CS 450
Assignment 2

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1 Theoretical Problems

Each problem has 20 points.

Question-1: Find the matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ containing independent columns of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} \quad \boldsymbol{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1}$$

Question-2: In general, for any vector x, does it hold that $\|x\|_1 \geq \|x\|_2 \geq \|x\|_\infty$?

Question-3: Consider a non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, use the definition of condition number to give a step-by-step proof showing that the matrix condition number is given by the following

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\| \tag{2}$$

where $\|\boldsymbol{A}\|$ is the matrix norm. Furthermore, show that

$$\|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\| = \left(\max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \right) \cdot \left(\min_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \right)^{-1}. \tag{3}$$

In addition, prove that $\kappa(\boldsymbol{A}) \geq 1$.

Question-4: Given an $n * n$ square matrix, show that:
(a) The $L-1$ matrix norm is $\|\boldsymbol{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$
(b) The $L-\infty$ matrix norm is $\|\boldsymbol{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$
(c) $\|\boldsymbol{A}\|_1 = \left\|\boldsymbol{A}^\top\right\|_\infty$

(d) $\|\boldsymbol{A}\|_2 \leqslant \|\boldsymbol{A}\|_1 \cdot \|\boldsymbol{A}\|_\infty$

Question-5: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an invertable matrix, and $\boldsymbol{x}$, $\boldsymbol{x}+\Delta\boldsymbol{x}$ be the solutions to the following systems

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{4}$$

$$(\boldsymbol{A}+\Delta\boldsymbol{A})(\boldsymbol{x}+\Delta\boldsymbol{x}) = \boldsymbol{b}. \tag{5}$$

Consider $\boldsymbol{b} \neq \boldsymbol{0}$, show the following:
(a) The following inequality holds

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}+\Delta\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A})\frac{\|\Delta\boldsymbol{A}\|}{\|\boldsymbol{A}\|}. \tag{6}$$

(b) The following inequality holds

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(\boldsymbol{A})\frac{\|\Delta\boldsymbol{A}\|}{\|\boldsymbol{A}\|}\left(\frac{1}{1-\|\Delta\boldsymbol{A}\|\cdot\|\boldsymbol{A}^{-1}\|}\right). \tag{7}$$

Question-6: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Show that the function

$$\|\boldsymbol{x}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}} \tag{8}$$

defines a norm on $\mathbb{R}^n$ (i.e., it satisfies the three defining properties of a norm). This vector norm is said to be *induced* by $\boldsymbol{A}$.

Moreover, what if $\boldsymbol{A}$ is positive semi-definite, does it still hold?

Question-7: Show that $\boldsymbol{A}^{\mathbf{T}}\boldsymbol{A}$ has the same nullspace as $\boldsymbol{A}$. Here is one approach : First, if $\boldsymbol{A}\boldsymbol{x}$ equals zero then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ equals ____. This proves $\mathbf{N}(\boldsymbol{A}) \subset \mathbf{N}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)$. Second, if $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \|\boldsymbol{A}\boldsymbol{x}\|^2 = 0$. Deduce $\mathbf{N}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right) = \mathbf{N}(\boldsymbol{A})$.

Question-8: Given a matrix $\boldsymbol{A}$, prove the following:
(a) If $\boldsymbol{A}$ is an $n * n$ matrix, and $rank(\boldsymbol{A}) = n$, then

$$\kappa(\boldsymbol{A}) = \frac{\lambda_{\max}}{\lambda_{\min}} \tag{9}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximal and minimal eigenvalues of $\boldsymbol{A}$, respectively. Notice that both of them are positive.

(b) If $\boldsymbol{A}$ is an $m * n$ matrix, where $m \geq n$ and $rank(column(\boldsymbol{A})) = n$, then

$$\kappa(\boldsymbol{A}) = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{10}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ denote the maximal and minimal singular values of $\boldsymbol{A}$, respectively. Notice that both of them are positive.

Question-9: Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}. \tag{11}$$

(a) What is the determinant of $\boldsymbol{A}$?

(b) In floating-point arithmetic, for what range of values of ϵ will the computed value of the determinant be zero?

(c) What is the LU factorization of $\boldsymbol{A}$?

(d) In floating-point arithmetic, for what range of value of ϵ will be the computed value of U (in the LU factorization) be singular?

2 Programming Problems

(a) Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.8 & 0.9 \end{bmatrix}$$

is singular. Describe the set of solutions to the system $\boldsymbol{Ax} = \boldsymbol{b}$ if

$$\boldsymbol{b} = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.5 \end{bmatrix}$$

(b) If we were to use Gaussian elimination with partial pivoting to solve this system using exact arithmetic, at what point would the process fail? (c) Because some of the entries of $\boldsymbol{A}$ are not exactly representable in a binary floating-point system, the matrix is no longer exactly singular when entered into a computer; thus, solving the system by Gaussian elimination will not necessarily fail. Solve this system on a computer using a library routine for Gaussian elimination. Compare the computed solution with your description of the solution set in part a. If your software includes a condition estimator, what is the estimated value for cond $(\boldsymbol{A})$? How many digits of accuracy in the solution would this lead you to expect?