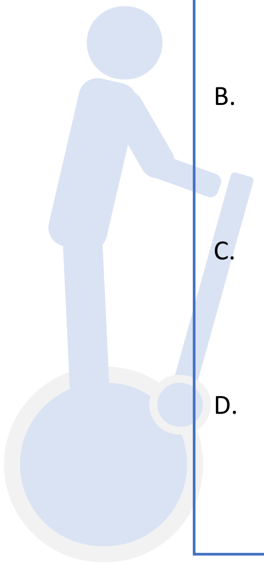
ECE 486 Control Systems

Lecture 03: Linear Systems & Dynamic Response

Liangjing Yang
Assistant Professor, ZJU-UIUC Institute
liangjingyang@intl.zju.edu.cn

Announcement

Reminders:
<u>NO</u> lab this Friday; need volunteers to move to Friday PM slots
Graduate Student: 15min slot with me to discuss your project before Oct 13



<u>Control Systems Project (Graduates)</u>
You will have a choice of doing **ONE** of the following options:

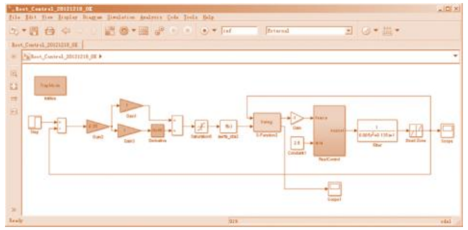
A. A <u>literature review</u> on Control Systems relevant to your field of interest
   - historical development, existing state-of-the-art technology and an analysis of the development prospect
   - draw relevance towards your field of interest
   - apply understanding in control systems towards your area(s) of expertise.

B. A <u>simulation-based project</u> related to control systems
   - control theory for **engineering application** or
   - **scientific methodology** in analytical studies related to control systems.

C. A <u>prototype development project</u> related to control systems
   - control theory for **engineering application** or
   - **scientific methodology** in analytical studies related to control systems (Subject to availability of resources)

D. An <u>educational/ walk-through</u>
   - **tutorial or demo** video (approx. 10~15min)
   - explain one or more **concepts in Modern Control Theory**

<u>**Lab Sessions (Undergrads)**</u>
Lab 0: Matlab Simulation
Lab 1: Analog simulation & circuit prototyping
Lab 2: Digital simulation
Lab 3: Digital simulation of closed loop-system
Lab 4: DC motor & PID Control
Lab 5: Control Design using Frequency Response Method
Lab 6: Control Design using State-Space Model



Checklist

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

Wk	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	**Linear systems and their dynamic response**	**Section 3.1, Appendix A**
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	National Holiday Week	
4	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	Review A	
	Term Test A	
7	Introduction to Root Locus design method	Ch. 5
	Root Locus continued; introduction to dynamic compensation	
8	Lead and lag dynamic compensation	Ch. 5
	Lead and lag continued; introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling: Wk 1–4; Analysis: Wk 4–5; Root Locus: Wk 7–8

Wk	Topic	Ref.
9	Bode plots for three types of transfer functions	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	Control design using frequency response	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Frequency Response: Wk 9–11; State-Space: Wk 13–15

Recap: Lecture 02

$m\ddot{x} = -f_s - f_d + f$

$m\ddot{x} = -kx - b\dot{x} + f$

- **Dynamic Systems** consist of components with inputs and outputs related to time varying function
- **State-space form** represents systems of ODEs (of various order) as a larger system of first order ODEs
- The process of **linearization** linearizes a non-linear model about an operating point (equilibrium point with known initial conditions)

Input, $r(t)$ → Dynamic System → Output, $x_c(t)$

$$\dot{x}(t) = f(x(t), u(t))$$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{cases}$$

where

A: ~~state~~ Dynamic (system) matrix	**x:** state vector
B: input control matrix	**$\dot{\mathbf{x}}$:** state change
C: output matrix	**u:** input
D: feedforward matrix	**y**: output

Non-linear $\dot{x}(t) = f(x(t), u(t))$

↓

Linear $\dot{x}(t) = Ax(t) + Bu(t)$

Quick Overview: System Representation & Analysis

Mathematical Representation

$x_1 = x$

$x_2 = \dot{x}$

State space model:

State Equation $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$

Output Equation $y = \begin{pmatrix}b_0 & b_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

Configuration form

Equations of Motion
$$\begin{cases}\ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\\ \quad \dots\\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\end{cases}$$

$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} + f/m$

Initial Conditions
$$\begin{cases}q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0}\\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0}\end{cases}$$

Transfer Function:

$$\frac{O(s)}{I(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_o}{s^n + a_{n-1}s^{n-1} + \dots + a_o}$$

