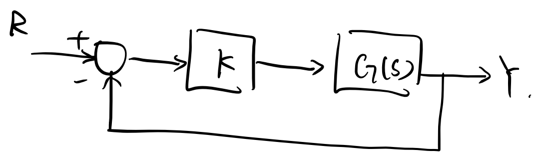
Review: Frequency Domain Design Method

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given K is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given
    - K is stabilizing from frequency response data

Nyquist criterion- A frequency-domain substitute for the Routh–Hurwitz criterion

Nyquist Stability Criterion.



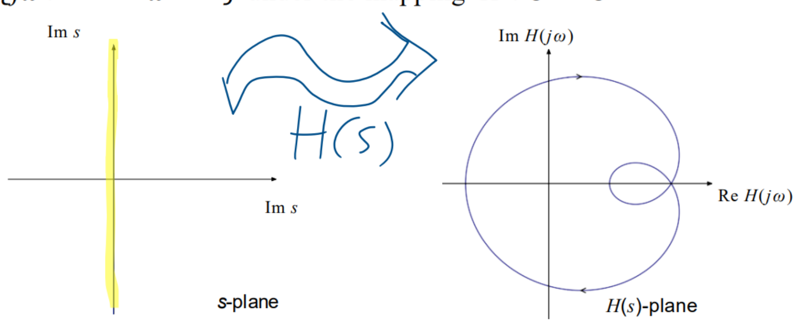
Goal : count # of RHP poles of $H_{cl} = \frac{KG}{1+KG}$.

Nyquist plot: $\text{Im}\{H(s)\}$ v.s. $\text{Re}\{H(s)\}$; for $s=j\omega$, (ω from $-\infty$ to ∞.)

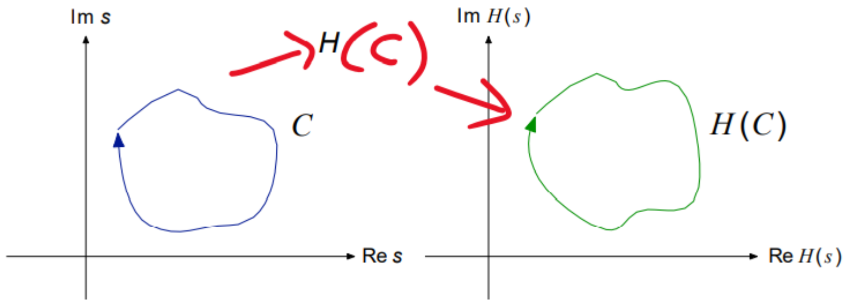
Nyquist Plot: Mapping of the s-Plane

- View the Nyquist plot of H as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \to \mathbb{C}$

★ For ω from $-\infty$ to ∞: Nyquist plot symmetric about real axis.

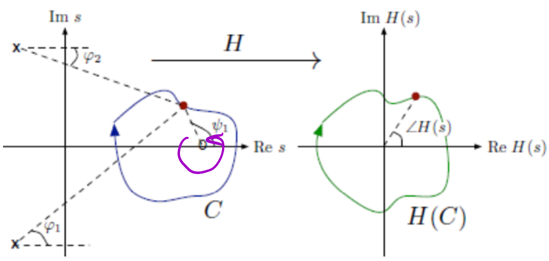


If we choose any closed curve (or *contour*) C on the left, it will get mapped by H to some other curve (contour) on the right:



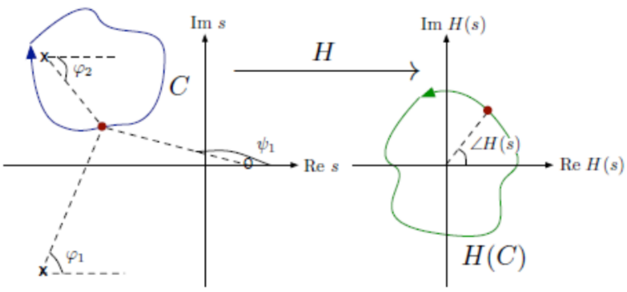
Phase of H: $\angle H(s) = \angle \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = \underbrace{\sum_{i}^{m} \psi_i}_{\angle zero} - \underbrace{\sum_{j}^{n} \varphi_j}_{\angle pole} = \sum \angle zeros - \sum \angle poles.$

I. Contour encircles a zero:



1. φ_1, φ_2 return to original value.
2. ψ_1 net change of $-360°$.
   $\Rightarrow \angle H(s)\ -360°$.
   ψ_1, $\angle H(s)$ same direction.

