Linear system: / State-space model.

$x_1 \equiv x,\ x_2 \equiv \dot{x},\ x_3 \equiv \ddot{x},\ \cdots$

dynamics, control, input

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

measured output, state, sensor

Causal: output not affected by future time.

Laplace: $F(s) = \int_{\mathbb{R}} f(t) e^{-st} dt$.

$s = e^{j\omega} = \cos\omega + j\sin\omega$.

State-Space transfer function:

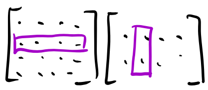
$G(s) = C(Is - A)^{-1}B + D$

Eigenvalue: λ such that $\det(A - \lambda I) = 0$.

Eigenvector: $\vec{\alpha}$ such that $A\vec{\alpha} = \lambda\vec{\alpha}$, where $A \in \mathbb{R}^{n\times n}$ and $\lambda = eig(A)$.

Rank: 非零子式的最高阶数.

AB =



Inverse of 2x2 Mx:

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \Longrightarrow M^{-1} = \frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Full-state feedback with observer

System: $\dot{x} = Ax + Bu$

$y = Cx$

Observer: $\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu$

Controller: $u = -K\hat{x}$

$\Longrightarrow$ overall o/c system.

$$\begin{cases} \dot{\hat{x}} = (A - LC)\hat{x} + Ly + B(-K\hat{x}) \\ \quad = (A - LC - BK)\hat{x} + Ly \\ u = -K\hat{x} \end{cases}$$

dynamic output feedback

$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A-LC-BK \end{pmatrix}\begin{pmatrix} x \\ \hat{x} \end{pmatrix}$.

$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A-BK & BK \\ 0 & A-LC \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix}$.

CL characteristic polynomial: $= \det\begin{pmatrix} Is-A+BK & -BK \\ 0 & Is-A+LC \end{pmatrix}$

$= \det(Is - A + BK)\cdot\det(Is - A + LC)$.

Separation Principle:

CL eigenvalues are {controller poles (roots of $\det(Is - A + BK)$)}

$\cup$ {observer poles (roots of $\det(Is - A + LC)$)}

if the system is controllable and observable.

if $u_1 \to y_1$, $u_2 \to y_2$ $\Rightarrow u_1 + u_2 = y_1 + y_2$.

Impulse response. $\int_{-a}^{a} \delta(t) dt = 1$.

$\int_{\mathbb{R}} \delta(t - \tau) f(t) dt = f(\tau)$

Characteristic Polynomial: $P(\lambda) = \det(A - \lambda I)$, λ = eigen value for $P(\lambda) = 0$.

i.e. $\lambda = eig(A)$.

Vector division: $\vec{z}_1 = |z_1| e^{j\phi_1}$, $\vec{z}_2 = |z_2| e^{j\phi_2}$

$\Rightarrow \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\phi_1 - \phi_2)}$.

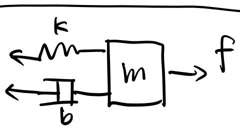
Transient response: ① vanishes at $t \to \infty$. ② $\mathcal{L}$ gives transient rsps.

DC gain: $= y(t \to \infty)$. where $u(t) = 1(t)$.

FVT: (all) Poles of $sY(s)$ lies in OLHP $\Leftrightarrow Re(s) < 0$: $y(t \to \infty) = \lim_{s \to 0} sY(s)$.

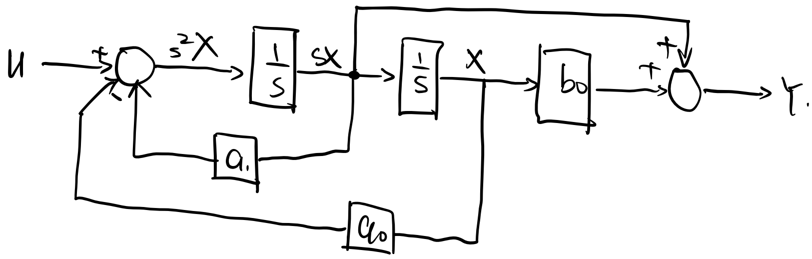
or satisfies R-H Criterion.