ICs= 0

Systematic Modeling

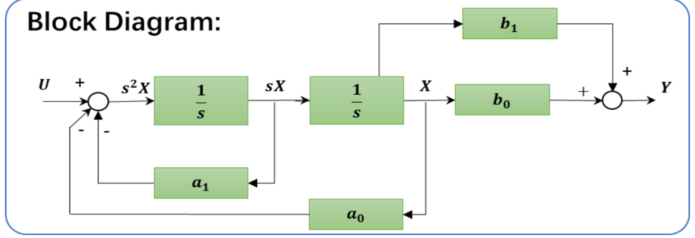
Input | Dynamic System | Output

$u(\#) \rightarrow$ LTI $\rightarrow y(\#)$

$i(t) \rightarrow * h(t) \rightarrow o(t)$

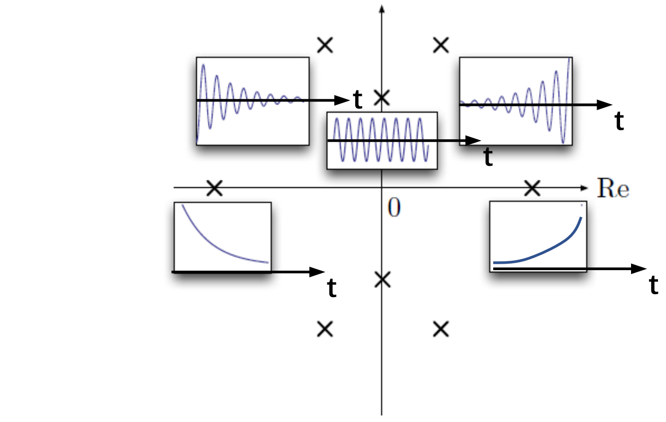
$I(s) \rightarrow H(s) \rightarrow O(s)$

Block Diagram:



Analysis of Systems

Analyzing effect of poles and zeros



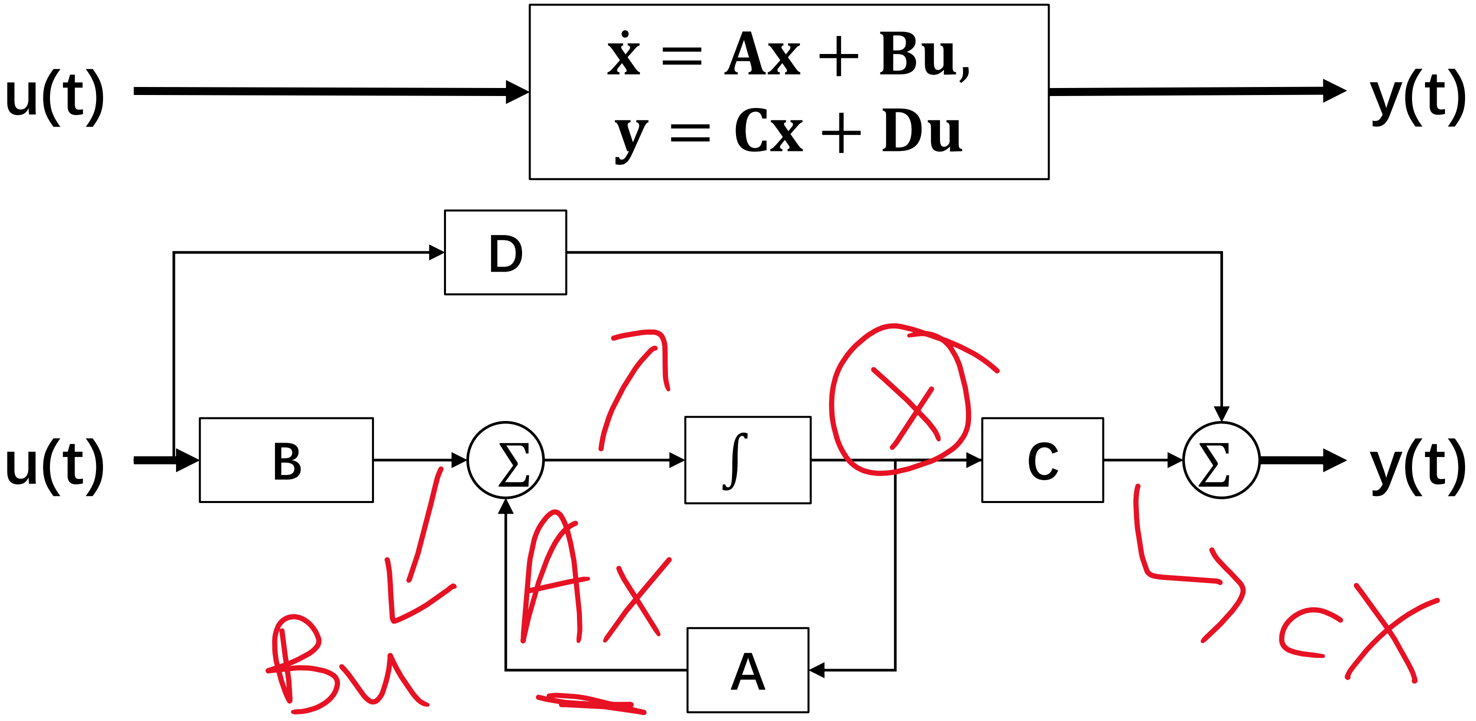
- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Stability Analysis

