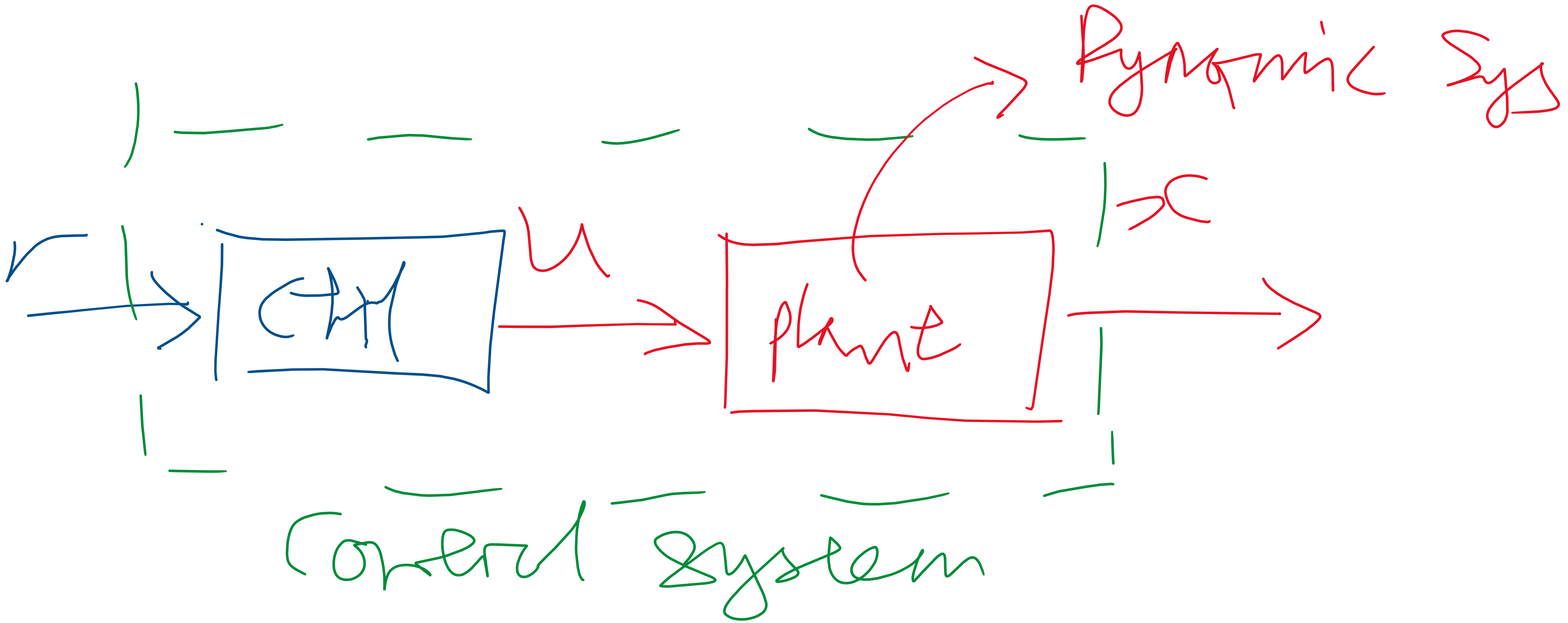
Dynamic Sys
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Control System

Control Systems

ECE 486

Liangjing Yang
Assistant Professor, ZJU-UIUC Institute
liangjingyang@intl.zju.edu.cn

ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 486 Control Systems

Lecture 02: Dynamic Systems & Linearization

Liangjing Yang

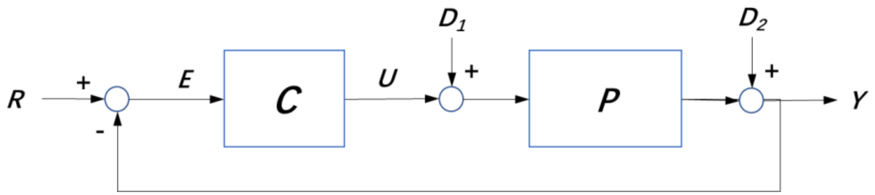
Assistant Professor, ZJU-UIUC Institute

liangjingyang@intl.zju.edu.cn

Recap: Lecture 01

- A **control system** is designed to achieve a targeted output by generating the appropriate inputs in a dynamical environment (within specified performance criteria)
- **Closed loop** control incorporate feedback to achieve reliability and accuracy at the expense of implementation cost and design complexity compared to **open loop control**
- **Feedback control** aims to minimize error between desired state (reference) and actual state (output) of plant by feeding sensor measurement to controller designed with the control law to generate plant input

	Open		Closed	
Cost	Economical	🙂	Expensive	☹
Design	Simple	🙂	Complex	☹
Accuracy	Erroneous	☹	Accurate	🙂
Reliability	Unreliable	☹	Reliable	🙂