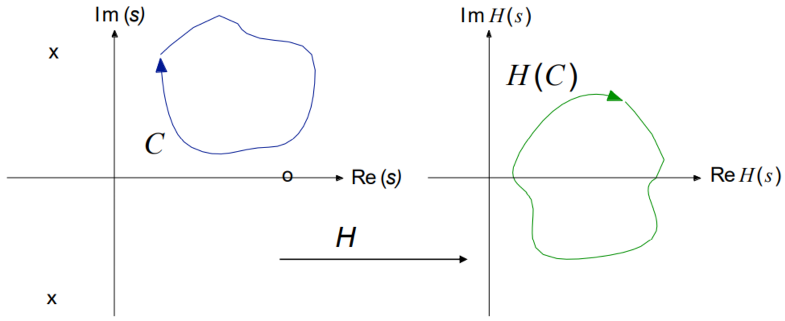
I. Contour encircles a pole.



1. φ_1, φ_2 return.
2. ψ net change: $-360°$
$\Rightarrow \angle H(s)$ $-360°$.

H(C) encircles the origin once counterclockwise.

II. Contour encircles no poles, no zeros.



- ϕ_1, ϕ_2, ψ_1 all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin

The Argument Principle.

The Argument Principle. Let C be a closed, clockwise ↻ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of C under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$

Then:

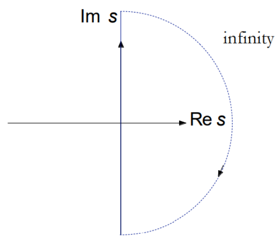
H(s) on Nyquist plot.

#(clockwise encirclements ↻ of 0 by $H(C)$)
$= $ #(zeros of $H(s)$ inside C) $-$ #(poles of $H(S)$ inside C).

More succinctly,

$$N = Z - P$$

- If $N < 0$, it means that $H(C)$ encircles the origin counterclockwise (↺).
- We do not want C to pass through any pole of H because then $H(C)$ would not be defined.
- We also do not want C to pass through any zero of H because then $0 \in H(C)$, so #(encirclements) is not well-defined.



C: the whole Im axis + path around infinity.

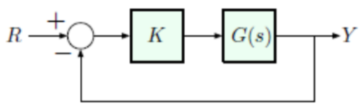
$H(C)$ = Nyquist plot of H

By Argument Principle:

N = # ① (CW ↻ of 0 by Nyquist plot of $1+KG(s)$).
② ↻ of -1 by Nyquist Plot of $KG(s)$.
③ CW ↻ of $-1/K$ by Nyquist plot of $G(s)$.

$\boxed{N = Z - P.}$

Z = # ① (zeros of $1+KG(s)$ inside C)
② # (RHP zeros of $1+KG$).
③ # (RHP closed-loop poles)



$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \le \deg(p)$$

$$1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}$$

$$\text{closed-loop t.f.} = \frac{KG(s)}{1+KG(s)} = \frac{Kq(s)}{p(s)+Kq(s)}$$

P = # ① (poles of $1+KG(s)$ inside C).
② # (RHP poles of $1+KG(s)$)
③ # (RHP roots of $p(s)$)
④ # (RHP open-loop poles).

The Nyquist Thm: N = Z − P.

Nyquist Stability Criterion:

The Nyquist Plot of $G(s)$ CCW $\circlearrowleft$ $-\frac{1}{K}$ for P times, where $P = Z - N$.

$$\Updownarrow$$

$$\boxed{\text{stable.}}$$

$$\Updownarrow$$

$$N = -P$$

Example 1

$$\boxed{G(s) = \frac{1}{(s+1)(s+2)}}$$ (no open-loop RHP poles)

Characteristic equation:

$$(s+1)(s+2) + K = 0 \quad \Longleftrightarrow \quad s^2 + 3s + K + 2 = 0$$

From Routh, we already know that the closed-loop system is stable for $K > -2$.

We will now reproduce this answer using the Nyquist criterion.

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph $\text{Im}\, G(j\omega)$ vs. $\text{Re}\, G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

$$(\text{Re}\, G(j\omega), \text{Im}\, G(j\omega)), \qquad -\infty < \omega < \infty$$

- Symmetry:

$$G(-j\omega) = \overline{G(j\omega)}$$

— Nyquist plots are always *symmetric w.r.t. the real axis*!!

