

# Circulations

**Circulation**  
Demands  
Lower bounds

**Examples**  
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Image segmentation



## **1** Circulation

### Circulation

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## **2** Examples

# Circulation with demands

- We introduce another flow problem, to deal with supply and demand inside a network.
- Instead of having a pair source/sink the new setting consider a producer/consumer scenario.
- Some nodes are able to produce a certain amount of flow.
- Some nodes are willing to consume flow.
- The question is whether it is possible to route “all” the produced flow to the consumers. When possible the flow assignment is called a **circulation**

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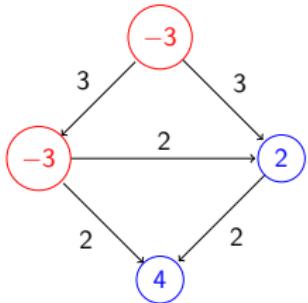
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# Network with demands

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A **network with demands**  $\mathcal{N}$  is a tuple  $(V, E, c, d)$  where  $c$  assigns a positive capacity to each edge, and  $d$  is a function associating a demand  $d(v)$ , to  $v \in V$ .



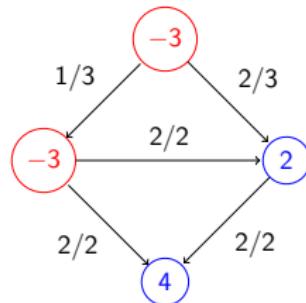
- When  $d(v) > 0$ ,  $v$  can receive  $d(v)$  units of flow more than it sends,  $v$  is a **sink**.
- If  $d(v) < 0$ ,  $v$  can send  $d(v)$  units of flow more than it receives,  $v$  is a **source**.
- If  $d(v) = 0$ ,  $v$  is neither a source or a sink.
- Define  $S$  to be the set of sources and  $T$  the set of sinks.

# Network with demands: circulation

Given a network  $\mathcal{N} = (V, E, c, d)$ , a **circulation** is a flow assignment  $f : E \rightarrow \mathbb{R}^+$  s.t.

- 1 **capacity:** For each  $e \in E$ ,  $0 \leq f(e) \leq c(e)$ ,
- 2 **conservation:** For each  $v \in V$ ,

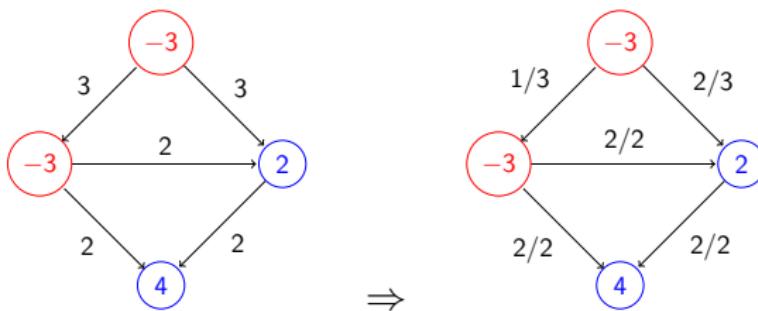
$$\sum_{(u,v) \in E} f(u,v) - \sum_{(v,z) \in E} f(v,z) = d(v).$$



Take into account that a circulation might not exist.

# Network with demands: circulation problem

**Circulation problem:** Given  $\mathcal{N} = (V, E, c, d)$  with  $c > 0$ , obtain a circulation provided it does exists.



# A first conditions for a circulation to exists

If  $f$  is a circulation for  $\mathcal{N} = (V, E, c, d)$ ,

$$\sum_{v \in V} d(v) = \sum_{v \in V} \left( \underbrace{\sum_{(u,v) \in E} f(u,v)}_{\text{edges to } v} - \underbrace{\sum_{(v,z) \in E} f(v,z)}_{\text{edges out of } v} \right).$$

For  $e = (u, v) \in E$ ,  $f(e)$  appears in the sum of edges to  $v$  and in the sum of edges out of  $u$ . Both terms cancel!

Then,  $\sum_{v \in V} d(v) = 0$ .

# A first conditions for a circulation to exists

If there is a circulation, then  $\sum_{v \in V} d(v) = 0$ .

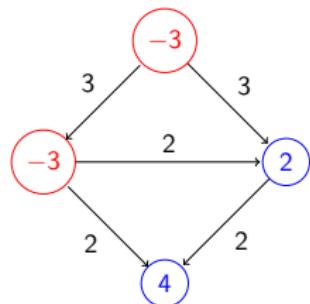
Recall that

$$S = \{v \in V | d(v) < 0\} \text{ and}$$

$$T = \{v \in V | d(v) > 0\}.$$

Define  $D = -\sum_{v \in S} d(v) = \sum_{v \in T} d(v)$ .

