Completions of Metric Spaces

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1 Complete Metric Spaces and Completions of Metric Spaces

1.1 Definition of Complete Metric Spaces

The incentive for defining complete metric spaces is to guarantee a Cauchy sequence actually converge to some point within the metric space. We have seen a basic example of this before, where we arrive at $\mathbb{R}$ when trying to patching all the "holes" in $\mathbb{Q}$. But it remains unclear how we expand the notion of convergence to a general metric space, since it might not be ordered. The lack of order suggests confining relative distances, which is something every metric space does have, would be the most generic approach. Therefore, we can define complete metric spaces using Cauchy sequences once we extend the definition of a Cauchy sequence to a general metric space (M, d):

Definition 1.1. Given metric space (M, d), a sequence $\{a_n\}$ in M is a ***Cauchy sequence*** if for every $\epsilon > 0$, there exists a positive integer N such that if $m, n \geq N$, we have $d(a_m, a_n) < \epsilon$.

Generally, if a sequence is Cauchy, it might not converge to a point within its metric space. To illustrate this, consider the following example:

Example 1.2. In metric space $((0,1), ||\cdot||)$, consider the sequence $\{a_n\}$ with $a_n = \frac{1}{n+1}$. We know $\{a_n\}$ is convergent in $\mathbb{R}$, so it is Cauchy in $\mathbb{R}$. Since we are still using the Euclidean metric, $\{a_n\}$ remains Cauchy in $(0,1)$. However, $\lim a_n = 0$ is not an element of $(0,1)$ so this is an example where a Cauchy sequence converges to some point outside of its metric space.

To address such "holes" in a metric space, we introduce the concept of complete metric spaces as defined by:

Definition 1.3. If every Cauchy sequence in a metric space (M, d) converges to some point in M, then (M, d) is a ***complete*** metric space.

It is possible to directly prove some metric space is complete by checking against the definition, and the followings provide a tangible example by examining the case of $\mathbb{R}^n$:

Example 1.4. The metric space $(\mathbb{R}^n, ||\cdot||)$ is complete. This mainly follows from the fact that $\mathbb{R}$ is complete, and that coordinate convergence in finite $\mathbb{R}^n$ leads to overall convergence, which can be achieved by manipulating triangle inequalities of the sum of all the coordinate sequences.

1.2 Definition of Completion of A Metric Space

However, not all subsets of $\mathbb{R}^n$ are complete. To generalize the transformation of an arbitrary (M, d) into its complete counterpart, we need to introduce the concept completion:

Definition 1.5. Given a metric space (M, d), we say $(\widehat{M}, \widehat{d})$ is a ***completion*** of (M, d) if:

1. $(\widehat{M}, \widehat{d})$ is a complete metric space;

2. there is an injective mapping $i : M \mapsto \widehat{M}$ such that for any point x, y in M, there is $d(x,y) = \widehat{d}(i(x), i(y))$ and that the image of this mapping satisfies $\overline{i(M)} = \widehat{M}$[1].

The injection $i : M \mapsto \widehat{M}$ is often called an ***isometric injection*** because it preserves the distance relationships between any pair of elements, and it is often denoted as $i : M \hookrightarrow \widehat{M}$. Moreover, since the closure of $i(M)$ is $\widehat{M}$, we say $i(M)$ is ***dense*** in $\widehat{M}$.

2 Properties of Completions

It would appear that by definition, a completion is linked to a closed image of the original set. Recall when introducing closed subsets in metric spaces, we often discuss the relationship between compactness and uniform continuity. Here, let us explore the continuity of functions on the original metric spaces given its image being complete:

Theorem 2.1. Given metric space (M_1, d_1) and complete metric space (M_2, d_2), let $f : X \mapsto M_2$ be a uniformly continuous function where $\overline{X} = M_1$. Then, there is a unique $g : M_1 \mapsto M_2$ that satisfies the following two property as the ***unique uniformly continuous extension*** of f:

1. g is uniformly continuous on M_1;
2. $g|_X = f$.

Proof: For any point x in M_1, since $M_1 = \overline{X}$, we can find some sequence $\{x_n\}$ in X that converges towards x. Therefore, the following g can be defined on the entire M_1:

$$\forall x \in M_1 : g(x) = \lim_{n\to\infty} f(x_n) \text{ with } \lim_{n\to\infty} x_n = x \tag{2.1}$$

Notice that $\lim_{n\to\infty} f(x_n)$ always exists because M_2 is complete. The remaining tasks are establishing g as a qualified extension and proving that g is the unique extension:

(i) g is a qualifying extension. To show that g is a qualifying extension, by definition there are two properties we need to check:

(1) First we check if g satisfies property (2). If we were to confine g on X, then every $x \in X$ there is a constant sequence $\{x_n\}$ with $x_n = x$ to fulfill g's definition. In other words, if we substitute this constant sequence into g's definition (2.1), we can write:

$$\forall x \in X : g(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x) = f(x) \tag{2.2}$$

Therefore, $g|_X = f$ is satisfied;

(2) Now we need to show that g satisfies property (1). Let $\epsilon > 0$. Since f is uniformly continuous on X, there exists a positive δ_0 such that:

$$\forall \epsilon > 0, \exists \delta_0 > 0 : d_1(x,y) < \delta_0 \Rightarrow d_2(f(x), f(y)) < \frac{\epsilon}{3} \tag{2.3}$$

Since $\overline{X} = M_1$, we know by property of closed sets that there is a sequence $\{x_n\}$ in X that has $\lim_{n\to\infty} x_n = x$ with $x \in M_1$. By definition of convergence, using the δ_0 in (2.3) we know there is a positive number N_1 satisfies:

$$\exists N_1 \in \mathbb{N}, \forall n > N_1 : d_1(x_n, x) < \frac{\delta_0}{3} \tag{2.4}$$

[1] Note that this deviates from JP Definition 46.6 in the sense that JP does not require $\widehat{M} = \overline{i(M)}$ but rather requires a weaker $M \subset i(M)$.

