Max-flow and min-cut problems

Dinic and Edmonds-Karp algorithm

J.Edmonds, R. Karp: *Theoretical improvements in algorithmic efficiency for network flow problems*. Journal ACM 1972.

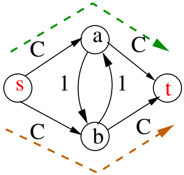
Yefim Dinic: *Algorithm for solution of a problem of maximum flow in a network with power estimation*. Doklady Ak.N. 1970

Choosing a good augmenting path can lead to a faster algorithm.
Use BFS to find an augmenting paths in G_f.

Edmonds-Karp algorithm

FF algorithm but using BFS: choose the augmenting path in G_f with the smallest length (number of edges).

Edmonds-Karp(G, c, s, t)
For all $e = (u, v) \in E$ let $f(u, v) = 0$
$G_f = G$
while there is an $s \rightsquigarrow t$ path in G_f
do
  $P = \text{BFS}(G_f, s, t)$
  $f = \text{Augment}(f, P)$
  Compute G_f
return f



The BFS in EK will choose: ⟶ or ⟶

BFS paths on G_f

For $\mathcal{N} = (V, E, c, s, t)$ and a flow f in $\mathcal{N}$, assuming that G_f has an augmenting path, let f' be the next flow after executing one step of the EK algorithm.

- The path from s to t in a BFS traversal starting at s, is a path $s \rightsquigarrow t$ with minimum number of edges, i.e., a shortest length path.
- For $\in V$, let $\delta_f(s, v)$ denote length of a shortest length path from s to v in G_f.

Some properties of G_f and $G_{f'}$

How can we have $(u, v) \in E_{f'}$ but $(u, v) \notin E_f$?

- (u, v) is a forward edge saturated in f and not in f''.
- (u, v) is a backward edge in G_f and $f(v, u) = 0$

In any of the two cases, the augmentation must have modified the flow from v to u, so (u, v) must form part of the augmenting path.

EK and the shortest length distances

Lemma

If the EK-algorithm runs on $\mathcal{N} = (V, E, c, s, t)$, for all vertices $v \neq s$, $\delta_f(s, v)$ increases monotonically with each flow augmentation.

Proof. By contradiction.

Let f be the first flow such that, for some $u \neq s$,

$$\delta_{f'}(s, u) < \delta_f(s, u).$$

Proof (cont)

Let v be the vertex with the minimum $\delta_{f'}(s, v)$ whose distance was decreased.

- Let $P : s \rightsquigarrow u \to v$ be a shortest length path from s to v in $G_{f'}$
- Then, $\delta_{f'}(s, v) = \delta_{f'}(s, u) + 1$ and $\delta_{f'}(s, u) \geq \delta_f(s, u)$.
- If $(u, v) \in E_f$, $\delta_f(s, v) \leq \delta_f(s, u) + 1 \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v)$
- So, $(u, v) \notin E_f$

EK and the shortest length distances

Proof (cont)

How can we have ?

- $(u, v) \in E_{f'}$ but $(u, v) \notin E_f$
- If so, (v, u) appears in the augmenting path.
- Then, the shortest length path from s to u in G_f has (v, u) as it last edge.
  $\delta_f(s, v) \leq \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 1 - 1$
- which contradicts $\delta_{f'}(s, v) < \delta_f(u, v)$. $\square$

Some properties of G_f and $G_{f'}$

Let P be an augmenting path in G_f.

$(u, v) \in P$ is critical if $b(P) = c_f(u, v)$.

Critical edges do not appear in $G_{f'}$.

- (u, v) forward, $f'(u, v) = c(u, v)$
- (u, v) backward, $f'(v, u) = 0$

EK and critical edges

Lemma

In the EK algorithm, each one of the edges can become critical at most $|V|/2$ times.

Proof:

- Let $(u, v) \in E$, when (u, v) is critical for the first time, $\delta_f(s, v) = \delta_f(s, u) + 1$
- After this step (u, v) disappears from the residual graph until after the flow in (u, v) changes.
- At this point, (v, u) forms part of the augmenting path in $G_{f'}$, and $\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1$,

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 \geq \delta_f(s, u) + 2$$

- So, the distance has increased by at least 2. $\square$

Complexity of Edmonds-Karp algorithm

Theorem

The EK algorithms runs in $O(mn(n+m))$ steps. Therefore it is a polynomial time algorithm.

