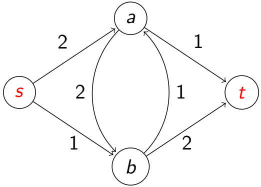
Max-flow and min-cut problems

Flow Network

A network $\mathcal{N} = (V, E, c, s, t)$ is formed by

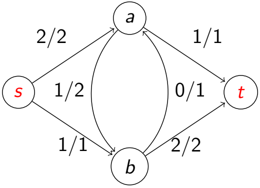
- a digraph $G = (V, E)$,
- a source vertex $s \in V$
- a sink vertex $t \in V$,
- and edge capacities $c : E \to \mathbb{R}^+$



A flow in a network

Given a network $\mathcal{N} = (V, E, c, s, t)$
A Flow is an assignment $f : E \to \mathbb{R}^+ \cup \{0\}$ that follows the Kirchoff's laws:

- $\forall (u, v) \in E$, $0 \leq f(u, v) \leq c(u, v)$,
- (Flow conservation) $\forall v \in V - \{s, t\}$, $\sum_{u \in V} f(u, v) = \sum_{z \in V} f(v, z)$



$f(e)/c(e)$

The value of a flow f is

$$|f| = \sum_{v \in V} f(s, v) = f(s, V) = f(V, t).$$

with value 3.

A flow in a network

Given a network $\mathcal{N} = (V, E, c, s, t)$
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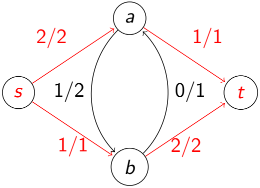


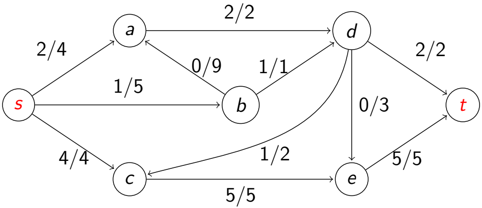
The value of a flow f is

$$|f| = \sum_{v \in V} f(s, v) = f(s, V) = f(V, t).$$

saturated

The Maximum flow problem

INPUT: A network $\mathcal{N} = (V, E, c, s, t,)$
QUESTION: Find a flow of maximum value on $\mathcal{N}$.



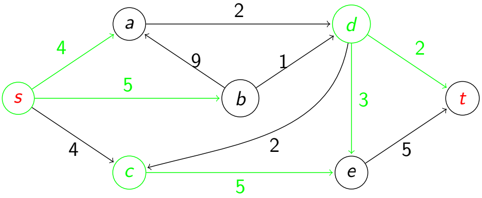
The value of the flow is 7 $= 4 + 1 + 2 = 5 + 2$.

As t cannot receive more flow, this flow is a maximum flow.

The (s, t)-cuts

Given $\mathcal{N} = (V, E, c, s, t)$ a (s, t)-cut is a partition of $V = S \cup T$ $(S \cap T = \emptyset)$, with $s \in S$ and $t \in T$.

The capacity of a cut (S, T) is the sum of weights leaving S, i.e.,
$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$

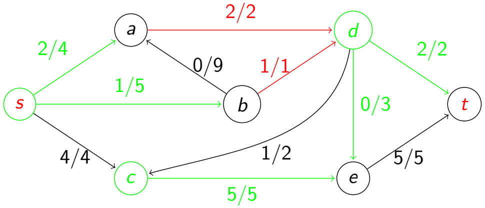


S={s,c,d}
$T = \{a, b, e, t\}$
$c(S, T) = 19$
$(4 + 5) + 5 + (3 + 2)$

The (s, t)-cuts

Given $\mathcal{N} = (V, E, c, s, t)$ a (s, t)-cut is a partition of $V = S \cup T$ $(S \cap T = \emptyset)$, with $s \in S$ and $t \in T$.

The flow across the cut:
$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{v \in T} \sum_{u \in S} f(v, u)$.