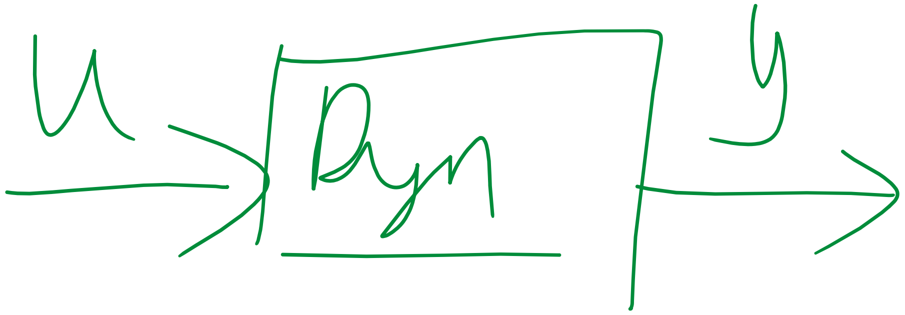


Lecture Overview

- Study **dynamic systems** and get comfortable with few examples
- Represent dynamical systems using **state-space model**
- **Linearization** for solving non-linear problems

Dynamic Systems

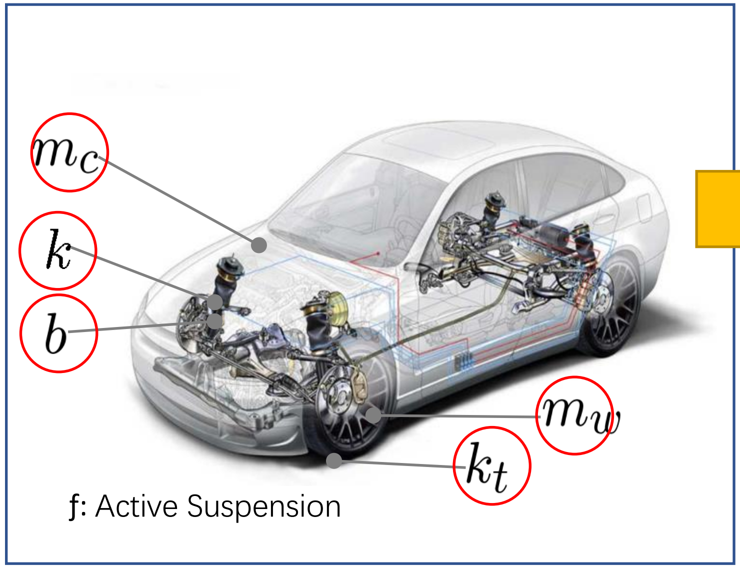
- **Dynamic system**: consists of <u>components</u> with <u>inputs and outputs</u> <u>which are time-dependent functions</u>.
  - Components like amplifiers, delay lines, integrators, etc
  - The time dependent functions are signal
  - Input/output relationship governed by law of physics

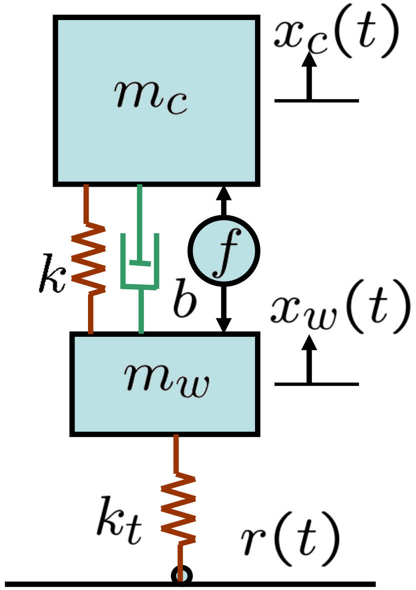


Dynamic Systems

- **Dynamic system**: consists of <u>components</u> with <u>inputs and outputs</u> related to <u>time varying function</u>

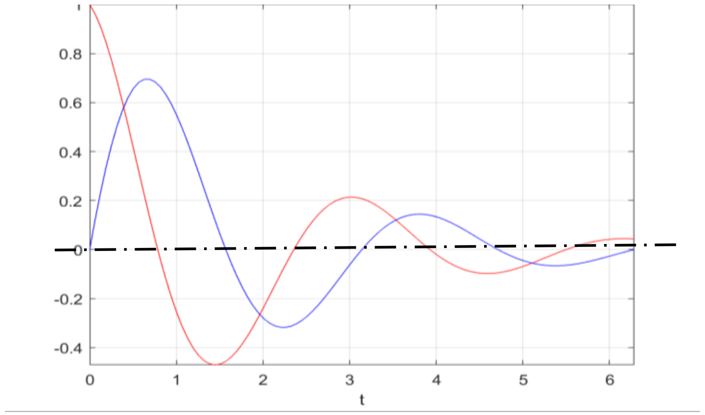
Example of Dynamic System:
Mechanical System for active suspension





$$\boxed{\begin{aligned} m_c\ddot{x}_c + k(x_c - x_w) + b(\dot{x}_c - \dot{x}_w) &= f \\ m_w\ddot{x}_w + k(x_w - x_c) + k_t(x_w - r) + b(\dot{x}_w - \dot{x}_c) &= -f \end{aligned}}$$

