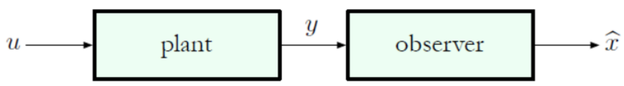
Observer.

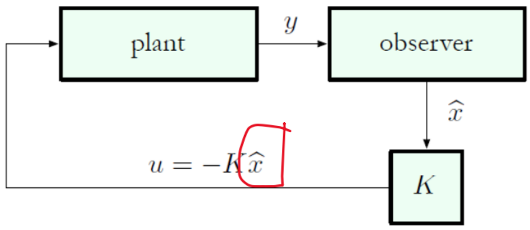
- When full state feedback is unavailable, the observer is used to estimated the state x.



The idea is to design the observer in such a way that the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} (\widehat{x}_i(t) - x_i(t))^2} \xrightarrow{t\to\infty} 0$$

If we are successful, then we can try estimated state feedback:



Observability:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \Rightarrow \quad \mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

System is observable $\leftrightarrow$ $\mathcal{O}(A,C)$ is invertible. (SO case).

Observer Canonical Form.

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & * \\ 1 & 0 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & * \\ 0 & 0 & \dots & 0 & 1 & * \end{pmatrix} = A_{CCF}^T, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\det(Is - A) = \det[(Is - A)^T] = \det(Is - A^T) = \det(Is - A_{CCF}).$$

Fact: A system in OCF is *always observable*!!

☆ observability is preserved under invertible coordinate transf. the $\mathcal{O}$ mx changes.

The Luenberger Observer

For a system $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$, there is a state $\hat{x}$: $\boxed{\dot{\hat{x}} = (A - LC)\hat{x} + Ly}$.

observer

output injection matrix.

Assumption: L s.t. $A-LC$ is Hurwitz. (all eigenvalues in LHP).

$$\Downarrow$$

$$\|x(t) - \hat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t\to\infty} 0.$$

state estimation error $e = x - \hat{x}$.

$$\dot{e} = \dot{x} - \dot{\hat{x}}$$
$$= Ax - [(A-LC)\hat{x} + LCx]$$
$$= \dots = (A-LC)e.$$

For fast convergence $\rightarrow$ eigenvalues of $A-LC$ far into LHP.

observer poles.

☆ OP should be stable & fast.

Observer Pole Placement in OCF.

$$\dot{\hat{x}} = Ax, \qquad y = Cx, \qquad \dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 1 & 0 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

☆ each observer gain only affects one of the coef. of the characteristic polynomial.

Eigenvalues of $A - LC$ are the roots of the characteristic polynomial

$$\det(Is - A + LC)$$
$$= s^n + (a_1 + \ell_n)s^{n-1} + \ldots + (a_{n-1} + \ell_2)s + (a_n + \ell_1)$$

Procedures:

General procedure for any *observable* system:

1. Convert to OCF: $T = \underbrace{\mathcal{O}(\bar{A}, \bar{C})^{-1}}_{\text{new}} \underbrace{[\mathcal{O}(A, C)]}_{\text{old}}$
2. Find $\bar{L}$, such that $\bar{A} - \bar{L}\bar{C}$ has desired eigenvalues.
3. Convert back to original coordinates: $L = T^{-1}\bar{L}$.

The resulting observer is

$$\dot{\hat{x}} = (A - T^{-1}\bar{L}C)\hat{x} + T^{-1}\bar{L}y$$

☆ Controllability - Observability Duality.

The system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable if and only if the system

$$\dot{x} = A^T x + C^T u$$

is controllable.

i.e. CCF controllable.

$\Updownarrow$

OCF observable.

Note:

$$\mathcal{C}(A^T, C^T) = [C^T | A^T C^T | \cdots | (A^T)^{n-1} C^T]$$
$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T = [\mathcal{O}(A, C)]^T$$

Use C/O duality for observer pole placement.

Given an observable pair (A, C):

1. For $F = A^T$, $G = C^T$, consider the system $\dot{x} = Fx + Gu$ (this system is controllable).
2. Use our earlier procedure to find K, such that
$$F - GK = A^T - C^T K$$
has desired eigenvalues.
3. Then
$$\text{eig}(A^T - C^T K) = \text{eig}(A^T - C^T K)^T = \text{eig}(A - K^T C),$$
so $L = K^T$ is the desired output injection matrix.

Final answer: use the observer

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly$$
$$= (\underline{A - K^T C})\hat{x} + \underline{K^T}y.$$