CS450 Assignment 3 Jiadong Hong

Question 1

Question-1: Given the linear system

$$2x_1 - 6\alpha x_2 = 3,$$
$$3\alpha x_1 - \quad x_2 = \frac{3}{2}.$$

(a) Find value(s) of α for which the system has no solutions.
(b) Find value(s) of α for which the system has an infinite number of solutions.
(c) Assuming a unique solution exists for a given α, find the solution.

Question 1

(a) System: $\begin{bmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$ no solution.

$\Rightarrow \det\left(\begin{bmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{bmatrix}\right) = 0$ ① if $\alpha = \frac{1}{3} \Rightarrow$ infinite sols.

$\Rightarrow -2 + 18\alpha^2 = 0 \quad \alpha = \pm\frac{1}{3}$ ② if $\alpha = -\frac{1}{3} \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$

$\Rightarrow$ no solution $\Rightarrow \alpha = -\frac{1}{3}$

(b) $\begin{bmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{3}{2} \end{bmatrix}$ infinite solution

$\Rightarrow \begin{bmatrix} 2 \\ 3\alpha \end{bmatrix} = k\begin{bmatrix} -6\alpha \\ -1 \end{bmatrix} \Rightarrow \frac{3\alpha}{2} = \frac{1}{6\alpha} \Rightarrow 18\alpha^2 = 2 \quad \alpha = \frac{1}{3}$

(c) $\begin{cases} 2x_1 - 6\alpha x_2 = 3 \\ 3\alpha x_1 - x_2 = \frac{3}{2} \end{cases} \Rightarrow \begin{cases} 2x_1 - 6\alpha x_2 = 3 \\ 18\alpha^2 x_1 - 6\alpha x_2 = 9\alpha \end{cases} \Rightarrow (2-18\alpha^2)x_1 = 3 - 9\alpha$

$\Rightarrow x_1 = \frac{3-9\alpha}{2-18\alpha^2} = \frac{3\cancel{(1-3\alpha)}}{2\cancel{(1-3\alpha)}(1+3\alpha)} = \frac{3}{2(1+3\alpha)}$

$\Rightarrow \frac{3}{1+3\alpha} - 6\alpha x_2 = 3$

$\Rightarrow \frac{\cancel{3-3}-9\alpha}{1+3\alpha} = 6\alpha x_2 \Rightarrow x_2 = \frac{-3}{2(1+3\alpha)} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2(1+3\alpha)} \\ \frac{-3}{2(1+3\alpha)} \end{bmatrix}$

Question 2

Question-2: Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear systems, and compare the approximations to the actual solution.

(a)

$$
\begin{aligned}
0.03x_1 + 58.9x_2 &= 59.2 \\
5.31x_1 - 6.10x_2 &= 47.0
\end{aligned}
$$

Actual solution $[10, 1]$.

(b)

$$
\begin{aligned}
3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\
-3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\
6.11x_1 - 14.2x_2 + 21x_3 &= -139
\end{aligned}
$$

Actual solution $\left[0, 10, \frac{1}{7}\right]$.

(a) LU Factorization: $\begin{bmatrix} 1 & 0 \\ 177 & 1 \end{bmatrix} \begin{bmatrix} 0.03 & 58.9 \\ 0 & -1.04\times10^{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow \begin{bmatrix} 0.03 & 58.9 \\ 0 & -1.04e4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 59.2 \\ -1.04e^4 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$

(b) after elimination:

$\begin{bmatrix} 1 & -2.32 & 3.44 & 22.7 \\ & 1 & 0.675 & 10.1 \\ & & 1 & 0.143 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2.71e{-3} \\ 9.99 \\ 1.43e{-1} \end{bmatrix}$

Question 3

Question-3: Let $\boldsymbol{x}$ be the solution to the linear least squares problem $\boldsymbol{A}\boldsymbol{x} \cong \boldsymbol{b}$, where

$$
\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}
$$

$$A^T r = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \neq 0$$

$$A^T r = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$\Rightarrow$ C is possible.

$$A^T r = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Question 4

Question-4: Let $\boldsymbol{a}$ be any nonzero vector. If $\boldsymbol{v} = \boldsymbol{a} - \alpha \boldsymbol{e}_1$, where $\alpha = \pm\|\boldsymbol{a}\|_2$, and

$$\boldsymbol{H} = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}$$

show that $\boldsymbol{H}\boldsymbol{a} = \alpha \boldsymbol{e}_1$.

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proof. $Hx = (I - 2xx^T)x$

$= x - 2xx^Tx = -x.$

Suppose $x = \dfrac{v}{\|v\|_2}$

$\Rightarrow Hx = -x$ (proof in lect 13 page 37)

$Ha = H(v + \alpha e_1) = \|v\|_2 Hx + \alpha He_1$

$= -v + \alpha He_1$

$\Rightarrow$ proof $-v + \alpha He_1 = \alpha e_1$

$\Leftrightarrow \alpha He_1 = v + \alpha e_1.$

$\alpha He_1 = a$

$He_1 = \dfrac{a}{\alpha} \Leftrightarrow$ unit vector for a direction

Since H is house hold transform matrix

e_1 is symmetric with unit vector in a direction

$\Rightarrow$ Q.E.D.