Mass-Damping:



$f = m\ddot{x} + b\dot{x} + kx$. $\Leftrightarrow F = s^2X + sbX + kX$.

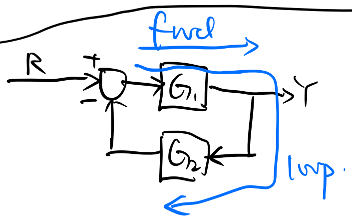
2nd-Order Damping System.



$$\begin{cases} s^2X = U - a_1 sX - a_0 X \\ Y = b_1 sX + b_0 X. \end{cases}$$

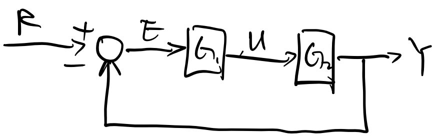
i.e. $\begin{cases} \frac{U}{X} = s^2 + a_1 s + a_0 \\ \frac{Y}{X} = b_1 X + b_0 \end{cases}$ $\Rightarrow \frac{Y}{U} = \frac{Y}{X}\cdot\frac{X}{U} = \frac{b_1 X + b_0}{s^2 + a_1 s + a_0}$.

For Feedback System:



$\text{Gain} = \frac{\text{fwd gain}}{1 + \text{loop gain}} = \frac{G_1}{1 + G_1 G_2}$

Unity Feedback



$\text{gain} = \frac{G_1 G_2}{1 + G_1 G_2}$

rise: 10% ~ 90%.

$t_r \doteq \frac{1.8}{\omega_n}$, exact at $\zeta = 0.5$

$t_p = \frac{\pi}{\omega_d}$

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$t_s \doteq \frac{3}{\sigma}$.

A damping system: $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ poles $s = -\omega_n(\zeta \pm \sqrt{\zeta^2 - 1})$,

$\frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2}$. $y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right]$

ζ: damping coef. / damping ratio

ω_n: natural freq.

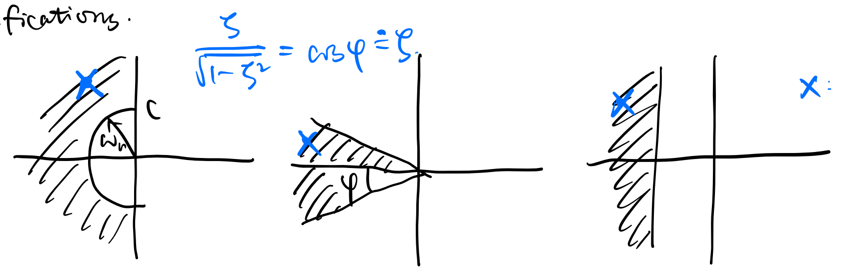
ω_d: damped freq.

ω_r: resonant freq.

$\omega_{BW} = \omega_n \underbrace{\sqrt{(1 - 2\zeta^2) + \sqrt{(1 - 2\zeta^2)^2 + 1}}}_{=1 \text{ for } \zeta = 1/\sqrt{2}}$.

Specifications.

$\frac{\zeta}{\sqrt{1-\zeta^2}} = \cos\varphi = \zeta$



Rise. M_p t_s

× : pole (σ, ω_d).

$\begin{cases} \sigma = \zeta\omega_n \\ \omega_d = \omega_n \cdot \sqrt{1-\zeta^2} \end{cases}$

$\omega_n = \sigma^2 + \omega_d^2$.

$M_r \downarrow \leftrightarrow$ better damping.

$\omega_{BW}\uparrow \leftrightarrow \omega_n \uparrow \leftrightarrow t_r \downarrow$.

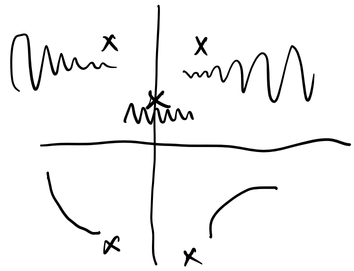
Effects of zeros.

