$\frac{20}{20}$

CRISTIAN B. JETOMO

MATH 174 B2L

Exercise 3.1: Piecewise Polynomial Interpolation

1. For this exercise, we wish to interpolate the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ over the interval $[-3, 3]$ using the piecewise linear interpolant $s(x)$. We construct s using the seven linearly spaced abscissas over the interval $[-3, 3]$ given below:

$$x_1 = -3, \quad x_2 = -2, \quad x_3 = -1$$
$$x_4 = 0, \quad x_5 = 1, \quad x_6 = 2, \quad x_7 = 3.$$

On each sub-interval $[x_i, x_{i+1}]$ for $i = 1,2, \dots, 6$, we construct the linear polynomial s_i having x_i and x_{i+1} as endpoints. That is,

$$s_i(x) = a_i + b_i(x - x_i)$$

where

$$a_i = f(x_i) \quad \text{and} \quad b_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Using MATLAB, we obtained the following values of $f(x_i)$ for each of the abscissas x_i for $i = 1, 2, \dots, 7$ ~~which are the following~~, respectively:

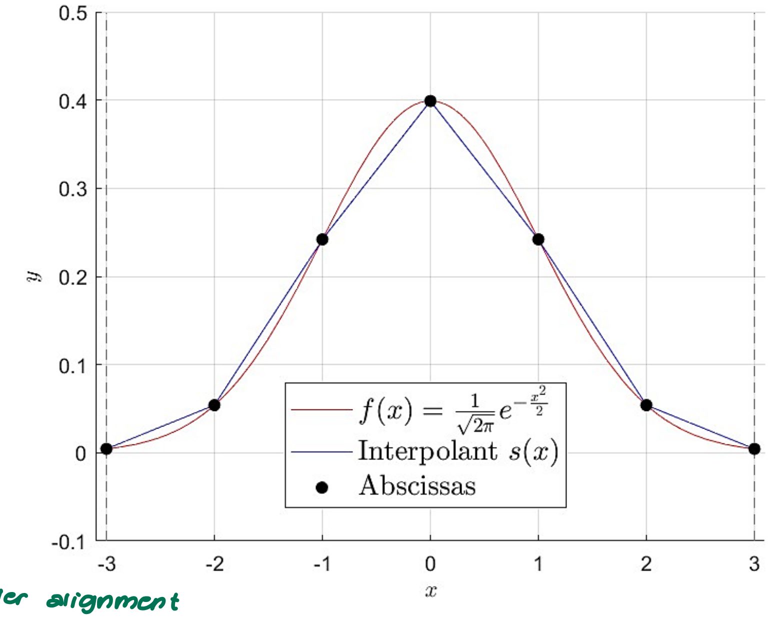
```
0.004431848411938
0.053990966513188
0.241970724519143
0.398942280401433
0.241970724519143
0.053990966513188
0.004431848411938
```

a. Using these values, ~~to obtain the linear polynomials s_i~~, we obtain the piecewise linear interpolant s to be

(6pts)

$$s(x) = \begin{cases} 0.1531 + 0.04956x, & -3 \le x \le -2 \\ 0.4300 + 0.1880x, & -2 \le x \le -1 \\ 0.3989 + 0.1570x, & -1 \le x \le 0 \\ 0.3989 - 0.1570x, & 0 \le x \le 1 \\ 0.4300 - 0.1880x, & 1 \le x \le 2 \\ 0.1531 - 0.04956x, & 2 \le x \le 3 \end{cases}$$

Having this interpolant, we obtain the figure below.



center alignment

Figure 3.1.1. Graph of the function $f(x)$, the interpolant $s(x)$, and the abscissas $(x_i, f(x_i))$ for $i = 1,2,\ldots,7$ over the interval $[-3, 3]$.

only pertains to x_i

passes through

We observe from the plot that the interpolant $s(x)$ ~~interpolates~~ all the abscissas ~~relative to it~~, hence the constructed piecewise function for $s(x)$ is correct. Further more, we see that $s(x)$ consists of linear functions for each sub-interval $[x_i, x_{i+1}]$ that relatively approximates the graph of the function $f(x)$.

