ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 486 Control Systems

Lecture 09: Term Test 1 Review

Liangjing Yang

Assistant Professor, ZJU-UIUC Institute

liangjingyang@intl.zju.edu.cn

Checklist

Modeling | Analysis | Design

Root Locus
Frequency Response
State-Space

Wk	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	National Holiday Week	
4	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	**Review A**	
	Term Test A	
7	Introduction to Root Locus design method	Ch. 5
	Root Locus continued; introduction to dynamic compensation	Root Locus
8	Lead and lag dynamic compensation	Ch. 5
	Lead and lag continued; introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Wk	Topic	Ref.
9	Bode plots for three types of transfer functions	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	Control design using frequency response	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	State-Space Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Lecture Overview

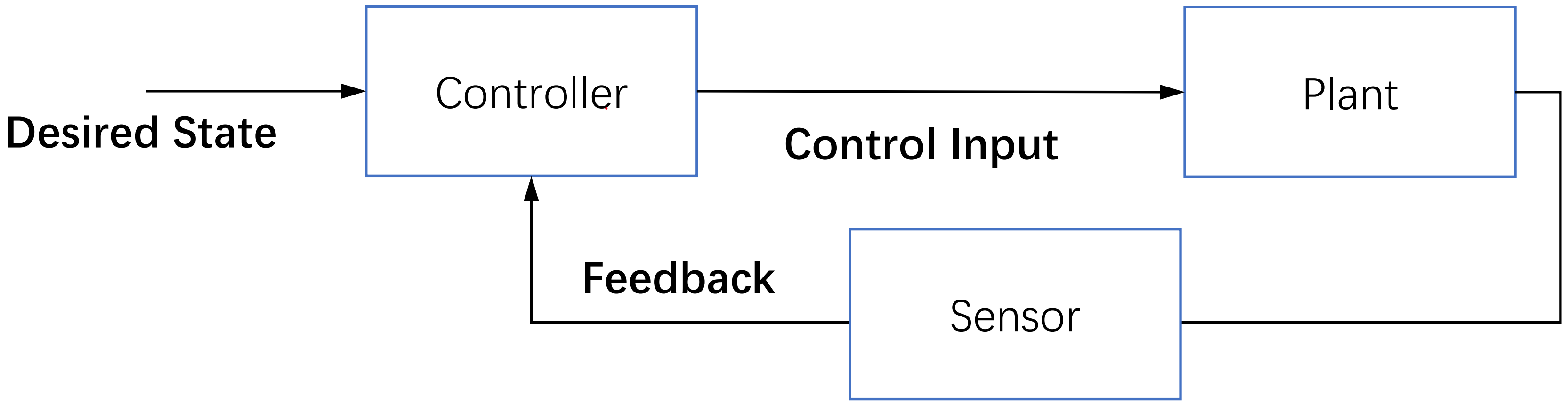
- Recap Lecture08:
  - Basic properties and benefits of feedback control
  - PID control
- Review Week 01/2
  - State-space models of systems; linearization
  - Linear systems and their dynamic response
  - Transient and steady-state dynamic response with arbitrary initial conditions
- Review Week 04/5
  - System modeling diagrams; prototype second-order system
  - Transient response specifications

Recap Week05: Open vs Closed Loop

Open-loop

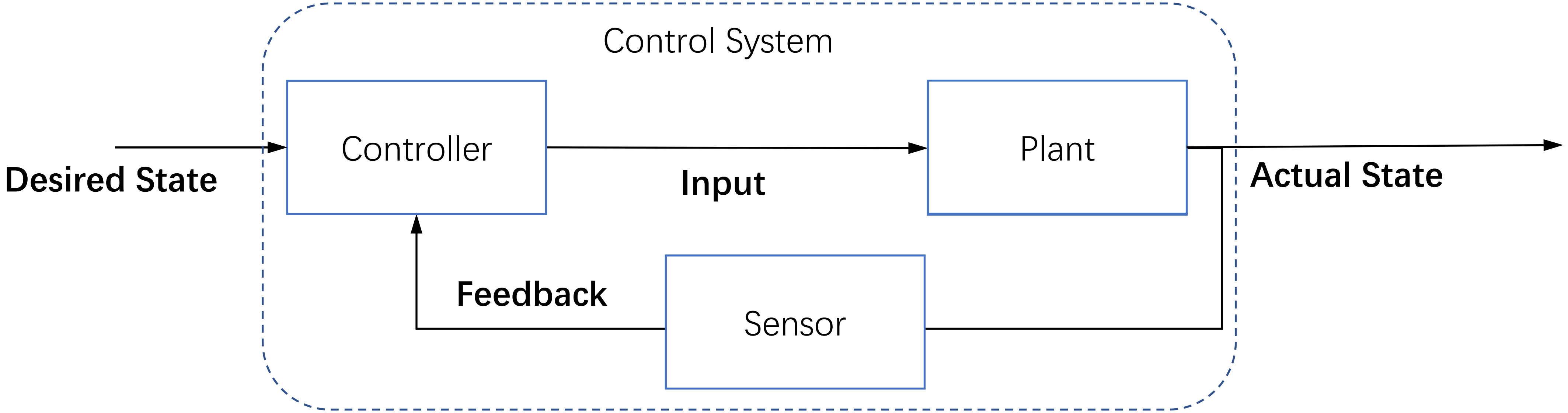
Through preestablished model, generate the command that will achieve the desired state.

