ECE 486 Control Systems

Lecture 04: Dynamic Response with Arbitrary IC

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Recap: Lecture 03

- The **dynamic response**, describing the behavior of a system overtime, consists of **transient and steady-state** response
- The **transfer function**, defined as the ratio of the Laplace transforms of the output and input, assuming zero ICs, maps the input to the output in the frequency domain

Dynamic Response (Review)

- System response describes the behavior of a dynamic system
- Free and Forced Response
  - Total Response $x(t)=x_h\,(t)+\,x_p(t)$
  - Free Response: x_h is the solution
  - Forced Response: x_p is determined by the forcing function f
- Transient and Steady-State
  - Total Response $x(t)=x_{tr}\,(t)+\,x_{ss}(t)$
  - $x_{\mathrm{tr},}$ Transient State: component that decays towards zero
  - x_{ss}, Steady State: component that remains after the x_t decays towards 0

Recap: Dynamic System (Overview)

- consists of <u>components (or subsystems)</u> with (<u>inputs &) outputs</u> of <u>time related function</u>
- a system that evolves over time considering causality (for this course)

s-domain $H(s) = \frac{O(s)}{I(s)}$

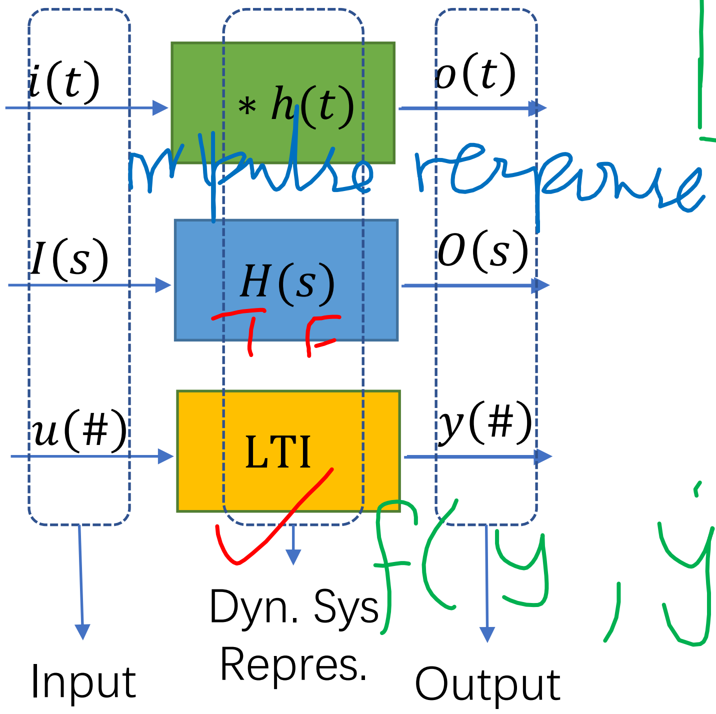
t-domain

$$\boxed{I(s)\,H(s) = O(s)}$$

$$o(t) = [i * h](t)$$

$$\boxed{H(s) = O/I}$$

$$\boxed{* \, h(t)}$$



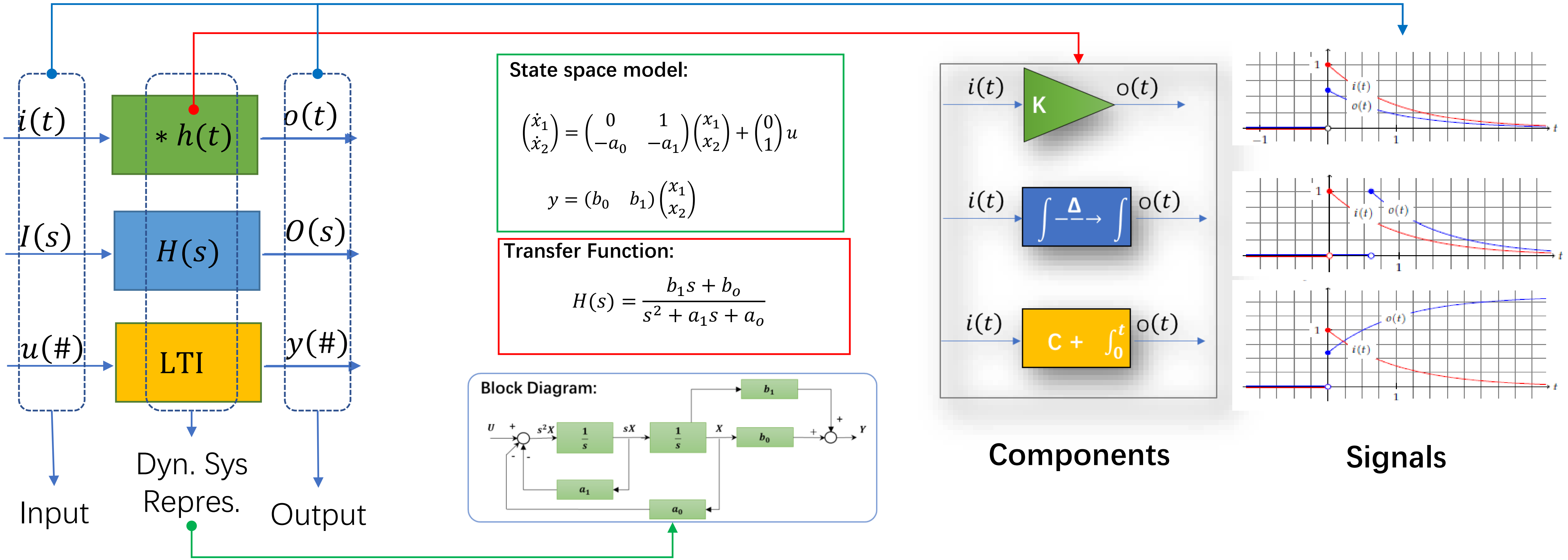
$f(y, \dot{y}, \ddot{y})$

$$\boxed{m\ddot{y} + b\dot{y} + ky = f}$$

governing eq^n

Recap: Dynamic System (Overview)

- consists of components (or subsystems) with (inputs &) outputs of time related function
- a system that evolves over time considering causality (for this course)



Transfer Functions (Review)

