CS450 Assignment 2 Jiadong Hong

Part 1 Theoretical Problems

Question 1

Question-1: Find the matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ containing independent columns of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} \quad \boldsymbol{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1}$$

$$C_1 = \{[1,3,2]^T\}$$
$$C_2 = \{[1,4,7]^T, [2,5,8]^T\}$$

Question 2

In general, for any vector x, does it hold that

$$\| x \|_1 \geq \| x \|_2 \geq \| x \|_\infty$$

Since the concept of the norm is:

- L-p norm: $\|\boldsymbol{\beta}\|_p = \left(\sum_{j=1}^d \beta_j^p\right)^{1/p}$, $p \geq 1$
- $p = 1$, $\|\boldsymbol{\beta}\|_1 = \sum_j |\beta_j|$
- $p = 2$, $\|\boldsymbol{\beta}\|_2 = \sqrt{\sum_j |\beta_j|^2}$
- $p = \infty$, $\|\boldsymbol{\beta}\|_\infty = \max_j |\beta_j|$
- The L-0 norm: Counts the number of non-zero entries, e.g., if $\boldsymbol{\beta} = (10, 0, 2, 0.01, 0, 1)^T$, then $\|\boldsymbol{\beta}\|_0 = 4$

Proof: Let x be an arbitrary vector.

$\|x\|_1 \geq \|x\|_\infty$:

The L_1 norm ($\|x\|_1$) is defined as the sum of the absolute values of the vector's components.

The L_∞ norm ($\|x\|_\infty$) is defined as the maximum absolute value of any component in the vector.

Since the sum of absolute values is always greater than or equal to the maximum absolute value, we have $\|x\|_1 \geq \|x\|_\infty$.

$\|x\|_2 \geq \|x\|_\infty$:

The L_2 norm ($\|x\|_2$) is defined as the square root of the sum of the squares of the vector's components.

The L_∞ norm ($\|x\|_\infty$) is still defined as the maximum absolute value of any component in the vector.

Since the square root of a positive value is always greater than or equal to the value itself, we have $\|x\|_2 \geq \|x\|_\infty$. Hence, for any vector x, $\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty$.

QED

Question 3

Question-3: Consider a non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, use the definition of condition number to give a step-by-step proof showing that the matrix condition number is given by the following

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\| \tag{2}$$

where $\|\boldsymbol{A}\|$ is the matrix norm. Furthermore, show that

$$\|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\| = \left(\max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}\right) \cdot \left(\min_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}\right)^{-1}. \tag{3}$$

In addition, prove that $\kappa(\boldsymbol{A}) \geq 1$.

(1) $Ax=b$

$\Rightarrow \dfrac{\|\delta x\|}{\|x\|} \leq \mathrm{cond}(A)\cdot\dfrac{\|\delta b\|}{\|b\|}$

$\delta x = A^{-1}\delta b$

$\Rightarrow \|\delta x\| \leq \|A^{-1}\|\,\|\delta b\|$ ①

$\because \|b\| = \|A\cdot x\| \leq \|A\|\cdot\|x\|$ ②

$\Rightarrow \dfrac{\|\delta x\|}{\|x\|} \leq \|A\|\,\|A^{-1}\|\,\dfrac{\|\delta x\|}{\|A\|}$

(2) According to defination.

$\|A\| = \max\limits_{x\neq 0}\dfrac{\|Ax\|}{\|x\|}$

$\|A^{-1}\| = \max\limits_{x\neq 0}\dfrac{\|A^{-1}x\|}{\|x\|}$

$= \sqrt{\lambda_{max}}$

$\because \lambda_{max}$ for A^{-1}

$= \dfrac{1}{\lambda_{min}}$ for A

$\Rightarrow \|A^{-1}\| = (\min\limits_{x=1}\|A\|)^{-1}$

$= \left(\min\limits_{x\neq 0}\dfrac{\|Ax\|}{\|x\|}\right)^{-1}$

$\Rightarrow$ Q.E.D.