LHP: ① increased overshoot. ② little influence on settling time

③ $a \to \infty$ yields less significant effect.

RHP: ① delays the response.

② creates an undershoot. (when a is small enough)



Extra Poles: ☆ extra LHP poles. → real parts 5× that of dominant LHP poles.

$y = \Sigma C_k e^{\boxed{\lambda_k} t}$. $\quad$ $\mathrm{Re}(\text{pole}) = \lambda_k$.

Pole locations: ① RHS → unstable. ③ Im axis $\begin{cases} \text{impulse / step : unstable if } \omega=0. \\ \omega=0 : \begin{cases} \text{impulse}: Y = \frac{1}{s},\ y = 1(t) \to \text{stable} \\ \text{step}: Y = \frac{1}{s^2},\ y = t. \to \text{unit ramp}. \end{cases} \end{cases}$

② LHS → stable.

Stability. $\overline{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$

necessary: $a_0, a_1, \cdots, a_n > 0$.

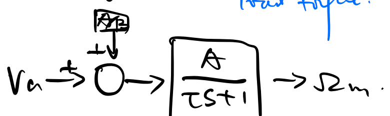
R-H. For lower order 2nd: $s^2 + a_1 s + a_2$ is stable iff $a_1, a_2 > 0$

3rd: $s^3 + a_1 s^2 + a_2 s + a_3$ is stable iff $\begin{cases} a_1, a_2, a_3 > 0 \\ a_1 a_2 > a_3. \end{cases}$

$H = \frac{Y}{R} = \frac{\text{Fwd gain}}{1 + \text{loop gain}} = \frac{\cdots}{\text{polynomial}}$ ← check if it's stable.

DC Motor:

T_e disturbance/load torque.



$\Omega_m = \frac{A}{\tau s+1} V_a + \frac{B}{\tau s+1} T_e$.

Disturbance Rejection: see coefficient of disturbance.

open-loop motor: $\omega_m(\infty) = \omega_{ref} + BT_e$ poor rejection.

closed: $\omega_m(\infty) = \frac{AK_{cl}}{1+AK_{cl}} \omega_{ref} + \frac{B}{1+AK_{cl}} T_e$.

As K_{cl} large, this item → 0. → good rejection.

Sensitivity: $S = \frac{\partial T/T}{\partial A/A}$. $\quad S_{ol} = 1$, $\quad S_{cl} = \frac{1}{1+AK_{cl}}$.

P-ctrl: always unstable. k_P

D-ctrl: lack of causality + noise amplification + cannot be implemented directly. $k_D s$ (introducing LHP zeros)

I-ctrl: disturbance rejection. k_I/s

PD: allow arbitrary pole placement, imperfect ss tracking.

P: does not introduce stability. D: introduce an OL zero into LHP.

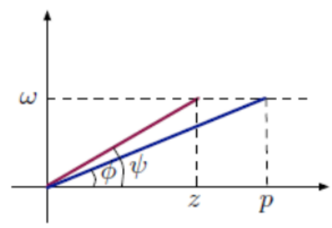
$\angle \frac{j\omega+z}{j\omega+p} = \angle(j\omega+z) - \angle(j\omega+p)$

$K \cdot \frac{s+z}{s+p}$:

lead ⟺ $z<p$

lag ⟺ $z>p$

- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



Root Locus: Characteristic Eq $\boxed{1+KL(s)=0}$ $\longleftrightarrow$ Phase Condition: $L(s)=-1/K$, $\angle L = 180°$

$$L(s)=\frac{b(s)}{a(s)}=\frac{s^m+b_1s^{m-1}+\dots+b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\dots+a_{n-1}s+a_n}$$

m: #(OL zeros), n: #(OL poles).

Rules: (A) #branches = deg(a).

(B) start: OL poles.

(C) stop: OL zeros.

OL tf: $H_{OL}=KL(s)$.

CL tf: $H_{cl}=\frac{KL(s)}{1+KL(s)}$.

★ if s on RL, $\sum\psi - \sum\varphi_i = 180°$ — phase condition.