Similarly, there is also N_2 satisfies $d_1(y_n, y) < \frac{\delta_0}{3}$ if $n > N_2$. Take N_3 to be $\max\{N_1, N_2\}$ then we can write the following using triangle inequality of d_1:

$$\exists N_3 \in \mathbb{N}, \forall n > N_3 : d_1(x,y) \leq d_1(x,x_n) + d_1(x_n,y_n) + d_1(y_n,y) < \frac{\delta_0}{3} + \frac{\delta_0}{3} + \frac{\delta_0}{3} = \delta_0 \tag{2.5}$$

Since $\lim_{n\to\infty} x_n = x$, by definition of g we have $\lim_{n\to\infty} f(x_n) = g(x)$. We can exploit this convergence to know there is a positive integer N_4 satisfying:

$$\exists N_4 \in \mathbb{N} : d_2(f(x_n), g(x)) < \frac{\epsilon}{3} \tag{2.6}$$

Similarly, we can find $N_5 \in \mathbb{N}$ satisfying $d_2(f(y_n), g(y)) < \frac{\epsilon}{3}$ if $n > N_5$. Taking N_6 to be $\max\{N_4, N_5\}$, we can use triangle inequality to write the following:

$$\exists N_6 \in \mathbb{N}, \forall n > N_6 : d_2(g(x),g(y)) \leq d_2(g(x),f(x_n)) + d_2(f(x_n),f(y_n)) + d_2(f(y_n),g(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{2.7}$$

Define N_7 as $\max\{N_3, N_6\}$, and if we constrain if $n > N_7$ then we would simultaneously have $d_1(x,y) < \delta_0$ and $d_2(g(x), g(y)) < \epsilon$. Thus, g is uniformly continuous on M_1;

Combining (1) and (2), we know g is a uniformly continuous extension of f;

(ii) g is the unique extension. Assume for contradiction that there is another extension $h \neq g$. Since by definition, g and h must be the same on original domain X, i.e. $g|_X = h|_X = f$ by property (2), we know they can only be different at points $M_1 - X$. Since $M_1 = \overline{X}$, we know $M_1 - X = \partial X$. Assume their function values are different at $x \in \partial X$, i.e.

$$\exists x \in \partial X : h(x) \neq g(x) \tag{2.8}$$

Since $M_1 = \overline{X}$, there must be some sequence $\{x_n\}$ in X that converges towards x. Given f is uniformly continuous,

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x) \tag{2.9}$$

Since h and g are also uniformly continuous, we have:

$$g(x) = \lim_{n\to\infty} g(x_n) = f(x) \qquad h(x) = \lim_{n\to\infty} h(x_n) = f(x) \tag{2.10}$$

Thus, (2.10) contradicts with (2.8) so the assumption is false and g is unique.

Combining (i) and (ii), we know g is the unique uniformly continuous extension function. ■

3 Constructing Completions of Metric Spaces

3.1 Pseudometrics

As stated and proved in the previous text, not all metric spaces are complete, and we have come up with the completion operation to find a complete counterpart for a general metric space. However, do all metric spaces have completions? In this chapter, we answer this question by providing a method for constructing the completion for any metric space, thus revealing an affirmative answer to this question. We begin by building a pseudometric for a general metric space:

Definition 3.1. Given a metric space (M, d), a function $\widetilde{d} : M \times M \mapsto [0, \infty)$ is a ***pseudometric*** on M if:

1. $\widetilde{d}(x, x) = 0$ for every $x \in M$;

2. $\widetilde{d}(x,y) = \widetilde{d}(y,x)$ for every $x, y \in M$;

3. $\widetilde{d}(x,y) \leq \widetilde{d}(x,z) + \widetilde{d}(z,y)$ for every $x, y, z \in M$.

Notice that the only distinction between our pseudometric and a genuine metric lies in the first requirement where we cannot state "$\widetilde{d}(x,y) = 0$" $\Rightarrow$ "$x = y$" for pseudometric. In other words, this suggests there could be "equivalent pairs" in M with zero distance under pseudometric, but these pairs are still distinct elements in M, which makes $(M, \widetilde{d})$ not a metric space but rather a pseudometric space. To transform the pseudometric space $(M, \widetilde{d})$ into an actual metric space, we need to trim out these "equivalent pairs", so that our pseudometric will become a metric. In specific, we wish to build a difference set between the original M and the set of equivalent pairs in the form of $M^* := M - \widetilde{M}$ where the set $\widetilde{M}$ is the set generated by selecting all the elements that has pseudometric distance zero but are distinct. In particular, we consider using the pseudometric introduced in the following lemma:

Lemma 3.2. Given a metric space (M, d), and M_C as the collection of Cauchy sequences in (M, d). If $\tilde{d} : \tilde{M} \times \tilde{M} \mapsto \mathbb{R}$ is defined as:

$$\forall \{a_k\}, \{b_k\} \in M_C, \widetilde{d}(\{a_k\}, \{b_k\}) = \lim_{k\to\infty} d(a_k, b_k) \tag{3.1}$$

Then $\tilde{d}(\{a_k\}, \{b_k\})$ is a pseudometric on M_C.

Proof: To show that $\widetilde{d}$ is a pseudometric, first we need to show that the limit given in (3.1) exists, and then we need to check if $\widetilde{d}$ satisfies the three properties we defined for a pseudometric.

