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MATH 175 - H4L

Exercise 5.2: On Singular and Rectangular Systems

5.2.1 Introduction to the Problem

In this problem, we wish to solve a given matrix equation or system given by

$$Ax = b \tag{5.2.1}$$

where A is a 32×32 square matrix and b is a 32×1 vector. For this, we first verify if A is ill-conditioned. After, we find the least-squares solution x^* of the system. We then speculate on how much a 1% perturbation on b causes a deviation from the solution x^*. To verify this, we generate a noise vector ε of the appropriate size and measure its relative size to b. Finally, we solve the system with the noise vector given by

$$Ax = b + \varepsilon \tag{5.2.2}$$

by finding the Tikhonov solution x_α using the L-curve.

5.2.2 Checking Ill-Conditioned Matrix

As stated, we first determine if the given matrix A is ill-conditioned. To do this, we use the common formula for the condition number K given by

$$K(A) = \|A\|_2 \cdot \left\|A^+\right\|_2 \tag{5.2.3}$$

where A^+ is the Moore-Penrose pseudoinverse of A. Note that the bigger the value of $K(A)$ is, the more sensitive the solution is to perturbations in the right-hand side vector. It is also noted that

$$\frac{\|\tilde{x} - x^*\|_2}{\|x^*\|_2} \leq K(A)\frac{\|\epsilon\|_2}{\|b\|_2} \tag{5.2.4}$$

which tells that a perturbation on the right-hand side causes a relative change in the solution of at most $K(A)$ times of the perturbation in b.
To find $K(A)$, we first determine the pseudoinverse of A denoted by A^+. Suppose that A^* denotes the conjugate transpose of A. Theorem 9.2 then states that

$$A^+ = \begin{cases} (A^*A)^{-1}A^* & \text{, if } A^*A \text{ is invertible} \\ A^*(AA^*)^{-1} & \text{, if } AA^* \text{ is invertible} \end{cases} \tag{5.2.5}$$

However, we note the function *pinv* from MATLAB which obtains this exact quantity. That is, $\text{pinv}(A) = A^+$. Hence, using MATLAB, we obtain from Equation 5.2.3 that

$$K(A) = \|A\|_2 \cdot \|\text{pinv}(A)\|_2 = 5.457316 \times 10^{13}$$

This states that A is ill-conditioned due to the very high value of $K(A)$.

5.2.3 Least Square Solution of Equation 5.2.1

Note that the least squares solution of a given matrix system given by

$$x^* = A^+b \tag{5.2.6}$$

minimizes $\|Ax - b\|_2$ over the space of all 32×1 matrix x. That is, x^* is the best solution with respect to the l_2-norm. Using MATLAB, we obtain the solution as shown in Figure 5.2.1.

```
0.1242
0.1799
0.2610
0.3422
0.4551
0.5720
0.6860
0.7972
0.8965
0.9650
0.9951
0.9913
0.9539
0.8829
0.7875
0.6908
0.6158
0.5897
0.6444
0.8124
1.0966
1.4500
1.7835
1.9969
2.0155
1.8185
1.4590
1.0481
0.6761
0.3794
0.1956
0.0880
```

Figure 5.2.1: Obtained Values of the Least Square Solution x^*.

We check the accuracy of the solution using the formula

$$\frac{\|Ax^* - b\|_2}{\|b\|_2} \tag{5.2.7}$$

Using MATLAB, we obtain that this quantity is equal to 4.5788×10^{-5}. This states that the obtained x^* is an accurate measure of the actual solution having a relative error that is very small.

5.2.4 Perturbations on Right-Hand Side

We now speculate on the caused "noise" in the solution by applying a 1% perturbation in b. We can do this by solving for the vector $b_1 = 0.01b$ and solving for the matrix system $Ax = b + b_1$ by finding its least squares solution x_1^*. This is verified by the result from MATLAB that

$$\frac{\|b_1\|_2}{\|b\|_2} = 0.0100$$

This value can be interpreted as a 1% measurement error or a 1% noise in the data. Solving the solution then implies that

$$\frac{\|x_1^* - x^*\|_2}{\|x^*\|_2} = 0.009969$$

This implies that a 1% perturbation on b causes the solution to deviate by 0.9969% from the original least square solution.

