9.7
10

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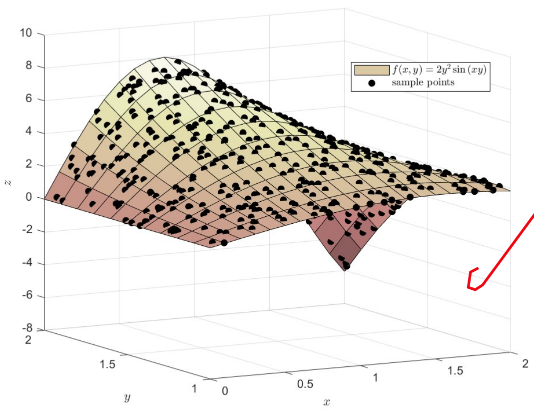
MATH 174 - B2L

Problem Set 2.3

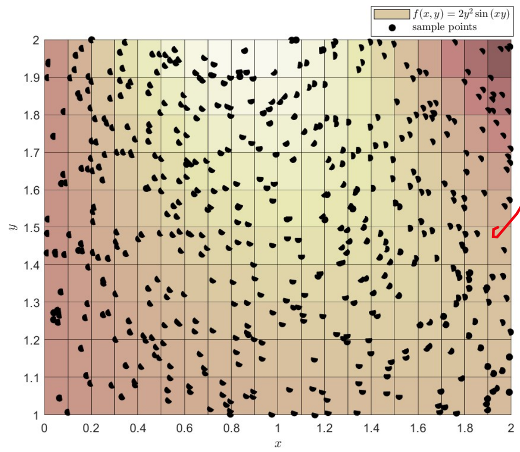
In this problem, we wish to approximate the definite integral

$$A = \int_0^2 \int_1^2 f(x,y)dydx \tag{2.3.1}$$

where $f(x,y) = 2y^2 \sin(xy)$ using the Monte Carlo integration technique. To do this, we shall generate $N = 500$ random points (x_i, y_i) on the rectangle $\Omega = [0,2] \times [1,2] \subset \mathbb{R}^2$. We then plot the generated random points together with the integrand $f(x,y)$. Using MATLAB, we obtain the figure below.



(a)



(b)

Figure 2.3.1: Scatter plot of 500 random data points on $f(x,y) = 2y^2 \sin(xy)$ over Ω; (a) 3D plot, (b) projection on the xy-plane.

Using the random data points we generated, we approximate the given definite integral A in Equation 2.3.1 using the formula

$$A \approx I = \frac{V}{N} \sum_{i=1}^{N} f(x_i, y_i) \tag{2.3.2}$$

where $V = |\Omega| = 2$ and where $f(x_i, y_i)$'s are the function values of f evaluated at each of the random data point. We repeat this simulation nine (9) more times such that for each simulation, we use a different set of 500 random data points. For each of these set of data points, we obtain the respective estimates using Equation 2.3.2.

Summarized in Table 2.3.1 are the estimates and their respective relative errors which we obtained from the multiple Monte Carlo simulations performed. Note that the relative errors of the estimates are computed using

-0.3

$$\text{relative error} = \frac{|\text{exact} - \text{estimate}|}{|\text{exact}|} \tag{2.3.3}$$

where the exact value is $A = 5.541650809387662$ as determined using MATLAB.

Table 2.3.1: Estimates obtained from ten (10) Monte Carlo simulations with $N = 500$ random data points and their respective relative errors.

Simulation #	Estimate	Relative Error
1	5.735244159782328	3.493423838014392
2	5.553644946652034	0.216436178982164
3	5.479317984288539	1.124806077523504
4	5.501877633953709	0.717713490113372
5	5.577059550955720	0.638956563413843
6	5.469604657054410	1.300084664504729
7	5.289324584297877	4.553268218602637
8	5.598872763117641	1.032579563350411
9	5.330148015259575	3.816602694810661
10	5.773136027704600	4.177188824759540

Notice from the table that the best estimate obtained among the ten (10) simulations performed is $A \approx 5.553644946652034$ which has a relative error of approximately 0.22%. This is a great estimate to the actual value of A. However, the probabilistic nature of the technique used is still evident in the large range in value of the relative errors, and consequently the estimates, in Table 2.3.1. With this, we take the average of the ten (10) estimates we obtained and analyze its accuracy. As computed, the average estimate is

$$A \approx I_{ave} = 5.530823032306643$$

with relative error that is approximately 0.20%. From this, we can infer that our estimate is a great approximation of the exact value of A considering that only 500 random data points are used and only ten (10) simulations are performed. It can also be inferred that Monte Carlo simulation can be an efficient technique to solve the given definite integral.

APPENDIX A

Program Used in MATLAB

```
clc
clear
close all
format long

%% Initializing Constants

% integrating bounds
a = [0 1];  % lower bounds
b = [2 2];  % upper bounds

% volume of omega
V = abs(b(1)-a(1)) * abs(b(2)-a(2));

% no. of data points
N = 500;

% no. of trials
trials = 10;
t = 1;  % counter

% function and estimates
func = @(x,y) 2*y.^2 .* sin(x.*y);  % vectorized func
est = zeros(trials, 1);

while t <= trials
    seed = zeros(N, 2);

    % random points between bounds
    seed(:,1) = (b(1)-a(1))*rand(N,1) + a(1);
    seed(:,2) = (b(2)-a(2))*rand(N,1) + a(2);

    % change of variables
    X = seed(:,1);
    Y = seed(:,2);

    % function values
    F = 2*Y.^2 .* sin(X.*Y);

    % t^th trial estimate
    est(t) = V * mean(F);

    if t == 1
        % Plotting
        [X_3d, Y_3d] = meshgrid(a(1):0.1:b(1), a(2):0.1:b(2));
        Z_3d = 2*Y_3d.^2 .* sin(X_3d.*Y_3d);  % plot of f(x,y) in 3d

        surf(X_3d, Y_3d, Z_3d)
        colormap("pink")
        hold on
        scatter3(X,Y,F,'ko','filled')
```

```matlab
        legend('$f(x,y) = 2y^2 \sin{(xy)}$', 'sample points', ...
            'Interpreter', 'latex')
        xlabel('$x$', 'Interpreter', 'latex')
        ylabel('$y$', 'Interpreter', 'latex')
        zlabel('$z$', 'Interpreter', 'latex')
        zlim([-8 10])
        hold off
    end
    t = t+1;
end
disp('Estimates = ')
disp(est)

%% Relative Error
exact = integral2(func, a(1), b(1), a(2), b(2));  % int(func, xmin, xmax, ymin, ymax)

RelError = abs((exact-est)/exact) * 100;
disp('Relative errors (in %) = ')
disp(RelError)

%% Average estimate and Relative Error
est_ave = mean(est);
relE = abs((exact-est_ave)/exact) * 100;
disp('average estimate =')
disp(est_ave)
disp('with relative error (in %) =')
disp(relE)
```