ECE 486 Control Systems

Lecture 18: Introduction to State Space Method

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Checklist

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	Root Locus
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Modeling Analysis Design

Root Locus

Frequency Response

State-Space

Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	**Control design using frequency response: PD and Lead**	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	**Nyquist stability criterion**	Ch. 6
	Nyquist stability; gain and phase margins from Nyquist plots	Ch. 6
12	**Review B**	
	Term Test B	
13	**Introduction to state-space design**	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	State-Space
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Admin. Announcement

- Mini-Symposium
  - Grads to present project
  - Undergrads as audiences, evaluation form earns you bonus
- Bonus Points
  - Additional points towards final grade until A-
  - No limit for Grad Students

Lecture Overview

- **Review**: Frequency domain-based approach so far
- **Today's topic:** Introduction to State Space Method
- **Learning Goal**: introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

Reading: FPE, Chapter 7

Review: Frequency domain-based approach

- Root Locus
- Frequency Response
- Nyquist Criterion

Review: Frequency domain-based approach

- Root Locus $s = \sigma + j\omega$
- Routh Analysis H_{CL}
- Frequency Response $s = j\omega$
- Nyquist Criterion G_{OL}

Quick Overview: System Representation & Analysis

Mathematical Representation

f

Configuration form

Equations of Motion
$$\begin{cases} \ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \dots \\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \end{cases}$$

Initial Conditions
$$\begin{cases} q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0} \\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0} \end{cases}$$

State space model:

State Equation
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Output Equation
$$y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Transfer Function:

$$\frac{O(s)}{I(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_o}{s^n + a_{n-1} s^{n-1} + \dots + a_o}$$

ICs= 0

Systematic Modeling

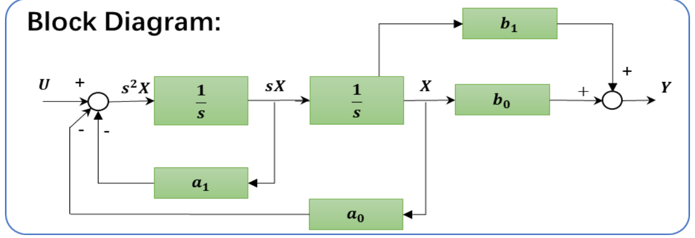
Input — Dynamic System — Output

$u(\#) \rightarrow$ LTI $\rightarrow y(\#)$

$i(t) \rightarrow * h(t) \rightarrow o(t)$

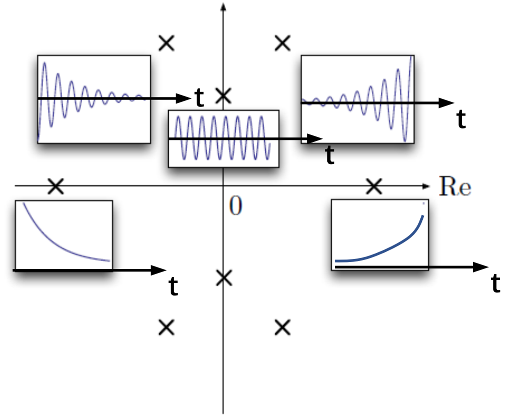
$I(s) \rightarrow H(s) \rightarrow O(s)$

Block Diagram:



Analysis of Systems

Analyzing effect of poles and zeros



- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Stability Analysis

Dynamic Response Specification

Design Methods

Quick Overview: System Representation & Analysis

Mathematical Representation

Configuration form

Equations of Motion

$$\begin{cases} \ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \quad \dots \\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \end{cases}$$

Initial Conditions

$$\begin{cases} q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0} \\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0} \end{cases}$$

State space model:

State Equation $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Output Equation $$y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Transfer Function:

$$\frac{O(s)}{I(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_o}{s^n + a_{n-1} s^{n-1} + \cdots + a_o}$$

ICs= 0

$$u \rightarrow \boxed{\text{LTI}} \rightarrow Y$$

$$H(s) = \frac{Y(s)}{U(s)}$$

$$u(t) \rightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}} \rightarrow y(t)$$

Introduction to State-Space

- introduce basic notions of state-space control: different state-space realizations of the same transfer function; several canonical forms of state-space systems; controllability matrix.