Example 1

$$= \frac{1}{(j\omega)^2+3(j\omega)+2} = \frac{1}{2}\cdot\frac{1}{\left(\frac{j\omega}{\sqrt{2}}\right)^2+\frac{3}{2}(j\omega)+1}, \quad \omega_n=\sqrt{2}. \quad 2\zeta\omega_n=\frac{3}{2} \Rightarrow \zeta=\frac{3}{4}\times\frac{1}{\sqrt{2}} = \frac{3}{8}\sqrt{2}.$$

$$\boxed{G(s) = \frac{1}{(s+1)(s+2)}}$$

(no open-loop RHP poles)

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph $\operatorname{Im} G(j\omega)$ vs. $\operatorname{Re} G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

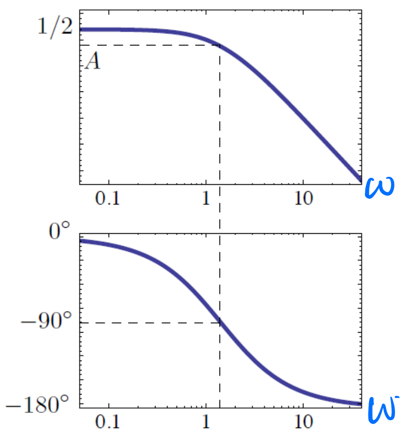
$$(\operatorname{Re} G(j\omega), \operatorname{Im} G(j\omega)), \qquad -\infty < \omega < \infty$$

- Symmetry:

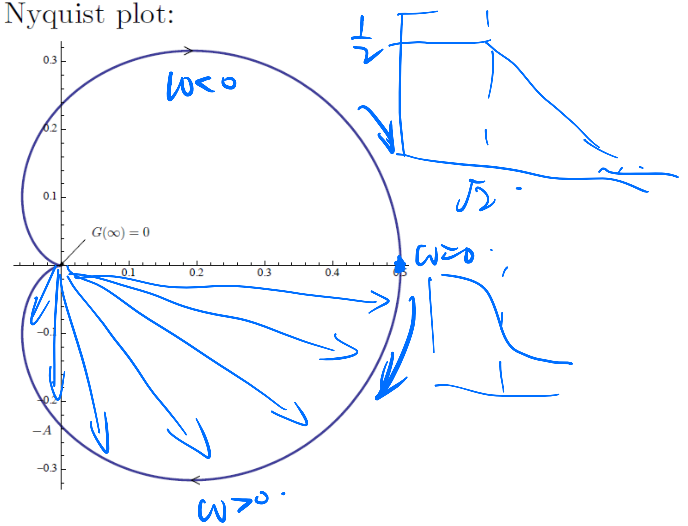
$$G(-j\omega) = \overline{G(j\omega)}$$

— Nyquist plots are always *symmetric w.r.t. the real axis*!!

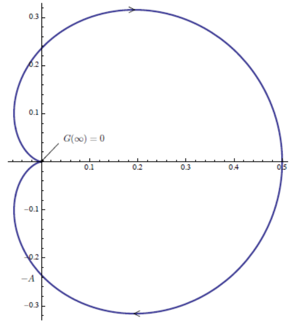
Bode plot:



Nyquist plot:



Nyquist plot:



$\#(\circlearrowright \text{ of } -1/K)$

$= \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=0}$

$\Longrightarrow K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowright \text{ of } -1/K) = 0$$

- If $K > 0$, $\#(\circlearrowright \text{ of } -1/K) = 0$
- If $0 < -1/K < 1/2$, $\#(\circlearrowright \text{ of } -1/K) > 0 \Longrightarrow$ closed-loop stable for $K > -2$

Example 2

$$G(s) = \frac{1}{(s-1)(s^2+2s+3)} = \frac{1}{s^3+s^2+s-3}$$

$\#(\text{RHP open-loop poles}) = 1 \qquad \text{at } s = 1$

Routh: the characteristic polynomial is

$$s^3 + s^2 + s + K - 3 \qquad \text{— 3rd degree}$$

— stable if and only if $K - 3 > 0$ and $1 > K - 3$.