Input, $r(t)$ → Mechanical System → Output, $x_c(t)$



ODE: Mechanical Example

u, input

Newton's 2nd Law $\sum F = m\ddot{x}$

Net total Force= Drag force+ Spring force+ External Force

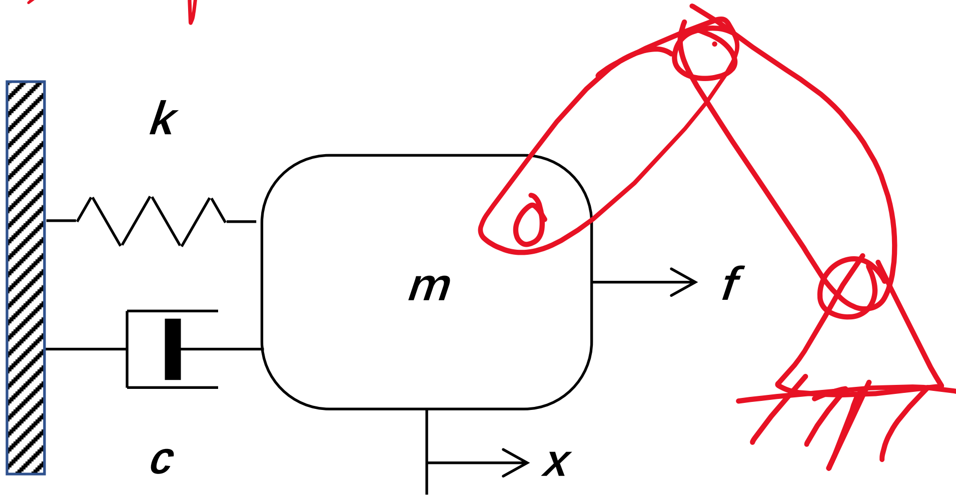
$F_{drag} = -c\dot{x}$

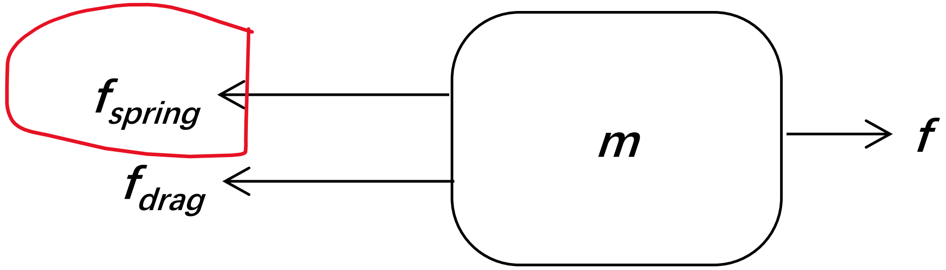
Stokes's law

$F_{Spring} = -kx$

Hooke's law

$$\boxed{m\ddot{x} = -c\dot{x} - kx + f(t)}$$





Free Body Diagram

ODE: Electrical

Kirchhoff's 2nd Law $\sum V_j = 0$

$$\sum V_k - V_s = 0, \quad \sum V_k = V_s$$

j^{th} circuit element *k^{th} component of dynamic system* *s ideal source*

Ideal Voltage Source= V_Inductor+ V_Resistor+ V_Capacitor

Ideal Voltage Source

$V_L = -L\ddot{q}$

Lenz's Law law

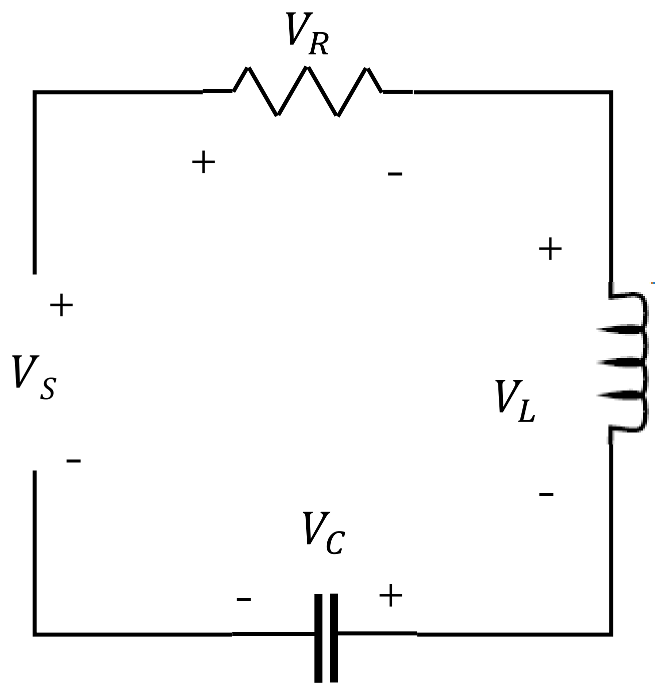
$V_R = R\dot{q}$

Ohm's law

$V_C = \dfrac{1}{C} q$

Gauss's law

$$\boxed{V = L\ddot{q} + R\dot{q} + \frac{1}{C} q}$$

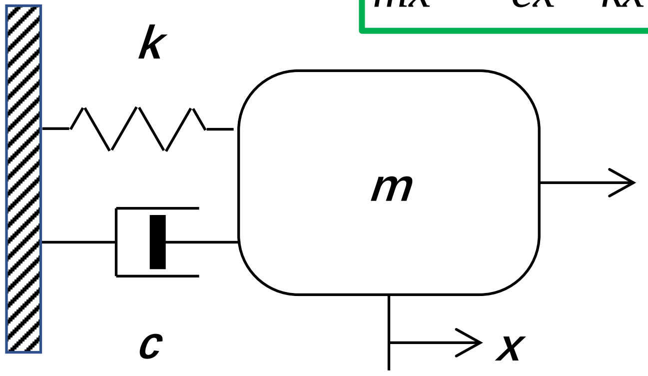


State-space form

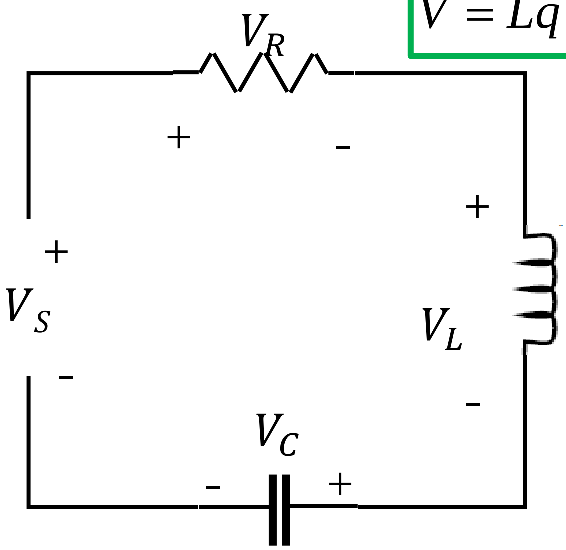
- Dynamic systems are generally modeled as systems of ODEs
- ODEs not always convenient for the purpose of analysis, simulation and control
- One option is to represent a systems of ODEs (of various order) as a larger system of first order ODEs

Dynamic Systems

$$\boxed{m\ddot{x} = -c\dot{x} - kx + f(t)}$$



$$\boxed{V = L\ddot{q} + R\dot{q} + \tfrac{1}{C}q}$$



State-space form

- State variables
  - smallest set of independent variables that completely describe the state of a system
  - Not unique for a given system
  - # of state variables = # of initial conditions needed to solve the system model
  - exactly those variables for which initial conditions are required

State-space form

- State variable equations
  - # of state variable equation = # of state variables
  - Each state variable equation is a first order ODE
  - Left side is a first-derivative of a state variable