Closed-loop

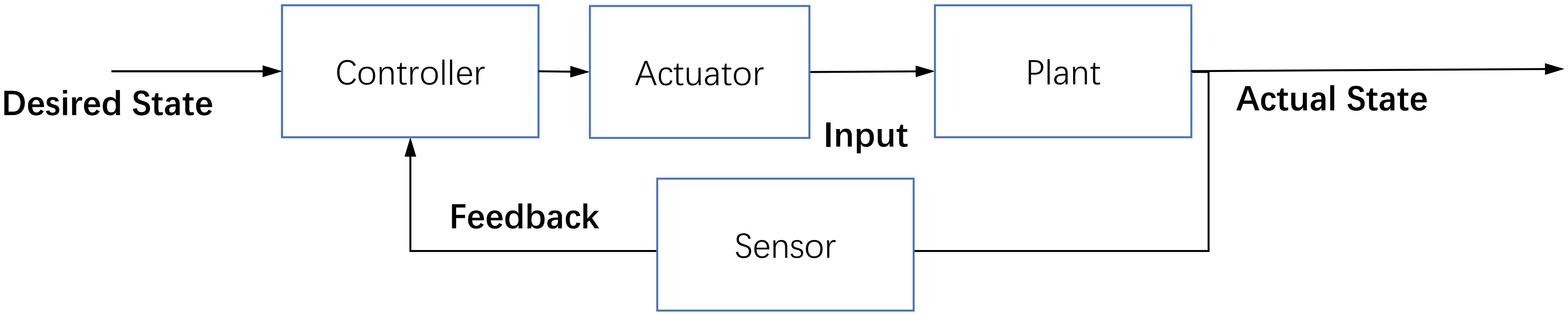


Through sensors, we are able to **feedback** the measurement to produce the command that will minimize the error between desired and actual targeted profile.

Recall: Feedback Control



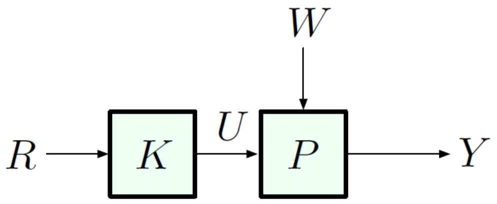
Recall: Terminology



- **Plant** is the system being controlled
- **Sensors** measure the quantity that is subject to control
- **Actuators** act on the plant
- **Controller** processes the sensor signals and drives the actuators
- **Control law** is the rule for mapping sensor signals to actuator signals

Recap Week05: Open vs Closed Loop

- Open-loop control



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

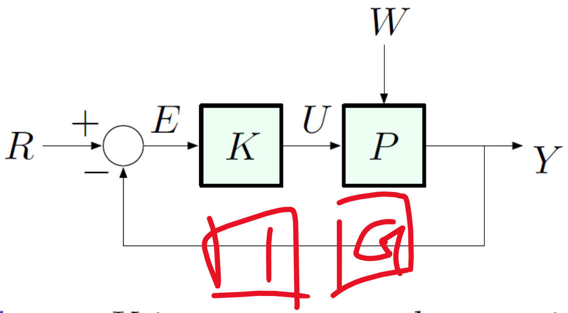
  e.g., if both K and P are stable (all poles in OLHP),

  $$\frac{Y}{R} = KP$$

  is also stable:

  $$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

  $$\frac{Y}{R} = \frac{KP}{1 + KP}$$

  has new poles, which may be unstable

$$\frac{KP}{1+GKP}$$

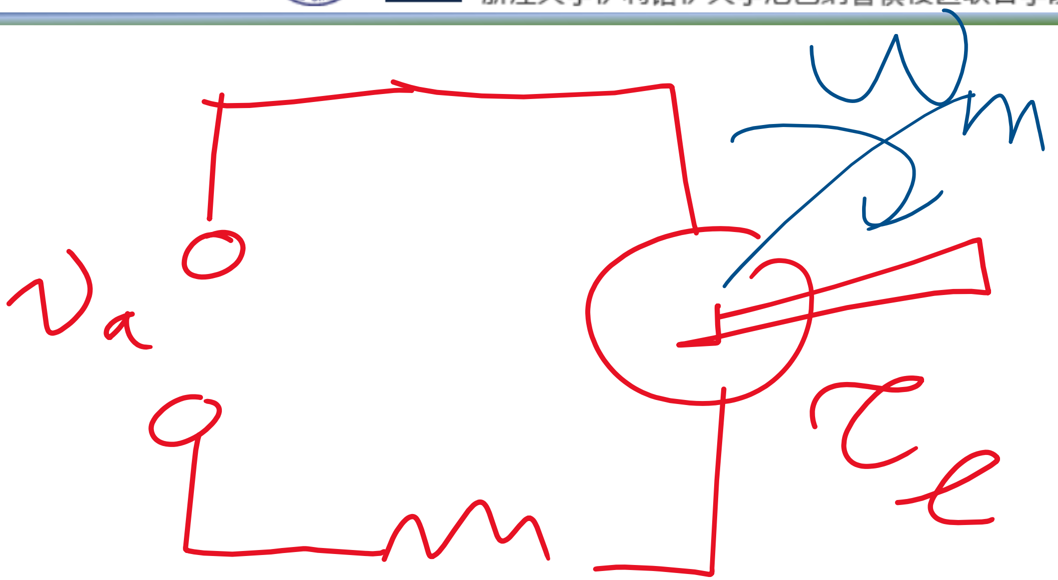
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

- track a given reference
- reject disturbances
- meet performance specs

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Case Study: DC Motor



Inputs: v_{a} – input voltage

τ_{e} – load/disturbance torque

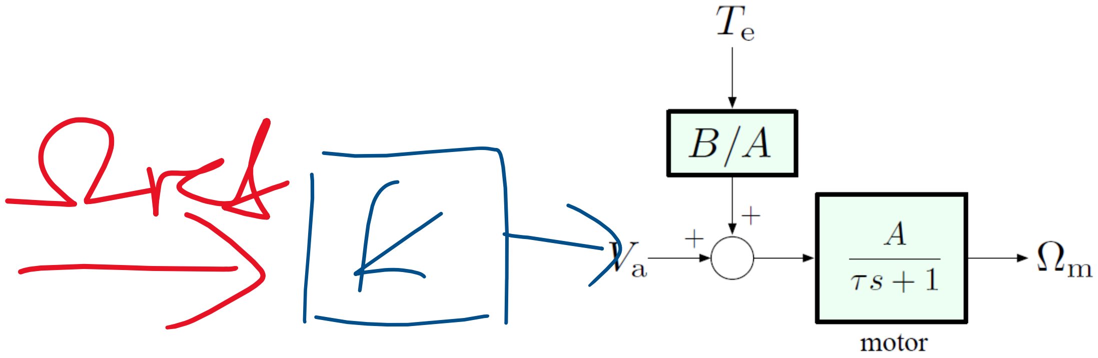
Outputs: ω_{m} – angular speed of the motor

Transfer function:

$$\Omega_{\mathrm{m}} = \frac{A}{\tau s + 1} V_{\mathrm{a}} + \frac{B}{\tau s + 1} T_{\mathrm{e}}$$

