Homework 10

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Problem 1.(Revisit 3-Dimensional Matching).

Recall that in the last assignment we proved that 3-dimensional matching problem is NP-complete. Now let's consider the following maximization version: Given **disjoint** sets X, Y, and Z, and a set $T \subseteq X \times Y \times Z$ of ordered triples, we want to find out the maximum size $|M|$ of a 3-dimensional matching $M \subseteq T$.

Give a 3-approximation algorithm. Prove its approximation ratio and give its running time. ◀

Solution. 考虑如下的近似算法:

Algorithm 1 3-Approximation Algorithm for 3-Dimensional Matching

Input: Disjoint sets X, Y, Z and a set $T \subseteq X \times Y \times Z$ of ordered triples

Output: A 3-dimensional matching $M \subseteq T$

1: $M \leftarrow \varnothing$, $U_X \leftarrow X, U_Y \leftarrow Y, U_Z \leftarrow Z$
2: **while** $T \neq \varnothing$ **do**
3: 随机选取一个三元组 $(x, y, z) \in T$;
4: 将 (x, y, z) 加入 M;
5: 从 U_X, U_Y, U_Z 中删除 x, y, z;
6: 从 T 中删除所有包含 x, y, z 的三元组.
7: **return** M

第 5-6 行保证了选取的三元组是合法的. 每次循环用时 $O(|T|)$, 总的时间复杂度为 $O(|T|^2)$. 下面我们证明这个算法的近似比为 3.

设最优解为 M^*, 假设 $|M^*| > 3|M|$. 对于任意一个三元组 $(x, y, z) \in M$, 那么在 M^* 至多有 3 个对应的三元组 $(x, \cdot, \cdot), (\cdot, y, \cdot), (\cdot, \cdot, z)$. 由于 $|M^*| > 3|M|$, 那么在 M^* 存在一个三元组 $(\bar{x}, \bar{y}, \bar{z})$ 使得 $\bar{x}, \bar{y}, \bar{z}$ 都不在 M 中, 但由我们的算法 $(\bar{x}, \bar{y}, \bar{z})$ 应当被加入 M 中, 矛盾. 故 $3|M| \geq |M^*|$, 即近似比为 3.

一个紧实例为:

$$X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}, Z = \{z_1, z_2, z_3\},$$
$$T = \{(x_1, y_2, z_3), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$$

最优解为 $M^* = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$, 而我们的算法可能得到 $M = \{(x_1, y_2, z_3)\}$. ◁

Problem 2. (Bounded Subset Sum).

Suppose you are given a list of N **positive** integers $L = [a_1, a_2, \ldots, a_N]$, and a **positive** integer C. The problem is to find a subset $S \subseteq \{1, 2, \ldots, N\}$ such that

$$T(S) = \sum_{i \in S} a_i \leq C$$

and $T(S)$ is as large as possible.

(a) Prof. Luo proposes the following greedy algorithm for obtaining an approximate solution to this maximization problem:

Algorithm 2 Prof. Luo's Greedy Algorithm

Initialize $S \leftarrow \varnothing, T = 0$
for $i = 1, 2, \ldots, N$ **do**
    if $T + a_i \le C$ **then**
        $S \leftarrow S \cup \{i\}$
        $T \leftarrow T + a_i$
return S

Show that Prof. Luo's algorithm is not a ρ-approximation algorithm for any fixed value ρ. (Use the convention that $\rho > 1$.)

(b) Describe a 2-approximation algorithm for this maximization problem that runs in $O(N)$ time. Prove its approximation ratio. ◀

Solution. (a) 给任意的 $\rho > 0$, 设最优解为 $T(S^*)$, 我们考虑如下的反例:

$$L = \left\{ \left\lfloor \frac{C}{\rho} \right\rfloor - 1, C \right\}, \text{ 其中取 } C \text{ 使得 } \left\lfloor \frac{C}{\rho} \right\rfloor - 1 \ge 1.$$

那么

$$T(S^*) = C, \quad T(S) = \left\lfloor \frac{C}{\rho} \right\rfloor - 1 \implies \frac{T(S^*)}{T(S)} > \frac{C}{C/\rho} > \rho.$$

故对任意的 $\rho > 1$, 罗老师的算法都不是 ρ-近似算法.

