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MATH 174 – B2L

10/10 Excellent!

Problem Set 1.3

Consider the following data relating the score x of randomly selected MATH 174 students in PS1 last year to their final rating y. The values are displayed in %.

Table 1.3.1. The scores (in %) in PS1 and the corresponding final rating (in %) of ten randomly selected MATH 174 students last year.

x	72.00	81.67	90.33	90.83	91.67	92.50	93.33	95.00	97.50	97.67
y	80.57	85.20	83.57	87.37	95.68	99.71	95.79	98.00	94.77	94.40

a. Find the natural cubic spline interpolating the given data.
b. Plot a graph of the natural cubic spline you got.
c. What is your expected score in this problem set? Express your answer in percent. i.e., use a number between 0 and 100. Based on the model you obtained from above, what is your predicted final rating?

In this problem, we wish to construct the natural cubic spline interpolating the given

$n+1=10$ data points in the table. That is, we shall construct a cubic spline interpolant $s(x)$ relative to the abscissas or data points (x_i, y_i) for $i = 1,2,\dots,n+1$ given by

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

for $i = 1,2,..,n$ and where $x \in [72.00, 97.67]$. Furthermore, let $h_i = x_{i+1} - x_i$ for $i = 1,2,..,n$. Then by definition of cubic splines,

$$\begin{aligned} a_i &= y_i, \\ b_i &= \frac{a_{i+1} - a_i}{h_i} - \frac{2c_i + c_{i+1}}{3} h_i, \\ d_i &= \frac{c_{i+1} - c_i}{3h_i}, \end{aligned}$$

and the values of c_i are obtained by solving for x in the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} & & & ??? & & & \\ h_1 & 2(h_1+h_2) & h_2 & 0 & 0 & \dots & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & 0 & \dots & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & h_{n-1} & 2(h_{n-1}+h_n) & h_n \\ & & & ??? & & & \end{bmatrix}, \quad x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \\ c_{n+1} \end{bmatrix},$$

$$\text{and } b = \begin{bmatrix} ??? \\ \frac{3(a_3-a_2)}{h_2} - \frac{3(a_2-a_1)}{h_1} \\ \frac{3(a_4-a_3)}{h_3} - \frac{3(a_3-a_2)}{h_2} \\ \vdots \\ \frac{3(a_{n+1}-a_n)}{h_n} - \frac{3(a_n-a_{n-1})}{h_{n-1}} \\ ??? \end{bmatrix}$$

Since we want to construct a natural cubic spline $s(x)$, then we shall impose the conditions

$$s''(x_1) = s''(x_{n+1}) = 0$$

Since $s_i''(x) = 2c_i + 6d_i(x - x_i)$ and $d_i = \frac{c_{i+1} - c_i}{3h_i}$, then

$$\begin{aligned} s''(x_1) &= s_1''(x_1) \\ &= 2c_1 + 6d_1(x_1 - x_1) \\ &= 2c_1 = 0 \end{aligned}$$

and

$$\begin{aligned} s''(x_{n+1}) &= s_n''(x_{n+1}) \\ &= 2c_n + 6d_n(x_{n+1} - x_n) \\ &= 2c_n + 6d_n h_n \\ &= 2c_n + 6\left(\frac{c_{n+1} - c_n}{3h_n}\right) h_n \\ &= 2c_n + 2\left(c_{n+1} - c_n\right) \\ &= 2c_{n+1} = 0 \end{aligned}$$

Hence, we obtain the two equations $2c_1 = 0$ and $2c_{n+1} = 0$ from the imposed conditions. These equations will then be used to fill the missing rows for matrices A and b.

So, to find $c_i's$, we solve for the entries of the matrix x using the matrix equation $Ax = b$ given by

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_1 & 2(h_1+h_2) & h_2 & 0 & 0 & \cdots & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \cdots & h_{n-1} & 2(h_{n-1}+h_n) & h_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \\ c_{n+1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ \frac{3(a_3-a_2)}{h_2} - \frac{3(a_2-a_1)}{h_1} \\ \frac{3(a_4-a_3)}{h_3} - \frac{3(a_3-a_2)}{h_2} \\ \vdots \\ \frac{3(a_{n+1}-a_n)}{h_n} - \frac{3(a_n-a_{n-1})}{h_{n-1}} \\ 0 \end{bmatrix}$$

Using MATLAB, we solve the constants a_i's and the increments h_i's using the given abscissas. Also, we solve for the constants c_i's by solving for the matrix $x = A^{-1}b$. Lastly, b_i's and d_i's can be obtained using their respective formulas which requires values of a_i, h_i, and c_i. The summary of the obtained values of the constants are tabulated as follows:

Table 1.3.2. Values of constants a_i, b_i, c_i, and d_i (rounded to 4 decimal places) required by the formula for $s_i(x)$ for $i = 1,2, \ldots, 9$.

