, we will employ the methods with deflation process to reduce the degrees of the polynomials and find the roots easier.
Consider the function

$$f(x) = x^4 + \frac{12x^3}{5} + \frac{111x^2}{25} + \frac{36x}{5} + \frac{311}{25} \tag{3.1}$$

We wish to find the roots of f using Muller's Method applied with deflation process. We use the jump-based halting criterion with maximum iteration $N_{\text{max}} = 100$ and error tolerance $\epsilon = 10^{-6}$. Let's first do a quick analysis of the function f. We do this by observing its graph in Figure 3.1. From the plot, we observe that the graph did not cross the x axis yet the Fundamental Theorem of Algebra guarantees that f has 4 roots. This implies that we can expect our estimated roots to all be complex.

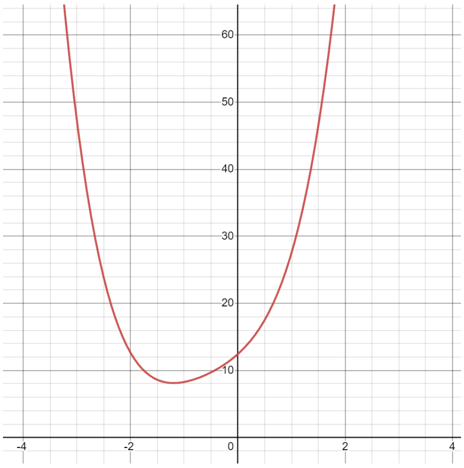

Figure 3.1: Plot of $f(x)$ over the interval $[-4, 4]$.

We now implement the Muller's Method. For this implementation, we use the initial (ordered) triple $(0, 1, 2)$. Using MATLAB, we obtain the result of our first run of the method.

```
n           x_n               rel jump
0     0.000000 +0.000000 i    NaN
1     1.000000 +0.000000 i    NaN
2     2.000000 +0.000000 i    NaN
3     -0.707547 -1.559916 i   1.562382E+00
4     -0.454031 -0.382899 i   7.029154E-01
5     -0.769247 -0.757626 i   8.244637E-01
6     -0.753197 -0.943373 i   1.726783E-01
7     -0.733573 -0.945148 i   1.632281E-02
8     -0.733724 -0.945734 i   5.057187E-04
9     -0.733724 -0.945734 i   5.305190E-07
Root is -7.3372445192e-01 -9.4573395036e-01 i after 7 iterations.
Note that the absolute function value at the estimate is 1.122734E-05.
```

Using our defined initial triple, we obtained the first approximate root $r_1^* = -7.3372445192 \times 10^{-1} - 9.4573395036 \times 10^{-1}i$. This is a good estimate since $|f(r_1^*)| = 1.122734 \times 10^{-5}$ which is of magnitude of order $\times 10^{-5}$. We then perform the deflation process by dividing f by $x - r_1^*$. Since r_1^* is close to a root r, the continuity of f guarantees that we can ordinarily run the Muller's method on $f_{-1} = \dfrac{f(x)}{x - r_1^*}$.

We obtain f_{-1} to be the function

```
Our deflated function becomes f(x) =
x^3 - x^2*(3.13372 + 0.945734i) + x*(5.84488 + 3.65758i) + (6.37057 - 8.21135i)
```

Implementing Muller's Method on f_{-1} and using also our initial triple $(0, 1, 2)$, we have the results in MATLAB given below.

```
Performing Muller's Method on f, we have
n          x_n              rel jump
0     0.000000 +0.000000 i     NaN
1     1.000000 +0.000000 i     NaN
2     2.000000 +0.000000 i     NaN
3     -0.121674 +1.606576 i    1.330657E+00
4     -1.563885 +0.326409 i    1.196901E+00
5     -0.727414 +1.159192 i    7.388294E-01
6     -0.749852 +0.942307 i    1.593269E-01
7     -0.733438 +0.945635 i    1.390805E-02
8     -0.733724 +0.945734 i    2.528949E-04
9     -0.733724 +0.945734 i    7.266632E-08
Root is -7.3372356454e-01 +9.4573463451e-01 i after 7 iterations.
Note that the absolute function value at the estimate is 3.995217E-06.
```

After 7 iterations, we obtained another root $r_2^* = -7.3372445192 \times 10^{-1} + 9.4573395036 \times 10^{-1} i$ which is actually the complex conjugate of r_1^*. This is also a good estimate since $|f(r_2^*)| = 3.995217 \times 10^{-6}$. Performing another deflation process to find $f_{-2} = \dfrac{f_{-1}}{x - r_2^*}$, we have $f_{-2} =$

```
x^2 - x*(3.86745 - 6.84146e-7i) + (8.68251 - 0.00000230865i)
```

Finally, since f_{-2} is of degree 2, we can easily apply the quadratic formula on the function.

```
Applying the quadratic formula on f, we have the roots
1.933724 +2.223336 i and 1.933724 -2.223337 i.
```

This implies that the estimated roots r_i^* for $i = \overline{1,4}$ of our given function f in Equation 3.1 are

```
-0.733724 -9.457340e-01 i
-0.733724 +9.457346e-01 i
1.933724 +2.223336e+00 i
1.933724 -2.223337e+00 i
```

and their respective corresponding function values $f(r_j^*)$ are

```
4.415716e-06 +1.032253e-05 i
-1.562646e-06 +1.494474e-05 i
9.177191e-07 +2.649783e-05 i
1.497232e-05 +1.563143e-05 i
```

which are very small values. Therefore, our obtained roots are good estimates for the roots of f.

