$\xrightarrow{u} \boxed{H} \xrightarrow{y}$ state: $\vec{x}(t) \in \mathbb{R}^n$.

$\dddot{x} + a_1\ddot{x} + a_2\dot{x} + a_3 x + a_4 = 0$. $\Rightarrow$ $\boxed{\text{let } z_i = x^{(i-1)},\ \Rightarrow \dot{z}_i = x^{(i)}}$.

$\dot{z}_1 = \dot{x} = z_2$

$\dot{z}_2 = \ddot{x} = z_3$

$\dot{z}_3 = \dddot{x} = -a_1\ddot{x} - a_2\dot{x} - a_3 x - a_4$.

$$\Rightarrow \dot{z} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix}}_{\text{cononical form.}} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -a_4 \end{pmatrix}.$$

State-Space Model.

dynamics mx. control mx.

$$\begin{cases} \dot{x} = Ax + Bu. \leftarrow \text{input} \\ y = Cx. \end{cases} \quad x, y \in ?^n.$$

output. sensor mx. state.

Impulse Response

(do convolution of u.)

given $u(t)$, get y.

$u \to \boxed{\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx \end{matrix}} \to y$.

Linear: $\boxed{\begin{matrix} \text{if } u_1(t) \to y_1(t), \\ u_2(t) \to y_2(t). \\ \Rightarrow u_1 + u_2 = y_1 + y_2. \end{matrix}}$

① $\delta(t) = 0$, $t \neq 0$

② $\int_{-a}^{a} \delta(t)\,dt = 1$, $a > 0$.

③ $\int_{\mathbb{R}} \delta(t-z) f(t)\,dt = f(z)$. $\leftarrow$ sifting property. $\Rightarrow u(t) = \int_{\mathbb{R}} u(\tau)\,\delta(t-\tau)\,d\tau$, superposition principle.

$= \sum_{t<z} \delta(t-z) f(t) + \sum_{t>z} \delta(t-z) f(t) + \delta(z-z) f(t) = 1 \times f(t) = f(t)$.

☆① $u(t) = \delta(t-\tau) \xrightarrow{x(0)=0,\ LTI} y(t) = h(t-\tau)$.

impulse response of the system.

② $\underbrace{u(t) = \int_{\mathbb{R}} u(\tau)\,\delta(t-\tau)\,d\tau}_{\text{superpos of input}} \xrightarrow[x(0)=0.]{LTI} \underbrace{y(t) = \int_{\mathbb{R}} u(\tau)\,h(t-\tau)\,d\tau}_{\text{superpos of output.}} = u(t) * h(t)$.

$\Rightarrow$ convolution of u, h.

Laplace Transf.

$$F(s) = \int_{\mathbb{R}} f(\tau) e^{-s\tau} d\tau, \quad s = e^{j\omega} = \cos\omega + j\sin\omega \in \mathbb{C}.$$

$$y = h * u \iff Y = HU. \Rightarrow H = \frac{Y}{U}.$$

- Transfer function: $H(s) = \int_{\mathbb{R}} h(\tau) e^{-s\tau} d\tau$, ← laplace.
- Causal: only consider $t > 0$, output not affected by future time.

Basic eq: $u \longrightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}} \longrightarrow y$

laplace: $\mathcal{L}(Ax + Bu) = \mathcal{L}(Ax) + \mathcal{L}(Bu)$

$= A\mathcal{L}(x) + B\mathcal{L}(u)$.

$\mathcal{L}(\dot{x}) = sx. \Rightarrow sX = AX + BU.$

Laplace: $Y = CX$.

$\Rightarrow (Is - A) X = BU.$

$\Rightarrow X = (Is - A)^{-1} BU$

$\Rightarrow H = \frac{Y}{X} = \ldots$

$$\begin{cases} H(s) = C(Is - A)^{-1} B & (\text{mx inversion}). \\ h(t) = C e^{At} B, \quad t \geq 0 & (\text{matrix exp}) \end{cases}$$

Note: if can model with ODE, can get $H(s)$ using $\mathcal{L}$.

$\boxed{Av = \lambda v}$ → eigen vector.

$A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$.

For input e^{st}, $y(t) = e^{st} H(s)$.

Get transfer function: use $\mathcal{L}$ $\begin{cases} \text{linearity} \\ \text{differentiation.} \end{cases}$

eg. $\dot{y} = -ay + u \iff sY = -aY + U.$

$\Rightarrow \frac{Y}{U} = \frac{1}{s+a}$