(3) Since

$\mathrm{cond}(A) = \dfrac{|\lambda_{max}|}{|\lambda_{min}|}$

$\because \lambda_{max} > \lambda_{min}$

$\Rightarrow \mathrm{cond}(A) > 1$

Question 4

Question-4: Given an $n * n$ square matrix, show that:

(a) The $L-1$ matrix norm is $\|\boldsymbol{A}\|_1 = \max\limits_j \sum_{i=1}^{n} |a_{ij}|$

(b) The $L-\infty$ matrix norm is $\|\boldsymbol{A}\|_\infty = \max\limits_i \sum_{j=1}^{n} |a_{ij}|$

(c) $\|\boldsymbol{A}\|_1 = \left\|\boldsymbol{A}^\top\right\|_\infty$

$$(d)\ \|\boldsymbol{A}\|_2 \leqslant \|\boldsymbol{A}\|_1 \cdot \|\boldsymbol{A}\|_\infty$$

(a) $\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{\|x\|_1 = 1} \|Ax\|_1$

$\Rightarrow$ For x: $\sum_j |x_j| = 1$

$\|Ax\|_1 = \|b\|_1 = \sum_j |b_j|$

$= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} \cdot x_j|$

$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\, |x_j| = \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}| \cdot |x_j|$

$\leqslant \max \left(\sum_{i=1}^{n} |a_{ij}| \right) \cdot \sum_{j=1}^{n} |x_j|$

As $\sum_j |x_j| = 1 \Rightarrow \|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |a_{ij}|$ Q.Z.D.

(b) let $x = (x_1, x_2, \cdots, x_n) \neq 0$

let $A \neq 0$

let $u = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|$

$\|Ax\|_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij} x_j|$

$\leqslant \max |x_i| \max \sum_j |a_{ij}| = \mu \|x\|_\infty$

assume $u = \sum_{j=1}^{n} |a_{i_0, j}|$

there is a vector: $x_0 = (x_1, x_2, \cdots, x_n)^T$

$x_j = \mathrm{sign}(a_{i_0, j})$ $(j = 1, 2, 3, \cdots n)$

obviously, $\|x_0\|_\infty = 1$

$Ax_0 = \sum_{i=0}^{n} |a_{i_0, j}|\, x_j = \sum_{i=1}^{n} |a_{i_0, j}|$

$\Rightarrow \|Ax_0\|_\infty = \max \left| \sum_{i=1}^{n} a_{ij} x_j \right|$

$= \sum_{j=1}^{n} |a_{i_0, j}| = \mu$

(C) Since $\|A^T\|_\infty$ is $\max \sum_j |a_{ij}|$ for A^T

$\Leftrightarrow \max \sum_i |a_{ij}|$ for A

$\Leftrightarrow \|A\|_1$

$\Rightarrow$ Q.Z.D.

(e)

$\|A\|_2^2 = \max_{\|x\| = 1} \left[\sum_{i=1}^{n} (A_i x_j)^2 \right]$

Set $P(A)$ as the spectral radius

$\Rightarrow \|A\|_2^2 = P(A \cdot A^T) \leqslant \|A^T A\|_1 \leqslant \|A\|_1 \cdot \|A^T\|_1 = \|A\|_1 \cdot \|A\|_\infty$

Question 5

Question-5: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an invertable matrix, and $\boldsymbol{x}$, $\boldsymbol{x}+\Delta\boldsymbol{x}$ be the solutions to the following systems

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{4}$$

$$(\boldsymbol{A}+\Delta\boldsymbol{A})(\boldsymbol{x}+\Delta\boldsymbol{x}) = \boldsymbol{b}. \tag{5}$$

Consider $\boldsymbol{b} \neq \boldsymbol{0}$, show the following:
(a) The following inequality holds

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}+\Delta\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A})\frac{\|\Delta\boldsymbol{A}\|}{\|\boldsymbol{A}\|}. \tag{6}$$

(b) The following inequality holds

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A})\frac{\|\Delta\boldsymbol{A}\|}{\|\boldsymbol{A}\|}\left(\frac{1}{1-\|\Delta\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|}\right). \tag{7}$$

1(a).

$(A+\Delta A)(x+\Delta x) = b$

$\Rightarrow \cancel{Ax} + A\Delta x + \Delta A x + \Delta A \Delta x = \cancel{b}$

$\Rightarrow A\Delta x + \Delta A x + \Delta A \Delta x = 0$

$\Rightarrow -\Delta A(x+\Delta x) = A\Delta x$

$\|\Delta A (x+\Delta x)\| \leq \|\Delta A\| \cdot \|(x+\Delta x)\|$

$\| A \cdot \Delta x\| \leq \|A\| \, \|\Delta x\|$

$\Rightarrow \|A\|\,\|A^{-1}\| \frac{\|\Delta A\|}{\|A\|} \|x+\Delta x\| = k(A) \frac{\|\Delta A\|}{\|A\|} \|x+\Delta x\| \geq \|A \Delta x\| \cdot \|A^{-1}\| \geq \|A\|\,\|A^{-1}\| \cdot \|\Delta x\|$

$\because \|A\|\,\|A^{-1}\| = k(A) > 1 \Rightarrow \|A\|\,\|A^{-1}\|\,\|\Delta x\| > \|\Delta x\|$

$\Rightarrow k(A)\frac{\|\Delta A\|}{\|A\|} \geq \frac{\|\Delta x\|}{\|x+\Delta x\|}$

to prove.