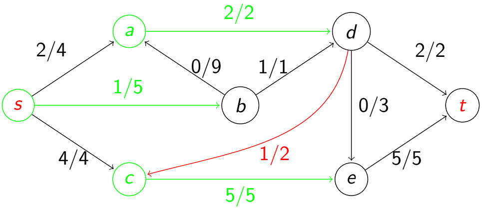
S={s,c,d}
$T = \{a, b, e, t\}$
$c(S, T) = 19$
$f(S, T) = 10 - 3 = 7$

Due to the capacity constrain: $f(S, T) \leq c(S, T)$

Another (s,t)-cut

Given $\mathcal{N} = (V, E, c, s, t)$ a (s,t)-cut is a partition of $V = S \cup T$ $(S \cap T = \emptyset)$, with $s \in S$ and $t \in T$.

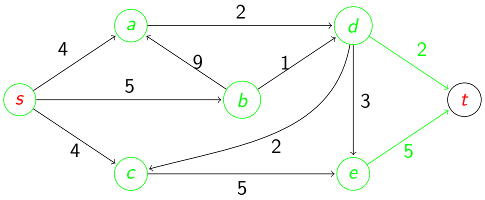
The flow across the cut:
$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u)$.



S={s,a,c}
$T = \{b, d, e, t\}$
$c(S,T) = 12$
$f(S,T) = 8 - 1 = 7$

The Minimum Cut problem

INPUT: A network $\mathcal{N} = (V, E, c, s, t,)$
QUESTION: Find a (s, t)-cut of minimum capacity in $\mathcal{N}$.

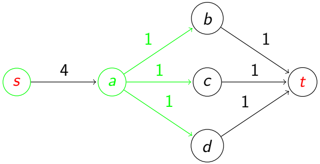


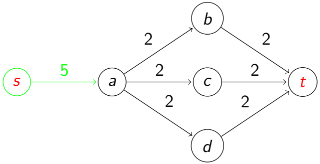
MinCut
S={s,a,b,c,d,e}
$T = \{t\}$
$c(S, T) = 7$

Changing weights effect on min cuts

Given a network $\mathcal{N} = (V, E, s, t, c)$ assume that (S, T) is a min (s, t)-cut.

If we change the input by adding $c > 0$ to the capacity of every edge, then it may happen that (S, T) is not longer a min (s, t)-cut.

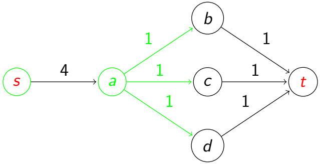


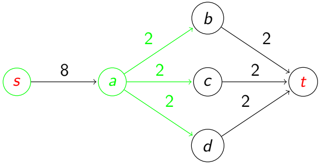


Changing weights effect on Min-Cut and Max-Flow

Given a network $\mathcal{N} = (V, E, s, t, c)$.

If we change the network by multiplying by $c >$ the capacity of every edge, the capacity of any (s, t)-cut in the new network is c times its capacity in the original network.





Notation

Let $\mathcal{N} = (V, E, s, t, c)$ and f a flow in $\mathcal{N}$

For $v \in V$, $U \subseteq V$ and $v \notin U$.

- $f(v, U)$ flow $v \to U$ i.e. $f(v, U) = \sum_{u \in U} f(v, u)$,
- $f(U, v)$ flow $U \to v$ i.e. $f(U, v) = \sum_{u \in U} f(u, v)$,

For a (s, t)-cut (S, T) and $v \in S$

- $S' = S \setminus \{v\}$ and $T' = T \cup \{v\}$
- $f_{-v}(S, T) = \sum_{u \in S'} \sum_{w \in T} f(u, w) - \sum_{w \in T} \sum_{u \in S'} f(w, u)$ i.e, the contribution to $f(S, T)$ from edges not incident with v.

Flow conservation on (s, t)-cuts

Theorem

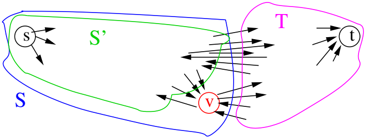
Let $\mathcal{N} = (V, E, s, t, c)$ and f a flow in $\mathcal{N}$. For any (s, t)-cut (S, T), $f(S, T) = |f|$.