Dynamic Response Specification

Design Methods

Recall State-space form



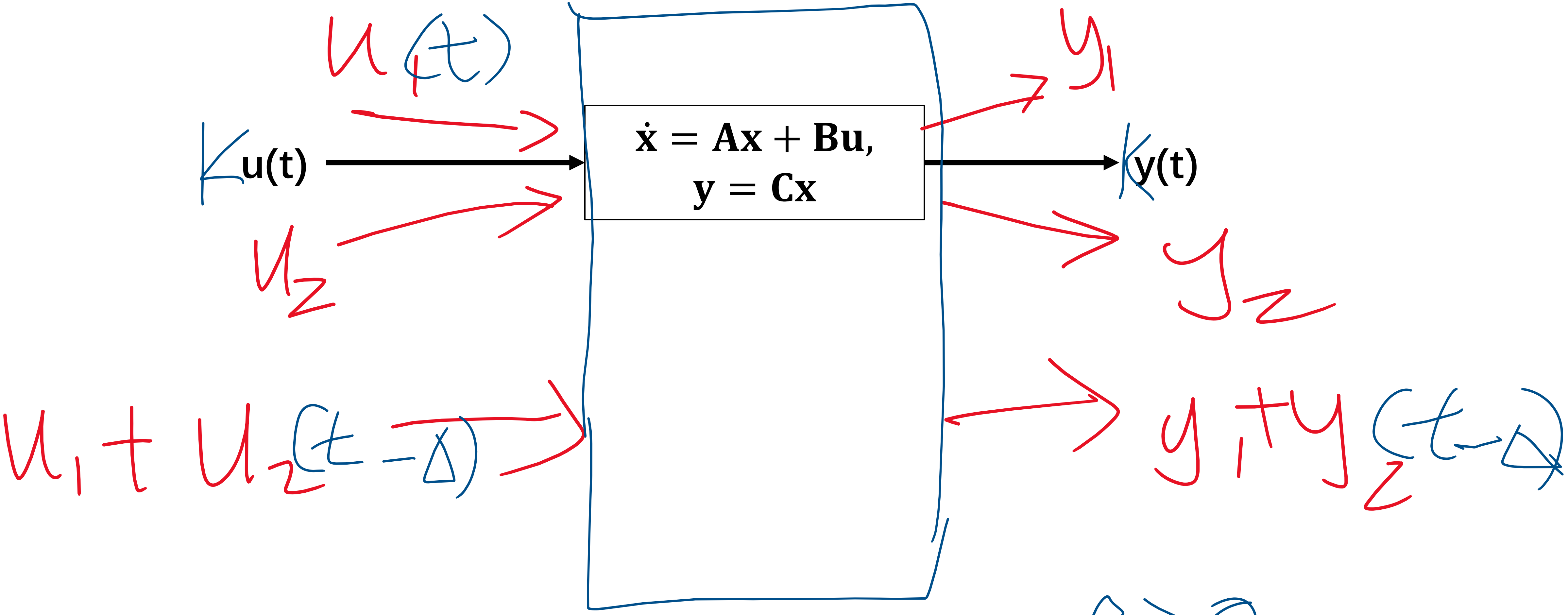
A: system (dynamic) matrix
B: input (control) matrix
C: output (sensor) matrix
D: feedforward matrix

x: state vector
ẋ: state change
u: input
y: output

Lecture Overview

$$u(t) \longrightarrow \boxed{\begin{array}{c} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx} \end{array}} \longrightarrow y(t)$$

- Characterize the output of a specific system given an input
- Considering only SISO Single input Single o/p
- Linear Time-Invariant Causal System



$\epsilon > 0$

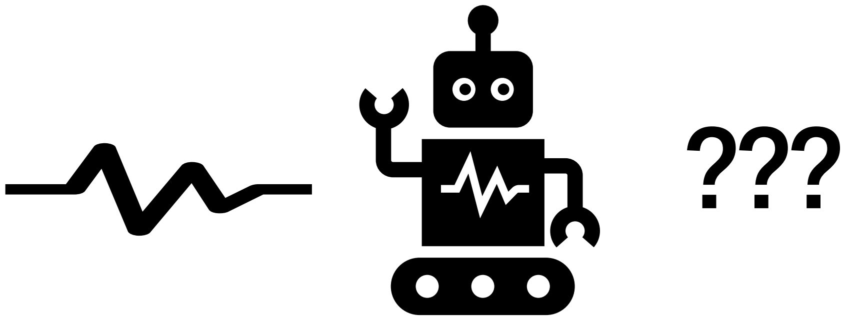
Lecture Overview

$$\mathbf{u}(t) \longrightarrow \boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}} \longrightarrow \mathbf{y}(t)$$

