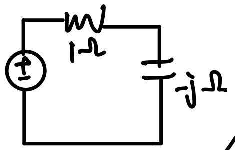
Lect 12. Periodic signals.

Example: low pass.



$H(w) = \frac{1}{wj+1}$

$f(t) = \cos(0.5t) + \cos(\pi t)$ V

$e^{j\theta} \cdot e^{j0.5t}$

$y(t) = |H(0.5)| \cos(0.5t + \angle H(0.5)) + |H(\pi)| \cos(\pi t + \angle H(\pi))$

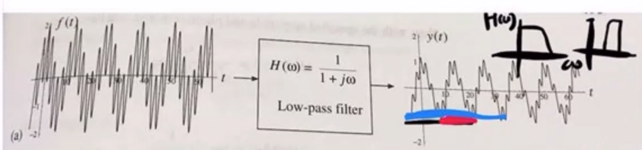
complex number

$|H(w)| = \frac{1}{\sqrt{1^2 + 0.5^2}} = \left| \frac{1}{1+0.5j} \right| \quad \angle H(0.5) = -\tan^{-1} \frac{0.5}{1}$

结论

$|H(\pi)| = \frac{1}{\sqrt{1+\pi^2}} \quad \angle H(\pi) \approx -72.34°$

*
$$\boxed{H(w) = \frac{1}{wj+1} \quad \text{low pass filter.}}$$



Periodic: signal $f(t)$ if $f(t-t_0) = f(t)$

$f(t-kt_0) = f(t)$ — integer number

$f(t-t_0-t_0) = f(t-t_0)$

$f(t-2t_0) = f(t-t_0)$

- periodic signal. delayed by integer multiples of some time interval t_0. they are indistinguishable from undelay version.
- periodic signals consist of replicas, repeated in time.
- Smallest non-zero value of t_0 that satisfies the condition $f(t-t_0) = f(t)$ said to be the period of $f(t)$ (T)
- $\sin\omega t$, $\cos\omega t$ and $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$ periodic signals $T = \frac{2\pi}{w}$

$e^{jnw_0 t} = \cos(nw_0)t + j\sin(nw_0 t)$

$T = \frac{2\pi}{n_0 w}$

a weighted linear sum of signal $e^{jn\omega_0 t}$

$$f(t)=\sum_{n=-\infty}^{n=\infty} F_n e^{jn\omega_0 t}$$

F_n ↳ Constant coeffecients.

· Periodic T

· fourier series $f(t-t_0)$

$$\rightarrow F_n=\frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt$$

↳ come back

Trigonometric form:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos n\omega_0 t+b_n\sin n\omega_0 t$$

$$a_n=F_n+F_{-n}$$
$$b_n=j(F_n-F_{-n})$$

verify Trigo form

$$f(t)=\sum_{-\infty}^{\infty}F_n\cdot e^{jn\omega_0 t}=F_0+\sum_{m=1}^{\infty}(F_m e^{jm\omega t}+F_{-m}e^{-jm\omega t})$$

化为三角即得证.

verify the trig form.

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}=F_0+\sum_{m=1}^{\infty}(F_m e^{jm\omega_0 t}+F_{-m}e^{-jm\omega_0 t})$$

$$=F_0+\sum_{m=1}^{\infty}\left[F_m\cos m\omega_0 t+jF_m\sin m\omega_0 t+F_{-m}\cos m\omega_0 t-jF_{-m}\sin m\omega_0 t\right]$$

$$=F_0+\sum_{m=1}^{\infty}\left[\underbrace{(F_m+F_{-m})}_{a_n}\cos m\omega_0 t+\underbrace{j(F_m-F_{-m})}_{b_n}\sin m\omega_0 t\right]$$

$F_0 \rightarrow a_0$

结论.

$\frac{a_0}{2}$

$\ast$ $f(t)\rightarrow$ real value function $F_{-n}=F_n^{*}\leftarrow$ Conjugate Symmetric

proof

$F_n+F_{-n}=F_n+F_n^{*}$ real

$F_n-F_{-n}=F_n-F_n^{*}=+2j(\cdot)$

$=\pm 2j\,\mathrm{img}(F_n)$

$\frac{a_0}{2}$

$\ast$ $f(t)\rightarrow$ real value function $F_{-n}=F_n^{*}\leftarrow$ Conjugate Symmetric compact form of Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = F_0 + \sum_{m=1}^{\infty} |F_m| \left[e^{j(m\omega_0 t + \angle F_m)} + e^{-j(m\omega_0 t + \angle F_m)} \right]$$

$$= F_0 + \sum_{n=1}^{\infty} 2|F_m| \cos(m\omega_0 t + \angle F_m)$$

- Fourier Series $f(t)$ are sum of an infinite number of periodic signals
- longest period, corresponding to the lowest freq. $\omega = n\omega_0$, sets the period of series, the interval over which every signal repeats.

Part. 2.

- all periodic signals in the lab can be expressed as Fourier Series.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_T f(t) e^{-jn\omega_0 t} dt$$ $< \infty \Rightarrow f_n$ bounded.
$\Rightarrow$ convergence of Fourier Series.

$\int_T \rightarrow$ integration over one period.

C'

By analogy, a convergent Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

相当于空间的基

as infinite weighted sum of orthogonal basis functions.

$\int e^{jnw_0 t} e^{jmw_0 t} dt = 0$ $\quad e^{jnw_0 t}$ $\quad -\infty < n < \infty$

F_n of $f(t)$ is the projection of $f(t)$ along the basis function $e^{jnw_0 t}$

$$F_n = \int_T f(t) \cdot e^{-jnw_0 t} dt \quad f(t)$$

$$\int_T f(t) e^{-jnw_0 t} dt$$

$$\int_T \sum_{n=-\infty}^{\infty} F_n e^{jnw_0 t} \cdot e^{-jmw_0 t} dt \quad m \neq n$$

$$= \frac{1}{T} F_{m=n} \int_T e^{jnw_0 t} \cdot e^{-jnw_0 t} dt$$

$$e^0 \quad 1 \quad dt \quad T$$

$$= \frac{1}{T} F_n \cdot T$$

$$= F_n$$

- all sum of co-sinusoidal sinusodal signal periodic $\quad n \geq 0 \quad 0 \ \infty$

$p(t) = 2\cos(\pi t) + 4\cos(2t)$ periodic ?

not periodic, ratios of the two freq $\frac{\pi}{2}$ not a rational number.

$$g(t) = \sum_{k=1}^{\infty} C_k \cos(w_k t + \theta_k)$$

$f(t) \to F_n \qquad g(t) \to G_n$

$f(t) \to F_n \qquad g(t) \to G_n$

1. Scaling constant k $\quad kf(t) \to kF_n$

2. addition $\quad f(t) \leftrightarrow F_n \quad f(t)+g(t) \to F_n + G_n$
   $g(t) \leftrightarrow G_n$

3. Time shift $\quad f(t-t_0) \to \underline{f(t-t_0) \leftrightarrow F_n e^{-jn\omega_0 t_0}}$

4. Derivative $\quad \frac{df(t)}{dt} \to \frac{df(t)}{dt} \leftrightarrow jn\omega_0 F_n$

5. Hermitian $\quad$ Real $f(t)$ $\quad F_{-n} = F_n^*$ $\quad$ Conjugate Symmetric

6. even function $\quad f(-t) = f(t) \quad f(t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t)$

7. odd function $\quad f(-t) = -f(t) \quad f(t) = \sum_{n=0}^{\infty} a_n \sin(n\omega_0 t)$