  - Right side is an algebraic function of the state variables,
  - Suppose a dynamic system has n state variables $x_1, x_2, \ldots, x_n$ and m inputs $u_1, u_2, \ldots, u_m$

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \dot{x}_2 = f_2(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \qquad \ldots \\ \dot{x}_n = f_n(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \end{cases}$$

State-space form

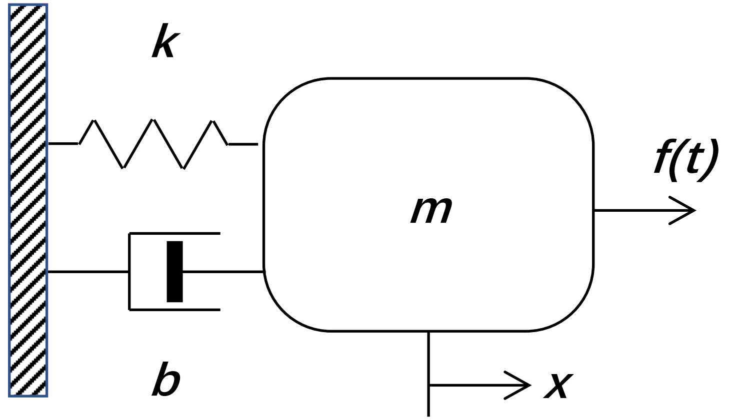
Mechanical Dynamic Systems

$$\ddot{x} = \frac{f}{m} - \frac{b\dot{x}}{m} - \frac{kx}{m}$$

$$\boxed{m\ddot{x} = -c\dot{x} - kx + f(t)}$$

- State variable equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$



$x(0)$ and $\dot{x}(0)$ are two required initial conditions to solve the differential equation

$x(0), \dot{x}(0)$

Chosen **state variables** $\boxed{(x_1,\ x_2) = (x,\ \dot{x})}$

$x_1 = x$ and $x_2 = \dot{x}$

but we are interested to write in the form, $\begin{cases} \dot{x}_1 = \boxed{x_2} \\ \dot{x}_2 = \boxed{\ddot{x}} \end{cases}$

Hence, **state-variable equations** can be written as:

$$\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}[-bx_2 - kx_1 + f(t)] \end{cases}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} =$$

$x(t=0) = 0$

$\dot{x}(t=0) = 0$

State-space form

Mechanical Dynamic Systems

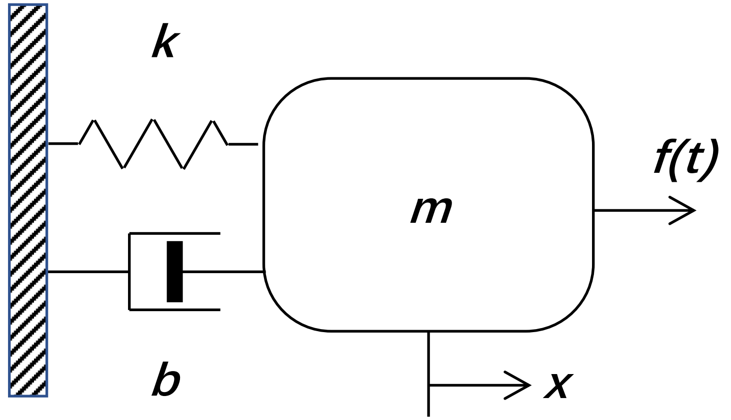
$\dot{x}_1 = 0x_1 + 1x_2 + 0f(t)$

- State equation

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

$\dot{x}(0)$ and $\dot{x}(0)$ are two required initial conditions to solve the differential equation

$$\boxed{m\ddot{x} = -c\dot{x} - kx + f(t)}$$



Chosen **state variables** $(\boxed{x_1,\ x_2) = (x,\ \dot{x})}$
i.e. $x_1 = x$ and $x_2 = \dot{x}$
but we are interested to write in the form, $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} \end{cases}$