ψ: $\angle$(s to zeros), φ: $\angle$(s to poles)

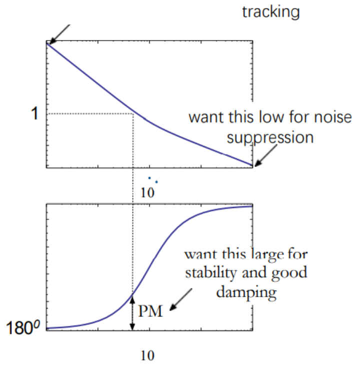
(D) if $s\in\mathbb{R}$, s is on RL iff. odd #(real OL poles and zeros) to the right of s. (s右侧 zeros, poles 数量和为奇.)

(E) asymptotes: if $\angle s^{m-n}=180° \Rightarrow \angle s=\frac{180°+\ell\cdot 360°}{n-m}$

Near ∞, $\angle s \approx \frac{(2\ell+1)\cdot 180°}{n-m}$.

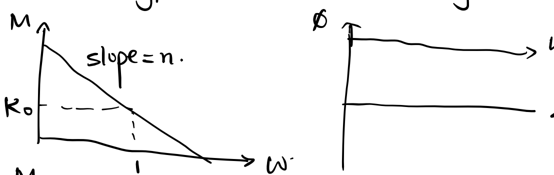
(F) $j\omega$-crossings: First solve $K_{critical}$ using R-H criterion. then find the corresponding ω_0. Crossings at $\pm j\omega_0$ on Im axis.

- PD control
  - provides stability, allows us to shape transient response specs;
  - replaces noncausal D-controller Ks with a causal, stable **lead** controller $K\frac{s+z}{s+p}$, where $p>z$;
  - introduces a zero in Left Half Plane at $-z$, hence pulls the root locus into LHP; but the shape of root locus differs depending on how large p is.
- PI control
  - provides stability and perfect steady-state tracking of constant references;
  - replaces unstable I-controller $\frac{K}{s}$ with a stable **lag** controller $K\frac{s+z}{s+p}$, where $p<z$; and this does not change the shape of root locus compared to PI.



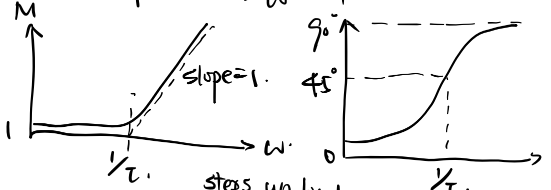
Bode plot (OL). ★ $M=1 \Longleftrightarrow M=\log_{10}1 = 0$ dB. all in log scale.

Type I. $K_0(j\omega)^n$. (low freq asymptote).



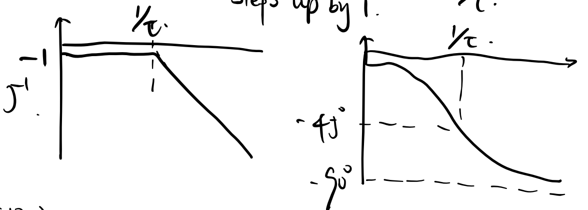
Type II. $(j\omega\tau+1)^{\pm1}$.

$(j\omega\tau+1)$



stable LR zero: steps up by 90°.

$(j\omega\tau+1)^{-1}$



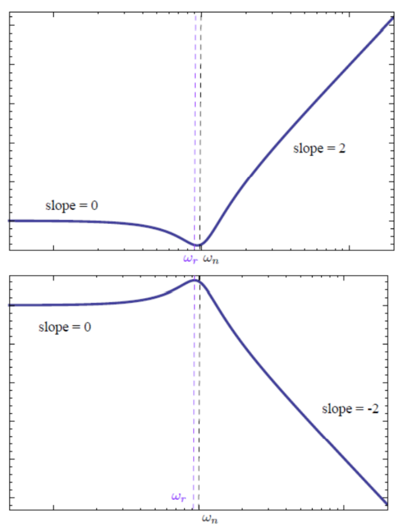
$M_p=\exp(-\pi\zeta/\sqrt{1-\zeta^2})$ (overshoot).