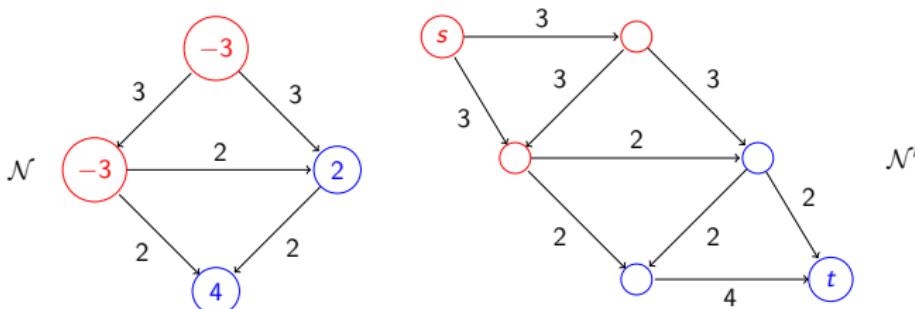
$D$  is the total amount of extra flow that has to be transported from the sources to the sinks.



# Circulation problem: reduction to Max-flow

From  $\mathcal{N} = (V, E, c, d)$ , define a flow network  
 $\mathcal{N}' = (V', E', c', s, t)$ :

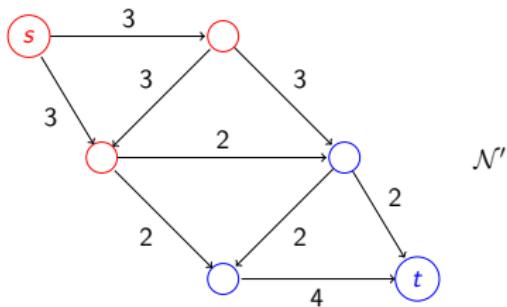
- $V' = V \cup \{s, t\}$ , we add a source  $s$  and a sink  $t$ .
- For  $v \in S$  ( $d(v) < 0$ ), add  $(s, v)$  with capacity  $-d(v)$ .
- For  $v \in T$  ( $d(v) > 0$ ), add  $(v, t)$  with capacity  $d(v)$ .
- Keep  $E$  and, for  $e \in E$ ,  $c'(e) = c(e)$ .



# Circulation problem: reduction to Max-flow

1.- Every flow  $f'$  in  $\mathcal{N}'$  verifies  $|f'| \leq D$

The capacity  $c'(\{s\}, V) = D$ , by the capacity restriction on flows,  $|f'| \leq D$ .



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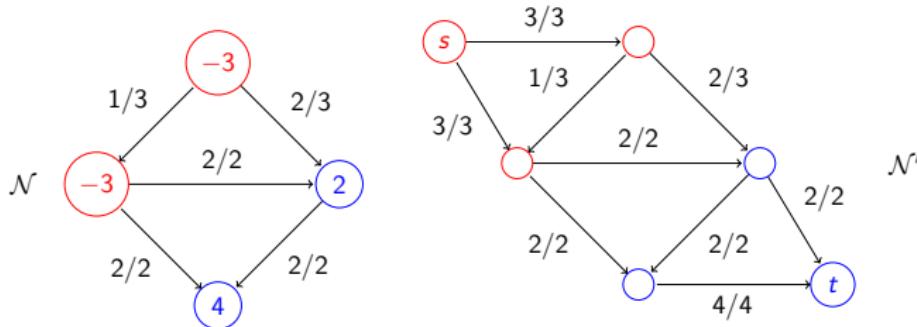
# Circulation problem: reduction to Max-flow

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2.- If there is a circulation  $f$  in  $\mathcal{N}$ , we have a max-flow  $f'$  in  $\mathcal{N}'$  with  $|f'| = D$ .

Extend  $f$  to a flow  $f'$ , assigning  $f'(s, v) = -d(v)$ , for  $v \in S$ , and  $f'(u, t) = d(u)$ , for  $u \in T$ .

By the circulation condition,  $f'$  is a flow in  $\mathcal{N}'$ . Furthermore,  $|f'| = D$ .

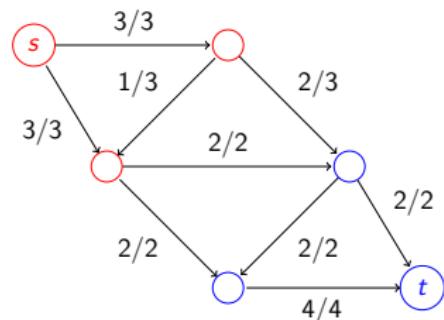


# Analysis

3.- If there is a flow  $f'$  in  $\mathcal{N}'$  with  $|f'| = D$ ,  $\mathcal{N}$  has a circulation

For  $e \in E$ , define  $f(e) = f'(e)$ .

- As  $|f'| = D$ , all edges  $(s, v) \in E'$  and  $(u, t) \in E'$  are saturated by  $f'$ .
- By flow conservation,  $f$  satisfies  $d(v) = \sum_{(u,v) \in E} f(u, v) - \sum_{(v,z) \in E} f(v, z)$ .
- So,  $f$  is a circulation for  $\mathcal{N}$ .



# Circulation: main results

From the previous discussion, we can conclude:

**Theorem (Necessary and sufficient condition)**

*There is a circulation for  $\mathcal{N} = (V, E, c, d)$  iff the maxflow in  $\mathcal{N}'$  has value  $D$ .*

**Theorem (Circulation integrality theorem)**

*If all capacities and demands are integers, and there exists a circulation, then there exists an integer valued circulation.*

Sketch Proof Max-flow formulation + integrality theorem for max-flow



# Circulation: main results

## Theorem

*There is a polynomial time algorithm to solve the circulation problem.*

The cost of the algorithm is the same as the cost of the algorithm used for the MaxFlow computation.