- A single-input-single-output (SISO) system with amplifiers, zero-initial-value integrators, splitting and summing junctions can be represented with a multiplication by a **transfer function** $H(s)$
- Such a dynamic system is called a **convolution**

Frequency

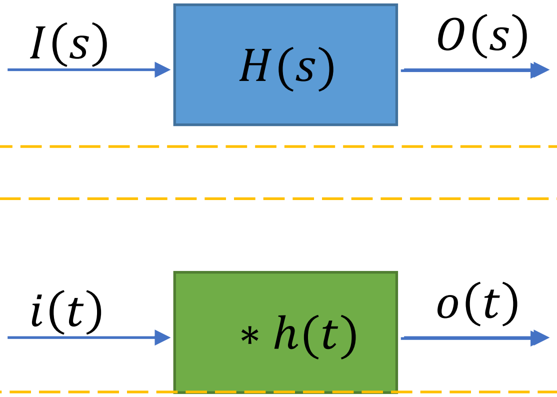
$$\mathcal{L}(o(t)) = \mathcal{L}(i(t)) \cdot H(s)$$

$$O(s) = I(s) \cdot H(s)$$

Equivalent

Time

$$o(t) = i(t) * h(t)$$



where $H(s) = \mathcal{L}\big(h(t)\big)$

Lecture Overview

- Continue to look at the methodology of characterizing the output of a given system with a given input

$$u \longrightarrow \boxed{\text{Dynamics}} \longrightarrow y$$

Dynamic Response (Recap)

$$\mathbf{u(t)} \longrightarrow \boxed{\;* \, h\;} \longrightarrow \mathbf{y(t)}$$

Dynamic Response (Recap)

$$\mathbf{u(t)} \longrightarrow \boxed{* \, h} \longrightarrow \mathbf{y(t)}$$

Response

- Transient — *Diminishes over time*
- Steady State — *Dominates over time*

Dynamic Response (Recap)

$$u(t) \longrightarrow \boxed{\ast\, h} \longrightarrow y(t)$$

We are interested in computing the response y of a given input u under a given set of ICs

The total response consists of:

- Transient response
  - dependent on the IC
- Steady-state response
  - dominating factor when the effect of IC fade away

Reminder: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}\mathrm{d}\tau, \qquad s \in \mathbb{C}$$

time domain	frequency domain
$u(t)$	$U(s)$
$h(t)$	$H(s)$
$y(t)$	$Y(s)$

convolution in time domain $\longleftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \longleftrightarrow \quad Y(s) = H(s)U(s)$$

The Laplace transform of the impulse response

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau,$$

is called the transfer function of the system.

$$H = \frac{Y}{u}$$

Take an Example (1st order ODE)