State-Space Methods

- the state-space approach reveals internal system architecture for a given transfer function
- the mathematics is different: heavy use of linear algebra
- this is just a short introduction

State-Space Methods

Frequency-Domain vs. State-Space

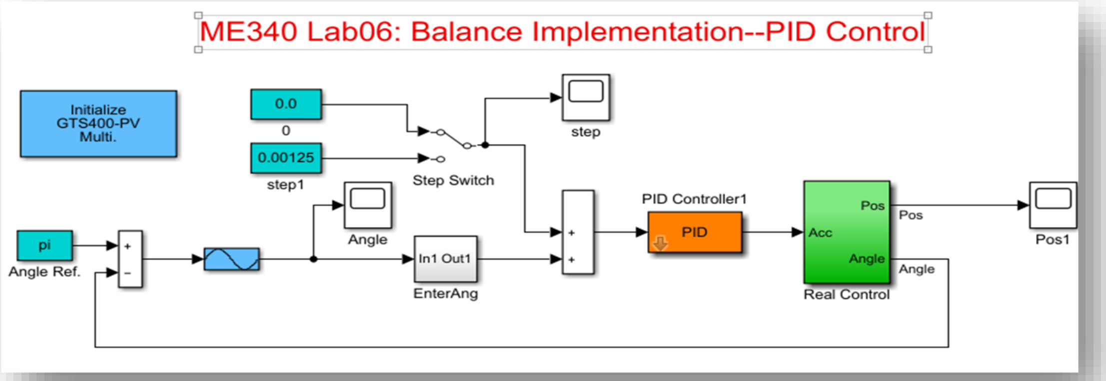
- 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
- 90% of current research in systems and control is in the state-space framework

