Parameter spaces & moduli functors

One of the most fundamental problems in algebraic geometry is the classification problem. For examples,

- classify all smooth projective varieties with given numerical inv. up to birational equivalence
- classify all vector bundles on a fixed variety.

Here the moduli space is the solution to the classification problem

What is a moduli problem?

It consists of two things:

(1) specify a class of certain types of objects

(2) choose an equivalence relation on objects.

What is a moduli space?

$*$: certain types of algebro-geometric objects

"Def" A moduli space of $*$ is a space M such that

$$\{\text{pts of } M\} \xleftrightarrow{1:1} \{\text{isom. classes of } *\}$$

Ex1 $*$ = smooth, connected, projective curves of genus g

the moduli space of smooth curves of genus g, denoted by M_g

Ex2 $*$ = degree d plane curves $C \subset \mathbb{P}^2$

$$M = \{C \subset \mathbb{P}^2 \text{ of deg } d\} / \underset{eq.}{\sim}$$

Where $C \underset{eq.}{\sim} C'$ if they are projectively equivalent.

that is, $\exists\ \sigma \in \mathrm{Aut}(\mathbb{P}^2)$ such that
$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, "\sigma"] & \mathbb{P}^2 \\ C \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sigma|_C"] & C' \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

Other choices for $\underset{eq}{\sim}$:

(a) C and C' are abstractly isomorphic

(b) C and C' are equal as subschemes of $\mathbb{P}^2$

Ex3 Hurwitz moduli space $\mathrm{Hur}_{d,g}$

Instead of studying C in $\mathbb{P}^N$, study

branched covers $C \xrightarrow{f} \mathbb{P}^1$ of degree d, $g(C) = g$

We say $(C \xrightarrow{f} \mathbb{P}^1) \sim (C' \xrightarrow{f'} \mathbb{P}^1)$ if $\exists$ an isomorph. $C \xrightarrow{\varphi} C$

over $\mathbb{P}^1$, i.e.
$$\begin{tikzcd} C \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & \mathbb{P}^1 \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$$\mathrm{Hur}_{d,g} := \left\{ C \xrightarrow[\text{finite morph.}]{\deg d} \mathbb{P}^1 \;\middle|\; \begin{array}{l} C \text{ smooth, Connected,} \\ \text{Projective curve of genus } g \end{array} \right\} \Big/ \sim$$

<u>Ex4</u> Vector bundles on a curve

Fix a sm, conn. proj curve $C/\mathbb{C}$

Fix integers $r \geq 0$, d.

$\mathcal{E}$: vectors bundle (i.e. locally free $\mathcal{O}_C$-module of finite rank) of rank r and deg d

$$M_{C,r,d} := \left\{ \begin{array}{l} \text{all vector bundles } \mathcal{E} \text{ on } C \\ \text{of rank } r,\ \deg d \end{array} \right\} \Big/ \underset{\text{isom.}}{\sim}$$

Application : number of moduli of M_g

<u>Riemann (1857)</u> : The "number of moduli" of smooth curves of genus g is $\underline{3g-3}$.

assume $d \gg 0$ (in fact, explicitly $d > 2g$ is enough)

Here's the one poof of Riemann

$$\begin{tikzcd} & {[C \to \mathbb{P}^1]} \arrow[r, mapsto] & \text{branched points} \\ \mathrm{Hur}_{d,C} \arrow[dr] \arrow[r, phantom, "\subseteq"] & \mathrm{Hur}_{d,g} \arrow[r, "\text{dense image}", "\text{finite fibres}"'] \arrow[d] & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1 \\ & {[C] \in M_g} & \end{tikzcd}$$

① $C \xrightarrow[\deg = d]{f} \mathbb{P}^1$ Riemann-Hurwitz formula

$\cup$

R ramification divisor

$$2g(C) - 2 = d\,(2g(\mathbb{P}^1) - 2) + \deg R$$

$$K_C \sim f^* K_{\mathbb{P}^1} + R$$

$\Rightarrow$ # brand points $= \deg R$
$= 2d + 2g - 2$

$[C \xrightarrow{d} \mathbb{P}^1]$ determined by branched points Completely.