Conservation of Mass

$$\dot m_{tank} = \dot m_{in} - \dot m_{out}$$

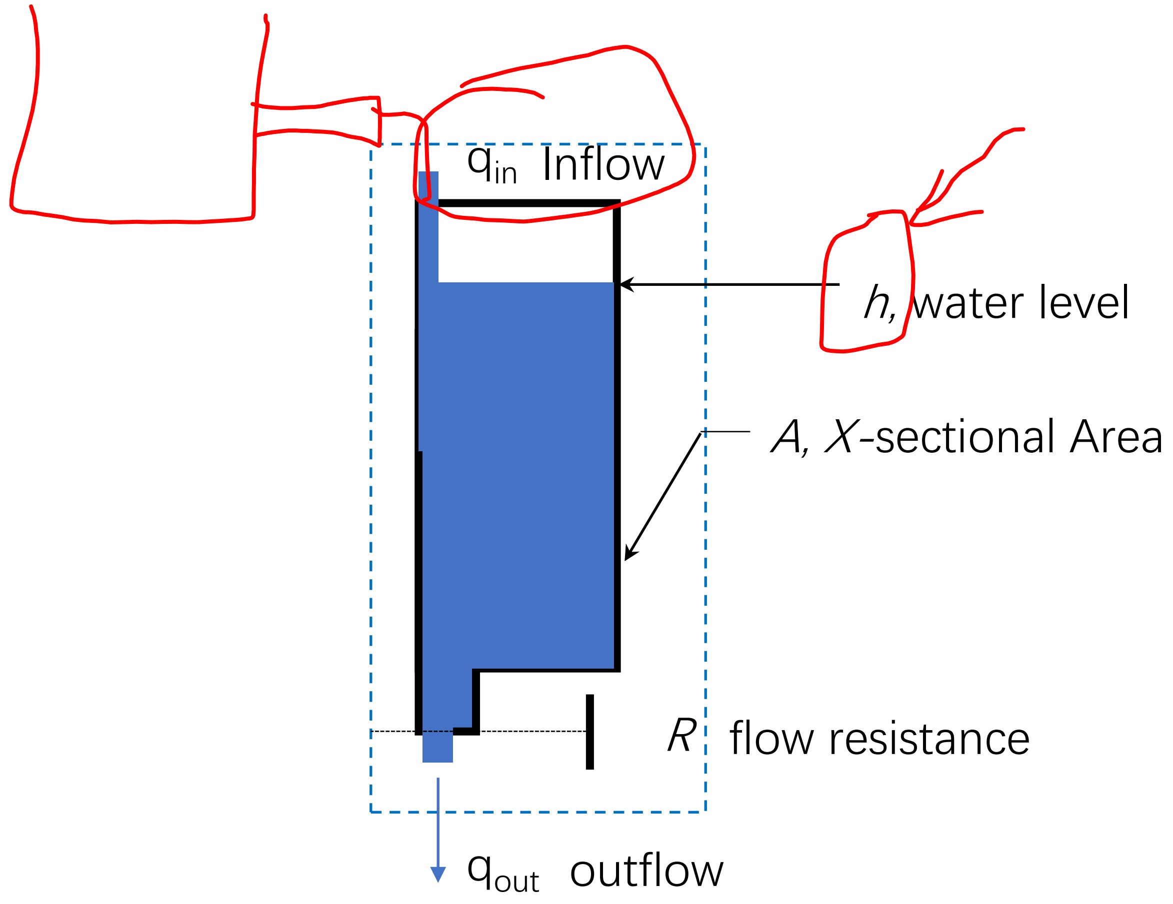
$$\frac{d}{dt}(\rho V) = \rho \dot V = \rho \dot q_{in} - \rho \dot q_{out}$$

$$A\dot h = \dot q_{in} - \dot q_{out} \qquad \dot q_{out} = \frac{gh}{R}$$

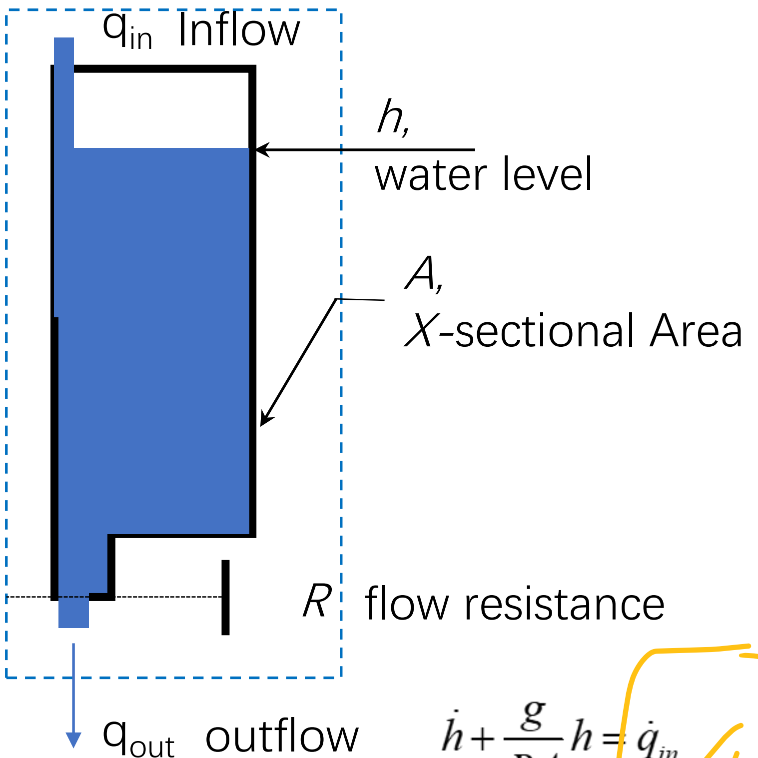
$$A\dot h + \frac{g}{R} h = \dot q_{in}$$