$\boldsymbol{i}$	$\boldsymbol{a_i}$	$\boldsymbol{b_i}$	$\boldsymbol{c_i}$	$\boldsymbol{d_i}$
1	80.5700	1.6939	-0.0000	-0.0130
2	85.2000	-1.9514	-0.3770	0.0670
3	83.5700	6.6026	1.3647	1.2604
4	87.3700	8.9125	3.2552	-2.4859
5	95.6800	9.1191	-3.0093	-2.5634
6	99.7100	-1.1742	-9.3923	6.1648
7	95.7900	-4.0246	5.9580	-1.6501
8	98.0000	2.0694	-2.3089	0.3857
9	94.7700	-2.2427	0.5840	-1.1452

Using these constants, we obtain the natural cubic spline interpolating the given data to be

$$s(x) = \begin{cases} -0.01299x^3 + 2.807x^2 - 200.4x + 4.809 \times 10^3 & 72.00 \leq x \leq 81.67 \\ 0.06704x^3 - 16.80x^2 + 1.401 \times 10^3 x + 3.879 \times 10^4 & 81.67 \leq x \leq 90.33 \\ 1.260x^3 - 340.2x^2 + 3.061 \times 10^4 x - 9.183 \times 10^5 & 90.33 \leq x \leq 90.83 \\ -2.486x^3 + 680.6x^2 - 6.211 \times 10^4 x + 1.889 \times 10^6 & 90.83 \leq x \leq 91.67 \\ -2.563x^3 + 702.0x^2 - 6.406 \times 10^4 x + 1.949 \times 10^6 & 91.67 \leq x \leq 92.50 \\ 6.165x^3 - 1.720 \times 10^3 x^2 + 1.600 \times 10^5 x - 4.959 \times 10^6 & 92.50 \leq x \leq 93.33 \\ -1.650x^3 + 468.0x^2 - 4.424 \times 10^4 x + 1.394 \times 10^6 & 93.33 \leq x \leq 95.00 \\ 0.3857x^3 - 112.2x^2 + 1.088 \times 10^4 x - 3.516 \times 10^5 & 95.00 \leq x \leq 97.50 \\ -1.145x^3 + 335.6x^2 - 3.278 \times 10^4 x + 1.067 \times 10^6 & 97.50 \leq x \leq 97.67 \end{cases}$$

Furthermore, using this equation, we obtain its plot to be as follows:

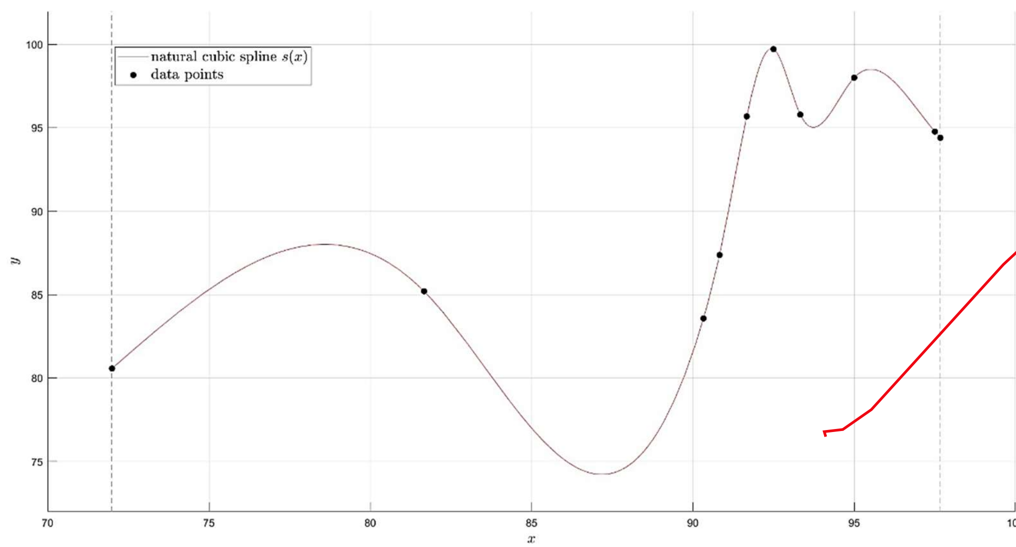

Figure 1.3.1. Graph of the obtained piecewise function for the natural cubic spline $s(x)$ over the interval $[72.00, 97.67]$ and scatter plot of the given data points in Table 1.3.1.

