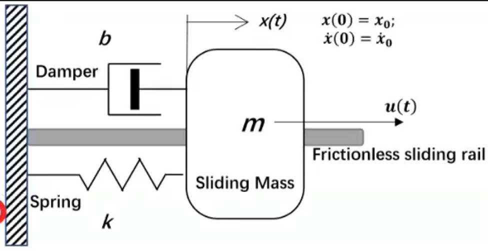
Recap: System Response

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = u(t)$ $s^2X - sx_0 - \dot{x}_0$



Transient response analysis

Laplace Transform (account for ICs!)

$$[s^2X(s) - sx_0 - \dot{x}_0] + 2\zeta\omega_n[sX(s) - x_0] + \omega_n{}^2X(s) = U(s)$$

$$s^2X + 2\zeta\omega_n sX + \omega_n{}^2X - (sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0) = U$$

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

$$\boxed{X(s) = \frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}}$$

Inverse Laplace Transform

$$\boxed{x(t) = L^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}\right\} + L^{-1}\left\{\frac{U(s)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}\right\}}$$

Scale & Phase

$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = M_1 \cdot M_2$, $\angle[(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})] = \phi_1 + \phi_2$.

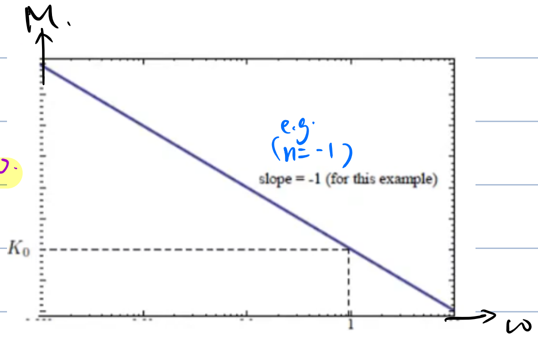
$\Rightarrow \log(M_1 M_2) = \log M_1 + \log M_2$.

Decibel Scale $(M)_{dB} = 20\log_{10} M$, one decade = 20dB.

Bode Plot

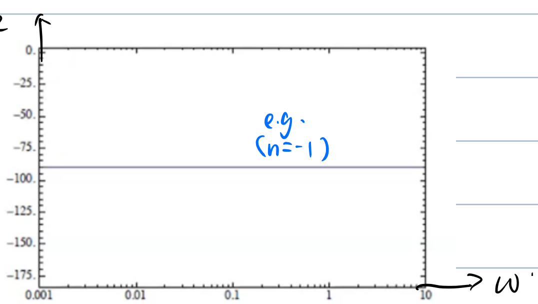
※ Type I: $K_0(j\omega)^n$.

Magnitude



$\log M = \log|K_0(j\omega)^n| = \log|K_0| + n\log\omega$.

Phase

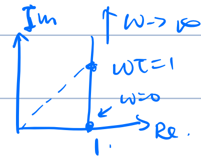


$\angle K_0(j\omega)^n = \angle(j\omega)^n = n\angle j\omega = n \cdot 90°$

※ Type 2: ① $j\omega\tau + 1$. (stable zeros).

Magnitude

Nyquist plot:



Phase.

small $\omega \to \phi = 0$.

large $\omega \to \phi = 90°$

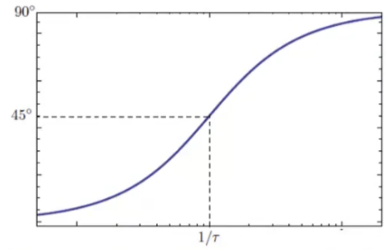
break-pt $\to \phi = \angle(j+1)$.
($\omega\tau = 1$).

$\omega\tau \ll 1 \Rightarrow j\omega\tau + 1 \approx 1$.

$\gg 1 \Rightarrow j\omega\tau + 1 \approx j\omega\tau$.

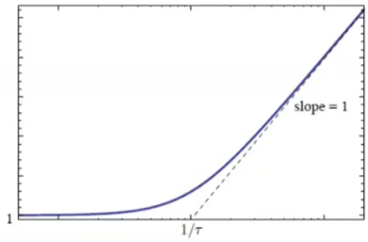
> transition at $\omega\tau = 1$ i.e. $\omega = \frac{1}{\tau}$.

