Approximate PI via Dynamic Compensation.

$K\frac{s+z}{s} \longrightarrow K\frac{s+z}{s+p}$ $(p<z)$ ← lag compensation.

Lag compensation *does not* give perfect tracking (indeed, it does not change system type), but we can get as good a tracking as we want by playing with K, z, p. On the other hand, unlike PI, lag compensation gives a stable controller.

Ex1 Pole Placement Via RL

$s^2+2\zeta\omega_n s+\omega_n^2$

$2\zeta\omega_n = 4$

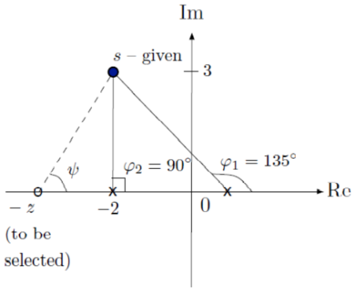
$\zeta = \frac{2}{\sqrt{13}}$

Let $G_p(s) = \frac{1}{s-1}$, $G_c(s) = K\frac{s+z}{s+p}$

Problem: given $p=2$, find K and z to place poles at $-2 \pm 3j$.

Desired characteristic polynomial:

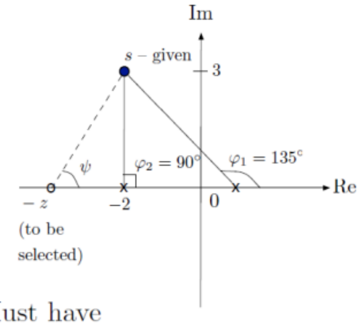
$(s+2)^2 + 9 = s^2 + 4s + 13$, damping ratio $\zeta = \frac{2}{\sqrt{13}} \approx 0.555$



Must have

$$\underbrace{\psi}_{\substack{\text{angle from}\\ s \text{ to zero}}} - \sum_i \underbrace{\varphi_i}_{\substack{\text{angles from}\\ s \text{ to poles}}} = 180^\circ$$

So, we want $\psi = 180^\circ + \sum_i \varphi_i$



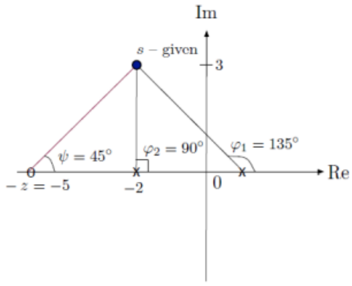
We have
$\varphi_1 = 135^\circ$,
$\varphi_2 = 90^\circ$

We want $\psi = 180^\circ + \sum_i \varphi_i$

Must have

$$\begin{aligned}\psi &= 180^\circ + 135^\circ + 90^\circ\\ &= 405^\circ\\ &= 45^\circ \mod 360^\circ\end{aligned}$$

Thus, we should have $z=-5$



- resulting characteristic polynomial:

$$(s-1)(s+2) + K(s+5)$$
$$s^2 + (K+1)s + 5K - 2$$

- compare against desired characteristic polynomial:

$$s^2+4s+13 \implies K+1=4,\ 5K-2=13$$

so we need $K=3$

- compute s.s. tracking error: $\left|\frac{1}{1-\frac{Kz}{p}}\right| = \frac{1}{6.5} \approx 15\%$

Summary: PI & PD.

PD control:

- provides stability, allows to shape transient response specs
- replace noncausal D-controller Ks with a causal, stable lead controller $K\frac{s+z}{s+p}$, where $p > z$
- this introduces a zero in LHP (at $-z$), pulls the root locus into LHP
- shape of RL differs depending on how large p is

PI control:

- provides stability and perfect steady-state tracking of constant references
- replace unstable I-controller K/s with a stable lag controller $K\frac{s+z}{s+p}$, where $p < z$
- this does not change the shape of RL compared to PI

Frequency Response Design Method.

① Sinusoidal :

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \xrightarrow{\text{set } s=j\omega} M\sin(\omega t+\phi).$$

Derivation:

1. $u(t) = e^{st} \longmapsto y(t) = G(s)e^{st}$
2. Euler's formula: $\sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$
3. By linearity,

$$\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \qquad G(j\omega) = M(\omega)e^{j\phi(\omega)}$$

$$= \frac{M(\omega)e^{j(\omega t+\phi(\omega))} - M(\omega)e^{-j(\omega t+\phi(\omega))}}{2j}$$

$$= M(\omega)\sin(\omega t + \phi(\omega))$$

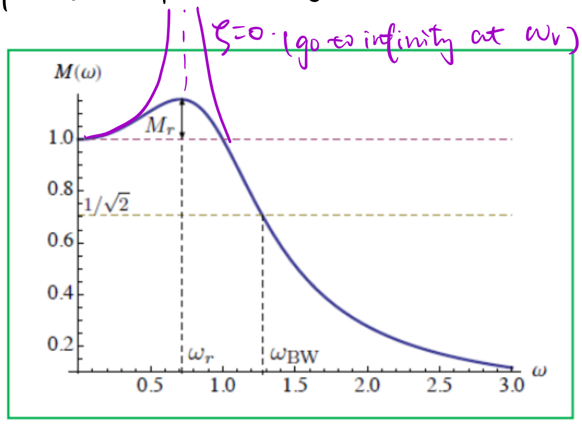
② 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Rightarrow M(\omega) = G(j\omega) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

Frequency Response Magnitude Plot.

$\zeta = 0$. (go to infinity at ω_r).



ω_r – resonant frequency

M_r – resonant peak

ω_{BW} – bandwidth

$M_r \downarrow \Rightarrow$ Better Damping.

$\omega_{BW} \uparrow \Rightarrow \omega_n \uparrow \Rightarrow t_r \downarrow$.

We can get the following formulas using calculus:

$$\begin{cases} \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \quad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}};\ \text{for } \zeta \geq \frac{1}{\sqrt{2}},\ \omega_r = 0\right)$$

$$\omega_{\text{BW}} = \omega_n\sqrt{\underbrace{(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2+1}}_{=1 \text{ for } \zeta=1/\sqrt{2}}}$$

— so, if we know $\omega_r, M_r, \omega_{\text{BW}}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

$\zeta = 0 \Rightarrow$ no damping.

Scale Convention for Bode Plots.

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Bode Form.

const. term in each factor $=1$, i.e. lump all DC gains into one number in the front

e.g.

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

rewrite as
$$\left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$