(i) $\lim_{k\to\infty} d(a_k, b_k)$ exists. Let $\epsilon > 0$ be some arbitrary positive number, and denote $c_k = d(a_k, b_k)$. If $\{c_k\}$ is Cauchy, then $\lim_{k\to\infty} c_k$ exists due to $\mathbb{R}$ being complete[2] and consequently $\widetilde{d}(\{a_k\}, \{b_k\})$ exists. Since d is a metric on M, for any $a_m, b_m, a_n, b_n \in M$ we can write the following by triangle inequality property:

$$\begin{aligned}\forall m, n : d(a_m, b_m) - d(a_n, b_n) &\leq d(a_m, a_n) + d(a_n, b_m) - d(a_n, b_n) \\ &\leq d(a_m, a_n) + d(b_m, b_n) + d(b_n, a_n) - d(a_n, b_n) \\ &\leq d(a_m, a_n) + d(b_n, b_m)\end{aligned} \tag{3.2}$$

Symmetrically, we can also write $d(a_n, b_n) - d(a_m, b_m) \leq d(a_n, a_m) + d(b_n, b_m)$. Combining these two inequalities together, we can write:

$$\forall m, n : |d(a_n, b_n) - d(a_m, b_m)| \leq d(a_n, a_m) + d(b_n, b_m) \tag{3.3}$$

Given that $\{a_k\}$ is Cauchy, we know there is a positive integer N_1 such that the difference between any term past N is less than 0.5ϵ:

$$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}, \forall m, n > N_1 : d(a_m, a_n) < 0.5\epsilon \tag{3.4}$$

Symmetrically, given that $\{b_k\}$ is Cauchy, we know there is a positive integer N_2 such that the difference between any term past N is less than 0.5ϵ, i.e. $d(b_m, b_n) < 0.5\epsilon$. Choose N as the bigger integer from the previous two, i.e. $N = \max\{N_1, N_2\}$, then given d is a metric on M, we can use triangle inequality property to write:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : d(a_m, b_n) + d(b_m, b_n) < 0.5\epsilon + 0.5\epsilon = \epsilon \tag{3.5}$$

[2] Recall that we have established $\mathbb{R}^n$ is complete as conclusion of Example 1.4, so $\mathbb{R}$ being complete is just a specific case of that conclusion when $n = 1$.

Plug (3.5) into (3.3) and we know for arbitrary ϵ, there exists a positive integer to qualify $\{c_k\}$ as Cauchy, i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : d(c_n, c_m) = |d(a_n, b_n) - d(a_m, b_m)| \leq d(a_m, b_n) + d(b_m, b_n) < \epsilon \tag{3.6}$$

Since $\{c_k\}$ is a Cauchy sequence in the complete $\mathbb{R}$, we know that $\lim\limits_{k\to\infty} c_k$ exists.

(ii) $\widetilde{d}$ satisfies the requirements (1)∼(3):

1. To check $\widetilde{M}(\{a_k\}, \{a_k\}) = 0$, we simply plug the arguments into definition (3.1) and will have:

$$\forall \{a_k\} \in M_C : \widetilde{d}(\{a_k\}, \{a_k\}) = \lim_{k\to\infty} d(a_k, a_k) \tag{3.7}$$

   Since d is a metric, we know we know for any $a_k \in M$, there will be $d(a_k, a_k) = 0$ so $\lim\limits_{k\to\infty} d(a_k, a_k) = 0$ as well;

2. Similarly, by plug the arguments into the definition (3.1), we have:

$$\forall \{a_k\}, \{b_k\} \in M_C : \widetilde{d}(\{a_k\}, \{b_k\}) = \lim_{k\to\infty} d(a_k, b_k) \tag{3.8}$$

   Since d is a metric, we know for any $a_k, b_k \in M$, there will be $d(a_k, b_k) = d(b_k, a_k)$ so $\lim\limits_{k\to\infty} d(a_k, b_k) = \lim\limits_{k\to\infty} d(b_k, a_k) = \widetilde{d}(\{b_k\}, \{a_k\})$;

3. Recall that d is a metric on M, so by triangle inequality we have:

$$\forall \{a_k\}, \{b_k\}, \{c_k\} \in M_C, \forall k : d(a_k, c_k) + d(c_k, b_k) \geq d(a_k, b_k) \tag{3.9}$$

   Since the right-hand side of (3.9) has been proved to be Cauchy in (3.6), we can take limits of both sides of (3.9) to arrive at:

$$\forall \{a_k\}, \{b_k\}, \{c_k\} \in M_C : \lim_{k\to\infty} (d(a_k, c_k) + d(c_k, b_k)) = \lim_{k\to\infty} (d(a_k, c_k)) + \lim_{k\to\infty} (d(c_k, b_k)) \geq \lim_{k\to\infty} (d(a_k, b_k)) \tag{3.10}$$

   By definition (3.1), we can rewrite (3.10) to reveal that $\widetilde{d}$ satisfies triangle inequality:

$$\forall \{a_k\}, \{b_k\}, \{c_k\} \in M_C : \widetilde{d}(\{a_k\}, \{c_k\}) + \widetilde{d}(\{c_k\}, \{b_k\}) \geq \widetilde{d}(\{a_k\}, \{b_k\}) \tag{3.11}$$

Combining (1)∼(3), we know $\widetilde{d}$ satisfies the properties of pseudometric.

Since (i) has showed that $\widetilde{d}$ is well-defined and (ii) has showed that $\widetilde{d}$ satisfies all the three properties, we know $\widetilde{d}$ is a pseudometric defined on M_C. ∎

3.2 Building Blocks for the Completion

Now that we have a pseudometric $\widetilde{d}$, we can use it to find all the equivalent pairs. An organized way to trim different equivalent pairs is to put them into buckets, where each bucket can be considered an "equivalent class", and some literatures refer to this relationship as "co-Cauchy":

Definition 3.3. Given metric space (M, d) and the pseudometric $\widetilde{d}$ defined in (3.1), two sequences in (M, d) are ***co-Cauchy*** under $\widetilde{d}$ if $\widetilde{d}(\{a_k\}, \{b_k\}) = 0$.