- Characterize the output of a specific system given an input
- Considering only SISO
  - A **linear** SISO dynamic system satisfies the **superposition principle**
  - A **time-invariant** SISO dynamic system satisfies the **time-shift principle**
  - A **causal** SISO dynamical system satisfies the **causality principle**

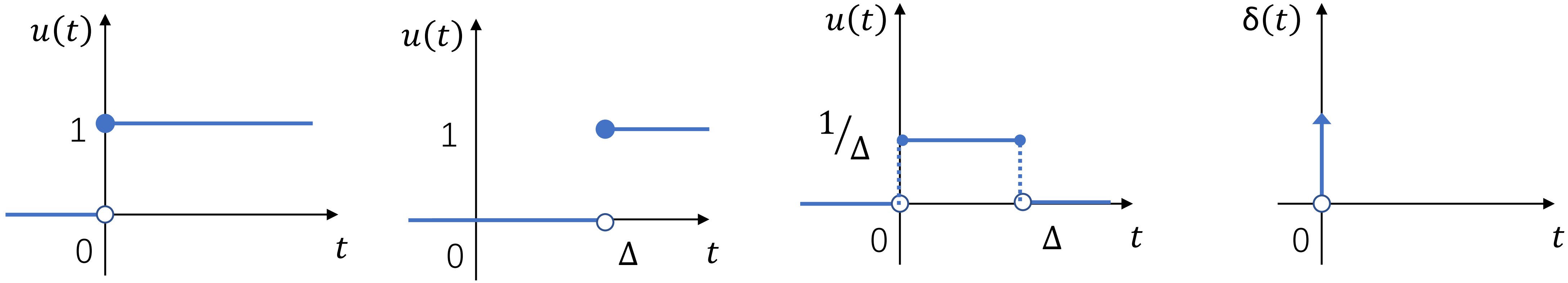
➢ Reading: Section 3.1 of Franklin, Powell, and Emami-Naeini, Feedback Control of Dynamic Systems

How will the system respond for a given type of input?



Input Signal

- unit step $u(t)$ is a signal that equals 1 for t > 0 and 0 otherwise
- delayed unit step signal $u(t-\Delta)$ equals 1 for t $\geq \Delta$ and 0 otherwise
- $\frac{u(t)-u(t-\Delta)}{\Delta}$ is a step pulse with pulse width Δ , amplitude $^1/_\Delta$
- Unit Impulse $\delta\ (t)$ acts in convolution with the exponentially decaying signal $e^{-\alpha t}$ as multiplication by 1

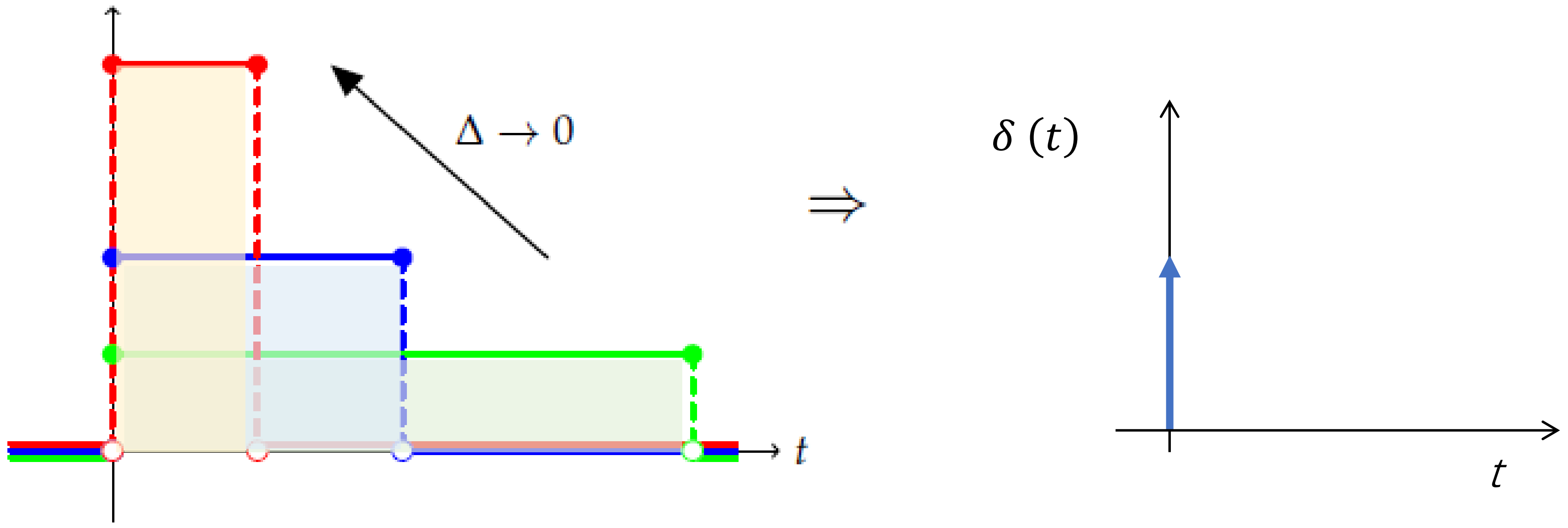


Unit Impulse

Unit Impulse $\delta\ (t)$ acts in convolution with the exponentially decaying signal $e^{-\alpha t}$ as multiplication by 1