$$\boxed{\dot h + \frac{g}{RA} h = \frac{\dot q_{in}}{A}}$$

$$\dot y + \alpha y = f, \quad y(0) = c$$



Take an Example (1st order ODE)



$$\dot{h} + \frac{g}{RA} h = \frac{q_{in}}{A}$$

$$\dot{y} + \alpha y = f, \quad y(0) = c$$

This corresponds to an IVP in the form of

$$\dot{y} + \alpha y = f, \qquad y(0) = c$$

with y and f denoting the output and input, respectively

Solution:

$$y(t) = ce^{-\alpha t} + \int_0^t f(\tau) e^{-\alpha(t-\tau)}\, d\tau$$

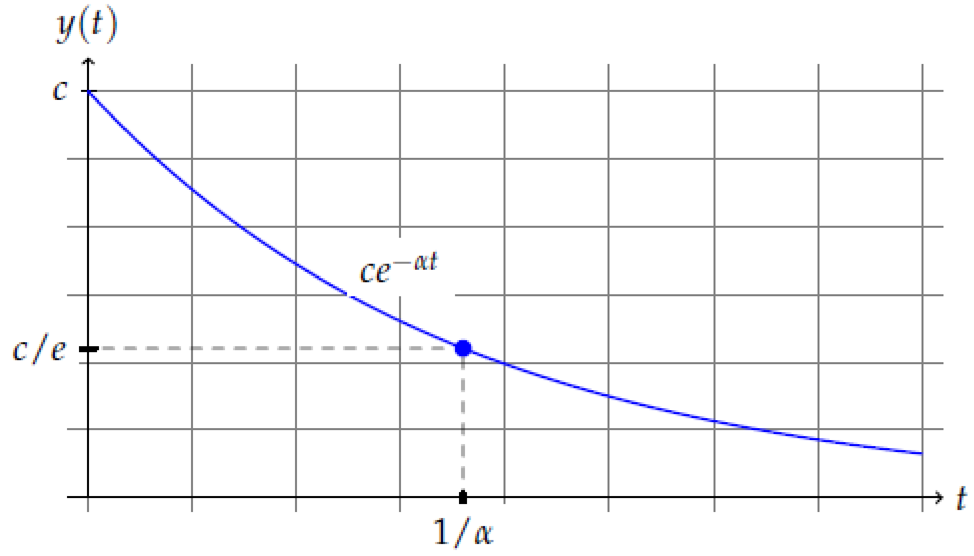
If $\alpha = 0$,

$$y(t) = c + \int_0^t f(\tau)\, d\tau$$

If $f(t) = 0$, i.e. free response

$$y(t) = ce^{-\alpha t}$$

Take an Example (1st order ODE)



Output $y(t)$ decays to a fraction $1/e$ of its original value after $1/\alpha$, the time constant of the system

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$$y(t) = ce^{-\alpha t}$$

Take an Example (1st order ODE)

Recalling that

$$\frac{d}{dt}\int_0^t g(\tau,t)d\tau = \int_0^t \frac{\partial g(\tau,t)}{\partial t}d\tau + g(t)$$

Verifying solution by differentiating

$$\dot{y} = -\alpha c e^{-\alpha t} - \alpha\int_0^t f(\tau)e^{-\alpha(t-\tau)}\,d\tau + f(t)$$

$$\dot{y} = -\alpha\left(ce^{-\alpha t} + \int_0^t f(\tau)e^{-\alpha(t-\tau)}\,d\tau\right) + f(t)$$

$\dot{y} = -\alpha y + f(t)$ (Verified)

This corresponds to an IVP in the form of

$$\dot{y} + \alpha y = f, \qquad y(0) = c$$

with y and f denoting the output and input, respectively

Solution:

$$\mathrm{y}(t) = ce^{-\alpha t} + \int_0^t f(\tau)e^{-\alpha(t-\tau)}\,d\tau$$

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If $f(t) = 0$, i.e. free response

$$\mathrm{y}(t) = ce^{-\alpha t}$$

Take an Example (1st order ODE)

Convolution of the function f and g

$$(f * g) = \int_0^{t-\Delta} f(\tau)g(t-\tau)\ d\tau$$

Convolution of input signal f with an exponentially decaying signal $e^{-\alpha t}$

$$\left(f(\#) * e^{-\alpha\#}\right)(t) = \underline{\int_0^{t-\Delta} f(\tau)e^{-\alpha(t-\tau)}\, d\tau}$$

Therefore solution of the IVP can be written as

$$\mathrm{y}(t) = \underbrace{ce^{-\alpha t}}_{y_p} + \underbrace{\boxed{\left(f(\#) * e^{-\alpha\#}\right)(t)}}_{y_h}$$

