A General State-Space Model.

State: $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

output: $y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p$

Input: $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$

System mx. input mx.

State-Space Model.

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

output mx. feedthrough mx.

$\Rightarrow$ $sX - x(0) = AX + BU$; $(Is - A)X = x(0) + BU$. $X = (Is - A)^{-1}(x(0) + BU)$.

$Y = CX + DU$.

$\Rightarrow Y = C(Is - A)^{-1}(x(0) + BU) + DU$.

$= C(Is - A)^{-1}BD + DU$

Transfer function: $\boxed{G(s) = C(Is - A)^{-1}B + D}$

uncontrollable.

★ 1. $G(s)$ undefined when $\mathbf{I}s - \mathbf{A}$ is singular, i.e. $\det(\mathbf{I}s - \mathbf{A}) = 0$

2. Roots of $(\mathbf{I}s - \mathbf{A} = 0)$ are the eigenvalues of A.

3. Claim: The state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{C} = B^T$$

has the same transfer function as the original model with (A, B, C).

one tf $\leftrightarrow$ ∞ # of ss models

Note: for a 2x2 mx, $\boxed{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \Longrightarrow M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Canonical Forms.

① Controller Canonical Form: $A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ $\Leftarrow$ always controllable.

e.g. For $G(s) = \frac{s+1}{s^2+5s+6}$, $A = \begin{bmatrix} 0 & 1 \\ \boxed{-6 \quad -5} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Claim.

$$\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

— the last row of the A matrix in CCF consists of the coefficients of the characteristic polynomial, in reverse order, with "–" signs.

② Observer Canonical Form: $\begin{cases} \dot{x} = \bar{A}x + \bar{B}u \\ y = \bar{C}x + \bar{D}u \end{cases}$ where $\bar{A} = A^T$, $\bar{B} = B^T$, $\bar{C} = C^T$, $\bar{D} = D^T$.

in CCF

③ Modal Canonical Form:

$$A = \begin{bmatrix} p & 0 & \cdots & 0 \\ 0 & p & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Controllability.

Controllability Matrix: $\mathcal{C}(A, B) = [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B]$.

System is controllable $\Leftrightarrow$ $\det \mathcal{C}(A, B) \neq 0$. (for SI system).

$\Leftrightarrow$ $\mathcal{C}(A, B)$ is invertible.