ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 486 Control Systems

Lecture 21: Introduction to State Space Method

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Schedule check

Week	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
4	National Holiday Week	
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control	Section 4.1, lab manual
6	Introduction to Proportional-Integral-Derivative (PID) control	Sections 4.1-4.3, lab manual
	Review A	
7	Term Test 1	
	Introduction to Root Locus design method	Ch. 5
8	Root Locus continued; introduction to dynamic compensation	Ch. 5
	Lead and lag dynamic compensation	Ch. 5
9	Introduction to frequency-response design method	Sections 5.1-5.4, 6.1
	Bode plots for three types of transfer functions	Section 6.1

Root Locus

Frequency Response

Week	Topic	Ref.
10	Stability from frequency response; gain and phase margins	Section 6.1
	Control design using frequency response	Ch. 6
11	Control design using frequency response continued; PI and lag, PID and lead-lag	Ch. 6
	Nyquist stability criterion	Ch. 6
12	Gain and phase margins from Nyquist plots	Ch. 6
	Introduction to state-space design (Review B)	
13	**Term Test II**	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback I; separation principle	Ch. 7
	Dynamic output feedback II (Review C)	Ch. 7
16	END OF LECTURES	
	Finals	

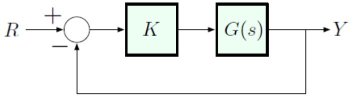
State-Space

Lecture Overview

- **Review**: Frequency domain-based approach so far
- **Today's topic:** Introduction to State Space Method
- **Learning Goal**: introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

Reading: FPE, Chapter 7

Review: Argument Principle → Nyquist Criterion



We now examine the Nyquist plot of $H(s) = 1 + KG(s)$.

By the argument principle,

$$N = Z - P,$$

where $N = \#(\circlearrowright$ encirclements of 0 by Nyquist plot of $1 + KG(s))$,

$Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C)$,

$P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)$

Now we extract information about RHP roots of $1 + KG(s)$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

for stability ⇒ 0
(C-L)

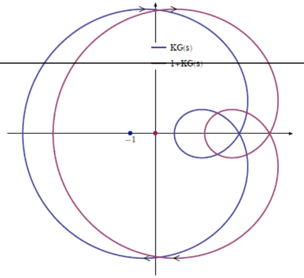
unstable O-L

$N = Z - P$

$\#(\circlearrowright$ of $-1/K$ by Nyquist plot of $G(s))$

$= \boxed{\#(\text{RHP closed-loop poles})} - \#(\text{RHP open-loop poles})$

Unstable C-L Poles



$N = \#(\circlearrowright$ encirclements of 0 by Nyquist plot of $1 + KG(s))$

$= \#(\circlearrowright$ encirclements of -1 by Nyquist plot of $KG(s))$

$= \boxed{\#(\circlearrowright \text{ encirclements of } -1/K \text{ by Nyquist plot of } G(s))}$

$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \le \deg(p)$$

$$1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}$$

$$\text{closed-loop t.f.} = \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)}$$

Therefore:

$Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C)$

$= \#(\text{RHP zeros of } 1 + KG(s))$

$= \boxed{\#(\text{RHP closed-loop poles})}$

$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \le \deg(p)$$

$$1 + KG(s) = 1 + K\frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}$$

Therefore:

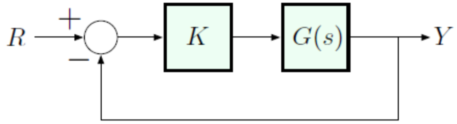
$P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)$

$= \#(\text{RHP poles of } 1 + KG(s))$

$= \#(\text{RHP roots of } p(s))$

$= \boxed{\#(\text{RHP open-loop poles})}$

Review: The Nyquist Stability Criterion



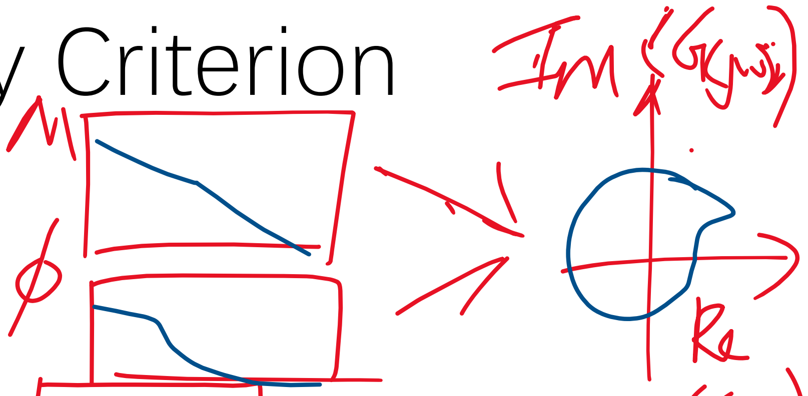
$$\underbrace{N}_{\#(\circlearrowleft \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

$$Z = N + P$$

$$Z = 0 \quad \Longleftrightarrow \quad N = -P$$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

How?