Phase plot:



For a stable real zero, the phase "steps up by 90°" as we go past the break-point.

Magnitude plot:

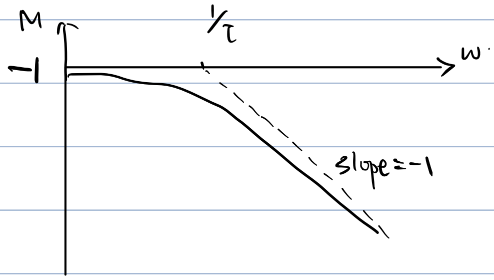


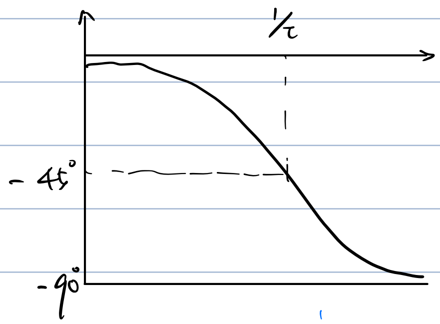
For a stable real zero, the magnitude slope "steps up by 1" at the break-point.

② $(j\omega\tau+1)^{-1}$ (stable poles).

Magnetude: $\log\left|\frac{1}{j\omega\tau+1}\right| = -\log|j\omega\tau+1|$

Phase $\angle\frac{1}{j\omega\tau+1} = -\angle(j\omega\tau+1)$



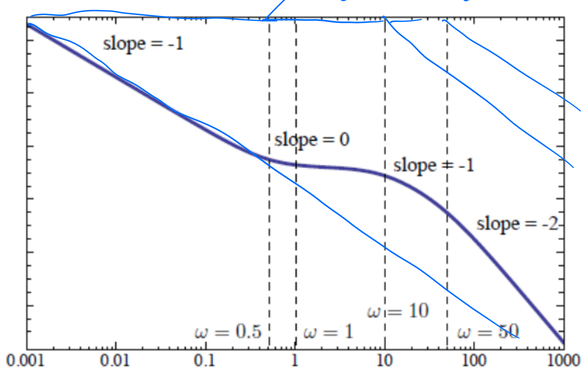


Ex.

Bode Plot (Magnitude)

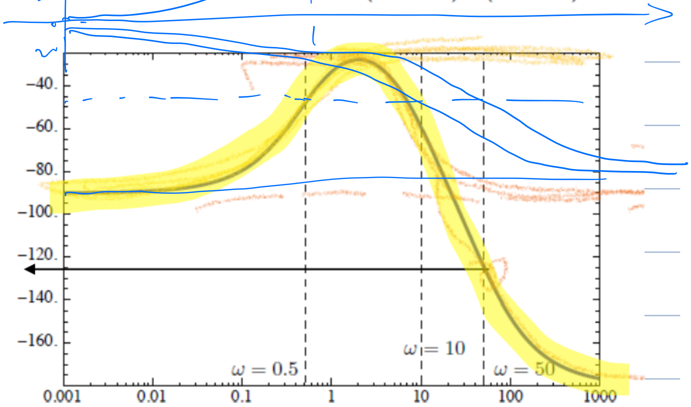
$k_0 = 2, n = -1$.

$$KG(j\omega) = \frac{2}{j\omega}\left(\underbrace{\frac{j\omega}{0.5}}_{\tau = \frac{1}{0.5} = 2} + 1\right) \cdot \frac{1}{\left(\underbrace{\frac{j\omega}{10}}_{\tau = 0.1} + 1\right)\left(\underbrace{\frac{j\omega}{50}}_{\tau = 0.02} + 1\right)}$$



Bode Plot (Phase)

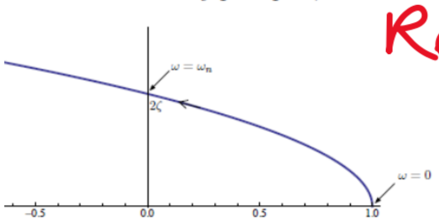
$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



Type 3. ① zeros: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$

Cartesian Form: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = \underbrace{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}_{Re} + \underbrace{2\zeta\frac{\omega}{\omega_n}}_{Im.} j$

And here is the Nyquist plot, for $0 < \omega < \infty$:



Re

$$(R(\omega), I(\omega)) = \left(\boxed{1 - \left(\frac{\omega}{\omega_n}\right)^2}, \boxed{2\zeta\frac{\omega}{\omega_n}}\right)$$

$\omega \ll \omega_n$: $M \approx 1$.