1. $\delta\ (t)$=0 for all t≠0

2. $\int_{-a}^{a} \delta(t)\mathrm{d}t = 1$ for all a>0



Step Pulse

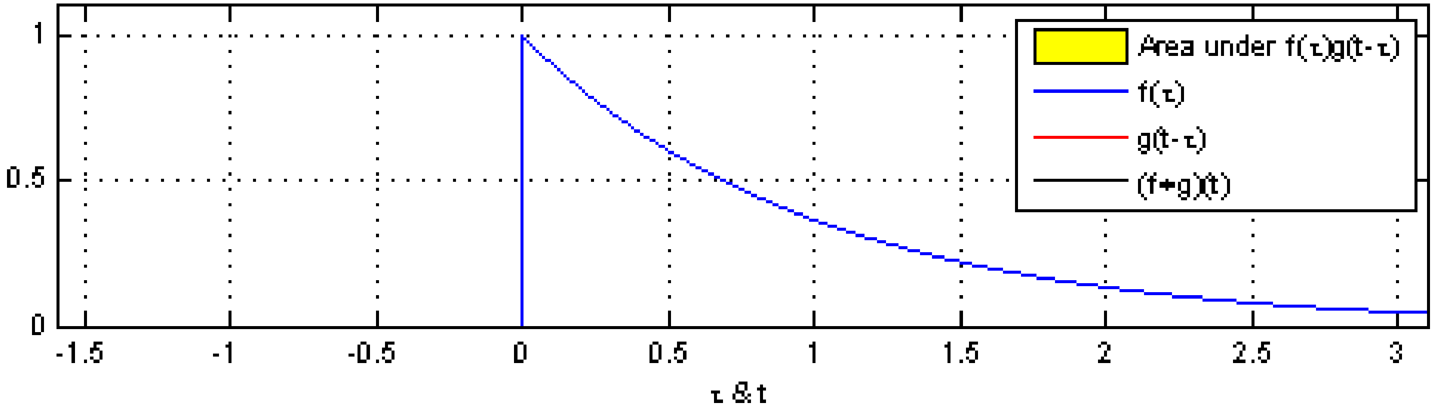
For $f(t) = \frac{u(t)-u(t-\Delta)}{\Delta}$ and with $c = 0$, the solution of the IVP for $t \geq \Delta$ becomes

- IVP $\quad y(0) = c, \quad \frac{dy(t)}{dt} + \alpha\, y(t) = f(t)$

$$y(t) = ce^{-\alpha t} + \int_0^{t-\Delta} f(\tau)e^{-\alpha(t-\tau)}\, d\tau$$

$$y(t) = \frac{1}{\Delta}\int_0^{\Delta} 1.\, e^{-\alpha(t-\tau)}\, d\tau = \frac{e^{-\alpha t}}{\Delta}\int_0^{\Delta} e^{\alpha\tau}\, d\tau$$

$$y(t) = \frac{e^{-\alpha t}}{\alpha\Delta} e^{\alpha\tau}\Big|_0^{\Delta} = \frac{e^{\alpha\Delta}-1}{\alpha\Delta}\; e^{-\alpha\tau}$$



Impulse Response

$h(t-\tau)$

$$u(t) \longrightarrow \boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu},\\ \mathbf{y} &= \mathbf{Cx}\end{aligned}} \longrightarrow y(t)$$

input

response

Zero initial condition: $\boxed{x(0)=0}$

$$u(t) = \delta(t-\tau) \xrightarrow{x(0)=0;\ \text{LTI system}} y(t) = h(t-\tau)$$

Questions to consider:

1. If we know h, how can we find the system's response to other (arbitrary) inputs?
2. If we don't know h, how can we determine it?

We will start with Question 1.

Impulse Response

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}} \longrightarrow y(t)$$

Zero initial condition: x(0)=0

Knowing *h*, how to find system's response to other (arbitrary) inputs?

Recall the *sifting property* of the δ-function: for any function f which is "well-behaved" at $t = \tau$,

$$\int_{-\infty}^{\infty} f(t)\boxed{\delta(t-\tau)}\mathrm{d}t = \boxed{f(\tau)}$$

— any *reasonably regular* function can be represented as an integral of impulses!!

Impulse Response

$$\mathbf{u(t)} \longrightarrow \boxed{\begin{array}{c}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\\ \mathbf{y} = \mathbf{C}\mathbf{x}\end{array}} \longrightarrow \mathbf{y(t)}$$

Zero initial condition: x(0)=0

Knowing *h*, how to find system's response to other (arbitrary) inputs?