$$\Rightarrow \dim \mathrm{Hur}_{d,g} = 2d + 2g - 2$$

$$\| $$

$$\underbrace{\dim M_g}_{\text{Want to know}} + \underbrace{\dim \mathrm{Hur}_{d,C}}_{?}$$

② Calculate $\dim \mathrm{Hur}_{d,C}$

For a fixed curve C, $C \xrightarrow{f} \mathbb{P}^1$ finite map of deg d determined by an effective divisor $D := f^{-1}(0) = \sum_i P_i \in \mathrm{Sym}^d C$ & a section $t \in H^0(C, \mathcal{O}(D))$ so that $f(p) = [s(p) : t(p)] \in \mathbb{P}^1$ where $s \in H^0(C, \mathcal{O}(D))$ $\mathrm{div}(s) = D$. Note $H^1(C, \mathcal{O}(D)) \overset{\text{Serre}}{=} H^0(C, \underbrace{K_C - D}_{\text{negative deg}})^\vee = 0$

by Riemann-Roch

$$\underset{\substack{\| \\ h^0(D)}}{\chi(\mathcal{O}_C(D))} = \deg D + 1 - \underset{\substack{\| \\ d+1-g}}{g(C)}$$

$D \in \mathrm{Sym}^d C$

$\Rightarrow$ the # moduli of $\mathrm{Hur}_{d,C}$ = # parametes deteriminig D + # of sections t

$= d + (d+1-g)$

$= 2d - g + 1$

$\Rightarrow$ # of moduli of M_g = # of moduli of $\mathrm{Hur}_{d,g} - \dim \mathrm{Hur}_{d,C}$

$= (2d+2g-2) - (2d-g+1)$

$= 3g-3$

$\square$

Main Theorem

(Theorem) The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper, irreducible Deligne-Mumford stack of dim $3g-3$ which admits a projective coarse moduli space $\overline{M}_g$ (proper alg space)

$\overline{\mathcal{M}}_g \longrightarrow M_g$

Moduli functor

Grothendieck's idea : Spaces are functors

(study a scheme X by studying all maps to it)

Sch : category of schemes

Sets : category of sets

Consider the contravariant functor

$$F : \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Sets}}$$

$$B \longmapsto S(B)/\sim$$

$S(B) = \{$all families of $*$ over a scheme $B\}$ / $\sim$ eq.

Call F the moduli functor of the moduli problem classfying $*$ up to an equiv relation

- take M_g as an example

B : scheme

$f : B \longrightarrow M_g$ a map of sets.

$\Rightarrow$ for $\forall$ point $b \in B$, $f(b) \in M_g$ corresponds to an isom. class of a genus g curve C_b.

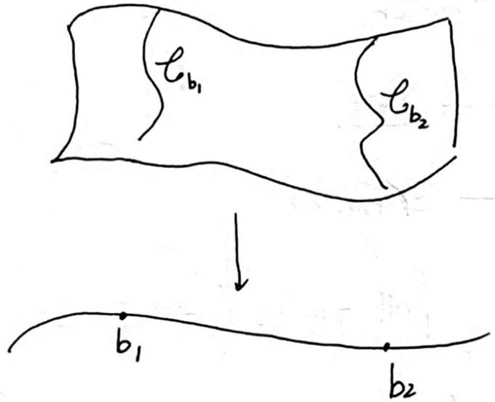
If M_g is a topological space, wish f continous

scheme , wish f algebraic

that is, the curves C_b varying continuously (resp. algebraically) as $b \in B$ moves

A nice way of packaging this is ~~Vatt~~ via "families of curves"

i.e. a smooth, proper morphism $\mathcal{C} \longrightarrow B$ such that each fibre $\mathcal{C}_b$ is a genus g curve.



$$\begin{tikzcd} \mathcal{C} \arrow[d] \\ B \end{tikzcd}$$

$\Rightarrow M_g : \underline{Sch} \longrightarrow \underline{Sets}$ a functor!

$$B \longmapsto \left\{ \begin{array}{c} \text{families of curves over } B \\ \mathcal{C} \longrightarrow B \end{array} \right\}$$

Recall a Category $\mathcal{C}$ consists of the following

(1) Objects: a class of elements

(2) morphisms: for $\forall$ pair A, B of objects, a set $Hom_{\mathcal{C}}(A,B)$

elements called morphisms/arrows from A to B.

satisfying
- Composition of morphisms $Hom(A,B) \times Hom(B,C) \to Hom(A,C)$
- $\exists$ identity morphism $1_A \in Hom_{\mathcal{C}}(A,A)$ for $\forall$ object A
- associativity axiom holds $(h \circ g) \circ f = h \circ (g \circ f)$; identity axiom. $1 \circ f = f$ $g \circ 1 = g$.