τ – time constant

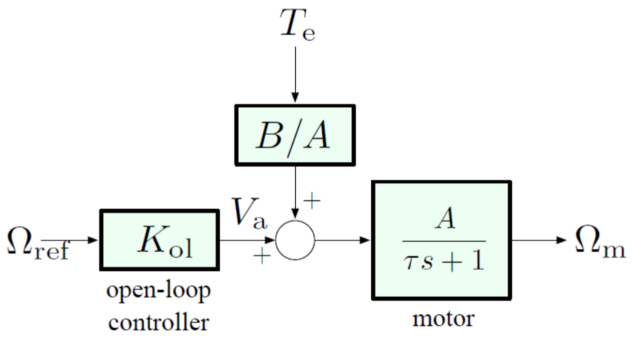
A, B – system gains



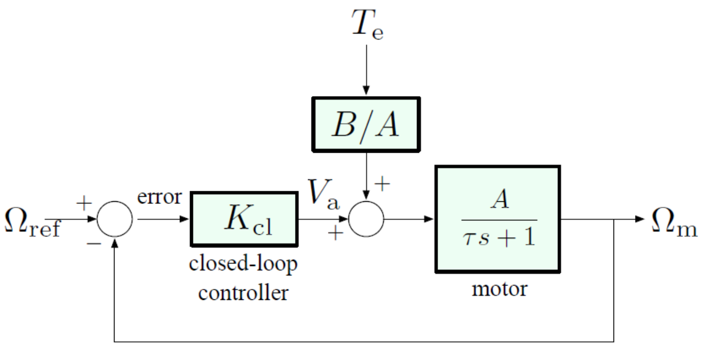
Objective: have Ω_{m} approach and track a given reference $\boxed{\Omega_{\mathrm{ref}}}$ in spite of disturbance T_{e}.

