RELATIVE SPECTRAL CORRESPONDENCE FOR PARABOLIC HIGGS BUNDLES AND DELIGNE-SIMPSON PROBLEM

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Abstract. In this paper, we generalize the spectral correspondence for parabolic Higgs bundles established by Diaconescu–Donagi–Pantev to the relative setting. We show that the relative moduli space of $\vec{\xi}$-parabolic Higgs bundles on a curve can be realized as the relative moduli space of pure dimension-one sheaves on a family of holomorphic symplectic surfaces. This leads us to formulate the image of the relative moduli space under the Hitchin map in terms of linear systems on the family of surfaces. Then we explore the connections between the geometry of these linear systems and the so-called OK conditions introduced by Balasubramanian–Distler–Donagi in the context of six-dimensional superconformal field theories. As applications, we obtain (a) the non-emptiness of the moduli spaces and (b) the Deligne–Simpson problem (and its higher genus analogue). In particular, we prove a conjecture proposed by Balasubramanian–Distler–Donagi that the OK condition is sufficient for solving the Deligne–Simpson problem.

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1. Introduction

1.1. **Overview.** In the 1980s, Nigel Hitchin introduced the moduli space of Higgs bundles (often called Hitchin system) and showed that it carries an integrable system structure via the so-called Hitchin fibration. The generic fibers of the Hitchin fibration can be described by the Beauville-Narasimhan-Ramanan (BNR) [BNR89] or spectral correspondence, which identifies Higgs bundles with line bundles (or more generally, torsion-free sheaves) on spectral curves lying in the cotangent bundle of the base curve.

In fact, the spectral correspondence extends beyond the generic fibers [Sim94]: it can be extended to an isomorphism between the moduli space of Higgs bundles and the moduli space of pure dimension-one sheaves on the total space of the cotangent bundle. This perspective of Hitchin systems in terms of moduli of sheaves on a surface has led to many applications, for example, it has been used in [CDP14][DDP18][Chu+20][MS23][KK24] to connect topological invariants of the Hitchin system to enumerative geometry.

In this work, we focus on a variant of Higgs bundles—parabolic Higgs bundles—which originated from the tame non-abelian Hodge correspondence (tame NAHC) developed by Simpson [Sim90]. The additional parabolic structures allows for the specification of ordered eigenvalues of the residues of the Higgs field at fixed marked points. We are interested in the relative moduli space of parabolic Higgs bundles as these ordered eigenvalues vary over a base. This relative viewpoint has been useful in recent work, for example in the study of the topological [Hau+22] [MSY23] or motivic [MSY24] aspects of the Hitchin systems through specialization arguments.

The first goal of this work is to generalize the parabolic spectral correspondence, originally established by Diaconescu–Donagi–Pantev [DDP18], to the setting of relative moduli space of parabolic Higgs bundles, which we call the *relative spectral correspondence*. This leads to a family of surfaces together with a family of linear systems on it. Then we explore the connections between the geometry of these linear systems and the so-called OK conditions introduced by Balasubramanian–Distler–Donagi [BDD22], motivated by six-dimensional superconformal field theories. As a by-product, we found a new perspective on the classical multiplicative Deligne–Simpson problem through these surfaces and linear systems. In particular, our approach provides a geometric realization of the OK condition and shows its sufficiency for the Deligne–Simpson problem, as conjectured in *loc. cit.*

1.2. **Relative spectral correspondence.** Let C be a smooth complex curve of genus g and $D = p_1 + \cdots + p_n$ be a reduced divisor. We reserve r and d for a rank and degree of a vector bundle over C, respectively. To each point p_i, we assign a partition of r, $\underline{m}_i = (\underline{m}_{i,1}, \cdots, \underline{m}_{i,\ell(i)})$ satisfying $\underline{m}_{i,1} \geq \cdots \geq \underline{m}_{i,\ell(i)}$. We encode this data as $\vec{m} = (\underline{m}_1, \cdots, \underline{m}_n)$. To such data, we define a *(quasi-)parabolic Higgs bundle* that is a Higgs bundle $(E, \Phi : E \to E \otimes K_C(D))$ over C with a filtration associated to $\underline{m}_i$ on each restriction E_{p_i} preserved by $\Phi|_{E_{p_i}}$ (see Definition 2.1).

Consider a collection of sections $\vec{\xi} = (\underline{\xi}_1, \cdots, \underline{\xi}_n)$ where $\underline{\xi}_i = (\xi_{i,1}, \cdots, \xi_{i,\ell(i)}) \in H^0(p_i, K_C(D)_{p_i})^{\times \ell(i)}$, satisfying the residue condition $\sum_{i,j} m_{i,j} \mathrm{res}_{p_i}(\xi_{i,j}) = 0$. The collection of such sections is denoted by $\mathcal{N}(\vec{m})$. Then for $\vec{\xi} \in \mathcal{N}(\vec{m})$, we say a parabolic Higgs bundle is *$\vec{\xi}$-parabolic* if the induced Higgs field on the j-th associated graded pieces of the filtration on E_{p_i} is the identity map scaled by $\xi_{i,j}$ (see Definition 2.5). When $\xi_{i,j} = 0$ for all i, j, they are usually known as strongly parabolic Higgs bundles. Therefore, $\vec{\xi}$-parabolic Higgs bundles can be seen as a variation of strongly parabolic Higgs bundles by deforming the nilpotent residue condition.

In our previous work [LL24], we construct a family of moduli space of (irregular) $\vec{\xi}$-parabolic Higgs bundles over $\mathcal{N}(\vec{m})$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{N}}} : \boldsymbol{\mathcal{H}}(\vec{m}) \to \mathcal{N}(\vec{m})$. Moreover, we show that this is a smooth family (see [LL24, Section 2]).[1] If it is clear from context, we simply write $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\vec{m})$ and $\mathcal{H}(\vec{m})_{\vec{\xi}}$ for the fiber of $\boldsymbol{\pi}_{\boldsymbol{\mathcal{N}}}$ at $\vec{\xi} \in \mathcal{N}(\vec{m})$.

When all values $\xi_{i,j}(p_i)$'s are distinct and non-zero for each $i \in I$, a condition that we refer to *generic*, Diaconescu–Donagi–Pantev [DDP18] established the spectral correspondence for $\vec{\xi}$-parabolic Higgs bundles. For simplicity, we often abuse to write $\xi_{i,j} = \xi_{i,j}(p_i)$. To elaborate, consider the total space of the projectivization of the twisted canonical line bundle $\pi : M := \mathbb{P}(K_C(D) \oplus \mathcal{O}_C) \to C$, and blow up M at each point $(p_i, \xi_{i,j})$ once. Since the $\xi_{i,j}$'s are distinct, the order of the blow-ups does not matter; for concreteness, we perform them in increasing lexicographic order of (i,j) starting from $(i,j) = (1,1)$. Denote the exceptional divisor appearing in the (i,j)-th blow-up by $E_{i,j}$. Let $Z_{\vec{\xi}}$ be the resulting surface and $p_{\vec{\xi}} : Z_{\vec{\xi}} \to M$ be the blow-up map. Finally, we remove the strict transform of the fibers M_{p_i} and the infinity section C_∞ from $Z_{\vec{\xi}}$, and the resulting open surface is denoted by $S_{\vec{\xi}}$.

To define the moduli of pure dimension one sheaves, consider the curve class $\Sigma(\vec{m})_{\vec{\xi}} := rp_{\vec{\xi}}^* C_0 - \sum_{i,j} m_{i,j} E_{i,j}$, where C_0 is the strict transform of the zero section of π. An effective curve $\Sigma \in |\Sigma(\vec{m})_{\vec{\xi}}|$ is characterized as a branched $r:1$ cover of C_0 lying in $S_{\underline{\xi}}$ that intersects each exceptional divisor $E_{i,j}$ exactly $m_{i,j}$ times. One can then define a moduli space $\mathcal{M}(\vec{m})_{\vec{\xi}} = \mathcal{M}(S_{\vec{\xi}}, \Sigma(\vec{m})_{\vec{\xi}})$ of pure dimension-one sheaves on $S_{\vec{\xi}}$ whose support is $\Sigma(\vec{m})_{\vec{\xi}}$, after choosing a suitable stability condition. Then the spectral correspondence in [DDP18] says that, for generic $\vec{\xi}$, there is an isomorphism of moduli spaces:

$$\mathcal{M}(\vec{m})_{\vec{\xi}} \cong \mathcal{H}(\vec{m})_{\vec{\xi}}.$$

On the other hand, there is also a spectral correspondence introduced by Su–Wang–Wen [SWW22a][SWW22b] for strongly parabolic Higgs bundles with integral spectral curves. This is in fact the extreme opposite of the case of generic $\vec{\xi}$ studied in [DDP18], since all $\xi_{i,j} = 0$. While the work of [SWW22b] is not formulated in terms of surfaces, one can still spot some similarities between the constructions of [DDP18] and [SWW22b] which involve some pattern of blow-ups. Hence, a natural question is to establish a family version of this spectral correspondence over the whole base $\mathcal{N}(\vec{m})$ which specializes to both of their constructions, and one of the main results of this article is to address this question.

Question 1.1. Is there a family of surfaces of holomorphic symplectic surfaces over $\mathcal{N}(\vec{m})$ such that the relative moduli spaces of sheaves on this family of surfaces is isomorphic to $\boldsymbol{\mathcal{H}}(\vec{m})$ over $\mathcal{N}(\vec{m})$?

Building on the work of [DDP18] for generic $\vec{\xi}$, we need to construct the right surfaces and curve classes for non-generic $\vec{\xi}$ that fit into a family over $\mathcal{N}(\vec{m})$. In this case, since the eigenvalues $\xi_{i,j}$ can be repeated, the blow-up process must be handled with care. A natural approach is to construct these objects as limits of those associated with generic $\vec{\xi}$. Therefore, the desired family of surfaces will be constructed as a sequence of blow-ups starting from the trivial family of surfaces. A key observation is that the order of blow-ups becomes crucial, and the choice of the curve class must be adjusted accordingly when we perform the blow-up in a family. To illustrate this idea, we assume $i = 1$ and simplify the notation by dropping the subscript. For instance, let $D = p$ and temporarily write $\vec{m} = \underline{m}_1 = \underline{m}$ and $\vec{\xi} = \underline{\xi}_1 = \underline{\xi}$.

We perform the blow-up construction globally by starting from the trivial family of surfaces, $\pi_{\mathcal{N}} : \boldsymbol{M} := M \times \mathcal{N}(\vec{m}) \to \mathcal{N}(\vec{m})$, and blowing up the ℓ tautological sections $\boldsymbol{\xi}_j := \{(p, \xi_j(p) \times$

[1] In this article, we only focus on the regular parabolic case where the divisor D is reduced.

$\underline{\xi} \mid \underline{\xi} \in \mathcal{N}(\vec{m})\}$ iteratively, as follows: First, blow up $\boldsymbol{\xi}_1$, and let $\boldsymbol{p}_1 : \boldsymbol{M}_1 \to \boldsymbol{M}$ be the corresponding blow-up map. Next, blow up $\boldsymbol{M}_1$ along the strict transform of $\boldsymbol{\xi}_2$, and denote the resulting map by $\boldsymbol{p}_2^1 : \boldsymbol{M}_2 \to \boldsymbol{M}_1$. We continue this construction iteratively, yielding the following chain of blow-ups:

$$\begin{tikzcd} \boldsymbol{Z} = \boldsymbol{M}_\ell \arrow[r, "\boldsymbol{p}_\ell^{\ell-1}"] & \boldsymbol{M}_{\ell-1} \arrow[r] & \cdots \arrow[r, "\boldsymbol{p}_2^1"] & \boldsymbol{M}_1 \arrow[r, "\boldsymbol{p}_1"] & \boldsymbol{M} \arrow[d, "\pi_{\mathcal{N}}"] \\ & & & & \mathcal{N}(\vec{m}) \end{tikzcd}$$

We denote the composition $\boldsymbol{p}_j^i \circ \boldsymbol{p}_k^j$ by $\boldsymbol{p}_k^i$. Also, let $\boldsymbol{E}_i$ be the exceptional divisor appearing at the i-th blow up. In the end, we remove the strict transform of $M_p \times \mathcal{N}(\vec{m})$ and $C_\infty \times \mathcal{N}(\vec{m})$ from $\boldsymbol{Z}$ to obtain a family of holomorphic symplectic surfaces $\boldsymbol{S} \subset \boldsymbol{Z}$.

To construct a relative moduli of sheaves, we choose the relative divisor class in $\boldsymbol{S}$,

$$\boldsymbol{\Sigma}(\underline{m}) = r\boldsymbol{f}^*\boldsymbol{C}_0 - \sum_{j=1}^{\ell} m_j \boldsymbol{\Xi}_j, \quad \text{where } \boldsymbol{\Xi}_{\boldsymbol{j}} = (\boldsymbol{p}_\ell^j)^* \boldsymbol{E}_j$$

By specializing the pair $(\boldsymbol{S}, \boldsymbol{\Sigma}(\underline{m}))$ to any (including non-generic) $\underline{\xi} \in \mathcal{N}(\vec{m})$, this yields the natural pair for establishing the corresponding spectral correspondence (see Figure 1 and Figure 2 for examples). By choosing a suitable stability condition and numerical data, one can construct a relative moduli space $\boldsymbol{\mathcal{M}}(\vec{m}) := \boldsymbol{\mathcal{M}}(\boldsymbol{S}, \boldsymbol{\Sigma}(\underline{m})) \to \mathcal{N}(\vec{m})$ of pure dimension one sheaves on $\boldsymbol{S}$ whose support is $\boldsymbol{\Sigma}(\underline{m})$.

In the general setup where $D = p_1 + \cdots + p_n$, the above construction of the holomorphic symplectic surface and the relative moduli space extends naturally. These are described in Sections 3.1 and 4.1, respectively. When the context is clear, we will simply denote this moduli space by $\boldsymbol{\mathcal{M}}$ and $\mathcal{M}(\vec{m})_{\vec{\xi}}$ for the fiber of $\boldsymbol{\mathcal{M}}(\vec{m}) \to \mathcal{N}(\vec{m})$ at $\vec{\xi} \in \mathcal{N}(\vec{m})$

Theorem 1.2 (Relative Spectral Correspondence, Theorem 4.2)**.** *There is an isomorphism of relative moduli spaces*

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}(\vec{m}) \arrow[rr, "\cong"] \arrow[dr] & & \boldsymbol{\mathcal{H}}(\vec{m}) \arrow[dl] \\ & \mathcal{N}(\vec{m}) & \end{tikzcd} \tag{1}$$

The strategy of the proof is to explicitly construct the relative morphisms $\boldsymbol{Q} : \boldsymbol{\mathcal{M}}(\vec{m}) \to \boldsymbol{\mathcal{H}}(\vec{m})$ and $\boldsymbol{R} : \boldsymbol{\mathcal{H}}(\vec{m}) \to \boldsymbol{\mathcal{M}}(\vec{m})$ and show that they are inverse of each other. Moreover, one can compare the spectral correspondences arising from different choices of partitions $\vec{m}$. This comparison behaves as expected, and we study the functoriality of the spectral correspondence in Section 4.3.

Furthermore, the spectral correspondence is compatible with two important structural morphisms on each side. On the Higgs side, for each $\vec{\xi} \in \mathcal{N}(\vec{m})$, there is the Hitchin map $h : \mathcal{H}(\vec{m})_{\vec{\xi}} \to A := \bigoplus_{\mu=1}^r H^0(C, K_C(D)^{\otimes \mu})$ that sends a $\underline{\xi}$-parabolic Higgs bundle to the characteristic polynomial of its Higgs field. Clearly, this cannot be surjective as there are constraints on the eigenvalues over each p_i. For strongly parabolic Higgs bundles ($\vec{\xi} = 0$), the image of the Hitchin map has been studied before by Baraglia–Kamgarpour [BK18] and Su–Wang–Wen [SWW22a]. Consider the subsheaves of $(K_C(D))^{\otimes \mu}$:

$$L(\vec{m})_\mu := (K_C(D))^{\otimes \mu} \otimes \mathcal{O}_C\left(-\sum_{i \in I} \gamma_{P^i}(\mu)\, p_i\right) \tag{2}$$

where $\gamma_{P^i}(\mu)$ is the level function associated to the partition $\underline{m}_i = P^i$ (we write P^i to emphasize its role as a partition) for $i \in I$. Then they show that the image of strongly parabolic Hitchin map is given by

$$A(\vec{m})_0 = \bigoplus_{\mu=1}^{r} H^0\left(C, L(\vec{m})_\mu\right) \subset A.$$

When $\underline{m}_i = (1, \ldots, 1)$ for all i, the image of the Hitchin map for $\vec{\xi} \neq 0$ can be described as the fibers of $A \to A/A(\vec{m})_0$, where each $\underline{\xi}_i$ is identified as a set of unordered eigenvalues which sits in $A/A(\vec{m})_0$ (see e.g. [Mar94][LM10]) However, this description does not generalize directly for partitions $\underline{m}_i \neq (1, \ldots, 1)$. Instead, we characterize the image of $\mathcal{H}(\vec{m})_{\vec{\xi}} \to A$ from the viewpoint of $\mathcal{M}(\vec{m})_{\vec{\xi}}$.

On the surface side, for each $\vec{\xi} \in \mathcal{N}(\vec{m})$, there is the Fitting support map $\mathcal{M}(\vec{m})_{\vec{\xi}} \to B(\vec{m})_{\vec{\xi}}$, which sends a pure dimension-one sheaf to its Fitting support. Here, $B(\vec{m})_{\vec{\xi}}$ is defined as the affine subspace of effective curves in $|\Sigma(\vec{m})_{\vec{\xi}}|$ lying in $S_{\vec{\xi}}$. In the relative setting, this extends to a family of base spaces $\boldsymbol{B}(\vec{m})$ over $\mathcal{N}(\vec{m})$, which gives rise to the relative support map

$$\boldsymbol{supp} : \boldsymbol{\mathcal{M}}(\vec{m}) \to \boldsymbol{B}(\vec{m})$$

To better understand the supports represented by curves in $B(\vec{m})_{\vec{\xi}}$, we provide a detailed characterization in terms of curves with prescribed multiplicities (determined by $\vec{m}$) at the centers (determined by $\vec{\xi}$) of the sequence of blow-ups (Section 3.2). In turn, this leads to another characterization of $B(\vec{m})_{\vec{\xi}}$ in terms of the Hitchin base A (Proposition 3.12), which will be used later in our study of non-emptiness and the Deligne-Simpson problem. This characterization gives rise to a map over $\mathcal{N}(\vec{m})$:

$$\boldsymbol{\iota} : \boldsymbol{B}(\vec{m}) \hookrightarrow \boldsymbol{A} := A \times \mathcal{N}(\vec{m}).$$

As a consistency check, we verify that the image $A(\vec{m})_0$ of the strongly parabolic Hitchin map described in [BK18][SWW22a] is compatible with the base $B(\vec{m})_0$ (Corollary 3.13).

Now, we can compare the two structural morphisms on both sides.

Corollary 1.3. The relative spectral correspondence (1) identifies the Hitchin map with the support map. In other words, there is a commutative diagram over $\mathcal{N}(\vec{m})$,

$$\begin{tikzcd}
\boldsymbol{\mathcal{M}}(\vec{m}) \arrow[r, "\cong"] \arrow[d, "\boldsymbol{supp}"'] & \boldsymbol{\mathcal{H}}(\vec{m}) \arrow[d, "\boldsymbol{h}"] \\
\boldsymbol{B}(\vec{m}) \arrow[r, hook, "\boldsymbol{\iota}"'] & \boldsymbol{A}
\end{tikzcd}$$

Hence, it is natural to refer to $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ as the *family of parabolic Hitchin bases associated to $\vec{m}$.*

1.3. **The *OK* conditions, flatness, non-emptiness.** Although the bases $B(\vec{m})_{\vec{\xi}}$ have been defined and characterized in various ways, its non-emptiness is not immediately clear from these descriptions. For example, the general question of whether there exists curves passing through certain points on a given surface with prescribed multiplicities is not a trivial question. Even if the non-emptiness is granted, it remains a priori unclear whether the dimension of $B(\vec{m})_{\vec{\xi}}$ behaves well as $\vec{\xi}$ varies, or whether it may jump in the non-generic case $\vec{\xi}$. Interestingly, these questions are closely related to the so-called *the "OK" condition* proposed by Balasubramanian–Distler–Donagi [BDD22] in the context of 6d superconformal field theories (SCFT).

In the study of 4d $\mathcal{N} = 2$ theories arising from compactification of 6d $(2, 0)$ SCFTs on a punctured Riemann surface (base curve), the Coulomb branch of the theory is described by the base of a Hitchin system defined on the base curve. In [BDD22], they consider the setup where the base curves vary over the Deligne-Mumford moduli space of stable pointed curves. In this setting, they show that the OK conditions ensure that the family of Hitchin bases (hence Coulomb branches) vary in a well-behaved manner, fitting together into a vector bundle over the moduli space of curves.

Definition 1.4. *We say that the collection of line bundles $L(\vec{m})_\mu$, $\mu = 2, \ldots, r$ satisfies the OK* condition *if*

$$H^1(C, L(\vec{m})_\mu) = 0. \tag{3}$$

If it is clear from the context, we simply say the OK condition holds.

While our base curve is fixed and the varying parameters are the eigenvalues, we also find that the OK condition guarantees the family of parabolic Hitchin bases to behave well in a family. It would be interesting to explore the connection between our work and theirs (also the more recent work [DH24]).

Proposition 1.5 (Lemma 3.11 and Proposition 3.14)**.** *Suppose that the expected dimension* $\operatorname{expdim}(B(\vec{m})_{\vec{\xi}}) \geq 0$ *and $B(\vec{m})_{\vec{\xi}}$ is non-empty for every $\vec{\xi} \in \mathcal{N}(\vec{m})$. If the OK condition holds, then the family of parabolic Hitchin bases $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ associated to $\vec{m}$ forms an affine bundle.*

In fact, the OK condition is also sufficient for the non-emptiness of $B(\vec{m})_{\vec{\xi}}$.

Theorem 1.6 (Theorem 5.13)**.** *Suppose that the OK condition holds. Then $B(\vec{m})_{\vec{\xi}} \neq \emptyset$ for any $\vec{\xi} \in \mathcal{N}(\vec{m})$. In particular, $B(\vec{m})_{\vec{\xi}} \neq \emptyset$ in the following cases:*

(1) When $n \geq 3$ and $g = 0$, if the inequalities

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 2 \tag{4}$$

hold for $\mu = 2, \ldots, r$.

(2) When $n \geq 1$ and $g = 1$, if at least one of the partitions P^i is not the singleton partition $m_1 = r$.

(3) When $n \geq 1$, $g \geq 2$.

Having established the non-emptiness of $B(\vec{m})_{\vec{\xi}}$, we can now use this result to deduce the non-emptiness of the moduli space $\mathcal{H}(\vec{m})_{\vec{\xi}}$. Specifically, to obtain an element in the $\mathcal{H}(\vec{m})_{\vec{\xi}}$, it suffices to find an integral curve Σ in $B(\vec{m})_{\vec{\xi}}$. By applying the spectral correspondence $\mathcal{M}(\vec{m})_{\vec{\xi}} \to \mathcal{H}(\vec{m})_{\vec{\xi}}$ to a line bundle on Σ, we obtain a stable $\vec{\xi}$-parabolic Higgs bundle in $\mathcal{H}(\vec{m})_{\vec{\xi}}$. Given the advantage of having the parabolic Hitchin bases $B(\vec{m})_{\vec{\xi}}$ in families, we begin by first constructing such an element for the strongly parabolic case, where $\vec{\xi} = 0$, and then deforming it to an integral curve in $B(\vec{m})_{\vec{\xi}}$ for other $\vec{\xi} \neq 0$. However, in order to guarantee the existence of integral curves, we need to slightly strengthen the OK conditions. See also Remark 5.18 for a variation of the following proposition.

Proposition 1.7 (Proposition 5.17)**.** *Let $\vec{\xi} \in \mathcal{N}(\vec{m})$. Then the moduli space of stable $\vec{\xi}$-parabolic Higgs bundle $\mathcal{H}(\vec{m})_{\vec{\xi}}$ is non-empty in the following cases*

(1) When $n \geq 3$ and $g = 0$, if the inequalities

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 1 \tag{5}$$

hold for $\mu = 2, \ldots, r$

(2) When $n \geq 2$ and $g = 1$, if at least two of the partitions P^i are not the singleton partition $m_{i,1} = r$.

(3) When $n \geq 1$ and $g \geq 2$.

1.4. **Multiplicative Deligne–Simpson problem.** As another application of the non-emptiness of $B(\vec{m})_{\vec{\xi}}$, we obtain a new approach to the classical multiplicative Deligne–Simpson problem (DSP for short), which asks the following question:

Question 1.8. Given conjugacy classes C_i in $GL_r(\mathbb{C})$ for $i = 1, \ldots, n$, when does there exist irreducible solutions to the equation $T_1 \cdots T_n = Id_r$ with $T_i \in C_i$? Here, an irreducible solution means that the matrices T_j have no common invariant subspace.

This problem has been studied by many authors, starting from Simpson, who first obtained a sufficient and necessary condition, which we recall in Appendix B, in the case when one of the C_i is regular i.e. semisimple with distinct eigenvalues [Sim91]. Katz later studied the rigid case by middle convolution [Kat96]. A sufficient and necessary condition in the case of multiplicatively generic eigenvalues (see Definition 5.28) was given by Kostov (see [Kos04] for a survey). More generally, Crawley-Boevey and Shaw gave a sufficient (conjecturally necessary) condition in terms of quivers without the genericity assumption on the eigenvalues via representations of certain multiplicative preprojective algebra [CS06].

Note that the DSP can also be reformulated as finding an irreducible local system on $\mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}$ whose monodromy transformation T_i around the puncture p_i has the prescribed conjugacy class C_i for $i = 1, \ldots, n$. So, a natural generalization of DSP is to replace $\mathbb{P}^1$ by curves of higher genus. This higher genus analogue of DSP has been addressed in the work of Hausel–Letellier–Rodriguez–Villegas [HLR13, Section 5.2] [Let15, Corollary 3.15] for multiplicatively generic eigenvalues (see Definition 5.28), based on the quiver-theoretic approach of Crawley-Boevey and Shaw.

In [BDD22, Appendix B], it is conjectured that the OK condition is sufficient for solving the DSP ($g = 0$) and the authors verify that the OK condition is numerically equivalent to Simpson's criterion in the special case when one of the C_i is semisimple with distinct eigenvalues. For completeness, we include a direct combinatorial proof of this numerical equivalence in Appendix B. However, the conjecture is expected to hold more generally without the assumption on one of the C_i. Moreover, another challenge is to understand how the OK condition leads to actual solutions of the DSP from a geometric viewpoint. In our approach, the key observation is that the OK condition guarantees the non-emptiness of $B(\vec{m})_{\vec{\xi}}$, which in turn provides the necessary geometric input to construct such solutions.

If we view the DSP as the problem of finding irreducible local systems on $\mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}$ with prescribed monodromies, the tame non-abelian Hodge correspondence (tame NAHC) developed by Simpson [Sim90] converts the DSP into the existence problem of stable parabolic Higgs bundles of parabolic degree 0 with prescribed residues data of the Higgs field. In fact, the original approach of the DSP by Simpson in [Sim91] proceeds by constructing parabolic Higgs bundles explicitly through systems of Hodge bundles. We proceed by constructing parabolic Higgs bundles through spectral correspondence.

As in our approach to the non-emptiness of $\mathcal{H}(\vec{m})_{\vec{\xi}}$, whenever $B(\vec{m})_{\vec{\xi}}$ is non-empty and contains at least an integral member $\Sigma \subset S_{\vec{\xi}}$, pushing a line bundle over Σ forward to the base curve C yields a stable parabolic Higgs bundle on C. Crucially, the geometry of the surface $S_{\vec{\xi}}$ together with the choice of curve class defining $B(\vec{m})_{\vec{\xi}}$ uniquely determines the conjugacy classes of the residues of the Higgs field at the marked points p_i. This is the necessary step to ensure that the corresponding local system has the prescribed conjugacy classes of monodromies. The local analysis of the conjugacy classes is studied in Lemma A.3. However, the construction of parabolic structures in this case differs from that of the relative spectral correspondence (Theorem 1.2) and some technical modification are required to achieve the desired conjugacy classes.

Following this approach, we provide a complete answer to the question posed in [BDD22]: we show that the OK condition (3) is sufficient for solving the DSP under some genericity assumptions on the eigenvalues of the conjugacy classes (Theorem 5.30 and Remark 5.31). Moreover, by slightly strengthening the OK condition, we are able to remove the genericity assumption (Theorem 5.26). Finally, aside from a few exceptional cases in $g = 1$, this approach also recovers the result in [HLR13][Let15] concerning the higher genus analogue of DSP (Theorem 5.27) for $g \geq 1$. As the statement involves further background from the tame non-abelian Hodge correspondence, we refer the reader to Section 5.4 for the precise formulation.

Remark 1.9. The approach to solving the DSP in fact can be used to show the existence of stable filtered local systems with generic eigenvalues and parabolic weights. This implies that the OK condition is also sufficient for the existence of stable filtered local systems with prescribed residue diagrams.

Remark 1.10. A spectral correspondence approach to the DSP when all the eigenvalues are zero was previously proposed by Wen [Wen21], building on the spectral correspondence developed in [SWW22a].

Further Directions.

(1) (Kostov's work) In the case when one of the conjugacy classes has distinct eigenvalues, Simpson's criterion (and hence the OK condition) provides a sufficient and necessary for the solvability of DSP. However, in the more general setting of multiplicatively generic eigenvalues, Kostov gives a full necessary and sufficient criterion for DSP, in which the OK condition arises as a special case (Remark 5.32). So, additional input is needed to solve DSP in this broader setting. We plan to explore this direction in future work.
(2) (Irregular DSP) The original parabolic spectral correspondence in [DDP18] is established for irregular parabolic Higgs bundles. By suitably generalizing their spectral correspondence, we expect our approach here should also provide solutions to the irregular version of DSP which addresses the existence of Stokes data with prescribed local data, as posed by Boalch in [Boa14, Section 9.4].