Workflow:

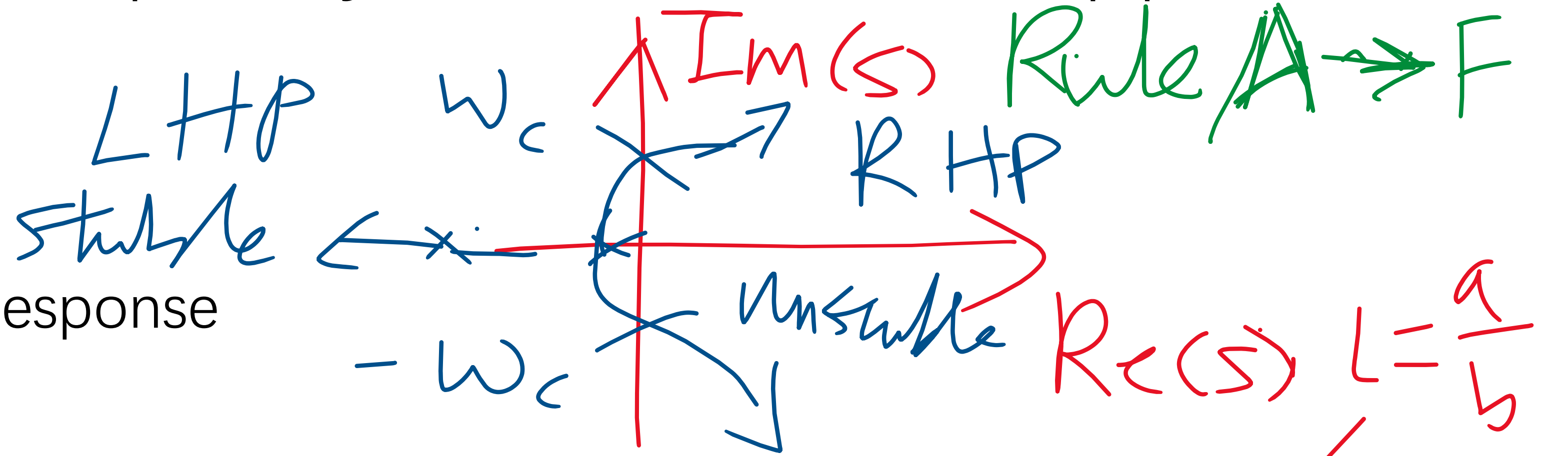
$$\boxed{\text{Bode } M \text{ and } \phi\text{-plots}} \longrightarrow \text{Nyquist plot}$$

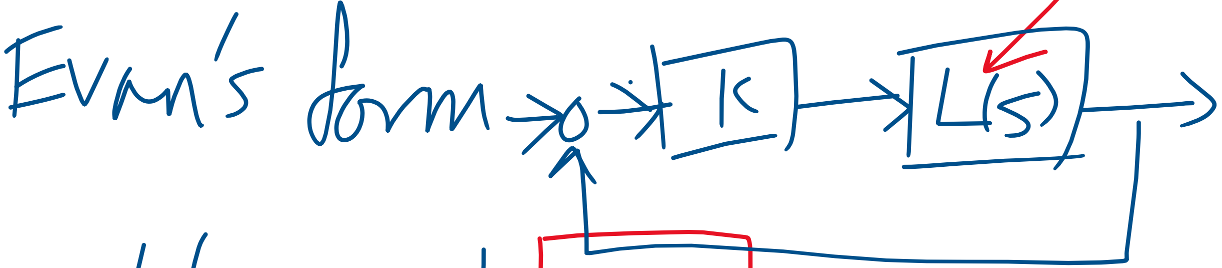
Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)

Review: Frequency domain-based approach

- Root Locus
- Frequency Response
- Nyquist Criterion





$$H_{CL} = \frac{K\,\boxed{L(s)}}{1 + K\,\boxed{L(s)}}$$

Review: Frequency domain-based approach

- Root Locus
- Frequency Response
- Nyquist Criterion

Mag Phase

GM, PM

Question 1

a) A system can be represented by the block diagram shown in Figure 4.

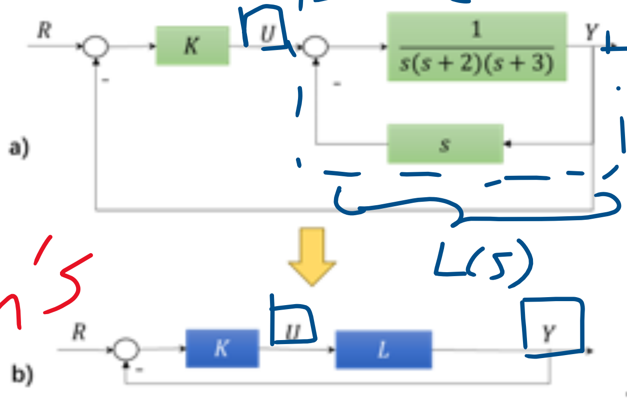
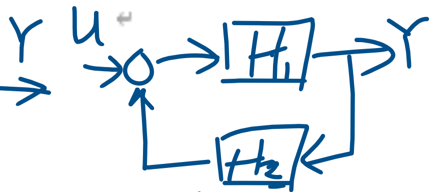