Proof:

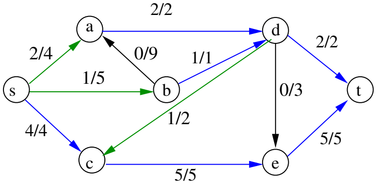
- Need time $O(m+n)$ to find the augmenting path using BFS.
- By the previous Lemma, there are $O(mn)$ augmentations.
  $\square$

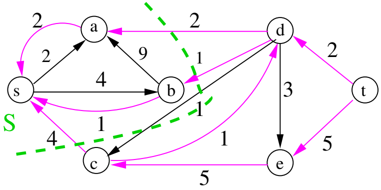
Finding a min-cut

Given (G, s, t, c) to find a min-cut:

1. Compute the max-flow f^* in G.
2. Obtain G_{f^*}.
3. Find the set $S = \{v \in V | s \rightsquigarrow v\}$ in G_{f^*}.
4. Output the cut $(S, V - \{S\}) = \{(v, u) | v \in S and u \in V - \{S\}\}$ in G.

The running time is the same than the algorithm to find the max-flow.





The max-flow problems: History

- Ford-Fulkerson (1956) $O(mC)$, where C is the max flow.
- Dinic (1970) (blocking flow) $O(n^2 m)$
- Edmond-Karp (1972) (shortest augmenting path) $O(nm^2)$
- Karzanov (1974), $O(n^2 m)$ Goldberg-Tarjant (1986) (push re-label preflow + dynamic trees) $O(nm \lg(n^2/m))$ (uses parallel implementation)
- King-Rao-Tarjan (1998) $O(nm \log_{m/n \lg n} n)$.
- J. Orlin (2013) $O(nm)$ (clever follow up to KRT-98)

So: Maximum flows can be computed in $O(nm)$ time!

Applications: Generalized assignment problems

- Consider a generalized assignment problem $\mathcal{GP}$ where, we have as input d finite sets $X_1, \ldots, X_d$, each representing a different set of resources.
- Our goal is to chose the "largest" number of d-tuples, each d-tuple containing exactly one element from each X_i, subject to the constrains:
  - For each $i \in [d]$, each $x \in X_i$ can appears in at most $c(x)$ selected tuples.
  - For each $i \in [d]$, any two $x \in X_i$ and $y \in X_{i+1}$ can appear in at most $c(x, y)$ selected tuples.
  - The values for $c(x)$ and $c(x, y)$ are either in $\mathbb{Z}^+$ or ∞.
- Notice that only pairs of objects between adjacent X_i and X_{i+1} are constrained.

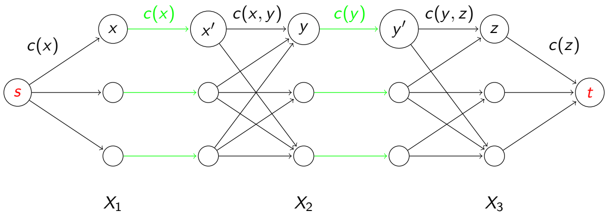
Applications: Generic reduction to Max-Flow

Make the reduction from $\mathcal{GP}$ to the following network $\mathcal{N}$:

- V contains a vertex x, for each element x in each X_i, and a copy x', for each element $x \in X_i$ for $1 \leq i < d$.
- We add vertex s and vertex t.
- Add an edge $s \to x$ for each $x \in X_1$ and add an edge $y \to t$ for every $y \in X_d$. Give capacities $c(s,x) = c(x)$ and $c(y,t) = c(y)$.
- Add an edge $x' \to y$ for every pair $x \in X_i$ and $y \in X_{i+1}$. Give a capacity $c(x,y)$. Omit the edges with capacity 0.
- For every $x \in X_i$ for $1 \leq i < d$, add an edge $x \to x'$ with $c(x,x') = c(x)$.

Every path $s \rightsquigarrow t$ in $\mathcal{N}$ identifies a feasible d-tuple, conversely every d-tuple determines a a path $s \rightsquigarrow t$.