Hence, **state-variable equations** can be written as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m}[-bx_2 - kx_1 + f(t)] \end{cases}$$

Matrix Form

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f(t)$$

State equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{u}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad \boldsymbol{u} = f(t)$$

State-space form

- Example 8.1: State variable equation

Given *K*= 10; *b*=0.5; *m*=2

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

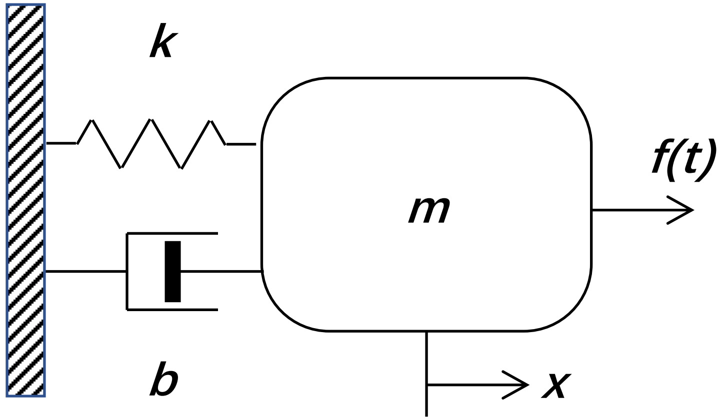
$$\boxed{2\ddot{x} + 0.5\dot{x} + 10x = f(t)}$$

$\dot{x}(0)$ and $\dot{x}(0)$ are two required initial conditions to solve the differential equation

Chosen state variables $\boxed{(x_1,\ x_2) = (x,\ \dot{x})}$

$x_1 = x$ and $x_2 = \dot{x}$ but we are interested to write in the form,

$$\begin{cases} \dot{x}_1 = \cdots \\ \dot{x}_2 = \cdots \end{cases}$$



$$\boxed{\dot{x}_2 = \ddot{x} \underset{\text{equation of motion}}{\overset{\text{Using the}}{=}} \tfrac{1}{2}\left[-0.5\dot{x} - 10x + F(t)\right] \underset{x_1=x,\, x_2=\dot{x}}{\overset{\text{Use state variables}}{=}} \tfrac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right]}$$

With this, the state-variable equations can be written as

$$\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \tfrac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right] \end{cases}}$$

State-space form

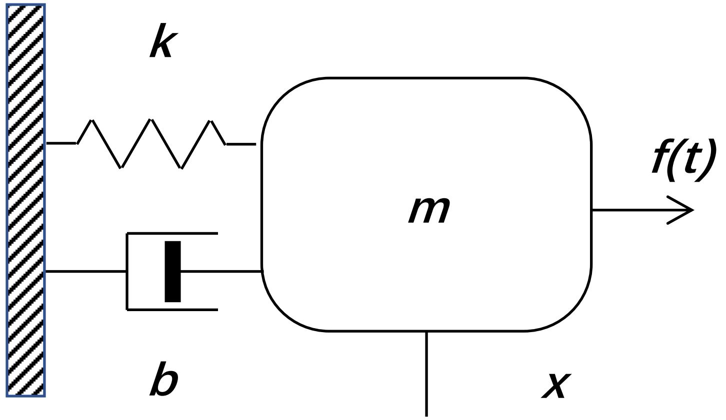
- Example 8.1: State equation

Given *K*= 10; *b*=0.5; *m*=2

$m\ddot{x} + b\dot{x} + kx = f(t)$

$\boxed{2\ddot{x} + 0.5\dot{x} + 10x = f(t)}$

$\dot{x}(0)$ and $\dot{x}(0)$ are two required initial conditions to solve the differential equation



$\dot{x}(0)$ and $\dot{x}(0)$ are two required initial conditions to solve the differential equation

Chosen state variables $\boxed{(x_1,\ x_2) = (x,\ \dot{x}),}$

$$\dot{x}_2 = \ddot{x} \underset{\text{equation of motion}}{\overset{\text{Using the}}{=}} \tfrac{1}{2}\left[-0.5\dot{x} - 10x + F(t)\right] \underset{x_1=x,\ x_2=\dot{x}}{\overset{\text{Use state variables}}{=}} \tfrac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right]$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{1}{4} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} F(t)$$

State variable equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right] \end{cases}$$

State equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,\ \ \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix},\ \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{1}{4} \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},\ \ u = F(t)$$

State-space form

- Output equation
  - **(Measured) Outputs** are physical quantity that are being measured

$$\begin{cases} y_1 = g_1(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ y_2 = g_2(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ \dots \\ y_n = g_n(x_1, \dots, x_n; u_1, \dots, u_m; t) \end{cases}$$