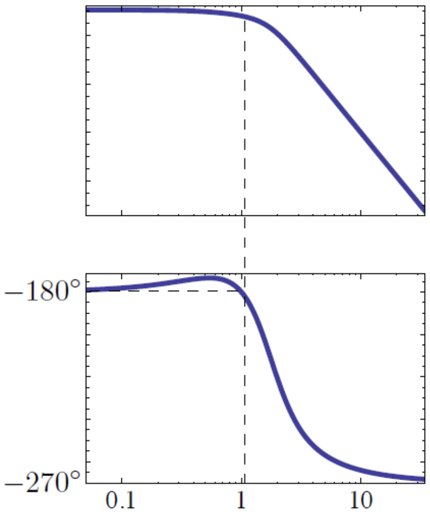
Stability range: $3 < K < 4$

Let's see how to spot this using the Nyquist criterion ...

Example 2

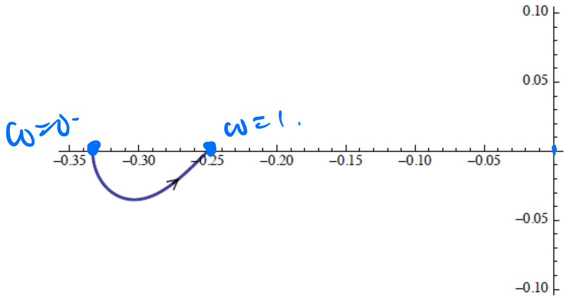
$$G(s) = \frac{1}{(s-1)(s^2+2s+3)} \qquad \text{(1 open-loop RHP pole)}$$

Bode plot:



Nyquist plot:

$$\begin{array}{ll} \omega = 0 & M = 1/3,\ \phi = -180^\circ \\ \omega = 1 & M = 1/4,\ \phi = -180^\circ \\ \omega \to \infty & M \to 0,\ \phi \to -270^\circ \end{array}$$



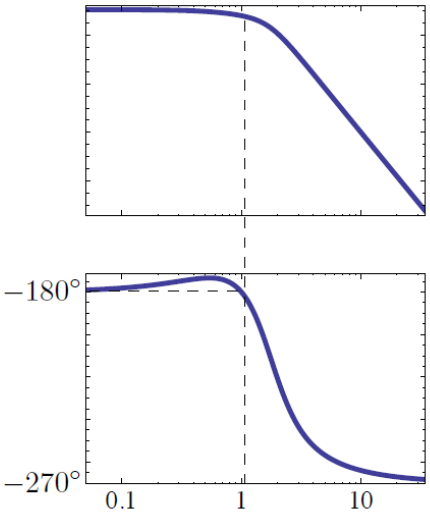
Example 2

$= (j\omega - 1)^{-1} \cdot (s^2+2s+3)^{-1}$

$$G(s) = \frac{1}{(s-1)(s^2+2s+3)}$$

(1 open-loop RHP pole)

Bode plot:

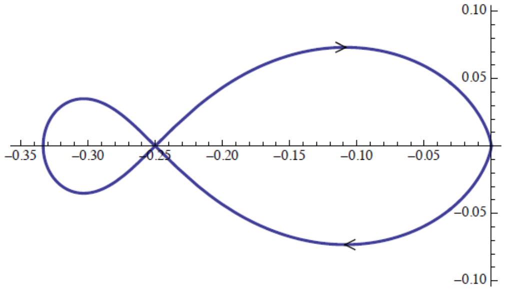


Nyquist plot:

$\omega = 0 \quad M = 1/3,\ \phi = -180°$

$\omega = 1 \quad M = 1/4,\ \phi = -180°$

$\omega \to \infty \quad M \to 0,\ \phi \to -270°$



$K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowleft \text{ of } -1/K) = -1$$

Which points $-1/K$ are encircled once $\circlearrowleft$ by this Nyquist plot?