This corresponds to an IVP in the form of

$$\dot{y} + \alpha y = f, \qquad y(0) = c$$

with y and f denoting the output and input, respectively

Solution:

$$\mathrm{y}(t) = ce^{-\alpha t} + \underline{\int_0^{t} f(\tau)e^{-\alpha(t-\tau)}\, d\tau}$$

If $\alpha = 0$,

$$\mathrm{y}(t) = c + \int_0^{t} f(\tau)\, d\tau$$

If $f(t) = 0$, i.e. free response

$$\mathrm{y}(t) = ce^{-\alpha t}$$

Take an Example (1st order ODE)

Solution of the IVP, $\dot{y} + \alpha y = f, \quad y(0) = c$

can be written as

$$y(t) = \underbrace{\boxed{ce^{-\alpha t}}}_{\text{Homogenous Solution}} + \underbrace{\boxed{\left(f(\#) * e^{-\alpha\#}\right)(t)}}_{\text{Particular Solution}}$$

We see in last lecture that we can use Laplace transform to simply our operation

Laplace Transforms Revisited

Reminder: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}\mathrm{d}\tau, \qquad s \in \mathbb{C}$$

time domain	frequency domain
$u(t)$	$U(s)$
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is called the transfer function of the system.

Laplace Transforms Revisited

One-sided (or unilateral) Laplace transform:

$$\mathscr{L}\{f(t)\} \equiv F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \qquad \text{(really, from } 0^-\text{)}$$

— for simple functions f, can compute $\mathscr{L}f$ by hand.

Example: unit step

$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathscr{L}\{1(t)\} = \int_0^\infty e^{-st}\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad (\text{pole at } s = 0)$$

— this is valid provided $\mathrm{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t\to+\infty} 0$.

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\begin{aligned}\mathscr{L}\{\cos t\} &= \mathscr{L}\left\{\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\right\} && \text{(Euler's formula)}\\ &= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} && \text{(linearity)}\end{aligned}$$

$$\begin{aligned}\mathscr{L}\{e^{jt}\} &= \int_0^\infty e^{jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{(j-s)t}\mathrm{d}t = \frac{1}{j-s}e^{(j-s)t}\Big|_0^\infty\\ &= -\frac{1}{j-s} \qquad \text{(pole at } s=j\text{)}\end{aligned}$$

$$\begin{aligned}\mathscr{L}\{e^{-jt}\} &= \int_0^\infty e^{-jt}e^{-st}\mathrm{d}t = \int_0^\infty e^{-(j+s)t}\mathrm{d}t = -\frac{1}{j+s}e^{-(j+s)t}\Big|_0^\infty\\ &= \frac{1}{j+s} \qquad \text{(pole at } s=-j\text{)}\end{aligned}$$

— in both cases, require $\mathrm{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

$e^{\theta j} = \cos\theta + j\sin\theta$ Euler's Formula $e^{\pi j} = -1$ Euler's Identity

