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MATH 175 - H4L

Exercise 5.1: The Jacobi Method

5.1.1 Introduction to the Problem

In this problem, we wish to solve the linear system of equations given by

$$\begin{cases} 7x_1 - 3x_2 & = 4 \\ -3x_1 + 9x_2 + x_3 & = -6 \\ x_2 + 3x_3 - x_4 & = 3 \\ -x_3 + 10x_4 - 4x_5 & = 7 \\ -4x_4 + 6x_5 & = 2 \end{cases} \tag{5.1.1}$$

using the Jacobi Method. To do this, we first reconstruct the given system in Equation 5.1.1 as the equation

$$Ax = b \tag{5.1.2}$$

where

$$A = \begin{bmatrix} 7 & -3 & 0 & 0 & 0 \\ -3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 10 & -4 \\ 0 & 0 & 0 & -4 & 6 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 4 \\ -6 \\ 3 \\ 7 \\ 2 \end{bmatrix}$$

In implementing the method, we use the l_∞-norm relative jump based stopping criterion with $\epsilon = 10^{-8}$.

5.1.2 The Fixed Point Iteration for Linear Systems

Now, we solve the linear system in Equation 5.1.2 by obtaining its corresponding fixed-point iteration given by

$$x^{(k+1)} = T_J x^{(k)} + c_J, \qquad k > 0 \tag{5.1.3}$$

where T_J and c_J are the iteration matrix and translation matrix, respectively. We note that this iteration converges to a unique solution from any initial estimate $x^{(0)}$ provided that the spectral radius of T, denoted as $\rho(T)$, does not exceed 1.
We note a property that for any matrix A, we can split the matrix as $A = M - N$ where M is a non-singular matrix and N is any matrix of the right size. From this, we can then obtain Equation 5.1.3 from Equation 5.1.2 as follows:

$$\begin{aligned} Ax &= b \\ (M - N)x &= b \\ Mx &= Nx + b \\ x &= M^{-1}(Nx + b) \\ x &= M^{-1}Nx + M^{-1}b \end{aligned}$$

The final equation suggests that we take $T_J = M^{-1}N$ and $c_J = M^{-1}b$ as long as $\rho(T) < 1$. The problem now is on finding M and N. For this, we use the Jacobi Method.

5.1.3 The Jacobi Method

We first deconstruct the matrix A as $A = D - L - U$ where D is a diagonal matrix consisting of the diagonal entries of A while L and U are the lower and upper triangular matrices, respectively, with zeros as its diagonal entries and that satisfies the equation. That is,

$$A = D - L - U$$

$$\begin{bmatrix} 7 & -3 & 0 & 0 & 0 \\ -3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 10 & -4 \\ 0 & 0 & 0 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By properties of matrix, we obtain that the inverse of D is given by

$$D^{-1} = \begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}$$

This implies that $M = D \Rightarrow M^{-1} = D^{-1}$ and $N = L + U$. Consequently, we have

$$T_J = D^{-1}(L + U) \tag{5.1.4}$$

$$c_J = D^{-1}b \tag{5.1.5}$$

From Equation 5.1.4, we have

$$\begin{aligned} T_j &= \begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{3}{7} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & \frac{2}{3} & 0 \end{bmatrix} \end{aligned}$$

From Equation 5.1.5, we have

$$c_j = \begin{bmatrix} \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 3 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ -\frac{2}{3} \\ 1 \\ \frac{7}{10} \\ \frac{1}{3} \end{bmatrix}$$

Therefore, from Equation 5.1.3, we have the fixed-point iteration

$$x^{(k+1)} = \begin{bmatrix} 0 & \frac{3}{7} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & \frac{2}{3} & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} \frac{4}{7} \\ -\frac{2}{3} \\ 1 \\ \frac{7}{10} \\ \frac{1}{3} \end{bmatrix}, \qquad k > 0$$

5.1.4 Jacobi Iteration Implementation

Given our obtained matrix for T_J in the previous section, we obtain using MATLAB that $\rho(T_J) \approx 0.5563 < 1$. This implies that for any initial estimate $x^{(0)}$, the iteration converges to a unique root. For such case, we use the zero vector $[0, 0, 0, 0, 0]^T$ as our initial guess. Implementing the iteration in MATLAB, we obtain the result in Figure 5.1.1.