Then, we can use co-Cauchyness to collect all the equivalent Cauchy sequences in (M, d) into equivalent classes:

Definition 3.4. An ***equivalence class*** X on (M, d) under $\widetilde{d}$ consists of any two sequences $\{x_n\}$ and $\{y_n\}$ in (M, d) that are co-Cauchy under $\widetilde{d}$.

Internally, each equivalent class is consist of all the co-Cauchy sequences in the original metric space, so on the equivalent class level, the original metric space has already been bucketized, which motivates us to define the set $\widetilde{M}^*$ as:

$$\widetilde{M}^* := \{X | X \text{ is an equivalent class of } (M, d) \text{ under } \widetilde{d}\} \tag{3.12}$$

To equip $\widetilde{M}^*$ with a metric, we need to assign a distance function between different equivalent classes. Intuitively, since each single equivalent class has sequences of the same $\widetilde{d}$ properties, we can consider the following function $\widetilde{d}^*$ that measure the distance between two equivalent classes via one "representative" sequence from each class:

$$\widetilde{d}^*(X, Y) := \lim_{n\to\infty} d(x_n, y_n) \tag{3.13}$$

where $\{x_n\} \in X$, $\{y_n\} \in Y$ are sequences of these two equivalent classes respectively. Thus, we have finished preparing the building blocks of our completion and we move on to prove $(\widetilde{M}^*, \widetilde{d}^*)$ is a completion.

3.3 Proof of the Completion

We gradually establish $(\widetilde{M}^*, \widetilde{d}^*)$ as the completion by first showing that it is a metric space, then showing that it is complete, and finally proving that there is an isometric injection $i : M \mapsto \widetilde{M}^*$ whose closure of image $\overline{i(M)}$ equal our $\widetilde{M}^*$.

Lemma 3.5. $(\widetilde{M}^*, \widetilde{d}^*)$ is a metric space with $\widetilde{M}^*$ defined in (3.12) and $\widetilde{d}^*$ defined in (3.13).
Proof: To show that $(\widetilde{M}^*, \widetilde{d}^*)$ is a metric space is equivalent with showing that $\widetilde{d}^*$ is a metric. By definition, any two $X, Y \in \widetilde{M}^*$ will have:

$$\widetilde{d}^*(X, Y) = \lim_{n\to\infty} d(x_n, y_n) \tag{3.14}$$

where $\{x_n\} \in X$ and $\{y_n\} \in Y$ are sequences of these two equivalent classes respectively. Using the definition of $\widetilde{d}$ in (3.1), we can rewrite (3.14) as:

$$\widetilde{d}^*(X, Y) = \lim_{n\to\infty} d(x_n, y_n) = \widetilde{d}(\{x_n\}, \{y_n\}) \tag{3.15}$$

Recall that for $\widetilde{d}^*$ to be a metric, it needs to have three qualifying properties:

1. Any two elements X, Y of $\widetilde{M}^*$ satisfies $\widetilde{d}^*(X, Y) = 0$ if and only if $X = Y$: this statement consists of two sub-arguments, the first one of which is

   $$\widetilde{d}^*(X, Y) = 0 \Rightarrow X = Y \tag{3.16}$$

   Assume for contradiction that there is some $X \neq Y$ with $\widetilde{d}^*(X, Y) = 0$. Then, by definition of $\widetilde{d}^*$, there must be sequences $\{x_n\}$ from X and $\{y_n\}$ from Y satisfying:

   $$\exists \{x_n\} \in X, \{y_n\} \in Y : \lim_{n\to\infty} d(x_n, y_n) = 0 \tag{3.17}$$

   Applying (3.15), we can write:

   $$\exists \{x_n\} \in X, \{y_n\} \in Y : \widetilde{d}(\{x_n\}, \{y_n\}) = 0 \tag{3.18}$$

   Recall that if $\{x_n\}$ and $\{y_n\}$ satisfy $\widetilde{d}(\{x_n\}, \{y_n\}) = 0$, then $\{x_n\}, \{y_n\}$ belong to the same equivalent class. This contradicts with the hypothesis that $X \neq Y$, so (3.16) is true;

The other sub-argument is

$$X = Y \Rightarrow \widetilde{d}^*(X,Y) = 0 \tag{3.19}$$

By definition, $X = Y$ gives us:

$$\widetilde{d}^*(X,Y) = \lim_{n\to\infty} d(x_n, y_n) = \widetilde{d}(\{x_n\},\{y_n\}) \text{ with } \{x_n\} \in X, \{y_n\} \in Y \tag{3.20}$$

Since $X = Y$, by our definition of equivalent class we know $\widetilde{d}(\{x_n\},\{y_n\}) = 0$, thus $\widetilde{d}^*(X,Y) = 0$;

2. Any two elements X, Y of $\widetilde{M}^*$ satisfies $\widetilde{d}^*(X,Y) = \widetilde{d}^*(Y,X)$: by definition, we can substitute $\widetilde{d}^*$ for $\widetilde{d}$:

$$\widetilde{d}^*(X,Y) = \lim_{n\to\infty} d(x_n, y_n) = \widetilde{d}(\{x_n\},\{y_n\}) \tag{3.21}$$

Recall that $\widetilde{d}$ as a pseudometric satisfies $\widetilde{d}(\{x_n\},\{y_n\}) = \widetilde{d}(\{y_n\},\{x_n\})$, we can substitute it back to have:

$$\widetilde{d}^*(X,Y) = \widetilde{d}(\{x_n\},\{y_n\}) = \widetilde{d}(\{y_n\},\{x_n\}) = \widetilde{d}^*(Y,X) \tag{3.22}$$