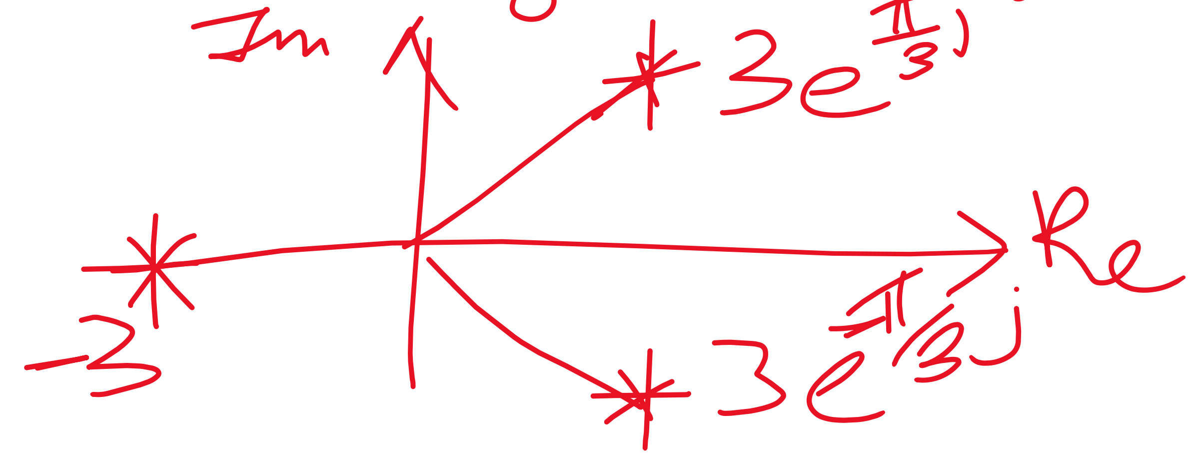
for example

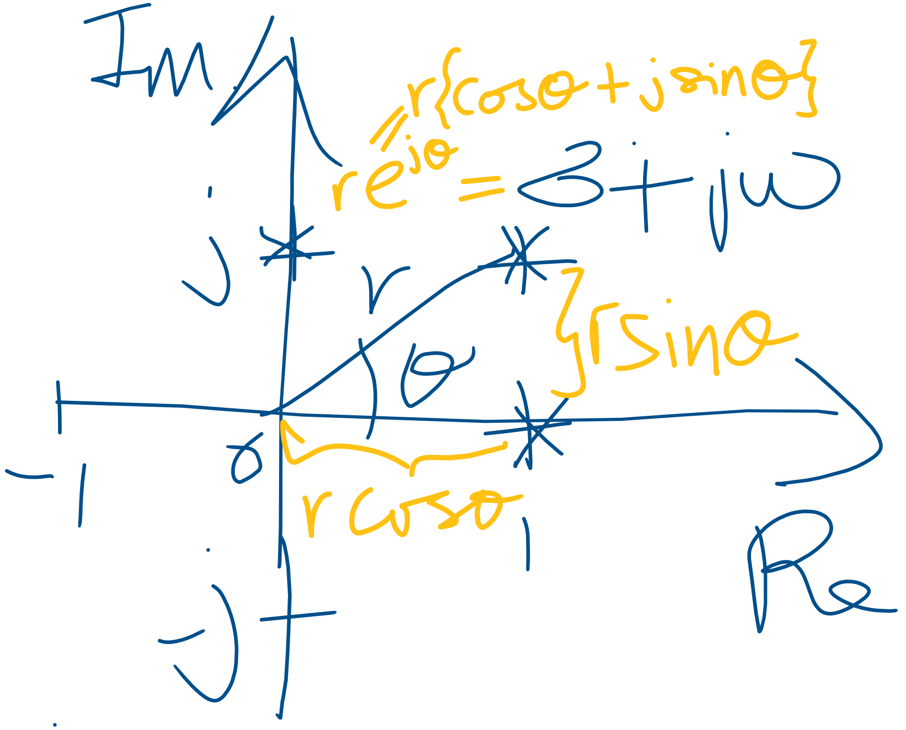
$x^4 = 16 \Rightarrow x = \pm 2, \pm 2j$

$x^4 - 16 = 0$

$(x-2)(x+2)(x-2j)(x+2j) = 0$

How about $y^3 = -27$?





$$\underbrace{re^{\theta j}}_{\text{polar form}} = \underbrace{r\cos\theta + jr\sin\theta}_{\text{rectangular form}}$$

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\begin{aligned}
\mathscr{L}\{\cos t\} &= \frac{1}{2}\mathscr{L}\{e^{jt}\} + \frac{1}{2}\mathscr{L}\{e^{-jt}\} \\
&= \frac{1}{2}\left(-\frac{1}{j-s} + \frac{1}{j+s}\right) \\
&= \frac{1}{2}\left(\frac{-\cancel{j} - s + \cancel{j} - s}{(j-s)(j+s)}\right) \\
&= \frac{1}{2}\left(\frac{-2s}{-1 + \cancel{js} - \cancel{js} - s^2}\right) \\
&= \frac{s}{s^2+1} \qquad \text{(poles at } s = \pm j\text{)}
\end{aligned}$$

for $\mathrm{Re}(s) > 0$

Proof

$\cos t = e^{jt} + e^{-jt}$

Euler's formula:

① $e^{jt} = \cos t + j\sin t$

② $e^{-jt} = \cos t - j\sin t$

$\left.\begin{matrix} ① \\ + \\ ② \end{matrix}\right\}$ $\cos t = \frac{1}{2}[e^{jt} + e^{-jt}]$

Laplace Transforms Revisited

Convolution: $\mathscr{L}\{f \star g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \qquad y(0) = 0$

Compute the response for $u(t) = \cos t$

We already know

$$H(s) = \frac{1}{s+1} \qquad \text{(from earlier example)}$$

$$U(s) = \frac{s}{s^2+1} \qquad \text{(just proved)}$$

Laplace Transforms Revisited

Convolution: $\mathscr{L}\{f \star g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \qquad y(0) = 0$

Compute the response for $u(t) = \cos t$

We already know

$$H(s) = \frac{1}{s+1} \qquad \text{(from earlier example)}$$
$$U(s) = \frac{s}{s^2+1} \qquad \text{(just proved)}$$

$$\Longrightarrow Y(s) = H(s)U(s) = \frac{s}{(s+1)(s^2+1)}$$
$$y(t) = \mathscr{L}^{-1}\{Y\}$$

Try Partial Fraction

— can't find $Y(s)$ in the tables. So how do we compute y?

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)(s^2+1)}\right\}$

This Laplace transform is not in the tables, but let's look at the table anyway. What do we find?

$$\frac{1}{s+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad (\#7)$$

$$\frac{1}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \qquad (\#17)$$

$$\frac{s}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad (\#18)$$

— so we see some things that are similar to $Y(s)$, but not quite.