1.5. **Notations.**

- C: a smooth projective curve of genus $g \geq 0$.
- $D = p_1 + \cdots + p_n$: a reduced effective divisor with $n \geq 1$.
- r is used to denote the rank of vector bundles.
- Parabolic data:

– $\vec{m} = (\underline{m}_1, \cdots, \underline{m}_n)$ where each component $\underline{m}_i = (m_{i,1}, \cdots, m_{i,\ell(i)})$ is a partition of r. In other words,

$$m_{i,1} \geq m_{i,2} \geq \cdots \geq m_{i,\ell(i)} \quad \text{and } \underline{m}_i = (m_{i,1}, \cdots, m_{i,\ell(i)})$$

When we wish to emphasize its role as a partition instead of a vector, we will write $P^i = \underline{m}_i$.

– $\vec{\xi} = (\underline{\xi}_1, \cdots, \underline{\xi}_n)$ where each component $\underline{\xi}_i = (\xi_{i,1}, \cdots, \xi_{i,\ell(i)})$ is a collection of eigenvalues.

- (Distinct part of $\vec{\xi}$) The entries in $\underline{\xi}_i$ might be repeated, we define the distinct part $\underline{\xi}_i^\circ = (\xi_{i,1}^\circ, \cdots, \xi_{i,e(i)}^\circ)$ to be the tuple of distinct entries in $\underline{\xi}_i$, listed in the order of their first appearance.
- (Partitions and Young diagrams) We often represent a partition $P = (m_1, \ldots, m_\ell)$ using the Young diagram. We call ℓ the number of rows in the Young diagram, m_1 the number of columns, and $|P| = r = \sum_j m_j$ the number of boxes in the Young diagram.
- (Conjugate partition) To each partition $P = (m_1, \ldots, m_\ell)$, we define the conjugate partition $\hat{P} := (n_1, \ldots, n_{m_1})$ by reflecting its Young diagram across the main diagonal. More formally, $n_j = \#\{k | m_k \geq j\}$ for $j = 1, \ldots, m_1$.
- (Level functions) To each Young diagram of P, we can write down two fillings of the boxes: (1) number the boxes in strictly increasing order starting from the top-left corner with 1, then proceed from top to bottom within each column and from left to right across columns (2) number the boxes in the j-th column by j. For example,

$$(1) \quad \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \qquad (2) \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

By pairing the numbers in the boxes (of the same position) of the two fillings, we define the level function $\gamma_P : \{1, \ldots, r\} \to \{1, \ldots, m_1\}, j \mapsto \gamma_P(j)$ associated to the Young diagram. In the example, we have

$$\gamma_P(1) = \gamma_P(2) = \gamma_P(3) = 1, \quad \gamma_P(4) = \gamma_P(5) = 2, \quad \gamma_P(6) = \gamma_P(7) = 3, \quad \gamma_P(8) = 4.$$

- (Level domain) Given a partition $P = (m_1, \ldots, m_\ell)$, we denote by

$$G(P) = \{(u, a) \in \mathbb{Z}^2 | 0 \leq u < |P|, 0 \leq a < \gamma_P(|P| - u)\}$$

We call $G(P)$ the level domain associated to the partition P.
- (Union of partitions) Given partitions P^1 and P^2, we define the union of the partitions $P^1 \cup P^2$ by combining all the parts of both partitions and arranging them in non-increasing order. For example, $P^1 = (2, 1), P^2 = (4, 3, 1, 1, 1)$ then $P^1 \cup P^2 = (4, 3, 2, 1, 1, 1, 1)$.
- For each i, the parabolic data $(\underline{m}_i, \underline{\xi}_i)$ determines a decomposition of the partition P^i into subpartitions indexed by $\underline{\xi}_i^\circ$. For each $\xi_{i,j}^\circ \in \underline{\xi}_i^\circ$, we define $P^{\xi_{i,j}^\circ}$ to be the subpartition of P^i consisting of the collection of $m_{i,k}$'s such that $\xi_{i,k} = \xi_{i,j}^\circ$. Then we have

$$P^i = P^{\xi_{i,1}^\circ} \cup \cdots \cup P^{\xi_{i,e(i)}^\circ}$$

- (Example) Suppose we have a single marked point (so $n = 1$). Let $\ell(1) = 6$ and define

$$\vec{m} = \underline{m} = (3, 2, 2, 1, 1, 1), \quad \vec{\xi} = \underline{\xi} = (\xi_1, \xi_1, \xi_2, \xi_3, \xi_2, \xi_1).$$

We also write $P = \underline{m} = (3, 2, 2, 1, 1, 1)$. The *distinct part* of $\underline{\xi}$ is

$$\underline{\xi}^\circ = (\xi_1^\circ, \xi_2^\circ, \xi_3^\circ) = (\xi_1, \xi_2, \xi_3),$$

So, $e(1) = 3$. The subpartitions of P corresponding to each $\xi_j^\circ \in \underline{\xi}^\circ$:
- $P^{\xi_1^\circ} = (m_1, m_2, m_6) = (3, 2, 1)$
- $P^{\xi_2^\circ} = (m_3, m_5) = (2, 1)$
- $P^{\xi_3^\circ} = (m_4) = (1)$

Then the full partition P is decomposed as:

$$P = P^{\xi_1^\circ} \cup P^{\xi_2^\circ} \cup P^{\xi_3^\circ}$$

The level domain $P^{\xi_1^\circ}$ associated to the partition $P^{\xi_1^\circ} = (3, 2, 1)$ is visualized as

$a = 2$	$\bullet$					
$a = 1$	$\bullet$	$\bullet$	$\bullet$			
$a = 0$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	$u = 0$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$

where each dot corresponds to a pair $(u, a) \in G(P^{\xi_i^\circ})$.

2. Moduli spaces of parabolic Higgs bundles

2.1. **Parabolic Higgs bundles.** Let C be a smooth projective curve of genus $g \geq 0$ and $D = p_1 + \cdots + p_n$ be a reduced effective divisor with $n \geq 1$. Throughout this paper, we reserve the letter r (resp. d) the rank (resp. degree) of Higgs bundles. Let $I = \{1, \cdots, n\}$ be the index set of marked points and for each point p_i, we denote the associated partition of r by $\underline{m}_i = (m_{i,1}, \cdots, m_{i,\ell(i)})$. We use poly-multivector notation and denote the collection of partitions of r by $\vec{m} = (\underline{m}_1, \cdots, \underline{m}_n)$. We write the index set of such partition by $J_i = \{1, \cdots, \ell(i)\}$ for $i \in I$.

Definition 2.1. *A parabolic Higgs bundle on C of type $\vec{m}$ with poles at D is a quadraple $(E, E_D^\bullet, \Phi, \vec{\alpha})$ where*

(1) A Higgs bundle (E, Φ) where $\Phi : E \to E \otimes K_C(D)$.

(2) A quasi-parabolic structure of type $\underline{m}_i$ at each p_i:

$$E_{p_i}^\bullet : 0 = E_{p_i}^{\ell(i)} \subset E_{p_i}^{\ell(i)-1} \subset ... \subset E_{p_i}^1 \subset E_{p_i}^0 = E_{p_i}$$

such that $\Phi_{p_i}(E_{p_i}^j) \subset E_{p_i}^j \otimes_D K_C(D)_{p_i}$ and $dim(E_{p_i}^{j-1}/E_{p_i}^j) = m_{i,j}$. We simply refer this condition as Φ_D preserving $E_D^\bullet$, where $E_D^\bullet$ is denoted as a collection of filtrations $E_{p_i}^\bullet$ for all $i \in I$.

(3) A collection of parabolic weights $\vec{\alpha} = (\underline{\alpha}_1, \cdots, \underline{\alpha}_n)$ where $\underline{\alpha}_i = (\alpha_{i,1}, ..., \alpha_{i,\ell(i)}) \in \mathbb{Q}^{\ell(i)}$:

$$1 > \alpha_{i,\ell(i)} > \alpha_{i,\ell(i)-1} > ... > \alpha_{i,1} \geq 0$$

for each $i \in I$.

When $\Phi_{p_i}(E_{p_i}^{j-1}) \subset E_{p_i}^j \otimes_D K_C(D)_{p_i}$ for all i, j in condition (2), we call it a *strongly parabolic Higgs bundle.*

Remark 2.2. We reverse the filtration indices used in our previous paper [LL24]. This modification aligns with the spectral correspondence argument in [DDP18] for the relative setting.

Recall that in [MY92, Section 1] and [Yok93, Section 1], the parabolic Euler characteristic and the (reduced) parabolic Hilbert polynomial of $E_* := (E, E_D^\bullet, \vec{\alpha})$ are defined as

$$\text{par-}\chi(E_*) = \chi(E) + \sum_{i=1}^{n} \sum_{j=1}^{\ell(i)} \alpha_{i,j} \chi(E_{p_i}^{j-1}/E_{p_i}^j), \qquad \text{par-}P_{E_*}(t) = \frac{\text{par-}\chi(E_*(t))}{\text{rank}(E)}$$

where $E_{p_i}^{j-1}/E_{p_i}^{j}$ is viewed as a torsion sheaf on the curve C in the expression and $E_*(t) = (E\otimes\mathcal{O}(t), E_D^\bullet\otimes\mathcal{O}(t), \vec{\alpha})$.

Definition 2.3. *A parabolic Higgs bundle (E_*,Φ) is said to be $\vec{\alpha}$-(semi)stable if for any nontrivial proper subbundle $0\subset F\subset E$ preserved by Φ, we have*

$$\text{par-}P_{F_*}(t) < \text{par-}P_{E_*}(t) \quad \text{for } t\gg 0 \quad (\text{resp. } \le)$$

where $F_* = (F, F_D^\bullet, \vec{\alpha})$ is defined by the induced filtration $F_D^\bullet = F_D\cap E_D^\bullet$ that is preserved by Φ_D.

Remark 2.4. In the literature, stability of parabolic Higgs bundles is often in terms of slope by replacing degree with the parabolic degree pardeg $:= \deg(E) + \alpha_{i,j}\sum\dim(E_{p_i}^{j-1}/E_{p_i}^{j})$. It can be checked that the two definitions are equivalent.

Due to the work of Yokogawa [Yok93, Theorem 4.6], there exists a coarse moduli space for $\vec{\alpha}$-stable parabolic Higgs bundles of rank r, degree d such that $\chi(E_{p_i}^{j-1}/E_{p_i}^{j}) = m_{i,j}$ for all $i\in I, j\in J_i$. We will denote this moduli space by $\mathrm{Higgs}^{\mathrm{par}}(\vec{m})$. Moreover, one can define the coarse moduli space of $\vec{\alpha}$-stable strongly parabolic Higgs bundles as a closed subscheme in $\mathrm{Higgs}^{\mathrm{par}}(\vec{m})$, which we denote by $\mathrm{Higgs}^{\mathrm{s-par}}(\vec{m})$.

One can fix the polar part Φ_D of the Higgs fields by choosing a collection of sections $\underline{\xi}_i = (\underline{\xi}_{i,1},\cdots,\underline{\xi}_{i,\ell(i)}) \in H^0(p_i, K_C(D)_{p_i})^{\times\ell(i)}$ for $i\in I$ and denote by $\vec{\xi} = (\underline{\xi}_1,\ldots,\underline{\xi}_n)$ the collection of such sections. We have the residue maps

$$\mathrm{res}_{p_i}: H^0(p_i, K_C(D)_{p_i}) \to H^0(D, K_C(D)|_D) \to H^1(C, K_C)\cong\mathbb{C}$$

We define the following:

$$\mathcal{N}(\vec{m}) = \mathcal{N}(\vec{m}, D) := \left\{ \vec{\xi} = (\underline{\xi}_i)_{i\in I} \,\middle|\, \xi_{i,j}\in H^0(p_i, M_{p_i}), \sum_{i=1}^{n}\sum_{j=1}^{\ell(i)} \mathrm{res}_{p_i}(\xi_{i,j}) = 0 \right\}$$

Definition 2.5. *[DDP18] For $\vec{\xi}\in\mathcal{N}(\vec{m})$, we call a parabolic Higgs bundle $(E, E_D^\bullet, \Phi, \vec{\alpha})$ a (regular) $\vec{\xi}$-parabolic Higgs bundle if the induced morphism of $\mathcal{O}_{p_i}$-modules $gr_j\Phi_{p_i} := \Phi_{p_i,j}: E_{p_i}^{j-1}/E_{p_i}^{j} \to E_{p_i}^{j-1}/E_{p_i}^{j}\otimes K_C(D)_{p_i}$ satisfies the following condition*

$$\Phi_{p_i,j} = \mathrm{Id}_{E_{p_i}^{j-1}/E_{p_i}^{j}}\otimes\xi_{i,j}, \quad 1\le j\le\ell(i), \quad i\in I. \tag{6}$$

Note that a $\vec{\xi}$-parabolic Higgs bundle $(E, E_D^\bullet, \Phi, \vec{\alpha})$ induces a rank one Higgs bundle $(\det(E), \Phi_0)$ where $\Phi_0 := \Phi\wedge\cdots\wedge\mathrm{Id} + \cdots + \mathrm{Id}\wedge\cdots\wedge\Phi \in H^0(C, K_C(D))$. By restricting Φ_0 to D, we see that the identity $\sum_{i=1}^{n}\sum_{j=1}^{\ell(i)}\mathrm{res}_{p_i}(\xi_{i,j}) = 0$ must hold. A priori, each $\xi_{i,j}\in H^0(p_i, K_C(D)_{p_i})$ is a section. When the context is clear, we will, by slight abuse of notation, identify $\xi_{i,j}$ with its residue $\mathrm{res}(\xi_{i,j})\in\mathbb{C}$.

In [LL24], we construct a relative moduli space of $\vec{\xi}$-parabolic Higgs bundles on C.

Theorem 2.6. *[LL24] Fix the numerical data: a rank $r\ge 1$, a degree $d\in\mathbb{Z}$ and a collection of partitions $\vec{m}$ with a parabolic structure $\vec{\alpha}$. There exists a relative coarse moduli scheme $\boldsymbol{\pi}_{\mathcal{N}}: \boldsymbol{\mathcal{H}}(C, D; r, d, \vec{\alpha}, \vec{m}) \to \mathcal{N}(\vec{m})$. In fact, every fiber of $\boldsymbol{\pi}_{\mathcal{N}}$ is smooth.*

If it is clear from the context, we abbreviate to write this moduli space as $\boldsymbol{\mathcal{H}}(\vec{m})$.

Examples 2.7. When $n = 1$ and $\underline{m}_1$ is the trivial flag ($\underline{m} = (r)$), the base $\mathcal{N}(\vec{m}) = \{0\}$ so that $\boldsymbol{\mathcal{H}}$ consists of stable holomorphic Higgs bundles (E,Φ).

Examples 2.8. When $\vec{m} = \vec{1}$, meaning that each $\underline{m_i}$ is the full flag, the moduli space $\boldsymbol{\mathcal{H}}$ becomes isomorphic to the moduli space of parabolic Higgs bundles $\mathrm{Higgs}^{\mathrm{par}}(\vec{1})$ [LL24, Proposition 2.8].

Remark 2.9. For a fixed partition $\vec{m}$, there is a morphism $G(\vec{m}) : \boldsymbol{\mathcal{H}}(\vec{m}) \to \mathrm{Higgs}^{\mathrm{par}}(\vec{m})$ defined to be the composition

$$G(\vec{m}) : \boldsymbol{\mathcal{H}}(\vec{m}) \hookrightarrow \mathrm{Higgs}^{\mathrm{par}}(\vec{m}) \times \mathcal{N}(\vec{m}) \twoheadrightarrow \mathrm{Higgs}^{\mathrm{par}}(\vec{m})$$

where the first morphism is a canonical inclusion and the second morphism is the projection. In general, the morphism $G(\vec{m})$ is not surjective because the block diagonal part of a Higgs field Φ_D over D may not be diagonal.

With respect to the partial order on the set of partitions of r, one can build a tower of the relative coarse moduli schemes. For the moment, suppose that $n = 1$ and we simply write $\vec{m} = (\underline{m})$. Consider two partitions $\underline{m} = (m_1, \cdots, m_\ell)$ and $\underline{m}' = (m'_1, \cdots, m'_{\ell'})$ of r. We say $\underline{m}'$ is a refinement of $\underline{m}$, denoted by $\underline{m}' \geq \underline{m}$, if there exists an increasing sequence of integers $0 = a_0 \leq a_1 \leq \cdots \leq a_\ell = \ell'$ such that $m_i = \sum_{a_{j-1} < i \leq a_j} m'_i$ for all $j = 1, \cdots, \ell$. For such a pair $\underline{m}' \geq \underline{m}$, there is a canonical inclusion

$$\begin{aligned} &\iota_{\underline{m}}^{\underline{m}'} : \mathcal{N}(\underline{m}) \hookrightarrow \mathcal{N}(\underline{m}') \\ &(\xi_1, \cdots, \xi_l) \mapsto (\underbrace{\xi_1, \ldots, \xi_1}_{a_1-\text{times}}, \underbrace{\xi_2, \ldots, \xi_2}_{a_2-a_1\text{times}}, \ldots, \underbrace{\xi_l, \ldots, \xi_\ell}_{a_\ell-a_{\ell-1}-\text{times}}) \end{aligned} \tag{7}$$

More generally, for two collections of partitions of r, $\vec{m} = (\underline{m}_1, \cdots, \underline{m}_n)$ and $\vec{m}' = (\underline{m}'_1, \cdots, \underline{m}'_n)$, we say $\vec{m}'$ is a refinement of $\vec{m}$ if $\underline{m}'_i \geq \underline{m}_i$ for all $i \in I$. There is a canonical inclusion

$$\iota_{\vec{m}}^{\vec{m}'} : \mathcal{N}(\vec{m}) \hookrightarrow \mathcal{N}(\vec{m}')$$

induced by (7).

Even though one chooses generic parabolic weights, there is no morphism between $\boldsymbol{\mathcal{H}}(\vec{m})$ and $\boldsymbol{\mathcal{H}}(\vec{m}')$ in general. Consider a closed subspace of $\boldsymbol{\mathcal{H}}(\vec{m}')$ consisting of $(E, E_D^\bullet, \Phi)$ such that the induced Higgs field Φ on $E_{p_i}^{a_{i,j-1}}/E_{p_i}^{a_{i,j}}$ is $\xi_{i,j} \otimes \mathrm{Id}$ for a given $\vec{\xi} \in \mathcal{N}(\vec{m})$. We denote it by $\boldsymbol{\mathcal{H}}(\vec{m}', \vec{m})$. Then there is a canonical forgetful map

$$\begin{tikzcd} \boldsymbol{\mathcal{H}}(\vec{m}') & \\ \boldsymbol{\mathcal{H}}(\vec{m}', \vec{m}) \arrow[u, hook] \arrow[r, "\mathbf{For}_{\vec{m}'}^{\vec{m}}"] & \boldsymbol{\mathcal{H}}(\vec{m}) \end{tikzcd} \tag{8}$$

over $\mathcal{N}(\vec{m})$. One can also see that over each $\vec{\xi}$, the fiber of this forgetful map is given by a product of flag varieties.

For later use, we introduce a relative moduli space of $\vec{\xi}$-meromorphic Higgs bundles, denoted by $\boldsymbol{\mathcal{H}}^{mero}(\vec{m})$. The definition is similar to the relative moduli scheme $\boldsymbol{\mathcal{H}}(\vec{m})$, but without imposing the (quasi-)parabolic structures. We say a Higgs bundle (E, Φ) is $\vec{\xi}$-*meromorphic* if there exists a quasi-parabolic structure $E_D^\bullet$ of type $\vec{m}$ such that the $\vec{\xi}$-parabolic condition (6) holds. A stability condition is given by the reduced Hilbert polynomial. Then by the similar argument for the proof of Theorem 2.6, one can show that there exists a relative coarse moduli scheme $\boldsymbol{\pi}_{\mathcal{N}} : \boldsymbol{\mathcal{H}}^{mero}(\vec{m}) \to \mathcal{N}(\vec{m})$.

Remark 2.10. In general, the $\vec{\xi}$-meromorphic condition is not same as the condition that the eigenvalues of Φ at D is $\vec{\xi}$ with multiplicities $\vec{m}$, up to reordering. This is because, we regard each $\underline{\xi}_i$ as an ordered set of eigenvalues. Therefore, similar to Remark 2.9, we have a morphism

$$G^{mero}(\vec{m}) : \boldsymbol{\mathcal{H}}^{mero}(\vec{m}) \hookrightarrow \mathrm{Higgs}^{mero}(\vec{m}) \times \mathcal{N}(\vec{m}) \twoheadrightarrow \mathrm{Higgs}^{mero}(\vec{m})$$

where Higgs^{mero} is a moduli space of meromorphic Higgs bundles. Note that this morphism is not surjective, except in the full flag case, where it is in fact an isomorphism.

By construction, for a suitable choice of parabolic weights, there exists a canonical forgetful map

$$\boldsymbol{For} : \boldsymbol{\mathcal{H}}(\vec{m}) \to \boldsymbol{\mathcal{H}}^{mero}(\vec{m})$$

over $\mathcal{N}(\vec{m})$. In particular, this is compatible with the forgetful maps in (8).

2.2. **Parabolic Hitchin maps and the OK conditions.** The Hitchin morphisms for the moduli of parabolic Higgs bundles have been considered before [Yok93], [LM10], [BK18], [SWW22a]. Define the Hitchin base

$$A = \bigoplus_{\mu=1}^{r} H^0(C, (K_C(D))^{\otimes \mu}). \tag{9}$$

The Hitchin map $h(\vec{m})^{\mathrm{par}} : \mathrm{Higgs}^{\mathrm{par}}(\vec{m}) \to A$ is defined by sending a parabolic Higgs bundle to the coefficients of the characteristic polynomial of the underlying Higgs bundle with coefficients in $K_C(D)$.

As in the introduction, consider the affine subspace

$$A(\vec{m})_0 = \bigoplus_{\mu=1}^{r} H^0\left(C, L(\vec{m})_\mu\right) \subset A.$$

where $L(\vec{m})_\mu$ is defined in (2). Recall that the set of line bundles $L(\vec{m})_\mu$, $\mu = 2, \ldots, r$ satisfies the *OK condition* if $H^1(C, L(\vec{m})_\mu) = 0$ for all $2 \leq \mu \leq r$. By Serre duality, one can see that this condition holds

(1) when $g = 0, n \geq 3$, $\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 2$;
(2) when $g \geq 1, n \geq 1$, except in the case when $g = 1$ and $\gamma_{P_i}(\mu) = \mu$ for all i .

Under this assumption, it is shown in the work of [BK18], [SWW22a] that the image of $\mathrm{Higgs}^{\mathrm{s-par}}$ under the Hitchin map is $A(\vec{m})_0$. We will denote the corresponding Hitchin map by $h(\vec{m})^{\mathrm{s-par}} : \mathrm{Higgs}^{\mathrm{s-par}}(\vec{m}) \to A(\vec{m})_0$.

Combining our previous discussion, we get the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{\mathcal{H}}(\vec{m}) \arrow[r, "G(\vec{m})"] \arrow[dd] & \mathrm{Higgs}^{\mathrm{par}}(\vec{m}) \arrow[d, "h^{\mathrm{par}}"] \\
 & A \arrow[d, two heads, "\mathrm{pr}_{\vec{m}}"] \\
\mathcal{N}(\vec{m}) \arrow[d, "\iota_{\vec{m}}"] & A/A(\vec{m})_0 \arrow[d, two heads, "\mathrm{pr}^{\vec{1}}_{\vec{m}}"] \\
\mathcal{N}(\vec{1}) \arrow[r, "\cong"] & A/A(\vec{1})_0
\end{tikzcd}$$

where pr's are the canonical projections and the the bottom isomorphism follows from the OK condition. For later use, we denote by $\mathrm{pr}_{\vec{1}}$ the composition $\mathrm{pr}^{\vec{1}}_{\vec{m}} \circ \mathrm{pr}_{\vec{m}}$.

3. Family of surfaces

3.1. **Construction of holomorphic symplectic surfaces.** The goal of this section is to introduce a family of surfaces associated to the data $(C, D, \vec{m}, \vec{\xi})$. Consider the projectivization $M := \mathbb{P}(K_C(D) \oplus \mathcal{O}_C)$ of the twisted canonical line bundle $K_C(D)$. We write $M^\circ = \mathrm{Tot}(K_C(D))$ for the total space of the twisted line bundle.[2] We will use the bold face to indicate the objects in families. For example, let $\boldsymbol{M} := M \times \mathcal{N}(\vec{m})$ (resp. $\boldsymbol{C} := C \times \mathcal{N}(\vec{m})$) be the trivial family of the projective surfaces (resp. the base curves) over $\mathcal{N}(\vec{m})$. We denote the projection to $\mathcal{N}(\vec{m})$ by $\boldsymbol{\pi}_{\mathcal{N}}$. We also write $\boldsymbol{\pi_C} : \boldsymbol{M} \to \boldsymbol{C}$ for the canonical projection over $\mathcal{N}(\vec{m})$. There are tautological sections $\{\boldsymbol{\xi}_{i,j} = \xi_{i,j}(p_i) \times \mathcal{N}(\vec{m}) | i \in I, j \in J_i\}$ of $\boldsymbol{\pi}_{\mathcal{N}}$. We often abuse notation by writing the image of the section $\xi_{i,j}$ at p_i simply as $\xi_{i,j}$. These are non-singular codimension two subschemes in $\boldsymbol{M}$ that are flat over $\mathcal{N}(\vec{m})$ whose fiber over $\vec{\xi}$ is $(\xi_{i,j}, \vec{\xi}) \in \boldsymbol{M}$. Then we construct a family of projective surfaces over $\mathcal{N}(\vec{m})$ as follows.

We begin with $\underline{\boldsymbol{\xi}_1} = (\boldsymbol{\xi}_{1,1}, \cdots, \boldsymbol{\xi}_{1,\ell(1)})$. Blow up $\boldsymbol{M}$ along the subscheme $\boldsymbol{\xi}_{1,1}$. Denote the resulting blow-up by $\boldsymbol{p}_{1,1} : \boldsymbol{M}_{1,1} \to \boldsymbol{M}$ and the exceptional divisor by $\boldsymbol{E}_{1,1}$. Next, we take the strict transform of $\boldsymbol{\xi}_{1,2}$ in $\boldsymbol{M}_{1,1}$ which we denote by $\boldsymbol{\xi}_{1,2}$ as well. Then we further blow up $\boldsymbol{M}_{1,1}$ along $\boldsymbol{\xi}_{1,2}$. Denote the resulting blow-up by $\boldsymbol{p}^{1,1}_{1,2} : \boldsymbol{M}_{1,2} \to \boldsymbol{M}_{1,1}$ and the exceptional divisor by $\boldsymbol{E}_{1,2}$. Let $\boldsymbol{p}_{1,2} = \boldsymbol{p}^{1,1}_{1,2} \circ \boldsymbol{p}_{1,1}$. We proceed this iteratively until the $\ell(1)$-th step and obtain the following sequence of blow-ups

$$\begin{tikzcd}
\boldsymbol{Z}_1 := \boldsymbol{M}_{1,\ell(1)} \arrow[r, "\boldsymbol{p}^{1,\ell(1)-1}_{1,\ell(1)}"] \arrow[drrr, "\boldsymbol{p}_{1,\ell(1)}"'] & \cdots \arrow[r, "\boldsymbol{p}^{1,2}_{1,3}"] & \boldsymbol{M}_{1,2} \arrow[r, "\boldsymbol{p}^{1,1}_{1,2}"] \arrow[dr, "\boldsymbol{p}_{1,2}"] & \boldsymbol{M}_{1,1} \arrow[d, "\boldsymbol{p}_{1,1}"] \\
 & & & \boldsymbol{M}
\end{tikzcd}$$

Next, we apply the same construction for $\underline{\boldsymbol{\xi}_2}$. Since $p_1 \neq p_2 \in C$, the strict transform of $\boldsymbol{\xi}_{2,j}$ along $\boldsymbol{p}_{1,\ell(1)}$ is the same with the pullback. Thus, we abuse to denote $\boldsymbol{\xi}_{2,j}$ its inverse image, and apply the same construction. Then we get a sequence of blow-ups

$$\begin{tikzcd}
\boldsymbol{Z}_2 := \boldsymbol{M}_{2,\ell(2)} \arrow[r, "\boldsymbol{p}^{2,\ell(2)-1}_{2,\ell(2)}"] \arrow[drrr, "\boldsymbol{p}^{1,\ell(1)}_{2,\ell(2)}"'] & \cdots \arrow[r, "\boldsymbol{p}^{2,2}_{2,3}"] & \boldsymbol{M}_{2,2} \arrow[r, "\boldsymbol{p}^{2,1}_{2,2}"] \arrow[dr, "\boldsymbol{p}^{1,\ell(1)}_{2,2}"] & \boldsymbol{M}_{2,1} \arrow[d, "\boldsymbol{p}^{1,\ell(1)}_{2,1}"] \\
 & & & \boldsymbol{M}_{1,\ell(1)}
\end{tikzcd}$$

We iteratively apply this construction for the rest of the tautological sections $\underline{\boldsymbol{\xi}}_3, \cdots, \underline{\boldsymbol{\xi}}_n$. Then we have a tower of blown-up surfaces

$$\boldsymbol{p}^{i,j}_{i',j'} : \boldsymbol{M}_{i',j'} \to \boldsymbol{M}_{i,j}$$

for $i' > i$ or $i' = i, j' > j$ such that $\boldsymbol{p}^{i',j'}_{i'',j''} \circ \boldsymbol{p}^{i,j}_{i',j'} = \boldsymbol{p}^{i,j}_{i'',j''}$ for all valid pairs. We also write a canonical projection to $\boldsymbol{M}$ as $\boldsymbol{p}_{i,j} : \boldsymbol{M}_{i,j} \to \boldsymbol{M}$ and $\boldsymbol{f}_{i,j} := \boldsymbol{\pi_C} \circ \boldsymbol{p}_{i,j}$. For later use, we distinguish the resulting surface by writing it as $\boldsymbol{Z} = \boldsymbol{Z}_n = \boldsymbol{M}_{n,\ell(n)}$ and the projections by $\boldsymbol{f} = \boldsymbol{f}_{n,\ell(n)} : \boldsymbol{Z} \to \boldsymbol{C}$ and $\boldsymbol{p} = \boldsymbol{p}_{n,\ell(n)} : \boldsymbol{Z} \to \boldsymbol{M}$.

Examples 3.1. (Generic ξ) Consider the case $n = 1$ with $D = p$. When $\vec{\xi} = \underline{\xi}$ is generic i.e. $\xi_1, \ldots, \xi_\ell$ are mutually distinct, the restriction of $\boldsymbol{p} : \boldsymbol{Z} \to \boldsymbol{M}$ to $\underline{\xi} \in \mathcal{N}(\vec{m})$, is given by the blow-up of M at the distinct points $\xi_1, \ldots, \xi_\ell$. See Figure 1.