By the sifting property, for a general input $u(t)$ we can write

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\mathrm{d}\tau.$$

Now we recall the *superposition principle:* the response of a linear system to a sum (or integral) of inputs is the sum (or integral) of the individual responses to these inputs.

Impulse Response , $h(t-\tau)$

$$u(t) \longrightarrow \boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu},\\ \mathbf{y} &= \mathbf{Cx}\end{aligned}} \longrightarrow y(t)$$

Zero initial condition: x(0)=0

Knowing *h*, how to find system's response to other (arbitrary) inputs?

The *superposition principle:* the response of a linear system to a sum (or integral) of inputs is the sum (or integral) of the individual responses to these inputs.

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\mathrm{d}\tau \quad \longrightarrow \quad y(t) = \int_{-\infty}^{\infty} u(\tau)\underbrace{h(t-\tau)}_{\substack{\text{response to}\\ \delta(t-\tau)}}\mathrm{d}\tau$$

— the integral that defines $y(t)$ is a convolution of u and h.

Impulse Response

$y = u * h$

$\mathbf{u(t)} \longrightarrow \boxed{\begin{array}{c}\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx}\end{array}} \longrightarrow \mathbf{y(t)}$

Zero initial condition: x(0)=0

Knowing *h*, how to find system's response to other (arbitrary) inputs?

Conclusion so far: for zero initial conditions, the output is the convolution of the input with the system impulse response:

$$y(t) = u(t) \star h(t) = h(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau) \mathrm{d}\tau$$

Q: Does this formula provide a *practical* way of computing the output y for a given input u?

Impulse Response

$$\mathbf{u(t)} \longrightarrow \boxed{\begin{array}{c}\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx}\end{array}} \longrightarrow \mathbf{y(t)}$$

Zero initial condition: x(0)=0

Knowing *h*, how to find system's response to other (arbitrary) inputs?

Conclusion so far: for zero initial conditions, the output is the convolution of the input with the system impulse response:

$$y(t) = u(t) \star h(t) = h(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

Q: Does this formula provide a *practical* way of computing the output y for a given input u?

A: Not directly (computing convolutions is not exactly pleasant), but ...we can use Laplace transforms.

Laplace Transforms & Transfer Functions

Reminder: the *two-sided* Laplace transform of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}\,\mathrm{d}\tau, \qquad s \in \mathbb{C}$$

$H = \frac{Y}{U}$

time domain	frequency domain
$u(t)$	$U(s)$
$h(t)$	$H(s)$
$y(t)$	$Y(s)$

convolution in time domain $\longleftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \longleftrightarrow \quad Y(s) = H(s)U(s)$$

The Laplace transform of the impulse response

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,\mathrm{d}\tau,$$

is called the transfer function of the system.

$u \longrightarrow \boxed{\star h} \longrightarrow y = u \star h$

$u \longrightarrow \boxed{H} \longrightarrow Y$

$UH = Y$

Laplace Transforms & Transfer Functions

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Limits of integration:

- We only deal with *causal* systems — output at time t is not affected by inputs at future times $t' > t$
- If the system is causal, then $h(t) = 0$ for $t < 0$ — $h(t)$ is the response at time t to a unit impulse at time 0
- We will take all other possible inputs (not just impulses) to be 0 for $t < 0$, and work with *one-sided* Laplace transforms:

$$y(t) = \int_0^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

$$H(s) = \int_0^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

Laplace Transforms & Transfer Functions

$$\boxed{Y(s) = H(s)U(s),} \qquad \text{where } \boxed{H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau}$$

implse response

Given $u(t)$, we can find $U(s)$ using tables of Laplace transforms or MATLAB. But how do we know $h(t)$ [or $H(s)$]?

- Suppose we have a state-space model:

$$u \longrightarrow \boxed{\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx \end{matrix}} \longrightarrow y$$

In this case, we have an exact formula:

$$H(s) = C(Is - A)^{-1}B \qquad \text{(matrix inversion)}$$

$$h(t) = Ce^{At}B, \; t \geq 0^{-} \qquad \text{(matrix exponential)}$$

— will not encounter this until much later in the semester.

Laplace Transforms & Transfer Functions

$$Y(s) = H(s)U(s), \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau$$

- So, how should we compute $H(s)$ in practice?

  Try injecting some specific inputs and see what happens at the output.

  Let's try $u(t) = e^{st}, t \geq 0$ (s is some fixed number)

$$\begin{aligned} y(t) &= \int_0^{\infty} h(\tau)u(t-\tau)\mathrm{d}\tau \qquad (\text{because } u \star h = h \star u) \\ &= \int_0^{\infty} h(\tau)e^{s(t-\tau)}\mathrm{d}\tau \\ &= e^{st}\int_0^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau \\ &= e^{st}H(s) \end{aligned}$$

  – so, $u(t) = e^{st}$ is multiplied by $H(s)$ to give the output.

