ECE 486 Control Systems

Lecture 19: Controllability

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Checklist

Modeling | Analysis | Design

Root Locus
Frequency Response
State-Space

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	Root Locus
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	**Control design using frequency response: PD and Lead**	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	**Nyquist stability criterion**	Ch. 6
	Nyquist stability; gain and phase margins from Nyquist plots	Ch. 6
12	**Review B**	
	Term Test B	
13	**Introduction to state-space design**	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	State-Space
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Lecture Overview

- **Review**: state-space notions: canonical forms, controllability
- **Today's topic:** controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.
- **Learning Goal**: explore the effect of pole-zero cancellations on internal stability; understand the effect of coordinate transformations on the properties of a given state-space model (transfer function; open-loop poles; controllability).

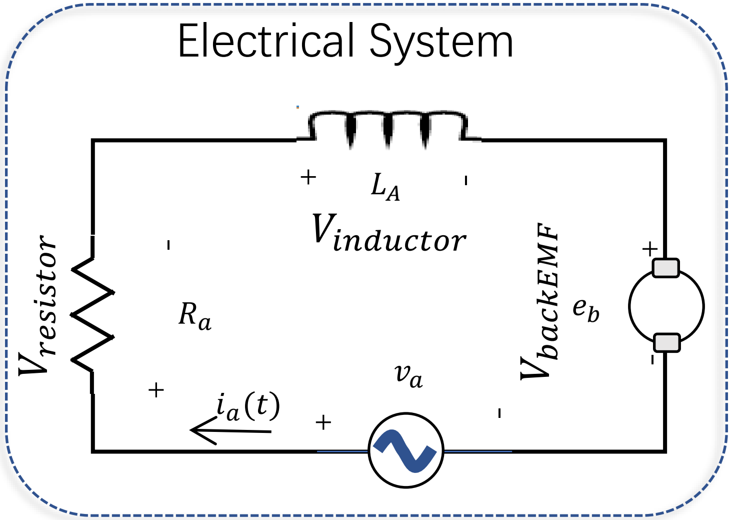
Reading: FPE, Chapter 7

State-Space Model: Example

$\dot{x} = Ax + Bu$

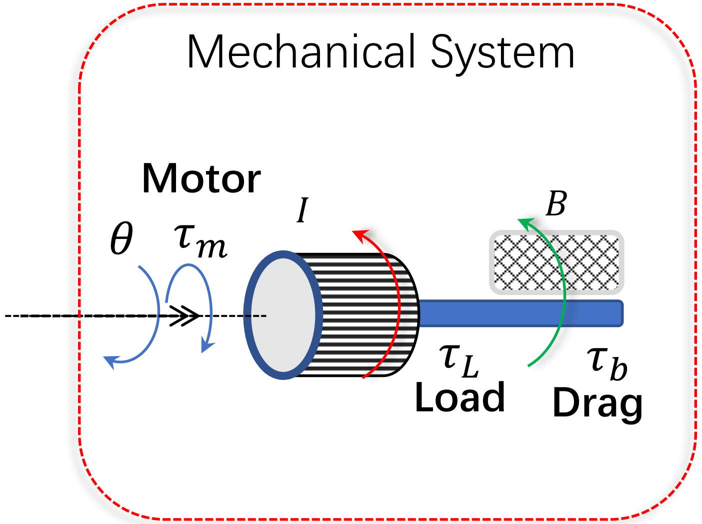
$y = Cx + Du$

Modeling of Dynamic System



$$V_{inductor} = L_a \frac{di_a}{dt};$$
$$V_{resistor} = R_a i_a;$$
$$V_{backEMF} = K_e \dot{\theta}$$

$$\tau_m = K_t i_a;$$
$$\tau_b = B\dot{\theta};$$



$x_1 = i_a$

$x_2 = \omega$

Kirchhoff's Law

$$\boxed{v_a} = V_{inductor} + V_{resistor} + V_{backEMF}$$
$$L_a \frac{di_a}{dt} + R_a i_a + K_e \dot{\theta} = v_a$$