3. Any three elements X, Y, Z of $\widetilde{M}^*$ satisfies $\widetilde{d}^*(X,Y) \le \widetilde{d}^*(X,Z) + \widetilde{d}^*(Y,Z)$: following the same substitution trick in (3.21) we have $\widetilde{d}*^(X,Y) = \widetilde{d}(\{x_n\},\{y_n\})$. Recall that as a pseudometric, $\widetilde{d}$ satisfies triangle inequality so:

$$\widetilde{d}^*(X,Y) = \widetilde{d}(\{x_n\},\{y_n\}) \le \widetilde{d}(\{x_n\},\{z_n\}) + \widetilde{d}(\{z_n\},\{y_n\}) = \widetilde{d}^*(X,Z) + \widetilde{d}^*(Z,Y) \tag{3.23}$$

Combining (1)∼(3), we know $\widetilde{d}^*$ is a metric on $\widetilde{M}^*$, so $(\widetilde{M}^*, \widetilde{d}^*)$ is a metric space. ∎

Before showing the completeness of $(\widetilde{M}^*, \widetilde{d}^*)$, we need to establish the measurement of a set:

Definition 3.6. Given metric space (M, d), the ***diameter*** of set $S \subset M$ is defined as:

$$\rho(S) = \sup\{d(x,y) | x, y \in S\} \tag{3.24}$$

Now, we prove a useful lemma regarding the sequences in the equivalent classes we constructed:

Lemma 3.7. In any equivalent class $X \in \widetilde{M}^*$, we can find a sequence $\{x_n\}$ whose element set satisfies $\rho(\{x_n\}) < \epsilon$ for arbitrary $\epsilon > 0$.
Proof: Consider any sequence $\{y_n\}$ in this equivalent class X: by definition, $\{y_n\}$ is Cauchy, which suggests that there is a positive integer N such that:

$$\forall n, m > N, d(y_n, y_m) < \epsilon \tag{3.25}$$

Therefore, the truncated sequence $\{y_{N+n}\}$ is a Cauchy sequence with diameter less than ϵ. We claim that $\{x_n\}$ defined by $x_n = y_{N+n}$ is an element of equivalent class X as well. We can first write:

$$\widetilde{d}(\{x_n\},\{y_n\}) = \lim_{k\to\infty} d(x_k, y_k) = \lim_{k\to\infty} d(y_{N+k}, y_k) \tag{3.26}$$

Since $\{y_n\}$ is Cauchy, $\lim_{k\to\infty} d(y_{N+k}, y_k) = 0$ so $\widetilde{d}(\{x_n\},\{y_n\}) = 0$. According to the definition of equivalent class, $\{x_n\}, \{y_n\}$ belong to the same X so $\{x_n\} \in X$. ∎

With the definition of the diameter of a set and Lemma 3.7, we have all we need to directly show the completeness of $(\widetilde{M}^*, \widetilde{d}^*)$:

Lemma 3.8. $(\widetilde{M}^*, \widetilde{d}^*)$ is a complete metric space with $\widetilde{M}^*$ defined in (3.12) and $\widetilde{d}^*$ defined in (3.13).
Proof: Let $\epsilon > 0$ and $\{X_n\}$ be any arbitrary Cauchy sequence in $(\widetilde{M}^*, \widetilde{d}^*)$. Our goal is to show that

$$\lim_{n\to\infty} X_n = X \in \widetilde{M}^* \tag{3.27}$$

To begin with, for every n, we know X_n is itself an equivalent class of Cauchy sequences from (M, d). By Lemma 3.7, there must be a Cauchy sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$ within every X_n with a diameter less than a positive number, and we pick that positive number to be $\frac{1}{n}$:

$$\forall n \in \mathbb{N}, \exists \{x_k^{(n)}\} \in X_n : \rho(\{x_k^{(n)}\}) < \frac{1}{n} \tag{3.28}$$

Given the $\{x_k^{(n)}\}$'s defined in (3.28), our intermediary goal is to show that every first term in $\{x_k^{(n)}\}$ form a Cauchy sequence in (M, d), i.e.

$$\{x_1^{(n)}\}_{n=1}^{\infty} \text{ is Cauchy in } (M, d) \tag{3.29}$$

To show that $\{x_1^{(n)}\}_{n=1}^{\infty}$ is Cauchy in (M, d), first recall that the sequence $\{X_n\}$ is Cauchy in $(\widetilde{M}^*, \widetilde{d}^*)$, therefore there exists a positive integer N_0 such that:

$$\exists N_0 \in \mathbb{N}, \forall n, m > N_0 : \widetilde{d}^*(X_n, X_m) < \frac{\epsilon}{3} \tag{3.30}$$

Also, by Archimedean property, there is a positive integer N_1 such that:

$$\exists N_1 \in \mathbb{N} : \frac{1}{N_1} < \frac{\epsilon}{3} \tag{3.31}$$

Now, choose N as the greater between N_0 and N_1 then:

$$N := \max\{N_0, N_1\} \Rightarrow \forall n, m > N : \widetilde{d}^*(X_n, X_m) < \frac{\epsilon}{3}, \frac{1}{n} < \frac{\epsilon}{3}, \frac{1}{m} < \frac{\epsilon}{3} \tag{3.32}$$

By definition of $\widetilde{d}^*$ we know that $\widetilde{d}^*(X_n, X_m)$ is the $\widetilde{d}$ distance between sequence $\{x_k^{(n)}\}$ within equivalent class X_n and sequence $\{x_k^{(m)}\}$ within equivalent class X_m:

$$\widetilde{d}^*(X_n, X_m) = \widetilde{d}(\{x_k^{(n)}\}, \{x_k^{(m)}\}) \tag{3.33}$$

By definition of $\widetilde{d}$ we know that $\widetilde{d}(\{x_k^{(n)}\}, \{x_k^{(m)}\})$ is the limit of pair-wise d distance between the two sequences:

$$\widetilde{d}(\{x_k^{(n)}\}, \{x_k^{(m)}\}) = \lim_{k\to\infty} d(x_k^{(n)}, x_k^{(m)}) \tag{3.34}$$

Now, combine (3.30) with (3.34) and we know:

$$\widetilde{d}^*(X_n, X_m) = \lim_{k\to\infty} d(x_k^{(n)}, x_k^{(m)}) < \frac{\epsilon}{3} \tag{3.35}$$