Proof (Induction on $|S|$)

- If $S = \{s\}$ then, by definition, $f(S, T) = |f|$.
- Assume it is true for $S' = S - \{v\}$ and $T' = T \cup \{v\}$, i.e. $f(S', T') = |f|$.

Flow conservation on (s,t)-cuts

Proof (cont.) (Induction on $|S|$)

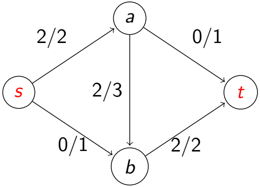


- IH: $f(S', T') = |f|$.
- Then, $f(S,T) = f_{-v}(S,T) + f(v,T) - f(T,v)$.
- But, $f(S',T') = f_{-v}(S,T) + f(S',v) - f(v,S')$ as $v \in T'$
- By flow conservation, $f(S',v) + f(T,v) = f(v,S') + f(v,T)$
- So, $f(S',v) - f(v,S') = f(v,T) - f(T,v)$
- Therefore, $f(S',T') = f(S,T) = |f|$ $\square$

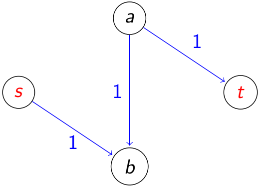
Residual graph

Given a network $\mathcal{N} = (V, E, s, t, c)$ together with a flow f.
The residual graph, $(G_f = (V, E_f, c_f)$ is a weighted digraph on the same vertex set and with edge set:

- if $c(u, v) - f(u, v) > 0$, then $(u, v) \in E_f$ and $c_f(u, v) = c(u, v) - f(u, v) > 0$ (forward edges)



$\mathcal{N}, f$

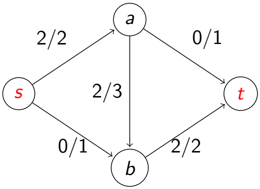


G_f forward edges

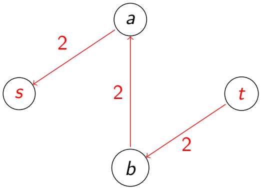
Residual graph

Given a network $\mathcal{N} = (V, E, s, t, c)$ together with a flow f on it, the residual graph, $(G_f = (V, E_f, c_f)$ is a weighted digraph on the same vertex set and with edge set:

- if $f(u, v) > 0$, then $(v, u) \in E_f$ and $c_f(v, u) = f(u, v)$ (backward edges).



$\mathcal{N}, f$

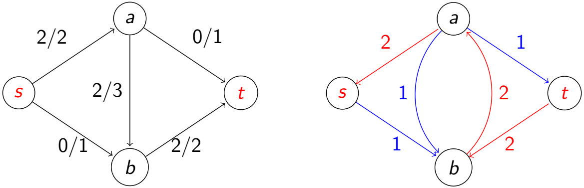


G_f backward edges

Residual graph

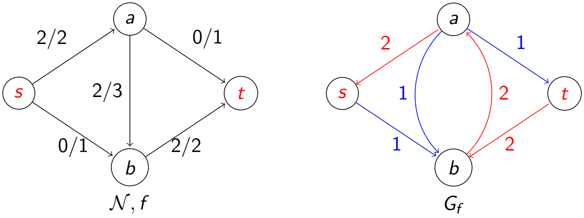
Given a network $\mathcal{N} = (V, E, s, t, c)$ together with a flow f on it, the residual graph, $(G_f = (V, E_f, c_f)$ is a weighted digraph on the same vertex set and with edge set:

- if $c(u,v) - f(u,v) > 0$, then $(u,v) \in E_f$ and $c_f(u,v) = c(u,v) - f(u,v) > 0$ (forward edges)
- if $f(u,v) > 0$, then $(v,u) \in E_f$ and $c_f(v,u) = f(u,v)$ (backward edges).