Figure 4



Evan's form :

$$L(s) = \frac{Y(s)}{U(s)}$$

$$L(s) = \frac{[H_1 \cancel{H_2}]}{[1 + H_1 H_2]}$$

$$= \frac{\left[\frac{1}{(s+2)(s+3)}\right]}{\left[1 + \frac{1}{(s+2)(s+3)}\right]} = \frac{\frac{1}{s(s+2)(s+3)}}{1 + \frac{s}{s(s+2)(s+3)}}$$

$$= \frac{1}{(s+2)(s+3) + 1} = \frac{1}{s(s+2)(s+3) + s}$$

$$= \frac{1}{\cancel{s^2 + 5s + 6}} = \frac{1}{s^3 + 5s^2 + 7s}$$

$$= \frac{1}{[s(s^2 + 5s + 7)]}$$

Characteristic eq^n : $[s(s^2+5s+7)] = 0$

i. Obtain the expression for L in the block diagram (b) reduced from (a). *(2 Points)*

ii. Write down the closed-loop transfer function of the system. *(2 Points)*

iii. Write down the characteristic equation. *(1 Points)*

$$H_{cl} = \frac{K\,L}{(1 + KL)} = 0$$ characteristic equation

$s = 0$,

b) Figure 5 is a plot of the root locus.

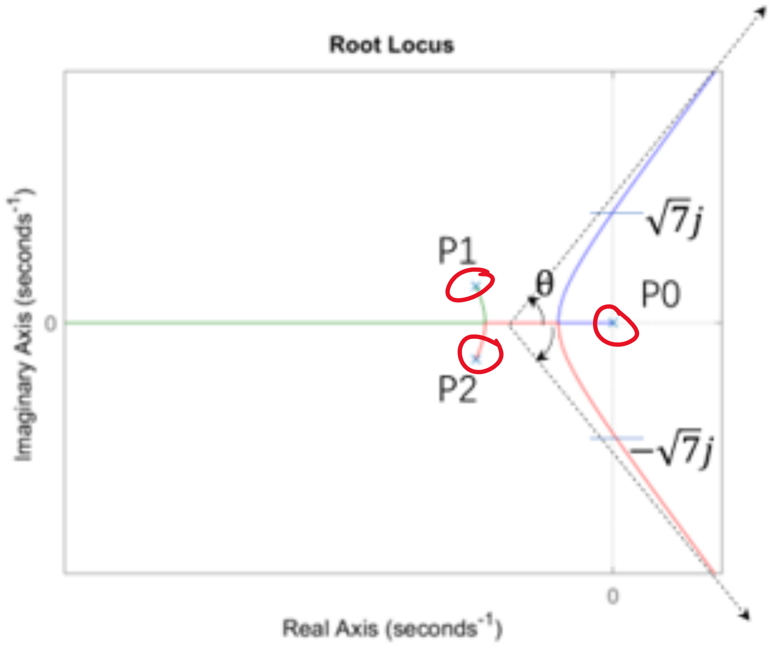


Figure 5

i. Obtain the range of values of K satisfying the Routh-Hurwitz Criteria. *(5 Points)*

ii. Obtain the value for P1 and P2 as indicated in the root locus plot. *(4 Points)*

iii. Obtain the value of θ. *(4 Points)*

iv. Validate using the Routh-Hurwitz Criteria that the $j\omega$-crossing is $\boxed{\pm\sqrt{7}j}$ *(2 Points)*

$$s(s^2+5s+7)=0$$

$$P_0 = S_0 = 0, \quad P_{1,2} = S_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{3}}{2}j$$

$$H_{CL} = \frac{KL}{1+KL} = \frac{k}{(s^3+5s^2+7s+k)} = 0$$

s^3	1	7
s^2	5	K
s^1	35-k	
s^0	K	

$K < 35$

$k > 0$

$0 < k < 35$

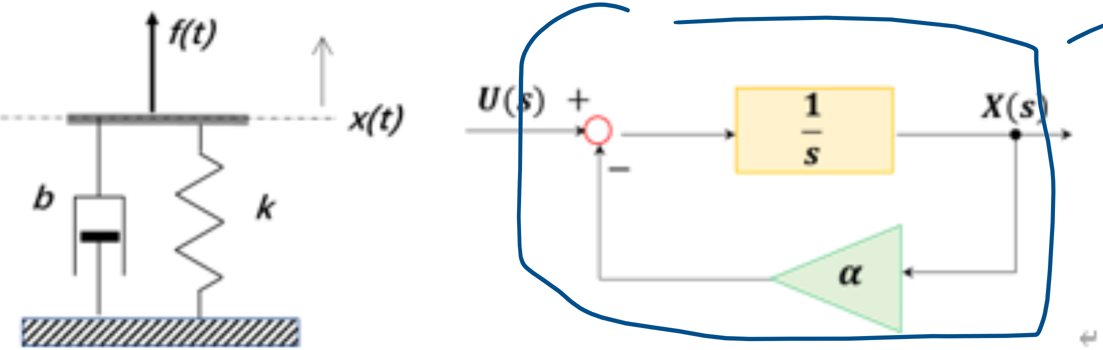
$K_{critical} = 35$

$$s^3+5s+7s+35=0$$

substitute $s = \pm\sqrt{7}j$

Question 2

a) A plate attached to a spring and damper with insignificant mass with zero-initial conditions is subjected to a force as shown.