1(b) $\Rightarrow \frac{\|\Delta x\|}{\|x\|} \leq \|A^{-1}\|\,\|\Delta A\| \left(\frac{1}{1-\|\Delta A\|\,\|A^{-1}\|}\right)$

$\Rightarrow (1-\|\Delta A\|\,\|A^{-1}\|)\|\Delta x\| \leq (\|\Delta A\|\,\|A^{-1}\|)\,\|x\|$,

$\Rightarrow \|\Delta x\| \leq (\|x\|+\|\Delta x\|)\,\|\Delta A\|\,\|A^{-1}\|$

$(\|x\|+\|\Delta x\|)\,\|\Delta A\|\,\|A^{-1}\| \geq \|\Delta A\|\,\|x+\Delta x\|\,\|A^{-1}\| \geq \|A^{-1}\|\,\|A\|\,\|\Delta x\| \geq \|\Delta x\|$

$\Rightarrow$ Q.E.D

Question-6: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Show that the function

$$\|\boldsymbol{x}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x}} \tag{8}$$

defines a norm on $\mathbb{R}^n$ (i.e., it satisfies the three defining properties of a norm). This vector norm is said to be *induced* by $\boldsymbol{A}$.

Moreover, what if $\boldsymbol{A}$ is positive semi-definite, does it still hold?

附 For all non-zero vector $\vec{z}$

$z^* M z \geq 0$

$\Rightarrow x^T A x \geq 0$

① $\|x\|_A \geq 0 \Rightarrow \sqrt{x^T A x} \geq 0$

② $\|\alpha x\|_A = \alpha \|x\|_A$ $\alpha \in \mathbb{R}$ $\Rightarrow \sqrt{\alpha x^T A x} = |\alpha| \sqrt{x^T A x}$

③ $\|x_1\|_A + \|x_2\|_A \geq \|x_1 + x_2\|_A \Rightarrow \sqrt{(x_1+x_2)^T A (x_1+x_2)}$

$\leq \sqrt{x_1^T A x_1} + \sqrt{x_2^T A x_2}$

∵ Positive semi-definite: $A = L \cdot L^T$

$\Rightarrow \sqrt{x^T A x} = \sqrt{(Lx)^T (Lx)}$

1) $\|x\|_A \geq 0 \Rightarrow$ ∵ $x^T A x \geq 0 \Rightarrow \sqrt{x^T A x} \geq 0 \Rightarrow$ satisfied.

2) since $\|\alpha x\|_A = \sqrt{\alpha x^T A \alpha x} = \sqrt{\alpha^2 x^T A x}$

$= |\alpha| \sqrt{x^T A x} \Rightarrow$ satisfied.

3) $\|x_1\|_A + \|x_2\|_A = y_1$ $\|x_1 + x_2\|_A = y_2$

$= \sqrt{x_1^T A x_1} + \sqrt{x_2^T A x_2}$

$y_1^2 = x_1^T A x_1 + x_2^T A x_2 + 2\sqrt{x_1^T A x_1 \cdot x_2^T A x_2}$

$y_2^2 = (x_1 + x_2)^T A (x_1 + x_2) \Rightarrow y_1^2 - y_2^2 = 2\sqrt{x_1^T A x_1 \cdot x_2^T A x_2} - x_1^T A x_2 - x_2^T A x_1$

$2\sqrt{x_1^TAx_1 \cdot x_2^TAx_2}$

$= 2\cdot\sqrt{\sum_i x_{1i}^2}\sqrt{\sum_i x_{2i}^2}$

$x_1^TAx_2 + x_2^TAx_1 = 2\sum_{i=1}^{n} x_{1i}x_{2i}$

$\Rightarrow 2\sqrt{x_1^TAx_1\, x_2^TAx_2} \geq x_1^TAx_2 + x_2^TAx_1$

$=$ 时 x_1, x_2 all positive.

Since semi positive.

$\Rightarrow$ holds.

$\Rightarrow$ satisfied, all holds to be a norm.

1) $\|x$

2) Sin

3) $\|$

$y_2 \geq y_1$

$= \sqrt{}$

y_1^2

y_2^2

Question-7: Show that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ has the same nullspace as $\boldsymbol{A}$. Here is one approach : First, if $\boldsymbol{Ax}$ equals zero then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax}$ equals ____. This proves $\mathbf{N}(\boldsymbol{A}) \subset \mathbf{N}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)$. Second, if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \mathbf{0}$ then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \|\boldsymbol{Ax}\|^2 = 0$. Deduce $\mathbf{N}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right) = \mathbf{N}(\boldsymbol{A})$.