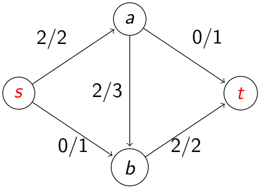
$\mathcal{N}, f$ $\qquad$ G_f

Residual graph

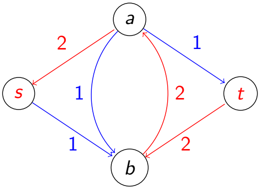


- Notice that, if $c(u, v) = f(u, v)$, then there is only a backward edge.
- c_f are called the residual capacity.

Residual graph



$\mathcal{N}, f$

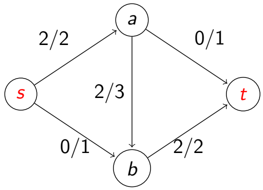


G_f

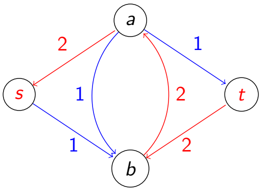
- forward edges: There remains capacity to push more flow through this edge.
- backward edges: there are units of flow that can be redirected through other links.

Augmenting paths

Let $\mathcal{N} = (V, E, c, s, t)$ and let f be a flow in $\mathcal{N}$,



$\mathcal{N}, f$

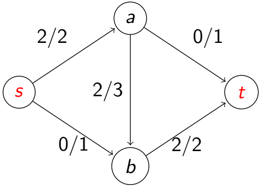


G_f

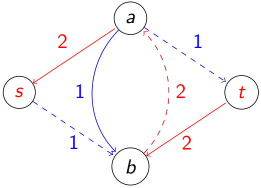
- An augmenting path P is any simple path P in G_f from s to t P might have forward and backward edges.
- For an augmenting path P in G_f, the bottleneck, $b(P)$, is the minimum (residual) capacity of the edges in P.
  In the example, for $P = (s, b, a, t)$, $b(P) = 1$.

Augmenting paths: increasing the flow

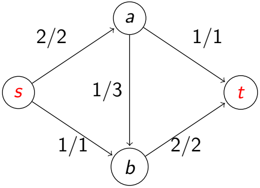
Augment(P, f)
b=bottleneck (P)
for each $(u,v) \in P$ **do**
  if (u,v) is a forward edge **then**
    Increase $f(u,v)$ by b
  else
    Decrease $f(v,u)$ by b
return f



$\mathcal{N}, f$



$G_f,\ P = (s,a,t),\ b(P) = 1$



$\mathcal{N}, f'$

Augmenting paths: increasing the flow

Lemma

Let f' =Augment(P, f), then f' is a flow in G.

Proof: We have to prove the two flow properties.

- Capacity law
  - Forward edges $(u, v) \in P$, we increase $f(u, v)$ by b, as $b \leq c(u, v) - f(u, v)$ then $f'(u, v) = f(u, v) + b \leq c(u, v)$.
  - Backward edges $(u, v) \in P$ we decrease $f(v, u)$ by b, as $b \leq f(v, u), f'(v, u) = f(u, v) - b \geq 0$.

Augmenting paths: increasing the flow

Lemma

Let $f' =$Augment(P, f), then f' is a flow in G.

Proof: We have to prove the two flow properties.

- Conservation law, $\forall v \in P \setminus \{s, t\}$ let u be the predecessor of v in P and let w be its successor.
- As the path is simple only the alterations due to (u, v) and (v, w) can change the flow that goes trough v. We have four cases:
  - (u, v) and (v, w) are backward edges, the flow in (v, u) and (w, v) is decremented by b. As one is incoming and the other outgoing the total balance is 0.
  - (u, v) and (v, w) are forward edges, the flow in (u, v) and (v, w) is incremented by b. As one is incoming and the other outgoing the total balance is 0.

Augmenting paths: increasing the flow

Lemma

Let f' =Augment(P, f), then f' is a flow in G.

Proof: We have to prove the two flow properties.

- Conservation law, $\forall v \in P \setminus \{s, t\}$ let u be the predecessor of v in P and let w be its successor.
- As the path is simple only the alterations due to (u, v) and (v, w) can change the flow that goes trough v. We have three cases:
  - (u, v) is forward and (v, w) is backward, the flow in (u, v) is incremented by b and the flow in (w, v) is decremented by b. As both are incoming, the total balance is 0.
  - (u, v) is backward and (v, w) is forward, the flow in (v, w) is incremented by b and the flow in (v, u) is decremented by b. As both are outgoing, the total balance is 0.

