ECE-210 Analog Signal Processing Spring 2022
Homework #9: Submission Deadline 27th April (10:00 PM)

1. Obtain the average power of the following signals:
   (a) $f(t) = 3 + 3e^{j4t} + 3e^{-j4t}$
   (b) $f(t) = 3 + 4\cos(2t) + 5\sin(5t)$

2. Consider the periodic function $f(t) = \begin{cases} \sin(\pi t), & \text{for } t \in [0, 2) \\ 0, & \text{for } t \in [2, 4) \end{cases}$, where the signal period is $T = 4\,\text{s}$. Its corresponding Fourier series in exponential form is given by:

$$f(t) = \frac{-j}{4}e^{j\pi t} + \frac{j}{4}e^{-j\pi t} + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{\pi(4-n^2)}e^{jn\frac{\pi}{2}t},$$

and in compact form:

$$f(t) = \frac{1}{2}\cos(\pi t - \frac{\pi}{2}) + \frac{4}{3\pi}\cos(\frac{\pi}{2}t) + \sum_{n=3, n \text{ odd}}^{\infty} \frac{4}{\pi(n^2-4)}\cos(\frac{n\pi}{2}t + \pi)$$

Let $f(t)$ be the input to an LTI system with frequency response $H(\omega) = \frac{\pi}{2}\left(1 + \frac{\pi^2}{4\omega^2}\right)e^{-j\omega}$ for $\omega \in [-7\pi/4, 7\pi/4]$ rad/s.

   (a) Obtain the corresponding steady state response $y_{ss}(t)$.
   (b) Is $y_{ss}(t)$ periodic? If so, obtain its fundamental frequency.
   (c) Calculate the ratio between the input signal power and the output signal power.

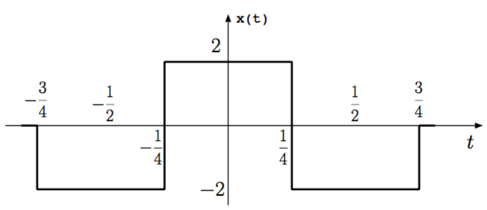
3. The input-output relation for a system with input $f(t)$ is given by

$$y(t) = 6f(t) + f^2(t) - f^3(t).$$

Obtain the total harmonic distortion (THD) of the system response to a pure cosine input of the form $f(t) = 2\cos(\omega_o t)$, where $\omega_o > 0$ and is a real positive constant. The following trigonometric identities can be useful: $\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)]$ and $\cos^3(\theta) = \frac{1}{4}[3\cos(\theta) + \cos(3\theta)]$.

4. Obtain the Fourier transform of
   (a) $f_1(t) = e^{-t}u(t-2)$.
   (b) $f_2(t) = \frac{1}{1+j(t-1)} + \frac{1}{1+j(t+1)}$.

5. Let $f(t) = \text{rect}(\frac{t}{2})$ and let $g(t) = f\left(t - \frac{1}{2}\right)$
   (a) Obtain $F(\omega)$ and plot $|F(\omega)|$ and $\angle F(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle F(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.
   (b) Obtain $G(\omega)$ and plot $|G(\omega)|$ and $\angle G(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle G(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.
   (c) Obtain the Fourier transform, $X(\omega)$, of the signal shown below



6. Let $f(t) = \frac{1}{2\pi}\text{sinc}\left(-\frac{t}{4}\right)\left[1 - 2\cos\left(-\frac{t}{2}\right)\right]$, with Fourier transform $F(\omega) = 2\text{rect}(2\omega) - 2\text{rect}(2\omega+1) - 2\text{rect}(2\omega-1)$.
   Let $G(\omega) = \text{sinc}\left(\frac{\omega}{4}\right)\left[1 - 2\cos\left(\frac{\omega}{2}\right)\right]$.

   (a) Obtain the inverse Fourier transform of $G(\omega)$, that is, obtain $g(t)$.

   (b) Plot $F(\omega)$ and $g(t)$.

