Cristian B. Jetomo

MATH 174 – B2L

9.7/10

Problem Set 1.2

Consider the function $f(x) = \sin(x^2)$. What should the value of h (the uniform increment) be so that the piecewise linear interpolant for f over the interval $[-1, 1]$ be as accurate (with respect to the L^∞ norm) as the algebraic interpolating polynomial for f relative to the 10 points that has the L^∞ norm.

Solution:

Let $s(x)$ be the piecewise linear interpolant for f over the interval $[-1, 1]$ relative to the data points $x_1, x_2, \dots, x_{n+1}$, where $x_1 = -1$ and $x_{n+1} = 1$, which has a uniform increment $h_i = h_{i+1} - h_i$ for $i = 1,2, \dots, n$.

Then by Theorem 4.1, the L^∞ norm of $s(x)$ is given by

$$\begin{aligned} \|s\|_\infty &= \max_{x\in[-1,1]} |f(x) - s(x)| \\ &\le \frac{1}{8}h^2 \max_{x\in[-1,1]} |f''(x)| \end{aligned}$$

Now, let $P_9(x)$ be the algebraic interpolating polynomial (AIP) for f over the interval $[-1, 1]$ relative to 10 data points where $P_9(x)$ has the minimum L^∞ norm as compared to all AIP of degree 9 relative to 10 data points.

Then as a consequence of Theorem 3.1, the 10 data points for P_9 are exactly the roots of the monic Chebyshev polynomial of degree 10 $(\hat{T}_{10})$.

Since $\left\|\hat{T}_{10}\right\|_\infty = 2^{1-10} = 2^{-9}$, this implies that the L^∞ norm of $P_9(x)$ is given by

-0.3

$$\begin{aligned} ||P_9 - f||_\infty &= \left\| \frac{f^{(10)}(\xi)}{10!}\omega(x) \right\|_\infty \\ &= \max_{\xi\in[-1,1]} \left| \frac{f^{(10)}(\xi)}{10!} \right| \|\omega\|_\infty \\ &= \max_{\xi\in[-1,1]} \left| \frac{f^{(10)}(\xi)}{10!} \right| \left\|\hat{T}_{10}\right\|_\infty \\ &= \max_{\xi\in[-1,1]} \left| \frac{f^{(10)}(\xi)}{10!} \right| \cdot 2^{-9} \\ &= \frac{1}{10!2^9} \max_{\xi\in[-1,1]} \left| f^{(10)}(\xi) \right| \end{aligned}$$

This is the absolute error!

Now, we wish to determine h such that the L^∞ norm of s is as accurate as the L^∞ norm of P_9. Hence,

$$\|s\|_\infty = \|P_9\|_\infty$$
$$\frac{1}{8}h^2 \max_{x\in[-1,1]} |f''(x)| = \frac{1}{10!2^9} \max_{x\in[-1,1]} \left|f^{(10)}(\xi)\right|$$
$$h^2 = \frac{8}{10!2^9} \cdot \frac{\max_{\xi\in[-1,1]} \left|f^{(10)}(\xi)\right|}{\max_{x\in[-1,1]} |f''(x)|}$$
$$h = \left|\sqrt{\frac{8}{10!2^9} \cdot \frac{\max_{\xi\in[-1,1]} \left|f^{(10)}(\xi)\right|}{\max_{x\in[-1,1]} |f''(x)|}}\right|$$

Given that $f(x) = \sin(x^2)$, the plot of the functions $|f''(x)|$ and $\left|f^{(10)}(x)\right|$ over the interval $[-1,1]$ are given as follows:

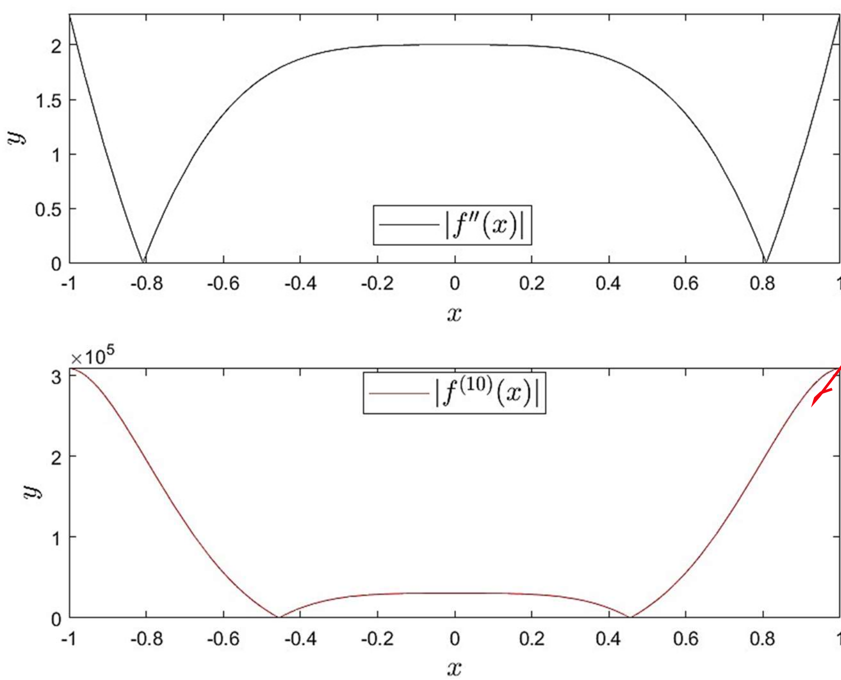

Figure 1.2.1. Graph of the functions $|f''(x)|$ and $\left|f^{(10)}(x)\right|$, respectively, over the interval $[-1,1]$ where $f(x) = \sin(x^2)$.

We can observe from the figure that both the functions $|f''(x)|$ and $\left|f^{(10)}(x)\right|$ are symmetric about the y-axis and are at their maximum whenever $x = \pm 1$. Hence, using MATLAB, we obtain that the increment h is given by

$$
\begin{aligned}
h &= \left| \sqrt{\frac{8}{10!2^9} \cdot \frac{f^{(10)}(1)}{f''(1)}} \right| \\
&= 0.024116146608070
\end{aligned}
$$

This implies that the uniform increment for the data points used in the piecewise linear interpolant s must be approximately 0.02412 so that the L^∞ error of s is as accurate as the L^∞ error of the AIP relative to 10 points that has the minimum L^∞ error.

Appendix A

Program Used in MATLAB for Problem Set 1.2

```
clc
clear
close all

syms x

%% Function and interval
f = sin(x^2);
interval = [-1, 1];

%% L-infty norm of AIP using monic Chebyshev polynomial roots
ChebDeg = 10;
omega_norm = 2^(1-ChebDeg);
f10Prime = diff(f, 10);

% Chebyshev nodes (not important in calculation)
X = zeros(1, ChebDeg);
for i = 1:ChebDeg
    root = cos((2*i-1)/(2*ChebDeg) * pi);
    X(i) = root;
end
Y = sin(X.^2);

% max value of f10Prime occurs at x=+-1
maxf10Prime = double(abs(subs(f10Prime, 1)));

% L-infty error of AIP
AIP_norm = (maxf10Prime/factorial(ChebDeg)) * omega_norm;
disp(AIP_norm)

%% L-infty norm of Piecewise Linear Interpolant
% 1/8 h^2 max(f2Prime)
f2Prime = diff(f, 2);

% max value of abs(f2Prime) occurs at x=+-1
maxf2Prime = double(abs(subs(f2Prime, 1)));

%% Plots of f2Prime and f10Prime together
t = tiledlayout(2, 1);
t.TileSpacing = 'compact';

% Tile 1
nexttile
fplot(abs(f2Prime), interval, 'k')
l1 = legend("$|f''(x)|$");
x1 = xlabel('$x$');
y1 = ylabel('$y$');

% Tile 2
nexttile
fplot(abs(f10Prime), interval, 'r')
```

```
l2 = legend('$|f^{(10)}(x)|$');
x2 = xlabel('$x$');
y2 = ylabel('$y$');
% labels
for i = [l1 l2 x1 x2 y1 y2]
    set(i, 'Interpreter', 'Latex', 'Fontsize', 14)
end

%% solving for increment h
% 1/8 h^2 max(f2Prime) = AIP_norm
h = abs(sqrt((8 * AIP_norm)/maxf2Prime));
disp(h)
```