Augmenting paths: incrementing the flow

Lemma

Consider f' =Augment(P, f), then $|f'| > |f|$.

Proof: Let P be the augmenting path in G_f. The first edge $e \in P$ leaves s, and as G has no incoming edges to s, e is a forward edge. Moreover P is simple $\Rightarrow$ never returns to s. Therefore, the value of the flow increases in edge e by b units. $\square$

Max-Flow Min-Cut theorem

Ford and Fulkerson (1954); Peter Elias, Amiel Feinstein and Claude Shannon (1956) (in framework of information-theory).

Theorem

For any $\mathcal{N}(G, s, t, c)$, the maximum of the flow value is equal to the minimum of the (S, T)-cut capacities.

$$\max_f \{|f|\} = \min_{(S,T)} \{c(S, T)\}.$$

Max-Flow Min-Cut theorem:Proof

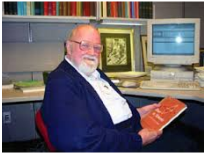
Proof:

- Let f^* be a flow with maximum value, $|f^*| = \max_f\{|f|\}$
- For any (s,t)-cut (S,T), $f^*(S,T) \leq c(S,T)$.
- G_{f^*} has no augmenting path. So, if $S_s = \{v \in V | \exists s \rightsquigarrow v \text{ in } G_{f^*}\}$, then $(S_s, V - \{S_s\})$ is a (s,t)-cut.
- For $e = (u,v) \in E$ with $u \in S_s$ and $v \notin S_s$, $(u,v) \notin E(G_{f^*}$, therefore $f^*(u,v) = c(u,v)$,
- Then, $c(S_s, V - \{S_s\}) = f^*(S_s, V - \{S_s\}) = |f^*|$
- $(S_s, V - \{S_s\})$ is a minimum capacity (s,t)-cut in G.

$\square$

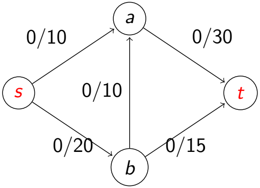
Ford-Fulkerson algorithm

L.R. Ford, D.R. Fulkerson: *Maximal flow through a network*. Canadian J. of Math. 1956.

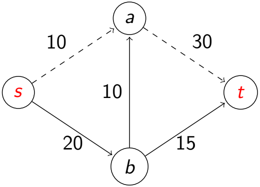


Ford-Fulkerson(G, s, t, c)
for all $(u, v) \in E$ set $f(u, v) = 0$
$G_f = G$
while there is an (s, t) path P in G_f **do**
  $f = \text{Augment}(P, G_f)$
  Compute G_f
return f