$\#(\circlearrowleft \text{ of } -1/K)$

$= \#(\text{RHP CL poles})$

$- \underbrace{\#(\text{RHP OL poles})}_{=1}$

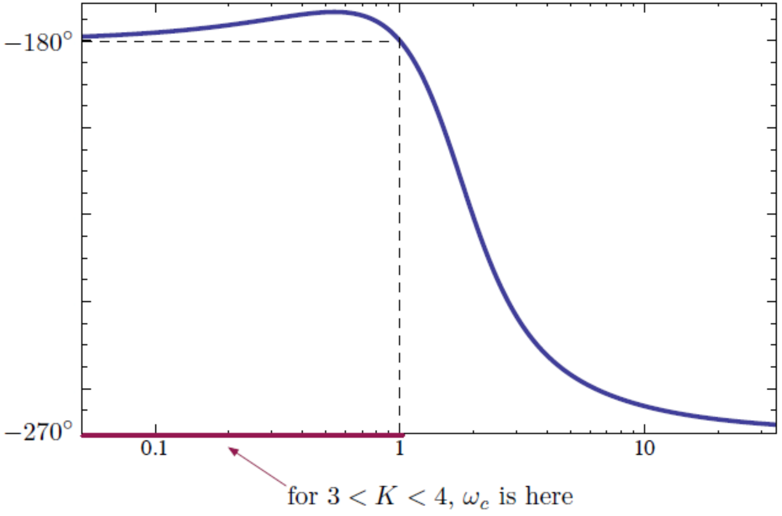
only $-1/3 < -1/K < -1/4$

$\Longrightarrow 3 < K < 4$

Example 2

Closed-loop stability range for $G(s) = \dfrac{1}{(s-1)(s^2+2s+3)}$ is $3 < K < 4$ (using either Routh or Nyquist).

We can interpret this in terms of phase margin:



So, in this case, stability $\Longleftrightarrow$ PM > 0 (typical case).

Example 3

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)} = \frac{s-1}{s^3+s^2-s+2}$$

Open-loop poles:

$$s = -2 \qquad \text{(LHP)}$$

$$s^2 - s + 1 = 0$$

$$\left(s - \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$s = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} \qquad \text{(RHP)}$$

$\therefore$ 2 RHP poles

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)} = \frac{s-1}{s^3+s^2-s+2}$$

Routh:

char. poly. $s^3 + s^2 - s + 2 + K(s-1)$

$s^2 + s^2 + (K-1)s + 2 - K$ (3rd-order)

— stable if and only if

$$K - 1 > 0$$
$$2 - K > 0$$
$$K - 1 > 2 - K$$

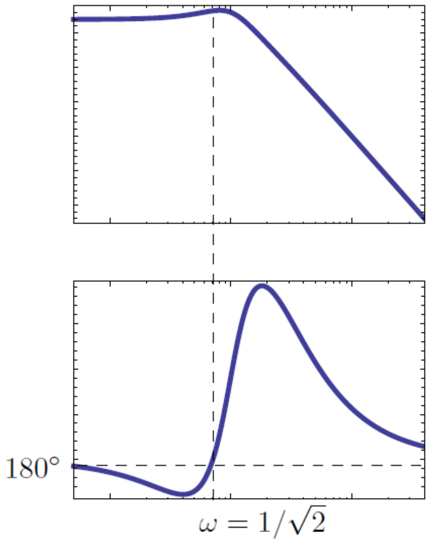
— stability range is $3/2 < K < 2$

Example 3

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)}$$

(2 open-loop RHP poles)

Bode plot (tricky, RHP poles/zeros)



$\phi = 180°$ when:

- $\omega = 0$ and $\omega \to 0$
- $\omega = 1/\sqrt{2}$:

$$\left.\frac{j\omega - 1}{(j\omega - 1)((j\omega)^2 - j\omega + 1)}\right|_{\omega = 1/\sqrt{2}}$$

$$= \frac{\frac{j}{\sqrt{2}} - 1}{\left(\frac{j}{\sqrt{2}} + 2\right)\left(-\frac{1}{2} - \frac{j}{\sqrt{2}} + 1\right)}$$

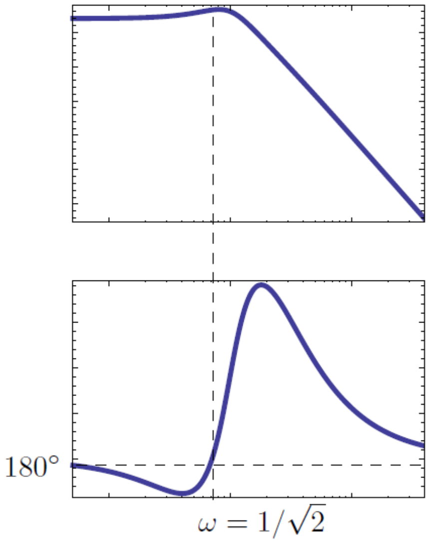
$$= \frac{\frac{j}{\sqrt{2}} - 1}{-\frac{3}{2}\left(\frac{j}{\sqrt{2}} - 1\right)} = -\frac{2}{3}$$