Newton's Law

$$I\ddot{\theta} = \tau_m - \tau_b - \tau_L$$
$$I\ddot{\theta} = K_t i_a - B\dot{\theta} - \tau_L$$
$$I\ddot{\theta} + B\dot{\theta} - K_t i_a = -\tau_L$$

modeling dyn. sys

State-space representation of Dynamic System

$$\left.\begin{aligned} \dot{x}_1 &= \frac{di_a}{dt} = -\frac{R_a}{L_a} i_a - \frac{K_e}{L_a}\omega + \frac{1}{L_a} v_a \\ \dot{x}_2 &= \dot{\omega} = \frac{K_t}{I} i_a - \frac{B}{I}\omega - \frac{1}{I}\tau_L \end{aligned}\right\} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boxed{y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

State-Space Model: Comparison with Transfer Function Approach

$$L_a \frac{di_a}{dt} + R_a i_a + K_e \omega = v_a$$

$$I\dot{\omega} + B\omega - K_t i_a = -\tau_L$$

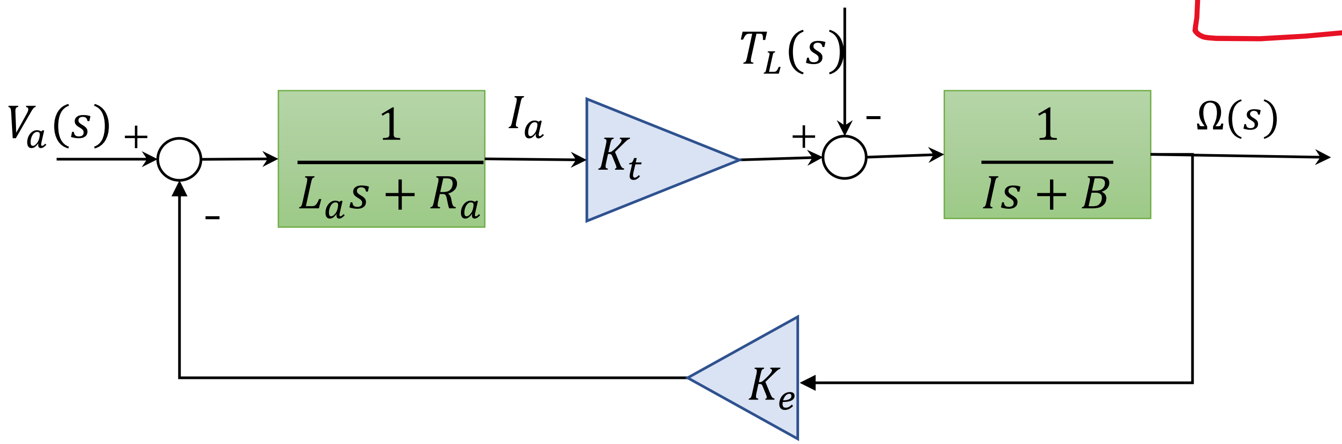
Laplace

$$L_a s I_a(s) + R_a I_a(s) = V_a(s) - K_e \Omega(s)$$

$$Is\Omega(s) + B\Omega(s) = -T_L(s) + K_t I_a(s)$$

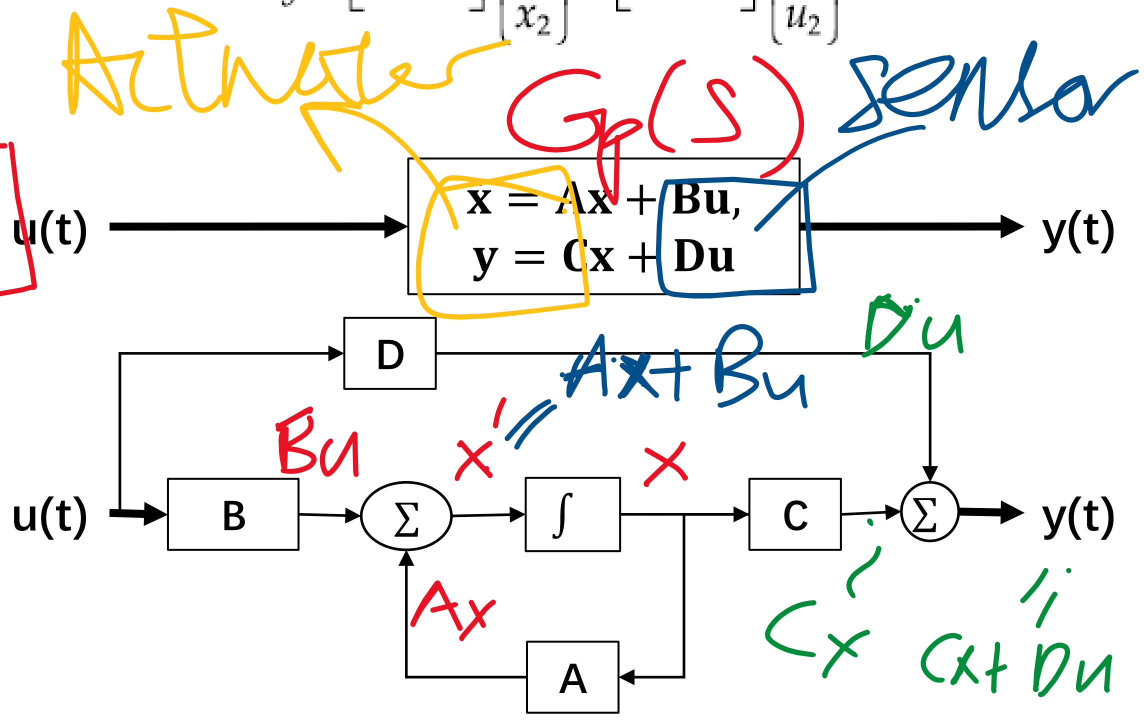
$$\frac{\Omega(s)}{V_a(s)} = \frac{(1/(L_a s + R_a))\cdot K_t \cdot (1/(Is+B))}{1 + (1/(L_a s + R_a))\cdot K_t \cdot (1/(Is+B))\cdot K_e} = \frac{K_t}{L_a I s^2 + \left(L_a B + R_a I\right)s + R_a B + K_t K_e}$$

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(1/(Is+B))}{1 - (1/(Is+B))\cdot(-K_e)\cdot(1/(L_a s + R_a))\cdot K_t} = -\frac{L_a s + R_a}{L_a I s^2 + (L_a B + R_a I)s + R_a B + K_t K_e}$$

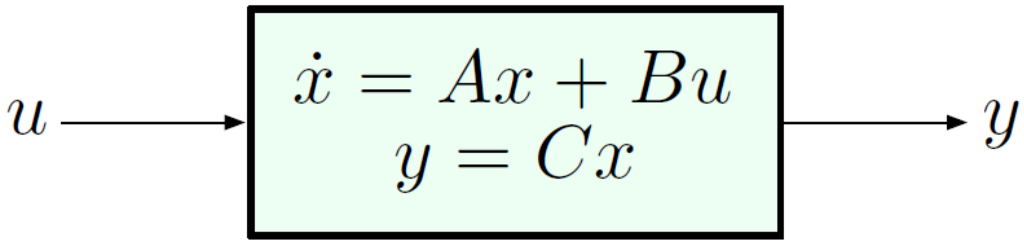


$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_e}{L_a} \\ \frac{K_t}{I} & -\frac{B}{I} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{I} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



State-space realizations



- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*

Review: Controllability

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = [B \,|\, AB \,|\, A^2B \,|\, \ldots \,|\, A^{n-1}B]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is $\boxed{\text{invertible}}$

- As we will see today, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

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Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Recall Example: Computing $\mathcal{C}$(A,B)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A,B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C} = \left[B \,\middle|\, AB \right] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & -5 \end{array} \right]$$

Recall Example: Computing $\mathcal{C}$(A,B)

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Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A,B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C}(A,B) = [B \,|\, AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\implies \mathcal{C}(A,B) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

Is this system controllable?

$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

Let's consider a general zero location $s = z$:

$$G(s) = \frac{s-z}{s^2+5s+6}$$

This gives us a CCF realization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.

OCF with Arbitrary Zeros

Start with the CCF

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} -z & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Convert to OCF: $\qquad (A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \qquad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We already know that this system realizes the same t.f. as the original system.

