



## Exercise 2: Math Background

### Exercise 1.1

**Notation.** We use the following notations in this exercise:

- Scalars are denoted with lowercase letters. E.g.  $x, \phi$
  - Vectors are denoted with bold lowercase letters. E.g.  $\mathbf{x}, \boldsymbol{\phi}$
  - Matrices are denoted with bold uppercase letters. E.g.  $\mathbf{X}, \boldsymbol{\Sigma}$
- a) Let  $\mathbf{x} \in \mathbb{R}^M, \mathbf{y} \in \mathbb{R}^N$ , function  $f : \mathbb{R}^M \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{A} \mathbf{y} + \mathbf{x}^\top \mathbf{B} \mathbf{x} - \mathbf{C} \mathbf{y} + \mathbf{D}$ . Compute the dimensions of the matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  for the function so that the mathematical expression is valid.
- b) Let  $\mathbf{x} \in \mathbb{R}^N, \mathbf{M} \in \mathbb{R}^{N \times N}$ . Express the function  $f(\mathbf{x}) = \sum_{i=1}^N \sum_{j=1}^N x_i x_j M_{ij}$  using only matrix-vector multiplications.
- c) Suppose  $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ , where  $\mathbf{V}$  is a vector space.  $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$  and  $\langle \mathbf{u}, \mathbf{v} \rangle = 1$ . Prove that  $\mathbf{u} = \mathbf{v}$ .

### Exercise 1.2

In this exercise we want to determine the gradients for a few simple functions, which will be helpful for the upcoming lectures.

- a) For  $x \in \mathbb{R}^n$ , let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f(x) = b^\top x$  for some known vector  $b \in \mathbb{R}^n$ . Determine the gradient of the function  $f$ .  
*Hint:* Use that  $f(x) = b^\top x = \sum_{i=1}^n b_i x_i$ .
- b) Now consider the quadratic function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f(x) = x^\top A x$  for a symmetric matrix  $A \in \mathbb{S}_n$ . Determine the gradient of the function  $f$ .  
*Hint:* A symmetric matrix  $A \in \mathbb{S}_n$  satisfies that  $A_{ij} = A_{ji}$  for all  $1 \leq i, j \leq n$ .
- c) Now let us go a step further and let us determine the derivative of the following function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with

$$f(x) = \|Ax - b\|_2^2 = (Ax - b)^\top (Ax - b)$$

where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ .

### Exercise 1.3

- a) Compute the derivatives for the following functions:  $f_i : \mathbb{R} \rightarrow \mathbb{R}, i \in \{1, 2, 3\}$
- $f_1 : f_1(x) = (x^3 + x + 1)^2$
  - $f_2 : f_2(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$
  - $f_3 : f_3(x) = (1 - x) \log(1 - x)$

b) For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the *gradient* is defined as  $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ . Calculate the gradients of the following functions:  $f_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $i \in \{4, 5\}$

- $f_4 : f_4(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|_2^2$
- $f_5 : f_5(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|_2$

c) For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the *Jacobian* is defined as

$$\mathbb{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Calculate the Jacobian matrix of the following functions:  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $i \in \{6, 7\}$

- $f_6 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $f_6(r, \varphi) = (r \cos \varphi, r \sin \varphi)^\top$
- $f_7 : \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $f_7(t) = (r \cos t, r \sin t)^\top$

d) For a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  the divergence is defined as  $\text{div} f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$ . Calculate the divergence for the following functions:  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i \in \{8, 9\}$

- $f_8 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $f_8(x, y) = (-y, x)^\top$
- $f_9 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $f_9(x, y) = (x, y)^\top$

### Exercise 1.4

In this exercise, we want to take a look at the softmax function which is a common activation function in neural networks in order to normalize the output of a network to a probability distribution over predicted output classes. We will discuss the softmax function later in this lecture in more detail.

The softmax function  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by

$$\sigma(z)_i = \frac{e^{z_i}}{\sum_{j=1}^n e^{z_j}}$$

for  $1 \leq i \leq n$  and  $z = (z_1 \ z_2 \ \dots \ z_n)^\top$ . In the expanded form, we write:

$$\hat{y} = \sigma(z_1, z_2, \dots, z_n) = \left[ \frac{e^{z_1}}{\sum_{k=1}^n e^{z_k}}, \frac{e^{z_2}}{\sum_{k=1}^n e^{z_k}}, \dots, \frac{e^{z_n}}{\sum_{k=1}^n e^{z_k}} \right].$$

Determine the derivative of the softmax function.

*Hint:* Deriving  $\sigma(z)$  with respect to  $z$  will lead to  $n \times n$  partial derivatives, i.e.  $\frac{\partial \sigma(z)_i}{\partial z_j}$  for  $1 \leq i, j \leq n$ . It is important to consider the two cases (1)  $i = j$  and (2)  $i \neq j$ .

### Exercise 1.5

#### a) Variance

We say that two random variables  $X, Y$  are independent if and only if the joint cumulative distribution function  $F_{X,Y}(x, y)$  satisfies  $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ . In the case of independence, the following property holds for these variables: Let  $f, g$  be two real-valued functions defined on the codomains of  $X, Y$ , respectively. Then  $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)]$ .

Assume that  $X, Y$  are two random variables that are independent and identical distributed (i.i.d.) with  $X, Y \sim \mathcal{N}(0, \sigma^2)$ . Prove that

$$\text{Var}(XY) = \text{Var}(X)\text{Var}(Y).$$

Remember this property as it will play an important role at a later point of the lecture, when we take a look at the initialization of the weights of a neural network (Xavier initialization).

#### b) Normal distribution

*Remark:* The family of random variables that are normally distributed is closed under linear transformation, that means if  $X$  is normally distributed, then for every  $a, b \in \mathbb{R}$  the random variable  $aX + b$  is normally distributed.

For this exercise, assume that the random variable  $X$  is normally distributed with mean  $\mu$  and variance  $\sigma^2$ , i.e.  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Let  $Z = \frac{X - \mu}{\sigma}$ . From the remark, we know that  $Z$  is again normally distributed. Determine the mean and the variance of the random variable  $Z$ .