CS 450
Assignment 3

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Question-1: Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3, \\ 3\alpha x_1 - \quad x_2 &= \frac{3}{2}. \end{aligned}$$

(a) Find value(s) of α for which the system has no solutions.
(b) Find value(s) of α for which the system has an infinite number of solutions.
(c) Assuming a unique solution exists for a given α, find the solution.

Question-2: Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear systems, and compare the approximations to the actual solution.

(a)

$$\begin{aligned} 0.03x_1 + 58.9x_2 &= 59.2 \\ 5.31x_1 - 6.10x_2 &= 47.0 \end{aligned}$$

Actual solution $[10, 1]$.
(b)

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\ 6.11x_1 - 14.2x_2 + 21x_3 &= -139 \end{aligned}$$

Actual solution $\left[0, 10, \frac{1}{7}\right]$.

Question-3: Let $\boldsymbol{x}$ be the solution to the linear least squares problem $\boldsymbol{Ax} \cong \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Let $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$ be the corresponding residual vector. Which of the following three vectors is a possible value for $\boldsymbol{r}$? Why?

$$(a) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} (b) \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} (c) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Question-4: Let $\boldsymbol{a}$ be any nonzero vector. If $\boldsymbol{v} = \boldsymbol{a} - \alpha\boldsymbol{e}_1$, where $\alpha = \pm\|\boldsymbol{a}\|_2$, and

$$\boldsymbol{H} = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}$$

show that $\boldsymbol{H}\boldsymbol{a} = \alpha\boldsymbol{e}_1$.

Question-5: Determine the Householder transformation that annihilates all but the first entry of the vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. Specifically, if

$$\left(\boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

what are the values of α and $\boldsymbol{v}$?

Question-6: Suppose that you are computing the QR factorization of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

by Householder transformations. (a) How many Householder transformations are required? (b) What does the first column of $\boldsymbol{A}$ become as a result of applying the first Householder transformation? (c) What does the first column then become as a result of applying the second Householder transformation? (d) How many Givens rotations would be required to compute the QR factorization of $\boldsymbol{A}$?

Question-7: we observed that the cross-product matrix $\boldsymbol{A}^T\boldsymbol{A}$ is exactly singular in floating-point arithmetic if

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$

where ϵ is a positive number smaller than $\sqrt{\epsilon_{\text{mach}}}$ in a given floating-point system. Show that if $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ is the reduced QR factorization for this matrix $\boldsymbol{A}$, then $\boldsymbol{R}$ is not singular, even in floatingpoint arithmetic.

Question-8: Let $c = \cos(\theta)$ and $s = \sin(\theta)$ for some angle θ. Give a detailed geometric description of the effects on vectors in the Euclidean plane $\mathbb{R}^2$ of each the following 2×2 orthogonal matrices.

(a) $\boldsymbol{G} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ (b) $\boldsymbol{H} = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}$

Question-9: Find a rotation matrix P with the property that PA has a zero entry in the second row and first column, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Question-10: Given a vector $\boldsymbol{a} = [2, 3, 4]^T$

1. Specify an elementary elimination matrix that annihilates the third component of $\boldsymbol{a}$
2. Specify a Householder transformation that annihilates the third component of $\boldsymbol{a}$
3. Specify the Givens rotation that annihilates the third component of $\boldsymbol{a}$