(4pts)

b. Now, we wish to calculate the values $s(0.5)$ and $s(1.75)$ using our obtained piecewise linear interpolant. Since $0.5 \in [0, 1]$ and $1.75 \in [1, 2]$, then we shall substitute $x = 0.5$ to the linear function $s_4(x) = 0.3989 - 0.1570x$ and $x = 1.75$ to $s_5(x) = 0.4300 - 0.1880x$, respectively. We obtain these values, using MATLAB, to be

```
s(0.5) ≈
     0.3205

s(1.75) ≈
     0.1010
```

c. We wish to determine the accuracy of the estimates $s(0.5)$ and $s(1.75)$ given that the approximate actual values are $f(0.5) = 0.352065326764300$ and $f(1.75) = 0.086277318826512$. We shall obtain the relative errors of these approximations using the formula

$$\text{relative error} = \frac{|f(x_i) - s(x_i)|}{|f(x_i)|}$$

From here, we obtain that

```
The relative error of s(0.5) is
    0.0898

or approximately (in percent)
8.978

and the relative error of s(1.75) is
    0.1705

or approximately (in percent)
17.05
```

(2pts)

2. Now, we know by Theorem 4.1 that the L^∞ norm of s, i.e., the piecewise linear interpolant for f relative to the abscissas over the interval $[-3,3]$, is given by

$$\max_{x\in[-3,3]} |f(x) - s(x)| \le \frac{1}{8}h^2 \max_{x\in[-3,3]} |f^{(2)}(x)|$$

where

$$h = \max_{i=1,2,\ldots,6} (x_{i+1} - x_i)$$

That is, h is the maximum length of interval (or increment) among the sub-intervals in the interval $[-3,3]$.
So, if we wish to have an approximation to f having an upper bound for the L^∞ error equal to 10^{-6}, then

$$\begin{aligned}\max_{x\in[-3,3]} |f(x) - s(x)| &\le \frac{1}{8}h^2 \max_{x\in[-3,3]} |f^{(2)}(x)| \\ &\le 10^{-6}\end{aligned}$$

That is,

$$\begin{aligned}\frac{1}{8}h^2 \max_{x\in[-3,3]} |f^{(2)}(x)| &\le 10^{-6} \\ h^2 &\le \frac{8\cdot 10^{-6}}{\max_{x\in[-3,3]} |f^{(2)}(x)|} \\ h &\le \left|\sqrt{\frac{8\cdot 10^{-6}}{\max_{x\in[-3,3]} |f^{(2)}(x)|}}\right|\end{aligned}$$

Using MATLAB, we can calculate $|f^{(2)}(x)|$ and determine its maximum value over the interval $[-3,3]$. To do this, we observe the illustration below.

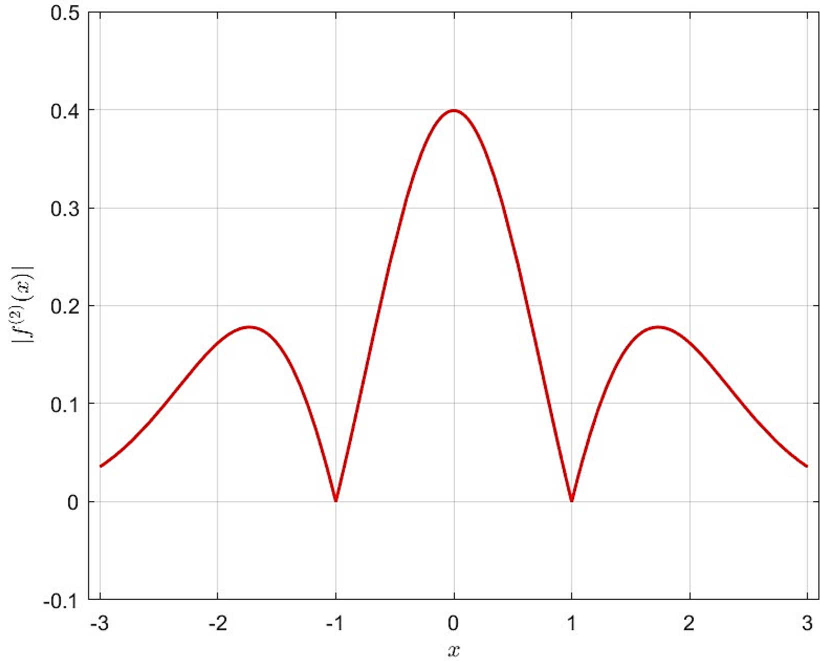


Figure 3.1.2. Graph of $\left|f^{(2)}(x)\right|$ over the interval $[-3, 3]$ where $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