$G(s)$

Figure 3b

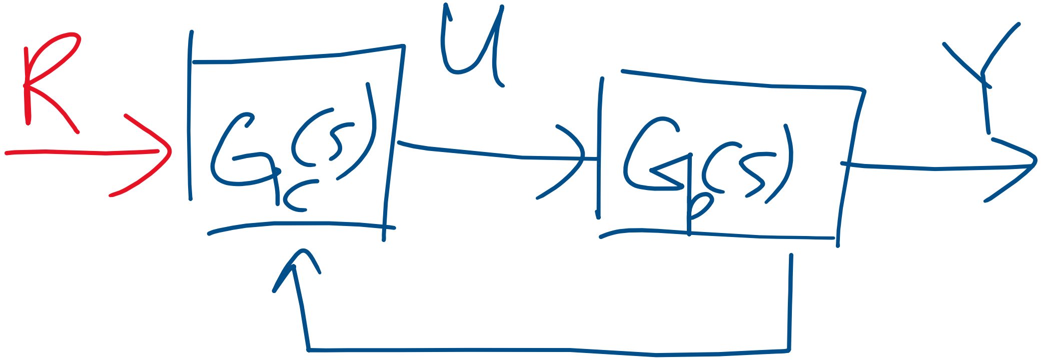
i) Show that the system can be represented with the given block diagram and provide the expressions of $U(s)$ and α (2 Points)

ii) Write down the frequency response function $G(j\omega)$ (2 points) $\rightarrow G(s)\big|_{s=j\omega}$

iii) Express $G(j\omega)$ in terms of its magnitude and phase given $k=b=1$. (2 points)

iv) Sketch the Bode diagrams representing gain $G(j\omega)$ (4 points)

v) Assuming significant plate mass $m=1$, $b=6$, $k=5$, rewrite the new transfer function of the plant $G_p(s)$ (1 Points)



A feedback control system is implemented as represented by the shown block diagram.

Figure 3c

vi) When K = 10, the bode plot is given by **Figure 3d**. Indicate the frequency values where there are changes in the magnitude slope. (4 Points)

vii) Given the Gain Margin (GM)=+8 dB, Phase Margin (PM)=+21°, on the bode plot on Figure 2, label the Gain Margin and Phase Margin. *(2 Points)*

viii) Comment on how changing the value of *K affect stability using the Bode plot. (3 points)*

A feedback control system is implemented as represented by the shown block diagram.

Figure 3c

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viii) Comment on how changing the value of *K affect stability using the Bode plot. (3 points)*

Governing Eq^n

a) i)

$$\boxed{\begin{gathered} f_{external} = f_{damper} + f_{spring} \\ f(t) = b\dot{x}(t) + kx(t) \\ \frac{f(t)}{b} = \dot{x}(t) + \frac{k}{b}x(t) \end{gathered}}$$

Letting $\frac{f(t)}{b} = u(t)$, $\frac{k}{b} = \alpha$,

$$u(t) = \dot{x}(t) + \alpha x(t)$$

With zero initial conditions and taking Laplace transform

$$U(s) = sX(s) + \alpha X(s)$$

ii) $G(s) = \frac{X(s)}{U(s)}$

$$G(j\omega) = \frac{1}{j\omega + \alpha}$$

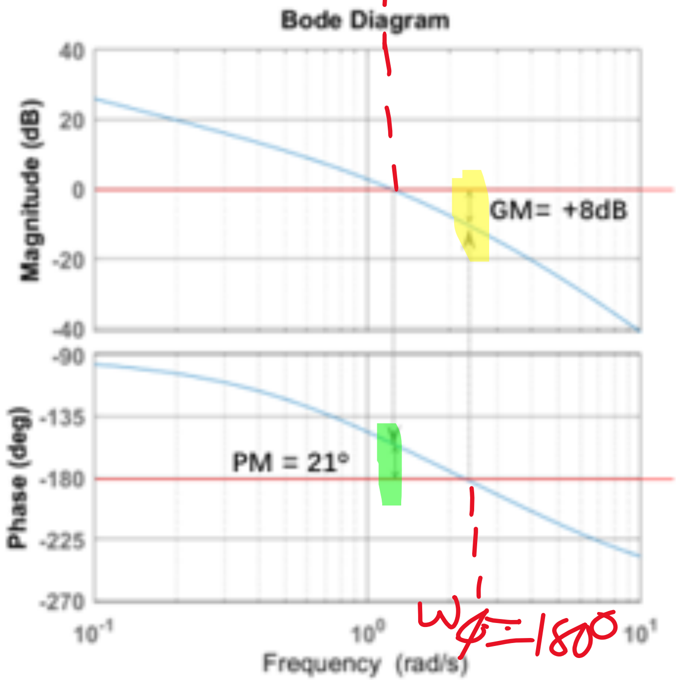
iii) $|G(j\omega)| = \frac{1}{\sqrt{\omega^2+1}}$ (iv) $\angle G(j\omega) = -\angle(\omega j + 1)$

Mag. Phase

d) vi) $\omega = 1,5$

v)