[2] While it is more common to denote $\mathrm{Tot}(K_C(D))$ by M and $\mathbb{P}(K_C(D) \oplus \mathcal{O}_C)$ by $\overline{M}$, we adopt a different convention here, as our focus is primarily on the projectivized space.

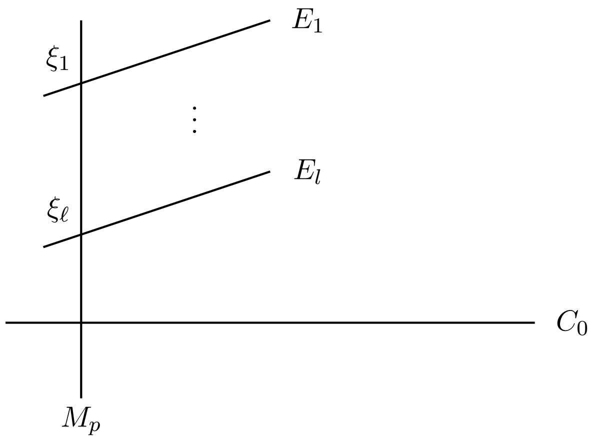


Figure 1. Surface $Z_{\underline{\xi}}$ for a generic $\underline{\xi}$

Examples 3.2. (Strongly parabolic $\xi = 0$) Consider the case $n = 1$ with $D = p$. When $\underline{\xi} = 0$, the restriction of $\boldsymbol{p} : \boldsymbol{Z} \to \boldsymbol{M}$ to $\underline{\xi} \in \mathcal{N}(\vec{m})$ is given by the successive blow-up of M along the strict transform of M_p and the previous exceptional divisor. See Figure 2.

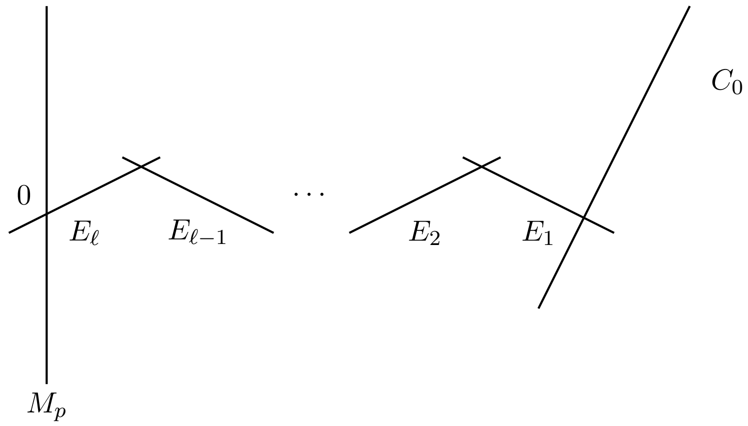


Figure 2. Surface $Z_{\underline{\xi}}$ for $\underline{\xi} = 0$

Similar as before, if we want to indicate the partition $\vec{m}$, we write it in the parenthesis. For example, $\boldsymbol{Z} = \boldsymbol{Z}(\vec{m})$ in the previous example. For a refinement $\vec{m}'$ of $\vec{m}$, we can perform the same construction to obtain $\boldsymbol{Z}(\vec{m}')$. Pulling back $\boldsymbol{Z}(\vec{m}')$ under the canonical inclusion $\iota_{\vec{m}}^{\vec{m}'} : \mathcal{N}(\vec{m}) \hookrightarrow \mathcal{N}(\vec{m}')$ (see (2.1)), and abusing notation to denote the pullback by the same symbol, we have a canonical blow-up map

$$\boldsymbol{f}_{\vec{m}'}^{\vec{m}} : \boldsymbol{Z}(\vec{m}') \to \boldsymbol{Z}(\vec{m}). \tag{10}$$

Now, we study the basic intersection theory of each fiber $Z_{\vec{\xi}}$. Recall that we have an exceptional divisor $\boldsymbol{E}_{i,j}$ in $\boldsymbol{M}_{i,j}$ for $i \in I, j \in J_i$. Define the pullback of such exceptional divisor to $\boldsymbol{Z}$ as $\boldsymbol{\Xi}_{i,j} = (\boldsymbol{p}_{n,\ell(n)}^{i,j})^* \boldsymbol{E}_{i,j}$. Denote the restrictions to $Z_{\vec{\xi}}$ by $E_{i,j,\vec{\xi}} = \boldsymbol{E}_{i,j}|_{Z_{\vec{\xi}}}$ and $\Xi_{i,j,\vec{\xi}} = \boldsymbol{\Xi}_{i,j}|_{Z_{\vec{\xi}}}$. We let C_0 and C_∞ be the zero section and the infinity section of $\pi : M \to C$, respectively. Also, denote by $F_{i,\vec{\xi}}$ and $\widetilde{C}_{\infty,\vec{\xi}}$ the strict transform of the fiber M_{p_i} and the infinity section C_∞ in $Z_{\vec{\xi}}$, respectively.

Lemma 3.3. For each $\vec{\xi} \in \mathcal{N}(\vec{m})$, we have

$$\mathrm{Num}(Z(\vec{\xi})) = \mathrm{Num}(M) \oplus \bigoplus_{i \in I, j \in I_i} \mathbb{Z}[\Xi_{i,j,\xi}]$$

Lemma 3.4. For each $\vec{\xi} \in \mathcal{N}(\vec{m})$, we have

$$\Xi_{i,j,\vec{\xi}} \cdot \Xi_{i',j',\vec{\xi}} = \begin{cases} -1 & \text{if } (i,j) = (i',j') \\ 0 & \text{if otherwise} \end{cases}, \quad \Xi_{i,j,\vec{\xi}} \cdot F_{k,\vec{\xi}} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}, \quad \Xi_{i,j,\vec{\xi}} \cdot p_{\vec{\xi}}^* C_0 = 0$$

Proof. For simplicity, we drop out $\vec{\xi}$ in the notation. We prove the first assertion. If $i \neq i'$, then it is clear that $\Xi_{i,j}$ does not intersect with $\Xi_{i',j'}$ If $i = i'$ and $j \neq j'$, say $j > j'$, we have

$$\Xi_{i,j} \cdot \Xi_{i',j'} = (p^{i,j})^* E_{i,j} \cdot (p^{i',j'})^* E_{i',j'} = E_{i,j} \cdot (p_{i,j}^{i',j'})^* E_{i',j'} = 0$$

If $(i,j) = (i',j')$, then clearly $(\Xi_{i,j})^2 = ((p^{i,j})^* E_{i,j})^2 = (E_{i,j})^2 = -1$. For the second computation, note that

$$\Xi_{i,j} \cdot F_k = \Xi_{i,j} \cdot \left(p^* M_{p_k} - \sum_{j'=1}^{\ell(k)} (p^{k,j'})^* E_{k,j} \right) = \Xi_{i,j} \cdot \left(p^* M_{p_k} - \sum_{j'=1}^{\ell(k)} \Xi_{k,j'} \right)$$

Due to the first assertion, this becomes 1 when $k = i$. The final equality is clear $\Xi_{i,j} \cdot p^* C_0 = E_{i,j} \cdot p_j^* C_0 = 0$. This lemma implies that the intersection pattern of the $\Xi_{i,j,\vec{\xi}}$'s is independent of $\vec{\xi}$ and behaves similarly to the generic case. □

For the purpose of establishing the spectral correspondence, we will be interested in the following family of (non-compact) holomorphic symplectic surfaces. Let $\boldsymbol{F}_i$ and $\widetilde{\boldsymbol{C}}_\infty$ denote the family of the strict transforms of M_{p_i} and C_∞, respectively. We define $\boldsymbol{S}$ to be the complement of the divisors $\boldsymbol{F}_i$'s and $\widetilde{\boldsymbol{C}}_\infty$ in $\boldsymbol{Z}$. The next proposition shows that $\boldsymbol{S}$ is a family of holomorphic symplectic surfaces.

Proposition 3.5. *For each $\vec{\xi} \in \mathcal{N}(\vec{m})$, the non-compact surface $Z_{\vec{\xi}} \backslash (F_{\vec{\xi}} \cup \widetilde{C}_{\infty,\vec{\xi}})$ is holomorphic symplectic.*

Proof. It is easy to check that the canonical divisor of $K_{Z_{\vec{\xi}}}$ is given by

$$K_{Z_{\vec{\xi}}} = -p_{\vec{\xi}}^* M_{p_1} - \cdots - p_{\vec{\xi}}^* M_{p_n} - 2p_{\vec{\xi}}^* C_\infty + \sum_{i \in I} \sum_{j \in J_i} \Xi_{i,j} = -F_{1,\vec{\xi}} - \cdots - F_{n,\vec{\xi}} - 2\widetilde{C}_{\infty,\vec{\xi}}$$

since $F_{i,\vec{\xi}} \sim p_{\vec{\xi}}^* M_{p_i} - \sum_{j \in J_i} \Xi_{i,j,\vec{\xi}}$ for $i \in I$. □

3.2. **Linear systems and Hitchin bases.** Since we will be considering pure dimension-one sheaves on the holomorphic symplectic surfaces $S_{\vec{\xi}}$ constructed by removing the divisors $F_{i,\vec{\xi}}$ and $\widetilde{C}_{\infty,\vec{\xi}}$ from $Z_{\vec{\xi}}$, the support of these sheaves must also be disjoint from these divisors. The next proposition characterizes the curve classes that are disjoint from these divisors.

Proposition 3.6. *A class Σ in $\mathrm{Num}(Z_{\vec{\xi}})$ that satisfies*

$$\Sigma \cdot \widetilde{C}_{\infty,\vec{\xi}} = 0, \qquad \Sigma \cdot F_{i,\vec{\xi}} = 0 \quad \textit{for } i \in I$$

is of the form $ap_{\vec{\xi}}^* C_0 + \sum_{i \in I} \sum_{j \in J_i} c_{i,j} \Xi_{i,j,\vec{\xi}}$ where $a = -\sum_{j \in J_i} c_{i,j}$ for any $i \in I$. Moreover, if we require

$$\Sigma \cdot \Xi_{i,j,\vec{\xi}} = m_{i,j}$$

for all $i \in I, j \in J_i$, then Σ is of the form $rp_{\vec{\xi}}^* C_0 - \sum_{i \in I} \sum_{j \in I_i} m_{i,j} \Xi_{i,j,\vec{\xi}}$, where $r = \sum_{j \in J_i} m_{i,j}$ for any $i \in I$.

Proof. A class $\Sigma \in \mathrm{Num}(Z_{\vec{\xi}})$ is of the form $ap_{\vec{\xi}}^*C_0 + bp_{\vec{\xi}}^*M_p + \sum_{i\in I}\sum_{j\in J_i} c_{i,j}\Xi_{i,j,\vec{\xi}}$ where $p \neq p_i$ for $i \in I$. Then by Lemma 3.4

$$0 = \Sigma \cdot \widetilde{C}_{\infty,\vec{\xi}} = \Sigma \cdot p_{\vec{\xi}}^* C_{\infty,\vec{\xi}} = b \cdot M_p \cdot C_{\infty,\vec{\xi}} \implies b = 0$$

Moreover,

$$0 = \Sigma \cdot F_{i,\vec{\xi}} = aC_0 \cdot M_{p_i} + \sum_{i\in I}\sum_{j\in J_i}(c_{i,j}\Xi_{i,j,\vec{\xi}} \cdot F_{i,\vec{\xi}}) = a + \sum_{j\in I} c_{i,j} = 0.$$

The last condition on $\Sigma \cdot \Xi_{i,j,\vec{\xi}}$ implies that $-c_{i,j} = m_{i,j}$. □

Hence, we will consider the following effective relative Cartier divisor

$$\boldsymbol{\Sigma}(\vec{m}) := r\boldsymbol{p}^*\boldsymbol{C}_0 - \sum_{i\in I}\sum_{j\in J_i} m_{i,j}\boldsymbol{\Xi}_{i,j}$$

on $\boldsymbol{Z}$ and the linear systems formed by its restrictions to $Z_{\vec{\xi}}$

$$\Sigma(\vec{m})_{\vec{\xi}} = rp_{\vec{\xi}}^*C_0 - \sum_{i\in I}\sum_{j\in J_i} m_{i,j}\Xi_{i,j,\vec{\xi}}$$

The complete linear system $|\Sigma(\vec{m})_{\vec{\xi}}|$ has a useful description as a subset of $|rC_0|$. To see this, recall the following elementary fact. Let X be a smooth projective surface and $D \subset X$ a divisor. Let $\pi : X' \to X$ be the blow-up of X at a point $x \in X$ and $E \subset X'$ be the exceptional divisor. Recall that we can identify the complete linear system $|\pi^*D - mE|$ as the sublinear system of $|D|$ consisting of divisors $D' \in |D|$ which pass through p with multiplicity $\geq m$ by mapping D' to $\pi^*D' - mE$ which is effective. Applying this iteratively, we can identify the linear system $|\Sigma(\vec{m})_{\vec{\xi}}|$ with a projective subspace in $|rC_0|$ for any given $\vec{m}$ and $\vec{\xi}$.

Examples 3.7. Consider the case $n = 1$ and $D = p$ for simplicity. Let $\vec{\xi} = \underline{\xi} = (\xi_1, \ldots, \xi_\ell) \in \mathbb{C}^{\times\ell}$ and $\vec{m} = \underline{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^{\times\ell}$. Then $Z_{\vec{\xi}}$ is constructed as a sequence of blow-ups

$$Z_{\vec{\xi}} = M_\ell \xrightarrow{p_\ell^{\ell-1}} \cdots \to M_1 \xrightarrow{p_1^0} M \quad \text{and the exceptional divisors } E_j \subset M_j$$

(here we simplify the notations). We also have a chain of complete linear systems: Let $L_0 := |rC_0|$ on M, $L_1 := |R_1|$ on M_1 where $R_1 := (p_1^0)^*(rC_0) - m_1E_1$ and for $j = 2, \ldots, \ell$, let $L_j = |R_j|$ on M_j where $R_j := (p_j^{j-1})^*R_{j-1} - m_jE_j$. Then we have

$$L_\ell \overset{i_\ell}{\hookrightarrow} \ldots \overset{i_2}{\hookrightarrow} L_1 \overset{i_1}{\hookrightarrow} L_0$$

where the inclusion $i_j : L_j \to L_{j-1}$ identifies L_j with the subset of L_{j-1} consisting of effective divisors in L_{j-1} that passes through the center of the blow-up $p_j^{j-1} : M_j \to M_{j-1}$ with multiplicity $\geq m_j$. In particular, $L_\ell = |\Sigma(\vec{m})_{\vec{\xi}}|$ can be seen as a projective subset of $|rC_0|$ consisting of effective divisors with prescribed multiplicities at the points in M_j determined by $\vec{\xi}$ and $\vec{m}$.

Examples 3.8. Continuing with the previous example above, suppose now that $\xi_1 = \cdots = \xi_\ell$. In this case, the center c_j of the blow-up $p_j^{j-1} : M_j \to M_{j-1}$ lies in the exceptional divisor E_j (see Figure 2). Let $\iota : |\Sigma(\vec{m})_{\vec{\xi}}| \to |rC_0|$ denote the inclusion described above. If $X \in i_1(L_1)$, then X passes through c_1 with multiplicity at least m_1. This implies that the total transform $(p_1^0)^*X = X^1 + m_1E_1$ where $X^1 \in L_1$ is a divisor in M_1. If, further, $X \in i_1 \circ i_2(L_2)$, then $X^1 \in i_2(L_2)$ and X^1 passes through c_2 with multiplicity at least m_2. Thus, the total transform $(p_2^0)^*X = (p_2^1)^*(X^1 + m_1E_1)$ passes through c_2 with multiplicity at least $m_1 + m_2$ where $p_2^0 = p_2^1 \circ p_1^0$. Proceed inductively, we find that if $X \in \iota(L_\ell)$, then for each $j = 2, \ldots, \ell$, the total transform of X in M_{j-1} passes through c_j with multiplicity at least $m_1 + \cdots + m_j$. In fact, this property can also be used as a characterization of $X \in \iota(|\Sigma(\vec{m})_{\vec{\xi}}|)$.

Examples 3.9. In general, the argument in Example 3.8 works for $\vec{\xi} = \underline{\xi} = (\xi_1, \ldots, \xi_l)$ with repeated entries since the conditions imposed by $|\Sigma(\vec{m})_{\vec{\xi}}|$ is local. Let $\{P^{\xi_1^\circ}, \ldots, P^{\xi_e^\circ}\}$ be the collection of partitions labeled by the reduced vector of distinct eigenvalues $(\xi_1^\circ, \ldots, \xi_e^\circ)$ determined by the parabolic $(\vec{m}, \vec{\xi})$. For example, if $P^{\xi_1^\circ} = (m_{j_1}, \ldots, m_{j_s})$ and let c_{j_a} be the center of the blow-up $M_{j_a} \to M_{j_a - 1}$ which lies in the exceptional divisor E_{j_a}, where $a = 1, \ldots, s$. Then $X \in \iota(|\Sigma(\vec{m})_{\vec{\xi}}|)$ satisfies the following local condition:

X passes through c_{j_1} with multiplicity at least m_{j_1} and, for $a = 1, \ldots, s$, the total transform of X in M_{j_a} passes through c_{j_a} with multiplicity at least $m_{j_1} + \cdots + m_{j_a}$.

The analogous condition holds for other ξ_j°, $j = 2, \ldots, e$. Hence, a divisor X in $|\Sigma(\vec{m})_{\vec{\xi}}|$ can be characterized completely by these local multiplicity conditions over the centers associated to each ξ_j°.

Definition 3.10. *For each $(\vec{m}, \vec{\xi})$, define $B(\vec{m})_{\vec{\xi}}$ as the subset of $|\Sigma(\vec{m})_{\vec{\xi}}|$ consisting of effective divisors in $Z_{\vec{\xi}}$ that are compactly supported away from $F_{i,\vec{\xi}}$ for all $i \in I$, as well as from $\widetilde{C}_{\infty,\vec{\xi}}$.*

Lemma 3.11. $B(\vec{m})_{\vec{\xi}}$ is an (open) affine space.

Proof. Recall that (see e.g. [KP95, Remark 1.1] and also the discussion after this proof) the subset $W \subset |rC_0|$ consisting of effective divisors that touch or contain C_∞ is a projective subspace of codimension 1 and the complement W^c of W is an affine space in the linear system $|rC_0|$, which can be identified with the usual Hitchin base A introduced in (9). Then an effective divisor $Q \in W$ if and only if $Q \cap C_\infty \neq \emptyset$ if and only if $C_\infty \subset Q$.

Let $\iota : |\Sigma(\vec{m})_{\vec{\xi}}| \to |rC_0|$ be the inclusion described above. Since the centers of the successive blow-up construction of $Z_{\vec{\xi}}$ are away from C_∞ (and its strict transforms), the subset of effective divisors that are disjoint from $\widetilde{C}_{\infty,\vec{\xi}}$ can be identified with $|\Sigma(\vec{m})_{\vec{\xi}}| \cap \iota^{-1}(W^c)$ which is an affine space. We claim that $|\Sigma(\vec{m})_{\vec{\xi}}| \cap \iota^{-1}(W^c) = B(\vec{m})_{\vec{\xi}}$. Clearly, we have $B(\vec{m})_{\vec{\xi}} \subset |\Sigma(\vec{m})_{\vec{\xi}}| \cap \iota^{-1}(W^c)$, so it suffices to check the other inclusion.

Let $X \in |\Sigma(\vec{m})_{\vec{\xi}}| \cap \iota^{-1}(W^c)$. If X does not contain any $F_{i,\vec{\xi}}$ as a component, then $X \cdot F_{i,\vec{\xi}} = \Sigma(\vec{m})_{\vec{\xi}} \cdot F_{i,\vec{\xi}} = 0$ implies that X is disjoint from $F_{i,\vec{\xi}}$, so $X \in B(\vec{m})_{\vec{\xi}}$. On the other hand, suppose that X contains $F_1 = F_1(\vec{\xi})$ as an irreducible component i.e. $X = F_1 + X'$ for some effective divisor X', then we can show that X' must contain $\widetilde{C}_{\infty,\vec{\xi}}$ as a component, which is a contradiction. Indeed, since $F_1 \sim p_{\vec{\xi}}^* M_{p_1} - \sum_{j \in J_1} \Xi_{1,j}$, we have

$$X' \sim r p_{\vec{\xi}}^* C_0 - \sum_{i \in I} \sum_{j \in J_i} m_{i,j} \Xi_{i,j} - F_1 \sim p_{\vec{\xi}}^* (rC_0 - M_{p_1}) - \sum_{i \in I} \sum_{j \in J_i} m_{i,j} \Xi_{i,j},$$

where $m'_{1,j} = m_{1,j} - 1$ and $m'_{i,j} = m_{i,j}$ for $i \neq 1$. So, we can write

$$X' + \sum_{i \in I} \sum_{j \in J_i} m'_{i,j} \Xi_{i,j} = p_{\vec{\xi}}^* Y$$

for an effective divisor $Y \in |rC_0 - M_{p_1}|$. Now, since $(Y + M_{p_1}) \cap C_\infty \neq \emptyset$, $Y + M_{p_1}$ lies in W and so $C_\infty \subset Y + M_p$. In particular, Y contains C_∞ as a component. Then it follows that X' also contains $C_{\infty,\vec{\xi}}$ as a component. The same argument holds for other components $F_{i,\vec{\xi}}$. $\square$

As the linear system $|\Sigma(\vec{m})_{\vec{\xi}}|$ can be described in terms of $|rC_0|$, $B(\vec{m})_{\vec{\xi}}$ also admits a description in terms of the Hitchin base A. First, let us recall the identification between

Hitchin base and the (open subset of) linear system $|rC_0|$ on the ruled surface M (see e.g. [KP95, Section 1.1] for details). Let $y \in H^0(M, \pi^*(K_C(D)) \otimes \mathcal{O}_M(1))$ be the zero section and $w \in H^0(M, \mathcal{O}_M(1))$ be the infinity section. So, we have $Div(y) = C_0$ and $Div(w) = C_\infty$. There are natural isomorphisms $H^0(M, rC_0) \cong H^0\left(M, \pi^*\left(K_C(D)\right)^{\otimes r} \otimes \mathcal{O}_M(r)\right)$ and

$$H^0\left(M, \pi^*\left(K_C(D)\right)^{\otimes r} \otimes \mathcal{O}_M(r)\right) \cong \bigoplus_{\mu=0}^{r} H^0(C, (K_C(D))^{\otimes \mu})$$
$$s_0 y^r + s_1 y^{r-1} w + \cdots + s_r w^r \longleftrightarrow (s_0, s_1, \ldots, s_r)$$

where we identify $H^0(C, (K_C(D))^{\otimes \mu})$ with $\pi^* H^0(C, (C, K_C(D))^{\otimes \mu}) \subset H^0(M, \pi^*(K_C(D))^{\otimes \mu})$ on the left-hand side. Note that $s_0 \neq 0$ defines the divisors that do not touch or contain C_∞. Denote by $W \subset |rC_0|$ the subset of divisors that touch or contain C_∞ as a component. This describes an identification between the Hitchin base A and divisors compactly supported in M°:

$$|rC_0| \setminus W \cong \bigoplus_{\mu=1}^{r} H^0(C, (K_C(D))^{\otimes \mu}) = A \tag{11}$$

$$Div(y^r + s_1 y^{r-1} + \cdots + s_r) \longleftrightarrow (s_1, \ldots, s_r) \tag{12}$$

where y is now treated as the tautological section of $\pi^*(K_C(D))$ on M°.

As $B(\vec{m})_{\vec{\xi}} \subset |rC_0| \setminus W$, we can provide another description in terms of A. For each $p_i \in D$, we choose a local trivialization of M around p_i i.e. $\mathrm{Spec}(\mathbb{C}[[x_i]][y])$ with horizontal coordinate x_i and vertical coordinate y. Then the evaluation of the local derivative at p_i, $s_\mu^{(a)}(p_i) := \frac{\partial}{\partial x_i^a}\Big|_0 s_\mu$, is well defined and we can define the evaluation maps

$$\mathrm{ev}_{u,a}(\xi_{i,j}) : A \to \mathbb{C}, \quad s = (s_1, \ldots, s_r) \mapsto \frac{\partial}{\partial y^u} \frac{\partial}{\partial x_i^a}\Big|_{(0,\xi_{i,j})} F_s(x_i, y)$$

where $F_s(x_i, y) = y^r + s_1(x_i) y^{r-1} + \cdots + s_r(x_i)$ and $u, a \in \mathbb{Z}_{\geq 0}$. Given the pair $(\vec{m}, \vec{\xi})$, let $\{P^{\xi^\circ_{i,1}}, \ldots, P^{\xi^\circ_{i,e(i)}}\}$ be the collection of partitions labeled by distinct eigenvalues $\underline{\xi}^\circ_i = (\xi^\circ_{i,1}, \ldots \xi^\circ_{i,e(i)})$. To specify orders of partial derivatives, we introduce the following notation: Given a partition $P = (m_1 \geq \cdots \geq m_\ell)$, we define

$$G(P) := \{(u, a) \in \mathbb{Z}^2 | 0 \leq u < |P|, 0 \leq a < \gamma_P(|P| - u)\}, \tag{13}$$

and call it the *level domain* associated to the partition P.

Proposition 3.12. *$B(\vec{m})_{\vec{\xi}}$ is isomorphic to the subspace of A defined by the vanishing of evaluations over the level domains associated to the partitions $P^{\xi^\circ_{i,j}}$, that is,*

$$B(\vec{m})_{\vec{\xi}} \cong \bigcap_{i \in I} \bigcap_{j=1}^{e(i)} \bigcap_{(u,a) \in G\left(P^{\xi^\circ_{i,j}}\right)} \mathrm{ev}_{u,a}(\xi^\circ_{i,j})^{-1}(0) \subset A.$$

Proof. Let $\vec{\xi}^\circ = (\underline{\xi}^\circ_1, \ldots, \underline{\xi}^\circ_n)$ be the distinct part of $\vec{\xi}$ where $\underline{\xi}^\circ_i = (\xi^\circ_{i,1}, \ldots, \xi^\circ_{i,e(i)})$. Apply the argument in Example 3.8 and its direct generalization to the multiple points case $D = p_1 + \cdots + p_n$, we can identify $B(\vec{m})_{\vec{\xi}}$ as the subset of $|rC_0| \setminus W$ which consists of effective divisors satisfying the local multiplicity conditions through the centers of blow-ups associated to each $\xi^\circ_{i,j}$. It remains to check that the local multiplicity conditions over each point p_i is equivalent to the vanishing of the evaluation maps under the isomorphism (11).

If $\xi^\circ_{i,j} = 0$, then the local multiplicity condition at $(p_i, 0)$ is equivalent to the vanishing of $\mathrm{ev}_{u,a}(0)$ by applying Lemma A.1. For $\xi^\circ_{i,j} \neq 0$, by doing a translation $y = \overline{y} + \xi^\circ_{i,j}$, $F'_s(x, \overline{y}) := F_s(x, \overline{y} + \xi_{i,j})$ vanishes at $(x, \overline{y}) = (0, 0)$ and

$$\mathrm{ev}_{u,a}(\xi^\circ_{i,j}) = \frac{\partial}{\partial y^u}\frac{\partial}{\partial x^a}\bigg|_{(0,\xi^\circ_{i,j})} F_s(x, y) = \frac{\partial}{\partial \overline{y}^u}\frac{\partial}{\partial x^a}\bigg|_{(0,\overline{y}=0)} F'_s(x, \overline{y}). \tag{14}$$

Again, by applying Lemma A.1, the local multiplicity condition at $(p_i, \xi^\circ_{i,j})$ is equivalent to the vanishing of $\mathrm{ev}_{u,a}(\xi^\circ_{i,j})$.

□

The affine space $B(\vec{m})_{\vec{\xi}}$ will later serve as the Hitchin base for the moduli space of $\vec{\xi}$-parabolic Higgs bundles later. For now, we verify that $B(\vec{m})_0$ matches the Hitchin base of the strongly parabolic Higgs bundles $A(\vec{m})_0$.

Corollary 3.13. $B(\vec{m})_0$ is isomorphic to $A(\vec{m})_0$.

Proof. This follows directly from the equivalence between (2) and (3) in Lemma A.1. □

As discussed in Example 3.7, the linear system $|\Sigma(\vec{m})_{\vec{\xi}}|$ can be described by imposing multiplicity conditions at points on the successive blown-up surfaces. In general, requiring a divisor to pass through a point with multiplicity $\geq k$ will impose $k(k+1)/2$ linear conditions. Since $\vec{\xi} \in \mathcal{N}(\vec{m})$, there is a linear relation between the eigenvalues $\xi_{i,j}$. So, the expected dimension of the linear system $|\Sigma(\vec{m})_{\vec{\xi}}|$ is

$$\mathrm{expdim}(|\Sigma(\vec{m})_{\vec{\xi}}|) := \dim(|rC_0|) - \sum_{i\in I}\sum_{j\in J_i} \frac{m_{i,j}(m_{i,j}+1)}{2} + 1.$$

We can compute $\dim(|rC_0|)$ simply by computing $\dim(A)$ since $|rC_0| \setminus W \cong A$. A direct computation via the Riemann-Roch theorem yields

$$\begin{aligned} \mathrm{expdim}(|\Sigma(\vec{m})_{\vec{\xi}}|) &= \left(r^2(g-1) + \frac{nr(r+1)}{2}\right) - \sum_{i\in I}\sum_{j\in J_i} \frac{m_{i,j}(m_{i,j}+1)}{2} + 1 \\ &= 1 + r^2(g-1) + \frac{nr^2 - \sum_{i\in I}\sum_{j\in J_i} m_{i,j}^2}{2}. \end{aligned} \tag{15}$$

Proposition 3.14. *Suppose that* $\mathrm{expdim}(|\Sigma(\vec{m})_{\vec{\xi}}|) \geq 0$ *and the family of linear systems* $|\boldsymbol{\Sigma}(\vec{m})|$ *(resp.* $\boldsymbol{B}(\vec{m})$*) over* $\mathcal{N}(\vec{m})$ *is non-empty over every* $\vec{\xi} \in \mathcal{N}(\vec{m})$*. If the OK condition holds, i.e.*

$$H^1(C, L(\vec{m})_\mu) = 0 \quad \text{for } \mu = 2, \ldots, r, \tag{16}$$

then the family $|\boldsymbol{\Sigma}(\vec{m})|$ *(resp.* $\boldsymbol{B}(\vec{m})$*) has constant dimension as* (15).