## Theorem

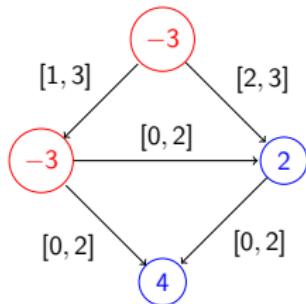
*If all capacities and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time  $O(Dm)$ .*

# Networks with demands and lower bounds

Generalization of the previous problem: besides satisfy demands at nodes, we want to force the flow to use certain edges.

Introduce a new constraint  $\ell(e)$  on each  $e \in E$ , indicating the min-value the flow must be on  $e$ .

A network  $\mathcal{N}$  with **demands and lower bounds** is a tuple  $(V, E, c, \ell, d)$  with  $c(e) \geq \ell(e) \geq 0$ , for each  $e \in E$ ,



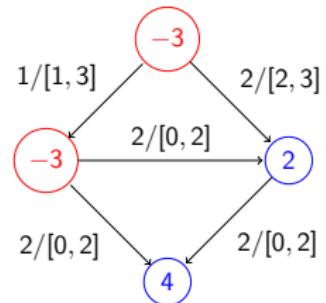
# Networks with demands and lower bounds: circulation

Given a network  $\mathcal{N} = (V, E, c, \ell, d)$  a **circulation** as a flow assignment  $f : E \rightarrow \mathbb{R}^+$  s.t.

**1 capacity:** For each  $e \in E$ ,  
 $\ell(e) \leq f(e) \leq c(e)$ ,

**2 conservation:** For each  $v \in V$ ,

$$\sum_{(u,v) \in E} f(u, v) - \sum_{(v,z) \in E} f(v, z) = d(v).$$



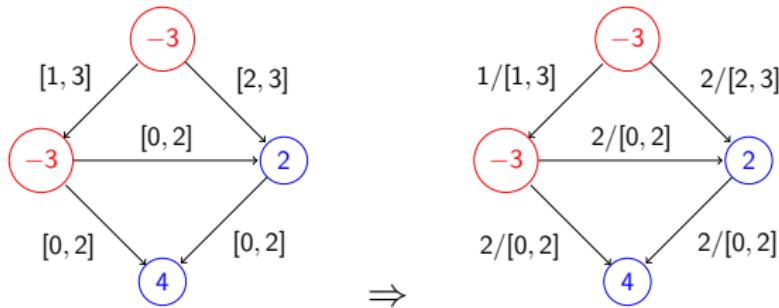
A circulation might not exist.

# Circulations with demands and lower bounds problem

**Circulation with demands and lower bounds problem:** Given  $\mathcal{N} = (V, E, c, \ell, d)$ , obtain a circulation for  $\mathcal{N}$ , provided it does exists

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We devise an algorithm to the problem by a reduction to a circulation with demands problem.

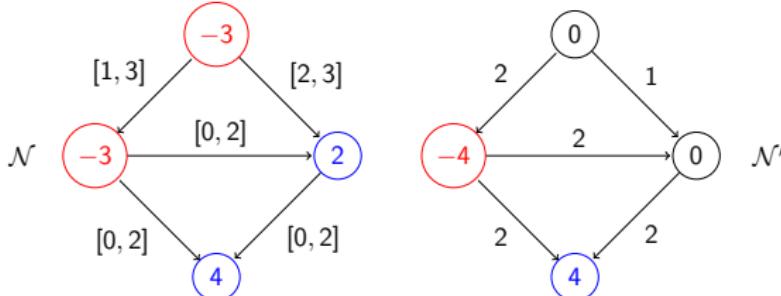
# Circulations with demands and lower bounds: the reduction

Let  $\mathcal{N} = (V, E, c, \ell, d)$ , construct a network  $\mathcal{N}' = (V, E, c', d')$  with only demands as follows:

Initially set  $c' = c$  and  $d' = d$ .

For each  $e = (u, v) \in E$ , with  $\ell(e) > 0$ :

- $c'(e) = c(e) - \ell(e)$ .
- Update the demands on both ends of  $e$ :  
 $d'(u) = d(u) + \ell(e)$  and  $d'(v) = d(v) - \ell(e)$



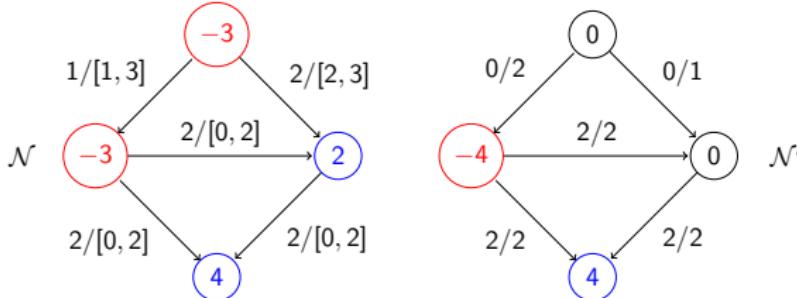
# Circulations with demands and lower bounds: the reduction

1.- If  $f$  is a circulation in  $\mathcal{N}$ ,  $f'(e) = f(e) - \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}'$ .