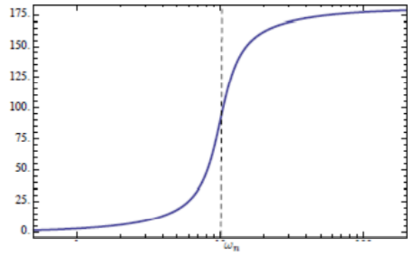
$$\begin{cases}\omega_r=\omega_n\sqrt{1-2\zeta^2}\\ M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}}-1\end{cases}\quad \text{(valid for } \zeta<\tfrac{1}{\sqrt2}\text{; for } \zeta\ge\tfrac{1}{\sqrt2},\ \omega_r=0)$$

$$\omega_{BW}=\omega_n\underbrace{\sqrt{(1-2\zeta^2)+\sqrt{(1-2\zeta^2)^2+1}}}_{=1\text{ for }\zeta=1/\sqrt2}$$ (0-freq response ~70.7%).

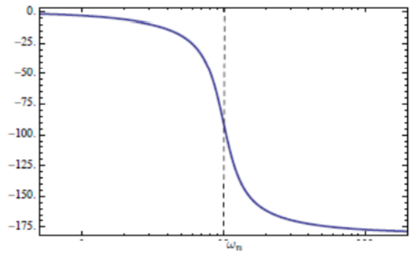
— so, if we know $\omega_r, M_r, \omega_{BW}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Type III. $\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\left(\frac{j\omega}{\omega_n}\right)+1$.





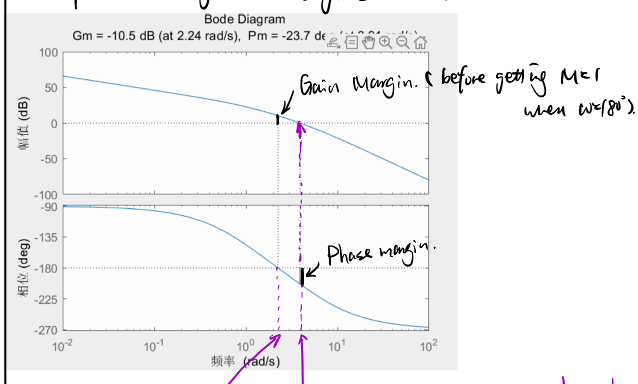
(stable complex zero — phase steps up by 180°)



(stable complex pole — phase steps down by 180°)

minimum-phase / non-~ zeros.

All tf with same magnitude plot, the one with only LHP zeros has the minimal net phase change as ω goes $0\to\infty$.

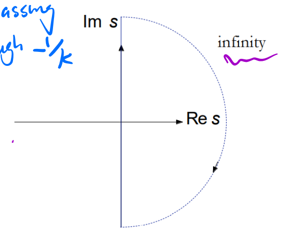


GM: ω s.t. magnitude = 0dB, i.e. $|KG(s)|=1$

PM: ω s.t. phase = 180°.

By Argument Principle:

The Nyquist Thm: $N = Z - P$ (no Im axis poles; plot not passing through $-1/k$)



$H(C)$ = Nyquist plot of H

N = # ① CW ⟳ of 0 by Nyquist plot of $1+KG(s)$.
② ⟳ of -1 by Nyquist Plot of $KG(s)$.
③ CW ⟳ of $-1/k$ by Nyquist plot of $G(s)$.

Z = # ① zeros of $1+KG(s)$ inside C)
② # (RHP zeros of $1+KG$).
③ # (RHP closed-loop poles).

P = # ① poles of $1+KG(s)$ inside C).
② # (RHP poles of $1+KG(s)$)
③ # (RHP roots of $p(s)$)
④ # (RHP open-loop poles).