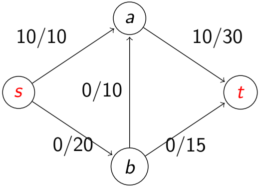
FF algorithm example



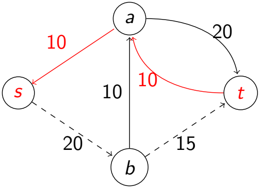
Initial f_0



G_{f_0}

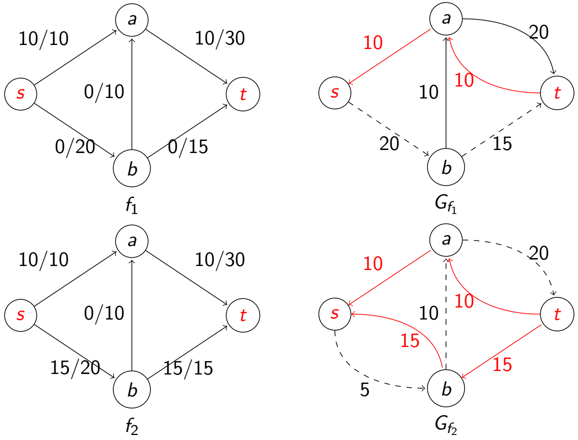


f_1



G_{f_1}

FF example



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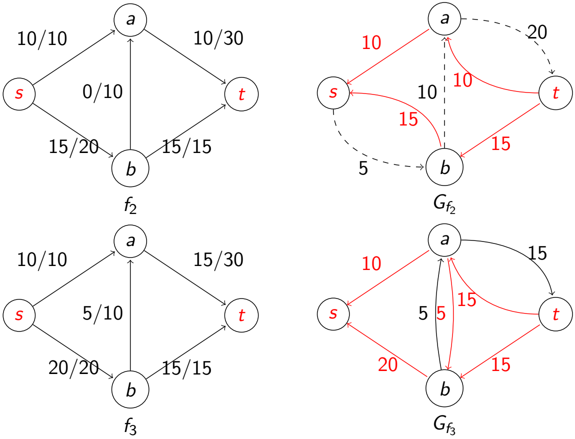
FF example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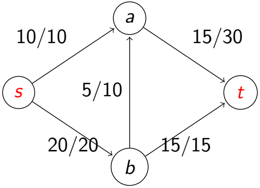
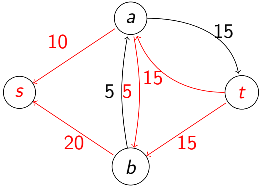
FF example



f_3



G_{f_3}

Flow with max value

$\{s\}, \{a, b, t\}$ is a min (s, t)-cut

Correctness of Ford-Fulkerson

Consequence of the Max-flow min-cut theorem.

Theorem

The flow returned by Ford-Fulkerson is the max-flow.

Networks with integer capacities

Lemma (Integrality invariant)

Let $\mathcal{N} = (V, E, c, s, t)$ where $c : E \to \mathbb{Z}^+$. At every iteration of the Ford-Fulkerson algorithm, the flow values $f(e)$ are integers.

Proof: (induction)

- The statement is true for the initial flow (all zeroes).
- Inductive Hypothesis: The statement is true after j iterations.
- At iteration $j + 1$: As all residual capacities in G_f are integers, then bottleneck $(P, f) \in \mathbb{Z}$, for the augmenting path found in iteration $j + 1$.
- Thus the augmented flow values are integers. □

Theorem (Integrality theorem)

Let $\mathcal{N} = (V, E, c, s, t)$ where $c : E \to \mathbb{Z}^+$. There exists a max-flow f^* such that $f^*(e)$ is an integer, for any $e \in E$.

Proof:
Since the algorithm terminates, the theorem follows from the integrality invariant lemma. $\square$

Networks with integer capacities: FF running time

Lemma

Let C be the min cut capacity (=max. flow value), Ford-Fulkerson terminates after finding at most C augmenting paths.

Proof: The value of the flow increases by ≥ 1 after each augmentation. $\square$

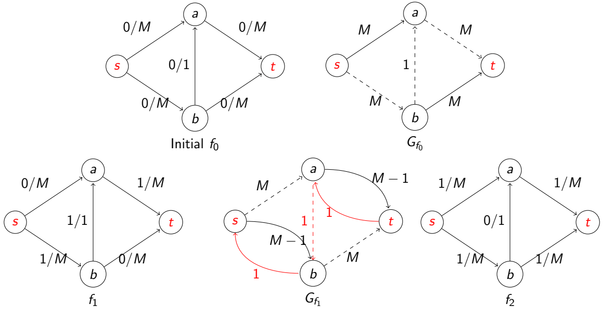
Networks with integer capacities: FF running time

- The number of iterations is $\leq C$. At each iteration:
- Constructing G_f, with $E(G_f) \leq 2m$, takes $O(m)$ time.
- $O(n+m)$ time to find an augmenting path, or deciding that it does not exist.
- Total running time is $O(C(n+m)) = O(Cm)$
- Is that polynomic? No, only pseudopolynomic

Max Flow and Min Cut
Properties of flows and cuts
Residual graph
Augmenting path
MaxFlow MinCut Thm
Ford Fulkerson alg
Maximum matching in Bip graphs
Disjoint paths problem

Networks with integer capacities: FF running time

The number of iterations of Ford-Fulkerson could be $\Theta(C)$

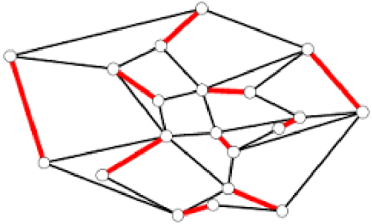


Ford-Fulkerson can alternate between the two long paths, and require $2M$ iterations. Taking $M = 10^{10}$, FF on a graph with 4 vertices can take time 210^{10}.