By construction of  $\mathcal{N}'$ ,  $f'$  verifies the capacity constraint.

Besides, for  $(u, v)$  with  $\ell(u, v) > 0$ , the flow out of  $u$  and the flow in  $v$  is decreased by  $\ell(u, v)$ .

$f$  is a circulation in  $\mathcal{N}$  so, the flow imbalance of  $f'$  matches the demand  $d'$  at each node.



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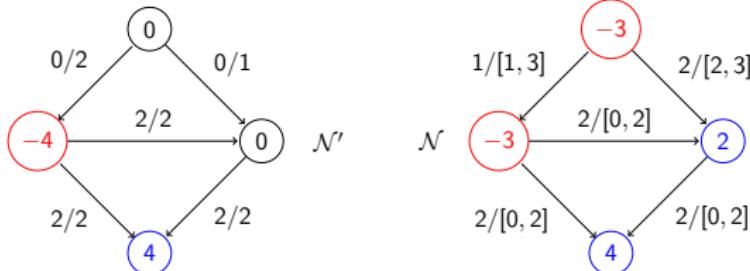
# Circulations with demands and lower bounds: the reduction

2.- If  $f'$  is a circulation in  $\mathcal{N}'$ ,  $f(e) = f'(e) + \ell(e)$ , for  $e \in E$ , is a circulation in  $\mathcal{N}$ .

$f'$  verifies the capacity constraint  $f'(e) \geq 0$ , so  $f'(e) \geq \ell(e)$ .

$f'$  is a circulation, the  $f'$  imbalance at  $u$  is  $d'(u)$ .

Therefore, for  $(u, v)$  with  $\ell(u, v) > 0$ , the increase of flow in  $(u, v)$  balances  $\ell(u, v)$  units of flow out of  $u$  with  $\ell(u, v)$  units of flow entering  $v$ . Thus the  $f$  imbalance at  $u$  is  $d(u)$ .



# Main result

## Theorem

*There exists a circulation in  $\mathcal{N}$  iff there exists a circulation in  $\mathcal{N}'$ . Moreover, if all demands, capacities and lower bounds in  $\mathcal{N}$  are integers, and  $\mathcal{N}$  admits a circulation, there is a circulation in  $\mathcal{N}$  that is integer-valued.*

The integer-valued circulation part is a consequence of the integer-value circulation Theorem for  $f'$  in  $G'$ .

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# Circulation with demands and lower bounds: main results

## Theorem

*There is a polynomial time algorithm to solve the circulation with demands and lower bounds problem.*

The cost of the algorithm is the same as the cost of the algorithm used for the circulation with demands computation.

## Theorem

*If all capacities, lower bounds, and demands are integers, and there exists a circulation, then we can obtain an integer valued circulation in time  $O((D + L)m)$  where  $L$  is the sum of all lower bounds.*

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## **1** Circulation

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## **2** Examples

# SURVEY DESIGN problem

Problem: Design a survey among customers of products  
(KT-7.8)

- Customer  $i$  can only be asked about a bought product and must receive a questionnaire for at least  $c_i$  such products, those values are determined as function of the purchased products.
- For each product  $j$ , we want to collect data from a minimum of  $p_j$  customers.
- The  $c$  and  $p$  values cannot

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# SURVEY DESIGN problem

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The input to the problem is:

A set  $C$  of  $n$  customers and a set  $P$  of  $m$  products.

- For each customer  $i \in C$ , a list of purchased products and the two values  $c_i \leq c'_i$ .
- For each product  $j \in P$ , two values  $p_j$  and  $p'_j$ .

Alternatively,

- The information about purchases can be represented as a bipartite graph  $G = (C \cup P, E)$ , where  $C$  is the set of customers and  $P$  is the set of products.
- $(i, j) \in E$  means  $i \in C$  has purchased product  $j \in P$ .

# SURVEY DESIGN: Input

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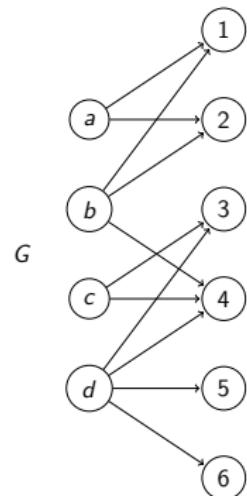
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Customers  $C = \{a, b, c, d\}$   
Products  $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	c
a	1,2	1
b	1,2,4	1
c	3,6	1
d	3,4,5,6	2

Prod.	1	2	3	4	5	6
d	1	1	1	1	0	1



# SURVEY DESIGN: Circulation with lower bounds formulation

We construct a network  $\mathcal{N} = (V', E', c, \ell)$  from  $G$  as follows:

- Nodes:  $V' = V \cup \{s, t\}$
- Edges:  $E'$  contains  $E$  and edges  $s \rightarrow \{C\}$ ,  $\{P\} \rightarrow t$ , and  $(t, s)$ .
- Capacities and lower bounds:
  - $c(t, s) = \infty$  and  $\ell(t, s) = 0$
  - For  $i \in C$ ,  $\ell(s, i) = c_i$  and  $c(s, i) =$  the number of purchased products.
  - For  $j \in P$ ,  $\ell(j, t) = p_j$  and  $c(j, t) =$  number of customers that purchased  $j$ .
  - For  $(i, j) \in E$ ,  $c(i, j) = 1$ , and  $\ell(i, j) = 0$ .