Proof. First, note that there is a natural $\mathbb{C}^*$-action on $\mathcal{N}(\vec{m})$ by scaling $(\xi_{i,j}) \mapsto (\lambda\xi_{i,j})$ where $\lambda \in \mathbb{C}^*$. There is also a $\mathbb{C}^*$-action on $\boldsymbol{Z}$ induced by the $\mathbb{C}^*$-action on M via scaling on fibers over C (fixing C_∞). Then it is easy to see that the family of surfaces $\boldsymbol{Z} \to \mathcal{N}(\vec{m})$ is $\mathbb{C}^*$-equivariant with respect to these actions. In particular, a non-zero $\lambda \in \mathbb{C}^*$ defines an isomorphism between $Z_{\vec{\xi}}$ and $Z_{\lambda\vec{\xi}}$ and induces an isomorphism between $|\Sigma(\vec{m})_{\vec{\xi}}|$ and $|\Sigma(\vec{m})_{\lambda\vec{\xi}}|$. For each non-zero $\vec{\xi} \in \mathcal{N}(\vec{m})$, if we restrict the family of linear systems to the line through the origin and $\vec{\xi}$, then by upper semicontinuity, the dimensions of the family of linear systems $|\Sigma(\vec{m})_{\lambda\vec{\xi}}|$ can only jump at $\lambda = 0$ i.e. $\dim(|\Sigma(\vec{m})_0|) \geq \dim(|\Sigma(\vec{m})_{\lambda\vec{\xi}}|)$ for $\lambda \in \mathbb{C}^*$. By

Proposition 3.13, we can compute the dimension of $|\Sigma(\vec{m})_0|$ by computing the dimension of $A(\vec{m})_0$ via a Riemann–Roch computation:

$$\dim(|\Sigma(\vec{m})_0|) = \dim(A(\vec{m})_0) = 1 + r^2(g-1) + \frac{nr(r+1)}{2} - \sum_{i=1}^{n}\sum_{\mu=1}^{r} \gamma_{P^i}(\mu) \tag{17}$$

Note that this formula relies on the vanishing assumption (16), so the only non-trivial h^1-term in the Riemann-Roch computation is

$$h^1\left(C, (K_C(D)) \otimes \mathcal{O}\left(-\sum_{i=1}^{n} \gamma_{P^i}(1)p_i\right)\right) = h^1(C, K_C) = 1$$

i.e. $\mu = 1$ which contributes to the "$+1$" in (17).

By the following obvious identity for a partition $P^i = (m_{i,1}, \dots, m_{i,\ell(i)})$,

$$\sum_{\mu=1}^{r} \gamma_{P^i}(\mu) = \sum_{j=1}^{\ell(i)} \frac{m_{i,j}(m_{i,j}+1)}{2} \quad \text{for } i = 1, \dots, n,$$

we see that $\dim(|\Sigma(\vec{m})_0|) = \mathrm{expdim}(|\Sigma(\vec{m})_{\vec{\xi}}|)$ computed in (15). Since the expected dimension is always lower than the actual dimension, we deduce that

$$\dim(|\Sigma(\vec{m})_0|) \geq \dim(|\Sigma(\vec{m})_{\vec{\xi}}|) \geq \mathrm{expdim}(|\Sigma(\vec{m})_{\vec{\xi}}|) = \dim(|\Sigma(\vec{m})_0|)$$

Hence, $\dim(|\Sigma(\vec{m})_0|) = \dim(|\Sigma(\vec{m})_{\lambda\vec{\xi}}|)$ for $\lambda \in \mathbb{C}^*$ and the dimension of the family of linear systems is constant everywhere.

□

Remark 3.15. It follows from Proposition 3.14 that $(\boldsymbol{\pi}_{\mathcal{N}} \circ \boldsymbol{f})_*\mathcal{O}(\boldsymbol{\Sigma}(\vec{m}))$ is a vector bundle on $\mathcal{N}(\vec{m})$ whose fiber is $H^0(Z_{\vec{\xi}}, \mathcal{O}(\Sigma(\vec{m})_{\vec{\xi}}))$, we denote its projectivization by $\overline{\boldsymbol{B}(\vec{m})}$. By Lemma 3.11, the open subset of $\overline{\boldsymbol{B}(\vec{m})}$ consisting of divisors compactly supported away from $F_{\vec{\xi}}$ and $\widetilde{C}_{\infty,\vec{\xi}}$ for each $\vec{\xi}$ forms an affine bundle $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ whose fiber over each $\vec{\xi}$ is the affine space $B(\vec{m})_{\vec{\xi}}$. We will call

$$\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$$

the family of Hitchin bases associated to $\vec{m}$.

Recall that by definition we have the inclusion $\iota : \mathcal{O}(\boldsymbol{\Sigma}(\vec{m})) \to \mathcal{O}(r\boldsymbol{f}^*\boldsymbol{C_0})$ induced by the section $\mathcal{O} \to \mathcal{O}(\sum_{i\in I}\sum_{j\in J_i} m_{i,j}\boldsymbol{\Xi}_{i,j})$. Since $(\boldsymbol{\pi}_{\mathcal{N}} \circ \boldsymbol{f})_*\mathcal{O}(r\boldsymbol{f}^*\boldsymbol{C_0}) \cong H^0(C, rC_0) \times \mathcal{N}(\vec{m})$, the projectivization of the inclusion ι induces the morphism $\overline{\boldsymbol{B}(\vec{m})} \to |rC_0| \times \mathcal{N}(\vec{m})$. Restricting to the open subsets of divisors away from the infinity divisors, we get a morphism $\boldsymbol{B}(\vec{m}) \to A \times \mathcal{N}(\vec{m})$.

The advantage of having a constant dimension for the family of Hitchin bases is that we can deform smooth or integral curves from the fiber at $0 \in \mathcal{N}(\vec{m})$ to nearby $\vec{\xi} \in \mathcal{N}(\vec{m})$. This property will be useful later when we study the non-emptiness of the moduli spaces in Section 5.3.

Proposition 3.16. *Suppose that the family of Hitchin bases $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ is non-empty for every $\vec{\xi} \in \mathcal{N}(\vec{m})$ and has constant dimension (e.g. by Proposition 3.14). Then, whenever $B(\vec{m})_0$ contains a smooth (resp. integral) member, $B(\vec{m})_{\vec{\xi}}$ also contains a smooth (resp. integral) member for every $\vec{\xi} \in \mathcal{N}(\vec{m})$.*

Proof. As explained in Remark 3.15, the equi-dimensional family of Hitchin bases $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ is an affine bundle, which is irreducible because $\mathcal{N}(\vec{m})$ is itself irreducible. Since $B(\vec{m})_0 \subset \boldsymbol{B}(\vec{m})$ contains a smooth member and smoothness is an open condition, there exists a non-empty open subset $U \subset \boldsymbol{B}(\vec{m})$ of smooth curves. As $\boldsymbol{B}(\vec{m}) \to \mathcal{N}(\vec{m})$ is a surjective morphism between irreducible varieties, the image of U in $\mathcal{N}(\vec{m})$ is also an open subset $V \subset \mathcal{N}(\vec{m})$ containing $0 \in \mathcal{N}(\vec{m})$. Now, recall that, as explained in the proof of Proposition 3.14, there is a $\mathbb{C}^*$-action on the family of surfaces $\boldsymbol{Z} \to \mathcal{N}(\vec{m})$ which identifies the curves in $B(\vec{m})_{\vec{\xi}}$ and $B(\vec{m})_{\lambda\vec{\xi}}$ for $\lambda \in \mathbb{C}^*$. In particular, the $\mathbb{C}^*$-action should preserves the smoothness of the curves. Hence, V is $\mathbb{C}^*$-invariant and it follows that $V = \mathcal{N}(\vec{m})$. The same argument goes for the case of integral curves since the family of surfaces is smooth and so the family of curves (which are Cartier divisors) parametrized by $\boldsymbol{B}(\vec{m})$ are all Cohen-Macaulay. In this case, the locus of integral curves is open. □

4. Relative spectral correspondence

4.1. **Relative moduli of pure dimension-one sheaves.** We first recall the absolute case, particularly the definition of β-twisted A-Gieseker semistablity condition first introduced in [MW97] for torsion-free sheaves. For pure dimension one sheaves, it is due to Yoshioka [Yos03]. Let X be a smooth projective surface and $\beta, A \in NS(X)_{\mathbb{Q}}$ with A ample. We define the β-twisted Chern character of $F \in \mathrm{Coh}(X)$ to be $\mathrm{ch}^{\beta}(F) := \mathrm{ch}(F) \cup \exp(\beta) \in H^*(X, \mathbb{Q})$. Then the β-twisted Hilbert polynomial is defined to be

$$P_\beta(F,t) = \int_X \mathrm{ch}^{\beta}(F(t))\mathrm{Td}_X.$$

Note that when F is a torsion sheaf i.e. $\mathrm{ch}_0(F) = 0$, we have

$$P_\beta(F,t) = \int_X \mathrm{ch}(F(t))\mathrm{Td}_X + \int_X \beta \cdot \mathrm{ch}_1(F(t)) = P(F,t) + \int_X \beta \cdot \mathrm{ch}_1(F(t))$$

where $P(F,t)$ denotes the usual Hilbert polynomial. In particular, $P_\beta(F,t) = P(F,t)$ when $\beta = 0$. Let $l(F)/d!$ be the leading coefficient of $P_\beta(F,t)$. Define the reduced β-twisted Hilbert polynomial as $p_\beta(F,t) = P_\beta(F,t)/l(F)$. Note that the leading coefficients of $P_\beta(F,t)$ and $P(F,t)$ are the same.

Definition 4.1. *A coherent sheaf F on X is said to be β-twisted A-Gieseker (semi)stable if it has support of pure dimension and for any proper subsheaf $F' \subset F$, one has*

$$p_\beta(F',t) < p_\beta(F,t) \quad \textit{for } t \gg 0, \quad (\textit{resp. } \leq)$$

As we are considering the relative moduli problems, we will need to make a choice of a rational Neron-Severi class $\boldsymbol{\beta}$ and an ample line bundle $\boldsymbol{A}$ on $\boldsymbol{Z}$ both relative to $\boldsymbol{\pi}_{\mathcal{N}}$.

- (A choice of $\boldsymbol{\beta}$) For a set of rational numbers $\{\beta_{i,j} | i \in I, j \in J_i\}$ with $0 \leq \beta_{i,1} < \cdots < \beta_{i,\ell(i)} < 1$ for $i \in I$, we choose $\boldsymbol{\beta} = \sum_{i \in I} \sum_{j \in J_i} \beta_{i,j} \boldsymbol{\Xi}_{i,j}$ on $\mathrm{NS}_{\boldsymbol{\pi}}(Z)_{\mathbb{Q}}$.
- (A choice of $\boldsymbol{A}$) One can prove that there exists a relative ample divisor $\boldsymbol{A}$ on $\boldsymbol{Z}$ which can be written as $\boldsymbol{A} = \kappa(\sum_{i \in I} \sum_{j \in J_i} \boldsymbol{\Xi}_{i,j}) = \boldsymbol{f}^*(\kappa D)$ for sufficently large enough κ (e.g. [LL24, Section 2.2])

With the fixed data: $r \geq 1$, $c \in \mathbb{Q}$, $d \in \mathbb{Z}$, $\boldsymbol{\beta}$, $\boldsymbol{A}$, and $\boldsymbol{\Sigma}_{\vec{m}}$, we define the relative moduli stack $\overline{\boldsymbol{\mathfrak{M}}}$ over the base $\mathcal{N}(\vec{m})$ of $\boldsymbol{\beta}$-twisted $\boldsymbol{A}$-Gieseker semistable pure dimension one sheaves $\boldsymbol{\mathcal{F}}$ on the family $\boldsymbol{Z} \to \mathcal{N}(\vec{m})$ with Chern character $(0, \boldsymbol{\Sigma}(\vec{m}), c)$. Fiberwise, for each $\vec{\xi} \in \mathcal{N}(\vec{m})$, let $\mathcal{F}_{\vec{\xi}}$ denote the restriction of $\boldsymbol{\mathcal{F}}$ to $Z_{\vec{\xi}}$, then $\mathcal{F}_{\vec{\xi}}$ is $\beta_{\vec{\xi}}$-twisted and $A_{\vec{\xi}}$-Gieseker stable and satisfies

$$(\mathrm{ch}_0(\mathcal{F}_{\vec{\xi}}), \mathrm{ch}_1(\mathcal{F}_{\vec{\xi}}), \mathrm{ch}_2(\mathcal{F}_{\vec{\xi}})) = (0, \Sigma(\vec{m})_{\vec{\xi}}, c), \quad \text{for all } \vec{\xi} \in \mathcal{N}(\vec{m})$$

Here $\beta_{\vec{\xi}}$ and $A_{\vec{\xi}}$ are the restriction of $\boldsymbol{\beta}$ and $\boldsymbol{A}$ to $Z_{\vec{\xi}}$.

Recall that $\boldsymbol{S} \to \mathcal{N}(\vec{m})$ is the family of open holomorphic symplectic surfaces in $\boldsymbol{Z} \to \mathcal{N}(\vec{m})$. We define the open substack $\boldsymbol{\mathfrak{M}} \subset \overline{\boldsymbol{\mathfrak{M}}}$ of sheaves compactly supported on $\boldsymbol{S} \to \mathcal{N}(\vec{m})$.

Equivalently, for each $\vec{\xi} \in \mathcal{N}(\vec{m})$, the restriction $\mathcal{F}_{\vec{\xi}}$ is a pure dimension one sheaf on $S_{\vec{\xi}} \subset Z_{\vec{\xi}}$ with compact support.

By [Yos03, Theorem 2.8], the relative moduli stack $\overline{\boldsymbol{\mathfrak{M}}(\vec{m})} \to \mathcal{N}(\vec{m})$ admits a relative coarse moduli space $\overline{\boldsymbol{\mathcal{M}}(\vec{m})} \to \mathcal{N}(\vec{m})$, which is projective over $\mathcal{N}(\vec{m})$. There is a natural morphism defined by taking the Fitting support morphism:

$$\begin{tikzcd} \overline{\boldsymbol{\mathcal{M}}(\vec{m})} \arrow[rr, "\overline{\boldsymbol{\phi}}"] \arrow[dr] & & \overline{\boldsymbol{B}(\vec{m})} \arrow[dl] \\ & \mathcal{N}(\vec{m}) & \end{tikzcd}$$

As $\overline{\boldsymbol{\mathcal{M}}(\vec{m})}$ is proper over $\mathcal{N}(\vec{m})$ and $\overline{\boldsymbol{B}(\vec{m})}$ is clearly separated over $\mathcal{N}(\vec{m})$, it follows that the Fitting support morphism $\overline{\boldsymbol{\phi}}$ is also proper. By definition, the preimage $\boldsymbol{\mathcal{M}}(\vec{m}) := \overline{\boldsymbol{\phi}}^{-1}(\boldsymbol{B}(\vec{m}))$ becomes the relative coarse moduli space of $\boldsymbol{\mathfrak{M}}(\vec{m})$. Restricting $\overline{\boldsymbol{\phi}}$ to the open subset $\boldsymbol{B}(\vec{m}) \subset \overline{\boldsymbol{B}(\vec{m})}$, we have

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}(\vec{m}) \arrow[rr, "\boldsymbol{\phi}"] \arrow[dr] & & \boldsymbol{B}(\vec{m}) \arrow[dl] \\ & \mathcal{N}(\vec{m}) & \end{tikzcd}$$

4.2. **Relative spectral correspondence.** The goal is to prove the spectral correspondence over $\mathcal{N}(\vec{m})$.

Theorem 4.2 (Relative spectral correspondence)**.** *Fix the numerical data: $r \geq 1, d \in \mathbb{Z}$. There is a commutative diagram*

$$\begin{tikzcd} \boldsymbol{\mathcal{M}} \arrow[rr, "\cong"] \arrow[dr] & & \boldsymbol{\mathcal{H}} \arrow[dl] \\ & \mathcal{N}(\vec{m}) & \end{tikzcd} \tag{18}$$

where $\boldsymbol{\mathcal{H}}$ is the relative coarse moduli scheme of semistable $\vec{\xi}$-parabolic Higgs bundles on C of rank r, degree d, parabolic weights $\vec{\alpha} = (\underline{\alpha}_1, \cdots, \underline{\alpha}_n)$, flag type $\vec{m} = (\underline{m}_1, \cdots, \underline{m}_n)$; such that $\alpha_{i,j} = \beta_{i,j}, d = c + r(g-1)$.

Our construction is based on the spectral correspondence of [DDP18] which is formulated for generic $\vec{\xi} \in \mathcal{N}(\vec{m})$. The main point in this section is to demonstrate that their construction extends naturally to a relative version over the entire base $\mathcal{N}(\vec{m})$ when the necessary geometric data is chosen appropriately. In particular, the family of open holomorphic symplectic surfaces $\boldsymbol{S} \subset \boldsymbol{Z}$ and the linear systems $\boldsymbol{\Sigma}(\vec{m})$ constructed earlier are crucial for the extension from generic $\vec{\xi}$ to the whole $\mathcal{N}(\vec{m})$.

In what follows, we describe the constructions at the level of closed points for ease of notation. While they extend naturally to flat families, doing so would require heavier notation. Before getting into the proof, we describe the relevant morphisms.

Construction of the morphism $\boldsymbol{Q} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{H}}$ over $\mathcal{N}(\vec{m})$.

Let $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{M}}$ be a family of $\boldsymbol{\beta}$-stable pure dimension one sheaves over $\boldsymbol{S}$ relative to $\mathcal{N}(\vec{m})$. We define the Higgs bundle as follows:

- $\boldsymbol{\mathcal{E}} := \boldsymbol{f}_* \boldsymbol{\mathcal{F}}$. This is a flat family of vector bundles of rank r over C.
- Let $\boldsymbol{s} : \mathcal{O}_{\boldsymbol{S}} \to \mathcal{O}_{\boldsymbol{S}}(\boldsymbol{p}^* \boldsymbol{C}_0)$ be the section obtained by pulling back the family of tautological sections $\mathcal{O}_{\boldsymbol{M}} \to \mathcal{O}_{\boldsymbol{M}}(\boldsymbol{C}_0)$. By tensoring with $\boldsymbol{\mathcal{F}}$ and taking the pushforward

along $\boldsymbol{f}$, the Higgs field is defined as

$$\boldsymbol{\Phi} : \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{E}} \otimes_{\boldsymbol{S}} \boldsymbol{K_C(D)}$$

where we use the projection formula $\boldsymbol{f}_*(\boldsymbol{\mathcal{F}} \otimes_{\boldsymbol{S}} \mathcal{O}_{\boldsymbol{S}}(\boldsymbol{p}^*\boldsymbol{C}_0)) = \boldsymbol{\mathcal{E}} \otimes_{\boldsymbol{S}} \boldsymbol{K_C(D)}$.

From now on, we fix $i \in I$. To obtain a quasi-parabolic structure, we define a filtration of pure dimension one sheaves on $\boldsymbol{\mathcal{F}}$ as follows:

$$\boldsymbol{\mathcal{F}}^{i,k} := \ker(\boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}} \otimes \mathcal{O}_{\sum_{j=1}^k \Xi_{i,j}})$$

Note that $\boldsymbol{\mathcal{F}}^{i,\ell(i)} = \boldsymbol{\mathcal{F}}(-\boldsymbol{p}^*\boldsymbol{M}_{p_i})$. Also over $\boldsymbol{S}$, the projection $\boldsymbol{f} := \boldsymbol{\pi} \circ \boldsymbol{p} : \boldsymbol{S} \to \boldsymbol{M}^\circ \to \boldsymbol{C}$ is a relative affine morphism so that the pushforward $\boldsymbol{f}_*(\boldsymbol{\mathcal{F}}^{i,k})$ of each $\boldsymbol{\mathcal{F}}^{i,k}$ is locally free. Therefore, we have the induced filtration of locally free subsheaves on $\boldsymbol{\mathcal{E}}$:

$$\boldsymbol{\mathcal{E}}^{i,\bullet} : \boldsymbol{\mathcal{E}}(-\boldsymbol{p_i}) = \boldsymbol{\mathcal{E}}^{i,\ell(i)} \subset \boldsymbol{\mathcal{E}}^{i,\ell(i)-1} \subset \cdots \subset \boldsymbol{\mathcal{E}}^{i,1} \subset \boldsymbol{\mathcal{E}}$$

where the relation $\boldsymbol{\mathcal{E}}(-\boldsymbol{p_i})$. Also, we have $\chi(\boldsymbol{\mathcal{E}}^{i,k}/\boldsymbol{\mathcal{E}}^{i,k+1}) = m_{i,k}$.

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-\sum_{j=1}^k \Xi_{i,j}) \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, "\boldsymbol{s}"] & \mathcal{O}_{\sum_{j=1}^k \Xi_{i,j}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(\boldsymbol{p}^*\boldsymbol{C}_0 - \sum_{j=1}^{k+1} \Xi_{i,j}) \arrow[r] & \mathcal{O}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & \mathcal{O}_{\sum_{j=1}^k \Xi_{i,j}}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & 0
\end{tikzcd}$$

where the two rows are exact. Tensoring by $\boldsymbol{\mathcal{F}}$, we get

$$\begin{tikzcd}
\boldsymbol{\mathcal{F}}(-\sum_{j=1}^k \Xi_{i,j}) \arrow[r, "i_1"] \arrow[d] & \boldsymbol{\mathcal{F}} \arrow[r] \arrow[d, "\boldsymbol{s}"] & \boldsymbol{\mathcal{F}}_{\sum_{j=1}^k \Xi_{i,j}} \arrow[r] \arrow[d] & 0 \\
\boldsymbol{\mathcal{F}}(\boldsymbol{p}^*\boldsymbol{C}_0 - \sum_{j=1}^k \Xi_{i,j}) \arrow[r, "i_2"] & \boldsymbol{\mathcal{F}}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & \boldsymbol{\mathcal{F}}_{\sum_{j=1}^k \Xi_{i,j}}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & 0
\end{tikzcd}$$

Note that the images of i_1, i_2 are $\boldsymbol{\mathcal{F}}^{i,k}, \boldsymbol{\mathcal{F}}^{i,k}(\boldsymbol{p}^*\boldsymbol{C}_0)$ respectively. Therefore, we arrive at the commutative diagram

$$\begin{tikzcd}
\boldsymbol{\mathcal{F}}^{i,k} \arrow[r] \arrow[d] & \boldsymbol{\mathcal{F}} \arrow[d, "\boldsymbol{s}"] \\
\boldsymbol{\mathcal{F}}^{i,k}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & \boldsymbol{\mathcal{F}}(\boldsymbol{p}^*\boldsymbol{C}_0)
\end{tikzcd}$$

In other words, the tautological section $\boldsymbol{s} : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}}(\boldsymbol{p}^*\boldsymbol{C}_0)$ satisfies $\boldsymbol{s}(\boldsymbol{\mathcal{F}}^{i,j}) \subset \boldsymbol{\mathcal{F}}^{i,j}(\boldsymbol{p}^*\boldsymbol{C}_0)$. By pushing forward this diagram, we see that the filtration $\boldsymbol{\mathcal{E}}^{i,\bullet}$ is indeed a quasi-parabolic structure. Similarly, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{\mathcal{F}}^{i,k} \arrow[r] \arrow[d] & \boldsymbol{\mathcal{F}}^{i,k-1} \arrow[r] \arrow[d, "\boldsymbol{s}"] & \boldsymbol{\mathcal{F}}^{i,k-1}/\boldsymbol{\mathcal{F}}^{i,k} \arrow[r] \arrow[d, "\boldsymbol{s}_{i,k}"] & 0 \\
0 \arrow[r] & \boldsymbol{\mathcal{F}}^{i,k}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & \boldsymbol{\mathcal{F}}^{i,k-1}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & \boldsymbol{\mathcal{F}}^{i,k-1}/\boldsymbol{\mathcal{F}}^{i,k}(\boldsymbol{p}^*\boldsymbol{C}_0) \arrow[r] & 0
\end{tikzcd}$$

where $\boldsymbol{s}_{i,k}$ is denoted as the induced map on the quotient. Since $\boldsymbol{\mathcal{F}}^{i,k-1}/\boldsymbol{\mathcal{F}}^{i,k}$ is supported on $\Xi_{i,k}$ where the value of $\boldsymbol{s}$ over $\vec{\xi} \in \mathcal{N}(\vec{m})$ is the constant $\xi_{i,k}$, the map $\boldsymbol{s}_{i,k}$ is just the scalar multiplication by $\xi_{i,k}$ for $\vec{\xi} \in \mathcal{N}(\vec{m})$. As the pushforward $\boldsymbol{f}_*$ is exact on sheaves supported on $\boldsymbol{S}$, we also get

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{\mathcal{E}}^{i,k} \arrow[r] \arrow[d] & \boldsymbol{\mathcal{E}}^{i,k-1} \arrow[r] \arrow[d] & \boldsymbol{\mathcal{E}}^{i,k-1}/\boldsymbol{\mathcal{E}}^{i,k} \arrow[r] \arrow[d, "\boldsymbol{\Phi}_{i,k}"] & 0 \\
0 \arrow[r] & \boldsymbol{\mathcal{E}}^{i,k}(\boldsymbol{K_C}(\boldsymbol{D})) \arrow[r] & \boldsymbol{\mathcal{E}}^{i,k-1}(\boldsymbol{K_C}(\boldsymbol{D})) \arrow[r] & \boldsymbol{\mathcal{E}}^{i,k-1}/\boldsymbol{\mathcal{E}}^{i,k}(\boldsymbol{K_C}(\boldsymbol{D})) \arrow[r] & 0
\end{tikzcd}$$

The map $\boldsymbol{\Phi}_{i,k}$ is induced from $\boldsymbol{s}_{i,k}$ and so it follows that $\boldsymbol{\Phi}$ satisfies the $\vec{\xi}$-parabolic condition.

In summary, we define $\boldsymbol{Q}(\boldsymbol{\mathcal{F}}) = (\boldsymbol{\mathcal{E}} := \boldsymbol{f}_*\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{E}}^{\bullet}_{\boldsymbol{D}}, \boldsymbol{\Phi}, \vec{\alpha})$ where $\alpha_{i,j} = \beta_{i,j}$ for $i \in I, j \in J_i$. The stability condition follows from the same argument in [DDP18] and also [LL24, Appendix A], we refer the reader to *loc. cit.* for details.

Construction of the morphism $\boldsymbol{R} : \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{M}}$ over $\mathcal{N}(\vec{m})$

Let's take $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^{\bullet}_{\boldsymbol{D}}, \boldsymbol{\Phi}, \vec{\alpha}) \in \boldsymbol{\mathfrak{H}}$, a family of $\vec{\xi}$-parabolic Higgs bundles over $\boldsymbol{C}$ relative to $\mathcal{N}(\vec{m})$. Our construction is essentially the same with the recursive construction in [DDP18]. Note that we have constructed $\boldsymbol{Z}$ by iteratively blowing up, with intermediate spaces $\boldsymbol{Z}_i$ for $i = 1, \ldots, n-1$, and $\boldsymbol{Z} = \boldsymbol{Z}_n$. Following this process, we first construct a pure dimension-one sheaf on $\boldsymbol{Z}_1$, then iteratively apply the same argument to obtain a pure dimension-one sheaf on $\boldsymbol{Z}$. Let $\boldsymbol{y}$ be a tautological section of $\boldsymbol{\pi}^*\boldsymbol{M}$ over $\boldsymbol{M}$. Consider the diagram

$$\begin{tikzcd}
\boldsymbol{\pi}^*\boldsymbol{\mathcal{E}} \otimes_{\boldsymbol{M}} \boldsymbol{\pi}^*\boldsymbol{M}^{-1} \arrow[r, "\boldsymbol{y}\mathrm{I}_{\boldsymbol{\pi}^*\boldsymbol{\mathcal{E}}} - \boldsymbol{\pi}^*\boldsymbol{\Phi}"] \arrow[d] & \boldsymbol{\pi}^*\boldsymbol{\mathcal{E}} \arrow[r] \arrow[d] & \boldsymbol{G} \arrow[d, two heads] \\
\boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}}) \otimes_{\boldsymbol{M}} \boldsymbol{\pi}^*\boldsymbol{M}^{-1} \arrow[r, "\boldsymbol{y}\mathrm{I}_{\boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}})} - \boldsymbol{\pi}^*\boldsymbol{\Phi}"] & \boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}}) \arrow[r] & \boldsymbol{R}_{1,1}
\end{tikzcd}$$

where $\boldsymbol{G}$ and $\boldsymbol{R}_{1,1}$ are defined as the cokernel of the top and bottom horizontal morphism, respectively. Note that there is a surjection $\boldsymbol{G} \twoheadrightarrow \boldsymbol{R}_{1,1}$. We pullback this morphism under $\boldsymbol{p}_{1,\ell(1)}$, and define a pure dimension one sheaf over $\boldsymbol{Z}_1$ as follows:

$$\boldsymbol{\mathcal{F}}_{1,1} := \ker(\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{G} \twoheadrightarrow \boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,1}).$$

Since $\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,1}$ is a locally free sheaf of rank $m_{1,1}$ on $\boldsymbol{\Xi}_{1,1}$ relative to $\mathcal{N}(\vec{m})$, the sheaf $\boldsymbol{\mathcal{F}}_{1,1}$ is of the class $|r\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{C}_0 - m_{1,1}\boldsymbol{\Xi}_{1,1}|$.