But is it *controllable*?

OCF with Arbitrary Zeros

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}}_{\bar{A}=A^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -z \\ 1 \end{pmatrix}}_{\bar{B}=C^T} u, \qquad y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{\bar{C}=B^T} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's find the controllability matrix:

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \,|\, \bar{A}\bar{B}] \qquad \bar{A}\bar{B} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} -z \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -z-5 \end{pmatrix}$$

$$\therefore \mathcal{C}(\bar{A}, \bar{B}) = \begin{pmatrix} -z & -6 \\ 1 & -z-5 \end{pmatrix}$$

$$\det \mathcal{C} = z(z+5) + 6 = z^2 + 5z + 6 = 0 \quad \text{for } z = -2 \text{ or } z = -3$$

The OCF realization of the transfer function $G(s) = \dfrac{s-z}{s^2+5s+6}$ is not controllable when $z = -2$ or -3, even though the CCF is always controllable.

Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

$$G(s) = \frac{s-z}{s^2+5s+6}$$

is not controllable when $z = -2$ or -3, even though the CCF is always controllable.

Let's examine $G(s)$ when $z = -2$:

$$G(s) = \left.\frac{s-z}{s^2+5s+6}\right|_{z=-2} = \frac{\cancel{s+2}}{(\cancel{s+2})(s+3)} = \frac{1}{s+3}$$

— pole-zero cancellation!

For $z = -2$, $G(s)$ is a first-order transfer function, which can always be realized by this 1st-order controllable model:

$$\dot{x}_1 = -3x_1 + u, \; y = x_1 \quad \longrightarrow \quad G(s) = \frac{1}{s+3}$$

We can look at this from another angle: consider the t.f.

$$G(s) = \frac{1}{s+3}$$

We can realize it using a one-dimensional controllable state-space model

$$\dot{x}_1 = -3x_1 + u, \quad y = x_1$$

or a noncontrollable two-dimensional state-space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

— certainly not the best way to realize a simple t.f.!

Thus, even the *state dimension* of a realization of a given t.f. is not unique!!

Beware of Pole-Zero Cancellations!

Here is a really bad realization of the t.f.

$$G(s) = \frac{1}{s+3}.$$

Use a two-dimensional model:

$$\begin{aligned}\dot{x}_1 &= -3x_1 + u \\ \dot{x}_2 &= 100x_2 \\ y &= x_1\end{aligned}$$

- x_2 is not affected by the input u (i.e., it is an uncontrollable mode), and not visible from the output y
- does not change the transfer function
- ... and yet, horrible to implement: $x_2(t) \propto e^{100t}$

The transfer function can mask undesirable internal state behavior!!

Pole-Zero Cancellation and Stability

- In case of a pole-zero cancellation, the t.f. contains *much less* information than the state-space model because some dynamics are "hidden."
- These dynamics can be either good (stable) or bad (unstable), but we cannot tell from the t.f.
- Our original definition of stability (no RHP poles) is flawed because there can be RHP eigenvalues of the system matrix A that are canceled by zeros,yet they still have dynamics associated with them.

Definition of Internal Stability (State-Space Version): a state-space model with matrices (A, B, C, D) is *internally stable* if all eigenvalues of the A matrix are in LHP.