(7) blank: zero.

零空间：在线性映射（即矩阵）的背景下

$Ax = 0$

$\Rightarrow A^TAx = 0$

$\Rightarrow A^TAx = 0 \Rightarrow N(A) \subset N(AA^T)$

$\Rightarrow x^TA^TAx = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow N(AA^T) \subset N(A)$

$\Rightarrow N(A^TA) = N(A)$.

Question-8: Given a matrix $\boldsymbol{A}$, prove the following:
(a) If $\boldsymbol{A}$ is an $n * n$ matrix, and $rank(\boldsymbol{A}) = n$, then

$$\kappa(\boldsymbol{A}) = \frac{\lambda_{\max}}{\lambda_{\min}} \tag{9}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximal and minimal eigenvalues of $\boldsymbol{A}$, respectively. Notice that both of them are positive.
(b) If $\boldsymbol{A}$ is an $m * n$ matrix, where $m \geq n$ and $rank(column(\boldsymbol{A})) = n$, then

$$\kappa(\boldsymbol{A}) = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{10}$$

(a) Since $\|A\| = \lambda_{max}$ $\quad \|A^{-1}\| = \frac{1}{\lambda_{min}}$

$\Rightarrow K(A) = \frac{\lambda_{max}}{\lambda_{min}}$.

(b) $A = U\Sigma V^T$ and $(A^TA)v = \lambda v$.

$\|U\Sigma V^T\| = \max_{\|x\|=1} \|U\Sigma V^T x\| \leq \|U\| \|V^T\| \|\Sigma x\|$

$\|U\| \|V^T\| \geq \|UV^T\| = 1$. $\quad \max \|\Sigma x\| = \lambda_{max}$

$\Rightarrow K(A) = \|A\| \|A^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}} = \frac{\sigma_{max}}{\sigma_{min}}$.

Part 2: Programming Problems

(a) Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.9 \end{bmatrix}$$

is singular. Describe the set of solutions to the system $\boldsymbol{Ax} = \boldsymbol{b}$ if

$$\boldsymbol{b} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.5 \end{bmatrix}$$

det(A) = 0

The determinant of matrix A is equal to 0, which means that A is singular.

When a matrix is singular, it implies that the system of linear equations Ax = b may have infinitely many solutions or no solution at all, depending on the specific values of b.

b = [0.1] [0.3] [0.5]

Since A is singular, there are either no solutions or infinitely many solutions to the system Ax = b, depending on whether the vector b is in the column space of A.

In this case, we can represent b as a linear combination of the columns of A (assuming it's possible):

b = 0.1 * [0.1] + 0.3 * [0.4] + 0.5 * [0.7]

This equation is consistent, which means that there are infinitely many solutions to the system Ax = b. In other words, there are infinitely many x vectors that satisfy the equation Ax = b. These x vectors can be found by varying the weights (0.1, 0.3, and 0.5) in the linear combination above.

(b) If we were to use Gaussian elimination with partial pivoting to solve this system using exact arithmetic, at what point would the process fail? (c) Because some of the entries of $\boldsymbol{A}$ are not exactly representable in a binary floating-point system, the matrix is no longer exactly singular when entered into a computer; thus, solving the system by Gaussian elimination will not necessarily fail. Solve this system on a computer using a library routine for Gaussian elimination. Compare the computed solution with your description of the solution set in part a. If your software includes a condition estimator, what is the estimated value for cond $(\boldsymbol{A})$? How many digits of accuracy in the solution would this lead you to expect?

(b) Numerical Instability: In exact arithmetic, Gaussian elimination should theoretically work for non-singular matrices. However, in practice, especially with large or ill-conditioned matrices, it might suffer from numerical instability due to the accumulation of round-off errors, especially when the pivots are very small relative to other matrix elements.

(c)

```python
# Define the matrix A and vector b
A = np.array([[0.1, 0.2, 0.3], [0.4, 0.5, 0.6], [0.7, 0.8, 0.9]])
b = np.array([0.1, 0.3, 0.5])

# Solve the system using NumPy's built-in Gaussian elimination
x = np.linalg.solve(A, b)

# Compute the condition number of A
condition_number = np.linalg.cond(A)

print("Solution (x):", x)
print("Condition Number (cond(A)):", condition_number)
```

```
Solution (x): [ 0.06327986  0.87344029 -0.27005348]
Condition Number (cond(A)): 2.37588029981422e+16
```

The condition number of this system is actually too large to convince the answer is correct.