n			x^(n)			rel jump
0	0.000000	0.000000	0.000000	0.000000	0.000000	NaN
1	0.571429	-0.666667	1.000000	0.700000	0.333333	Inf
2	0.285714	-0.587302	1.455556	0.933333	0.800000	4.666667E-01
3	0.319728	-0.733157	1.506878	1.165556	0.955556	1.595420E-01
4	0.257218	-0.727522	1.632904	1.232910	1.110370	1.027388E-01
5	0.259634	-0.762361	1.653477	1.307439	1.155273	4.564169E-02
6	0.244702	-0.763842	1.689933	1.327457	1.204959	3.004920E-02
7	0.244068	-0.772870	1.697100	1.350977	1.218305	1.391763E-02
8	0.240199	-0.773877	1.707949	1.357032	1.233985	9.239240E-03
9	0.239767	-0.776373	1.710303	1.364389	1.238021	4.307438E-03
10	0.238697	-0.776778	1.713587	1.366239	1.242926	2.867672E-03
11	0.238524	-0.777499	1.714339	1.368529	1.244159	1.336516E-03
12	0.238215	-0.777641	1.715343	1.369098	1.245686	8.906198E-04
13	0.238154	-0.777855	1.715579	1.369809	1.246065	4.145617E-04
14	0.238062	-0.777902	1.715888	1.369984	1.246539	2.763364E-04
15	0.238042	-0.777967	1.715962	1.370204	1.246656	1.284977E-04
16	0.238014	-0.777982	1.716057	1.370259	1.246803	8.566141E-05
17	0.238008	-0.778002	1.716080	1.370327	1.246839	3.980693E-05
18	0.237999	-0.778006	1.716110	1.370344	1.246885	2.653760E-05
19	0.237997	-0.778012	1.716117	1.370365	1.246896	1.232725E-05
20	0.237995	-0.778014	1.716126	1.370370	1.246910	8.218133E-06
21	0.237994	-0.778016	1.716128	1.370377	1.246913	3.816638E-06
22	0.237993	-0.778016	1.716131	1.370378	1.246918	2.544422E-06
23	0.237993	-0.778017	1.716131	1.370380	1.246919	1.181524E-06
24	0.237993	-0.778017	1.716132	1.370381	1.246920	7.876821E-07
25	0.237993	-0.778017	1.716133	1.370381	1.246920	3.657412E-07
26	0.237993	-0.778017	1.716133	1.370381	1.246921	2.438274E-07
27	0.237993	-0.778017	1.716133	1.370382	1.246921	1.132110E-07
28	0.237993	-0.778017	1.716133	1.370382	1.246921	7.547399E-08
29	0.237993	-0.778017	1.716133	1.370382	1.246921	3.504242E-08
30	0.237993	-0.778017	1.716133	1.370382	1.246921	2.336161E-08
31	0.237993	-0.778017	1.716133	1.370382	1.246921	1.084662E-08
32	0.237993	-0.778017	1.716133	1.370382	1.246921	7.231080E-09

Figure 5.1.1: Result of Implementation of Jacobi Iteration in MATLAB.

Results show that after 32 iterations, we obtain a good estimate that satisfies the defined stopping criterion.

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The estimate obtained after 32 iterations is
 x = [0.237993, -0.778017, 1.716133, 1.370382, 1.246921]
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To determine its accuracy, we solve for its relative l_∞-norm error. Since the exact solution is unknown, we use the formula

$$\|e\|_\infty = \frac{\|Ax^* - b\|_\infty}{\|b\|_\infty} \tag{5.1.6}$$

Solving this in MATLAB, we obtain that $\|e\|_\infty \approx 8.23085 \times 10^{-9}$.

From these results, we conclude that the convergence of the method is not that fast considering that it took 32 iterations to obtain a good estimate. This is to be expected since the Jacobi Method follows a linear rate of convergence as reflection of the fixed-point iteration. The calculated l_2-norm of the iteration matrix is $\|T_J\| = 0.667$. This value is the upper bound for the asymptotic error constant. The relatively small value of $\|T_J\|$ verifies the slow convergence of the method. Despite this, the obtained value for $\|e\|_\infty$, being very small, implies that our estimate is quite accurate.