Laplace Transforms & Transfer Functions

Example

$$\dot{y} = -ay + u \qquad \text{(think } y = x\text{, full measurement)}$$
$$u(t) = e^{st} \qquad \text{(always assume } u(t) = 0 \text{ for } t < 0\text{)}$$
$$y(t) = H(s)e^{st} \qquad \text{— what is } H\text{?}$$

Let's use the system model:

$$\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(H(s)e^{st}\right) = sH(s)e^{st}$$

Substitute into $\dot{y} = -ay + u$:

$$sH(s)\cancel{e^{st}} = -aH(s)\cancel{e^{st}} + \cancel{e^{st}} \qquad (\forall s; t > 0)$$
$$sH(s) = -aH(s) + 1$$

$$H(s) = \frac{1}{s+a} \qquad \Longrightarrow \qquad y(t) = \frac{e^{st}}{s+a}$$

$$\dot{y} = -ay + u$$
$$H(s) = \frac{1}{s+a}$$

Now we can fund the impulse response $h(t)$ by taking the inverse Laplace transform — from tables,

$$h(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Determining Impulse Response

$$u \longrightarrow \boxed{h} \longrightarrow y$$

$$u(t) = e^{st},\ t \geq 0 \qquad \xrightarrow{x(0)=0;\ \text{LTI system}} \qquad y(t) = e^{st}H(s)$$

Back to our two questions:

1. If we know h, how can we find y for a given u?
2. If we don't know h, how can we determine it?

We have answered Question 1. Now let's turn to Question 2.

One idea: inject the input $u(t) = e^{st}$, determine $y(t)$, compute

$$H(s) = \frac{y(t)}{u(t)};$$

repeat for all s of interest. Q: Is this a good idea?

Determining Impulse Response

$$u(t) = e^{st} \longrightarrow \boxed{h} \longrightarrow y(t) = e^{st}H(s)$$

compute $H(s) = \dfrac{y(t)}{u(t)}$, repeat for as many values of s as necessary

Q: Is this likely to work *in practice*?

A: No — e^{st} blows up very quickly if $s > 0$, and decays to 0 very quickly if $s < 0$.

So we need *sustained, bounded signals* as inputs.

This is possible if we allow s to take on *complex values*.

Frequency Response

$$u \longrightarrow \boxed{h} \longrightarrow y$$

$u(t) = A\cos(\omega t)$ $\quad$ A – amplitude; ω – (angular) frequency, rad/s

From Euler's formula:

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

By linearity, the response is

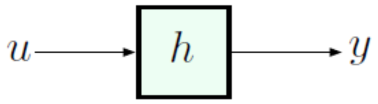
$$y(t) = \frac{A}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

$$\text{where } H(j\omega) = \int_0^\infty h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau$$

$$H(-j\omega) = \int_0^\infty \underbrace{h(\tau)e^{j\omega\tau}}_{\substack{\text{complex}\\ \text{conjugate}}}\,\mathrm{d}\tau = \overline{H(-j\omega)}$$

(recall that $h(\tau)$ is real-valued)

Frequency Response



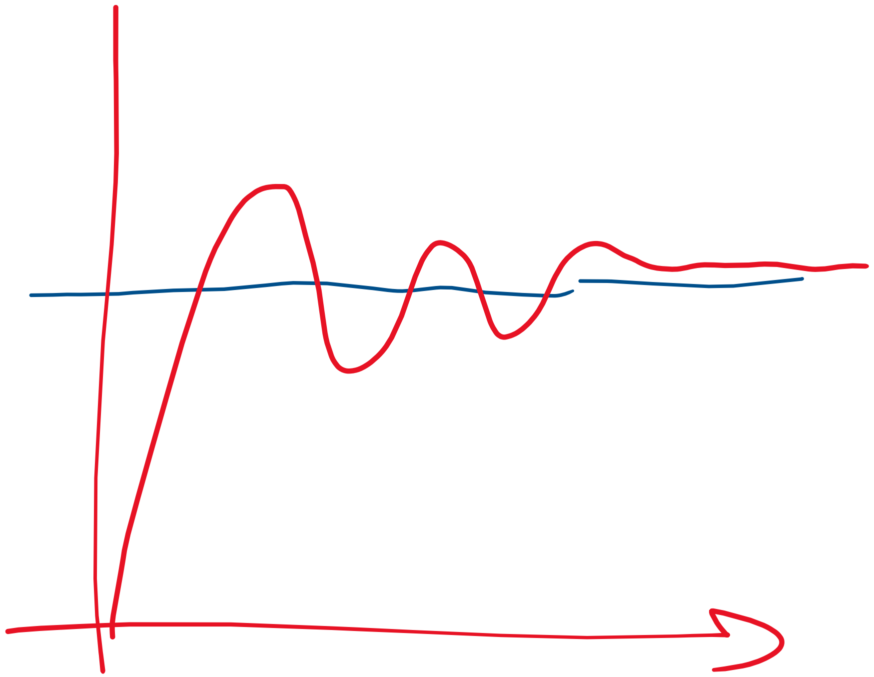
$$u(t) = A\cos(\omega t) \quad \longrightarrow \quad y(t) = \frac{A}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