$\omega_{M=0}$

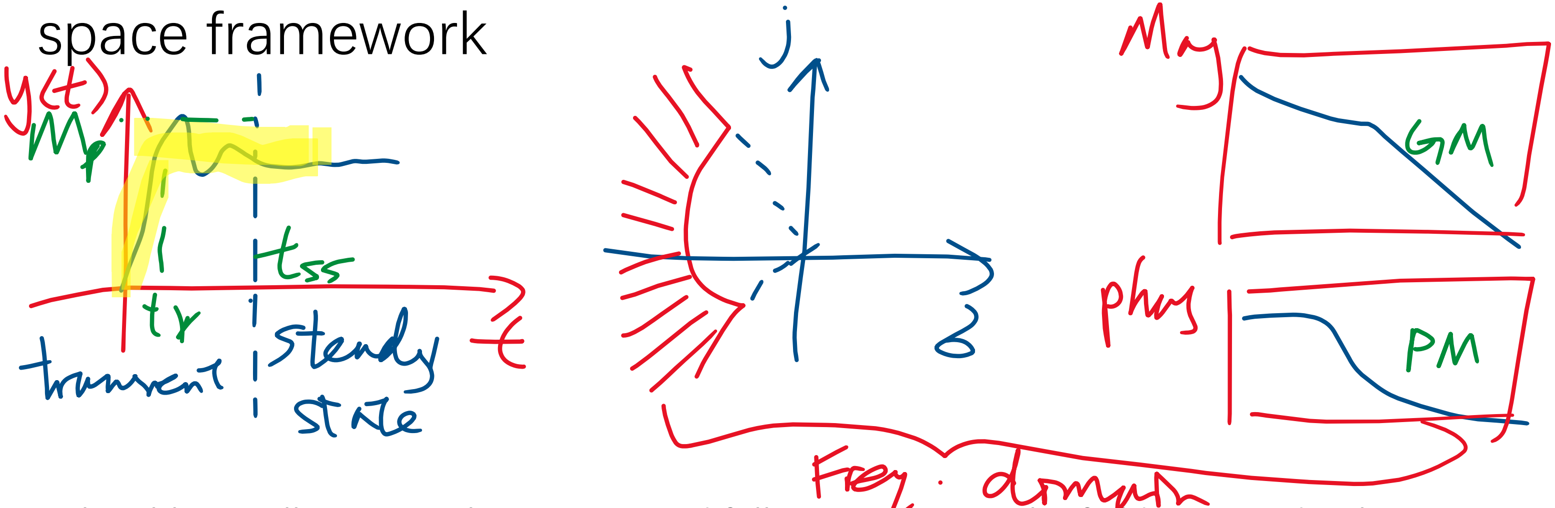


$\omega_{\phi=-180^\circ}$

viii) since increasing K shift the magnitude plot downwards but does not change the phase plot, the gain margin will be reduced and eventually become negative and unstable.

Frequency-Domain vs. State-Space

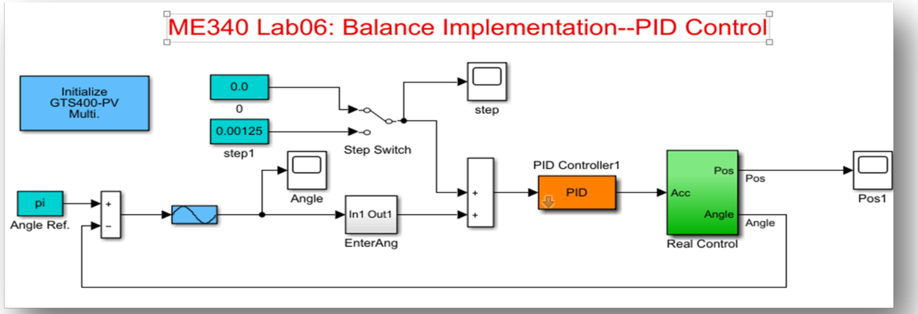
- 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
- 90% of current research in systems and control is in the state-space framework