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# SURVEY DESIGN: Circulation with lower bounds formulation

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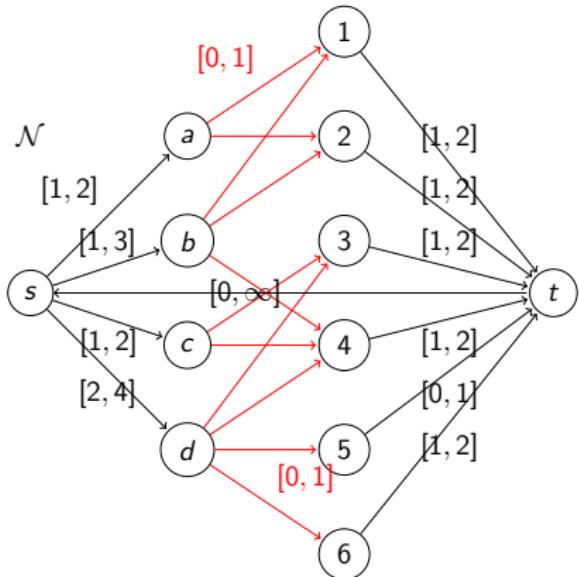
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Customers  $C = \{a, b, c, d\}$   
Products  $P = \{1, 2, 3, 4, 5, 6\}$

Customer	Bought	c
a	1,2	1
b	1,2,4	1
c	3,6	1
d	3,4,5,6	2

Prod.	1	2	3	4	5	6
d	1	1	1	1	0	1

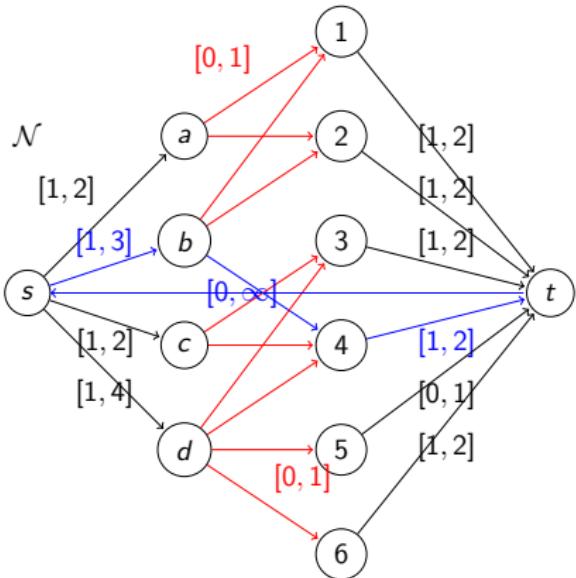


# SURVEY DESIGN: Circulation interpretation

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If  $f$  is a circulation in  $\mathcal{N}$ :

- one unit of flow circulates  
 $s \rightarrow i \rightarrow j \rightarrow t \rightarrow s$ .
- $f(i, j) = 1$  means ask  $i$  about  $j$ ,
- $f(s, i)$  # products to ask  $i$  for opinion,
- $f(j, t)$  # customers to be asked to review  $j$ ,
- $f(t, s)$  is the total number of questionnaires.

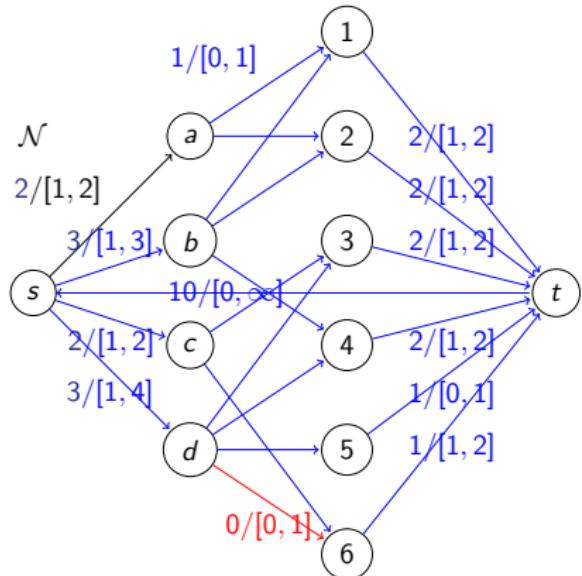


# SURVEY DESIGN: Circulation vs solutions

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## A solution

- Ask *a* about 1, 2.
- Ask *b* about 1, 2, 4.
- Ask *c* about 3, 6.
- Ask *d* about 3, 4, 5.



# Main result

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**Theorem**  $\mathcal{N}$  has a circulation iff there is a feasible way to design the survey.

**Proof** if there is a feasible way to design the survey:

- if  $i$  is asked about  $j$  then  $f(i, j) = 1$ ,
- $f(s, i) = \text{number questions asked to } i (\geq c_i)$ .
- $f(j, t) = \text{number of customers who were asked about } j (\geq d_j)$ ,
- $f(t, s) = \text{total number of questions}$ .
- easy to verify that  $f$  is a circulation in  $\mathcal{N}$

If there is an integral circulation in  $\mathcal{N}$ :

- if  $f(i, j) = 1$  then  $i$  will be asked about  $j$ ,
- the constraints will be satisfied by the capacity rule.