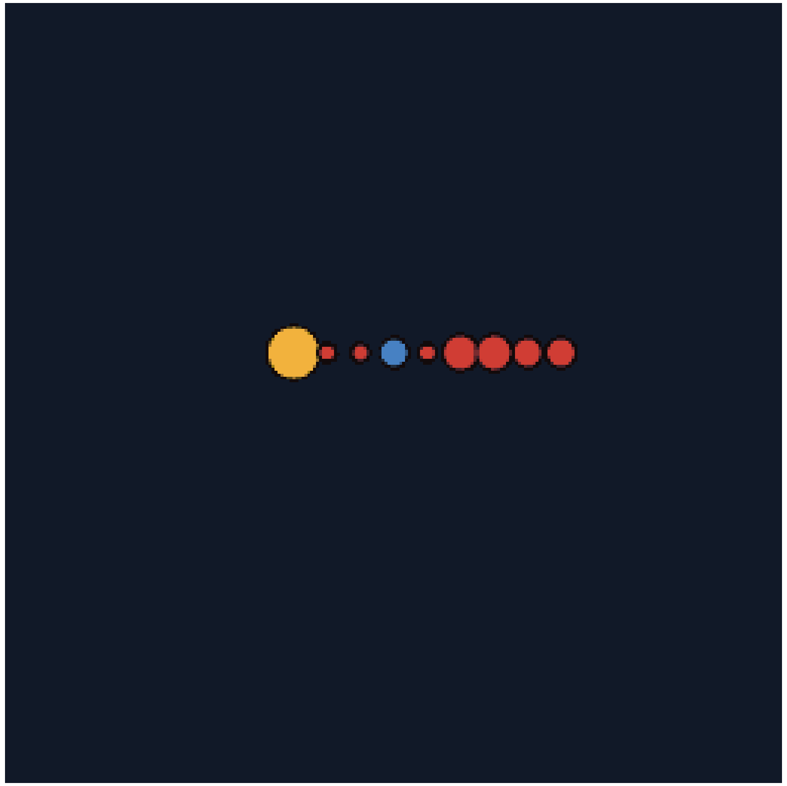
This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.

Any systematic approach to generate desired realization?

