Problem Given a set $A = \{a_1, a_2, ...a_n\}$ of n integers how many non-zero unique subsets can be selected such that the sum of the subset is even. In the event a set contains two numbers which are the same, the numbers will be considered independent and unique, such that if $a_i = a_j$ then choosing a_i and choosing $a_j$will be considered as different sets.

Decomposition

1. $A = \{\}$
2. A_e = even elements only
3. A_{eo} = even number of odd elements only
4. A_{oo} = odd number of odd elements only
5. A_u= $A_e \cup A_{eo} \cup A_{oo}$ Generic case

Math We will now consider each decomposition.

1. In the event of $A = \{\}$ the answer is 1.
2. For A_e, let $C^n_r = \binom{n}{r}$.Calculate $\sum_{r=1}^{n} \binom{n}{r} = 2^n - 1$ by the Bionomial Theorem.[1]
   For example, $A_e = \{2, 4, 6, 8\}$ produces the following combinations all 15 of which have even sums:
   $$\begin{aligned}\{2\}, \{4\}, \{6\}, \{8\},\\ \{2,4\}, \{2,6\}, \{2,8\}, \{4,6\}, \{4,8\}, \{6,8\},\\ \{2,4,6\}, \{2,4,8\}, \{2,6,8\}, \{4,6,8\},\\ \{2,4,6,8\}\end{aligned}$$
   $\sum_{r=1}^{n} \binom{n}{r} = 2^n - 1 = 2^4 - 1 = 16 - 1 = 15$. There are 15 possible subsets.
3. For A_{eo}, each subset of even length produces an even sum.
   For example, $A_{eo} = \{1, 3, 5, 7\}$produces the following combinations of which 7 have even sums:
   $$\begin{aligned}\{1\}, \{3\}, \{5\}, \{7\}, = C^4_1\\ \{1,3\}, \{1,5\}, \{1,7\}, \{3,5\}, \{3,7\}, \{5,7\}, = C^4_2\\ \{1,3,5\}, \{1,3,7\}, \{1,5,7\}, \{3,5,8\}, = C^4_3\\ \{1,3,5,7\} = C^4_4\end{aligned}$$
   For A_{eo} with $|A_{eo}|$ is even, the number of even non-empty subsets is calculated by $\sum_{r=1}^{n} \binom{n}{r} - 1, r \in \{2, 4, 6 \ldots, n\} = 2^{n-1} - 1 = 2^{4-1} - 1 = 7$ by combinatorial identity[2].

[1] Chong & Meng, Principles and Techniques in Combinatorics p.71, 2.3.1
[2] Chong & Meng, p.72 2.2.3

4. For A_{oo}, still each subset of even length produces and even sum.
   For example, $A_{oo} = \{1, 3, 5\}$produces the following combinations of which 3 have even sums:

$$\begin{aligned} \{1\}, \{3\}, \{5\} &= C^3_1 \\ \{1, 3\}, \{1, 5\}, \{3, 5\} &= C^3_2 \\ \{1, 3, 5\} &= C^3_3 \end{aligned}$$

   It follows that in the event of an odd number of odd elements, $2^{n-1} - 1 = 2^{3-1} - 1 = 3$ is the same result as A_{eo}.

5. For the generic case consider $A_e \cup A_{eo}$ and by the multiplication principle [3] the number of even summed where $|A_e| = i$ and $|A_{eo}| = j$ sets is given by $\sum_{r=0}^{e} \binom{n}{r} \sum_{r=0}^{j} \binom{n}{r}$. Removing the empty set the result is $2^n \cdot 2^{n-1} - 1$.

Solution Given a set $A = \{a_1, a_2, ...a_n\}$ the number of even non-empty subsets is computed by counting the number of even a, call e and the number of odd a, call o, computing the combinations as per above, and multiplying.

$$2^e \cdot 2^{o-1} - 1 \blacksquare$$

[3]Chong & Meng, p.3