Thus, there exists a positive integer K such that:

$$\exists N \in \mathbb{N}, \forall m, n > N, \exists K \in \mathbb{N} : d(x_K^{(n)}, x_K^{(m)}) < \frac{\epsilon}{3} \tag{3.36}$$

Given the N defined in (3.32), we can write by triangle inequality that:

$$\begin{aligned}\exists N \in \mathbb{N}, \forall m,n > N : d(x_1^{(m)}, x_1^{(n)}) &\leq d(x_1^{(m)}, x_K^{(m)}) + d(x_K^{(m)}, x_K^{(n)}) + d(x_K^{(n)}, x_1^{(n)}) \\ &< d(x_1^{(m)}, x_K^{(m)}) + \frac{\epsilon}{3} + d(x_K^{(n)}, x_1^{(n)})\end{aligned} \tag{3.37}$$

Recall that by design, we have constrained in (3.28) that $\rho(\{x_k^{(n)}\}_{k=1}^{\infty}) < \frac{1}{n}$ so:

$$d(x_1^{(m)}, x_K^{(m)}) \leq \rho(\{x_k^{(m)}\}_{k=1}^{\infty}) < \frac{1}{m} \qquad d(x_1^{(n)}, x_K^{(n)}) \leq \rho(\{x_k^{(n)}\}_{k=1}^{\infty}) < \frac{1}{n} \tag{3.38}$$

Plug (3.38) into (3.37) and we have:

$$\exists N \in \mathbb{N}, \forall m,n > N : d(x_1^{(m)}, x_1^{(n)}) < \frac{1}{m} + \frac{\epsilon}{3} + \frac{1}{n} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{3.39}$$

Recall that in (3.32), we have $\frac{1}{n} < \frac{\epsilon}{3}, \frac{1}{m} < \frac{\epsilon}{3}$, so we can rewrite (3.39) into:

$$\exists N \in \mathbb{N}, \forall m,n > N : d(x_1^{(m)}, x_1^{(n)}) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \tag{3.40}$$

Therefore, $\{x_1^{(n)}\}_{n=1}^{\infty}$ is Cauchy. By construction, we know there must be some equivalent class X^* containing $\{x_1^{(n)}\}_{n=1}^{\infty}$, and our following goal is to establish that

$$\{x_1^{(n)}\}_{n=1}^{\infty} \in X^* \Rightarrow \lim_{n\to\infty} X_n = X^* \tag{3.41}$$

Consider the limit of the $\widetilde{d}^*$ distance between X_n and the equivalent class X^* containing $\{x_1^{(n)}\}_{n=1}^{\infty}$:

$$\lim_{n\to\infty} \widetilde{d}^*(X_n, X^*) = \lim_{n\to\infty} \widetilde{d}(\{x_k^{(n)}\}, \{x_1^{(n)}\}) = \lim_{n\to\infty} \lim_{k\to\infty} d(x_k^{(n)}, x_1^{(n)}) \tag{3.42}$$

Recall that when constructing X_n in (3.28), we have constrained the diameter of all $\{x_n^{(k)}\}$'s in X_n to be $\leq \frac{1}{n}$, so $d(x_k^{(n)}, x_1^{(n)}) \leq \frac{1}{n}$. Using this, we can derive $\lim\limits_{n\to\infty} \widetilde{d}^*(X_n, X^*) \leq \lim\limits_{n\to\infty} \frac{1}{n} = 0$ from (3.42). Since $\widetilde{d}^*(X_n, X^*)$ is also bounded below by 0, we know $\lim\limits_{n\to\infty} \widetilde{d}^*(X_n, X^*) = 0$ using squeeze lemma of limits. Notice that $\lim\limits_{n\to\infty} \widetilde{d}^*(X_n, X^*) = 0$ suggests $\lim\limits_{n\to\infty} X_n = X^*$. Hence, every Cauchy sequence $\{X_n\}$ in $(\widetilde{M}^*, \widetilde{d}^*)$ converges to some element $X^* \in \widetilde{M}^*$. Per definition, $(\widetilde{M}^*, \widetilde{d}^*)$ is complete. ■

To continue justifying $(\widetilde{M}^*, \widetilde{d}^*)$ as the completion of (M, d), we need to define a mapping that takes in $x \in M$ and outputs $X \in \widetilde{M}^*$. In particular, let us consider the mapping i defined by:

$$\forall x \in M : i(x) = X \text{ where } \begin{cases} \forall n \in \mathbb{N} : x_n = x \text{ is a constant sequence} \\ \{x_n\} \in \text{ equivalent class } X \end{cases} \tag{3.43}$$

Lemma 3.9. The mapping $i : M \mapsto \widetilde{M}^*$ defined in (3.43) is an isometric injection.
Proof: An isometric injection will carry the metric value for any two points in the original metric space to the new $\widetilde{M}^*$. In particular, consider two points x, y from M:

$$\forall x, y \in M : \widetilde{d}^*(i(x), i(y)) = \widetilde{d}^*(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n) \tag{3.44}$$

Recall that by definition, $\{x_n\}, \{y_n\}$ are constant sequence with any $x_n = x$ and $y_n = y$. Therefore, we could rewrite (3.44) into:

$$\widetilde{d}^*(i(x),i(y)) = \lim_{n\to\infty} d(x,y) = d(x,y) \tag{3.45}$$

Thus, i is an isometric injection. ■

Lemma 3.10. The mapping $i : M \mapsto \widetilde{M}^*$ defined in (3.43) satisfies $\overline{i(M)} = \widetilde{M}^*$.