State-space form

- Output equation
  - **(Measured) Outputs** are physical quantity that are being measured

$$\begin{cases} y_1 = g_1(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ y_2 = g_2(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ \quad\quad\quad \cdots \\ y_n = g_n(x_1, \dots, x_n; u_1, \dots, u_m; t) \end{cases}$$

If at least one of the functions g_1, g_2, … , g_p is nonlinear, the output equation is expressed in vector form

$$\mathbf{y} = \boldsymbol{g}(\mathbf{x}, \mathbf{u}, t)$$

where,

$$\mathbf{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdots \\ y_p \end{Bmatrix}_{p\times 1}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix}_{m\times 1}, \quad \mathbf{g} = \begin{Bmatrix} g_1 \\ g_2 \\ \cdots \\ g_p \end{Bmatrix}_{p\times 1}$$

State-space form

- Output equation
  - **(Measured) Outputs** are physical quantity that are being measured

$$\begin{cases} y_1 = g_1(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ y_2 = g_2(x_1, \dots, x_n; u_1, \dots, u_m; t) \\ \dots \\ y_n = g_n(x_1, \dots, x_n; u_1, \dots, u_m; t) \end{cases}$$

However, if all are linear, we can express in,

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\begin{cases} y_1 = c_{11}x_1 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1m}u_m \\ y_2 = c_{21}x_1 + \cdots + c_{2n}x_n + d_{21}u_1 + \cdots + d_{2m}u_m \\ \dots \\ y_p = c_{p1}x_1 + \cdots + c_{pm}x_n + d_{p1}u_1 + \cdots + d_{pm}u_m \end{cases}$$

where **C** is the output matrix and **D** is the feedforward (direct transmission) matrix

State-space form

- State Equation and Output Equation combines to represents the **State-space form**

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

where

A: state (system) matrix
B: input (control) matrix
C: output matrix
D: feedforward matrix

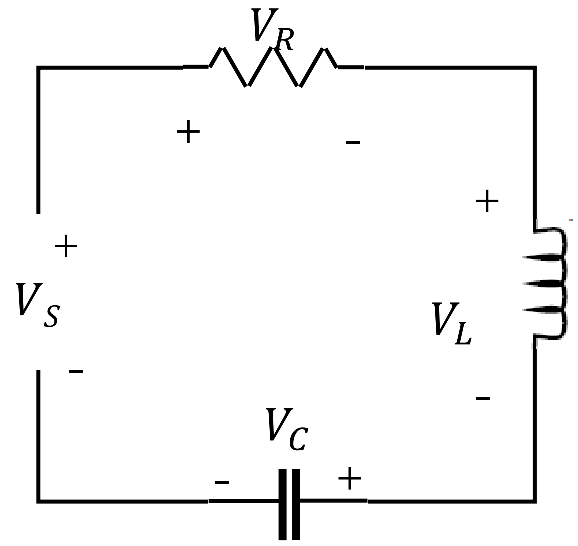
x: state vector
$\dot{\mathbf{x}}$**:** state change
u: input
y: output

State-space form

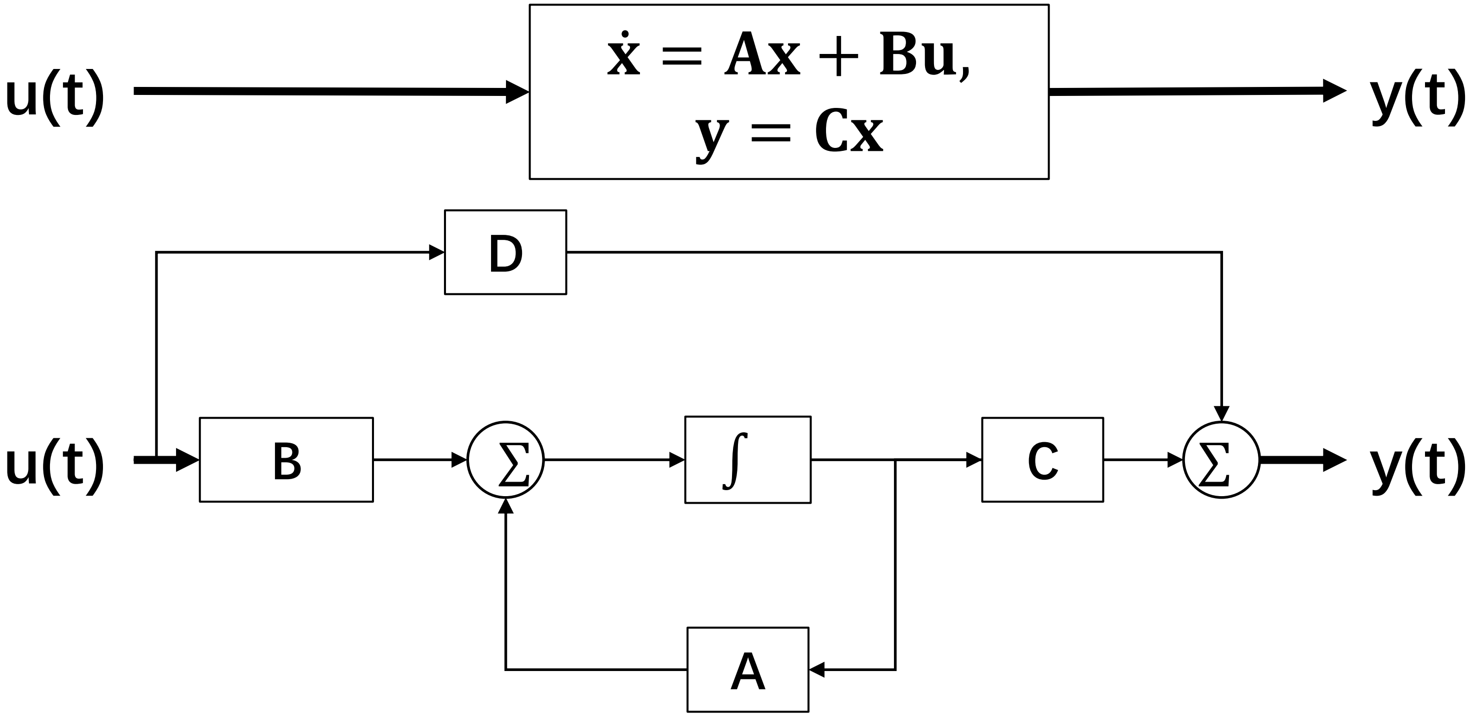
- Try representing the state-space form for the electrical domain