# Cost of the algorithm

- $\mathcal{N}$  has  $N = n + m + 2$  vertices and  $E = n + m + nm$  edges
- $L = \sum_e \ell(e) \leq nm$ .
- Obtain  $\mathcal{N}$  and extract the information from the circulation has cost  $O(nm)$ .
- FF analysis, the cost of obtaining a circulaton  $O(L(N + M)) = O(n^2m^2)$ .
- EK analysis, the cost of obtaining a circulaton  $O(NM(N + M)) = O((n + m)n^2m^2)$ .
- The algorithm has cost  $O(n^2m^2)$ .

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# Arrodoniment amb restriccions

Considerem una matriu  $A = (a_{ij})$  amb dimensions  $n \times n$ , on cada  $a_{ij} \in \mathbb{R}^+ \cup \{0\}$  i on la suma de cada fila i columna de  $A$  és un enter. Volem arrodonir cada valor  $a_{ij}$  per  $\lfloor a_{ij} \rfloor$  o  $\lceil a_{ij} \rceil$  sense modificar la suma de les files/columnes.

$$\begin{pmatrix} 10.9 & 2.5 & 1.3 & 9.3 \\ 3.8 & 9.2 & 2.2 & 11.8 \\ 7.9 & 5.2 & 7.3 & 0.6 \\ 3.4 & 13.1 & 1.2 & 6.3 \end{pmatrix} F \rightarrow \begin{pmatrix} 11 & 3 & 1 & 9 \\ 4 & 9 & 2 & 12 \\ 7 & 5 & 8 & 1 \\ 4 & 13 & 2 & 6 \end{pmatrix}$$

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# Arrodoniment amb restriccions

El problema consisteix en donada una matriu  $A$ , produir un algorisme eficient per a determinar si es possible arrodonir  $A$  i, si es possible, produir la matriu arrodonida.

Demostreu la correctesa del vostre algorisme. Quina es la complexitat del vostre algorisme?.

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## Notem que

- Els elements d' $A$  que són enters no s'han de modificar.
- Sigui  $r_i = \sum_{j=1}^n (a_{ij} - \lfloor a_{ij} \rfloor)$  i  $c_j = \sum_{i=1}^n (a_{ij} - \lfloor a_{ij} \rfloor)$
- si les files i columnes d' $A$  sumen un enter, aleshores  $r_i$  i  $c_j$  son enters.
- A més  $\sum_i r_i = \sum_j c_j$ .

Per a resoldre el problema farem una reducció d'aquest problema a un problema de circulació.

Una unitat de flux la podem interpretar com una part decimal que s'arrodoneix a 1.

Construir una xarxa amb demandes  $\mathcal{N} = (V, E, c, d)$  on:

- **Vertexs:**  $V = \{x_i, y_j | 1 \leq i \leq n\}$ . Els vèrtexs  $x$  representen les files i els  $y$  les columnes.
- **Arestes:**  $E = \{(x_i, y_j) | 1 \leq i, j \leq n \text{ i } a_{i,j} \notin \mathbb{Z}\}$
- **Capacitats:**  $c(x_i, y_j) = 1$ .
- **Demandes:**  $d(x_i) = -r_i$ ,  $1 \leq i \leq n$ , i  $d(y_j) = c_j$ ,  $1 \leq j \leq n$ .

$\mathcal{N}$  té  $O(n)$  vèrtexs i  $O(n^2)$  arestes.

Si existeix un arrodoniment d' $A$ ,  $\mathcal{N}$  té una circulació amb valors enters (0,1).

- Sigi  $B$  un arrodoniment d' $A$ , definim una nova matriu  $D$  on

$$d_{ij} = \begin{cases} 1 & \text{si } b_{ij} > a_{ij} \\ 0 & \text{altrament} \end{cases}$$

- Com  $B$  es un arrodoniment,  $\sum_j d_{ij} = r_i$  i  $\sum_i d_{ij} = c_i$ .
- Llavors, el flux  $f(i,j) = d_{ij}$  és una circulació a  $\mathcal{N}$ .

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Si  $\mathcal{N}$  té una circulació amb valors enters  $(0,1)$ , existeix un arrodoniment d' $A$ .

- Sigi  $f$  una circulació a  $\mathcal{N}$ ,
- definim la matriu  $B$  com

$$b_{i,j} = \begin{cases} a_{ij} & \text{si } a_{ij} \in \mathbb{Z} \\ \lceil a_{i,j} \rceil & \text{si } a_{ij} \notin \mathbb{Z} \text{ i } f(i,j) = 1 \\ \lfloor a_{i,j} \rfloor & \text{altrament} \end{cases}$$

- Com  $f$  es una circulació,  $\sum_j b_{ij} = \sum_j a_{ij}$  i  $\sum_i b_{ij} = \sum_i a_{ij}$ .
- Llavors,  $B$  és un arrodoniment vàlid d' $A$ .

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La construcció de  $\mathcal{N}$  té una complexitat de  $O(n^2)$ .

Ford-Fulkerson funciona en  $O(D|E|)$ , on  $D$  és la suma de les demandes positives, i.e.  $D = \sum r_i = O(n^2)$  i com que  $|E| = O(n^2)$ , el nombre total de passos és  $O(n^4)$ .