Define $\boldsymbol{R}_{1,2}$ to be the cokernel of

$$\boldsymbol{y}\mathrm{I}_{\boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^2_{\boldsymbol{p-1}})} : \boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^2_{\boldsymbol{p_1}}) \otimes_{\boldsymbol{M}} \boldsymbol{\pi}^*\boldsymbol{M}^{-1} \to \boldsymbol{\pi}^*(\boldsymbol{\mathcal{E}}^1_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^2_{\boldsymbol{p_1}}).$$

By construction, there is a surjective morphism

$$\ker(\boldsymbol{G} \twoheadrightarrow \boldsymbol{R}_{1,1}) \twoheadrightarrow \boldsymbol{R}_{1,2}.$$

Therefore, we have the induced surjective morphism over $\boldsymbol{Z}_1$, $\boldsymbol{\mathcal{F}}_{1,1} \twoheadrightarrow \boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,2} \twoheadrightarrow \boldsymbol{p}^*_{1,\ell(1)}(\boldsymbol{R}_{1,2}|_{\boldsymbol{\Xi}_{1,2}})$ and denote its kernel by $\boldsymbol{\mathcal{F}}_{1,2}$. One can think of the further restriction to $\boldsymbol{\Xi}_{1,2}$ as the pullback the proper transform of $\boldsymbol{M}_{p_1}$ in the $\boldsymbol{M}_{1,1}$ to $\boldsymbol{Z}_1$. Clearly, we have a short exact sequence

$$0 \to \boldsymbol{\mathcal{F}}_{1,2} \to \boldsymbol{\mathcal{F}}_{1,1} \to (\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,2})|_{\boldsymbol{\Xi}_{1,2}} \to 0$$

which implies that the support lies in the class $|r\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{C}_0 - m_{1,1}\boldsymbol{\Xi}_{1,1} - m_{1,2}\boldsymbol{\Xi}_{1,2}|$.

Proceeding recursively, one then constructs a sequence $\boldsymbol{\mathcal{F}}_{1,1}, \boldsymbol{\mathcal{F}}_{1,2}, \cdots, \boldsymbol{\mathcal{F}}_{1,\ell(1)}$ of pure dimension one sheaves on $\boldsymbol{Z}_1$ and they fit into short exact sequences

$$0 \to \boldsymbol{\mathcal{F}}_{1,k+1} \to \boldsymbol{\mathcal{F}}_{1,k} \to (\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,k})|_{\boldsymbol{\Xi}_{1,k}} \to 0, \qquad 1 \le k \le \ell(1) - 1$$

The curve class of the support of $\boldsymbol{\mathcal{F}}_{1,\ell(1)}$ is given by $|r\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{C}_0 - m_{1,1}\boldsymbol{\Xi}_{1,1} - \cdots - m_{1,\ell(1)}\boldsymbol{\Xi}_{1,\ell(1)}|$. Moreover, the pushforward of the term $(\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,k})|_{\boldsymbol{\Xi}_{1,k}}$ to $\boldsymbol{C}$ becomes $\boldsymbol{\mathcal{E}}^{k-1}_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^{k}_{\boldsymbol{p_1}}$. This is because,

$$(\boldsymbol{p}^*_{1,\ell(1)}\boldsymbol{R}_{1,k})|_{\boldsymbol{\Xi}_{1,k}} = (\boldsymbol{p}^{1,k-1}_{1,\ell(1)})^*(\boldsymbol{p}^*_{1,k-1}((\boldsymbol{\mathcal{E}}^{k-1}_{\boldsymbol{p_1}}/\boldsymbol{\mathcal{E}}^{k}_{\boldsymbol{p_1}})|_{\boldsymbol{M_{p_1}}})) \tag{19}$$

It implies that $(\boldsymbol{f}_{1,\ell(1)})_*\boldsymbol{\mathcal{F}}_{1,\ell(1)} \cong \boldsymbol{\mathcal{E}}(-\boldsymbol{p_1})$.

The rest of the construction can be done by applying same argument for other components of the divisor $\boldsymbol{D}$. The resulting pure dimension one sheaf, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{n,\ell(n)}$ has the support that belongs to $|r\boldsymbol{p}^*\boldsymbol{C}_0 - \sum_{i\in I}\sum_{j\in J_i} m_{i,j}\boldsymbol{\Xi}_{i,j}|$. Also, the pushfoward to $\boldsymbol{C}$ is given by $\boldsymbol{f}_*\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{E}}(-\boldsymbol{D})$. Therefore, we define

$$\boldsymbol{R}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^\bullet_{\boldsymbol{D}}, \boldsymbol{\Phi}) = \boldsymbol{\mathcal{F}}$$

with $\beta_{i,j} = \alpha_{i,j}$ for $i \in I, j \in J_i$. The stability follows from the same argument in [DDP18, Section 3.3].

Remark 4.3 (Elementary modification). The equation (19) implies that the recursive construction of $\boldsymbol{\mathcal{F}}_{1,k}$'s can be interpreted as an iterative elementary modification. For example, one can start with the sheaf $\boldsymbol{\mathcal{F}}'_{1,1} = \ker(\boldsymbol{G} \twoheadrightarrow \boldsymbol{R}_{1,1})$ on $\boldsymbol{M}$, then define a sheaf $\boldsymbol{\mathcal{F}}'_{1,2}$ on $\boldsymbol{M}_1$ via elementary modification

$$\boldsymbol{\mathcal{F}}'_{1,2} := \ker(\boldsymbol{p}^*_{1,1}\boldsymbol{\mathcal{F}}'_{1,1} \to \boldsymbol{p}^*_{1,1}\boldsymbol{R}_{1,2} \twoheadrightarrow \boldsymbol{p}^*_{1,1}\boldsymbol{R}_{1,2}|_{\boldsymbol{M_{p_1}}})$$

Proceeding recursively, we get pure dimension one sheaves $\boldsymbol{\mathcal{F}}'_{1,k}$ over $\boldsymbol{M}_{1,k}$ such that $\boldsymbol{\mathcal{F}}_{1,k} = \boldsymbol{p}^{1,k}_{1,\ell(1)}\boldsymbol{\mathcal{F}}'_{1,k}$.

Proof of Theorem 4.2. We show that $\boldsymbol{Q}$ and $\boldsymbol{R}$ are inverse to each other. First, we prove that $\boldsymbol{R} \circ \boldsymbol{Q} = \mathrm{Id}$. This direction is clear from the construction. For $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{M}}$, the support of $\boldsymbol{f}^*\boldsymbol{f}_*\boldsymbol{\mathcal{F}}$ lies in $|r\boldsymbol{p}^*\boldsymbol{C}_0|$. The mapping $\boldsymbol{R}$ describes the strict transform of $\boldsymbol{f}_*\boldsymbol{\mathcal{F}}$, which becomes isomorphic to $\boldsymbol{\mathcal{F}}(-\boldsymbol{p}^*\boldsymbol{M_D})$.

Next, we show that $\boldsymbol{Q} \circ \boldsymbol{R} = \mathrm{Id}$. Let $\boldsymbol{R}(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^\bullet_{\boldsymbol{D}}, \boldsymbol{\Phi}) = \boldsymbol{\mathcal{F}}$. We show that the filtration on the constructed pure dimension one sheaf $\boldsymbol{\mathcal{F}}$ recovers the parabolic structure on $\boldsymbol{\mathcal{E}}(-\boldsymbol{D})$.

Recall that for each $i \in I$ we have the filtration

$$0 \subset \boldsymbol{\mathcal{F}}^{i,\ell(i)} \subset \boldsymbol{\mathcal{F}}^{i,\ell(i)-1} \subset \cdots \subset \boldsymbol{\mathcal{F}}^{i,1} \subset \boldsymbol{\mathcal{F}}^{i,0} = \boldsymbol{\mathcal{F}}$$

where $\boldsymbol{\mathcal{F}}^{i,k} := \ker(\boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}} \otimes \mathcal{O}_{\sum_{j=1}^k \boldsymbol{\Xi}_{i,k}})$. We first look at the $\boldsymbol{\mathcal{F}}^{1,1}$. Consider the short exact sequence of sheaves over $\boldsymbol{Z}$,

$$0 \to \mathcal{O}(-\boldsymbol{\Xi}_{1,1}) \to \mathcal{O} \to \mathcal{O}_{\boldsymbol{\Xi}_{1,1}} \to 0.$$

By tensoring with $\boldsymbol{\mathcal{F}}$, we get the right exact sequence

$$\cdots \to \boldsymbol{\mathcal{F}}(-\boldsymbol{\Xi}_{1,1}) \to \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}} \otimes \mathcal{O}_{\boldsymbol{\Xi}_{1,1}} \to 0$$

which becomes left exact when the support of $\boldsymbol{\mathcal{F}}$ does not have $\boldsymbol{\Xi}_1$ as a component. If we push forward this sequence to $\boldsymbol{M}_{1,1}$ via $\boldsymbol{p}^{1,1}_{n,\ell(n)} : \boldsymbol{Z} \to \boldsymbol{M}_{1,1}$, then it becomes

$$\cdots \to \boldsymbol{\mathcal{F}}_{1,1}(-\boldsymbol{E}_{1,1}) \to \boldsymbol{\mathcal{F}}_{1,1} \to \boldsymbol{\mathcal{F}}_{1,1} \otimes \mathcal{O}_{\boldsymbol{E}_{1,1}} \to 0 \tag{20}$$

because $(\boldsymbol{p}^{1,1}_{n,\ell(n)})_*\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_{1,1}$ and $(\boldsymbol{p}^{1,1}_{n,\ell(n)})^*\mathcal{O}_{\boldsymbol{E}_{1,1}} = \mathcal{O}_{\boldsymbol{\Xi}_{1,1}}$. The surjectivity follows from the fact that the higher direct image of $\mathcal{O}_{\boldsymbol{\Xi}_{1,1}}$ vanishes. Then we have $(\boldsymbol{p}^{1,1}_{n,\ell(n)})_*\boldsymbol{\mathcal{F}}^1 = \ker(\boldsymbol{\mathcal{F}}_{1,1} \twoheadrightarrow \boldsymbol{\mathcal{F}}_{1,1} \otimes \mathcal{O}_{\boldsymbol{E}_{1,1}}) = \mathrm{Im}(\boldsymbol{\mathcal{F}}_{1,1}(-\boldsymbol{E}_{1,1}) \to \boldsymbol{\mathcal{F}}_{1,1})$. If we pushfoward the last two term of the sequence (20), then it becomes $\boldsymbol{G}_{1,1} \to \boldsymbol{G}_{1,1}|_{\boldsymbol{p_1}} \to 0$ where we recall that $\boldsymbol{G}_{1,1} = \ker(\boldsymbol{G} \twoheadrightarrow \boldsymbol{R}_{1,1})$. Due to the construction, the pushfoward of this sequence under $\boldsymbol{\pi} : \boldsymbol{M} \to \boldsymbol{C}$ becomes $\boldsymbol{\mathcal{E}}^{1,1} \to \boldsymbol{\mathcal{E}}^{1,1}_{\boldsymbol{p_1}} \to 0$. Therefore, $(\boldsymbol{f})_*\boldsymbol{\mathcal{F}}^{1,1} = \boldsymbol{\mathcal{E}}^{1,1}(-\boldsymbol{p_1})$.

For the next step, we note that $\boldsymbol{\mathcal{F}}^{1,2}$ can be viewed as the kernel of the map $\boldsymbol{\mathcal{F}}^{1,1} \to \boldsymbol{\mathcal{F}}^{1,1} \otimes \mathcal{O}_{\Xi_{1,2}}$. It means that we can apply the previous argument again to achieve the right exact sequence of sheaves over $\boldsymbol{M}_{1,2}$;

$$\cdots \boldsymbol{\mathcal{F}}_{1,2}(-\boldsymbol{E}_{1,2}) \to \boldsymbol{\mathcal{F}}_{1,2} \to \boldsymbol{\mathcal{F}}_{1,2} \otimes \mathcal{O}_{\boldsymbol{E}_{1,2}} \to 0 \tag{21}$$

because $(\boldsymbol{p}^{1,2}_{n,\ell(n)})_* \boldsymbol{\mathcal{F}}^{1,1} = \boldsymbol{\mathcal{F}}_{1,2}$ and $(\boldsymbol{p}^{1,2}_{n,\ell(n)})^*(\boldsymbol{E}_{1,2}) = \Xi_{1,2}$. The two rightmost term is push-forwarded to $\boldsymbol{\mathcal{E}}^{1,2} \to \boldsymbol{\mathcal{E}}^{1,2}_{\boldsymbol{p_1}} \to 0$, hence we have $(\boldsymbol{f})_* \boldsymbol{\mathcal{F}}^{1,2} = \boldsymbol{\mathcal{E}}^{1,2}(-\boldsymbol{p_1})$.

Applying the same argument inductively, we recover the induced filtration $\boldsymbol{\mathcal{E}}^{\bullet}(-\boldsymbol{p_1})$ on $\boldsymbol{\mathcal{F}}(-\boldsymbol{p_1})$. Furthermore, by continuing this process for the other components of $\boldsymbol{D}$, we obtain the induced filtration $\boldsymbol{\mathcal{E}}^{\bullet}(-\boldsymbol{D})$ on $\boldsymbol{\mathcal{F}}(-\boldsymbol{D})$.

The Higgs field on $\boldsymbol{\mathcal{E}(-D)}$ is also induced by the multiplication of the tautological section, hence we have

$$(\boldsymbol{Q} \circ \boldsymbol{R})((\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^{\bullet}_{\boldsymbol{D}}, \boldsymbol{\Phi})) = (\boldsymbol{\mathcal{E}}(-\boldsymbol{D}), \boldsymbol{\mathcal{E}}_{\boldsymbol{D}}(-\boldsymbol{D})^{\bullet}, \boldsymbol{\Phi})$$

Since $(\boldsymbol{\mathcal{E}}(-\boldsymbol{D}), \boldsymbol{\mathcal{E}}_{\boldsymbol{D}}(-\boldsymbol{D})^{\bullet}, \boldsymbol{\Phi}) \otimes \mathcal{O}(\boldsymbol{D}) = (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}^{\bullet}_{\boldsymbol{D}}, \boldsymbol{\Phi})$, we get the conclusion. □

Corollary 4.4 (Fitting support morphism vs Hitchin morphism)**.** The following diagram commutes:

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}(\vec{m}) \arrow[r] \arrow[d] & \mathrm{Higgs}^{\mathrm{par}}(\vec{m}) \times \mathcal{N}(\vec{m}) \arrow[d, "h^{\mathrm{par}}(\vec{m})"] \\ \boldsymbol{B}(\vec{m}) \arrow[r, hook, "\iota"] & \boldsymbol{A} \end{tikzcd}$$

where ι is defined in Remark 3.15.

Proof. The commutativity follows from the definition of $\boldsymbol{Q}$. □

4.3. **Relations with other spectral correspondences.** We can also study functorial properties of the spectral correspondences with respect to a refinement of the given partition. Let $\vec{m}$ be a partition of r and $\vec{m}'$ be its refinement. Recall that there exists a canonical embedding $\iota^{\vec{m}'}_{\vec{m}} : \mathcal{N}(\vec{m}) \hookrightarrow \mathcal{N}(\vec{m}')$. We abuse to write $\boldsymbol{\mathcal{H}}(\vec{m}') = (\iota^{\vec{m}'}_{\vec{m}})^* \boldsymbol{\mathcal{H}}(\vec{m}')$ and $\boldsymbol{\mathcal{M}}(\vec{m}') = (\iota^{\vec{m}'}_{\vec{m}})^* \boldsymbol{\mathcal{M}}(\vec{m}')$. Recall that there exists an increasing sequence of integers $0 = a_{i0} \leq a_{i1} \leq \cdots \leq a_{i\ell} = \ell'(i)$ such that $m_{ik} = \sum_{a_{i,j-1} < k \leq a_{i,j}} m'_{ik}$ for all $i \in I, j \in J_i$.

On the Hitchin side, we introduced a closed relative subspace $\boldsymbol{\mathcal{H}}(\vec{m}', \vec{m}) \subset \boldsymbol{\mathcal{H}}(\vec{m}')$ that consists of $\vec{\xi}'$-parabolic Higgs bundles satisfying $\vec{\xi}$-parabolic condition (see (8)). Note that it admits a forgetful map to $\boldsymbol{\mathcal{H}}(\vec{m})$ over $\mathcal{N}(\vec{m})$. On the surface side, we define a closed relative subspace $\boldsymbol{\mathcal{M}}(\vec{m}', \vec{m}) \subset \boldsymbol{\mathcal{M}}(\vec{m}')$ that consists of relative pure dimensional one sheaves with the restricted support $\boldsymbol{\Sigma}_{\vec{m}}$ satisfying the following property: For each $i \in I$, the induced morphism $\boldsymbol{s}$ on the successive quotient $\boldsymbol{\mathcal{F}}^{i,a_{ik}} / \boldsymbol{\mathcal{F}}^{i,a_{i,k-1}}$ is a scalar multiplication by $\xi_{i,k}$ for every $\vec{\xi} \in \mathcal{N}(\vec{m})$. Then there is a pushforward map

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}(\vec{m}') & \\ \boldsymbol{\mathcal{M}}(\vec{m}', \vec{m}) \arrow[u, hook] \arrow[r, "\boldsymbol{f}^{\vec{m}}_{\vec{m}'}"] & \boldsymbol{\mathcal{M}}(\vec{m}) \end{tikzcd} \tag{22}$$

The following result comes from Theorem 4.2.

Theorem 4.5. *For any pair of partitions $\vec{m}' \geq \vec{m}$, there is a commutative diagram of relative moduli spaces over $\mathcal{N}(\vec{m})$,*

$$
\begin{tikzcd}
\boldsymbol{\mathcal{H}}(\vec{m}') \arrow[rrr, "\cong"] \arrow[ddd, "\boldsymbol{For}"'] & & & \boldsymbol{\mathcal{M}}(\vec{m}') \arrow[ddd, "\boldsymbol{f_*}"] \\
& \boldsymbol{\mathcal{H}}(\vec{m}', \vec{m}) \arrow[r, "\cong"] \arrow[ul] \arrow[d, "\boldsymbol{For}^{\vec{m}}_{\vec{m}'}"] & \boldsymbol{\mathcal{M}}(\vec{m}', \vec{m}) \arrow[ur, hook] \arrow[d, "(\boldsymbol{f}^{\vec{m}}_{\vec{m}'})_*"] & \\
& \boldsymbol{\mathcal{H}}(\vec{m}) \arrow[r, "\cong"] \arrow[dl, "\boldsymbol{For}"] & \boldsymbol{\mathcal{M}}(\vec{m}) \arrow[dr, "\boldsymbol{f_*}"'] & \\
\boldsymbol{\mathcal{H}}^{mero} \arrow[rrr, "\cong"] & & & \boldsymbol{\mathcal{M}}^{mero}
\end{tikzcd}
\tag{23}
$$

where the horizontal arrows are the spectral correspondences, the left-hand side diagram is in (8)*, and the right-hand side diagram is in* (22)*.*

In the commutative diagram (23), the preimage over $\boldsymbol{For}$ parametrizes flags of type $\vec{m}$. We explain how this can be seen in the geometry of surfaces. For simplicity, let's work over non-relative setting by fixing $\vec{\xi} \in \mathcal{N}(\vec{m})$. Consider the diagram

$$
\begin{tikzcd}
\mathcal{M}_{\Sigma(\vec{m})}(S) \arrow[d, "p_*"] & \\
\mathcal{M}_{rC_0}(M^\circ) \arrow[r, "p^*"] & \mathcal{M}_{rp^*C_0}(S)
\end{tikzcd}
$$

where the subscript indicates the curve class, and we abuse notation by omitting $\vec{\xi}$. First, the spectral correspondence for the meromorphic case tells us that for any sheaf $\mathcal{F}' \in \mathcal{M}_{rC_0}(M^\circ)$, $\mathcal{F}' \cong \pi^*\pi_*\mathcal{F}' \in \mathcal{M}_{rC_0}(M^\circ)$ where $\pi = \pi_C$. Let $\mathcal{F} \in \mathcal{M}_{\Sigma_{\vec{m}}}(S)$. Then $p_*\mathcal{F}$ is a pure dimension one sheaf over M. Note that the spectral correspondence for the parabolic case (Theorem 4.2) tells that the strict transform of $p_*\mathcal{F}$ (more precisely $f_*\mathcal{F}$) is isomorphic to $\mathcal{F}$ ($R \circ Q = id$). Therefore, for any $\mathcal{F}' \in \mathcal{M}_{rC_0}(M^\circ)$, every element of $(p_*)^{-1}(\mathcal{F}')$ is obtained by a strict transform of $\mathcal{F}'$ (more precisely the a strict transform of $\pi_*\mathcal{F}'$). Since such strict transform is determined by a flag of $\pi_*\mathcal{F}'$ of type $\vec{m}$ at D whose associated Higgs bundle is $\vec{\xi}$-parabolic, we can see that the preimage $(p_*)^{-1}(\mathcal{F}')$ is parametrized by flags in $\mathcal{F}'$ at D.

Note that there is a surjective morphism

$$\bigcup For(\vec{m}) : \bigcup \mathcal{H}(\vec{m}) \twoheadrightarrow \mathcal{H}^{mero}.$$

This is because for any meromorphic Higgs bundle $(\mathcal{E}, \Phi)$, there exists a flag of $\mathcal{E}_D$ of some type $\vec{m}$ such that the block diagonal part of Φ_D is indeed diagonal. On the surface side, it implies that for any $\mathcal{F}' \in \mathcal{M}_{rC_0}(M^\circ)$, there exists a blown up surface $p : Z \to M$ and pure dimension one sheaf $\mathcal{F}$ on S with prescribed support so that $p_*\mathcal{F} \cong \mathcal{F}'$.

5. Non-emptiness results

5.1. **Controllability of line bundles.**

Recall that the non-emptiness problem of $B(\vec{m})_{\vec{\xi}}$ can be formulated as the existence of sections of line bundles whose local derivatives satisfy certain system of linear equations. We begin by some simple criteria which guarantee the existence of sections with prescribed local derivatives.

Definition 5.1. *Given a set of distinct points $p_1, \cdots, p_n$, we say that a line bundle L is controllable up to order $(t(p_1), \ldots, t(p_n))$ at $(p_1, \cdots, p_n)$ if the restriction $H^0(C, L) \to H^0(C, L|_{t(p_1)p_1 + \cdots + t(p_n)p_n})$ is surjective.*

In other words, if L is controllable up to order $(t(p_1), \dots, t(p_n))$ at $(p_1, \cdots, p_n)$, there exists a section $s \in H^0(C, L)$ and local charts around p_i's such that the local derivatives of s at p_i, $s^{(a)}(p_i)$, can be any values in $\mathbb{C}$ for $a < t(p_i)$. In this paper, we will only apply this notion to the line bundles $L = (K_C(D))^{\otimes \mu}$ where $D = p_1 + \cdots + p_n$ and $\mu = 1, \dots, r$. In this case, we always have $H^1(C, L) = 0$.

Lemma 5.2. Suppose that $H^1(C, L) = 0$ and et $L' = L\left(-\sum_{i=1}^n t(p_i)p_i\right)$. Then L is controllable up to order $(t(p_1), \dots, t(p_n))$ at $(p_1, \cdots, p_n)$ if and only if

$$H^1\left(C, L'\right) = 0$$

Proof. The result follows directly from the following long exact sequence

(24)

$$0 \to H^0(C, L') \to H^0(C, L) \to H^0(C, L|_{t(p_1)p_1 + \cdots + t(p_n)p_n}) \to H^1(C, L') \to H^1(C, L) \to \cdots$$

□

Lemma 5.3 (Controllability inequality). Suppose that $H^1(C, L) = 0$. If the following inequality holds

$$\sum_{i=1}^n t(p_i) < \deg(L) - (2g - 2), \tag{25}$$

then L is controllable up to order $(t(p_1), \dots, t(p_n))$ at $(p_1, \dots, p_n)$.

Proof. By Serre duality, we have $H^1(C, L') \cong H^0(C, (L')^\vee \otimes K_C)^\vee$. In particular, when $\deg((L')^\vee \otimes K_C) < 0$ which is equivalent to the inequality (25), we have $H^1(C, L') = 0$ and hence the controllability of L by Lemma 5.2. □

Remark 5.4. Suppose that $H^1(C, L) = 0$ holds. When $L' \cong K_C$, or equivalently, $L \cong K_C(\sum_{i=1}^n t(p_i)p_i)$ we have $H^1(C, L') = H^0(C, \mathcal{O})^\vee = \mathbb{C}$. In this case, L is not controllable up to order $(t(p_1), \dots, t(p_n))$ at $(p_1, \dots, p_n)$. By the long exact sequence (24), the image of $H^0(C, L) \to H^0(C, L|_{t(p_1)p_1 + \cdots + t(p_n)p_n})$ is described by the kernel of $f : H^0(C, L|_{t(p_1)p_1 + \cdots + t(p_n)p_n}) \to H^1(C, L')$. For example, when $t(p_1) = \cdots = t(p_n) = 1$ and $L \cong K_C(\sum_{i=1}^n p_i)$, an element $(b_1, \dots, b_n) \in H^0(C, L|_{p_1 + \cdots + p_n})$ comes from a section $s \in H^0(C, L)$ if and only if $f(b_1, \dots, b_n) = b_1 + \cdots + b_n = 0$.

Remark 5.5 (The *OK* conditions). In the setting of parabolic Higgs bundles, we are given $\vec{m} = (\underline{m}_1, \dots, \underline{m}_n)$. Let $\gamma_{P^i}(\mu)$ denote the level function associated to the partition $P^i = \underline{m}_i$. Applying Lemma 5.2 to the line bundle $L = (K_C(D))^{\otimes \mu}$ with $t(p_i) = \gamma_{P^i}(\mu)$, we find that the condition that $(K_C(D))^{\otimes \mu}$ is controllable up to order $(\gamma_{P^1}(\mu), \dots, \gamma_{P^n}(\mu))$ at $(p_1, \dots, p_n)$ for $\mu = 2, \dots, r$ is equivalent to the *OK* condition for the collection of line bundles $L_{\vec{m}}(\mu)$, $\mu = 2, \dots, r$.

5.2. **Non-emptiness of Hitchin bases.** In this section, we will study the non-emptiness problem of the subset $B(\vec{m})_{\vec{\xi}} \subset |rC_0|$ for any given pair $(\vec{m}, \vec{\xi})$. In general, the problem of showing the existence of divisors of a linear system with prescribed multiplicity conditions at some points on a surface is non-trivial. In our case, we will use the alternative description of $B(\vec{m})_{\vec{\xi}}$ in terms of the subspace of A defined by the vanishing of evaluations over the level domains associated to the partitions $P^{\xi^\circ_{i,j}}$ (Proposition 3.12)

Recall from Section 3.2 that for each $p_i \in D$, after choosing a local trivialization of M around p_i, we have the evaluations of local derivatives of a section s_μ at p_i, $s_\mu^{(a)}(p_i) := \frac{\partial}{\partial x_i^a}\Big|_0 s_\mu$. By Proposition 3.12, an element in $B(\vec{m})_{\vec{\xi}}$ is characterized by $s = (s_1, \dots, s_r) \in A$ satisfying the vanishing of the evaluation maps, which can be written out explicitly as follows: for $i = 1, \ldots, n$,

$$
\begin{cases}
\frac{\partial}{\partial y^u}\frac{\partial}{\partial x_i^a}\Big|_{(0,\xi^\circ_{i,1})} F_s(x_i, y) = 0 & (u, a) \in G(P^{\xi^\circ_{i,1}}) \\
\quad\vdots & \\
\frac{\partial}{\partial y^u}\frac{\partial}{\partial x_i^a}\Big|_{(0,\xi^\circ_{i,e(i)})} F_s(x_i, y) = 0 & (u, a) \in G(P^{\xi^\circ_{i,e(i)}})
\end{cases}
\tag{26}
$$

where $F_s(x_i, y) = y^r + s_1(x_i)y^{r-1} + \cdots + s_r(x_i)$ and $a, b \in \mathbb{Z}_{\geq 0}$. These are all linear equations that the evaluations $s^{(a)}_\mu(p_i)$ have to satisfy.

Examples 5.6. Let $n = 1$, $\vec{m} = \underline{m} = (3, 2, 1)$ and $\vec{\xi} = \underline{\xi} = (\xi_1, \xi_2, \xi_3)$ with $\xi_1 = \xi_3$. So, the distinct part $\underline{\xi}^\circ = (\xi^\circ_1, \xi^\circ_2) = (\xi_1, \xi_2)$ and

$$
P^{\xi^\circ_1} = (3, 1), \quad P^{\xi^\circ_2} = (2)
$$

The corresponding level domains are shown below

$$
G(P^{\xi^\circ_1}) = \begin{array}{c|cccc}
a=2 & \bullet & & & \\
a=1 & \bullet & \bullet & & \\
a=0 & \bullet & \bullet & \bullet & \bullet \\
\hline
 & u=0 & u=1 & u=2 & u=3
\end{array}
\qquad
G(P^{\xi^\circ_2}) = \begin{array}{c|cc}
a=1 & \bullet & \\
a=0 & \bullet & \bullet \\
\hline
 & u=0 & u=1
\end{array}
$$

Then the equations indexed by $G(P^{\xi^\circ_1})$ are explicitly given by

$$
\begin{array}{rcrcrcrcrcrcl}
(\xi^\circ_1)^5 s_1(p) & + & (\xi^\circ_1)^4 s_2(p) & + & (\xi^\circ_1)^3 s_3(p) & + & (\xi^\circ_1)^2 s_4(p) & + & (\xi^\circ_1) s_5(p) & + & s_6(p) & = & -(\xi^\circ_1)^6 \\
5(\xi^\circ_1)^4 s_1(p) & + & 4(\xi^\circ_1)^3 s_2(p) & + & 3(\xi^\circ_1)^2 s_3(p) & + & 2(\xi^\circ_1) s_4(p) & + & s_5(p) & & & = & -6(\xi^\circ_1)^5 \\
20(\xi^\circ_1)^3 s_1(p) & + & 12(\xi^\circ_1)^2 s_2(p) & + & 6(\xi^\circ_1) s_3(p) & + & 2s_4(p) & & & & & = & -30(\xi^\circ_1)^4 \\
60(\xi^\circ_1)^2 s_1(p) & + & 24(\xi^\circ_1) s_2(p) & + & 6s_3(p) & & & & & & & = & -120(\xi^\circ_1)^3 \\
(\xi^\circ_1)^5 s^{(1)}_1(p) & + & (\xi^\circ_1)^4 s^{(1)}_2(p) & + & (\xi^\circ_1)^3 s^{(1)}_3(p) & + & (\xi^\circ_1)^2 s^{(1)}_4(p) & + & (\xi^\circ_1) s^{(1)}_5(p) & + & s^{(1)}_6(p) & = & 0 \\
5(\xi^\circ_1)^4 s^{(1)}_1(p) & + & 4(\xi^\circ_1)^3 s^{(1)}_2(p) & + & 3(\xi^\circ_1)^2 s^{(1)}_3(p) & + & 2(\xi^\circ_1) s^{(1)}_4(p) & + & s^{(1)}_5(p) & & & = & 0 \\
(\xi^\circ_1)^5 s^{(2)}_1(p) & + & (\xi^\circ_1)^4 s^{(2)}_2(p) & + & (\xi^\circ_1)^3 s^{(2)}_3(p) & + & (\xi^\circ_1)^2 s^{(2)}_4(p) & + & (\xi^\circ_1) s^{(2)}_5(p) & + & s^{(2)}_6(p) & = & 0
\end{array}
$$

and the equations indexed by $G(P^{\xi^\circ_2})$ are explicitly given by

$$
\begin{array}{rcrcrcrcrcrcl}
(\xi^\circ_2)^5 s_1(p) & + & (\xi^\circ_2)^4 s_2(p) & + & (\xi^\circ_2)^3 s_3(p) & + & (\xi^\circ_2)^2 s_4(p) & + & (\xi^\circ_2) s_5(p) & + & s_6(p) & = & -(\xi^\circ_2)^6 \\
5(\xi^\circ_2)^4 s_1(p) & + & 4(\xi^\circ_2)^3 s_2(p) & + & 3(\xi^\circ_2)^2 s_3(p) & + & 2(\xi^\circ_2) s_4(p) & + & s_5(p) & & & = & -6(\xi^\circ_2)^5 \\
(\xi^\circ_2)^5 s^{(1)}_1(p) & + & (\xi^\circ_2)^4 s^{(1)}_2(p) & + & (\xi^\circ_2)^3 s^{(1)}_3(p) & + & (\xi^\circ_2)^2 s^{(1)}_4(p) & + & (\xi^\circ_2) s^{(1)}_5(p) & + & s^{(1)}_6(p) & = & 0
\end{array}
$$

Define the vector space

$$
V := \bigoplus_{\mu=1}^{r} H^0\left(C, (K_C(D))^{\otimes\mu}|_{\sum_{i=1}^n m_{i,1}p_i}\right).
$$

Given the choice of local trivializations around $p_i \in D$, there is a natural choice of basis $\{e_{\mu,a,i}\}$ of V such that the restriction of s_μ in $H^0(C, (K_C(D))^{\otimes\mu}|_{m_{i,1}p_i})$ can be written as $\sum_{a=0}^{m_{i,1}-1} s^{(a)}_\mu(p_i)e_{\mu,a,i}$. The index set of the basis is given by

$$
\Pi := \{(\mu, a, i) \in \mathbb{Z}^{\geq 0} | 1 \leq \mu \leq r, 0 \leq a < m_{i,1}, 1 \leq i \leq n\}.
$$

Hence, we can regard V as the set of relevant evaluations $s^{(a)}_\mu(p_i)$. We denote by $\Pi(a_0, i_0)$ the subset of Π where $a = a_0$ and $i = i_0$, i.e.,

$$
\Pi(a = a_0, i = i_0) = \{(\mu, a, i) \in \Pi \mid a = a_0, i = i_0\}.
$$

Similarly, we define $\Pi(\mu = \mu_0)$ as the subset of Π where $\mu = \mu_0$, i.e.,

$$\Pi(\mu = \mu_0) = \{(\mu, a, i) \in \Pi \mid \mu = \mu_0\},$$

and so on for other fixed entries.