7. Given that $f(t) = 5\triangle^2(\frac{t}{4})$, evaluate the Fourier transform $F(\omega)$ at $\omega = 0$.

* Average power & Parseval's Theorem

1. Obtain the average power of the following signals:

(a) $f(t) = 3 + 3e^{j4t} + 3e^{-j4t}$

(b) $f(t) = 3 + 4\cos(2t) + 5\sin(5t)$

$P = \frac{1}{T}\int_T |f(t)|^2 dt = \frac{1}{T}\int_T f(t)\cdot f^*(t)\,dt$

$\because f^*(t) = \sum_{-\infty}^{\infty} F_n^* e^{-jn\omega t}$

$P = \frac{1}{T}\int_T f(t)\cdot \sum_{-\infty}^{\infty} F_n^* e^{-jn\omega t}\,dt$

$= \frac{1}{T}\int_T \sum_{-\infty}^{\infty} F_n e^{jn\omega t} \cdot \sum_{-\infty}^{\infty} F_n^* e^{-jn\omega t}\,dt$

$\because \int_T e^{jn\omega t}\cdot e^{-jm\omega t}\,dt = 0$ (正交性)

$\therefore P = \sum_{-\infty}^{\infty} |F_n|^2 \Rightarrow P = \frac{C_0^2}{4} + \sum_{n=1}^{\infty}\frac{1}{2}C_n^2$

(a) $f(t) = 3 + 3e^{j4t} + 3e^{-j4t}$

$\Rightarrow \boxed{P = 3^2 + 2\times 3^2 = 27W}$

(b) $f(t) = 3 + 4\cos(2t) + 5\sin(5t)$

$\Rightarrow f(t) = 3 + 4\,\frac{e^{2jt}+e^{-2jt}}{2} + 5\,\frac{e^{5jt}-e^{-5jt}}{2}$

$\Rightarrow P = 3^2 + 2^2 + 2^2 + (\frac{5}{2})^2\times 2$

$\boxed{= 9 + 8 + 12.5 = 29.5W}$

$\Rightarrow$ Parseval Theorem $P = \frac{1}{T}\int |f(t)|^2 dt = \sum_{-\infty}^{\infty}|F_n|^2 = \frac{C_0^2}{4} + \sum_{n=1}^{\infty}\frac{1}{2}C_n^2$

2. Consider the periodic function $f(t) = \begin{cases} \sin(\pi t), & \text{for } t \in [0,2) \\ 0, & \text{for } t \in [2,4) \end{cases}$, where the signal period is $T = 4$ s. Its corresponding Fourier series in exponential form is given by:

$$f(t) = \frac{-j}{4}e^{j\pi t} + \frac{j}{4}e^{-j\pi t} + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{\pi(4-n^2)} e^{jn\frac{\pi}{2}t},$$

and in compact form:

$$f(t) = \frac{1}{2}\cos(\pi t - \frac{\pi}{2}) + \frac{4}{3\pi}\cos(\frac{\pi}{2}t) + \sum_{n=3, n \text{ odd}}^{\infty} \frac{4}{\pi(n^2-4)}\cos(\frac{n\pi}{2}t + \pi)$$

Let $f(t)$ be the input to an LTI system with frequency response $H(\omega) = \frac{\pi}{2}\left(1 + \frac{\pi^2}{4\omega^2}\right)e^{-j\omega}$ for $\omega \in [-7\pi/4, 7\pi/4]$ rad/s.

(a) Obtain the corresponding steady state response $y_{ss}(t)$.

(b) Is $y_{ss}(t)$ periodic? If so, obtain its fundamental frequency.

(c) Calculate the ratio between the input signal power and the output signal power.

$y(t) \Leftrightarrow Y(\omega) = H(\omega)F(\omega)$

Just steady state response.

(a)

$f(t) = \frac{-j}{4}e^{j\pi t} + \frac{j}{4}e^{-j\pi t} + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{\pi(4-n^2)} e^{jn\frac{\pi}{2}t}$, $H(\omega) = \frac{\pi}{2}\left(1 + \frac{\pi^2}{4\omega^2}\right)e^{-j\omega}$ for $\omega \in [-7\pi/4, 7\pi/4]$ rad/s.