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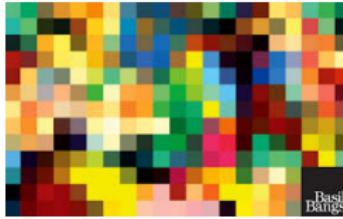
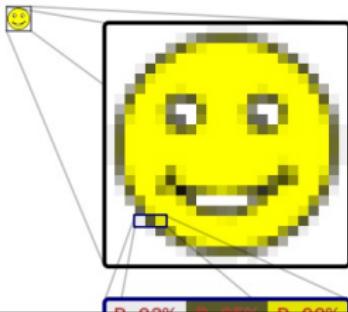
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# Pixels and digital image

- In digital imaging, a **pixel** is the smallest controllable element of a picture represented on the screen.
- Digital images are represented by a **raster graphics image**, a dot matrix data structure representing rectangular grid of pixels, or points of color
- The address of a pixel corresponds to its physical coordinates.



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# Image segmentation

Given a set of pixels classify each pixel as part of the main object or as part of the background.

Important problem in different techniques for image processing.

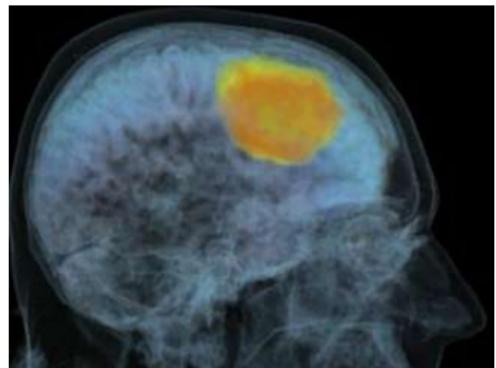
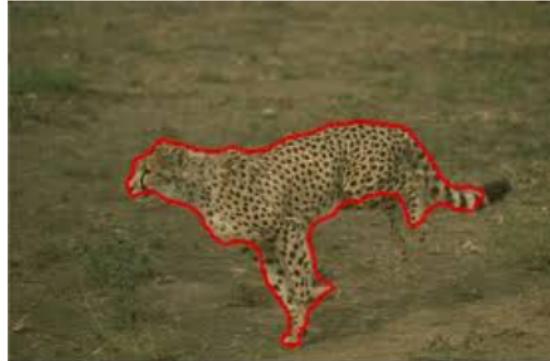
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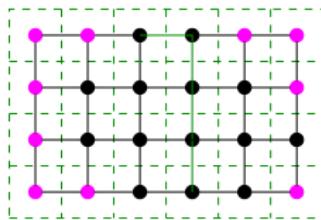
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# Foreground/background segmentation

- We aim to **label each pixel as belonging to the foreground or the background**
- Picture pixels as a grid of dots.
- Define the undirected graph  $G = (V, E)$ , where,  $V$  = set pixels in image,  $E$  = pairs of neighbors of pixels (in the grid)



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# Foreground/background segmentation

Given information:

- For each pixel  $i$ ,  $a_i \geq 0$  is likelihood that  $i$  is in the foreground and  $b_i \geq 0$  is likelihood that  $i$  is in the background.
- For each  $(i, j)$  of neighboring pixels, there is a separation penalty  $p_{ij} \geq 0$  for placing one in the foreground and the other in the background.

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# Foreground/background segmentation

## Goals:

- For  $i$  isolated, if  $a_i > b_i$  we prefer to label  $i$  as foreground (otherwise we label  $i$  as background)
- If many neighbors of  $i$  are labeled foreground we prefer to label  $i$  as foreground. This makes the labeling smoother by minimizing the amount of foreground/background

We want to partition  $V$  into  $A$  (set of foreground pixels) and  $B$  (set of background pixels), such that we maximize the objective function:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\{(i,j) \in E, i \in A, j \in B\}} p_{ij}$$

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# Formulate as a min-cut problem

Segmentation has the flavor of a cut problem, but

- it is a **maximization** different than the min-cut,
- $G$  is undirected,
- it does not have sink  $s$  and source  $t$  (but we can add them).

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# From maximization to minimization

Recall we want to maximize

$$\underbrace{\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\{(i,j) \in E, i \in A, j \in B\}} p_{ij}}_{(*)}$$

- Let  $Q = \sum_{i \in V} (a_i + b_i)$ , then

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - (\sum_{i \in A} b_i + \sum_{j \in B} a_j)$$

$$(*) = Q - (\sum_{i \in A} b_i - \sum_{j \in B} a_j) - \sum_{\{(i,j) \in E, i \in A, j \in B\}} p_{ij}$$

- Therefore, maximizing  $(*)$  is equivalent to minimize

$$\sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\{(i,j) \in E, i \in A, j \in B\}} p_{ij}.$$

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# Transforming $G$ into a network $\mathcal{N}$

We transform  $G = (V, E)$  to  $\mathcal{N} = (V', E', c, s, t)$  by

- Add a node  $s$  representing the foreground
- Add a node  $t$  representing the background
- $V' = V \cup \{s, t\}$
- For each  $(v, u) \in E$  create antiparallel directed edges  $(u, v)$  and  $(v, u)$  in  $E'$
- For each pixel  $i$  create directed edges  $(s, i)$  and  $(i, t)$
- $E' = \{(s, v) \cup (v, t)\}_{v \in E} \cup \{(u, v) \cup (v, u)\}_{(u, v) \in E}$