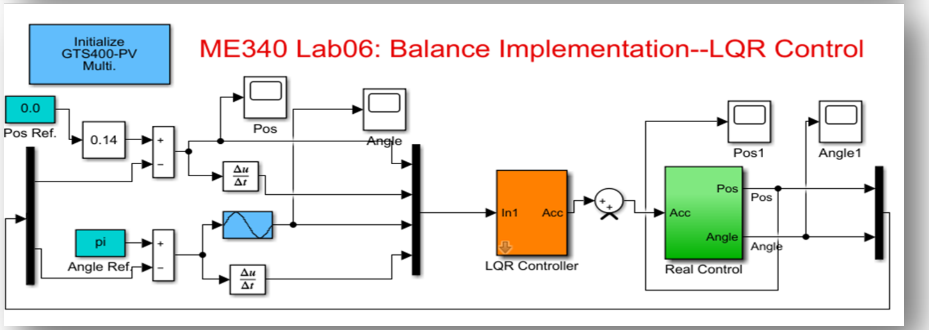


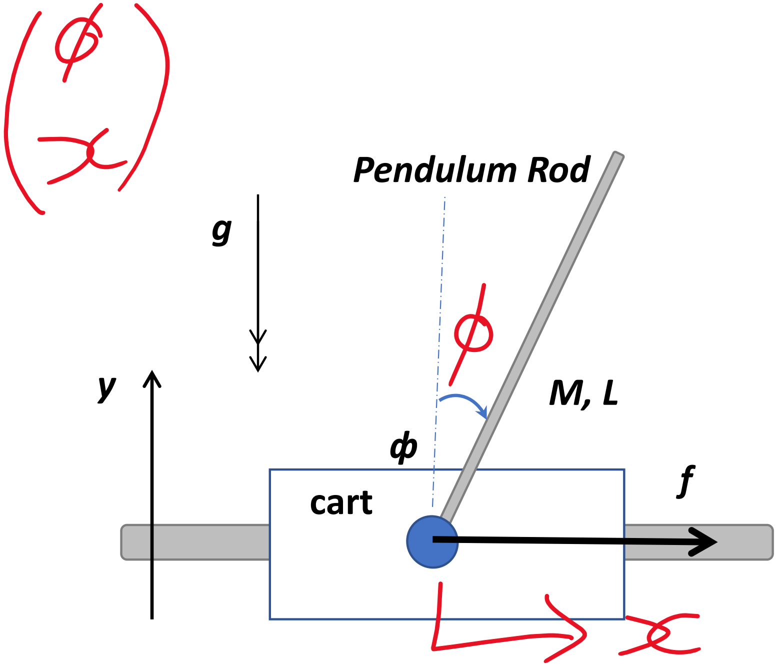
To be able to talk to control engineers and follow progress in the field, we need to know both methods and understand the connections between them.

Frequency-Domain vs. State-Space

- Frequency-domain methods: E.g., PID is a popular architecture
- State-space framework: E.g., LQR







Introduction to State-Space

- introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

Frequency-Domain vs. State-Space

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To be able to talk to control engineers and follow progress in the field, we need to know both methods and understand the connections between them.

State-Space Methods

- the state-space approach reveals internal system architecture for a given transfer function
- the mathematics is different: heavy use of linear algebra
- this is just a short introduction

A General State-space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

$$\text{output } y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p$$

state input

State eq^n
$$\dot{x} = A\boxed{x} + B\boxed{u}$$
How system changes

Output eq^n
$$y = C\boxed{x} + D\boxed{u}$$
observed output

where:

Dynamics

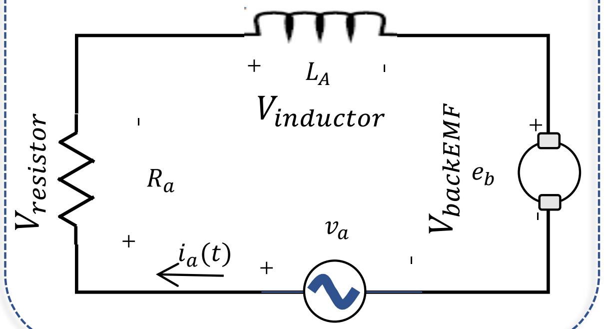
A – system matrix ($n \times n$) $\qquad$ B – input matrix ($n \times m$)

C – output matrix ($p \times n$) $\qquad$ D – feedthrough matrix ($p \times m$)

State-Space Model: Example

Modeling of Dynamic System

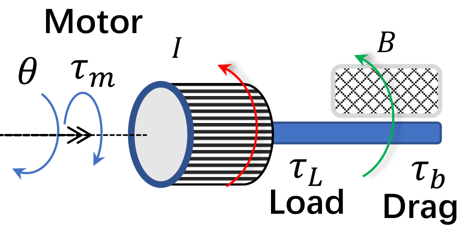
Electrical System



$$V_{inductor} = L_a \frac{di_a}{dt};$$
$$V_{resistor} = R_a i_a;$$
$$V_{backEMF} = K_e \dot{\theta}$$

$$\tau_m = K_t i_a;$$
$$\tau_b = B\dot{\theta};$$

Mechanical System



Kirchhoff's Law

$$v_a = V_{inductor} + V_{resistor} + V_{backEMF}$$
$$L_a \frac{di_a}{dt} + R_a i_a + K_e \dot{\theta} = v_a$$

Newton's Law

$$I\ddot{\theta} = \tau_m - \tau_b - \tau_L$$
$$I\ddot{\theta} = K_t i_a - B\dot{\theta} - \tau_L$$
$$I\ddot{\theta} + B\dot{\theta} - K_t i_a = -\tau_L$$

x

B

A

State-space representation of Dynamic System

output

$$\left.\begin{aligned} \dot{x}_1 &= \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_e}{L_a}\omega + \frac{1}{L_a} v_a \\ \dot{x}_2 &= \dot{\omega} = \frac{K_t}{I} i_a - \frac{B}{I}\omega - \frac{1}{I}\tau_L \end{aligned}\right\} \quad \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

State-Space Model: Comparison with Transfer Function Approach

$$L_a \frac{di_a}{dt} + R_a i_a + K_e \omega = v_a$$

$$I\dot{\omega} + B\omega - K_t i_a = -\tau_L$$

Laplace

$$L_a s I_a(s) + R_a I_a(s) = V_a(s) - K_e \Omega(s)$$

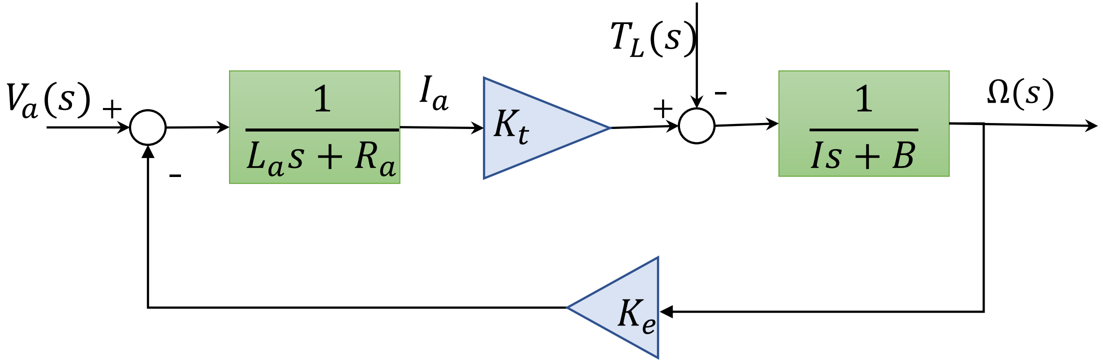
$$I s \Omega(s) + B\Omega(s) = -T_L(s) + K_t I_a(s)$$

