Homework 1

Name: 方嘉聪 ID: 2200017849

Problem 1. Given random variables $X \sim \mathcal{N}(0,1)$, for $t > 0$, define:

$$\bar{\Phi}(t) := \mathbb{P}[X \geq t] = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{\tau^2}{2}} \mathrm{d}\tau$$

Find elementary function $f(t)$, such that $\bar{\Phi}(t) \sim f(t)$, i.e.

$$\lim_{t \to \infty} \frac{\bar{\Phi}(t)}{f(t)} = 1$$

◀

Solution. Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, we can prove:

$$\left(\frac{x}{1+x^2}\right) \phi(x) < \bar{\Phi}(x) < \frac{\phi(x)}{x}.$$

1. For upper bound we have:

$$\bar{\Phi}(x) = \int_x^\infty \phi(t) \mathrm{d}t < \int_x^\infty \frac{t}{x} \phi(t) \mathrm{d}t = \int_{\frac{x^2}{2}}^\infty \frac{1}{x\sqrt{2\pi}} e^{-m} \mathrm{d}m = \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{\phi(x)}{x}.$$

2. For lower bound, using $\phi'(x) = -x\phi(x)$, we have:

$$\begin{aligned}
\left(1 + \frac{1}{x^2}\right) \bar{\Phi}(x) &= \int_x^\infty \left(1 + \frac{1}{x^2}\right) \phi(t) dt > \int_x^\infty \left(1 + \frac{1}{t^2}\right) \phi(t) dt \\
&= \int_x^\infty \frac{-\phi'(t)t + \phi(t)}{t^2} dt = -\left(\frac{\phi(t)}{t}\right)\Bigg|_x^\infty = \frac{\phi(x)}{x} \\
\implies \bar{\Phi}(x) &> \left(\frac{x}{1+x^2}\right) \phi(x)
\end{aligned}$$

Thus,

$$f(t) = \frac{\phi(t)}{t} = \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{t}\right).$$

Then we have

$$\left(\frac{t^2}{1+t^2}\right) < \frac{\bar{\Phi}(t)}{f(t)} < 1 \implies \lim_{t \to \infty} \frac{\bar{\Phi}(t)}{f(t)} = 1.$$

◁