Recall $\mathcal{A}, \mathcal{B}$ two categories

a functor $F : \mathcal{A} \to \mathcal{B}$ consists of the following:

(1) a map $F : Ob(\mathcal{A}) \longrightarrow Ob(\mathcal{B})$

$$A \longmapsto FA$$

(2) For any pair of objects $A, A' \in \mathcal{A}$,

a map $Hom_{\mathcal{A}}(A, A') \longrightarrow Hom_{\mathcal{B}}(FA, FA')$

$$A \xrightarrow{f} A' \longmapsto FA \xrightarrow{Ff} FA'$$

Such that for $\forall$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & A' \arrow[r, "g"] & A'' \end{tikzcd}$$

$$\left\{ \begin{array}{l} \begin{tikzcd} FA \arrow[r, "Ff"'] \arrow[rr, bend right, "F(g\circ f)"'] & FA' \arrow[r, "Fg"'] & FA'' \end{tikzcd} \\ \\ F(Id_A) = Id_{FA} \end{array} \right.$$

Example $X \in$ Category of schemes $= \underline{Sch}$

$$h_X : \underline{Sch} \longrightarrow \underline{Sets}$$

$$Y \longmapsto Hom_{Sch}(Y, X) = h_X(Y)$$

$$Y_1 \xrightarrow{f} Y_2 \longmapsto \underbrace{Hom(Y_2, X)}_{=h_X(Y_2)} \longrightarrow \underbrace{Hom(Y_1, X)}_{=h_X(Y_1)}$$

$$\varphi \longmapsto \varphi \circ f$$

$\rightsquigarrow$ It's a contravariant functor, called the functor of points of X.

• a functor $F : \mathcal{C} \longrightarrow \underline{\mathrm{Set}}$ is said to be <u>representable</u> if it is naturally isomorphic to h^A for some object $A \in \mathcal{C}$.

i.e. $\exists$ natural isom. $\Phi : h^A \longrightarrow F$

· for $\forall\, B \in \mathrm{Ob}\,\mathcal{C}$ $\quad \Phi(B) : h^A(B) \xrightarrow[\sim]{\text{bijection}} FB$

· for $\forall\, B \xrightarrow{f} B'$

$$\begin{tikzcd}
h^A(B) \arrow[r, "\Phi(B)"] \arrow[d, "h^A(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & FB \arrow[d, "Ff"] \\
h^A(B') \arrow[r, "\Phi(B')"'] & FB'
\end{tikzcd}$$

• $\mathcal{A}, \mathcal{B}$ two categories, $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ two funtors

a natural transformation $\alpha : F \Longrightarrow G$ is a class of morphisms $(\alpha_A : FA \longrightarrow GA)_{A \in \mathcal{A}}$ of $\mathcal{B}$ indexed by the objects of $\mathcal{A}$.

& such that for $\forall$ morphism $A \xrightarrow{f} A'$ in $\mathcal{A}$,

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowleft"] & GA \arrow[d, "Gf"] \\
FA' \arrow[r, "\alpha_{A'}"'] & GA'
\end{tikzcd}$$

a Contravariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of the following

(1) a map $\quad F : \mathrm{Ob}\,\mathcal{A} \longrightarrow \mathrm{Ob}\,\mathcal{B}$

$\qquad A \longmapsto FA \quad$ for $\forall\, A \in \mathrm{Ob}\,\mathcal{A}$

(2) For any pair of objects $A, A' \in \mathcal{A}$,

a map $\quad \mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FA', FA)$

$\qquad (A \xrightarrow{f} A') \longmapsto (FA' \xrightarrow{Ff} FA)$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & A' \arrow[r, "g"] \arrow[d, phantom, "\circlearrowleft"] & A'' & {} \arrow[r, mapsto, "F"] & {} & FA & FA' \arrow[l, "Ff"'] \arrow[d, phantom, "\circlearrowleft"] & FA'' \arrow[l, "Fg"'] \arrow[ll, bend left, "F(g \circ f)"] \\
& {} & & & & & {} &
\end{tikzcd}$$

a Contravariant functor $F : \mathcal{A} \to \mathcal{B}$ is <u>representable</u> if $F \overset{\text{nat. isom.}}{\cong} h_A$ for some $A \in \mathrm{Ob}\,\mathcal{A}$.