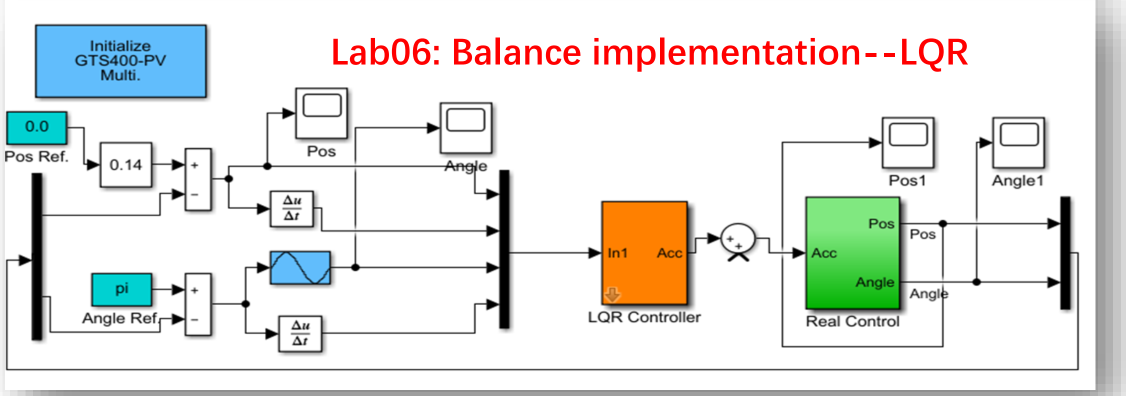
MIMO

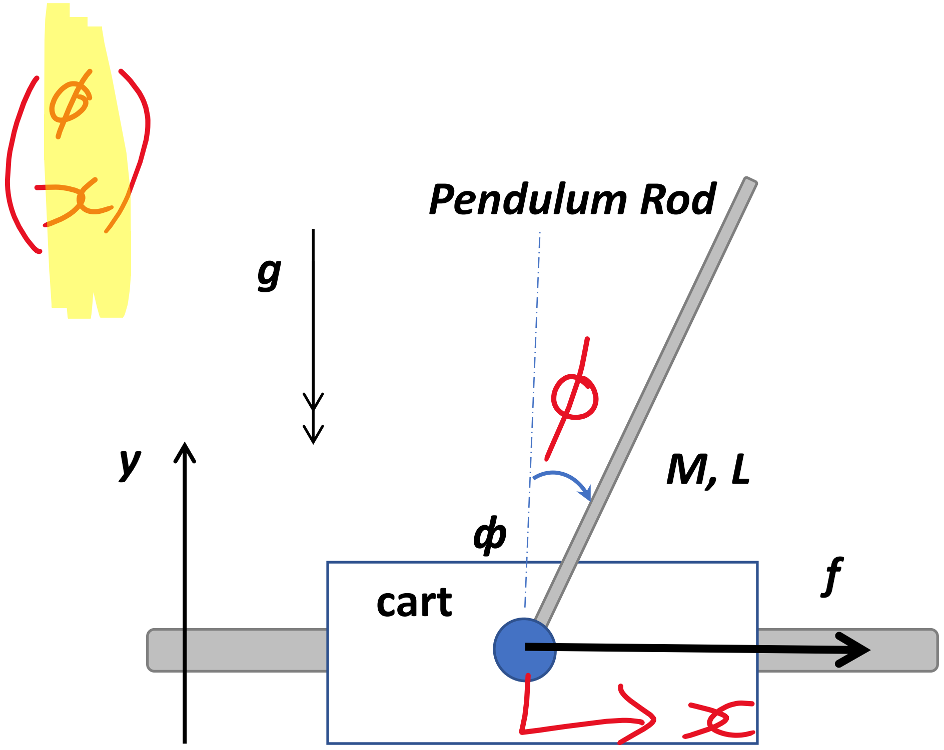
To be able to talk to control engineers and follow progress in the field, we need to know both methods and understand the connections between them.

Frequency-Domain vs. State-Space

- Frequency-domain methods: E.g., PID is a popular architecture
- State-space framework: E.g., LQR







A General State-space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

$$\text{output } y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p$$

$\dot{x} = f(x, u, t)$

state eqn $\longrightarrow$ $\dot{x} = \boxed{A}x + \boxed{B}u$

output eqn $\longrightarrow$ $y = Cx + Du$

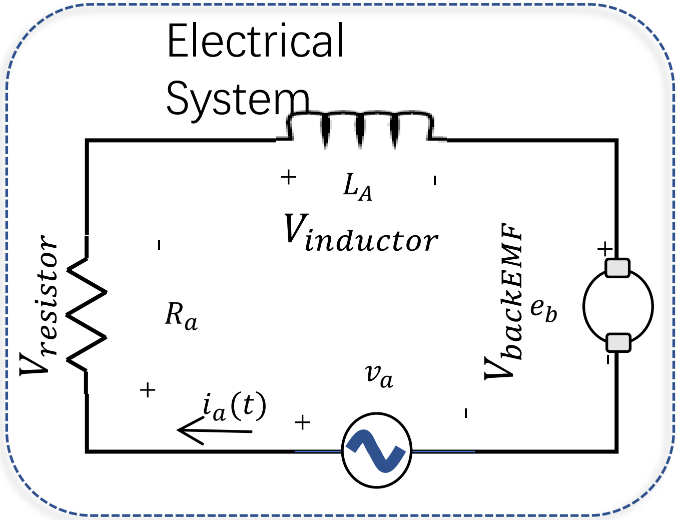
where:

A – system matrix $(n \times n)$ B – input matrix $(n \times m)$

C – output matrix $(p \times n)$ D – feedthrough matrix $(p \times m)$

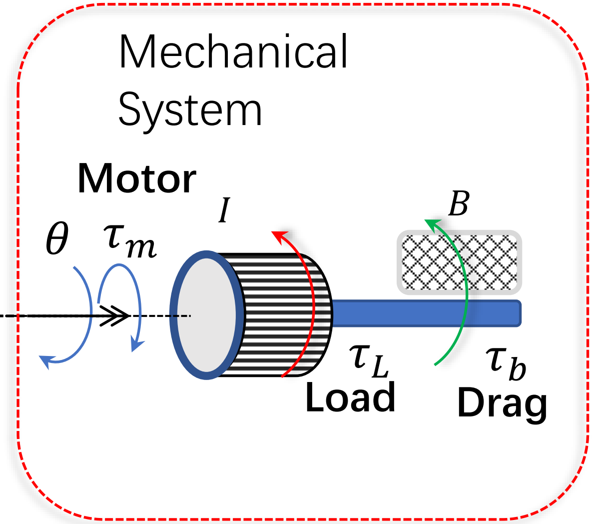
State-Space Model: Example

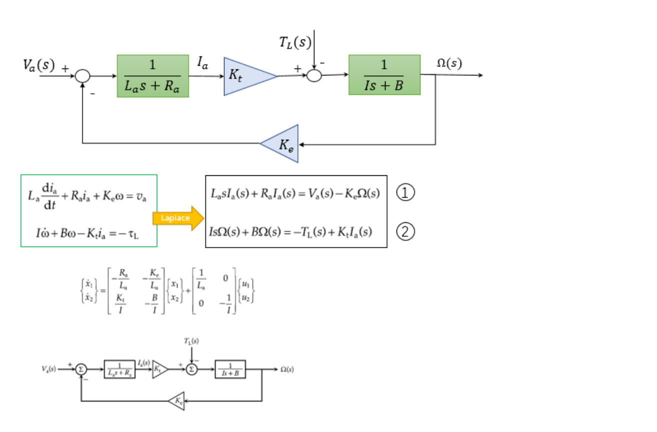
Modeling of Dynamic System



$$V_{inductor} = L_a \frac{di_a}{dt};$$
$$V_{resistor} = R_a i_a;$$
$$V_{backEMF} = K_e \dot{\theta}$$

$$\tau_m = K_t i_a;$$
$$\tau_b = B\dot{\theta};$$





Kirchhoff's Law

$$v_a = V_{inductor} + V_{resistor} + V_{backEMF}$$

$$L_a \frac{di_a}{dt} + R_a i_a + K_e \dot{\theta} = v_a$$

Newton's Law

$$I\ddot{\theta} = \tau_m - \tau_b - \tau_L$$
$$I\ddot{\theta} = K_t i_a - B\dot{\theta} - \tau_L$$
$$I\ddot{\theta} + B\dot{\theta} - K_t i_a = -\tau_L$$

State-space representation of Dynamic System

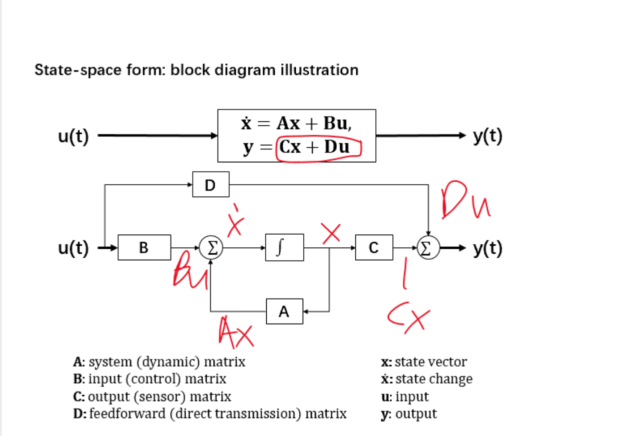
$$\dot{x}_1 = \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_e}{L_a}\omega + \frac{1}{L_a} v_a$$

$$\dot{x}_2 = \dot{\omega} = \frac{K_t}{I} i_a - \frac{B}{I}\omega - \frac{1}{I}\tau_L$$

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I}\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} + \begin{bmatrix}\frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

o/p eqn

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} + \begin{bmatrix}0 & 0\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$