$\omega \gg \omega_n$: $M = \left(\frac{\omega}{\omega_n}\right)^2$

$\Rightarrow \log M = \underbrace{2\log\omega} - 2\log\omega_n$.

magnitude slope steps up by 2

Some obvious points: $\omega = 0 \quad \to 1 + 0j$

$\omega = \omega_n \quad \to 0 + 2\zeta j$

② poles $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$.

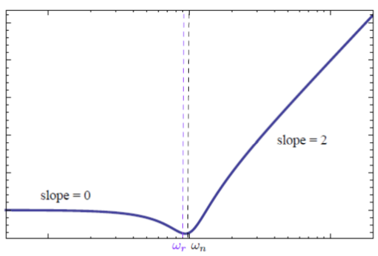
Magnitude $= -\log|\dots| = -M_{zero}$

phase $= -\angle|\dots| = -\angle zero$.

Phase for Type 3.

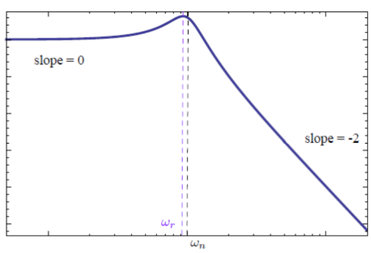
Magnitude for Type 3

Stable real zero $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]$

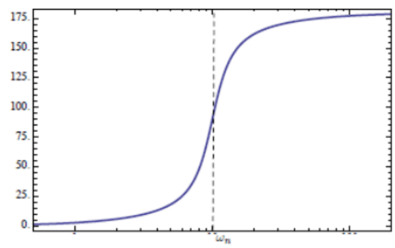


For a stable real zero, the magnitude slope "steps up by 2" at the break-point.

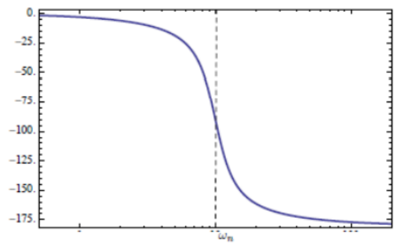
Stable real pole $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$



For a stable real pole, the magnitude slope "steps down by 2" at the break-point.



(stable complex zero — phase steps up by 180°)



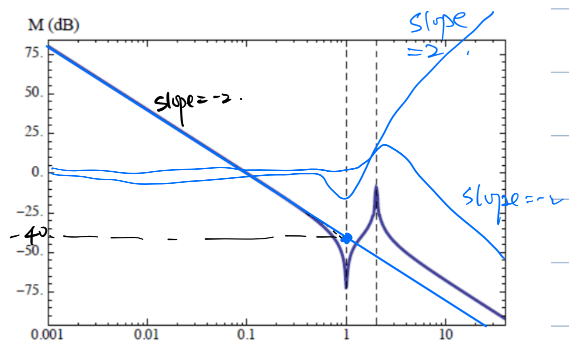
(stable complex pole — phase steps down by 180°)

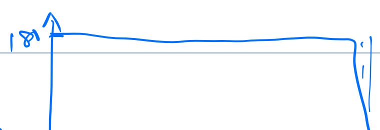
Ex.

$$KG(s) = \frac{0.01\,(s^2 + 0.01s + 1)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about magnitude?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — asymptote has slope $= -2$, passes through $(\omega = 1, M = 0.01)$
- complex zero with break-point at $\omega_n = 1$ and $\zeta = 0.005$ — slope up by 2; large resonant dip
- complex pole with break-point at $\omega_n = 2$ and $\zeta = 0.01$ — slope down by 2; large resonant peak





$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about phase?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$
  — phase starts at $n \times 90° = -180°$
- complex zero with break-point at $\omega_n = 1$ — phase up by 180°
- complex pole with break-point at $\omega_n = 2$ — phase down by 180°
- since ζ is small for both pole and zero, the transitions are very sharp

sharpness depending on ζ.