For term $\frac{-j}{4}e^{j\pi t}$ & $\frac{j}{4}e^{-j\pi t}$

$\Rightarrow F_1 = \frac{-j}{4}, \omega = \pi \Rightarrow H(\pi) = \frac{\pi}{2}(1 + \frac{\pi^2}{4\pi^2})e^{-j\pi} = -\frac{5}{8}\pi \Rightarrow f_1(t) = \frac{5\pi j}{32}e^{j\pi t}$

$F_2 = \frac{j}{4}, \omega = -\pi \quad H(-\pi) = \frac{\pi}{2}(1 + \frac{\pi^2}{4\pi^2})e^{j\pi} = -\frac{5}{8}\pi \quad f_2(t) = \frac{-5\pi j}{32}e^{-j\pi t}$

As $\pi, -\pi \in [-\frac{7}{4}\pi, \frac{7}{4}\pi]$

For $\sum_{-\infty}^{\infty}\frac{2}{\pi[4-(2n-1)^2]}e^{j(2n-1)\cdot\frac{\pi}{2}t}$ as $\omega \in [-\frac{7}{4}\pi, \frac{7}{4}\pi]$

and $w = (2n-1)\frac{\pi}{2}$

$\Rightarrow w = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \frac{2}{3\pi} e^{j\frac{\pi}{2}t} \quad \frac{2}{3\pi} e^{-j\frac{\pi}{2}t} \quad \frac{2}{-5\pi} e^{j\frac{3}{2}\pi t} \quad \frac{2}{-5\pi} e^{-j\frac{3}{2}\pi t}$

$$f(t) = \frac{-j}{4} e^{j\pi t} + \frac{j}{4} e^{-j\pi t} + \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{\pi(4-n^2)} e^{jn\frac{\pi}{2}t},$$

$\because H(w) = \frac{\pi}{2}(1 + \frac{\pi^2}{4w^2}) e^{-jw}$

$\therefore H(\frac{-3\pi}{2}) = \frac{\pi}{2}(1 + \frac{1}{9}) e^{\frac{3}{2}\pi j} = \frac{5\pi}{9} e^{\frac{3}{2}\pi j} \Rightarrow H(\frac{3\pi}{2}) = \frac{5\pi}{9} e^{-\frac{3}{2}\pi j}$

$H(\frac{-\pi}{2}) = \frac{\pi}{2} \cdot 2 e^{\frac{\pi}{2}j} = \pi e^{\frac{\pi}{2}j} \Rightarrow H(\frac{\pi}{2}) = \pi e^{-\frac{\pi}{2}j}$

$\Rightarrow f_3(t) = H(\frac{-3\pi}{2}) \cdot \frac{2}{-5\pi} e^{j\frac{3}{2}\pi t} = \frac{5\pi}{9} e^{\frac{3}{2}\pi j} \cdot \frac{2}{-5\pi} e^{j\frac{3}{2}\pi t} = \frac{2}{9} e^{j\frac{3}{2}\pi(1+t)}$

$f_4(t) = H(\frac{3\pi}{2}) \cdot \frac{2}{-5\pi} e^{-j\frac{3}{2}\pi t} = \frac{2}{9} e^{-j\frac{3}{2}\pi(t+1)}$

$f_5(t) = H(-\frac{\pi}{2}) \cdot \frac{2}{3\pi} e^{j\frac{\pi}{2}t} = \pi \cdot e^{\frac{\pi}{2}j} \cdot \frac{2}{3\pi} \cdot e^{j\frac{\pi}{2}t} = \frac{2}{3} e^{\frac{\pi}{2}j(t+1)}$

$f_6(t) = H(\frac{\pi}{2}) \cdot \frac{2}{3\pi} e^{-j\frac{\pi}{2}t} = \frac{2}{3} e^{-\frac{\pi}{2}j(t+1)}$

$\Rightarrow f(t) = \frac{5\pi j}{32} e^{j\pi t} + \frac{-5\pi j}{32} e^{-j\pi t} + \frac{2}{9} e^{j\frac{3}{2}\pi(1+t)} + \frac{2}{9} e^{-j\frac{3}{2}\pi(1+t)} + \frac{2}{3} e^{\frac{\pi}{2}j(t+1)} + \frac{2}{3} e^{-\frac{\pi}{2}j(t+1)}$

(b) periodic,

$W_1 = \pi, W_2 = \frac{3}{2}\pi, W_3 = \frac{\pi}{2} \Rightarrow T = T_{max} = 4$