米田
Yoneda's lemma ("h_A determines A")

(covariant version)

F covariant functor $F : \mathscr{C} \to \underline{Set}$

then for $\forall$ object $A \in \mathscr{A}$, $\exists$ a bijection

$$\theta_{F,A} : \mathrm{Nat}(h^A, F) \xrightarrow{\sim} \underset{\text{Set}}{\underline{FA}}$$
$$\alpha : h^A \Rightarrow F \longmapsto \alpha_A(1_A)$$

(Contravariant version)

$G : \mathscr{C}^{opp} \longrightarrow \underline{Set}$ contravariant functor

$\Rightarrow$ for $\forall A \in \mathrm{ob}\,\mathscr{A}$, $\exists$ bijection

$$\theta_{F,A} : \mathrm{Nat}(h_A, G) \xrightarrow{\sim} G(A)$$

Construction of the Hilbert scheme

$X \subset \mathbb{P}^r$ a closed subscheme (more generally, only need to assume X projective scheme)

define the Hilbert polynomial of $X \subset \mathbb{P}^r$ by

$$P_X(m) = \chi(X, \mathcal{O}_X(m)) \quad \text{for } m \gg 0$$
$$\qquad\qquad \| $$
$$\qquad\qquad h^0(X, \mathcal{O}_X(m))$$

Consider the contravariant functor

$$\mathcal{H}ilb_X : \underline{Sch} \longrightarrow \underline{Set}$$

$$B \longmapsto \mathcal{H}ilb_X(B) := \left\{ Z \subset X \times B \;\middle|\; \begin{array}{l} \cdot\ Z \text{ closed subscheme} \\ \cdot\ Z \hookrightarrow X \times B \text{ with } Z \xrightarrow[\text{flat}]{\pi} B,\ X\times B \xrightarrow{pr_2} B \end{array} \right\}$$

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\pi"', "\text{flat}"] & X \times B \arrow[d, "pr_2"] \\ & B \end{tikzcd}$$

$$B_1 \xrightarrow{f} B_2 \longmapsto \mathcal{H}ib_X(B_2) \longrightarrow \mathcal{H}ilb_X(B_1)$$

$$Z_2 \longmapsto (id_X \times f)^*(Z_2)$$

here, roughly speaking, a flat family means requiring that the fibres vary "continuously".

$$X \times B_1 \xrightarrow{id_X \times f} X \times B_2 \supset Z_2$$

Def. $f : X \longrightarrow Y$ a morphism of schemes.
$x \longmapsto y = f(x)$

$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X \rightsquigarrow$ local homo. $f^\#_x : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$

We say that f is flat if $\mathcal{O}_{x,X}$ is a flat $\mathcal{O}_{y,Y}$-module for $\forall x \in X$.

Fact | If B is connected & $Z \xrightarrow{\pi} B$ is flat.
then $P_{Z_b}(m) := \chi(\mathcal{O}_{Z_b} \otimes \mathcal{O}_X(m))$ (for $m \gg 0$)
the Hilbert polynomial of $Z_b := \pi^{-1}(b)$ is independent of $b \in B$

To obtain a well-behaved moduli functor, we need to restrict the family to be flat & having the constant Hilbert polynomial P. Hence

Consider the subfunctor of $\mathcal{H}ilb_X$

$$\mathcal{H}ilb_X^P : \underline{Sch} \longrightarrow \underline{Set}$$

$$B \longmapsto \mathcal{H}ilb_X^P(B) := \left\{ Z \subset X \times B \;\middle|\; \begin{array}{l} \cdot\ Z \text{ closed subsch.} \\ \cdot\ Z \hookrightarrow X \times B \text{ with } Z \xrightarrow[\text{flat}]{\pi} B,\ X\times B \xrightarrow{pr_2} B \\ \cdot\ P_{Z_b}(m) = P \text{ for } \forall b \in B \end{array} \right.$$