Question 5

Question-5: Determine the Householder transformation that annihilates all but the first entry of the vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. Specifically, if

$$\left(\boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

what are the values of α and $\boldsymbol{v}$?

Since Q4 have proof that

$$Ha = \alpha e_1$$

We can conclude that

$$\alpha = 2$$

$$v = a - 2e_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Where,

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 6

Question-6: Suppose that you are computing the QR factorization of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

by Householder transformations. (a) How many Householder transformations are required? (b) What does the first column of $\boldsymbol{A}$ become as a result of applying the first Householder transformation? (c) What does the first column then become as a result of applying the second Householder transformation? (d) How many Givens rotations would be required to compute the QR factorization of $\boldsymbol{A}$?

(a) There should be 3 householder transformation in QR process.

(b) [2, 0, 0, 0]^T

(c) [2, 0, 0, 0]^T

(d) there should be 6.

Question 7

Question-7: we observed that the cross-product matrix $\boldsymbol{A}^T\boldsymbol{A}$ is exactly singular in floating-point arithmetic if

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$

where ϵ is a positive number smaller than $\sqrt{\epsilon_{\text{mach}}}$ in a given floating-point system. Show that if $\boldsymbol{A} = \boldsymbol{QR}$ is the reduced QR factorization for this matrix $\boldsymbol{A}$,

then $\boldsymbol{R}$ is not singular, even in floatingpoint arithmetic.

$$\begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} + \sqrt{1+\epsilon^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \epsilon \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ \epsilon \\ 0 \end{bmatrix} \begin{bmatrix} 2 & \epsilon & 0 \end{bmatrix} / 4 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 4 & 2\epsilon & 0 \\ 2\epsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\epsilon & \\ -\epsilon & 1 & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -\epsilon & \\ -\epsilon & 1 & \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -\epsilon \\ 0 & \epsilon \end{bmatrix}$$

$$\Rightarrow \quad \text{R matrix is} \begin{bmatrix} -1 & -1 \\ & -\epsilon \end{bmatrix} \Rightarrow \text{R is non-singular Q.E.D.}$$

Rank (R) = 2

Question 8

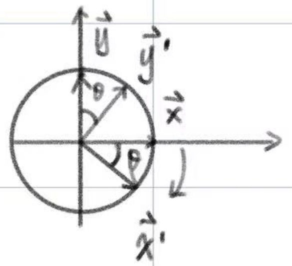
Question-8: Let $c = \cos(\theta)$ and $s = \sin(\theta)$ for some angle θ. Give a detailed geometric description of the effects on vectors in the Euclidean plane $\mathbb{R}^2$ of each the following 2×2 orthogonal matrices.

(a) $\boldsymbol{G} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ (b) $\boldsymbol{H} = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}$

linear.

We can imagine a rotation transform
Just rotate the orthogonal base vector by θ.

For. $G = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$

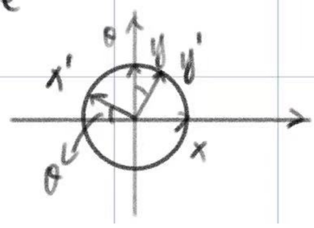


$\Leftrightarrow \begin{bmatrix} c \\ -s \end{bmatrix} x_1 + \begin{bmatrix} s \\ c \end{bmatrix} y$

clockwisely rotate the orthogonal base vector by θ.

For. H is the same

$\begin{bmatrix} -c & s \\ s & c \end{bmatrix}$



Just do a symmetric transform and rotate clockwise by θ.

Question 9

Question-9: Find a rotation matrix P with the property that PA has a zero entry in the second row and first column, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & c & -s \\ & s & c \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ c & 3c-s & -3 \\ s & 3s+c & 3c \end{bmatrix}$$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z})$

Question 10

Question-10: Given a vector $\boldsymbol{a} = [2, 3, 4]^T$

1. Specify an elementary elimination matrix that annihilates the third component of $\boldsymbol{a}$
2. Specify a Householder transformation that annihilates the third component of $\boldsymbol{a}$
3. Specify the Givens rotation that annihilates the third component of $\boldsymbol{a}$

Q_{10}: (a) $\begin{bmatrix} 1 & & \\ & 1 & \\ & -\frac{4}{3} & 1 \end{bmatrix}$

(b) For vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \sqrt{25} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$H_2 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - 2 \dfrac{\begin{bmatrix} -2 \\ 4 \end{bmatrix} \begin{bmatrix} -2 & 4 \end{bmatrix}}{20}$

$= \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} / 10 = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

$\Rightarrow H = \begin{bmatrix} 1 & & \\ & \frac{3}{5} & \frac{4}{5} \\ & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$

(c) $\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 3c-4s \\ 3s+4c \end{bmatrix} \Rightarrow 3s+4c=0$

$\Rightarrow \tan\theta = \frac{-4}{3}$

$\theta = \arctan\left(-\frac{4}{3}\right)$

$\begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix} \quad \theta = \tan^{-1}\left(-\frac{4}{3}\right)$