CS 450
Sample Problem Set

Howard Yang

Jan., 2024

0.1 Theoretical Problems

Question-1: Consider using the fixed-point iteration to find the root of the following equation

$$x^3 + x - 1 = 0. \tag{1}$$

One can construct different forms of fixed-point relationships, e.g.,

1) $x = g_1(x)$, with $g_1(x) = 1 - x^3$;

2) $x = g_2(x)$, with $g_2(x) = \sqrt[3]{1 - x}$;

3) $x = g_3(x)$, with $g_3(x) = \frac{1+2x^3}{1+3x^2}$.

Suppose the algorithm starts from the same initial point, do all the above fixed-point iterations converge? Which one converges the fastest, what are the respective convergence rates? Can you construct a fixed-point iteration that has the fastest convergence rate amongst all the possible candidates?

Question-2: Which of the following fixed-point iterations converge to $\sqrt{5}$?

1) $x = h_1(x)$, with $h_1(x) = \frac{4}{5}x + \frac{1}{x}$;

2) $x = h_2(x)$, with $g_2(x) = \frac{x}{2} + \frac{5}{2x}$;

3) $x = h_3(x)$, with $g_3(x) = \frac{x+5}{x+1}$.

Rank the ones converge from fastest and slowest.

Question-3: Consider using the Gradient Descent Method to solve the following optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \qquad f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x} \tag{2}$$

where $\boldsymbol{A}$ is a symmetric and positive definite matrix. Answer the following questions.

(a) Provide the steps for gradient descent algorithm.
(b) What would be the convergence rate?
(c) What happens if we use Newton's method to solve this problem?

Question-4: For Lagrange polynomial interpolation of m data points $(x_i, y_i), i = 1, \cdots, m$.

(a) What is the degree of each polynomial function $L_j(x)$ in the Lagrange basis?
(b) How many polynomials of degree $m+1$ interpolates these data points?
(c) How many polynomials of degree m interpolates these data points?
(d) How many polynomials of degree $m-1$ interpolates these data points?
(e) How many polynomials of degree $m-2$ interpolates these data points?