The set of all linear equations (26) can be divided into different systems of equations labeled by a and i, each of which only involves $s_1^{(a)}(p_i), \dots, s_r^{(a)}(p_i)$. For each eigenvalue $\xi^\circ_{i,j}$, the number of equations involving x-derivative of order a is

$$c(a, \xi^\circ_{i,j}) := \#\{u | \gamma_{P^{\xi^\circ_{i,j}}}(u) \geq a\}.$$

Hence, the number of equations involving $s_1^{(a)}(p_i), \dots, s_r^{(a)}(p_i)$ is $c(a,i) := \sum_{j=1}^{e(i)} c(a, \xi^\circ_{i,j})$ and we denote by $A_{a,i}X_{a,i} = B_{a,i}$ the system of equations of size $c(a,i) \times r$ and regard $X_{a,i} = (x_{\mu,a,i})^T_{1 \leq \mu \leq r}$ as variables where $x_{\mu,a,i} : V \to \mathbb{C}$ is the dual basis of $e_{\mu,a,i}$. Note that for $a > 0$, the system of equations is homogeneous i.e. $B_{a,i} = 0$. More explicitly, define the following $c \times r$ matrix where the first row is $(\xi^{r-1}, \xi^{r-2}, \dots, \xi, 1)$ and the k-th row is the k-th derivative of the first row (divided by $k!$):

$$R(\xi, c) = \begin{pmatrix} \xi^{r-1} & \dots & \dots & & \xi & 1 \\ (r-1)\xi^{r-2} & \dots & \dots & & 1 & 0 \\ & & \vdots & \iddots & & \\ \frac{(r-1)!}{(c-1)!}\xi^{r-c} & \dots & (r-c-1)! & 0 & \dots & 0 \end{pmatrix}$$

and similarly the $c \times 1$ matrix $K(\xi, c) = (\xi^r, r\xi^{r-1}, \dots, \frac{r!}{c!}\xi^{r-c+1})^T$. Then the coefficient matrix is given by

$$A_{a,i} = \begin{pmatrix} R(\xi^\circ_{i,1}, c(a, \xi^\circ_{i,1})) \\ \vdots \\ R(\xi^\circ_{i,e(i)}, c(a, \xi^\circ_{i,e(i)})) \end{pmatrix}$$

Meanwhile, we have $B_{a,i} = 0$ for $a > 0$ and

$$B_{0,i} = \begin{pmatrix} K(\xi^\circ_{i,1}, c(a, \xi^\circ_{i,1})) \\ \vdots \\ K(\xi^\circ_{i,e(i)}, c(a, \xi^\circ_{i,e(i)})) \end{pmatrix}$$

Remark 5.7. Note that we divide the equations in (26) by some factor of $k!$ to align with the form of a generalized matrix. The $c(a,i) \times c(a,i)$ submatrix $A'_{a,i}$ formed by the last $c(a,i)$ columns is a generalized Vandermonde matrix [Kal84]. A fundamental property about the generalized Vandermonde matrix $A'_{a,i}$ is that $A'_{a,i}$ is invertible if and only if the entries $\xi^\circ_{i,j}$ are pairwise distinct i.e. $\xi^\circ_{i,j} \neq \xi^\circ_{i,j'}$ for all $j \neq j'$ [Kal84, Page 19], which is true under our assumption of $\vec{\xi}^\circ$.

Thus, the systems of linear equations can be expressed succinctly in terms of a linear map

$$T : V \to \mathbb{C}^s, \quad \text{where} \quad s = \sum_{a,i} c(a,i)$$

such that the solution space is given by $T^{-1}(\beta)$ where β is the vector formed by all $B_{a,i}$ for $a \geq 0$.

Examples 5.8. Continuing with Example 5.6, the sets of equations are divided into systems of equations $A_a X_a = B_a$ where $a = 0, 1, 2$ (here $i = 1$, so we drop the subscript i) and the coefficient matrices are given by

$$A_0 = \begin{pmatrix} (\xi_1^\circ)^5 & (\xi_1^\circ)^4 & (\xi_1^\circ)^3 & (\xi_1^\circ)^2 & (\xi_1^\circ) & 1 \\ 5(\xi_1^\circ)^4 & 4(\xi_1^\circ)^3 & 3(\xi_1^\circ)^2 & 2(\xi_1^\circ) & 1 & 0 \\ 10(\xi_1^\circ)^3 & 6(\xi_1^\circ)^2 & 3(\xi_1^\circ) & 1 & 0 & 0 \\ 10(\xi_1^\circ)^2 & 4(\xi_1^\circ) & 1 & 0 & 0 & 0 \\ (\xi_2^\circ)^5 & (\xi_2^\circ)^4 & (\xi_2^\circ)^3 & (\xi_2^\circ)^2 & (\xi_2^\circ) & 1 \\ 5(\xi_2^\circ)^4 & 4(\xi_2^\circ)^3 & 3(\xi_2^\circ)^2 & 2(\xi_2^\circ) & 1 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -(\xi_1^\circ)^6 \\ -6(\xi_1^\circ)^5 \\ -15(\xi_1^\circ)^4 \\ -20(\xi_1^\circ)^3 \\ -(\xi_2^\circ)^6 \\ -6(\xi_2^\circ)^5 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} (\xi_1^\circ)^5 & (\xi_1^\circ)^4 & (\xi_1^\circ)^3 & (\xi_1^\circ)^2 & (\xi_1^\circ) & 1 \\ 5(\xi_1^\circ)^4 & 4(\xi_1^\circ)^3 & 3(\xi_1^\circ)^2 & 2(\xi_1^\circ) & 1 & 0 \\ (\xi_2^\circ)^5 & (\xi_2^\circ)^4 & (\xi_2^\circ)^3 & (\xi_2^\circ)^2 & (\xi_2^\circ) & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} (\xi_1^\circ)^5 & (\xi_1^\circ)^4 & (\xi_1^\circ)^3 & (\xi_1^\circ)^2 & (\xi_1^\circ) & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 \end{pmatrix}$$

On the other hand, we consider the subspace of V which comes from global sections i.e.

$$S := \operatorname{Im}\left(\bigoplus_{\mu=1}^{r} H^0\left(C, (K_C(D))^{\otimes \mu}\right) \to V \right) \subset V$$

As our goal is to find sections satisfying the constraints (26), the question can be formulated as follows: Does the affine subspace $T^{-1}(\beta)$ intersect the linear subspace S non-trivially in V?

Define the standard decomposition $\Pi = \Pi_{\text{pivot}} \sqcup \Pi_{\text{free}}$ where

$$\Pi_{\text{pivot}} := \{(\mu, a, i) \subset \Pi | r - c(a,i) + 1 \le \mu \le r\}, \quad \Pi_{\text{free}} := \Pi_{\text{pivot}}^c \subset \Pi. \tag{27}$$

Lemma 5.9. Let $V = V_{\text{free}} \oplus V_{\text{pivot}}$ be the direct sum decomposition induced by the standard decomposition (27). Then there is an affine map $H : V_{\text{free}} \to V_{\text{pivot}}$ whose graph is $T^{-1}(\beta)$.

Proof. Since $T^{-1}(\beta)$ is the solution set of the systems of equations $A_{a,i}X_{a,i} = B_{a,i}$. When $a = 0$, the matrix $A_{0,i}$ is always a square generalized Vandermonde matrix, which is invertible, so there is always a unique solution for $A_{0,i}X_{0,i} = B_{0,i}$.

For $a > 0$ and $1 \le i \le n$, in order to solve the homogeneous system of equations $A_{a,i}X_{a,i} = 0$, it suffices to find $c(a,i)$ columns which form an invertible submatrix of $A_{a,i}$ such that $A_{a,i}X_{a,i} = 0$ is equivalent to an expression of the pivot variables (corresponding to the choice of columns) in terms of the free variables. In our case, we take the submatrix $A'_{a,i}$ formed by the last $c(a,i)$ columns i.e. $(\mu, a, i) \in \Pi_{\text{pivot}}$ which is invertible by Remark 5.7. Hence, Π_{pivot} and Π_{free} are the index sets of the pivot variables and free variables respectively.

Therefore, we get an affine map $R : V_{\text{free}} \to V_{\text{pivot}}$ from the expression of pivot variables in terms of free variables and the solution set $T^{-1}(\beta)$ is then the graph of H.

□

Proposition 5.10. *Let $\Pi = \Pi_{pivot} \sqcup \Pi_{free}$ be the standard decomposition. Define*

$$t(\mu_0, i_0) := \#\{a | (\mu_0, a, i_0) \in \Pi_{pivot}\}$$

for each pair $1 \le \mu_0 \le r, 1 \le i_0 \le n$. Suppose that $\vec{\xi} \in \mathcal{N}(\vec{m})$ and $(K_C(D))^{\otimes \mu}$ is controllable up to order $(t(\mu,1), \ldots, t(\mu,n))$ at $(p_1, \ldots, p_n)$ for $\mu = 2, \ldots, r$. Then $S \cap T^{-1}(\beta) \ne \emptyset$.

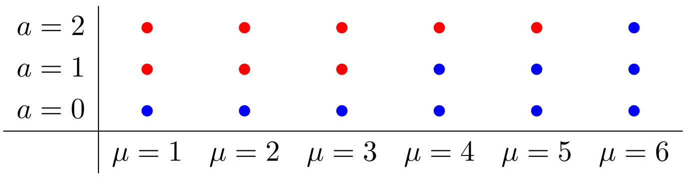
Proof. In terms of the vector spaces defined above, the assumptions that $\vec{\xi} \in \mathcal{N}(\vec{m})$ and the controllability of the line bundles $K_C(D)^{\otimes \mu}$ means that the composition $S \to V \to V_{\text{pivot}}$ is surjective. Moreover, if we write $S = \bigoplus_{\mu=1}^{r} S_\mu$ and $V_{\text{pivot},\mu} = \bigoplus_{\mu=1}^{r} V_{\text{pivot},\mu}$ the direct sum decomposition induced by $V = \bigoplus_{\mu=1}^{r} V_\mu$, then $S_\mu \to V_{\text{pivot},\mu}$ is surjective.

We will find an element $v = \sum_{(\mu,a,i)\in\Pi} c_{\mu,a,i} e_{\mu,a,i} \in S \cap T^{-1}(\beta)$ by induction. For simplicity, we call $c_{\mu,a,i}$ a pivot (resp. free) coefficient with $\mu = \mu_0$ if $(\mu, a, i) \in \Pi_{\text{pivot}}(\mu = \mu_0)$ (resp. $\Pi_{\text{free}}(\mu = \mu_0)$). Similarly, we call $c_{\mu,a,i}$ a pivot (resp. free) coefficient with $a = a_0, i = i_0$ if $(\mu, a, i) \in \Pi_{\text{pivot}}(a = a_0, i = i_0)$ (resp. $\Pi_{\text{free}}(a = a_0, i = i_0)$)

Since $A_{a,i}X_{a,i} = B_{a,i}$ always has a unique solution for $a = 0$, the coefficients $c_{\mu,a,i}$ are uniquely determined for $(\mu, a, i) \in \Pi(a = 0)$. These determine all the pivot coefficients with $\mu = 1$ which yield a vector in $V_{\text{pivot},\mu=1}$. Then we can lift this vector via the surjection $S_1 \twoheadrightarrow V_{\text{pivot},1}$ to a vector in S_1 which determines all the free coefficients with $\mu = 1$. Now, suppose all the coefficients are determined up to $\mu = k \leq r - 1$ and we want to determine the coefficients for $\mu = k + 1$. For each fixed a_0 and i_0, if c_{k+1,a_0,i_0} is a pivot coefficient, the set of coefficients $c_{k-\delta,a_0,i_0}$ with $\delta = 1, \ldots, k$ include all the free coeffcients with $a = a_0, i = i_0$ and are all determined by induction hypothesis. So, by applying the map $H : V_{\text{free}} \to V_{\text{pivot}}$ in Lemma 5.9, the pivot coefficient c_{k+1,a_0,i_0} is uniquely determined. Once all the pivot coefficients with $\mu = k+1$ are determined, we can use the surjective map $S_{k+1} \twoheadrightarrow V_{\text{pivot},\mu=k+1}$ to lift to a vector in S_{k+1} which determines the free coefficients with $\mu = k + 1$. Hence, all the coefficients $c_{\mu,a,i}$ are determined and the vector $v = \sum_{(\mu,a,i)\in\Pi} c_{\mu,a,i} e_{\mu,a,i}$ lies in both S and $T^{-1}(\beta) = \text{graph}(H)$ by construction.

□

Examples 5.11. Continuing with Example 5.8, we can concretely visualize the proof. In the table below, each dot represents a variable labeled by a pair (a, μ), forming the index set Π. The decomposition $\Pi = \Pi_{\text{free}} \sqcup \Pi_{\text{pivot}}$ is represented by dots of different colors: red dots for free variables Π_{free} (the free variables), and blue dots for the pivot variables Π_{pivot}.



The goal is to find a vector $v \in S \cap T^{-1}(\beta)$.

- ($v \in T^{-1}(\beta)$) To solve the systems of equations $A_a X_a = B_a$ for a fixed a, we first select values for the free variables (the red dots), which then determine the values of the pivot variables (the blue dots) in the same row.
- ($v \in S$) We also need to ensure that the choice of values for all the variables can be lifted to sections s_μ of the line bundles $(K_C(D))^{\otimes\mu}$. For a fixed μ, under the controllability assumption for $\mu > 1$ and $\vec{\xi} \in \mathcal{N}(\vec{m})$ for $\mu = 1$, a section s_μ can be found for any pivot variable values, which in turn determines the free variables. Pictorially, this means that the blue dots in a column determine the red dots in the same column. Here the values of the pivot variables for $a = 0$ are uniquely determined.

The idea of the proof then proceeds by moving from left to right across the columns. For each fixed μ, the blue dots determine the remaining red dots in the column. Once the red dots in a row are determined, they determine the remaining blue dots in that row. This process continues across all columns, and by the time we reach the final column, all the variables (both red and blue dots) are determined.

Proposition 5.12. *Continuing with the notation in Proposition 5.10, we have*

$$t(\mu, i) = \gamma_{P^i}(\mu) \quad \text{for } 1 \leq \mu \leq r, 1 \leq i \leq n.$$

Proof. In the standard decomposition $\Pi = \Pi_{\text{pivot}} \sqcup \Pi_{\text{free}}$, the elements in Π_{pivot} correspond to the set of pivot variables which is in (non-canonical) bijection with the set of linear equations where the a and i entries in $(\mu, a, i) \in \Pi_{\text{pivot}}$ denotes the variables involved in $A_{a,i}X_{a,i} = B_{a,i}$. The set of linear equations are in bijection with the set $G(P^{\xi^\circ_{i,k}})$ where an equation labeled by $(u, a) \in G(P^{\xi^\circ_{i,k}})$ belongs to the system of equations $A_{a,i}X_{a,i} = B_{a,i}$. So, we have a (non-canonical) bijection which preserves the a and i entries on both sides while matching the μ and u entries

$$\Pi_{\text{pivot}} \leftrightarrow \bigsqcup_{1 \leq i \leq n, 1 \leq k \leq e(i)} G(P^{\xi^\circ_{i,k}})$$

Since $t(\mu_0, i_0)$ counts the number of different a's in Π_{pivot} for each fixed μ_0 and i_0, we can count this number on the other side as well. On the other side, the number of different a's for each fixed u_0 and i_0 is given by the level function $\gamma_{P^{\xi^\circ_{i_0,k}}}(u_0)$.

By the definition of Π_{pivot}, $t(\mu, i_0)$ is increasing in $1 \leq \mu \leq r$ for fixed i_0. On the other side, arranging $\gamma_{P^{\xi^\circ_{i_0,k}}}(u_0)$ in an increasing order for a fixed i_0 is simply the level function associated to $P^i = \sum_{k=1}^{e(i_0)} P^{\xi^\circ_{i_0,k}}$. Hence, we have $t(\mu, i) = \gamma_{P^i}(\mu)$. ☐

Corollary 5.13. Suppose that the OK condition holds. Then for any $\vec{\xi} \in \mathcal{N}(\vec{m})$, the affine space $B(\vec{m})_{\vec{\xi}}$ is non-empty. In particular,

(1) when $n \geq 3$ and $g = 0$, if the inequalities

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 2 \tag{28}$$

hold for $\mu = 2, \ldots, r$, then $B(\vec{m})_{\vec{\xi}} \neq \emptyset$.

(2) when $n \geq 1$ and $g \geq 1$, if one of the partitions P^i is not the singleton partition $m_1 = r$, then $B(\vec{m})_{\vec{\xi}} \neq \emptyset$.

Proof. Combining Proposition 5.10 and Proposition 5.12, we see that $B(\vec{m})_{\vec{\xi}}$ is non-empty if $(K_C(D))^{\otimes \mu}$ is controllable up to order $(\gamma_{P^1}(\mu), \ldots, \gamma_{P^n}(\mu))$ at $(p_1, \ldots, p_n)$ for $\mu = 2, \ldots, r$. The controllability can be guaranteed by the controllability inequalities in Lemma 5.3:

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < \deg((K_C(D))^{\otimes \mu}) - (2g-2) = (2g-2)(\mu - 1) + n\mu \quad \text{for } \mu = 2, \ldots, r. \tag{29}$$

(1) When $g = 0$, the right-hand side of (29) becomes $(n-2)\mu + 2$.

(2) When $g \geq 1$, note that we always have the weak inequalities $\gamma_{P^i}(\mu) \leq \mu$ for $2 \leq \mu \leq r$. Hence, we always have

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) \leq \mu n \quad \text{for } 2 \leq \mu \leq r.$$

For $g \geq 2$, we have $\mu n < (2g-2)(\mu-1) + \mu n$, so the controllability inequalities always hold for all choices of partitions P^i. For $g = 1$, if one of the partitions P^{i_0} is not the singleton partition for some i_0, then $\gamma_{P^{i_0}}(\mu) < \mu$ is strict for $2 \leq \mu \leq r$, so the controllability inequalities also hold.

☐

5.3. **Non-emptiness of moduli spaces.** We can now combine the non-emptiness of $B(\vec{m})_{\vec{\xi}}$ and the relative spectral correspondence to construct stable $\vec{\xi}$-parabolic Higgs bundles. Before proceeding to the main theorem, we will need to slightly enhance the OK condition to guarantee that there exists an integral curve in $B(\vec{m})_{\vec{\xi}}$.

Lemma 5.14. Suppose $H^0(C, L(\vec{m})_r) \neq 0$. There exists $s = (s_1, \dots, s_r) \in A(\vec{m})_0$ such that the corresponding spectral curve $C_s \subset Tot(K_C(D))$ is integral.

Proof. Let $s_1 = \cdots = s_{r-1} = 0$ and a_s be a non-zero section in $H^0(C, L_r)$. So, the spectral curve is of the form of a cyclic cover $C_s = \{y^r + s_r = 0\} \subset Tot(K_C(D))$. Note that [BNR89, Remark 3.1] here we treat s_r as a section in $(K_C(D))^{\otimes r}$ so that $div(s_r) = Z + \sum_{i=1}^n \gamma_{P^i}(r)p_i$ for some effective divisor $Z \in |L(\vec{m})_r|$. Recall that a cyclic cover is integral if $div(s_r)$ is not of the form mZ' for some $Z' \in |(K_C(D))^{\otimes k}|$ where $m|r$. Since $|L(\vec{m})_r|$ is a linear system with base point, by [Har77, Remark 10.9.2], we can assume that Z is reduced and away from $\sum_{i=1}^n \gamma_{P^i}(r)p_i$. Then, it is clear that $div(s_r)$ is not of the form mZ'. Hence, C_s is integral. □

Remark 5.15. If $H^0(C, L_{\vec{m}}(r)) = 0$, we must have $s_r = 0$, so the spectral curve defined by $y^r + s_1 y^{r-1} + \cdots + s_{r-1}y = 0$ must be reducible as it always contains the zero section as an irreducible component. In particular, no spectral curve is integral in this case. For example, this happens when $C = \mathbb{P}^1$ and $\deg(L(\vec{m})_r) = r(n-2) - \sum_{i=1}^n \gamma_{P^i}(r) < 0$. When $g > 0$, it is easy to check that $\deg(L(\vec{m})_r) > 0$ for all choice of $\vec{m}$, so $\dim H^0(C, L(\vec{m})_r) > 0$.

Lemma 5.16. Suppose $H^1(C, L(\vec{m})_\mu \otimes \mathcal{O}(-p_i)) = 0$ for $\mu = 2, \dots, r$ and $i = 1, \dots, n$. Then there exists an integral curve $\Sigma \in B(\vec{m})_0$ contained in Z_0.

Proof. First, observe that $H^1(C, L(\vec{m})_r \otimes \mathcal{O}(-p_i)) = 0$ implies the assumption $H^0(C, L(\vec{m})_r \neq 0$ in Lemma 5.14. Indeed, consider the long exact sequence

(30)

$$H^0(C, L(\vec{m})_\mu \otimes \mathcal{O}(-p_i)) \to H^0(C, L(\vec{m})_\mu) \to H^0(p_i, L(\vec{m})_\mu|_{p_i}) \to H^1(C, L(\vec{m})_\mu \otimes \mathcal{O}(-p_i)) \to \dots$$

Setting $\mu = r$, the claim follows from the vanishing of the last term and the fact that $H^0(p_i, L(\vec{m})_\mu|_{p_i}) \neq 0$. Then Lemma 5.14 says that there is an open subset $U \subset A(\vec{m})_0$ representing integral curves in $\mathrm{Tot}(K_C(D))$. Let $s = (s_1, \dots, s_r) \in U$ with $C_s \subset \mathrm{Tot}(K_C(D))$. Let $p : Z_{\vec{\xi}} \to \mathrm{Tot}(K_C(D))$ be the composition of blow-ups. Under the identification $B(\vec{m})_0 \cong A(\vec{m})_0$ in Corollary 3.13,

$$C_s \longleftrightarrow \Sigma := p^* C_s - \sum_{i=1}^n \sum_{j=1}^{\ell(i)} m_{i,j} \Xi_{i,j}$$

If the curve Σ is exactly the strict transform of C_s in $Z_{\vec{\xi}}$, then Σ is also integral. So, it suffices to check that C_s and its total transform in the blow-up have multiplicty exactly $m_{i,j}$ through the blow-up center. By Remark A.2, we need to check that vanishing order of s_μ at p_i equals exactly $\gamma_{P_i}(\mu)$ for $\mu = 1, \dots, r$ and all $p_i \in D$ (in fact, it suffices to check the minimal indices μ). We claim that this is true for generic $(s_1, \dots, s_r) \in U$. Indeed, it follows again from the long exact sequence 30 and the vanishing of H^1 in the assumption that

$$\dim(H^0(C, L(\vec{m})_\mu \otimes \mathcal{O}(-p_i))) < \dim(H^0(C, L(\vec{m})_\mu)).$$

□

Proposition 5.17. *For any $\vec{\xi} \in \mathcal{N}(\vec{m})$, the moduli space of stable $\vec{\xi}$-parabolic Higgs bundle $\mathcal{H}(\vec{m})_{\vec{\xi}}$ is non-empty in the following cases:*

(1) *When $n \geq 3$ and $g = 0$, if the inequalities*

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 1 \tag{31}$$

hold for $\mu = 2, \ldots, r$

(2) *When $n \geq 2$ and $g = 1$, if at least two of the partitions P^i are not the singleton partition $m_{i,1} = r$.*

(3) *When $n \geq 1$ and $g \geq 2$.*

Proof of Proposition 5.17. Note that when $g = 0, 1$, the conditions are stronger than the one in Theorem 5.13. So, $B(\vec{m})_{\vec{\xi}} \neq \emptyset$ in all cases.

Moreover, similar to the argument in the proof of Theorem 5.13, the strengthened inequalities in this theorem are required to guarantee that $H^1(C, L(\vec{m})_\mu \otimes \mathcal{O}(-p_i)) = 0$. So, we can apply Lemma 5.16 which guarantees that there exists an integral curve $\Sigma_0 \in B(\vec{m})_0$ contained in Z_0. Then we can apply Proposition 3.16 to obtain an integral curve $\Sigma_{\vec{\xi}}$ in $B(\vec{m})_{\vec{\xi}}$ for all $\vec{\xi} \in \mathcal{N}(\vec{m})$. By choosing a line bundle L on Σ, it must be β-twisted A-Gieseker for any choice of parameters β and A. So, the pure dimension one sheaf on $Z_{\vec{\xi}}$ formed by L on $\Sigma_{\vec{\xi}}$ is an element in $\mathcal{H}(\vec{m})_{\vec{\xi}}$. We can now apply the spectral correspondence (Theorem 4.2) to obtain the desired stable $\vec{\xi}$-parabolic Higgs bundles. ∎

Remark 5.18. (1) The condition in Proposition 5.17 is not optimal. In particular, when $g = 0$, it suffices to check (31) only for $\mu \in J_{\min}$ (see Remark A.2 for the definition).

(2) If one asks about the non-emptiness of $\mathcal{H}(\vec{m})_{\vec{\xi}}$ for a particular $\vec{\xi} \in \mathcal{N}(\vec{m})$, rather than for all $\vec{\xi} \in \mathcal{N}(\vec{m})$, the conditions can be relaxed depending on $\vec{\xi}$. We use this observation to study the Deligne–Simpson problem in Section 5.4.

Remark 5.19. When $\vec{m} = (\underline{1}, \ldots, \underline{1})$, the non-emptiness of $\mathcal{H}(\vec{m})_{\vec{\xi}} \cong \mathrm{Higgs}^{\mathrm{par}}_{\vec{\xi}}$ follows from that of meromorphic Higgs bundles studied by Markman [Mar94]. Under the assumption that the linear system $(K_C(D))^{\otimes r}$ is very ample (which translates to some genus and number of marked points assumption), the generic spectral curve is smooth by Bertini's theorem. So, at least for some generic (more generic than our definition) $\vec{\xi}$, there exist $\vec{\xi}$-parabolic Higgs bundles. Also, since the Hitchin map $h : \mathrm{Higgs}^{\mathrm{par}} \to A$ is proper [Yok93] and its image must contain the proper dense subset of smooth spectral curves, we see that the Hitchin map is surjective.

However, when $\vec{m} \neq (\underline{1}, \ldots, \underline{1})$, no spectral curve in $\mathrm{Tot}(K_C(D))$ is smooth or integral. Therefore, we may not be able to apply the similar argument in this case.

5.4. **Multiplicative Deligne–Simpson problem.** As outlined in the introduction, we study the multiplicative Deligne–Simpson problem (DSP, for short) via the spectral correspondence. For historical background and motivation of DSP, we refer the reader to Section 1.4.

Definition 5.20. *Given conjugacy classes $C_1, \ldots, C_n$ in $GL_r(\mathbb{C})$, we say that the multiplicative DSP is solvable for the tuple of conjugacy classes $\{C_i\}$ if there exist irreducible solutions to the equation $T_1 \cdots T_n = Id_r$ with $T_i \in C_i$ where an irreducible solution means that the matrices T_j have no common invariant subspace.*

Let us recall some definitions in the tame non-abelian Hodge correspondnce (NAHC) of Simpson [Sim90].