Dynamic Systems with electrical components

$$\boxed{V_s = L\ddot{q} + R\dot{q} + \tfrac{1}{C}q}$$



State-space form: block diagram illustration



A: system (dynamic) matrix
B: input (control) matrix
C: output (sensor) matrix
D: feedforward matrix

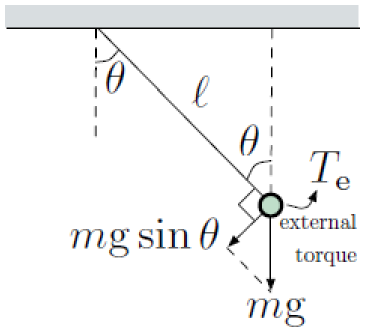
x: state vector
ẋ: state change
u: input
y: output

Congratulation on acquiring new a tool for modeling and analysis!

But……

Is this sufficient to solve real world problem?

Pendulum Example



Newton's 2nd law (rotational motion):

$$\underbrace{T}_{\text{total torque}} = \underbrace{J}_{\text{moment of inertia}} \underbrace{\alpha}_{\text{angular acceleration}}$$

$$= \text{pendulum torque} + \text{external torque}$$

$$\text{pendulum torque} = \underbrace{-mg\sin\theta}_{\text{force}} \cdot \underbrace{\ell}_{\text{lever arm}}$$

$$\text{moment of inertia } J = m\ell^2$$

$$-mg\ell\sin\theta + T_e = m\ell^2\ddot{\theta}$$

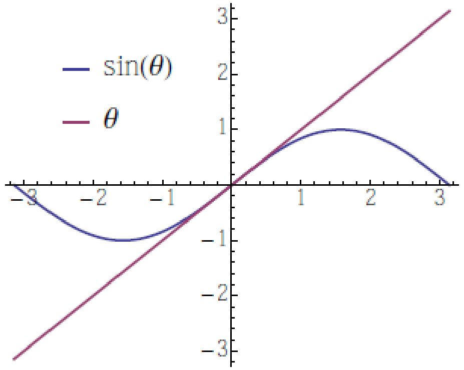
$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e$$

(nonlinear equation)

Pendulum Example

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e \qquad \text{(nonlinear equation)}$$

For *small* θ, use the approximation $\sin\theta \approx \theta$



$$\ddot{\theta} = -\frac{g}{\ell}\theta + \frac{1}{m\ell^2}T_e$$

State-space form: $\theta_1 = \theta,\ \ \theta_2 = \dot{\theta}$

$$\dot{\theta}_2 = -\frac{g}{\ell}\theta + \frac{1}{m\ell^2}T_e = -\frac{g}{\ell}\theta_1 + \frac{1}{m\ell^2}T_e$$

$$\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{g}{\ell} & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{m\ell^2}\end{pmatrix}T_e$$

Linearization

Non-linear $\dot{x}(t) = f(x(t), u(t))$

Linear $\dot{x}(t) = Ax(t) + Bu(t)$

Linearization

Taylor series expansion:

Linear Portion

$$f(x) = \boxed{f(x_0) + f'(x_0)(x - x_0)} + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

$$\approx f(x_0) + f'(x_0)(x - x_0) \qquad \text{linear approximation around } x = x_0$$