5.2.5 Noise Vector

We now generate a 32×1 noise vector ε of uniformly distributed random numbers from $(0, 1)$. We then measure its size relative to the original right-hand side b using

$$\frac{\|\varepsilon\|_2}{\|b\|_2}$$

Implementing this process in MATLAB, we obtain the noise vector ε

```
0.4868
0.4359
0.4468
0.3063
0.5085
0.5108
0.8176
0.7948
0.6443
0.3786
0.8116
0.5328
0.3507
0.9390
0.8759
0.5502
0.6225
0.5870
0.2077
0.3012
0.4709
0.2305
0.8443
0.1948
0.2259
0.1707
0.2277
0.4357
0.3111
0.9234
0.4302
0.1848
```

Figure 5.2.2: Values of Generated Noise Vector ε.

with a relative error of

$$\frac{\|\varepsilon\|_2}{\|b\|_2} \approx 0.2337006$$

This can be interpreted as a 23.37% measurement error or a 23.37% noise in the data.

5.2.6 Tikhonov Solution

We now implement a regularization technique to mitigate the effect of the noise to the solution. Particularly, we shall use the basic Tikhonov Regularization routine with the L-curve parameter. The resulting solution of this routine is given by

$$x_\alpha = (\alpha I + A^* A)^{-1} A^* b \tag{5.2.8}$$

The problem now is on finding an optimal value of α that balances the solution norm and the residual norm. To do this, we apply the L-curve technique. This technique entails plotting the solution norm versus the residual norm in a log-log scale and choosing the α that will produce the corner of the L-curve or plot.
In the implementation, we start by generating data points in the α-interval $[10^{-16}, 1]$ with step size 10^{-6}. This generates $10^6 + 1$ linearly spaced points or α values in the interval. For each of these values, we calculate their corresponding x_α values using Equation 5.2.8. We then record and plot the points $(\|Ax_\alpha - b\|_2, \|x_\alpha\|_2)$. From MATLAB, we have the plot in Figure 5.2.3.

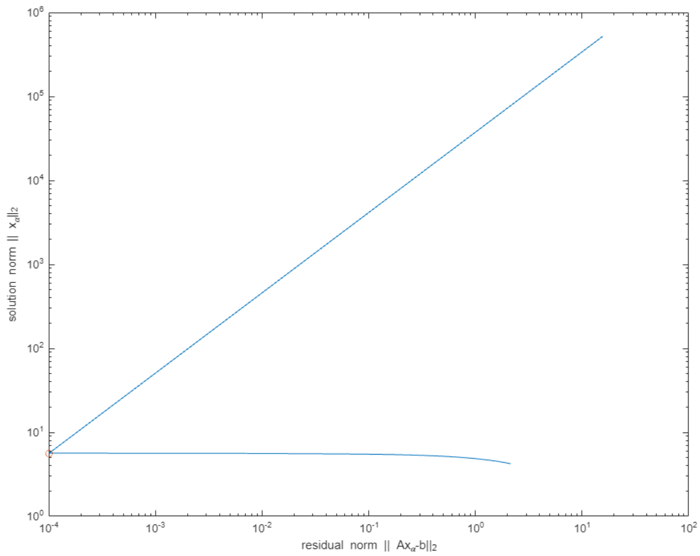

Figure 5.2.3: The L-curve of the Given Problem

We use MATLAB to extract the α value corresponding to the "corner" in the orange circle. Doing so reveals that $\alpha^* = 1.000000 \times 10^{-6}$. Substituting this to the Equation 5.2.8 we have $\tilde{x}$ equal to the values in Figure 5.2.4.

```
0.1350
0.1659
0.2351
0.3362
0.4591
0.5901
0.7139
0.8170
0.8920
0.9391
0.9642
0.9722
0.9597
0.9149
0.8280
0.7073
0.5904
0.5380
0.6074
0.8208
1.1466
1.5066
1.8052
1.9643
1.9478
1.7658
1.4628
1.0990
0.7334
0.4133
0.1714
0.0277
```

Figure 5.2.4: Values of the Tikhonov Solution x_α.

Moreover, we obtain from the process that the l_2-relative error is

$$\frac{\|Ax_\alpha - (b+\varepsilon)\|_2}{\|b+\varepsilon\|_2} \approx 0.1935$$

and the relative difference between x_α and x^*

$$\frac{\|x_\alpha - x^*\|_2}{\|x^*\|_2} \approx 0.0346$$

This implies that x_α is an accurate solution having a small l_2-relative error while still being close by a factor of approximately 3.46% difference with respect to the least square solution.