State-Space Model: Comparison with Transfer Function Approach

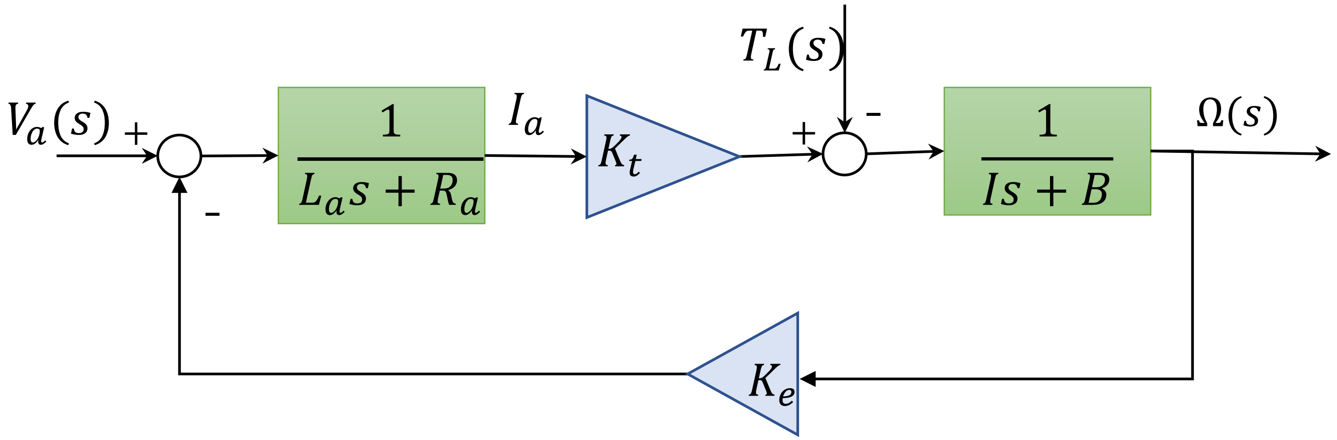
$$\begin{cases} L_a \dfrac{di_a}{dt} + R_a i_a + K_e\omega = v_a \\ I\dot{\omega} + B\omega - K_t i_a = -\tau_L \end{cases} \xrightarrow{\text{Laplace}} \begin{cases} L_a s I_a(s) + R_a I_a(s) = V_a(s) - K_e\Omega(s) \\ Is\Omega(s) + B\Omega(s) = -T_L(s) + K_t I_a(s) \end{cases}$$

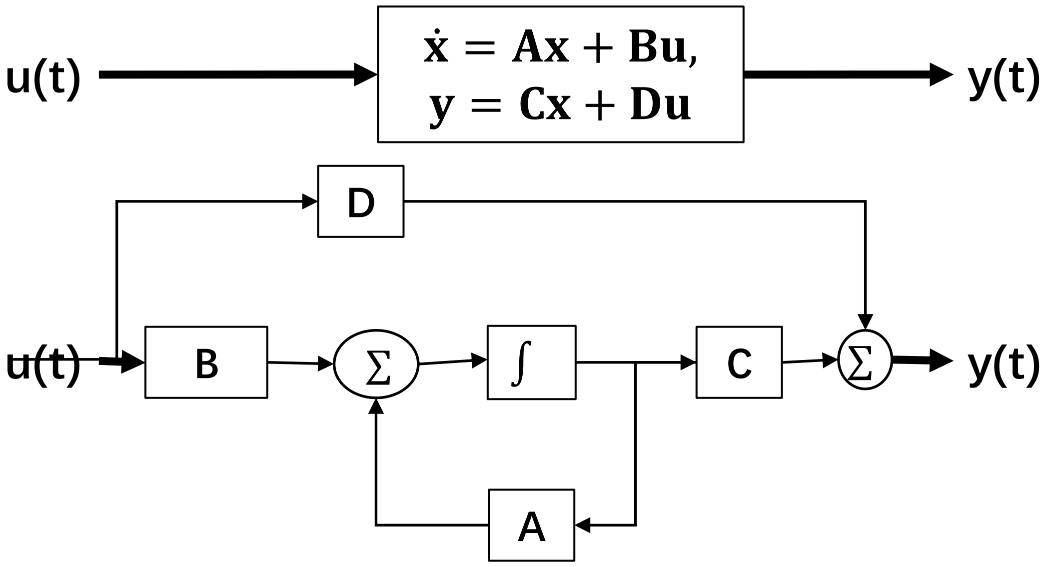
$$\frac{\Omega(s)}{V_a(s)} = \frac{(1/(L_a s + R_a))\cdot K_t \cdot (1/(Is+B))}{1 + (1/(L_a s + R_a))\cdot K_t \cdot (1/(Is+B))\cdot K_e} = \frac{K_t}{L_a I s^2 + \left(L_a B + R_a I\right)s + R_a B + K_t K_e}$$

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(1/(Is+B))}{1 - (1/(Is+B))\cdot(-K_e)\cdot(1/(L_a s + R_a))\cdot K_t} = -\frac{L_a s + R_a}{L_a I s^2 + (L_a B + R_a I)s + R_a B + K_t K_e}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\dfrac{R_a}{L_a} & -\dfrac{K_e}{L_a} \\ \dfrac{K_t}{I} & -\dfrac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{L_a} & 0 \\ 0 & -\dfrac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$





From State-Space to Transfer Function

Let us find the *transfer function* from u to y corresponding to the state-space model

$$u \longrightarrow \boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}} \longrightarrow y$$

- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$$

Recall matrix-vector multiplication:

$$\begin{aligned} \dot{x}_i &= (Ax)_i + (Bu)_i \\ &= \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k \end{aligned}$$

$$\begin{aligned} y_\ell &= (Cx)_\ell + (Du)_\ell \\ &= \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k \end{aligned}$$

From State-Space to Transfer Function

Now we take the Laplace transform:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k$$

$$\downarrow \mathscr{L}$$

$$sX_i(s) - x_i(0) = \sum_{j=1}^{n} a_{ij} X_j(s) + \sum_{k=1}^{m} b_{ik} U_k(s), \qquad i = 1, \dots, n$$

Write down in matrix-vector form:

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(Is - A)X(s) = x(0) + BU(s) \qquad (I \text{ is the } n \times n \text{ identity matrix})$$

$$X(s) = (Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)$$

$$y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k$$

$$\downarrow \mathscr{L}$$

$$Y_\ell(s) = \sum_{j=1}^{n} c_{\ell j} X_j(s) + \sum_{k=1}^{m} d_{\ell k} U_k(s), \qquad \ell = 1, \dots, p$$

Write down in matrix-vector form:

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C\left[(Is - A)^{-1} x(0) + (Is - A)^{-1} BU(s)\right] + DU(s) \\ &= C(Is - A)^{-1} x(0) + \left[C(Is - A)^{-1} B + D\right] U(s) \end{aligned}$$

To find the input-output t.f., set the IC to 0:

$$Y(s) = G(s)U(s), \qquad \text{where } G(s) = C(Is - A)^{-1} B + D$$

$$G(s) = \frac{Y(s)}{U(s)}$$

From State-Space to Transfer Function

The transfer function from u to y, corresponding to

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

is given by

$$\boxed{G(s) = C(Is - A)^{-1}B + D}$$

Observe that $G(s)$ contains information about the state-space matrices A, B, C, D!!

From State-Space to Transfer Function

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad & Y(s) &= G(s)U(s) \\ y &= Cx + Du & &= \left[C(Is - A)^{-1}B + D\right]U(s) \end{aligned}$$

Important!!

- $G(s)$ is *undefined* when the $n \times n$ matrix $Is - A$ is *singular* (or noninvertible), i.e., precisely when $\det(Is - A) = 0$
- since A is $n \times n$, $\det(Is - A)$ is a *polynomial* of degree n (the characteristic polynomial of A):

$$\det(Is - A) = \det \begin{pmatrix} s - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & s - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & s - a_{nn} \end{pmatrix},$$

  and its roots are the eigenvalues of A
- G is (open-loop) stable if all eigenvalues of A lie in LHP.

Example Compute *G(s)*

Consider the state-space model in Controller Canonical Form (CCF)*:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— this is a *single-input, single-output* (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

Let's compute the transfer function:

$$G(s) = C(Is - A)^{-1}B \qquad (D = 0 \text{ here})$$

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$$

* We will explain this terminology later.

Example Compute *G(s)*

$$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix} \quad \text{— how do we compute } (Is - A)^{-1}?$$

$$\underbrace{(Is - A)}_{M}^{-1} = \frac{1}{\det(M)} \operatorname{Adj}(M)$$

Review: Matrix Analysis

- Eigenvalue and Eigenvector

For a matrix $\boldsymbol{A}$, there exists a column matrix $\boldsymbol{v}$ such that $\boxed{\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}\lambda}$,
λ and $\boldsymbol{v}$ are the eigenvalue and the associated eigenvector of $\boldsymbol{A}$

determinant

Given matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if λ satisfies $\left|\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right| = 0$

such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \gamma \\ \eta \end{pmatrix} = \begin{pmatrix} \gamma \\ \eta \end{pmatrix}\lambda$,

$\boxed{(a-\lambda)(d-\lambda) - bc = 0 \text{ Characteristic equation}}$

then $\boxed{\begin{pmatrix} \gamma \\ \eta \end{pmatrix}} = \boldsymbol{v}$ is the eigenvector of $\boldsymbol{A}$ associated with the real $\boxed{\text{eigenvalue } \lambda}$

Example Compute *G(s)*

$Is - A = \begin{pmatrix} s & -1 \\ 6 & s+5 \end{pmatrix}$ — how do we compute $(Is - A)^{-1}$?