(4pts) We can observe from the figure that the graph of $\left|f^{(2)}(x)\right|$ is symmetric about the x-axis. Moreover, the minimum value of $\left|f^{(2)}(x)\right|$ occurs whenever $x = \pm 1$ and the maximum value occurs when $x = 0$. Hence, our upper bound for the largest increment h becomes

$$h \leq \left| \sqrt{\frac{8 \cdot 10^{-6}}{|f^{(2)}(0)|}} \right|$$

(3pts) Finally, using MATLAB, we calculate that in order to construct a piecewise linear interpolant that approximates f over the interval $[-3,3]$ with an L^{∞} error not exceeding 10^{-6}, then such piecewise linear interpolant must be relative to the abscissas which has the biggest increment that is approximately

```
0.0045
```

Appendix A
Program Used in MATLAB for Item 1

(1pt)

```
clc
clear
close all
syms S
syms x real
%% Abscissas
numNodes = 7;
T = linspace(-3, 3, numNodes);
interval = [min(T) max(T)];
F = zeros(1,numNodes);  % F(x)
f = 1/(sqrt(2*pi)) * exp(-(x^2/2));  % symbolic f(x)
for i = 1:numNodes
    den = sqrt(2*pi);
    n = (T(i)^2/2);
    F(i) = (1/den)*exp(-n);
end

disp(F')

%% Calculation of linear interpolants S_i
for i = 1:numNodes-1
    a = F(i);
    b = (F(i+1)-F(i))/(T(i+1)-T(i));
    S(i) = a + b*(x-T(i));
end
S = S';
disp(vpa(expand(S), 4))
%% Plotting the Piecewise Linear Interpolant s
s = piecewise( ...
    (T(1)<=x)&(x<=T(2)), S(1), ...
    (T(2)<=x)&(x<=T(3)), S(2), ...
    (T(3)<=x)&(x<=T(4)), S(3), ...
    (T(4)<=x)&(x<=T(5)), S(4), ...
    (T(5)<=x)&(x<=T(6)), S(5), ...
    (T(6)<=x)&(x<=T(7)), S(6));

figure
hold on
fplot(f, interval, 'r')
fplot(s, 'b')
scatter(T, F, 'filled', 'k')
grid
xlim([-3.1 3.1])
ylim([-0.1 0.5])
lgnd = legend('$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$', ...
    'Interpolant $s(x)$', ...
    'Abscissas', 'Interpreter', 'latex');
xlabel('$x$','Interpreter', 'latex')
ylabel('$y$','Interpreter', 'latex')
set(lgnd, 'Fontsize', 14)

%% Calculation of s(0.5) and s(1.75)
```

```
s50 = double(subs(S(4), 0.5));
s175 = double(subs(S(5), 1.75));
disp('s(0.5) =')
disp(s50)
disp('s(1.75) = ')
disp(s175)

%% Relative error of s(0.5)
exact50 = 0.352065326764300;
exact175 = 0.086277318826512;
rel50 = abs(exact50-s50)/abs(exact50);
rel175 = abs(exact175-s175)/abs(exact175);
disp('The relative error of s(0.5) is')
disp(rel50)
disp('or approximately (in percent)')
disp(vpa(rel50*100, 4))
disp('and the relative error of s(1.75) is')
disp(rel175)
disp('or approximately (in percent)')
disp(vpa(rel175*100, 4))
```

Appendix B
Program Used in MATLAB for Item 2

```
%% Item 2: Finding upper bound for increment h
f2Prime = diff(f, 2);

% plot f2Prime to find max
figure
fplot(abs(f2Prime), interval, 'r', 'LineWidth',1.5)
grid
xlim([-3.1 3.1])
ylim([-0.1 0.5])
xlabel('$x$','Interpreter', 'latex')
ylabel('$|f^{(2)}(x)|$','Interpreter', 'latex')

% calculation of upper bound
maxf2Prime = double(subs(abs(f2Prime), 0));
h_ub = abs(sqrt((8*10^(-6)/maxf2Prime)));
disp('Upper bound for increment is')
disp(h_ub)
```