Case Study: DC Motor

- Open-loop control

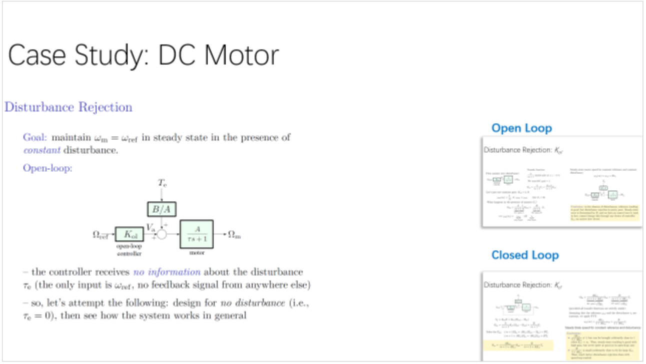


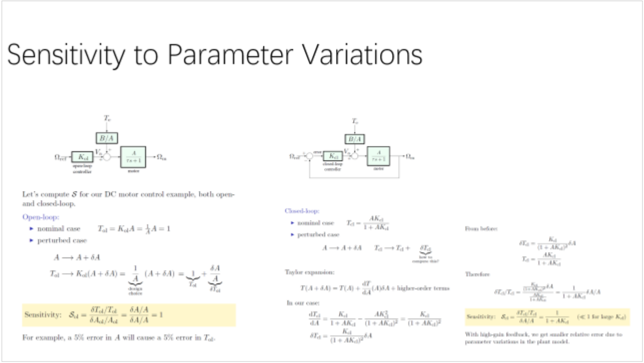
- Feedback (closed-loop) control

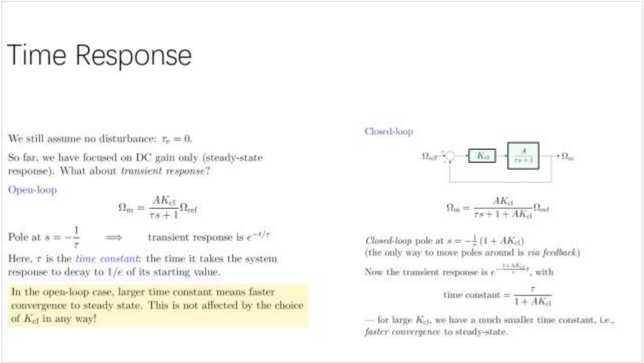


Summary

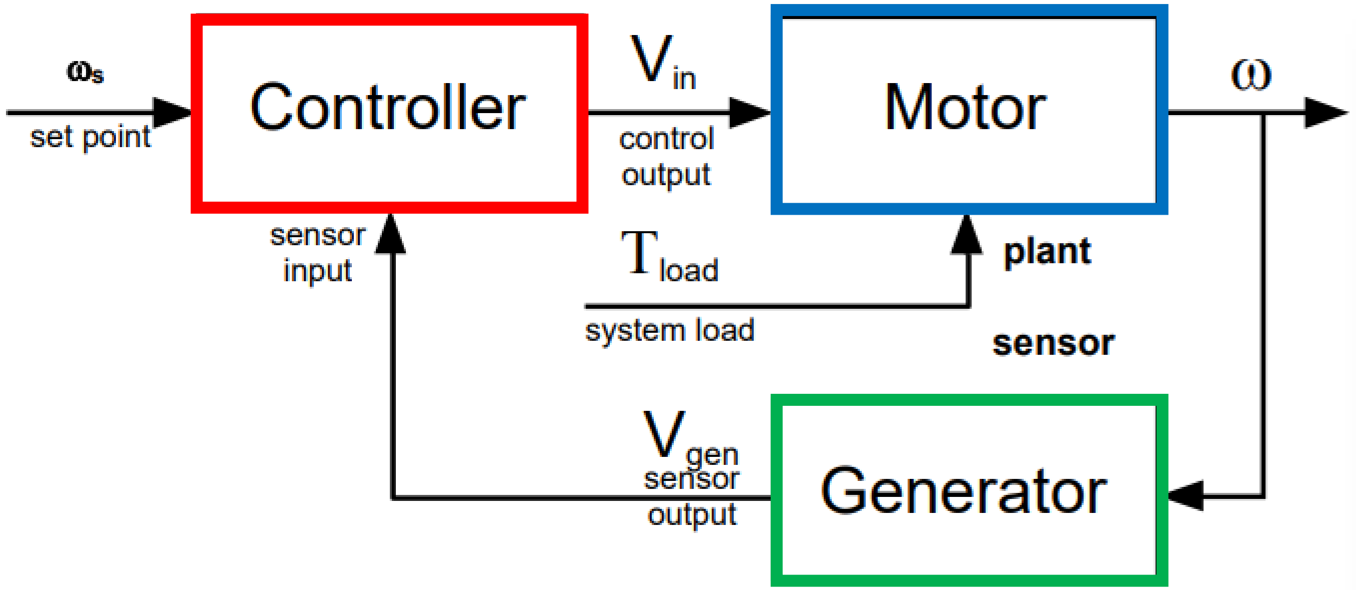
- Feedback Control:
  - reduces steady-state error to disturbances
  - reduces steady-state sensitivity to model uncertainty (parameter variations)
  - improves time response
- However, what we see so far works well for first order systems
  - static gain may cause underdamping or instability in higher order systems
- More sophisticated control: example PID