Observe from the figure that the natural cubic spline $s(x)$ passes through all the given data points in Table 1.3.1. Hence, we can infer that our constructed piecewise function $s(x)$ is correct.

Now, using this function, we shall predict the final rating of an expected score. Suppose that the expected score is 92.80%, based on the conditions of our piecewise function, we shall use the equation

$$s_6(x) = 6.165x^3 - 1.720 \times 10^3 x^2 + 1.600 \times 10^5 x - 4.959 \times 10^6$$

By substitution, we obtain that $s_6(92.80) \approx 98.68$. That is, based on our model, an expected score of 92.80 in this problem set will incur a final rating of 98.68 or uno (1.0).

Appendix A

Program Used in MATLAB for Problem Set 1.3

```
clc
clear
close all

syms x real
syms S
%% Abscissas
X = [72.00 81.67 90.33 90.83 91.67 92.50 93.33 95.00 97.50 97.67];
Y = [80.57 85.20 83.57 87.37 95.68 99.71 95.79 98.00 94.77 94.40
];
numPts = length(X);  % n+1

%% Computation of increment h and constant a
h = zeros(1, numPts-1);  % h(i) = X(i+1)-X(i) for i = 1 to n
for i = 1:length(h)
    h(i) = X(i+1)-X(i);
end
a = Y';

%% Computation of constant c using matrices
A = zeros(numPts, numPts);  % square matrix A (n+1 by n+1)
B = zeros(numPts, 1);  % column vector B (n+1 by 1)

for k = 2:numPts-1  % iterates over 2 to n
    % supplying middle rows of A
    A(k, k-1) = h(k-1);
    A(k, k) = 2*(h(k-1)+h(k));
    A(k, k+1) = h(k);

    % suppying middle rows of B
    left = (3*(a(k+1)-a(k)))/h(k);
    right = (3*(a(k)-a(k-1)))/h(k-1);
    B(k) = left-right;
end

% imposing free conditions (natural)
% s''(X(1)) = s''(X(n+1)) = 0
% => 2c(i) = 0 and 2(c(n+1)) = 0
for k = [1 numPts]
    A(k, k) = 2;
end

% computes for c (c = inv(A)*B)
c = A\B;

%% Computes for constants b and d
b = zeros(numPts-1, 1);  % column vector of size n
d = zeros(numPts-1, 1);  % column vector of size n
for k = 1:numPts-1
    % constant d
    d(k) = (c(k+1)-c(k))/(3*h(k));
```

```
    % constant b
    l = (a(k+1)-a(k))/h(k);
    r = (2*c(k)+c(k+1))/3 * h(k);
    b(k) = l-r;
end

%% Constructing Piecewise Cublic Splines
for i = 1:numPts-1
    S(i) = a(i) + b(i)*(x-X(i)) + c(i)*(x-X(i))^2 + d(i)*(x-X(i))^3;
end

disp(vpa(expand(S'), 4))

%% Plotting Cubic Splines
s = piecewise( ...
    (X(1)<=x)&(x<=X(2)), S(1), ...
    (X(2)<=x)&(x<=X(3)), S(2), ...
    (X(3)<=x)&(x<=X(4)), S(3), ...
    (X(4)<=x)&(x<=X(5)), S(4), ...
    (X(5)<=x)&(x<=X(6)), S(5), ...
    (X(6)<=x)&(x<=X(7)), S(6), ...
    (X(7)<=x)&(x<=X(8)), S(7), ...
    (X(8)<=x)&(x<=X(9)), S(8), ...
    (X(9)<=x)&(x<=X(10)), S(9));

figure
hold on
fplot(s, 'r')
scatter(X, Y, 'filled', 'ko')
grid
legend('natural cubic spline $s(x)$', 'data points', ...
    'Interpreter', 'Latex', 'Fontsize', 14)
xlabel('$x$', ...
    'Interpreter', 'Latex', 'Fontsize', 14)
ylabel('$y$', ...
    'Interpreter', 'Latex', 'Fontsize', 14)
xlim([70, 100])
ylim([72, 102])

%% Substituting Estimate
est = 92.80;
pred_rating = double(subs(S(6), est));
disp(pred_rating)
```