(b) 不妨设 $\forall i \in [N], a_i \in (0, C]$, 考虑如下的算法:

$S \leftarrow \varnothing, T \leftarrow 0$
for $i = 1, 2, \cdots, N$ **do**
    if $a_i \ge C/2$ **then**
        return $\{i\}$
Run Prof. Luo's Greedy Algorithm(2).

至多需扫描两遍数组 L, 时间复杂度: $O(N)$. 下面我们证明这个算法的近似比为 2.

设最优解为 S^*, 则 $T(S^*) \le C$. 那么:

- 若存在 i 使得 $a_i \ge C/2$, 那么直接返回 $\{i\}$, 故 $T(S) = C/2 \ge T(S^*)/2$, 近似比为 2.
- 若上述条件不成立, 那么调用了罗老师的算法. 若 L 中所有元素都被加入 T 中, 那么有 $T(S) = T(S^*)$. 否则, 设第一个不满足 $T + a_i \le C$ 的下标为 j, 由于 $a_j < C/2$, 那么

$$T + a_j > C \implies T > C - a_j \ge C/2 \implies T \ge T(S^*)/2 \implies T(S) \ge T \ge T(S^*)/2.$$

  故此时近似比为 2.

综上, 近似比为 2. 证毕. ◁

Problem 3. (Hitting Set).

We are given a set $A = \{a_1, \dots, a_n\}$ and a collection $B_1, \dots, B_m$ of subsets of A. Also, each element $a_i \in A$ has a weight $w_i \geq 0$. We call $H \subseteq A$ is a *hitting set* if $H \cap B_i$ is not empty for each i. Now the problem is to find a hitting set H that minimizes the total weight of the elements in H, $\sum_{a_i \in H} w_i$.

Let $b = \max_i |B_i|$. Give a b-approximation algorithm that runs in polynomial time. Prove the approximation ratio. (*Hint: consider LP rounding.*) ◀

Solution. 设 $\varphi(a_i) = 1$ 若 $a_i \in H$. 考虑如下的线性规划:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} w_i \varphi(a_i) \\ \text{s.t.} \quad & \varphi(a_i) \in \{0, 1\}, \quad \forall i = 1, 2, \cdots, n \\ & \sum_{a_i \in B_j} \varphi(a_i) \geq 1, \quad \forall j = 1, 2, \cdots, m \end{aligned}$$

考虑线性化松弛后的线性规划:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} w_i \varphi(a_i) \\ \text{s.t.} \quad & \varphi(a_i) \geq 0, \quad \forall i = 1, 2, \cdots, n \\ & \varphi(a_i) \leq 1, \quad \forall i = 1, 2, \cdots, n \\ & \sum_{a_i \in B_j} \varphi(a_i) \geq 1, \quad \forall j = 1, 2, \cdots, m \end{aligned}$$

考虑如下的近似算法:

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1: H ← ∅
2: 解上述线性规划, 得到 φ(a_i);
3: for i = 1, 2, ··· , n do
4:     if φ(a_i) ≥ 1/b then
5:         H.append(a_i)
6: return H
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对于任意一个 B_j, $\sum_{a_i \in B_j} \varphi(a_i) \geq 1 \implies \exists \varphi(a_i) \geq 1/|B_j| \geq 1/b$. 故上述算法得到的解是一个合法的 *hitting set*. 显然上述算法是多项式时间的. 下面我们证明这个算法的近似比为 b.

设最优解为 H^*, 记对应的权重代价为 $w(H^*)$(与原线性规划的最优解相等), 记线性松弛后的最优解为 z^*, 那么 $w(H^*) \geq z^*$. 设上述近似算法的到的权重代价为 $w(H)$, 那么:

$$\begin{aligned} z^* = \sum_{i=1}^{n} w_i \varphi(a_i) &= \sum_{\varphi(a_i) \geq 1/b} w_i \varphi(a_i) + \sum_{\varphi(a_i) < 1/b} w_i \varphi(a_i) \\ &\geq \sum_{\varphi(a_i) \geq 1/b} w_i \varphi(a_i) \geq \frac{1}{b} \sum_{\varphi(a_i) \geq 1/b} w_i = \frac{1}{b} w(H) \end{aligned}$$

故 $w(H^*) \geq z^* \geq w(H)/b \implies b \geq w(H)/w(H^*)$. 故近似比为 b, 证毕. ◁