Maximum Matching problem

Given an undirected graph $G = (V, E)$ a subset of edges $M \subseteq E$ is a matching if each node appears at most in one edge in M (a node may not appear at all).

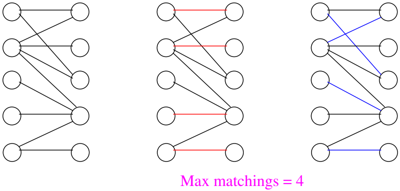
The Maximum Matching problem: Given a graph G, find a matching with maximum cardinality.



Maximum matching in bipartite graphs

A graph $G = (V, E)$ is said to be bipartite if there is a partition of V in L and R, ($L \cup R = V$ and $L \cap R = \emptyset$), such that every $e \in E$ connects a vertex in L with a vertex in R.

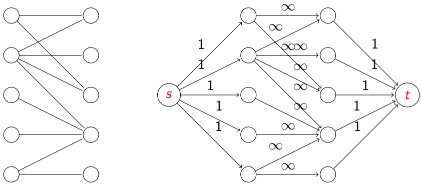
The MAXIMUM MATCHING BIPARTITE GRAPH problem: Given as input a bipartite graph $G = (L \cup R, E)$, find a maximum matching.



Maximum Matching: Network formulation

Given a bipartite graph $G = (L \cup R, E)$ construct $\mathcal{N} = (\hat{V}, \hat{E}, c, s, t)$:

- Add vertices s and t: $\hat{V} = L \cup R \cup \{s, t\}$.
- Add directed edges $s \to L$ with capacity 1. Add directed edges $R \to t$ with capacity 1.
- Direct the edges E from L to R, and give them capacity ∞.
- $\hat{E} = \{s \to L\} \cup E \cup \{R \to t\}$.



Maximum matching algorithm: Analysis

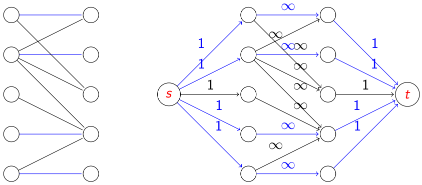
Theorem

Max flow in $\mathcal{N}$=Max bipartite matching in G.

Proof Matching as flows

Let M be a matching in G with k-edges, consider the flow f that sends 1 unit along each one of the k paths, $s \to u \to v \to t$, for $(u, v) \in M$.

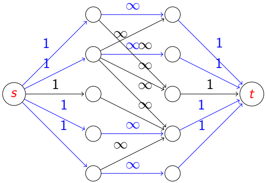
As M is a matching all these paths are disjoint, so f is a flow and has value k.

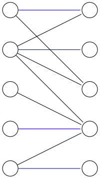


Maximum matching algorithm: Analysis

Flows as matchings

- Consider an integral flow f in $\hat{G}$. Therefore, for any edge e, the flow is either 0 or 1.
- Consider the cut $C = (\{s\} \cup L, R \cup \{t\})$ in $\hat{G}$.
- Let M be the set of edges in the cut C with flow=1, then $|M| = |f|$.
- Each node in L is in at most one $e \in M$ and every node in R is in at most one head of an $e \in F$
- Therefore, M is a matching in G with $|M| \leq |f|$





Maximum matching algorithm: Analysis

As $\mathcal{N}$ has integer capacities there is an integral maximum flow f^*, the associated matching is a maximum matching. $\square$

Maximum matching algorithm: Analysis

What is the cost of the algorithm?