This brings us to the method of partial fractions:

- boring (i.e., character-building), but *very useful*
- allows us to break up complicated fractions into sums of simpler ones, for which we know $\mathscr{L}^{-1}$ from tables

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$1(t)$
3	$\frac{1}{s^2}$	t
4	$\frac{2!}{s^3}$	t^2
5	$\frac{3!}{s^4}$	t^3
6	$\frac{m!}{s^{m+1}}$	t^m
7	$\frac{1}{(s+a)}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{(s^2+a^2)}$	$\sin at$
18	$\frac{s}{(s^2+a^2)}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

$$\frac{a(s^2+1)+(bs+c)(s+1)}{(s+1)(s^2+1)}$$

RHS

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad \text{(need } bs+c \text{ so that deg(num) = deg(den) − 1)}$$

- Find a: multiply by $s+1$ to isolate a

$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

— now let $s=-1$ to “kill” the second term on the RHS:

$$a = (s+1)Y(s)\Big|_{s=-1} = -\frac{1}{2}$$

$$0s^2 + 1s + 0 = \text{RHS}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad (\text{need } bs+c \text{ so that } \deg(\text{num}) = \deg(\text{den}) - 1)$$

- Find b: multiply by s^2+1 to isolate $bs+c$

$$(s^2+1)Y(s) = \frac{s}{s+1} = \frac{a(s^2+1)}{s+1} + bs + c$$

— now let $s = j$ to "kill" the first term on the RHS:

$$bj + c = (s^2+1)Y(s)\Big|_{s=j} = \frac{j}{1+j}$$

Match $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ parts:

$$c + bj = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{1}{2} + \frac{j}{2} \quad \Longrightarrow b = c = \tfrac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We found that

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

Now we can use linearity and tables:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{-\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}\right\} \\ &= -\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{(from tables)} \\ &= -\frac{1}{2}e^{-t} + \underline{\frac{1}{\sqrt{2}}\cos(t-\pi/4)} \quad (\cos(a-b) = \cos a\cos b + \sin a \sin b) \end{aligned}$$

$R\cos(t+\phi)$

$= A\cos t + B\sin t$

$$\boxed{\begin{aligned} R^2 &= A^2 + B^2 \\ \phi &= \tan^{-1}\left(\frac{B}{A}\right) \end{aligned}}$$

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u \qquad y(0) = 0$

$$u(t) = \cos t \quad \longrightarrow \quad y(t) = \underbrace{-\frac{1}{2}e^{-t}}_{\substack{\text{transient}\\ \text{response}}} + \underbrace{\frac{1}{\sqrt{2}}\cos(t - \pi/4)}_{\substack{\text{steady-state}\\ \text{response}}}$$

— transient response vanishes as $t \to \infty$ (we will see later why)

Let's compare against the frequency response formula:

$$H(s) = \frac{1}{s+1} \qquad \Longrightarrow \qquad H(j\omega) = \frac{1}{j\omega + 1}$$

$u(t) = \cos t$ has $A = 1$ and $\omega = 1$, so

$$\begin{aligned} y(t) &= M(1)\cos\left(t + \varphi(1)\right) \\ &= \frac{1}{\sqrt{2}}\cos\left(t - \pi/4\right) \end{aligned}$$

— the freq. response formula gives only the steady-state part!!

Transient and Steady-State Response

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$$\begin{aligned} y(t) &= M(1)\cos\left(t + \varphi(1)\right) \\ &= \frac{1}{\sqrt{2}}\cos\left(t - \pi/4\right) \end{aligned}$$

$$\boxed{u(t) = A\cos(\omega t) \quad \longrightarrow \quad y(t) = A \underbrace{M(\omega)}_{\substack{\text{amplitude}\\ \text{magnification}}} \cos\big(\omega t + \underbrace{\varphi(\omega)}_{\substack{\text{phase}\\ \text{shift}}}\big)}$$

— the freq. response formula gives only the steady-state part!!

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u \qquad y(0) = 0$

We computed the response to $u(t) = \cos t$ in two ways:

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4)$$

— using the method of partial fractions;

$$y(t) = \frac{1}{\sqrt{2}}\cos\left(t - \pi/4\right)$$

— using the frequency response formula.

Q: Which answer is correct? And why?

A: At $t = 0$, $\frac{1}{\sqrt{2}}\cos(t - \pi/4) = \frac{1}{2} \neq 0$, which is inconsistent with the initial condition $y(0) = 0$. The term $-\frac{1}{2}e^{-t}\Big|_{t=0} = -\frac{1}{2}$ cancels the steady-state term, so indeed $y(0) = 0$.

Therefore, the first formula is correct.