Definition 5.21. *A filtered local system on a punctured curve $C\setminus\{p_1,\dots,p_n\}$ is a local system $\mathbb{L}$ together with a decreasing, left-continuous filtration $\bigcup_{\beta\in\mathbb{R}}\mathbb{L}^\beta_{p_i}$ on the stalk $\mathbb{L}_{p_i}$, preserved by the local monodromy T_i at p_i, for each puncture p_i.*

One can define the filtered degree of a filtered local system and a stability condition. For our purposes, it suffices to note that when the filtrations are trivial i.e. when $\mathbb{L}^0_{p_i}=\mathbb{L}_{p_i}$ and $\mathbb{L}^\epsilon_{p_i}=0$ for $\epsilon>0$, then these stable filtered local systems of filtered degree zero are irreducible local systems.

There is also an analogous definition of filtered Higgs bundles, but it is equivalent to the definition of a parabolic Higgs bundle $(E,E^\bullet_D,\Phi,\vec{\alpha})$ on C we use in the paper. At each $p_i\in D$, one simply combines the quasi-parabolic structure $E^\bullet_{p_i}$ (indexed by $j=1,\dots,\ell(i)$) and the parabolic weights $\underline{\alpha}_i$ in Definition 2.1 into a decreasing, left-continuous filtration (indexed by the parabolic weights) of the fiber E_{p_i}:

$$E_{p_i}=E^0_{p_i}\supset E^{\alpha_{i,1}}_{p_1}\supset\cdots\supset E^{\alpha_{i,\ell(i)-1}}_{p_i}\supset E^{\alpha_{i,\ell(i)}}_{p_i}=0,\quad\text{where } E^{\alpha_{i,j}}_{p_i}=E^j_{p_i}.$$

Then Φ_i also preserves the filtration. Hence, we can equivalently work with parabolic Higgs bundles rather than filtered Higgs bundles.

At each p_i, both the filtered local systems and parabolic Higgs bundles contains the data of a filtered vector space and an endomorphism preserving the filtration.

Definition 5.22. *Let V be a finite-dimensional vector space over $\mathbb{C}$ equipped with a decreasing, left-continuous filtration $\bigcup_{\beta\in\mathbb{R}}V^\beta$, and let $T\colon V\to V$ be an endomorphism preserving the filtration i.e. $T(V^\beta)\subseteq V^\beta$ for all β. The residue of $(V,\bigcup_{\beta\in\mathbb{R}}V^\beta,T)$ is the graded vector space*

$$\operatorname{res}(V)=\bigoplus_\beta\operatorname{res}(V)_\beta$$

where $\operatorname{res}(V)_\beta=V^\beta/V^{\beta+\epsilon}$ for small $\epsilon>0$, together with the natural induced endomorphism $\operatorname{res}(T)$ acting on $\operatorname{res}(V)$.

Now, we can associate to the residue of $(V,\bigcup_{\beta\in\mathbb{R}}V^\beta,T)$ a collection of partitions as follows. A conjugacy class $C_0\subset GL_r(\mathbb{C})$ is determined by its Jordan normal form (JNF), we can identify C_0 with a collection of partitions $\{P^\lambda\}$ labeled by its eigenvalues λ, where each partition $P^\lambda=(n_1,\dots,n_\ell)$ records the sizes of Jordan blocks for eigenvalues λ. As the $\operatorname{res}(T)$ restricts to an endomorphism on $\operatorname{res}(V)_\beta$ for each β, we take $\{P^{\beta,\lambda}\}$ corresponding to the conjugacy class of this endomorphism on $\operatorname{res}(V)_\beta$. Then, we define the *residue diagram* of $(V,\bigcup_{\beta\in\mathbb{R}}V^\beta,T)$ to be the collection of partitions $\{P^{\beta,\lambda}\}$ labeled by the jumps β of the filtration and the eigenvalues λ.

Examples 5.23. When the filtrations of a filtered local system on $C\setminus\{p_1,\dots,p_n\}$ are trivial with $\beta=0$, then the residue diagrams $\{P^{0,\lambda}\}$ describe the conjugacy class of the local monodromy around p_i of the underlying local system.

Similarly, if the parabolic Higgs bundle $(E,F^\bullet_D,\Phi,\vec{\alpha})$ has trivial filtrations i.e. $F^0_{p_i}=E_{p_i},F^1_{p_i}=0$ and $\alpha_{i,1}=0$ for $i=1,\dots,n$, then the residue diagrams $\{P^{0,\xi}\}$ describe the conjugacy class of Φ_i.

Examples 5.24. A $\vec{\xi}$-parabolic Higgs bundle is the same as a parabolic Higgs bundle whose residue diagram at each p_i is given by $P^{\alpha_{i,j},\xi_{i,j}}=(1,\dots,1)$ (repeated $m_{i,j}$ times).

Now, we can state Simpson's tame NAHC [Sim90, Theorem, Page 718]: There is a one-to-one correspondence between stable filtered local systems on $C\setminus\{p_1,\dots,p_n\}$ of filtered degree zero and stable parabolic Higgs bundles on C of parabolic degree zero. Moreover, at each

$p_i \in D$, if we denote by $\{P^{\beta,\lambda}\}$ and $\{P^{\alpha,\xi}\}$ the residue diagram for the filtered local system and parabolic Higgs bundle in correspondence, respectively, then the residue diagrams are the same, with the labels permuted according to the following table [Sim90, page 719]:

TABLE 1. Simpson's table

	parabolic Higgs bundle	filtered local system
weight/jump	α	$\beta = -2b$
eigenvalue	$\xi = b + \sqrt{-1}c$	$\lambda = \exp(-2\pi\sqrt{-1}\alpha + 4\pi c)$

Hence, under the tame NAHC, the goal of producing irreducible local systems whose local monodromies have prescribed conjugacy classes is equivalent to producing parabolic Higgs bundles with prescribed residue diagrams. Recall that given parabolic data $(\vec{m}, \vec{\xi})$, we have a collection of partitions $\{P^{\xi^\circ_{i,j}}\}$ labeled by the unrepeated eigenvalues that decomposes the partitions $\{P^i\}$; see Section 1.5 for notation.

Proposition 5.25. *Fix the parabolic data $(\vec{m}, \vec{\xi})$. Suppose (E, Φ) is a Higgs bundle obtained from a line bundle L on an integral curve $\Sigma \subset Z_{\vec{\xi}}$ corresponding to a member in $B(\vec{m})_{\vec{\xi}}$ via pushing forward L from Σ. Then*

(1) (E, Φ) has no Higgs subbundle.

(2) If we equip (E, Φ) with the trivial filtrations $(E, F^\bullet_D, \Phi, \vec{\alpha})$, then the residue diagram $\{P^{0,\xi^\circ_{i,j}}\}$ of $(E, F^\bullet_D, \Phi, \vec{\alpha})$ is given by the conjugate partition of $P^{\xi^\circ_{i,j}}$

$$P^{0,\xi^\circ_{i,j}} = \widehat{P}^{\xi^\circ_{i,j}}.$$

Proof. Let $f_{\vec{\xi}} : Z_{\vec{\xi}} \to C$ be the composition of successive blow-ups and $q : \Sigma \hookrightarrow Z_{\vec{\xi}} \to M \to C$ be the projection map. Then $E = q_*L$ and Φ is obtained from the tautological section. Part (1) follows from the integrality of Σ.

For part (2), since the filtrations are trivial, each partition $P^{0,\xi^\circ_{i,j}}$ describes the Jordan normal form of Φ_i restricted to the generalized eigenspace of the eigenvalue $\xi^\circ_{i,j}$. Since Σ intersects the fiber $f^{-1}_{\vec{\xi}}(p_i)$ at the exceptional divisors of the blow-ups centered at the points $(p_i, \xi^\circ_{i,j})$, the restriction of Φ_i to its generalized eigenspaces of $\xi^\circ_{i,j}$ can be determined locally in terms of the restriction of L to the fiber over $\xi^\circ_{i,j}$. By doing a translation $\Phi - \xi^\circ_{i,j}$, we can further reduce the local analysis to the nilpotent case i.e $\xi^\circ_{i,j} = 0$. By applying Proposition A.3, the condition of $B(\vec{m})_{\vec{\xi}}$ implies that the Jordan normal form corresponding to eigenvalue $\xi^\circ_{i,j}$ is the conjugate of $P^{\xi^\circ_{i,j}}$, so $P^{0,\xi^\circ_{i,j}} = \widehat{P}^{\xi^\circ_{i,j}}$.

□

Let $C_i \subset GL_r(\mathbb{C})$ be a conjugacy class. Identify C_i with a collection of partitions $\{P^{\lambda_{i,1}}, \dots, P^{\lambda_{i,e(i)}}\}$, labeled by the eigenvalues $\lambda_{i,j}$ of C_i. Group the eigenvalues of C_i according to their absolute values. For each group of eigenvalues sharing the same absolute value v_a, define $R^{i,a}$ as the union of the corresponding partitions $P^{\lambda_{i,j}}$ with $|\lambda_{i,j}| = v_a$. Let $\widehat{R}^{i,a}$ denote the conjugate of $R^{i,a}$, and define the following partition of r

$$R^i = \widehat{R}^{i,1} \cup \cdots \cup \widehat{R}^{i,e'(i)}.$$

Theorem 5.26. *Let $n \geq 3$. Let $C_1, \dots, C_n \subset GL_r(\mathbb{C})$ be conjugacy classes. Suppose that the following two conditions hold:*

(1) $\prod_{i=1}^n \det(C_i) = 1$.

(2) $\sum_{i=1}^{n} \gamma_{R^i}(\mu) < (n-2)\mu + 1$ *for* $\mu = 2, \dots, r$.

Then the DSP is solvable for the tuple of conjugacy classes $(C_1, \dots, C_n)$*.*

Proof. By the preceding discussions, our goal is to find an irreducible local systems whose residue diagram is given by $\{P^{0,\lambda_{i,j}}\}$ where $P^{0,\lambda_{i,j}} = P^{\lambda_{i,j}}$. According to the tame NAHC, this amounts to finding a stable parabolic Higgs bundles of parabolic degree zero whose residue diagram is $\{P^{\alpha_{i,j},\tau_{i,j}}\}$ where

$$P^{\alpha_{i,j},\tau_{i,j}} = P^{0,\lambda_{i,j}} \quad \text{and} \quad \alpha_{i,j} = -\arg(\lambda_{i,j}), \tau_{i,j} = \sqrt{-1}\log|\lambda_{i,j}|/4\pi \tag{32}$$

according to Simpson's table 1. Let $\tau^\circ_{i,1}, \dots, \tau^\circ_{i,t(i)}$ be the set of distinct elements in $\{\tau_{i,j}\}$ ordered according to the first occurrence. In order to realize the residue diagrams $P^{\alpha_{i,j},\tau_{i,j}}$, we first realize $R^{i,a}$ (corresponding to an absolute value v_a) which, by definition, is the union of all partitions $P^{\alpha_{i,j},\tau_{i,j}}$ with $\tau_{i,j} = \sqrt{-1}\log(v_a)/4\pi = \tau^\circ_{i,a}$.

For each i, we can always choose a pair $(\underline{m}_i, \underline{\xi}_i)$ such that $\xi^\circ_{i,a} = \tau^\circ_{i,a}$ and the associated collection of partitions $\{P^{\xi^\circ_{i,a}}\}_{a=1}^{e(i)}$ satisfies $P^{\xi^\circ_{i,a}} = \widehat{R}^{i,a}$. With the choice of $(\underline{m}_i, \underline{\xi}_i)$ for $i = 1, \dots, n$, the two conditions guarantees that $B(\vec{m})_{\vec{\xi}}$ is non-empty by Corollary 5.13. Moreover, the inequality $\sum_{i=1}^{n} \gamma_{R^i}(\mu) < (n-2)\mu + 1$ is equivalent to $H^0(\mathbb{P}^1, L_{\vec{m}}(\mu)) = 0$, so Lemma 5.16 and Proposition 3.16 ensure that there exists an integral curve $\Sigma \in B(\vec{m})_{\vec{\xi}}$. Let L be a line bundle on Σ and (E, Φ) be the corresponding Higgs bundle. By Proposition 5.25, the residue diagrams of the Higgs bundle (treated as a parabolic Higgs bundle with trivial filtrations) are given by

$$P^{0,\xi^\circ_{i,a}} = \widehat{P}^{\xi^\circ_{i,a}} = R^{i,a}$$

In order to obtain the desired residue diagrams $P^{\alpha_{i,j},\tau_{i,j}}$, we construct a new filtration for the Higgs bundle (E, Φ). Without loss of generality, assume that Φ_i is in its Jordan normal form with respect to a suitable basis. As the Jordan blocks corresponding to each eigenvalue $\xi^\circ_{i,a} = \tau^\circ_{i,a}$ are described by $R^{i,a}$ which is the union of $P^{\alpha_{i,j},\tau_{i,j}}$ with $\tau_{i,j} = \tau^\circ_{i,a}$, we can explicitly construct a filtration $E^\bullet_{p_i}$ of E_{p_i} with parabolic weights $\alpha_{i,j}$ by choosing subsets of the basis vectors such that the residue diagram is exactly $\{P^{\alpha_{i,j},\tau_{i,j}}\}$. This produces a parabolic Higgs bundle $(E, E^\bullet_D, \Phi, \vec{\alpha})$ with the desired residue diagram (32).

It remains to check that $(E, E^\bullet_D, \Phi, \vec{\alpha})$ is stable of parabolic degree 0. Stability follows from the integrality of Σ and Proposition 5.25 which guarantees that (E, Φ) has no Higgs subbundles. Therefore, $(E, E^\bullet_D, \Phi, \vec{\alpha})$ is automatically stable as a parabolic Higgs bundle. Note that the condition (1) is clearly necessary for the existence of solution to DSP. According to Simpson's table, the condition (1) converts to the conditions on the Higgs bundles side that $\sum \tau_{i,j} = 0$ and the sum of $\alpha_{i,j}$ (counted with multiplicities) is an integer. Recall that the parabolic degree is defined as

$$\deg(E) + \alpha_{i,j} \sum \dim(E^{j-1}_{p_i}/E^j_{p_i}).$$

As $\chi(L) = \chi(E)$, the degree of E can be any value by choosing L to have an appropriate degree. Since $\alpha_{i,j} \sum \dim(E^{j-1}_{p_i}/E^j_{p_i})$ is an integer by the condition (1), we can always select $\deg(L)$ such that the parabolic degree is zero.

□

The same idea works for curves of higher genus.

Theorem 5.27. *Let* $n \geq 1$ *and* $g > 0$*. Let* $C_1, \dots, C_n \subset GL_r(\mathbb{C})$ *be conjugacy classes. Suppose that* $\prod_{i=1}^{n} \det(C_i) = 1$ *holds. For* $g = 1$*, assume that* $n \geq 2$ *and at least two of the* *partitions R^i are not the singleton partitions. Then there exist matrices $A_k, B_k, T_i \in GL_r(\mathbb{C})$ with $T_i \in C_i$ for $k = 1, \ldots, g, i = 1, \ldots, n$ such that*

$$\prod_{k=1}^{g} (A_k, B_k) \prod_{i=1}^{n} T_i = Id_r$$

where $(A_k, B_k) = A_k B_k A_k^{-1} B_k^{-1}$ and the matrices A_k, B_k, T_i have no common invariant subspace.

Proof. The existence of the matrices A_k, B_k, T_i is equivalent to the existence of irreducible local system on a genus $g > 0$ Riemann surface $C \backslash \{p_1, \ldots, p_n\}$ whose residue diagram is determined by the prescribed conjugacy classes C_i. We can proceed as in the proof of Theorem 5.26. $\square$

Note that Theorem 5.26 depends on the absolute values of the eigenvalues, since the construction of the partitions R^i depend on the absolute values. However, by assuming the genericity of the eigenvalues, one can obtain a sufficient criterion that is independent of the eigenvalues (depends only on the distribution of Jordan blocks).

Definition 5.28. *We say that a collection of eigenvalues $\{\lambda_{i,j} | i = 1, \cdots, n \text{ and } j = 1, \cdots, r\}$ is:*

- *Multiplicatively generic if for any non-empty subsets $T_1, \ldots, T_n \subset \{1, \ldots, r\}$ with $|T_1| = \cdots = |T_n| = m < n$, we have*

$$\prod_{i=1}^{n} \prod_{k \in T_i} \lambda_{i,k} \neq 1$$

- *Additively generic if for the same choice of subsets, we have*

$$\sum_{i=1}^{n} \sum_{k \in T_i} \lambda_{i,k} \neq 1$$

For each $\vec{\xi} = (\underline{\xi}_1, \ldots, \underline{\xi}_n) \in \mathcal{N}(\vec{m})$, the vector $\underline{\xi}_i = (\xi_{i,1}, \ldots, \xi_{i,\ell(i)})$ together with its associated multiplicities $\underline{m}_i = (m_{i,1}, \ldots, m_{i,\ell(i)})$ determines a vector of length r by repeating each entry $\xi_{i,j}$ exactly $m_{i,j}$ times. Then there is an open subset $\mathcal{N}(\vec{m})_{add} \subset \mathcal{N}(\vec{m})$ consisting of additively generic eigenvalues.

Lemma 5.29. Suppose that the OK condition holds i.e.

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 2 \tag{33}$$

holds for $\mu = 2, \ldots, r$. Then for a general choice of $\vec{\xi} \in \mathcal{N}(\vec{m})_{add}$, there exists an integral curve $\Sigma \in B(\vec{m})_{\vec{\xi}}$ contained in $Z(\vec{\xi})$.

Proof. Let $s \in A$ be a point that also lies in $B(\vec{m})_{\vec{\xi}} \subset A$, and let $C_s \subset \mathrm{Tot}(K_C(D))$ denote the corresponding spectral curve. Then as in the proof of Lemma 5.16, we have the identification

$$C_s \longleftrightarrow \Sigma := p^* C_s - \sum_{i=1}^{n} \sum_{j=1}^{\ell(i)} m_{i,j} \Xi_{i,j} \subset Z(\vec{\xi})$$

Since $\vec{\xi}$ is additively generic, all spectral curves C_s corresponding to points in $B(\vec{m})_{\vec{\xi}}$ are integral. As in the proof of Lemma 5.16, to conclude that Σ is integral, it suffices to check that C_s and its total transform in the blow-ups have multiplicities exactly $m_{i,j}$ through the blow-up centers, so that Σ is the strict transform of C_s. By Remark A.2, we need to guarantee that the the extra equations parametrized by $G(P^{\xi^\circ_{i,j}})_{\min}$ are not satisfied for each eigenvalue

$\xi^{\circ}_{i,j}$. If $\dim(B(\vec{m})_{\vec{\xi}}) > 0$, then one can choose a general element in $B(\vec{m})_{\vec{\xi}}$ to achieve this. When $\dim(B(\vec{m})_{\vec{\xi}}) = 0$, the condition that the extra equation is satisfied defines a closed subset of $\mathcal{N}(\vec{m})_{add}$, so we can choose a general $\vec{\xi}$ in $\mathcal{N}(\vec{m})_{add}$ so that this condition fails, as desired.

□

For each conjugacy class $C_i \subset GL(r, \mathbb{C})$, define the following partition of r

$$P^i = \widehat{P}^{\lambda_{i,1}} \cup \cdots \cup \widehat{P}^{\lambda_{i,e(i)}}.$$

where $\widehat{P}^{\lambda_{i,j}}$ is the conjugate partition of $P^{\lambda_{i,j}}$.

Theorem 5.30. *Let $n \geq 3$. Let $C_1, \ldots, C_n \subset GL_r(\mathbb{C})$ be a collection of conjugacy classes whose collection of eigenvalues is multiplicatively generic. Suppose that the following conditions hold:*

(1) $\prod_{i=1}^n \det(C_i) = 1$.
(2) $\sum_{i=1}^n \gamma_{P^i}(\mu) < (n-2)\mu + 2$ for $\mu = 2, \ldots, r$.

Then the DSP is solvable for the tuple of conjugacy classes $(C_1, \ldots, C_n)$.

Proof. The dependence on the absolute values of eigenvalues in the previous theorem arises from the coincidence of $\tau_{i,j} = \sqrt{-1}\log|\lambda_{i,j}|/4\pi$ for distinct $\lambda_{i,j}$. For each fixed i, in order to distinguish the eigenvalues, we assign $\beta_{i,j}$ to each $\lambda_{i,j}$ such that the modified values $\tau'_{i,j} = -\beta_{i,j}/2 + \sqrt{-1}\log|\lambda_{i,j}|/4\pi$ are distinct for distinct $\lambda_{i,j}$ and $\sum \tau'_{i,j} = 0$. Let $\vec{\beta} = (\underline{\beta}_1, \ldots, \underline{\beta}_n)$ be such a choice of $\beta_{i,j}$ where $\underline{\beta}_i = (\beta_{i,j})$.

By imposing the multiplicative genericity condition on the eigenvalues, every local system with the prescribed conjugacy classes will automatically be irreducible. In particular, if we can produce a $\vec{\beta}$-stable filtered local system with residue diagram $P^{\beta_{i,j},\lambda_{i,j}} = P^{\lambda_{i,j}}$, then the underlying local system $\mathbb{L}$ is also irreducible. Moreover, the conjugacy class of $\mathbb{L}$ is also determined by $\{P^{\beta_{i,j},\lambda_{i,j}}\}$ since we assume that, for each fixed i, the weights $\beta_{i,j}$ are distinct for distinct $\lambda_{i,j}$ which ensures that the generalized eigenspace corresponding to the eigenvalue $\lambda_{i,j}$ is isomorphic to the associated associated graded piece labeled by $\beta_{i,j}$ under the natural linear map.

In order to construct a $\vec{\beta}$-stable filtered local system via the tame NAHC, we construct a stable parabolic Higgs bundles of parabolic degree zero whose residue diagram is $\{P^{\alpha_{i,j},\tau'_{i,j}}\}$ where

$$P^{\alpha_{i,j},\tau'_{i,j}} = P^{\beta_{i,j},\lambda_{i,j}} \quad \text{and} \quad \alpha_{i,j} = -\arg(\lambda_{i,j}), \tau'_{i,j} = -\beta_{i,j}/2 + \sqrt{-1}\log|\lambda_{i,j}|/4\pi.$$

The construction of such a stable parabolic Higgs bundle follows the previous proof of Theorem 5.26, with two key modifications: first, we replace $R^{i,j}$ with $P^{\alpha_{i,j},\tau'_{i,j}}$; second, we can apply Lemma 5.29 to ensure the existence of an integral curve in $B(\vec{m})_{\vec{\xi}}$ because the multiplicative genericity of $\lambda_{i,j}$ converts to the additive genericity of $\vec{\xi}$ and we can also perturb $\beta_{i,j}$ to make $\vec{\xi}$ a general element in $\mathcal{N}(\vec{m})_{add}$ if necessary.

□

Remark 5.31 (Conjecture of Balasubramanian-Distler-Donagi)*.* In the setup of Theorem 5.30, let $\vec{m} = (\underline{m}_1, \ldots, \underline{m}_n)$ where $\underline{m}_i = P^i$. The inequalities in condition (2) are equivalent to the OK condition for the collection of line bundles $L_{\vec{m}}(\mu)$ on $C = \mathbb{P}^1$ for $\mu = 2, \ldots, r$. Therefore, Theorem 5.30 confirms the conjecture proposed in [BDD22]: the OK condition implies that multiplicative DSP is solvable for the tuple $(C_1, \ldots, C_n)$ under the assumption that $(C_1, \ldots, C_n)$ is multiplicatively generic.

In this sense, Theorem 5.26 extends the result to more general conjugacy classes, without the multiplicatively generic assumption, with the expense of imposing a strengthened version of the OK condition (inequalities) which also depends on the eigenvalues.

Moreover, except for the few exceptional cases in $g = 1$ stated in Theorem 5.27, the strengthened version of the OK condition also implies that DSP is always solvable when the genus $g(C) \geq 1$ without any assumption on the eigenvalues.

In Appendix B, we will show that this criterion is numerically equivalent to Simpson's criterion when one of the conjugacy classes has distinct eigenvalues.

Remark 5.32 (Comparison with Kostov's criterion for DSP). For multiplicatively generic eigenvalues, Kostov provided a sufficient and necessary condition for DSP [Kos04]. Formulated in terms of the notion of "defect" defined by Simpson in [Sim09, Section 2.7], Kostov's criterion takes the form of an iterative algorithm where the defect controls both the termination condition and the modification of conjugacy classes at each step. Roughly, Kostov's algorithm goes by iteratively verifying the inequalities (analogous to Simpson's criterion) for a given set of conjugacy classes; if the defect is negative, it modifies the conjugacy classes and repeats the verification until termination.

Analogous to Simpson's definition of defect, we define

$$\delta(\mu) = (n-2)r - \sum_{i=1}^{n} \gamma_{P^i}(\mu)$$

where P^i is as in Theorem 2. This differs from Simpson's original definition by incorporating conjugate partitions, as required by Proposition 5.25 which involves conjugate partitions. Then the inequalities in the third condition of Theorem 5.30 says that $\delta(\mu) > -2$ for $\mu = 2, \ldots, r$. This implies that the cases covered by Theorem 5.30 amounts to the situation where the negative defects must be -1 in Kostov's algorithm. The cases with higher defects in Kostov's algorithm will be taken up in future work.

Appendix A. Local behaviors

In this section, we study the local conditions imposed by the linear system $|\Sigma_{\vec{m}}(\vec{\xi})|$ by explicitly writing down the local charts of the successive blow-ups.

Let $M_0 = \mathbb{A}^2$ with coordinates (x, y). Alternatively, one may take $M_0 = \mathrm{Spec}(\mathbb{C}[[x]][y])$ in the context of ruled surface, the computations and results below remain unchanged in either setting. Let $l_x := \{x = 0\}$ in M_0 and M_1 be the blow-up of M_0 at the origin $c_1 := (0, 0)$. Let $M_\ell \to \cdots \to M_1 \to M_0$ be a sequence of blow-ups where $p_j^{j-1} : M_j \to M_{j-1}$ is the blow-up of M_j at the intersection c_j of the strict transform of l_x and the exceptional divisor from the previous blow-up. Let $\underline{m} = P = (m_1, \ldots, m_\ell)$ a partition of r and let $(n_1, \ldots, n_s)$ be the conjugate partition of P.

Lemma A.1. Let X be a plane curve in M_0, defined by the equation $F(x, y) = y^r + s_1(x)y^{r-1} + \cdots + s_r(x) = 0$ where $s_\mu(x) \in \mathbb{C}[x]$. Let $M_\ell \to \cdots \to M_1 \to M_0$ be as above. Then the following conditions are equivalent

1. X passes through c_1 with multiplicity at least m_1 and, for $j = 2, \ldots, \ell$, the total transform of X in M_{j-1} passes through c_j with multiplicity at least $m_1 + \cdots + m_j$;
2. Let $v(\mu)$ be the vanishing order of $s_\mu(x)$ at $x = 0$ for $1 \leq \mu \leq \ell$ and $\gamma(\mu)$ the level function associated to the partition P. Then $v(\mu) \geq \gamma(\mu)$ for all $1 \leq \mu \leq \ell$.
3. The partial derivative $\frac{\partial^u}{\partial y^u}\frac{\partial^a}{\partial x^a}\Big|_{(0,0)} F(x, y)$ vanishes for $(u, a) \in G(P) := \{(u, a) | 0 \leq a < \gamma(r - u)\}$.

Proof. (1) $\implies$ (2). Let us fix $1 \le \mu \le r$ and focus on the term $T_\mu = s_\mu(x)y^{r-\mu}$. Suppose X must pass through $c_1 = (0,0)$ with multiplicity at least m_1. If $r - \mu \le m_1$, then $s_\mu(x)$ must vanish at $x = 0$ with multiplicity at least $d_1 = m_1 - (r - \mu)$ in which case we can write $s_\mu = s_\mu^1(x)x^{d_1}$. For $r - \mu > m_1$, we can still write $s_\mu = s_\mu^1(x)x^{d_1}$ with $d_1 = 0$. Now, if we blow up at the origin to get M_1, we have $x = u_1 y$ (only this chart matters) and the term in the total transform $X_1 \subset M_1$ becomes

$$T_\mu^1 = s_\mu^1(u_1 y)u_1^{d_1}y^{r-\mu+d_1}.$$

Since X has multiplicity at least m_1 at $(0,0)$, the exceptional divisor must have at least multiplicity m_1, and so

$$r - \mu + d_1 \ge m_1.$$

Now, suppose X_1 passes through $c_2 := (u_1, y) = (0,0)$ with multiplicity at least $m_1 + m_2$. Following the same reasoning as before, the term T_μ^1 in the total transform $X_2 \subset M_2$ transforms into

$$T_\mu^2 := s_\mu^2(u_2 y^2)u_2^{d_1+d_2}y^{r-\mu+2d_1+2d_2-m_1-m_2}$$

for some $d_2 \ge 0$ where we let $u_1 = u_2 y$. Moreover, the inequality

$$r - \mu + 2d_1 + 2d_2 \ge m_1 + m_2$$

holds. Proceed inductively, for $j = 1, \dots, l$, the term in the total transform after the j-th blow-ups becomes

$$T_\mu^j := s_\mu^j(u_j y^j)u_j^{\Delta_\mu^j}y^{\nabla_j}, \quad \text{where} \quad \Delta_\mu^j = \sum_{a=1}^{j} d_a, \quad \nabla_j = r - \mu + j\sum_{a=1}^{j} d_a$$

for some $d_1, \dots, d_j \ge 0$. We also have that the inequality

$$\nabla_j \ge \sum_{a=1}^{j} m_a$$

holds. Our goal is to give a lower bound of Δ_μ^l in the last (l-th) blow-up since $s_\mu(x) = s_\mu^l(x)x^{\Delta_\mu^l}$. Note that each inequality $\nabla_j \ge \sum_{a=1}^{j} m_a$ implies

$$(34) \qquad (*_j) \qquad \Delta_\mu^l \ge \Delta_\mu^j \ge \frac{1}{j}\left(\sum_{a=1}^{j} m_a - (r-\mu)\right) \quad \text{for } j = 1, \dots, l$$

Denote by B_μ the lower bound $\frac{1}{j}\left(\sum_{a=1}^{j} m_a - (r-\mu)\right)$. Let $n_1 \ge \dots \ge n_{m_1}$ be the conjugate partition. Since we would like to relate the lower bound with the level function, we also view $1 \le \mu \le r$ as the integers in the filling (1) of the Young diagram introduced in 1.5. Let $m_{j+1} + 1 \le t \le m_j$ be the index of the columns where $j = 1, \dots, l$ and $m_{l+1} = 0$. Then, for $\mu = \sum_{b=1}^{t} n_b$, by counting the number of boxes, the inequality $(*_j)$ becomes

$$\Delta_\mu^l \ge \frac{1}{j}\left(\sum_{a=1}^{j} m_a - \sum_{t+1}^{m_1} n_b\right) = \frac{1}{j}(jt) = t$$

By the definition of level function, $t = \gamma(\mu)$ for $\mu = \sum_{b=1}^{t} n_b$. This yields the desired lower bound for the boxes at the bottom of each column in the Young diagram. Since B_μ is strictly increasing in μ and the value of B_μ between the two boxes at the bottom of two adjacent columns differ exactly by 1, all the boxes in the same column will have the value of B_μ which is precisely $\gamma(\mu)$. Hence, each $s_\mu(x)$ will have vanishing order at least $\gamma(\mu)$ at $x = 0$.