A useful formula for the inverse of a 2×2 matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \quad \Longrightarrow \quad M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying the formula, we get

$$\begin{aligned} (Is - A)^{-1} &= \frac{1}{\det(Is - A)} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \end{aligned}$$

Example Compute $G(s)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} G(s) &= C(Is - A)^{-1}B \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix} \\ G(s) &= \boxed{\frac{s+1}{s^2 + 5s + 6}} \end{aligned}$$

- the above state-space model is a *realization* of this t.f.
- note how coefficients 5 and 6 appear in both $G(s)$ and A!!

State-space Realizations of Transfer Functions

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{s+1}{s^2 + 5s + 6}$$

— at least in this example, information about the state-space model (A, B, C) is contained in $G(s)$.

Is this information *recoverable*? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

State-space Realizations of Transfer Functions

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$G(s) = \frac{s+1}{s^2 + 5s + 6}$$

— at least in this example, information about the state-space model (A, B, C) is contained in $G(s)$.

Is this information *recoverable*? — i.e., is there only one state-space realization of a given t.f.? Or are there many?

Answer: There are infinitely many!

State-space Realizations of Transfer Functions

CCF

Start with

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 1\\ -6 & -5\end{pmatrix}}_{A}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{B}u, \qquad y = \underbrace{\begin{pmatrix}1 & 1\end{pmatrix}}_{C}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

and consider a new state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T = \begin{pmatrix}0 & -6\\ 1 & -5\end{pmatrix}, \quad \bar{B} = C^T = \begin{pmatrix}1\\ 1\end{pmatrix}, \quad \bar{C} = B^T = \begin{pmatrix}0 & 1\end{pmatrix}$$

This is a different state-space model!

Claim: The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

Proof:

$$\begin{aligned}\bar{C}(Is - \bar{A})^{-1}\bar{B} &= B^T\left(Is - A^T\right)^{-1}C^T\\ &= B^T\left[(Is - A)^T\right]^{-1}C^T\\ &= B^T\left[(Is - A)^{-1}\right]^T C^T\\ &= \left[C(Is - A)^{-1}B\right]^T\\ &= C(Is - A)^{-1}B\end{aligned}$$

State-space Realizations of Transfer Functions

The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

But the state-space model is now in the Observer Canonical Form (OCF):

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{OCF}}$$

More Realizations

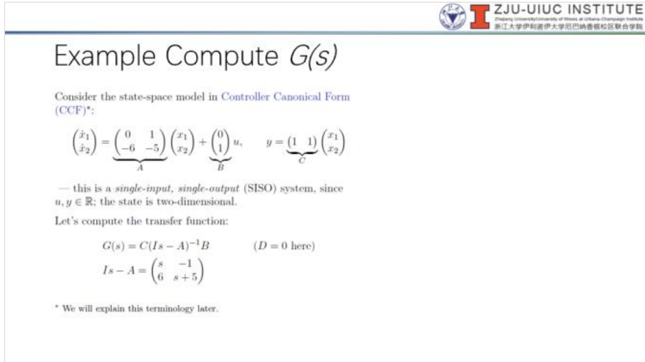
Yet another realization of $G(s) = \dfrac{s+1}{s^2+5s+6}$ can be extracted from the partial-fractions decomposition:

$$G(s) = \frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}.$$

This is the Modal Canonical Form (MCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} \text{Then} \quad C(Is-A)^{-1}B &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} s+3 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+3} \\ \frac{1}{s+2} \end{pmatrix} = \frac{2}{s+3} - \frac{1}{s+2} \end{aligned}$$



Bottom Line of State-Space Realization

- A given transfer function G(s) can be realized using infinitely many state-space models
- Certain properties make some realizations preferable
- One such property is **controllability**

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2B \,|\, \dots \,|\, A^{n-1}B\right]$$

— recall that A is $n \times n$ and B is $n \times 1$, so $\mathcal{C}(A, B)$ is $n \times n$;
— the controllability matrix only involves A and B, not C

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

(This definition is only true for the single-input case; the multiple-input case involves the *rank* of $\mathcal{C}(A, B)$.)

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B\right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}$(A,B)

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A,B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C}(A,B) = [B \mid AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies \mathcal{C}(A,B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.