$T_1 = 2 \quad T_2 = \frac{4}{3} \quad T_3 = 4 \quad \Rightarrow f = \frac{1}{4}$

(c) $P_{signal} = \frac{1}{T}\int_T |f(t)|^2 dt = \frac{1}{4}\int_0^2 \sin^2(\pi t)\, dt = \frac{1}{4}\int_0^2 \frac{1-\cos 2\pi t}{2} dt = \frac{1}{16\pi}\int_0^{4\pi} (1-\cos 2\pi t)\, d2\pi t$

$= \frac{1}{16\pi}(4\pi + \sin(2\pi t)|_0^{4\pi})$

$= \frac{1}{4}$

$P_{output} = (\frac{5\pi}{32})^2 \times 2 + (\frac{2}{9})^2 \times 2 + (\frac{2}{3})^2 \times 2$

$= \frac{25}{512}\pi^2 + \frac{8}{81} + \frac{8}{9} = 1.47$

$\Rightarrow P_{input} / P_{output} = 0.1701$

3. The input-output relation for a system with input $f(t)$ is given by

$$y(t) = 6f(t) + f^2(t) - f^3(t).$$

Obtain the total harmonic distortion (THD) of the system response to a pure cosine input of the form $f(t) = 2\cos(\omega_o t)$, where $\omega_o > 0$ and is a real positive constant. The following trigonometric identities can be useful: $\cos^2(\theta) = \frac{1}{2}[1+\cos(2\theta)]$ and $\cos^3(\theta) = \frac{1}{4}[3\cos(\theta) + \cos(3\theta)]$.

Harmonic distortion

S.H.D:

$y(t) = A\cos(w_0 t) + B\cos^2(w_0 t)$

$y(t) = \frac{B}{2} + A\cos(w_0 t) + \frac{B}{2}\cos(2w_0 t)$

$P = F_{rms}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

DC

AC Power in n-th harmonic

$P_y = \frac{B^2}{4} + \frac{A^2}{2} + \frac{B^2}{8}$

$\Rightarrow S.H.D = \frac{B^2}{4A^2}$

Simple Second.

$B \ll A \Rightarrow$ weakly linear.

Total harmonic distorsion.

T.H.D: $= \frac{\sum_{n=2}^{\infty}\frac{1}{2}C_n^2}{\frac{1}{2}C_1^2}$ (只计算一次)

*For: $y(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n\cos(n w_0 t + \theta_n)$

T.H.D $= \frac{\sum_{n=2}^{\infty} C_n^2}{C_1^2} \Rightarrow$ as $f(t) = 2\cos(w_0 t)$

$\Rightarrow y(t) = 12\cos(w_0 t) + 2(1+\cos(2w_0 t)) - 8 \times \frac{1}{4}[3\cos(w_0 t) + \cos(3w_0 t)]$

$= 6\cos(w_0 t) + 2 + 2\cos(2w_0 t) - 2\cos(3w_0 t)$

$\Rightarrow T.H.D = \frac{2^2+2^2}{6^2} = \frac{8}{36} = \frac{2}{9}$

4. Obtain the Fourier transform of

(a) $f_1(t) = e^{-t}u(t-2)$.

(b) $f_2(t) = \frac{1}{1+j(t-1)} + \frac{1}{1+j(t+1)}$.

Fourier Transformation:

$F(w) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\cdot e^{-jwt}\,dw$

Inverse Fourier Transformation:

$f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\cdot e^{jwt}\,dw$

(a) $\mathcal{F}(f_1(t)) = \int_{-\infty}^{\infty} e^{-t}u(t-2)\,e^{-jwt}\,dt$

$u(t-2) = \begin{cases} t>2 & u(t-2)=1 \\ t<2 & u(t-2)=0 \end{cases}$

$\int_{-\infty}^{\infty} e^{-t}u(t-2)\,e^{-jwt}\,dt$

$= \int_{2}^{\infty} e^{-t(1+jw)}\,dt = \frac{e^{-2(1+jw)}}{1+jw}$

time shift $f(t-t_0) = F(w)\cdot e^{-jwt_0}$

$f(t) = \frac{1}{e^2}(e^{-(t-2)}u(t-2)) \Rightarrow$

$\Rightarrow \mathcal{F}(f(t)) = \frac{1}{e^2(1+jw)}\cdot e^{-jw\cdot 2}$

$\Rightarrow \mathcal{F}(f(t)) = \frac{e^{-2-jw}}{1+jw}$

(b) $\mathcal{F}(f_2(t)) = \mathcal{F}(\frac{1}{1+j(t-1)}) + \mathcal{F}(\frac{1}{1+j(t+1)})$