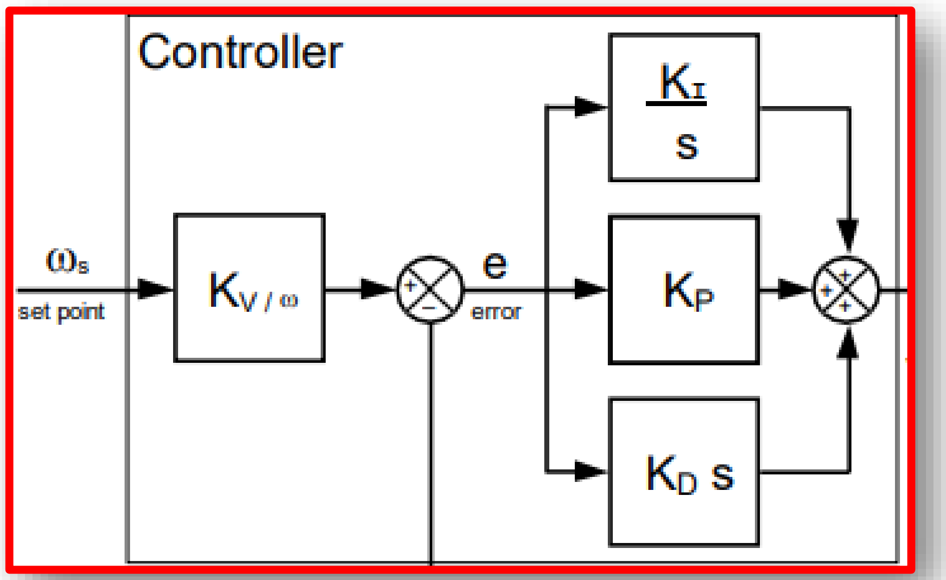






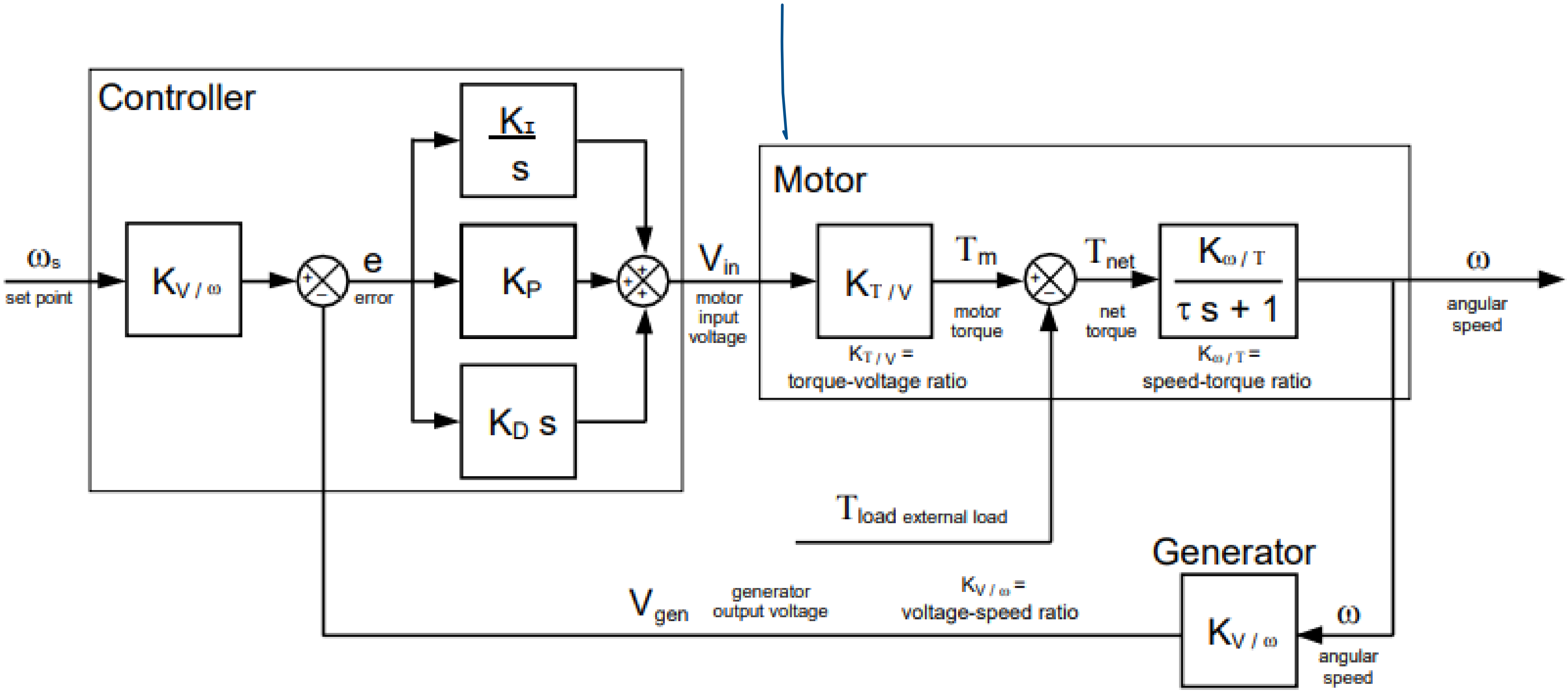
System Representation of DC motor (Plant) + Generator (Sensor) + Controller