We now consider another polynomial function given by

$$g(x) = x^6 - 2x^5 + 9x^4 - 8x^3 + 23x^2 - 6x + 7 \tag{3.2}$$

For this function, we will use the Bairstow's Method to approximate the roots of g. Similarly as above, we use the jump-based stopping criterion with maximum iteration $N_{\max} = 100$ and error tolerance $\epsilon = 10^{-6}$. Note that for each run of the method, we obtain two approximate roots. Hence, upon deflation, we expect our new function to be of 2 degrees less than the previous function. We analyze yet again the function g by observing its graph in Figure 3.2. Note that in the figure, g did not cross the x axis. Hence, we expect the roots to all be complex.

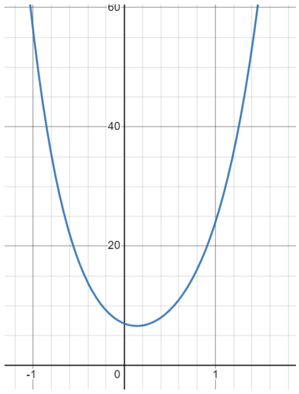

Figure 3.2: Plot of $g(x)$ over the interval $[-1, 2]$.

Performing a run of Bairstow's Method on g using the initial values $u_0 = 1$ and $u_0 = 2$, we have the result in MATLAB below.

```
n       u_n        rel jump (u)       v_n       rel_jump (v)
0     1.000000            NaN      2.000000     NaN
1     0.120267     8.797327E-01    1.427617     2.861915E-01
2    -1.146408     1.053217E+01   -0.076582     1.053643E+00
3    -0.646523     4.360449E-01    0.254362     4.321439E+00
4    -0.284670     5.596912E-01    0.384940     5.133531E-01
5    -0.207122     2.724116E-01    0.374061     2.826033E-02
6    -0.208255     5.469075E-03    0.373336     1.939863E-03
7    -0.208255     2.583204E-06    0.373336     3.723783E-07
8    -0.208255     4.909930E-13    0.373336     4.906761E-14
Our approximate quadratic factor is x^2 -0.208255 x  +0.373336
with roots 0.104127 +0.602074 i and 0.104127 -0.602074 i.

 The respective function values at these estimated roots are
 3.603050E-16 and 3.603050E-16.
```

After 8 iterations, we obtained two approximate roots $r_1^* = 0.104127 + 0.602074i$ and $r_2^* = 0.104127 - 0.602074i$. Note that r_1^* and r_2^* are complex conjugates of each other. These roots are also good approximates since their respective function values, i.e., $f(r_1^*)$ and $f(r_2^*)$ are both of magnitude of order 10^{-16}. Now, we apply the deflation process and obtain

$$g_{-2} = \frac{g(x)}{(x - r_1^*)(x - r_2^*)}$$

Using MATLAB, we obtain the reduced (deflated) function to be

```
x^4 - 1.79175*x^3 + 8.25353*x^2 - 5.61224*x + 18.7499
```

It can be noted that g_{-2} is of degree 4 which supports our assumption earlier. Performing another run of the Bairstow' Method on this updated function, we have the results in the next page.

```
Performing Bairstow's Method on g, we have
n        u_n       rel jump (u)        v_n      rel_jump (v)
0     1.000000            NaN     2.000000     NaN
1     0.415544     5.844558E-01     2.722693     3.613463E-01
2     0.269102     3.524106E-01     3.323862     2.207996E-01
3     0.298172     1.080269E-01     3.460827     4.120654E-02
4     0.299250     3.613924E-03     3.459626     3.470460E-04
5     0.299249     1.330271E-06     3.459628     5.615818E-07
6     0.299249     1.103362E-12     3.459628     3.454256E-13
Our approximate quadratic factor is x^2 +0.299249 x  +3.459628
with roots -0.149625 +1.853980 i and -0.149625 -1.853980 i.

 The respective function values at these estimated roots are
 -6.906475E-15 and -6.906475E-15.
```

From this run, we obtain another two approximate roots for g only after 6 iterations. These roots are $r_3^* = -0.149625 + 1.853980i$ and $r_4^* = -0.149625 - 1.853980i$ which are also complex conjugates of each other. These roots are also good estimates since $f(r_3^*)$ and $f_(r_4^*)$ are both of magnitude of order 10^{-15}. So, we can apply another deflation process and find

$$g_{-4} = \frac{g_{-2}}{(x - r_3^*)(x - r_4^*)}$$

Using MATLAB, we obtain this function to be

```
x^2 - 2.09099*x + 5.41963
```

which is of degree 2 as expected. Therefore, we can simply apply the qudratic formula on g_{-4} to find its roots and thus, the remaining roots of $g(x)$.

```
Applying the quadratic formula on g, we have the roots
1.045497 +2.080039 i and 1.045497 -2.080039 i.
```

This implies that the estimated roots r_j^* for $j = \overline{1, 6}$ of the function g given in Equation 3.2 are

```
0.104127 +6.020739e-01 i
0.104127 -6.020739e-01 i
-0.149625 +1.853980e+00 i
-0.149625 -1.853980e+00 i
1.045497 +2.080039e+00 i
1.045497 -2.080039e+00 i
```

and their respective corresponding function values $g(r_j^*)$ are

```
3.603050e-16 -3.194549e-16 i
3.603050e-16 +3.194549e-16 i
1.874224e-14 +1.365291e-14 i
1.874224e-14 -1.365291e-14 i
8.235598e-14 -3.207865e-14 i
8.235598e-14 +3.207865e-14 i
```

which are very small values. Therefore, our obtained roots are good estimates for the roots of g.

Remark: *The program used to solve this problem is made in MATLAB and is made solely by the author of this paper.*