Proof: By set theory, we need to establish the two-directional subset relationships respectively. First, we need to show that $\overline{i(M)} \subseteq \widetilde{M}^*$. Since $i : M \mapsto \widetilde{M}^*$, this is always true in the sense that the range of any function will always be a subset of its co-domain. Next we need $\widetilde{M}^* \subseteq \overline{i(M)}$. Consider any equivalent class X in $\widetilde{M}^*$: with in X let us say there is a Cauchy sequence $\{x_n\}$. Our goal is to show that $\{x_n\}$ is also in an equivalent class that is an element of $\overline{i(M)}$. Now, consider for each n, we can form a Cauchy sequence $\{c_n^{(k)}\}$ with $c_n^{(k)} = x_n$. In other words, $\{c_n^{(k)}\}_{k=1}^{\infty}$ is a constant sequence, but as n changes corresponds to different element of $\{x_n\}$. Since every $\{c_n^{(k)}\}$ is an element of $i(M)$ by our definition of i, we know the limit $\lim_{n\to\infty} C_n$ is in $\overline{i(M)}$ where C_n is an equivalent class containing $\{c_n^{(k)}\}$. On the other hand, $\lim_{n\to\infty} \widetilde{d}^*(C_n, X) = \lim_{n\to\infty}\lim_{k\to\infty} d(c_n^{(k)}, x_n) = 0$ so $\lim_{n\to\infty} C_n = X$. This suggests every $X \in \widetilde{M}^*$ is also in $\overline{i(M)}$ so $\widetilde{M}^* \subseteq \overline{i(M)}$. Combining this with $\overline{i(M)} \subseteq \widetilde{M}^*$, by set theory we have $\widetilde{M}^* = \overline{i(M)}$. ■

Theorem 3.11. Given a metric space (M,d), there always exists a completion $(\widetilde{M}^*, \widetilde{d}^*)$ per definition in (3.12) and (3.13).

Proof: Following Lemma 3.8, $(\widetilde{M}^*, \widetilde{d}^*)$ is a complete metric space. Following Lammata 3.9 and 3.10, there is an isometric injection $i : M \mapsto \widetilde{M}^*$ with $\overline{i(M)} = \widetilde{M}^*$. By definition, $(\widetilde{M}^*, \widetilde{d}^*)$ is a completion of (M,d). ■

4 Examples of Completions

4.1 Completion of subsets of $\mathbb{R}$

Consider the case where $A = (0,1)$. As discussed in Example 1.2, the metric space $(A, ||\cdot||)$ is not complete and $(\overline{A}, ||\cdot||)$ i.e. $([0,1], ||\cdot||)$ is complete. More generally, this is because of:

Theorem 4.1. Given complete (M,d), if subset $X \subseteq M$ is closed, then (X,d) is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in X. Since M is equipped with the same metric d, we know $\{x_n\}$ is also Cauchy in M. Since M is complete, there is $x \in M$ such that $\lim_{n\to\infty} x_n = x$. Recall that X is closed, so $x \in X$, therefore the Cauchy sequence converges in X.

4.2 p-adic metric on $\mathbb{Q}$

p-adic numbers form a classical non-Archimedean field, which plays increasing role in the research of the non-Kolmogorov theory of probability, which can be considered in physics, complex dynamic biochemical systems and computer sciences. It offers a more general way of treating rational numbers in a p-adic metric, where p is prime. We start by defining the p-adic norm:

Definition 4.2. Given a prime number $p \in \mathbb{N}$, one defines the $|\cdot|_p : \mathbb{Q} \mapsto [0,\infty)$ as the ***p-adic norm***:

$$|x|_p := \begin{cases} p^{-\epsilon} & \text{with } x \neq 0 \text{ and } x = p^{\epsilon}\cdot\frac{a}{b} \text{ with } a, b, \text{ and } p \text{ relatively prime}, \epsilon \in \mathbb{Z} \\ 0 & \text{with } x = 0 \end{cases} \tag{4.1}$$

Using the p-adic norm, we can define a function $d_p(x,y) : \mathbb{Q} \times \mathbb{Q} \mapsto [0, \infty)$ as the ***p-adic metric*** which can reflect the p-adic difference between two rationals:

$$d_p(x,y) := |x-y|_p \tag{4.2}$$

For instance, the 5-adic difference between 64 and 39 is $\frac{1}{25}$, the 7-adic difference between 3 and 2 is 1, and the 3-adic difference between 1 and $\frac{5}{9}$ is 9. To interpret the p-adic metric system, the norm formulation of p-adic metric is very helpful for tapping into known norm properties, and let us first use it to establish the triangle inequality property:

Lemma 4.3. For any $x, y \in \mathbb{Q}$, p-adic norm satisfies triangle inequality:

$$\forall x, y \in \mathbb{Q} : |x+y|_p \leq |x|_p + |y|_p \tag{4.3}$$

Proof: Notice that for any $x, y \in \mathbb{Q}$, we have:

$$x = p^{\epsilon_1} \cdot \frac{a_1}{b_1} \qquad y = p^{\epsilon_2} \cdot \frac{a_2}{b_2} \tag{4.4}$$

Where $\gcd(p, a_1) = \gcd(p, b_1) = \gcd(p, a_2) = \gcd(p, b_2) = 1$. Without loss of generality, let $\epsilon_1 \leq \epsilon_2$ and write:

$$x + y = p^{\epsilon_1} \cdot \frac{a_1}{b_1} + p^{\epsilon_2} \cdot \frac{a_2}{b_2} = p^{\epsilon_1} \cdot (\frac{a_1}{b_1} + p^{\epsilon_2 - \epsilon_1} \cdot \frac{a_2}{b_2}) = p^{\epsilon_1} \cdot \frac{a_1 b_2 + p^{\epsilon_2 - \epsilon_1} \cdot a_2 b_1}{b_1 b_2} \tag{4.5}$$

Then, we consider the two possibilities of ϵ_1, ϵ_2's relationships separately:

(i) If $\epsilon_1 < \epsilon_2$, then we know $p^{\epsilon_2 - \epsilon_1}$ is a multiplier of p. Also, recall that $\gcd(a_1, p) = \gcd(b_2, p) = 1$ so $(a_1 b_2 + p^{\epsilon_2 - \epsilon_1} a_2 b_1)$ will not be a multiplier of p. Therefore, the p-adic norm of $x + y$ would be the same as x:

$$|x+y|_p = |p^{\epsilon_1} \cdot \frac{a_1 b_2 + p^{\epsilon_2 - \epsilon_1} \cdot b_1}{b_1 b_2}|_p = |p^{\epsilon_1}|_p = |p^{\epsilon_1} \cdot \frac{a_1}{b_1}|_p = |x|_p \tag{4.6}$$

Since $|y|_p \geq 0$, we know $|x|_p + |y_p| \geq |x+y|_p$;

(ii) If $\epsilon_1 = \epsilon_2$ then $p^{\epsilon_2 - \epsilon_1} = 1$ and $(a_1 b_2 + p^{\epsilon_2 - \epsilon_1} a_2 b_1)$ might be a multiplier of p. Assume that in this case:

$$a_1 b_2 + p^{\epsilon_2 - \epsilon_1} a_2 b_1 = p^k \tag{4.7}$$

Since $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, we know $a_1 b_2 + a_2 b_1$ is an integer as well so $k \geq 0$. Then, we can observe the following for the p-adic norm of $x + y$:

$$|x+y|_p = |p^{\epsilon_1} \cdot \frac{p^k}{b_1 b_2}|_p = |p^{\epsilon_1 + k}|_p = p^{-\epsilon_1 - k} \tag{4.8}$$

Since $k \geq 0$, we know $p^{-\epsilon_1 - k} \leq p^{-\epsilon_1}$. Plug this back into (4.8) and we have:

$$|x+y|_p = p^{-\epsilon_1 - k} \leq p^{-\epsilon_1} |x|_p \tag{4.9}$$

Since $|y|_p \geq 0$, we know $|x|_p + |y_p| \geq |x+y|_p$;

Combining the sub-arguments (i) and (ii), we have showed the triangle inequality of p-adic norm.

Theorem 4.4. $(\mathbb{Q}, d_p)$ is a metric space.

Proof: To prove that $(\mathbb{Q}, d_p)$ is a metric space, we need to check if d_p has the three properties of a metric:

(1) $\forall x, y \in \mathbb{Q} : d_p(x, y) = 0 \Leftrightarrow x = y$:
The "$\Rightarrow$" logical derivation is true: assume for contradiction, some $x \neq y$ satisfies $d_p(x, y) = 0$, then by definition we will have $p^{\epsilon_1} = 0$ which is impossible for an integer ϵ_1. Thus, the assumption is false and the "$\Rightarrow$" argument is correct;
The "$\Leftarrow$" statement follows directly from definition, which assigns $d_p(x, x) = 0$.

(2) $\forall x, y \in \mathbb{Q} : d_p(x, y) = d_p(y, x)$:
If $x = y$ then using (1), we instantly know $d_p(x, y) = 0 = d_p(y, x)$;
Otherwise, $x \neq y$ so $d_p(x, y) \neq 0$. Assume for contradiction that there exists some x, y satisfying $D_1 = d_p(x, y) \neq d_p(y, x) = D_2$. By definition,

$$\begin{cases} x - y = p^{\epsilon_1} \cdot \frac{a_1}{b_1} \\ y - x = p^{\epsilon_2} \cdot \frac{a_2}{b_2} \end{cases} \Rightarrow \begin{cases} d_p(x, y) = p^{-\epsilon_1} \\ d_p(y, x) = p^{-\epsilon_2} \end{cases} \tag{4.10}$$

Taking the quotient of $x - y$ over $y - x$, we can write:

$$\frac{x - y}{y - x} = \frac{p^{\epsilon_1} \cdot \frac{a_1}{b_1}}{p^{\epsilon_2} \cdot \frac{a_2}{b_2}} \Rightarrow p^{\epsilon_1 - \epsilon_2} = -\frac{a_2 b_1}{a_1 b_2} \tag{4.11}$$

Since $\gcd(p, a_1, b_1) = \gcd(p, a_2, b_2) = 1$, the right-hand side of (4.11) must be 1, so $\epsilon_1 = \epsilon_2$. However, by hypothesis that $d_p(x, y) \neq d_p(y, x)$ we know $\epsilon_1 \neq \epsilon_2$. Therefore, we have reached a contradiction and the assumption is rejected;

(3) $\forall x, y, z \in \mathbb{Q} : d_p(x, y) \leq d_p(x, z) + d_p(z, y)$:
Recall that by Lemma 4.3, we have established that the p-adic norm satisfies triangle inequality. Therefore, we can write the argument of p-adic metric's triangle inequality in terms of the p-adic norms, i.e.

$$\forall x, y, z \in \mathbb{Q} : d_p(x, z) + d_p(z, y) = |x - z|_p + |z - y|_p \leq |(x - z) + (z - y)|_p \leq |x - y|_p = d_p(x, y) \tag{4.12}$$

Combining the properties (1)$\sim$(3), we have showed that the d_p qualifies as a metric. ■

Now that we have proved d_p is a metric, i.e. $(\mathbb{Q}, d_p)$ is a metric space, we can apply Theorem 3.9 to conclude that there exists a completion of $(\mathbb{Q}, d_p)$. This completion of $\mathbb{Q}$ is usually referred to as the ***p*-adic number** $\mathbb{Q}_p$. Just as a real number can be more simply characterized in terms of their decimal expansions instead of as Cauchy sequences of rational numbers, elements in $\mathbb{Q}_p$ can also be characterized in terms of certain expansions: an element $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{k=-n}^{\infty} a_k p^k \qquad \text{where } n \in \mathbb{N} \text{ and } a_k \in \mathbb{Z}. \tag{4.13}$$

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