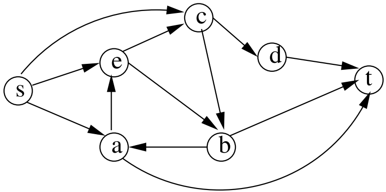
- The bipartite graph, has n vertices and m edges. The capacities are integers. We need an integral solution.
- The algorithm: (1) constructs $\mathcal{N}$, (2) runs FF on $\mathcal{N}$ to obtain a maxflow f, (3) from f obtains a maximum matching M.
- $\mathcal{N}$ has $n+2$ vertices and $m+2n$ edge, (1) takes $O(n+m)$
- The maximum value of a flow in $\mathcal{N}$ is at most n, (2) takes time $O(|f|(n+m)) = O(n(n+m))$
- (3) can be done in time $O(n+m)$.

We get a cost of $O(n(n+m))$.

Disjoint Path problem

Given a digraph $G = (V, E)$ and two vertices $s, t \in V$, a set of paths is edge-disjoint if their edges are disjoint (although they might share some vertex)

Disjoint Path problem: Given a digraph $G = (V, E)$ and two vertices $s, t \in V$, find a set of $s \rightsquigarrow t$ edge-disjoint paths of maximum cardinality



Disjoint Path problem

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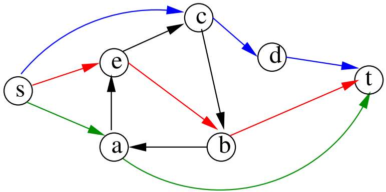
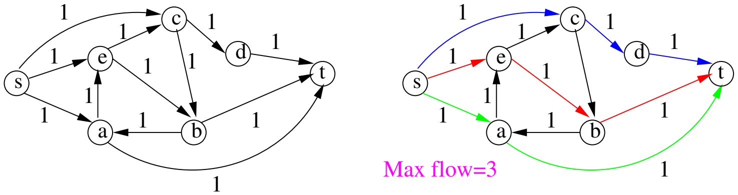


DISJOINT PATH: Max flow formulation

Thinking in terms of flow a path from s to t can be seen as a way of transporting a unit of flow.
We construct a network $\mathcal{N}$ assigning unit capacity to every edge



Theorem

The max number of edge disjoint paths $s \rightsquigarrow t$ is equal to the max flow value

DISJOINT PATH: Proof of the Theorem

Number of disjoints paths $\leq$ max flow

If we have k edge-disjoints paths $s \rightsquigarrow t$ in G then making $f(e) = 1$ for each e in a path, we get a flow with $|f| = k$

DISJOINT PATH: Proof of the Theorem

Number of disjoints paths $\geq$ max flow

- If the max flow value is k, there exists a 0-1 flow f^* with value k.
- Consider the graph $G^* = (V, E')$ where E' is formed by all edges e with $f(e) = 1$.
- We repeatedly compute a $s \rightsquigarrow t$ simple path in G^*, and remove its edges from G^*.
- Each time that we remove a path, the value of the flow in the network is reduced by one, so we can apply the process k times.
- None of the paths share an edge, so we get k disjoint paths. $\square$

Disjoint paths algorithm: Analysis

What is the cost of the algorithm?

- The graph, has n vertices and m edges. The capacities are integers. We need an integral solution.
- The algorithm: (1) constructs $\mathcal{N}$, (2) runs FF on $\mathcal{N}$ to obtain a max flow f, (3) from f obtains $|f|$ edge disjoint paths.
- $\mathcal{N}$ has n vertices and m edges, (1) takes $O(n+m)$
- The maximum value of a flow in $\mathcal{N}$ is at most n, (2) takes time $O(|f|(n+m)) = O(n(n+m))$
- (3) can be done in time $O(n+m)$ per path, i.e., $O(|f|(n+m))$.

We get a cost of $O(n(n+m))$.

Vertex disjoint paths

Can we do something similar to get the maximum number of vertex disjoint paths?

The case of undirected graphs

If we have an undirected graph, with two distinguised nodes u, v, how would you apply the max flow formulation to solve the problem of finding the max number of disjoint paths between u and v?

The case of undirected graphs