symmetry: $F(t) = 2\pi f(-w)$

time shift: $f(t-t_0) = F(w)e^{-jwt_0}$

$= \mathcal{F}(\frac{1}{1+jt})\cdot e^{-jw} + \mathcal{F}(\frac{1}{1+jt})e^{jw}$

$= 2\pi\cdot e^{w}u(-w)e^{-jw} + 2\pi e^{w}u(-w)e^{-jw}$

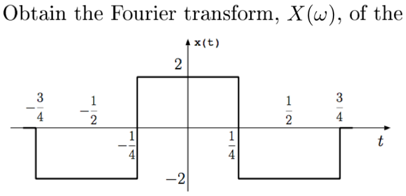
$= 2\pi u(-w)e^{w}(e^{jw} + e^{-jw})$

5. Let $f(t) = \text{rect}(\frac{t}{2})$ and let $g(t) = f\left(t - \frac{1}{2}\right)$

(a) Obtain $F(\omega)$ and plot $|F(\omega)|$ and $\angle F(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle F(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.

(b) Obtain $G(\omega)$ and plot $|G(\omega)|$ and $\angle G(\omega)$ in the frequency range $-3\pi < \omega < 3\pi$ rad/s. Remember to keep the phase $\angle G(\omega)$ in the interval $(-\pi, \pi]$ rad on the vertical axis.

(c) Obtain the Fourier transform, $X(\omega)$, of the signal shown below

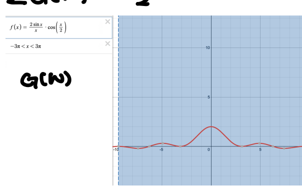


(b) time shift: $\mathcal{F}(f(t-t_0)) = F(w)\,e^{-jwt_0}$

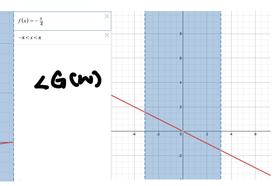
$\Rightarrow G(w) = 2\,\text{sinc}(w)\cdot e^{-jw\frac{1}{2}}$

$\angle G(w) = -\frac{w}{2}$

G(w)



∠G(w)



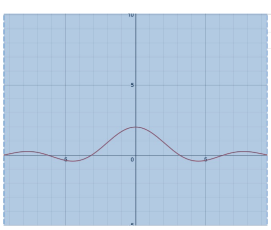
(a)

$\text{rect}(\frac{t}{\tau}) \Leftrightarrow \tau\,\text{sinc}(\frac{w\tau}{2})$

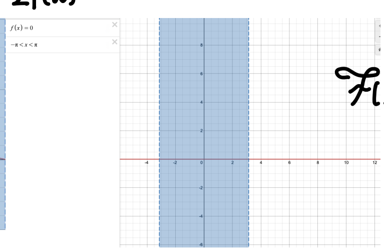
$F(w) = \mathcal{F}(\text{rect}(\frac{t}{2})) = 2\,\text{sinc}(\frac{w\cdot 2}{2}) = 2\,\text{sinc}(w)$

$\angle F(w) = 0$

F(w)



∠F(w)



(c) $\mathcal{F}(x(t)) = \int_{-\infty}^{\infty} f(t)\cdot e^{-jwt}\,dt$

$\mathcal{F}(x(t)) = \int_{-\frac{3}{4}}^{-\frac{1}{4}} -2e^{-jwt}\,dt + \int_{-\frac{1}{4}}^{\frac{1}{4}} 2e^{-jwt}\,dt + \int_{\frac{1}{4}}^{\frac{3}{4}} -2e^{-jwt}\,dt$

