Controllability Matrix: $\mathcal{C}(A,B) = [B\,|AB|\,A^2B|\cdots|A^{n-1}B]$

System is controllable $\Leftrightarrow$ $\det \mathcal{C}(A,B) \neq 0$. (for SI system)
$\Leftrightarrow$ $\mathcal{C}(A,B)$ is invertible.

Coordinate Transformation & State-Space Models

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ &G(s) \end{aligned}\right\} \xrightarrow{T} \left\{\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \\ &G(s). \end{aligned}\right.$$

, where $\begin{cases} \bar{A} = TAT^{-1} \\ \bar{B} = TB \\ \bar{C} = CT^{-1} \end{cases}$.

① TF unchanged.

② OL poles unchanged.

③ $\det(Is - \bar{A}) = \det(Is - A)$.
characteristic polynomial unchanged.

④ controllability unchanged;
controllability mx changed. $\mathcal{C}(\bar{A},\bar{B}) = T\,\mathcal{C}(A,B)$.
$\Updownarrow$
$T = \mathcal{C}(\bar{A},\bar{B})\,[\mathcal{C}(A,B)]^{-1}$.

Converting a Controllable System to CCF.

$Is - A =$

$$A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (C \text{ is immaterial})$$

Step 1: check for controllability.

$$\mathcal{C} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix} \qquad \det \mathcal{C} = -1 \qquad \text{– controllable}$$

Step 2: Determine desired $\mathcal{C}(\bar{A},\bar{B})$.

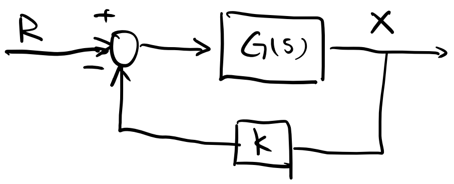
$$\mathcal{C}(\bar{A},\bar{B}) = [\bar{B}\,|\,\bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

Step 3: Compute T.

$$T = \mathcal{C}(\bar{A},\bar{B}) \cdot [\mathcal{C}(A,B)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}\begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Pole Placement

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu. \\ y &= x. \end{aligned}}$$



introducing a state feedback law: $u = -Ky \equiv -Kx = -(k_1\ k_2\ \cdots\ k_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = -(k_1x_1 + \cdots + k_nx_n)$.

CL system:
$$\begin{cases} \dot{x} = Ax - BKx = (A - BK)x \\ y = x. \end{cases}$$

$\Leftrightarrow$ $sX = (A - BK)X + BR, \quad Y = X$.

$Y = \underbrace{(Is - A + BK)^{-1}B}_{G}R$.

CL poles: eigenvalues of $A - BK$.

Pole Placement.

Beauty of CCF:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Claim.

$$\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

— the last row of the A matrix in CCF consists of the coefficients of the characteristic polynomial, in reverse order, with "–" signs.

Placement:

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

Closed-loop poles are the roots of the characteristic polynomial

$$\det(Is - A + BK)$$
$$= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1)$$
$$= (s - p_1)(s - p_2) \cdots (s - p_n).$$

General steps of pole placement:

① convert to CCF.

② $u = -\bar{k}\bar{x}$, place desired poles.

③ $u = -\bar{k}\bar{x} = -\underbrace{(\bar{k}T)}_{k}x$, $\quad k = \bar{k}T$.

convert back to the original coordinate