$y = \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x.$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

By the quadratic formula, the poles are:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The nature of the poles changes depending on ζ:

damping coefficient/ratio →

- $\zeta > 1$ both poles are real and negative (overdamped)
- $\zeta = 1$ one negative pole
- $\zeta < 1$ two complex poles with negative real parts (underdamped).

$\zeta = 0$: no damping.

$$s = -\sigma \pm j\omega_d$$

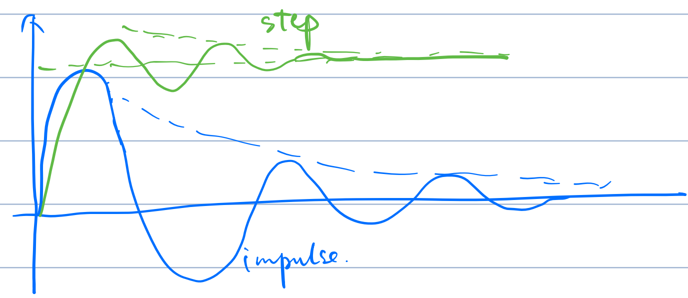
where $\sigma = \zeta\omega_n, \ \omega_d = \omega_n\sqrt{1 - \zeta^2}$

$$\Rightarrow H(s) = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$\mathcal{L}\{1(t)\} = \frac{1}{s}.$

Impulse response: $y(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t).$

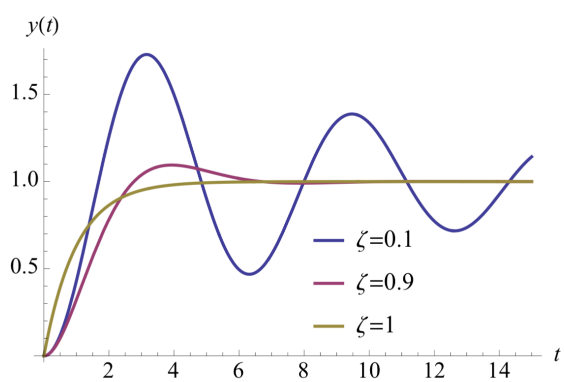
Step response: $y(t) = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right).$



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \quad \longrightarrow \quad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ (damped frequency)

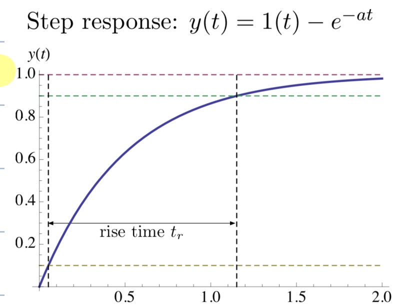


The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping ($\omega_d = \omega_n$)

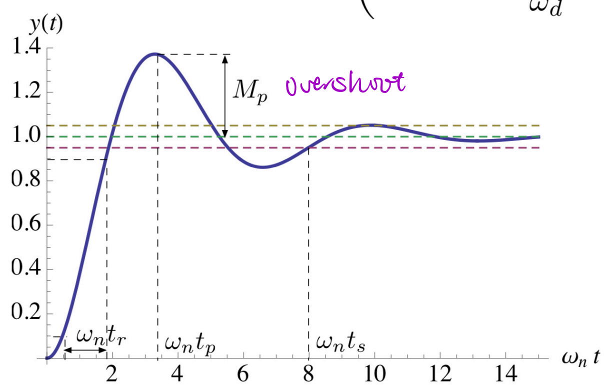
Rise time: t_r, y from 10% to 90% of steady-state time.

1st-order: $H(s) = \frac{s}{s+a}$, $t_r = \frac{\ln 0.9 - \ln 0.1}{a} \approx \frac{2.2}{a}$



2nd-order:

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



rise peak settling

① t_r, t_p, t_s should be small.

② small error, small M_p.