$= 2\left(\int_{-\frac{3}{4}}^{-\frac{1}{4}} -e^{-jwt}\,dt + \int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-jwt}\,dt - \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-jwt}\,dt\right)$

$= 2\left(\frac{1}{jwt}e^{-jwt}\Big|_{-\frac{3}{4}}^{-\frac{1}{4}} - \frac{1}{jwt}e^{-jwt}\Big|_{-\frac{1}{4}}^{\frac{1}{4}} + \frac{1}{jwt}e^{-jwt}\Big|_{\frac{1}{4}}^{\frac{3}{4}}\right)$

$= \frac{2}{jw}\left(e^{\frac{jw}{4}} - e^{\frac{3jw}{4}} - (e^{\frac{-jw}{4}} - e^{\frac{jw}{4}}) + e^{\frac{-3jw}{4}} - e^{\frac{-jw}{4}}\right)$

$= \frac{2}{jw}\left(2e^{jw/4} - e^{3jw/4} - 2e^{-jw/4} + e^{-3jw/4}\right)$

6. Let $f(t) = \frac{1}{2\pi}\text{sinc}\left(-\frac{t}{4}\right)\left[1 - 2\cos\left(-\frac{t}{2}\right)\right]$, with Fourier transform $F(\omega) = 2\text{rect}(2\omega) - 2\text{rect}(2\omega+1) - 2\text{rect}(2\omega-1)$. Let $G(\omega) = \text{sinc}\left(\frac{\omega}{4}\right)\left[1 - 2\cos\left(\frac{\omega}{2}\right)\right]$.

(a) Obtain the inverse Fourier transform of $G(\omega)$, that is, obtain $g(t)$.

(b) Plot $F(\omega)$ and $g(t)$.

$F(t) \Leftrightarrow 2\pi f(-w)$

(a) $f(t) \Leftrightarrow F(t)$

$\frac{1}{2\pi}\text{sinc}(-\frac{t}{4})[1-\cos(-\frac{t}{2})] \Leftrightarrow 2\text{rect}(2w) - 2\text{rect}(2w+1) - 2\text{rect}(2w-1)$

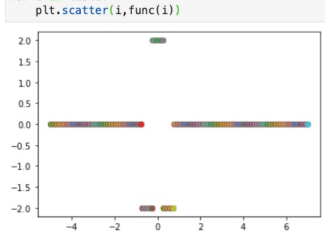
$\because G(w) = 2\pi \cdot f(-w)$

$\therefore 2\pi f(-w) \Leftrightarrow g(t) = F(t)$

$\Rightarrow \mathcal{F}^{-1}(G(w)) = F(t) = 2\text{rect}(2t) - 2\text{rect}(2t+1) - 2\text{rect}(2t-1)$

(b):

```
]: def rect(a,t=1):
       if ((a/t) > 1/2) or ((a/t) < -1/2):
           return 0
       else:
           return 1
   def func(omega):
       return 2*rect(2*omega)-2*rect(2*omega+1)-2*rect(2*omega-1)
   list1 = np.linspace(-5,7,1000)
   for i in list1:
       plt.scatter(i,func(i))
```

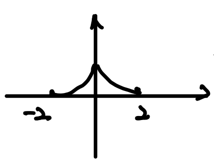


the two figure are the same.

7. Given that $f(t) = 5\triangle^2(\frac{t}{4})$, evaluate the Fourier transform $F(\omega)$ at $\omega = 0$.

$\Delta^2(\frac{t}{4}) = (1-2|\frac{t}{4}|)^2 = 1^2 + \frac{t^2}{4} - |t|$

$|\frac{t}{4}| \le \frac{1}{2} \Rightarrow t \in [-2, 2]$



$\Rightarrow \mathcal{F}(f(t)) = F(w) = \int_{-2}^{0} (\frac{t^2}{4}+t+1) \cdot e^{-jwt} dt + \int_{0}^{2} (\frac{t^2}{4}-t+1) e^{-jwt} dt$

$= \frac{4jw + te^{-2jw} - e^{2jw}}{2jw^3}$

$F(0) = (\int_{-2}^{0} (\frac{t^2}{4}+t+1) dt + \int_{0}^{2} \frac{t^2}{4} - t + 1 dt) \times 5$

$= \frac{20}{3}$