Nyquist Stability Criterion:
The Nyquist Plot of $G(s)$ CCW ⟲ $-\frac{1}{k}$ for P times,
where $P = Z - N$. ⟺ $\boxed{\text{stable}}$ ⟺ $N = -P$

Nyquist Stability Criterion. The CL system is stable iff. the Nyquist plot of $G(s)$ ⟳ $-1/k$ for P times CCW.
(Bode M, phase ⟹ Nyquist plot.)

Phase of H: $\angle H(s) = \angle \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = \sum^m \underbrace{\psi_i}_{\angle \text{zero}} - \sum^n \underbrace{\varphi_j}_{\angle \text{pole}} = \sum\angle\text{zeros} - \sum\angle\text{poles}$.

Controller Canonical Form: $A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

☆ characteristic polynomial does not change. controllability mx does not change.

$\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$

Controllability: $\mathcal{C}(A,B) = [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B]$.

controllable ⟺ SI: $\det \mathcal{C}(A,B) \neq 0$
general: $\mathcal{C}(A,B)$ invertible.

Convert to CCF: ① check for controllability. ② determine desired $\mathcal{C}(\bar{A},\bar{B})$. $= [\bar{B} \mid \bar{A}\bar{B}]$. ③ compute T.

$T = \mathcal{C}(\bar{A},\bar{B}) \cdot [\mathcal{C}(A,B)]^{-1}$.

$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots \end{bmatrix}$, $a_1 \sim a_n$ obtained from characteristic polynomial

$\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. (where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$).

Pole placement: $\dot{x} = (A - BK)x + Br, \quad y = Cx$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

Closed-loop poles are the roots of the characteristic polynomial

$$\det(Is - A + BK) = (s-p_1)(s-p_2)\cdots$$
$$= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1)$$

General steps of pole placement:
① convert to CCF.
② $u = -\bar{K}\bar{x}$, place desired poles.
③ $u = -\bar{K}\bar{x} = -\underbrace{(\bar{K}T)}_{K}x$, $K = \bar{K}T$.
convert back to the original coordinate.

Observer Canonical Form.

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_n \\ 1 & 0 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{pmatrix}, \quad C = (0 \quad 0 \quad \dots \quad 0 \quad 1)$$

☆ a system in OCF is always observable.

$$\det(Is - A) = \det((Is - A)^T) = \det(Is - A^T)$$
$$= s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

Pole placement: $\dot{x} = Ax, \quad y = Cx, \quad \dot{\hat{x}} = (A - LC)\hat{x} + Ly$

Eigenvalues of $A - LC$ are the roots of the characteristic polynomial

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 1 & 0 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

$$\det(Is - A + LC)$$
$$= s^n + (a_1 + \ell_n)s^{n-1} + \ldots + (a_{n-1} + \ell_2)s + (a_n + \ell_1)$$

Observability: $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$.

C-O Duality: $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$ observable ⟺ $\dot{x} = A^T x + C^T u$ controllable.

General procedure for any *observable* system:

1. Convert to OCF: $T = \underbrace{\mathcal{O}(\bar{A},\bar{C})^{-1}}_{\text{new}} \underbrace{[\mathcal{O}(A,C)]}_{\text{old}}$
2. Find $\bar{L}$, such that $\bar{A} - \bar{L}\bar{C}$ has desired eigenvalues.
3. Convert back to original coordinates: $L = T^{-1}\bar{L}$.

Given an observable pair (A, C):

1. For $F = A^T$, $G = C^T$, consider the system $\dot{x} = Fx + Gu$ (this system is controllable).
2. Use our earlier procedure to find K, such that
$$F - GK = A^T - C^T K$$
has desired eigenvalues.
3. Then
$$\mathrm{eig}(A^T - C^T K) = \mathrm{eig}(A^T - C^T K)^T = \mathrm{eig}(A - K^T C),$$
so $L = K^T$ is the desired output injection matrix.

Final answer: use the observer
$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly$$
$$= (A - K^T C)\hat{x} + K^T y.$$

The resulting observer is $\dot{\hat{x}} = (A - T^{-1}\bar{L}C)\hat{x} + T^{-1}\bar{L}y = (A - LC)\,\hat{x} + Ly$