(need to guess this, e.g., by mouseclicking in Matlab)

$$G(s) = \frac{s-1}{s^3+s^2-s+2}$$

(2 open-loop RHP poles)

Bode plot:

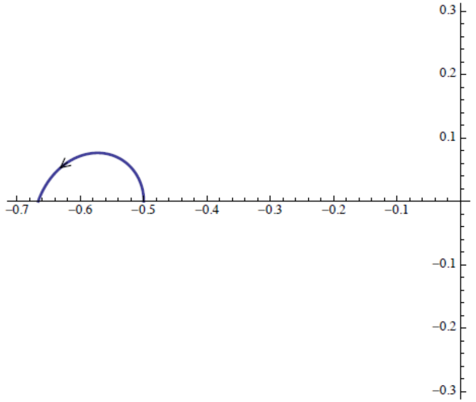


Nyquist plot:

$\omega = 0 \quad M = 1/2,\ \phi = 180°$

$\omega = 1/\sqrt{2} \quad M = 2/3,\ \phi = 180°$

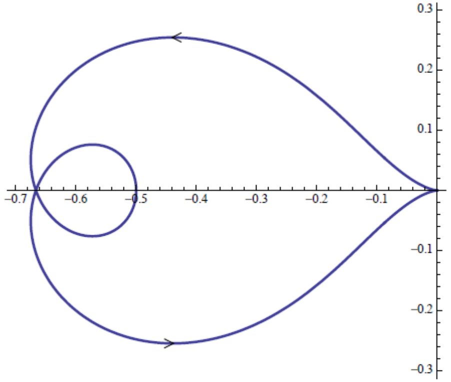
$\omega \to \infty \quad M \to 0,\ \phi \to 180°$



Example 3

$$G(s) = \frac{s-1}{s^3+s^2-s+2} \qquad \text{(2 open-loop RHP poles)}$$

Nyquist plot:



$$\#(\circlearrowright \text{ of } -1/K) = \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=2}$$

$K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowright \text{ of } -1/K) = -2$$

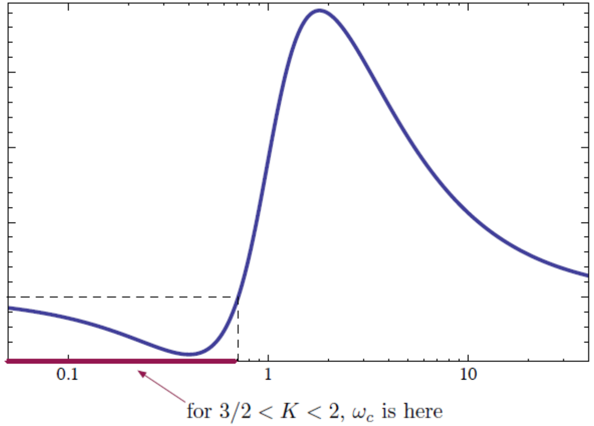
Which points $-1/K$ are encircled twice $\circlearrowleft$ by this Nyquist plot?

only $-2/3 < -1/K < -1/2$

$$\Longrightarrow \frac{3}{2} < K < 2$$

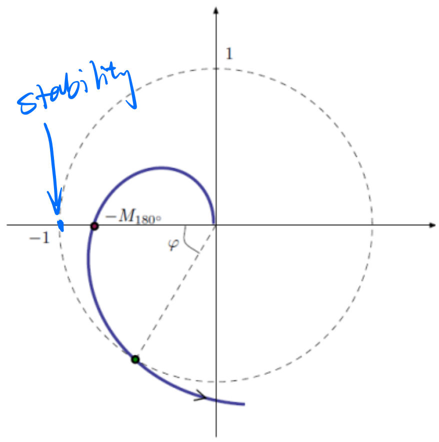
CL stability range for $G(s) = \dfrac{s-1}{s^3+s^2-s+2}$: $K \in (3/2, 2)$

We can interpret this in terms of phase margin:



So, in this case, stability $\Longleftrightarrow$ PM < 0 (atypical case; Nyquist criterion is the only way to resolve this ambiguity of Bode plots).

Stability Margin.



$$GM = \frac{1}{M_{180°}}.$$

$$\Rightarrow \frac{K}{M_{180°}} = 1.$$

$$PM = \varphi$$