Flow Network: The reduction



- To solve $\mathcal{GP}$, we construct $\mathcal{N}$, and then we find an integer maximum flow f^*.
- In the subgraph formed by edges with $f^*(e) > 0$, we find a (s, t) path P (a d-tuple), decrease in 1 the flow in each edge of P, remove edges with 0 flow.
- We repeat the procedure for $|f^*|$times. In this way we obtain a set of d-tuples with maximum size verifying all the restrictions.

Final's Scheduling

We have as input:

- n courses, each one with a final. Each exam must be given in one room. Each course c_i has $E[i]$ students.
- r rooms. Each r_j has a capacity $S[j]$,
- τ time slots. For each room and time slot, we only can schedule one final.
- p professors to watch exams. Each exam needs one professor in each class and time. Each professor has its own restrictions of availability and no professor can oversee more than 6 finals. For each p_ℓ and τ_k define a Boolean variable $A[k, \ell] = T$ if p_ℓ is available at τ_k.

Design an efficient algorithm that correctly schedules a room, a time slot and a professor to every final, or report that not such schedule is possible.

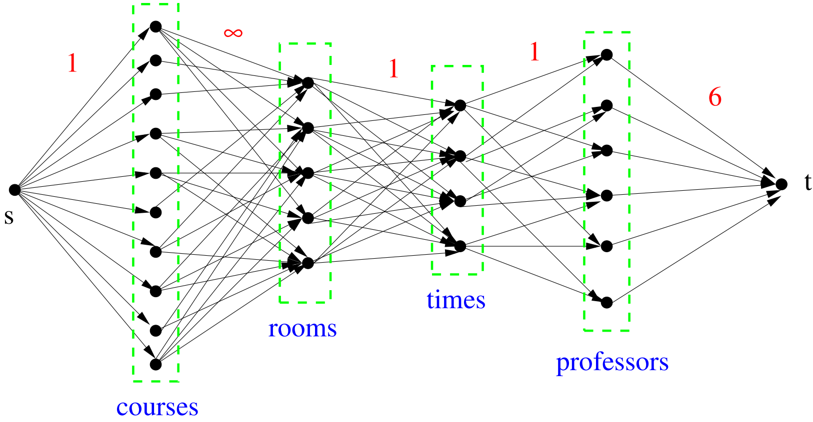
Construction of the network

Construct the network $\mathcal{N}$ with vertices $\{s, t, \{c_i\}, \{r_j\}, \{\tau_k\}, \{p_\ell\}\}$. Edges and capacities:

- (s, c_i) with capacity 1 (each course has one final)
- (c_i, r_j), if $E[i] \leq S[j]$, with capacity ∞
- $\forall j, k$, (r_j, τ_k), with capacity 1 (one final per room and time slot).
- (τ_k, p_ℓ), if $A[k, \ell] = T$, capacity 1
  (p can watch one final, if p is available at τ_k).
- (p_ℓ, t), capacity 6 (each p can watch ≤ 6 finals)

Notice that neither rooms nor time slots have individual restrictions.

FINAL'S SCHEDULING: Flow Network



Final's Scheduling

- Notice the input size to the problem is $N = n + r + \tau + p + 2$. and size of the network is $O(N)$ vertices and $O(N^2)$ edges, why?
- Every path $s \rightsquigarrow t$ is an assignment of room-time-professor to a final, and any assignment room-time-professor to a final can be represented by a path $s \rightsquigarrow t$.
- Every integral flow identifies a collection of $|f|$ (s,t)-paths leading to a valid assignment for $|f|$ finals and viceversa.

Final's Scheduling

- To maximize the number of finals to be given, we compute the max-flow f^* from s to t.
- If $|f^*| = n$, then we can schedule all finals, otherwise we can not.
- To recover the assignment we have to consider the edges with positive flow and extract assignment from the n (s, t)-paths
- Complexity:
  - To construct $\mathcal{N}$, we need $O(N^2)$.
  - As $|f^*| \leq n$ integral, we can use Ford-Fulkerson to compute f^*, with cost $O(nN^2)$.
  - The second part requires $O(N^2)$ time.
  - So, the cost of the algorithm is $O(nN^2) = O(n(n + r + \tau + p)^2)$.