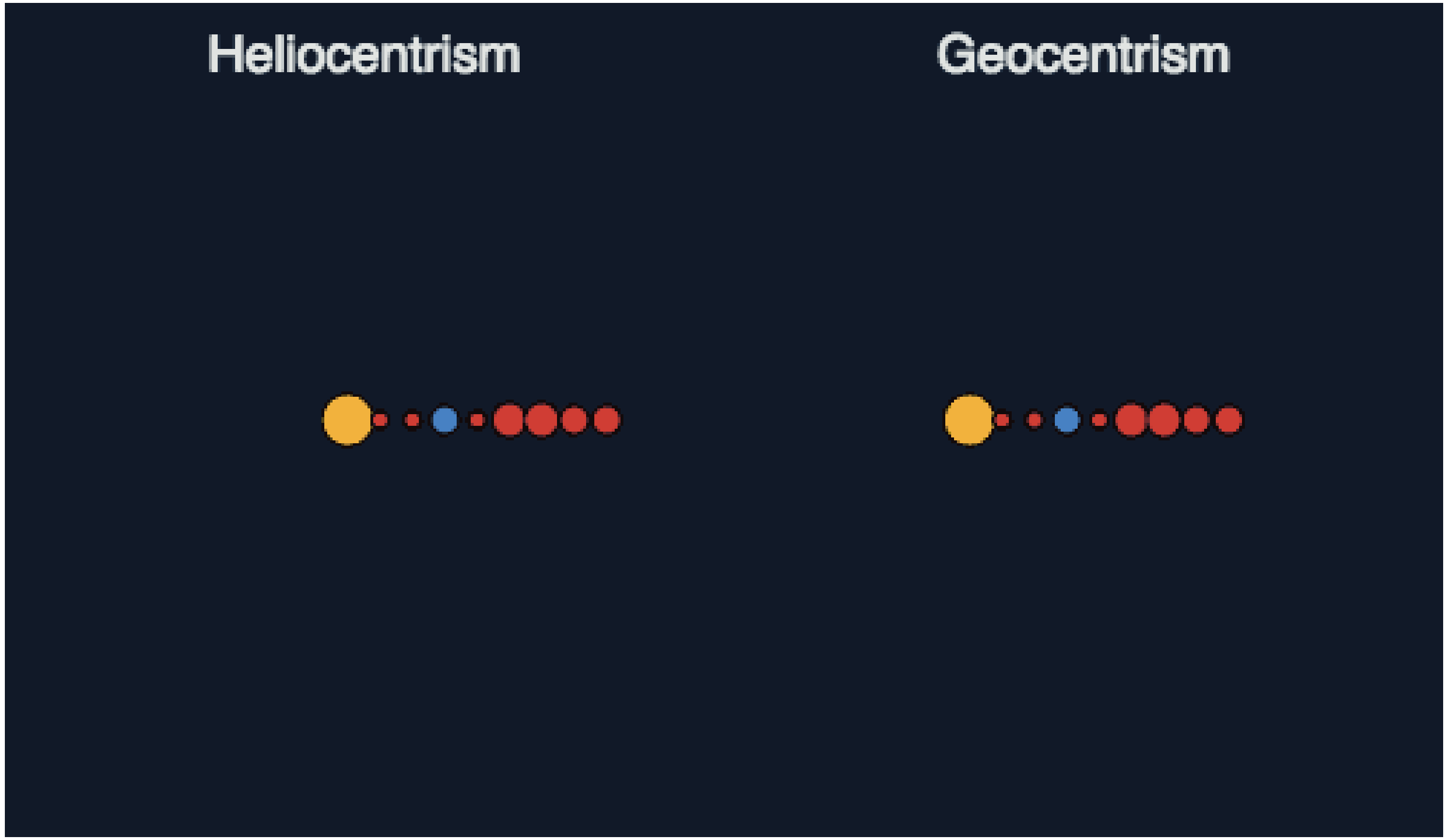
Coordinate Transformation

- A given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realization preferably with favorable properties (like controllability)
- One way is by means of coordinate transformations

Same system from a different coord. system

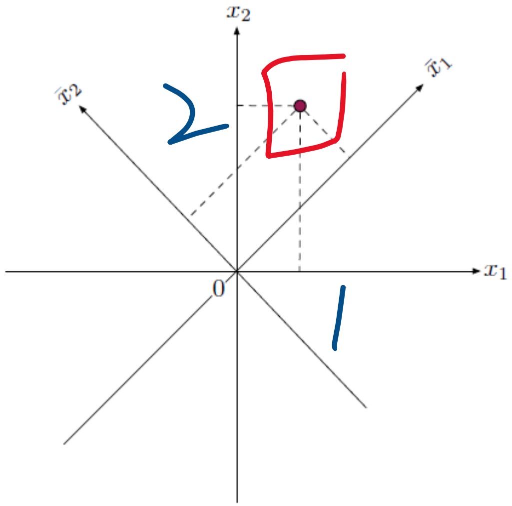


Same system from a different coord. system



Coordinate Transformation

$\to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 i + x_2 j$



$x \longmapsto \bar{x} = Tx,$ $\qquad T \in \mathbb{R}^{n\times n}$nonsingular

$x = T^{-1}\bar{x}$ $\qquad$ (go back and forth between the coordinate systems)

For example,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

This can be represented as

$$\bar{x} = Tx, \qquad \text{where } \boxed{T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The transformation is invertible: $\det T = -2$, and

$$T^{-1} = \frac{1}{\det T}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Or we can see this directly:

$$\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2$$

Coord. Transform and State-Space Models

Consider a state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

and a change of coordinates $\bar{x} = Tx$ (T invertible).

What does the system look like in the new coordinates?

$$\begin{aligned}
\dot{\bar{x}} &= \dot{\overline{Tx}} = T\dot{x} && \text{(linearity of derivative)} \\
&= T(Ax + Bu) \\
&= T(AT^{-1}\bar{x} + Bu) && (x = T^{-1}\bar{x}) \\
&= \underbrace{TAT^{-1}}_{\bar{A}}\bar{x} + \underbrace{TB}_{\bar{B}}u \\
y &= Cx \\
&= \underbrace{CT^{-1}}_{\bar{C}}\bar{x}
\end{aligned}$$

Coord. Transform and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{T} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$$

What happens to

- the transfer function?
- the controllability matrix?

Coord. Transform and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \xrightarrow{T} \qquad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\quad \bar{B} = TB$, $\quad \bar{C} = CT^{-1}$

Claim: The transfer function doesn't change.