$$H(j\omega) \in \mathbb{C} \quad \Longrightarrow \quad \begin{aligned} H(j\omega) &= M(\omega)e^{j\varphi(\omega)} \\ H(-j\omega) &= M(\omega)e^{-j\varphi(\omega)} \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \frac{A}{2}M(\omega)\left[e^{j(\omega t+\varphi(\omega))} + e^{-j(\omega t+\varphi(\omega))}\right] \\ &= AM(\omega)\cos\big(\omega t + \varphi(\omega)\big) \quad \text{(only true in } \textit{steady state}\text{)} \end{aligned}$$

The (steady-state) response to a cosine signal with amplitude A and frequency ω is still a cosine signal with amplitude $AM(\omega)$, same frequency ω, and phase shift $\varphi(\omega)$



Frequency Response

$x(0) = 0$

$u \longrightarrow \boxed{h} \longrightarrow y$

$$u \longrightarrow \boxed{h} \longrightarrow y$$

$$u(t) = A\cos(\omega t) \quad \longrightarrow \quad y(t) = A \underbrace{M(\omega)}_{\substack{\text{amplitude}\\ \text{magnification}}} \cos\big(\omega t + \underbrace{\varphi(\omega)}_{\substack{\text{phase}\\ \text{shift}}}\big)$$

Still an incomplete picture:

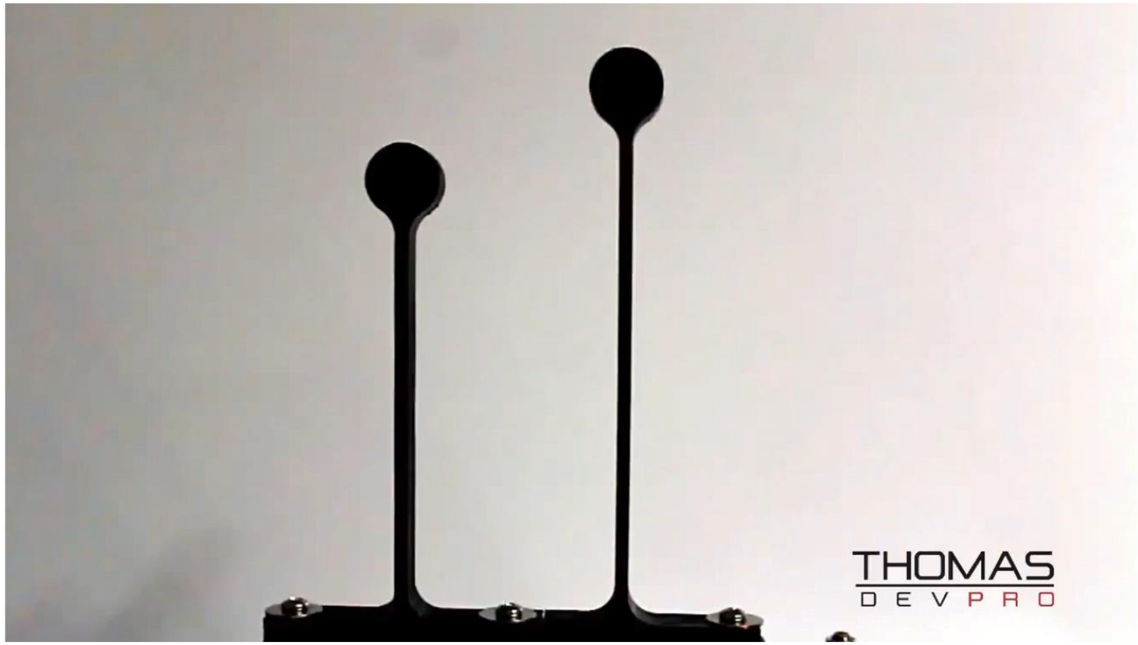
- What about response to general signals (not necessarily sinusoids)? — always given by $Y(s) = H(s)U(s)$
- What about response under *nonzero I.C.'s*?— we will see that, if *the system is stable*, then

$$\text{total response} = \begin{array}{c}\text{transient response}\\ \text{(depends on I.C.)}\end{array} + \begin{array}{c}\text{steady-state response}\\ \text{(independent of I.C.)}\end{array}$$

— need more on Laplace transforms

Natural Frequency

- **Natural (or modal) frequency** of a system is the angular frequency corresponding to a collection of initial values where the free response is harmonic



https://thomasdevpro.com/

Next Lecture

- dynamic response (transient and steady-state) with arbitrary I.C.'s