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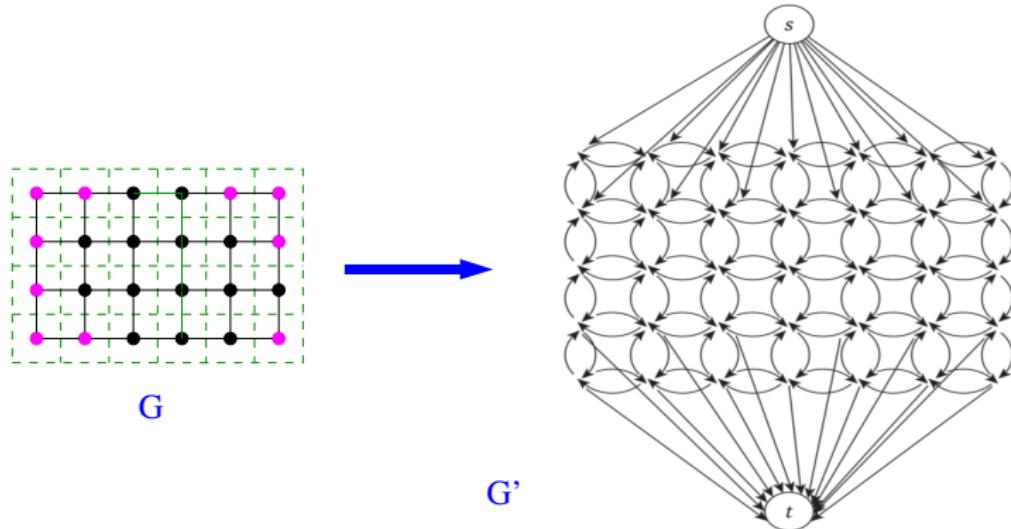
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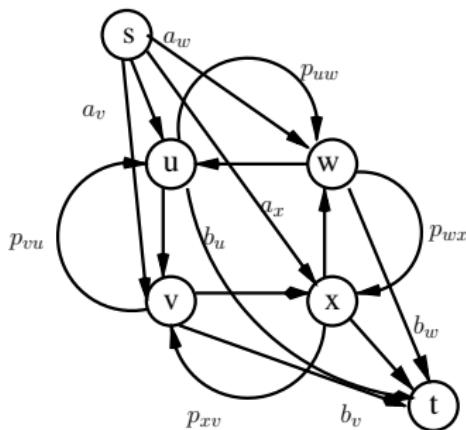
# The pixel graph $G$ and the graf $G' = (V', E')$

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# Adding capacities to the edges of $\mathcal{N}$

- For each  $i \in V$ ,  $c(s, i) = a_i$ ,  $c(i, t) = b_i$
- For each  $(i, j) \in E$ ,  $c(i, j) = c(j, i) = p_{ij}$



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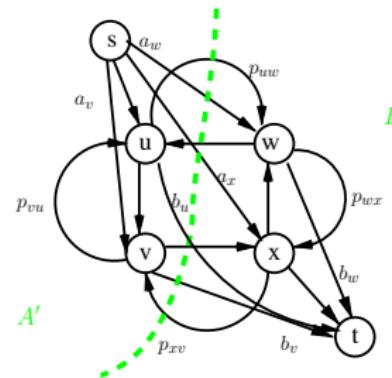
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# $(s, t)$ -cuts in $\mathcal{N}$

A  $(s, t)$ -cut  $(A', B')$  corresponds to a partition of the pixels into  $(A, B)$ , for  $A = A' - \{s\}$ , and  $B = B' - \{t\}\right).$

- An edge  $(s, j)$  with  $j \in B$  contributes with  $a_j$  to  $c(A', B')$ ,
- An edge  $(i, t)$  where  $i \in A$  contributes with  $b_i$  to  $c(A', B')$ ,
- An edge  $(i, j)$  where  $i \in A$  and  $j \in B$  contributes with  $p_{ij}$  to  $c(A', B')$

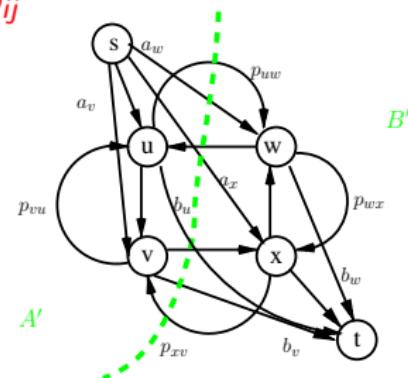


# $(s, t)$ -cuts in $\mathcal{N}$

Therefore,

$$c(A', B') = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\{(i, j) \in E, i \in A, j \in B\}} p_{ij}$$

We want to find a cut with the min value of the above quantity, which is equivalent to solve the min-cut problem in  $\mathcal{N}$



# Cost

The cost of the algorithm is determined by the cost of finding a min cut in the associated network, i.e., as both vertices and edges are  $O(nm)$ , the cost of the algorithm is  $O((nm)^3)$ .

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