Thm (Grothendieck 1960)

The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme $H_X^P = \mathrm{Hilb}_X^P$, that is $\mathcal{H}ilb_X^P \cong \mathrm{Hom}_{\underline{Sch}}(\cdot, H_X^P)$

Fine moduli space & coarse moduli space

moduli functor

$$F : \underline{Sch} \longrightarrow \underline{Set}$$

$$\begin{tikzcd} \mathcal{C}\times_{B_2}B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$

$$B \longmapsto S(B)/\sim$$

$$B_1 \to B_2 \longmapsto F(B_2) \to F(B_1)$$

<u>Recall</u> F is representable in the category of schemes if $\exists$ a scheme M & $\exists$ an isom. $\Psi : F \xrightarrow{\cong} h_M$

<u>Def</u> If F is representable by M, then we say that the scheme M is a <u>fine moduli space</u> for the moduli problem F.

If $\varphi : \mathcal{D} \to B$ is any family in $S(B)$, then

$$\Psi(B) : F(B) \xrightarrow[\sim]{\text{bijection}} h_M(B) = \mathrm{Hom}_{\underline{Sch}}(B, M)$$

$$\varphi \longleftrightarrow \partial : B \to M \quad \text{a morphism of schemes}$$

<u>Intuitively</u>, points of M classify the objects of moduli problem.

for $\forall\, b \in B$, $\partial(b) \in M$ determined by the fibre $\mathcal{D}_b$

$$\Psi(M) : F(M) \xrightarrow[\sim]{\text{bijection}} h_M(M) = \mathrm{Hom}_{\underline{Sch}}(M, M)$$

$$1 : \mathcal{U} \longrightarrow M \longleftrightarrow id_M$$

Then for $\forall$ morphism $\partial : B \to M$ as above

$$\begin{tikzcd} (\mathcal{D}\to B) & & \\ F(B) \arrow[r, "\Psi_B"] & h_M(B) & B \xrightarrow{\partial} M \\ F(M) \arrow[u, "\text{given by fibre-product}"] \arrow[r, "\Psi_M"'] & h_M(M) \arrow[u, "\partial\circ"'] & \\ (\mathcal{U} \xrightarrow{1} M) \arrow[r, leftrightarrow] & id_M \arrow[uur, bend right] & \end{tikzcd}$$

$\Rightarrow \exists$ a commutative Cartesian diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d, "\forall\varphi"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{U} \arrow[d, "1"] \\ B \arrow[r, dashed, "\exists!\,\partial"'] & M \end{tikzcd}$$

every family over B is the pullback of the universal family $\mathcal{U} \to M$ via a unique map $B \longrightarrow M$.

<u>Unfortunately</u>

few natural moduli functors are representable by scheme.

two solutions to this :

① look for a larger category, in which F can be represented.

$$(\text{Schemes}) \subset (\text{algebraic spaces}) \subset \begin{pmatrix}\text{Deligne-Mumford} \\ \text{Stacks}\end{pmatrix} \subset (\text{algebraic stacks})$$

② find a scheme M captures enough information in moduli functor :

"to ask only for a natural transformation $\Psi_M : F \longrightarrow h_M$ rather than an isom!"

i.e. Coarse moduli space.

Def (Coarse moduli space)

a scheme M and a natural transformation $\theta_M : F \longrightarrow h_M$ are called a coase moduli space for the functor F if

(1) The map $\theta_M(\mathrm{Spec}\,\mathbb{C}) : F(\mathrm{Spec}\,\mathbb{C}) \longrightarrow h_M(\mathrm{Spec}\,\mathbb{C})$ is a bijection of sets

$$h_M(\mathrm{Spec}\,\mathbb{C}) = \mathrm{Hom}_{\underline{Sch}}(\mathrm{Spec}\,\mathbb{C}, M)$$

(2) given another scheme M' & a natural transf. $\theta_{M'} : F \to h_{M'}$

then $\exists!$ morphism $M \xrightarrow{\pi} M'$ such that

$$\begin{tikzcd} F \arrow[r, "\theta_M"] \arrow[d, "\forall\theta_{M'}"'] \arrow[dr, phantom, "\circlearrowright"] & h_M \arrow[dl, dashed, "\pi\circ"] \\ h_{M'} & \end{tikzcd}$$