(2) $\implies$ (1). Suppose the vanishing order of $s_\mu(x)$ is at least $\gamma(\mu)$, so $s_\mu(x) = s'_\mu(x)x^{\gamma(\mu)}$ and $F(x,y) = y^r + s'_1 x^{\gamma(1)}y^{r-1} + \cdots + s'_r x^{\gamma(r)}$. To check that X passes through $(0,0)$ with multiplicity at least m_1, it suffices to show that the minimum of the sum of the exponents of each term is m_1 i.e. $\min_{1\le\mu\le r}\{\gamma(\mu)+r-\mu\} = m_1$. Note that when $\mu = r$, then $\gamma(r)+r-r = m_1$ by definition. It can be checked directly that m_1 is indeed the minimum. Proceed as in the first part, we see that the general expression for the pullback after the j-th blow-ups is

$$y^r + \sum_{\mu=1}^{r} s_\mu u_j^{\gamma(\mu)} y^{(j-1)\gamma(\mu)+r-\mu}$$

Similarly, in order to say that the pullback X_j of X passes through c_j with multiplicity at least $m_1 + \cdots + m_j$, it suffices to show that

$$\min_{1\le\mu\le r}\{j\gamma(\mu) + r - \mu\} = \sum_{a=1}^{j} m_a \tag{35}$$

Let $C_\mu := j\gamma(\mu) + r - \mu$. We claim that the minimum is attained at the bottom of the m_j-th column of the Young diagram where $\mu_{\min} = \sum_{b=1}^{m_j} n_b$ and $\gamma(\mu_{\min}) = m_j$. As

$$j\gamma(\mu_{\min}) - \sum_{a=1}^{j} m_a = -\sum_{b=m_j+1}^{m_1} n_b = -(r - \mu_{\min}),$$

it follows that $C_{\mu_{\min}} = \sum_{a=1}^{j} m_a$. To check that $\mu_{\min} := \sum_{b=1}^{m_j} n_b$ attains the minimum value, we write $\mu = \mu_{\min}+\delta$ where $1 \le \mu_{\min}+\delta \le r$. Note that if we write $g = \gamma(\mu_{\min} + \delta) - \gamma(\mu_{\min})$, then

$$C_\mu - \sum_{a=1}^{j} m_a = j\gamma(\mu_{\min} + \delta) + r - (\mu_{\min} + \delta) - \sum_{a=1}^{j} m_a = jg - \delta.$$

Note that g and δ have the same sign but $g \le \delta$. As $j = n_{m_j}$, it is easy to see that $jg - \delta \ge 0$. Hence, the equality holds.
(2) $\iff$ (3) Obvious. $\square$

Remark A.2. Let $J_{\min} = \{\mu | \mu = \sum_{b=1}^{m_j} n_b, \text{ for } j = 1, \ldots, \ell\}$ be the set of minimal indices in the proof of Lemma A.1. The implication (2) $\Rightarrow$ (1) further shows that if equalities are attained for the minimal indices in condition (2)—that is, if $v(\mu) = \gamma(\mu)$ for $\mu \in J_{\min}$—then the multiplicities are exact in condition (1): the curve X passes through c_1 with multiplicity exactly m_1, and for each $j = 2, \ldots, \ell$, the total transform of X in M_{j-1} passes through c_j with multiplicity exactly $m_1 + \cdots + m_j$.

Let $G(P)_{\min} = \{(u,a) | u \in J_{\min}, a = \gamma(r-u)\}$. Then the additional assumption on the minimal indices in condition (2) translates into condition (3) by requiring that the partial derivative $\frac{\partial^u}{\partial y^u}\frac{\partial^a}{\partial x^a}\Big|_{(0,0)} F(x,y) \neq 0$ for $(u,a) \in G(P)_{\min}$. Therefore, the non-vanishing of these partial derivatives at $G(P)_{\min}$ forces equality for minimal indices in condition (2), and thus exact multiplicity in condition (1).

Continuing with the notation and setup from Lemma 6.1. Consider the projection $\pi : M_0 \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[x])$ and denote by $p : M_\ell \to M_0$ and $f : M_\ell \to M_0 \to \mathbb{A}^1$ the composition. Let $E_j \subset M_\ell$ be the strict transform of the exceptional divisor of the blow-up $M_j \to M_{j-1}$ under $M_\ell \to M_j$. Let $\Xi_j = E_j + \cdots + E_\ell$.

For any $\mathcal{O}_{\Sigma_0}$-module F, multiplication by y induces an $\mathcal{O}_{M_0}$-module homomorphism $y : p_*F \to p_*F$. Pushing forward along π, we obtain a Higgs bundle (f_*F, Φ) where the Higgs field is given by

$$\Phi := \pi_* y : f_* F \to f_* F$$

This serves as the local model for the blow-ups of spectral curves, allowing us to analyze the local behavior of the induced Higgs field.

Proposition A.3. *Suppose that X satisfies one of the conditions in Lemma A.1 such that the divisor $\Sigma = p^*X - m_1\Xi_1 - \cdots - m_\ell\Xi_\ell$ is effective. Assume that Σ is integral. Let L be a line bundle on Σ. Then the Jordan normal form with eigenvalue 0 of Φ restricted to $0 \in \mathbb{A}^1$ is uniquely determined by the partition $(m_1, \ldots, m_\ell)$. More precisely, if we denote by $(n_1, \ldots, n_s)$ the partition corresponding to the Jordan normal form where each n_j represents a Jordan block of size n_j, then $(n_1, \ldots, n_s)$ is given by the conjugate partition of $(m_1, \ldots, m_\ell)$.*

Remark A.4. Recall that in linear algebra, given a vector space V of dimension r and a nilpotent operator $B : V \to V$, the Jordan normal form is determined by the sequence of subspaces:

$$0 \subset W_1 \subset \cdots \subset W_\ell = V, \quad \text{where } W_j := \ker(B^j)$$

for some l. Let $b_j = \dim(W_j/W_{j-1})$ for $j = 1, \ldots, \ell$ and u_j the number of Jordan blocks of size j, then u_j can be computed as follows:

$$\begin{aligned} u_1 + u_2 + \cdots + u_\ell &= b_1 \\ u_2 + \cdots + u_\ell &= b_2 \\ &\vdots \\ u_\ell &= b_\ell \end{aligned}$$

Note that $u_1 + 2u_2 + \cdots + \ell u_\ell = r = b_1 + b_2 + \cdots + b_\ell$ and the following partition of r is exactly the conjugate partition of r

$$\underbrace{\ell, \ldots, \ell}_{u_\ell-\text{copies}}, \ldots, \underbrace{1, \ldots, 1}_{u_1-\text{copies}} \tag{36}$$

Therefore, the Jordan normal form of B in the form of partition is given by (36) which is the conjugate partition of $(b_1, \ldots, b_\ell)$.

Note that this resembles the intersection numbers of the curve classes $\Sigma_{\vec{m}}(\vec{\xi})$ studied in Proposition 3.6:

$$\begin{aligned} E_1 \cdot \Sigma + E_2 \cdot \Sigma + \cdots + E_\ell \cdot \Sigma = \Xi_1 \cdot \Sigma &= m_1 \\ E_2 \cdot \Sigma + \cdots + E_\ell \cdot \Sigma = \Xi_2 \cdot \Sigma &= m_2 \\ &\vdots \\ E_\ell \cdot \Sigma = \Xi_\ell \cdot \Sigma &= m_\ell \end{aligned}$$

In fact, we will show that each $E_j \cdot \Sigma$ is realized as the number of Jordan blocks of size j.

Proof of Proposition A.3. To analyze the local behavior of the Higgs field, we work chart by chart on the successive blow-ups M_l. The variety M_l can be covered by 2ℓ open subsets $U_j = \mathrm{Spec} R_j, V_j = \mathrm{Spec} R'_j$, $j = 1, \ldots, \ell$ where

$$R_j = \mathbb{C}[u_{j-1}, v_j, y]/(y - u_{j-1}v_j), \quad R'_j = \mathbb{C}[u_j, y, u_{j-1}]/(u_{j-1} - u_j y)$$

where $u_0 = x$. Let $F_j(u_{j-1}, v_j)$ be the local equation of Σ_ℓ in U_j. Since L is locally free, we can represent the restriction of L to $\Sigma_\ell \cap U_j$ as an R_j-module as

$$L_j = \frac{\mathbb{C}[u_{j-1}, v_j]}{(F_j(u_{j-1}, v_j))}.$$

Let $N_0 = L_j/(x)$ be the scheme-theoretic restriction of L_j to the fiber over $x = 0$. Then $f_*L|_{x=0}$ is locally represented by N_0 as a module over $\mathbb{C}[x]/(x) \cong \mathbb{C}$. As Σ is assumed to be integral, it does not contain any E_j as an irreducible component, so N_0 is a finite dimensional vector space. The Higgs field Φ restricted to $f_*L|_{x=0}$ is induced by the action of multiplication by y on N_0.

Note that U_j is obtained from blowing up at the origin in V_{j-1} where we have the relation

$$x = u_0 = u_1 y = \cdots = u_{j-1} y^{j-1}$$

So, we also have the relation on U_j

$$y^j = y^{j-1} \cdot (u_{j-1} v_j) = x v_j.$$

This means that the action of multiplication by y is nilpotent of order j on N_0. Following the idea in Remark A.4, we would like to show that N_0 contains a vector subspace preserved by the operator y such that the Jordan normal form of y on it consists of exactly $e_j := E_j \cdot \Sigma_\ell$ Jordan blocks of size j. As we run the construction over all U_j for $j = 1, \dots, \ell$, we get the desired Jordan normal form of Φ restricted to the whole $f_*L|_{x=0}$

Consider $N_1 = N_0/(u_j) \cong L_j/(u_{j-1})$ the restriction of N_0 to the $E_j \cap U_j$. For simplicity, we assume that Σ_ℓ does not pass through the intersection $E_j \cap E_{j+1}$ (otherwise, we will need to consider more charts). Since L is a line bundle, the dimension of N_1 (as $\mathbb{C}$-vector space) is given exactly by e_j. After dividing $F_j(u_{j-1}, v_k)$ by its leading coefficient to make it monic, we can write

$$F_j(u_{j-1}, v_j) = v_j^{e_j} + u_{j-1} f + c \tag{37}$$

where $c \in \mathbb{C}$ is non-zero and f is a polynomial in v_j with coefficients in $\mathbb{C}[u_{j-1}]$ and degree $< e_j$. Hence, N_1 is always spanned by the basis $1, v_j, \dots, v_j^{e_j-1}$. Now, consider the following set of vectors in N_0 by repeatedly applying y to the basis vectors $1, v_j, \dots, v_j^{e_j-1}$

$$T = \{1, v_j, \dots, v_j^{e_j-1}, y, y v_j, \dots, y v_j^{e_j-1}, \dots, y^{j-1}, y^{j-1} v_j, \dots, y^{j-1} v_j^{e_j-1}\} \tag{38}$$

We claim that this set of vectors is linearly independent. Suppose

$$\sum_{a=0}^{j-1} \sum_{b=0}^{e_j-1} c_{ab} y^a v_j^b = \sum_{a=0}^{j-1} \sum_{b=0}^{e_j-1} c_{ab} u_{j-1}^a v_j^{a+b} = 0 \in N_0 \tag{39}$$

The linear combination (39) in N_1 is given by

$$c_{00} + c_{01} v_j + \cdots + c_{0(e_j-1)} v_j^{e_j-1} = 0$$

which implies that $c_{00} = c_{01} = \cdots = c_{0(e_j-1)} = 0$ since $1, v_j, \dots, v_j^{e_j-1}$ is a basis in N_1. Since $y = u_{j-1} v_j$, the linear combination (39) in $L_j/(u_{j-1}^2)$ is given by

$$c_{10} y + c_{11} y v_j + \cdots + c_{1(e_j-1)} y v_j^{e_j-1} = u_{j-1}(c_{10} v_j + c_{11} v_j^2 + \cdots + c_{1(e_j-1)} v_j^{e_j}) = 0 \in L_j/(u_{j-1}^2)$$

which implies that

$$u_{j-1}(c_{10} v_j + c_{11} v_j^2 + \cdots + c_{1(e_j-1)} v_j^{e_j}) = w u_{j-1}^2 \in L_j \quad \text{for some } w \in L_j.$$

Since the curve Σ_ℓ is assumed to be integral, L_j is a torsion-free module over $R_j/(F_j(u_{j-1}, v_j))$, so we obtain

$$c_{10}v_j + c_{11}v_j^2 + \cdots + c_{1(e_j-1)}v_j^{e_j} = wu_{j-1} \in L_j$$

Using the expression in 37, we get

$$c_{10}v_j + c_{11}v_j^2 + \cdots + c_{1(e_j-2)}v_j^{e_j-1} - c_{1(e_j-1)}c = w'u_{j-1} \in L_j \quad \text{for some } w \in L_j$$

As the left-hand side is a polynomial in v_j with degree $< e_j$ which is not divisble by u_{j-1} in L_j, it must be zero. It follows that $c_{10} = c_{11} = \cdots = c_{1(e_j-1)}c = 0$, and hence $c_{1(e_j-1)} = 0$. Proceed inductively for $L_j/(u_{j-1}^k)$, $k = 3, \ldots, e_{j-1} - 1$, we see that all $c_{ab} = 0$. Then it follows from the linear independence of the set of vectors 38 that if we rearrange the order of the vectors in T 38 and consider the subspace $\langle T \rangle \subset N_0$ they span, the operator y restricted to $\langle T \rangle$ is in its Jordan normal form with exactly e_j Jordan blocks of size j with respect to the basis vectors in T. $\square$

Appendix B. Comparison with Simpson's result

We shall now compare our result with Simpson's result [Sim91] when one of the conjugacy class has distinct eigenvalues.

In the setup of DSP, we are given a set of conjugacy classes $C_1, \ldots, C_n$. For each conjucacy class $C_i \in GL_r(\mathbb{C})$, we define

- d_i is the dimension of C_i. Note that $d_i = r^2 - \dim_{\mathbb{C}} Z(C_i)$ where $Z(C_i)$ is the group of centralizers of C_i.
- $R_i := \min_{\lambda \in \mathbb{C}} \operatorname{rank}(Y - \lambda I)$ for a matrix Y from C_i. Note that $r - R_i$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue.

Theorem B.1. *[Sim91] For generic eigenvalues and when one of the conjugacy classes, say C_{i_0}, has distinct eigenvalues, then the DSP is solvable if and only if*

(1) $d_1 + \cdots + d_n \geq 2r^2 - 2$
(2) $R_1 + \cdots + \widehat{R_i} + \cdots R_n \geq r \quad \forall i$

The folllowing numerical equivalence is observed in [BDD22].

Proposition B.2. *When one of the conjugacy classes, say C_{i_0}, has distinct eigenvalues. Then the following inequalities*

$$\begin{aligned} &(\alpha) \quad d_1 + \cdots + d_n \geq 2r^2 - 2 \\ &(\beta) \quad R_1 + \cdots + \widehat{R_i} + \cdots + R_n \geq r \quad \forall i \end{aligned} \tag{40}$$

hold if and only if

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu) < (n-2)\mu + 2 \quad \text{for } \mu = 2, \ldots, r. \tag{41}$$

Proof. ((41) $\implies$ (40)) Suppose C_{i_0} has distinct eigenvalues, which means that each $P^{\lambda_{i_0,j}}$ is the singleton partition of 1, and the partition $P^{i_0} = (\underline{1})$. Moreover, the maximal number of Jordan blocks corresponding to the same eigenvalue $r - R_i$ is simply the number of columns of P^i, or equivalently, $\gamma_{P^i}(r)$.

Since $\gamma_{P^{i_0}}(r) = 1$, we have $R(p_{i_0}) = r - 1$ which is the maximal value for R_i, so the second condition in Theorem B.1 is equivalent to the single inequality

$$R_1 + \cdots + \widehat{R_{i_0}} + \cdots R_n \geq r$$

Using the identification $r - R_i = \gamma_{P^i}(r)$ and $\gamma_{P^{i_0}} = 1$, we can write it as

$$(n-1)r - \sum_{i=1}^{n} \gamma_{P^i}(r) + \gamma_{P^{i_0}}(r) \geq r \iff \sum_{i=1}^{n} \gamma_{P^i}(r) < (n-2)r + 2 \tag{42}$$

Note that d_i can be computed in terms of the partition P^i (e.g. using [Kos04, Remark 14]), we have $d_i = r(r+1) - 2\sum_{\mu=1}^{r} \gamma_{P^i}(\mu)$. Then the inequality (α) is simply

$$nr(r+1) - 2\sum_{i=1}^{n}\sum_{\mu=1}^{r} \gamma_{P^i}(\mu) \geq 2r^2 - 2$$

A direct computation shows that this is a consequence of summing up all the inequalities in (41) for $\mu = 2, \dots, r$ and the equality $\sum_{i=1}^{n} \gamma_{P^i} = n$ on both sides.
((40) $\implies$ (41)) For this direction, we will make use a property of level functions.

Lemma B.3. For any partition P and any $2 \leq \mu \leq r$, if $\gamma_P(\mu-1) = \gamma_P(\mu) - 1$ and $\gamma_P(\mu-2) = \gamma_P(\mu-1) - 1$, then $\gamma_P(\mu+\epsilon) = \gamma_P(\mu) + \epsilon$ for $\epsilon \geq 1$.

Proof. Indeed, the number of columns is decreasing from left to right, so the number of columns will stabilize once if there is a column with only one block. ☐

Assume for simplicity that $i_0 = n$. Suppose that there exists $\mu_0 > 1$ such that

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0) \geq (n-2)\mu_0 + 2$$

$$(*_\mu) \qquad \sum_{i=1}^{n} \gamma_{P^i}(\mu) \leq (n-2)\mu + 1 \qquad \text{for } 1 < \mu < \mu_0$$

As each level function can only increase by 1, so the total increment from $\mu_0 - 1$ to μ_0 is bounded

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0) - \sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 1) \leq n - 1$$

and by combining with the inequality $(*_{\mu_0-1})$, we obtain

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0) \leq (n-1) + \sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 1) \leq (n-2)\mu_0 + 2$$

Hence, $\sum_{i=1}^{n} \gamma_{P^i}(\mu_0) = (n-2)\mu_0 + 2$ in which case the total increment from $\mu_0 - 1$ to μ_0 satisfies

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0) - \sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 1) = n - 1$$

and $\sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 1) = (n-2)(\mu_0 - 1) + 1$. In other words, $\gamma_{P^i}(\mu_0 - 1) = \gamma_{P^i}(\mu_0) - 1$ for $i = 1, \dots, n-1$.
Case 1: Suppose $\mu_0 - 2 > 1$. By the inequality $(*_{\mu_0-2})$, the total increment from $\mu_0 - 2$ to $\mu_0 - 1$ satisfies

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 1) - \sum_{i=1}^{n} \gamma_{P^i}(\mu_0 - 2) \geq n - 2$$

This implies that $\gamma_{P^i}(\mu_0 - 1) = \gamma_{P^i}(\mu_0 - 2) - 1$ for at least $n-2$ indices $i \in \{1, \dots, n-1\}$. By Lemma B.3, it follows that $\gamma_{P^i}(\mu_0 + \epsilon) = \gamma_{P^i}(\mu_0) + \epsilon$ for at least $n-2$ indices i, so the total increment from μ_0 to $\mu_0 + \epsilon$ satisfies

$$\sum_{i=1}^{n} \gamma_{P^i}(\mu_0 + \epsilon) - \sum_{i=1}^{n} \gamma_{P^i}(\mu_0) \geq (n-2)\epsilon \implies \sum_{i=1}^{n} \gamma_{P^i}(\mu_0 + \epsilon) \geq (n-2)(\mu_0 + \epsilon) + 2$$

In particular, when $\mu_0 + \epsilon = r$, the last inequality violates the hypothesis.
Case 2: Suppose $\mu_0 - 2 = 1$. In this case, we have $\sum_{i=1}^n \gamma_{P^i}(\mu_0 - 2) = \sum_{i=1}^n \gamma_{P^i}(1) = n$ by the definition of a level function. So, the total increment from $\mu_0 - 2$ to $\mu_0 - 1$ is

$$\sum_{i=1}^n \gamma_{P^i}(\mu_0 - 1) - \sum_{i=1}^n \gamma_{P^i}(\mu_0 - 2) = n - 3$$

By Lemma B.3, it follows that $\gamma_{P^i}(\mu_0 + \epsilon) = \gamma_{P^i}(\mu_0) + \epsilon$ for at least $n - 3$ indices i, which we can assume to be $i = 1, \dots, n-3$ and so $\gamma_{P^i}(\mu) = \mu$ for $i = 1, \dots, n-3$. The only freedom left are the choices of $\gamma_{P^i}(\mu)$ for $i = n-2, n-1$. We want to show that the sum of all $\gamma_{P^i}(\mu)$ violates (α). It suffices to consider the $\gamma_{P^i}(\mu)$ with the lowest growth for $i = n-1, n-2$. As $\gamma_{P^i}(1) = 1$ and $\gamma_{P^i}(2) = 1$ are fixed, the level function with the lowest growth is

$$\gamma_{P^i}(1) = 1, \gamma_{P^i}(2) = 1, \gamma_{P^i}(3) = 2, \gamma_{P^i}(4) = 2, \gamma_{P^i}(5) = 3, \gamma_{P^i}(6) = 3, \dots$$

i.e. the $\gamma_{P^i}(\mu) = \lceil \mu/2 \rceil$. Then we see that

$$\sum_{i=1}^n \gamma_{P^i}(\mu) = (n-3)\mu + 2\lceil \mu/2 \rceil + 1 \implies \sum_{i=1}^n \gamma_{P^i}(\mu) = \begin{cases} (n-2)\mu + 2 & \text{if } \mu \text{ is odd} \\ (n-2)\mu + 1 & \text{if } \mu \text{ is even} \end{cases}$$

A direct computation shows that the sum of $\sum_{i=1}^n \gamma_{P^i}(\mu)$ violates (α).

$\square$

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References

[BDD22] A. Balasubramanian, J. Distler, and R. Donagi. "Families of Hitchin systems and $\mathcal{N} = 2$ theories". In: *Adv. Theor. Math. Phys.* 26.6 (2022), pp. 1585–1667. URL: https://doi.org/10.4310/atmp.2022.v26.n6.a2.

[BK18] D. Baraglia and M. Kamgarpour. "On the image of the parabolic Hitchin map". In: *Q. J. Math.* 69.2 (2018), pp. 681–708. URL: https://doi.org/10.1093/qmath/hax055.

[BNR89] A. Beauville, M. S. Narasimhan, and S. Ramanan. "Spectral curves and the generalised theta divisor". In: *J. Reine Angew. Math.* 398 (1989), pp. 169–179. URL: https://doi.org/10.1515/crll.1989.398.169.

[Boa14] P. P. Boalch. "Geometry and braiding of Stokes data; fission and wild character varieties". In: *Ann. of Math. (2)* 179.1 (2014), pp. 301–365. URL: https://doi.org/10.4007/annals.2014.179.1.5.

[CDP14] W.-Y. Chuang, D.-E. Diaconescu, and G. Pan. "BPS states and the $P = W$ conjecture". In: *Moduli spaces.* Vol. 411. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2014, pp. 132–150.

[Chu+20] W.-Y. Chuang, D.-E. Diaconescu, R. Donagi, S. Nawata, and T. Pantev. "Twisted spectral correspondence and torus knots". In: *J. Knot Theory Ramifications* 29.6 (2020), pp. 2050040, 65. URL: https://doi.org/10.1142/S0218216520500406.

[CS06] W. Crawley-Boevey and P. Shaw. "Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem". In: *Adv. Math.* 201.1 (2006), pp. 180–208. URL: https://doi.org/10.1016/j.aim.2005.02.003.

[DDP18] D.-E. Diaconescu, R. Donagi, and T. Pantev. "BPS States, Torus Links and Wild Character Varieties". In: *Communications in Mathematical Physics* 359.3 (Feb. 2018), pp. 1027–1078. URL: `https://doi.org/10.1007%2Fs00220-018-3097-9`.

[DH24] R. Donagi and A. F. Herrero. *The meromorphic Hitchin fibration over stable pointed curves: moduli spaces*. 2024. arXiv: `2411.16912 [math.AG]`. URL: `https://arxiv.org/abs/2411.16912`.

[Har77] R. Hartshorne. *Algebraic geometry*. Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496.

[Hau+22] T. Hausel, A. Mellit, A. Minets, and O. Schiffmann. $P = W$ *via* H_2. 2022. arXiv: `2209.05429 [math.AG]`.

[HLR13] T. Hausel, E. Letellier, and F. Rodriguez-Villegas. "Arithmetic harmonic analysis on character and quiver varieties II". In: *Adv. Math.* 234 (2013), pp. 85–128. URL: `https://doi.org/10.1016/j.aim.2012.10.009`.

[Kal84] D. Kalman. "The generalized Vandermonde matrix". In: *Mathematics Magazine* 57.1 (1984), pp. 15–21. URL: `https://doi.org/10.2307/2690290`.

[Kat96] N. M. Katz. *Rigid local systems*. Vol. 139. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1996, pp. viii+223. URL: `https://doi.org/10.1515/9781400882595`.

[KK24] T. Kinjo and N. Koseki. "Cohomological χ-independence for Higgs bundles and Gopakumar–Vafa invariants". In: *Journal of the European Mathematical Society* (July 2024). URL: `http://dx.doi.org/10.4171/JEMS/1487`.

[Kos04] V. P. Kostov. "The Deligne-Simpson problem—a survey". In: *J. Algebra* 281.1 (2004), pp. 83–108. URL: `https://doi.org/10.1016/j.jalgebra.2004.07.013`.

[KP95] A. Kouvidakis and T. Pantev. "The automorphism group of the moduli space of semistable vector bundles". In: *Math. Ann.* 302.2 (1995), pp. 225–268. URL: `https://doi.org/10.1007/BF01444495`.

[LL24] J. Lee and S. Lee. *Generators for the cohomology of the moduli space of irregular parabolic Higgs bundles*. 2024. arXiv: `2402.05380 [math.AG]`.

[Let15] E. Letellier. "Character varieties with Zariski closures of GL_n-conjugacy classes at punctures". In: *Selecta Math. (N.S.)* 21.1 (2015), pp. 293–344. URL: `https://doi.org/10.1007/s00029-014-0163-9`.

[LM10] M. Logares and J. Martens. "Moduli of parabolic Higgs bundles and Atiyah algebroids". In: *J. Reine Angew. Math.* 649 (2010), pp. 89–116. URL: `https://doi.org/10.1515/CRELLE.2010.090`.

[Mar94] E. Markman. "Spectral curves and integrable systems". In: *Compositio Math.* 93.3 (1994), pp. 255–290. URL: `http://www.numdam.org/item?id=CM_1994__93_3_255_0`.

[MY92] M. Maruyama and K. Yokogawa. "Moduli of parabolic stable sheaves". In: *Math. Ann.* 293.1 (1992), pp. 77–99. URL: `https://doi.org/10.1007/BF01444704`.

[MW97] K. Matsuki and R. Wentworth. "Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface". In: *Internat. J. Math.* 8.1 (1997), pp. 97–148. URL: `https://doi.org/10.1142/S0129167X97000068`.

[MS23] D. Maulik and J. Shen. "Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles". In: *Geom. Topol.* 27.4 (2023), pp. 1539–1586. URL: https://doi.org/10.2140/gt.2023.27.1539.

[MSY23] D. Maulik, J. Shen, and Q. Yin. *Perverse filtrations and Fourier transforms*. 2023. arXiv: 2308.13160 [math.AG].

[MSY24] D. Maulik, J. Shen, and Q. Yin. *Algebraic cycles and Hitchin systems*. 2024. arXiv: 2407.05177 [math.AG]. URL: https://arxiv.org/abs/2407.05177.

[Sim09] C. Simpson. "Katz's middle convolution algorithm". In: *Pure Appl. Math. Q.* 5.2 (2009), pp. 781–852. URL: https://doi.org/10.4310/PAMQ.2009.v5.n2.a8.

[Sim90] C. T. Simpson. "Harmonic bundles on noncompact curves". In: *J. Amer. Math. Soc.* 3.3 (1990), pp. 713–770. URL: https://doi.org/10.2307/1990935.

[Sim91] C. T. Simpson. "Products of matrices". In: *Differential geometry, global analysis, and topology (Halifax, NS, 1990)*. Vol. 12. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1991, pp. 157–185.

[Sim94] C. T. Simpson. "Moduli of representations of the fundamental group of a smooth projective variety. II". In: *Inst. Hautes Études Sci. Publ. Math.* 80 (1994), pp. 5–79. URL: http://www.numdam.org/item?id=PMIHES_1994__80__5_0.

[SWW22a] X. Su, B. Wang, and X. Wen. "Parabolic Hitchin maps and their generic fibers". In: *Math. Z.* 301.1 (2022), pp. 343–372. URL: https://doi.org/10.1007/s00209-021-02896-3.

[SWW22b] X. Su, B. Wang, and X. Wen. *Topological Mirror Symmetry of Parabolic Hitchin Systems*. 2022. arXiv: 2206.02527 [math.AG]. URL: https://arxiv.org/abs/2206.02527.

[Wen21] X. Wen. *Parabolic Higgs bundles on projecitive line, quiver varieties and Deligne-Simpson problem*. 2021. arXiv: 2101.01913 [math.AG]. URL: https://arxiv.org/abs/2101.01913.

[Yok93] K. Yokogawa. "Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves". In: *J. Math. Kyoto Univ.* 33.2 (1993), pp. 451–504. URL: https://doi.org/10.1215/kjm/1250519269.

[Yos03] K. Yoshioka. "Twisted stability and Fourier-Mukai transform. II". In: *Manuscripta Math.* 110.4 (2003), pp. 433–465. URL: https://doi.org/10.1007/s00229-002-0340-6.

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