Transient and Steady-State Response

- Frequency response formula limited to steady state part of the response
- Inverse Laplace transform provide both steady-state and transient response

Next, how do we deal with nonzero IC

Laplace Transforms and Differentiation

Given a differentiable function f, what is the Laplace transform $\mathscr{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad \text{(integrate by parts)} \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \to 0$ as $t \to \infty$

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ — this is how we account for I.C.'s

Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Laplace Transforms and Differentiation

Given a differentiable function f, what is the Laplace transform $\mathscr{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad \text{(integrate by parts)} \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \to 0$ as $t \to \infty$

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ — this is how we account for I.C.'s

Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Consider the system

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \dot{y}(0) = 0$$

(need two I.C.'s for 2nd-order ODE's)

Let's compute the transfer function: $H(s) = \dfrac{Y(s)}{U(s)}$

— take Laplace transform of both sides (zero I.C.'s):

$$s^2Y(s) + 3sY(s) + 2Y(s) = U(s) \qquad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Consider the system

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \dot{y}(0) = 0$$

(need two I.C.'s for 2nd-order ODE's)

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$$s^2Y(s) + 3sY(s) + 2Y(s) = U(s) \qquad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$$

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

Compute the *step response*, i.e., response to $u(t) = 1(t)$

Caution!! $Y(s) = H(s)U(s)$ no longer holds if $\alpha \neq 0$ or $\beta \neq 0$

Again, take Laplace transforms of both sides, mind the I.C.'s:

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = U(s)$$

$U(s) = \mathscr{L}\{1(t)\} = 1/s$, which gives

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{\alpha s + (3\alpha+\beta) + \frac{1}{s}}{s^2+3s+2} = \frac{\alpha s^2 + (3\alpha+\beta)s + 1}{s(s+1)(s+2)}$$

Note: if $\alpha = \beta = 0$, then $Y(s) = \dfrac{1}{s(s+1)(s+2)} = H(s)U(s)$

Compute the step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

$$Y(s) = \frac{\alpha s^2 + (3\alpha+\beta)s + 1}{s(s+1)(s+2)} \qquad y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

Use the method of partial fractions:

$$\frac{\alpha s^2 + (3\alpha+\beta)s + 1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$$

— this gives $a = 1/2,\ b = 2\alpha + \beta - 1,\ c = -\alpha - \beta + 1/2$

$$Y(s) = \frac{1}{2s} + (2\alpha+\beta-1)\frac{1}{s+1} + \frac{-\alpha-\beta+1/2}{s+2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}1(t) + (2\alpha+\beta-1)e^{-t} + (1/2-\alpha-\beta)e^{-2t}$$

The step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

is given by

$$y(t) = \frac{1}{2}1(t) + (2\alpha+\beta-1)e^{-t} + (1/2-\alpha-\beta)e^{-2t}$$

What are the transient and the steady-state terms?

- The transient terms are e^{-t}, e^{-2t} (decay to zero at exponential rates -1 and -2)

  Note the poles of $H(s) = \dfrac{1}{(s+1)(s+2)}$ at $s = -1$ and $s = -2$

  — these are *stable poles* (both lie in LHP)
- the steady-state part is $\frac{1}{2}1(t)$ — converges to steady-state value of $1/2$

DC Gain

$$u \longrightarrow \boxed{h} \longrightarrow y$$

Definition: the steady-state value of the step response is called the *DC gain* of the system.

$$\text{DC gain} = y(\infty) = \lim_{t \to \infty} y(t) \qquad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain $= y(\infty) = 1/2$

Steady-State Value

$$u \longrightarrow \boxed{h} \longrightarrow y$$

$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

- $Y(s) = \dfrac{1}{s+a}$, $a > 0$ (pole at $s = -a < 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = 0$
- $Y(s) = \dfrac{1}{s+a}$, $a < 0$ (pole at $s = -a > 0$)
  $y(t) = e^{-at} \quad \Longrightarrow \quad y(\infty) = \infty$
- $Y(s) = \dfrac{1}{s^2+\omega^2}$, $\omega \in \mathbb{R}$ (poles at $s = \pm j\omega$, purely imaginary)
  $y(t) = \sin(\omega t) \quad \Longrightarrow \quad y(\infty)$ does not exist
- $Y(s) = \dfrac{c}{s}$ (pole at the origin, $s = 0$)
  $y(t) = c1(t) \quad \Longrightarrow \quad y(\infty) = c$

The Final Value Theorem

We can now deduce the Final Value Theorem (FVT):

If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\mathrm{Re}(s) < 0$, then

$$y(\infty) = \lim_{s \to 0} sY(s).$$

In our examples, multiply $Y(s)$ by s, check poles:

- $Y(s) = \dfrac{1}{s+a} \qquad sY(s) = \dfrac{s}{s+a}$
  if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer
- $Y(s) = \dfrac{1}{s^2+\omega^2} \qquad sY(s) = \dfrac{s}{s^2+\omega^2}$
  poles are purely imaginary (not in OLHP), FVT does not give correct answer
- $Y(s) = \dfrac{c}{s} \qquad sY(s) = c$
  poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain

$$u \longrightarrow \boxed{h} \longrightarrow y$$

Step response: $$Y(s) = \frac{H(s)}{s}$$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \to 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s)\Big|_{s=0} = \frac{3}{5}.$$