Proof:

$$\begin{aligned} \bar{G}(s) &= \bar{C}(Is - \bar{A})^{-1}\bar{B} \\ &= (CT^{-1})\left(Is - TAT^{-1}\right)^{-1}(TB) \\ &= CT^{-1}\left(TIT^{-1}s - TAT^{-1}\right)^{-1}TB \\ &= CT^{-1}\left[T\left(Is - A\right)T^{-1}\right]^{-1}TB \\ &= C\underbrace{T^{-1}T}_{I}(Is - A)^{-1}\underbrace{T^{-1}T}_{I}B \\ &= C\left(Is - A\right)^{-1}B \equiv G(s) \end{aligned}$$

Coord. Transform and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{T} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\quad \bar{B} = TB$, $\quad \bar{C} = CT^{-1}$

The transfer function doesn't change.

In fact:

- open-loop poles don't change
- characteristic polynomial doesn't change:

$$\begin{aligned} \det(Is - \bar{A}) &= \det(Is - TAT^{-1}) \\ &= \det\left[T(Is - A)^{-1}T^{-1}\right] \\ &= \det T \cdot \det(Is - A)^{-1} \cdot \det T^{-1} \\ &= \det(Is - A)^{-1} \end{aligned}$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{T} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\quad \bar{B} = TB$, $\quad \bar{C} = CT^{-1}$

Claim: Controllability doesn't change.

Proof: For any $k = 0, 1, \ldots,$

$$\bar{A}^k\bar{B} = (TAT^{-1})^k TB = TA^kT^{-1}TB = TA^kB \qquad \text{(by induction)}$$

Therefore, $\mathcal{C}(\bar{A}, \bar{B}) = [TB \,|\, TAB \,|\, \ldots \,|\, TA^{n-1}B]$

$$\begin{aligned} &= T[B \,|\, AB \,|\, \ldots \,|\, A^{n-1}B] \\ &= T\mathcal{C}(A, B) \end{aligned}$$

Since $\det T \neq 0$, $\det \mathcal{C}(\bar{A}, \bar{B}) \neq 0$ if and only if $\det \mathcal{C}(A, B) \neq 0$.

Thus, the new system is controllable if and only if the old one is.

$\Longleftrightarrow$

Coord. Transform and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{T} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}$, $\quad \bar{B} = TB$, $\quad \bar{C} = CT^{-1}$

Note: The *controllability matrix* does change:

$$\underbrace{\mathcal{C}(\bar{A},\bar{B})}_{\text{new}} = \underbrace{T}_{\substack{\text{coord.}\\ \text{trans.}}} \underbrace{\mathcal{C}(A,B)}_{\text{old}}$$

$$\Updownarrow$$

$$T = \mathcal{C}(\bar{A},\bar{B})\,[\mathcal{C}(A,B)]^{-1}$$

This is a recipe for going from one *controllable* realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to *convert a given controllable system to CCF* (useful for control design).

Converting a Controllable System to CCF

Note!! The way I do this is different from the textbook.

Consider $A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (C is immaterial).

Convert to CCF if possible.

Step 1: check for controllability.

$$AB = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix} \implies \mathcal{C} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$\det \mathcal{C} = -1$ – controllable

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$$\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out $\boxed{\bar{A} \text{ and } \bar{B}}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we need to find the coefficients a_1, a_2.

Recall: the characteristic polynomial does not change:

$$\det(Is - A) = \det(Is - \bar{A})$$

$$\det \begin{pmatrix} s+15 & -8 \\ 15 & s-7 \end{pmatrix} = \det \begin{pmatrix} s & -1 \\ a_2 & s+a_1 \end{pmatrix}$$

$$(s+15)(s-7) + 120 = s(s+a_1) + a_2$$

$$s^2 + 8s + 15 = s^2 + a_1 s + a_2$$

Step 2: Determine desired $\mathcal{C}(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \qquad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have just computed

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore, the new controllability matrix should be

$$\mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} \,|\, \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

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Step 3: Compute T.

Recall: $T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1}$

$$\boxed{\mathcal{C}(A, B) = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}}$$

$$[\mathcal{C}(A, B)]^{-1} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}^{-1}$$

$$= \frac{1}{-1}\begin{pmatrix} -8 & 7 \\ -1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix}}$$

$$\mathcal{C}(\bar{A}, \bar{B}) = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

$$\boxed{T} = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}\begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$T\,C(A,B) = \bar{C}(\bar{A},\bar{B})$$

In the next lecture, we will see why CCF is so useful.