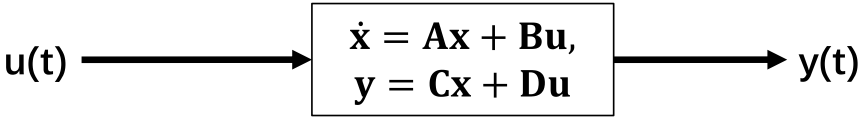
$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

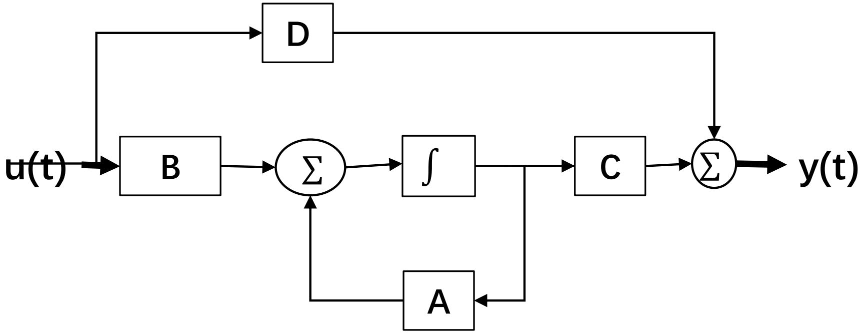
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\frac{\Omega(s)}{V_a(s)} = \frac{(1/(L_a s + R_a)) \cdot K_t \cdot (1/(Is+B))}{1 + (1/(L_a s + R_a)) \cdot K_t \cdot (1/(Is+B)) \cdot K_e} = \frac{K_t}{L_a I s^2 + \left(L_a B + R_a I\right)s + R_a B + K_t K_e}$$

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(1/(Is+B))}{1 - (1/(Is+B)) \cdot (-K_e) \cdot (1/(L_a s + R_a)) \cdot K_t} = -\frac{L_a s + R_a}{L_a I s^2 + (L_a B + R_a I)s + R_a B + K_t K_e}$$







From State-Space to Transfer Function

Let us find the *transfer function* from u to y corresponding to the state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$$

Recall matrix-vector multiplication:

$$\begin{aligned}\dot{x}_i &= (Ax)_i + (Bu)_i \\ &= \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k\end{aligned}$$

$$\begin{aligned}y_\ell &= (Cx)_\ell + (Du)_\ell \\ &= \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k\end{aligned}$$

From State-Space to Transfer Function

$U \rightarrow \boxed{G(s)} \rightarrow Y$

$\dot{x} = Ax + Bu$

Now we take the Laplace transform:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k$$

$$\downarrow \mathscr{L}$$

$$sX_i(s) - x_i(0) = \sum_{j=1}^{n} a_{ij} X_j(s) + \sum_{k=1}^{m} b_{ik} U_k(s), \qquad i = 1, \dots, n$$

Write down in matrix-vector form:

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \\ (Is - A)X(s) &= x(0) + BU(s) \qquad (I \text{ is the } n \times n \text{ identity matrix}) \\ X(s) &= (Is - A)^{-1}x(0) + (Is - A)^{-1}BU(s) \end{aligned}$$

output: $Y = Cx + Du$

$$y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k$$

$$\downarrow \mathscr{L}$$

$$Y_\ell(s) = \sum_{j=1}^{n} c_{\ell j} X_j(s) + \sum_{k=1}^{m} d_{\ell k} U_k(s), \qquad \ell = 1, \dots, p$$

Write down in matrix-vector form:

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C\left[(Is - A)^{-1}x(0) + (Is - A)^{-1}BU(s)\right] + DU(s) \\ &= C(Is - A)^{-1}x(0) + \boxed{\left[C(Is - A)^{-1}B + D\right]}\,U(s) \end{aligned}$$

0

G(s)

To find the input-output t.f., set the IC to 0:

$$Y(s) = G(s)U(s), \qquad \text{where } \boxed{G(s) = C(Is - A)^{-1}B + D}$$

$\frac{Y}{u} = G(s)$

From State-Space to Transfer Function

The transfer function from u to y, corresponding to

$$u \longrightarrow \boxed{\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}} \longrightarrow y$$

G(s)

is given by

T.F :
$$G(s) = C(Is - A)^{-1}B + D$$

Observe that $G(s)$ contains information about the state-space matrices A, B, C, D!!

feedforward
output
Dynamics, input

From State-Space to Transfer Function

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$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

is given by

$$G(s) = C(Is - A)^{-1}B + D$$

Observe that $G(s)$ contains information about the state-space matrices A, B, C, D!!

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad & Y(s) &= G(s)U(s) \\ y &= Cx + Du & &= \left[C(Is - A)^{-1}B + D\right]U(s) \end{aligned}$$

Important!!