Note: ① empirically, $\omega_n t_r \approx 1.8$ (exact when $\zeta = 0.5$). ← approx.

② $\dot{y}(t) = 0 \Rightarrow t_p = \frac{\pi}{\omega_d}$.

③ $M_p = y(t_p) - 1 = e^{-\sigma\pi/\omega_d} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ ← exact

④. need to define when is steady-state.

may vary.

$$t_s = \min\left\{ t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \le 0.05 \text{ for all } t' \ge t \right\}.$$

defined by designer.

$\approx \frac{3}{\sigma}$.

☆☆

$$t_r \approx \frac{1.8}{\omega_n}$$ (good when $\zeta = 0.5$).

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

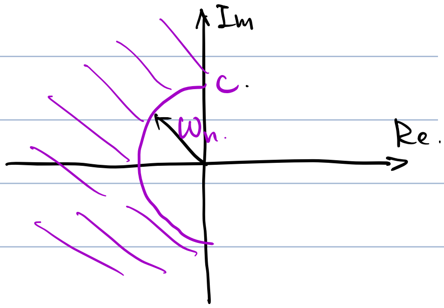
$$t_s \approx \frac{3}{\sigma}$$

Specifications in Frequency Domain

1. Rise: Δ want: $t_r \le c$ (c is a desired value).

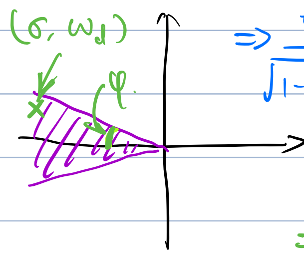
suppose.

$\Rightarrow \omega_n \geq \frac{1.8}{c}$.



$$\omega_n^2 = \sigma^2 + \omega_d^2$$
$$\zeta = \cos\varphi$$

2. overshoot: $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \le c$.



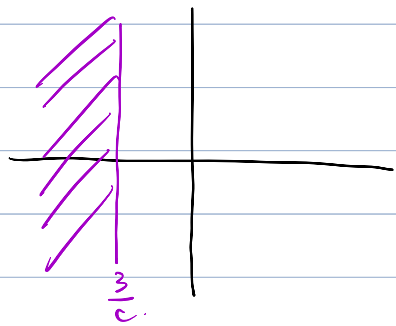
$\Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n \zeta}{\omega_n\sqrt{1-\zeta^2}}$

$= \frac{\sigma}{\omega_d} = \boxed{\cos\varphi \approx \zeta}$.

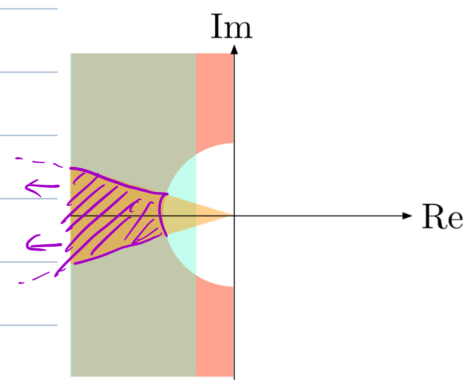
$\Rightarrow \varphi$ should be small, in which case $\boxed{\cos\varphi = \zeta}$.

3. settling time.

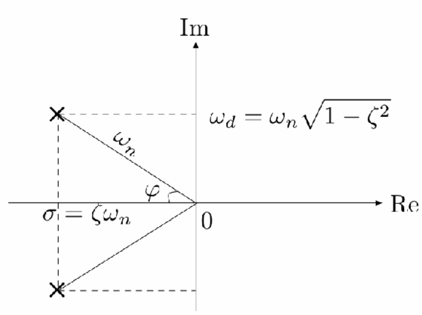
$t_s = \frac{3}{\sigma} \le c \quad \Rightarrow \sigma \geq \frac{3}{c}$.



4. combination of 1 ~ 3:



Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$
$$\zeta = \cos\varphi$$