Control systems are generally *nonlinear*:

$$\dot{x} = f(x, u) \qquad \text{nonlinear state-space model}$$

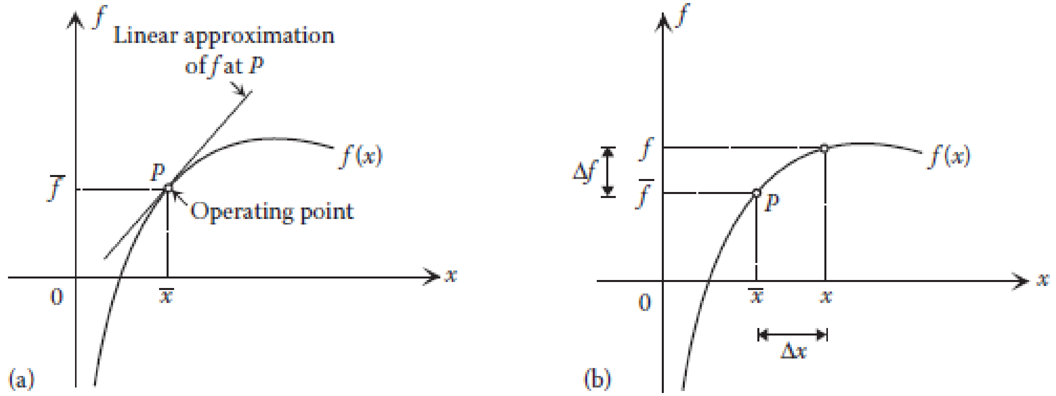
$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Assume $x = 0, u = 0$ is an *equilibrium point*: $f(0, 0) = 0$

This means that, when the system is at rest and no control is applied, the system does not move.

For a nonlinear function f of a single variable, the linearization of $f(x)$ will be done at on operating point P: $(\bar{x}, \bar{f})$ with incremental time-varying quantities $\Delta x(t)$ and $\Delta f(t)$

where $x(t) = \bar{x} + \Delta x(t)$ and $f(t) = \bar{f} + \Delta f(t)$ are nominal values of x and f, respectively.



Linearization

Linear approx. around $(x, u) = (0, 0)$ to all components of f:

$$\dot{x}_1 = f_1(x, u), \qquad \ldots, \qquad \dot{x}_n = f_n(x, u)$$

For each $i = 1, \ldots, n$,

$$f_i(x, u) = \underbrace{f_i(0,0)}_{=0} + \frac{\partial f_i}{\partial x_1}(0,0)x_1 + \ldots + \frac{\partial f_i}{\partial x_n}(0,0)x_n + \frac{\partial f_i}{\partial u_1}(0,0)u_1 + \ldots + \frac{\partial f_i}{\partial u_m}(0,0)u_m$$

Linearized state-space model:

$$\dot{x} = Ax + Bu, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=0\\u=0}}, \quad B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=0\\u=0}}$$

Important: since we have ignored the higher-order terms, this linear system is only an *approximation* that holds only for *small deviations* from equilibrium.

Linearization

Original nonlinear state-space model:

$$\dot\theta_1 = f_1(\theta_1, \theta_2, T_e) = \theta_2 \qquad \text{— already linear}$$

$$\dot\theta_2 = f_2(\theta_1, \theta_2, T_e) = -\frac{g}{\ell}\sin\theta_1 + \frac{1}{m\ell^2}T_e$$

Linear approx. of f_2 around equilibrium $(\theta_1, \theta_2, T_e) = (0, 0, 0)$:

$$\frac{\partial f_2}{\partial \theta_1} = -\frac{g}{\ell}\cos\theta_1 \qquad \frac{\partial f_2}{\partial \theta_2} = 0 \qquad \frac{\partial f_2}{\partial T_e} = \frac{1}{m\ell^2}$$

$$\left.\frac{\partial f_2}{\partial \theta_1}\right|_0 = -\frac{g}{\ell} \qquad \left.\frac{\partial f_2}{\partial \theta_2}\right|_0 = 0 \qquad \left.\frac{\partial f_2}{\partial T_e}\right|_0 = \frac{1}{m\ell^2}$$

Linearized state-space model of the pendulum:

$$\dot\theta_1 = \theta_2$$

$$\dot\theta_2 = -\frac{g}{\ell}\theta_1 + \frac{1}{m\ell^2}T_e$$

valid for *small* deviations from equ.

Linearization

Procedures

- Start from nonlinear state-space model

$$\dot{x} = f(x, u)$$

- Find equilibrium point (x_0, u_0) such that $f(x_0, u_0) = 0$

  Note: different systems may have different equilibria, not necessarily $(0, 0)$, so we need to shift variables:

$$\underline{x} = x - x_0 \qquad \underline{u} = u - u_0$$
$$\underline{f}(\underline{x}, \underline{u}) = f(\underline{x} + x_0, \underline{u} + u_0) = f(x, u)$$

  Note that the transformation is *invertible*:

$$x = \underline{x} + x_0, \qquad u = \underline{u} + u_0$$

- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$

$$\begin{aligned} \dot{\underline{x}} &= \dot{x} \qquad\qquad (x_0 \text{ does not depend on } t) \\ &= f(x, u) \\ &= \underline{f}(\underline{x}, \underline{u}) \end{aligned}$$

  — equivalent to original system

- The transformed system is in equilibrium at $(0, 0)$:

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

- Now linearize:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0 \\ u=u_0}}, \quad B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0 \\ u=u_0}}$$

Wrap up

- Dynamic Systems

Input, $r(t)$ → Dynamic System → Output, $x_c(t)$

$$\dot{x}(t) = f(x(t), u(t))$$

- State-space

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases}$$

where

A: state (system) matrix
B: input control matrix
C: output matrix
D: feedforward matrix

x: state vector
$\dot{\mathbf{x}}$**:** state change
u: input
y: output

- Linearization

Non-linear $\dot{x}(t) = f(x(t), u(t))$

Linear $\dot{x}(t) = Ax(t) + Bu(t)$