- $G(s)$ is *undefined* when the $n \times n$ matrix $Is - A$ is *singular* (or noninvertible), i.e., precisely when $\det(Is - A) = 0$
- since A is $n \times n$, $\det(Is - A)$ is a *polynomial* of degree n (the characteristic polynomial of A):

$$\det(Is - A) = \det \begin{pmatrix} s - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & s - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & s - a_{nn} \end{pmatrix},$$

  and its roots are the eigenvalues of A
- G is (open-loop) stable if all eigenvalues of A lie in LHP.

Example Compute *G(s)*

Consider the state-space model in Controller Canonical Form (CCF)*:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— this is a *single-input, single-output* (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

Let's compute the transfer function:

$$G(s) = C(Is - A)^{-1}B \qquad (D = 0 \text{ here})$$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$$

* We will explain this terminology later.

Example Compute *G(s)*

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix} \qquad \text{— how do we compute } (Is - A)^{-1}?$$

Review: Matrix Analysis

- Eigenvalue and Eigenvector

For a matrix $\boldsymbol{A}$, there exists a column matrix $\boldsymbol{v}$ such that $\boxed{\boldsymbol{Av} = \boldsymbol{v}\lambda,}$
λ and $\boldsymbol{v}$ are the eigenvalue and the associated eigenvector of $\boldsymbol{A}$

determinant

Given matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if λ satisfies $\left|\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right| = 0$

such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = \begin{pmatrix} \gamma \\ \eta \end{pmatrix}\lambda,$

$(a-\lambda)(d-\lambda) - bc = 0$
Characteristic equation

then $\boxed{\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = \boldsymbol{v}}$ is the eigenvector of $\boldsymbol{A}$ associated with the real $\boxed{\text{eigenvalue } \lambda}$

Example Compute *G(s)*

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$$ — how do we compute $(Is - A)^{-1}$?

A useful formula for the inverse of a 2×2 matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det M \neq 0 \quad \Longrightarrow \quad M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying the formula, we get

$$\begin{aligned} (Is - A)^{-1} &= \frac{1}{\det(Is - A)} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \end{aligned}$$

Example Compute *G(s)*

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 1\\ -6 & -5\end{pmatrix}}_{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{B}u, \qquad y = \underbrace{\begin{pmatrix}1 & 1\end{pmatrix}}_{C}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\begin{aligned}
G(s) &= C(Is - A)^{-1}B \\
&= \begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{s^2+5s+6}\begin{pmatrix}s+5 & 1\\ -6 & s\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} \\
&= \frac{1}{s^2+5s+6}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ s\end{pmatrix} \\
&= \frac{s+1}{s^2+5s+6}
\end{aligned}$$

- the above state-space model is a *realization* of this t.f.
- note how coefficients 5 and 6 appear in both $G(s)$ and A!!

State-space Realizations of Transfer Functions

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{s+1}{s^2 + 5s + 6}$$

— at least in this example, information about the state-space model (A, B, C) is contained in $G(s)$.

Is this information *recoverable*? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

State-space Realizations of Transfer Functions

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{s+1}{s^2 + 5s + 6}$$

— at least in this example, information about the state-space model (A, B, C) is contained in $G(s)$.

Is this information *recoverable*? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

Answer: There are infinitely many!

State-space Realizations of Transfer Functions

Start with

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and consider a new state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}, \quad \bar{B} = C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{C} = B^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

This is a different state-space model!

Claim: The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

Proof:

$$\begin{aligned} \bar{C}(Is - \bar{A})^{-1}\bar{B} &= B^T \left(Is - A^T\right)^{-1} C^T \\ &= B^T \left[(Is - A)^T\right]^{-1} C^T \\ &= B^T \left[(Is - A)^{-1}\right]^T C^T \\ &= \left[C(Is - A)^{-1}B\right]^T \\ &= C(Is - A)^{-1}B \end{aligned}$$

State-space Realizations of Transfer Functions

The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

But the state-space model is now in the Observer Canonical Form (OCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

More Realizations

Yet another realization of $G(s) = \dfrac{s+1}{s^2+5s+6}$ can be extracted from the partial-fractions decomposition:

$$G(s) = \frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}.$$

This is the Modal Canonical Form (MCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} \text{Then} \quad C(Is-A)^{-1}B &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} s+3 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} \\ \frac{1}{s+2} \end{pmatrix} = \frac{2}{s+3} - \frac{1}{s+2} \end{aligned}$$

Bottom Line of State-Space Realization

- A given transfer function G(s) can be realized using infinitely many state-space models
- Certain properties make some realizations preferable
- One such property is **controllability**

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A,B) = \left[B \,|\, AB \,|\, A^2B \,|\, \ldots \,|\, A^{n-1}B \right]$$

— recall that A is $n \times n$ and B is $n \times 1$, so $\mathcal{C}(A,B)$ is $n \times n$;
— the controllability matrix only involves A and B, not C

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A,B)$ is *invertible*.

(This definition is only true for the single-input case; the multiple-input case involves the *rank* of $\mathcal{C}(A,B)$.)

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = [B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}$(A,B)

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A,B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C}(A,B) = [B \,|\, AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies \mathcal{C}(A,B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad$ system is controllable

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.