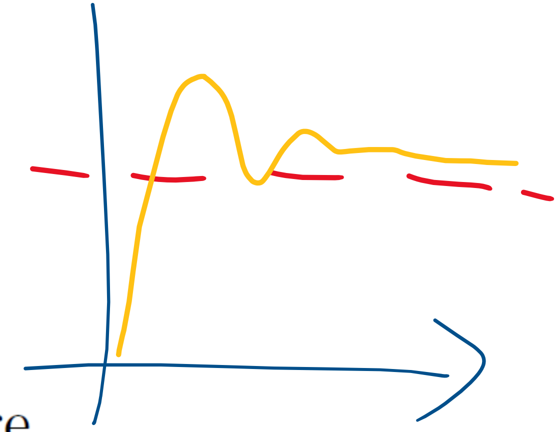


Control System

Representation: DC motor + Generator + Controller



PID Control: Summary & Further Comments



P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

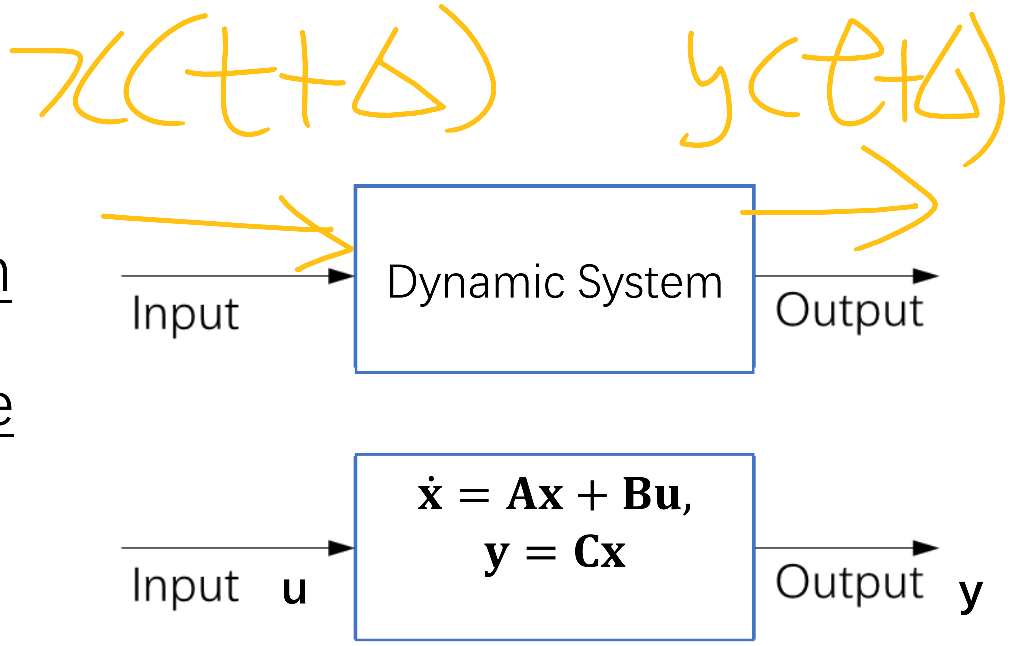
- arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

- but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (integrator wind-up)

Week 01/2 Take away

- **Dynamic Systems** consist of components with inputs-outputs related to time varying function
- **Control systems** are designed to achieve a targeted **output** by generating the appropriate **inputs** in a **dynamical environment** (within specified **performance criteria**)
- **Linear Time-Invariant Casual** Systems
- **State-space form** represents systems of ODEs (of various order) as a larger system of first order ODEs
- The process of **linearization** linearizes a non-linear model about an operating point (equilibrium point with known initial conditions)



Non-linear $\dot{x}(t) = f(x(t), u(t))$

Linear $\dot{x}(t) = Ax(t) + Bu(t)$

Quick Overview: System Representation & Analysis

Mathematical Representation

State space model:

State Equation $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$

Output Equation $y = (b_0 \quad b_1)\begin{pmatrix}x_1\\ x_2\end{pmatrix}$

Configuration form

Equations of Motion
$$\begin{cases}\ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\\ \dots\\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)\end{cases}$$

Initial Conditions
$$\begin{cases}q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0}\\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0}\end{cases}$$

Transfer Function:

$$\frac{O(s)}{I(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_o}{s^n + a_{n-1}s^{n-1} + \cdots + a_o}$$

ICs= 0

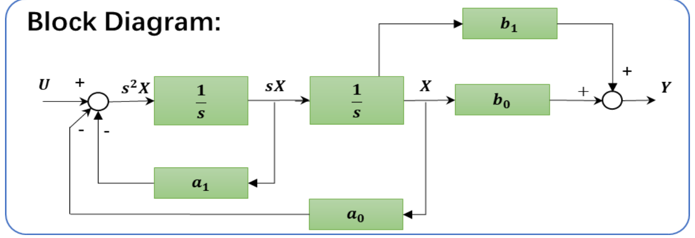
Systematic Modeling

Input | Dynamic System | Output

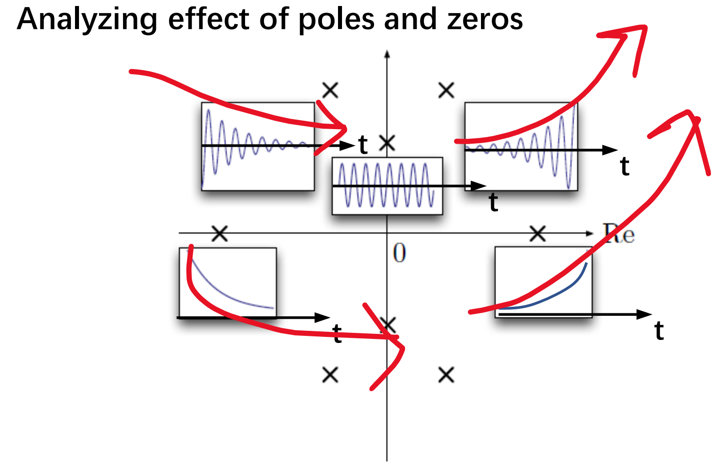
$u(\#)$ → LTI → $y(\#)$

$i(t)$ → $*\, h(t)$ → $o(t)$

$I(s)$ → $H(s)$ → $O(s)$



Analysis of Systems



- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Stability Analysis

Dynamic Response Specification

Design Methods

Dynamic Response

$$u(t) \longrightarrow \boxed{\ast\, h} \longrightarrow y(t)$$

We are interested in computing the response y of a given input u under a given set of ICs

The total response consists of:

- Transient response
  - dependent on the IC
- Steady-state response
  - dominating factor when the effect of IC fade away

Reminder: the *two-sided* $\boxed{\text{Laplace transform}}$ of a function $f(t)$ is

$$F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}\mathrm{d}\tau, \qquad s \in \mathbb{C}$$

time domain	frequency domain
$u(t)$	$U(s)$
$h(t)$	$H(s)$
$y(t)$	$Y(s)$

convolution in time domain $\longleftrightarrow$ multiplication in frequency domain

$$y(t) = h(t) \star u(t) \quad \longleftrightarrow \quad Y(s) = H(s)U(s)$$

The Laplace transform of the impulse response

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\mathrm{d}\tau,$$

is called the transfer function of the system.

Review: System Response

- System response describes the behavior of a dynamic system
- Free and Forced Response
  - Total Response $x(t) = \boxed{x_h(t)} + \boxed{x_p(t)}$
  - Free Response: x_h is the solution
  - Forced Response: x_p is determined by the forcing function f
- Transient and Steady-State
  - Total Response $x(t) = \boxed{x_{tr}(t)} + \boxed{x_{ss}(t)}$
  - x_{tr}, Transient State: component that decays towards zero
  - x_{ss}, Steady State: component that remains after the x_t decays towards 0

Laplace Transforms and Differentiation

Given a differentiable function f, what is the Laplace transform $\mathscr{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad \text{(integrate by parts)} \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \to 0$ as $t \to \infty$

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ — this is how we account for I.C.'s

Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

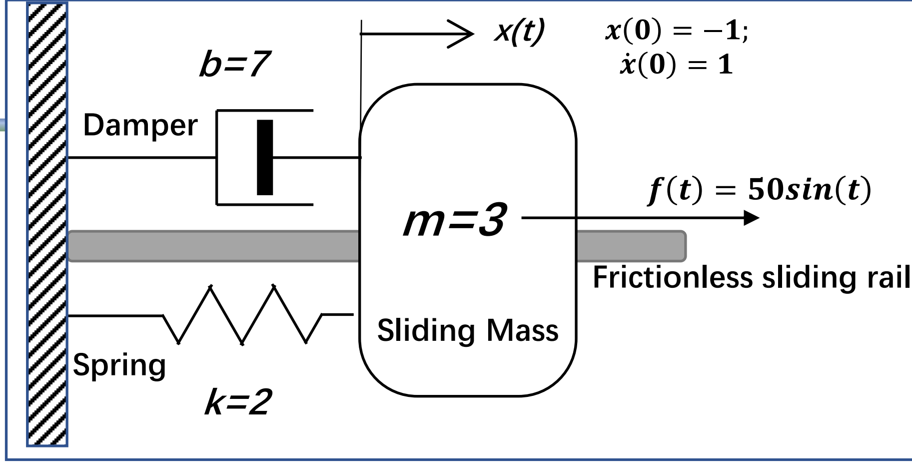
Laplace Transforms and Differentiation

- **(Inverse) Laplace transform** to obtain both transient and steady-state response as well as account for non-zero Ics
  - Inverse Laplace approach gives the total response (transient & steady-state)
  - Laplace of time derivative has an expression accounting for non-zeros ICs

$$\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} = s\mathscr{L}\{f'(t)\} - f'(0)$$
$$= s^2F(s) - \boxed{sf(0)} - \boxed{f'(0)}$$

Initial Conditions

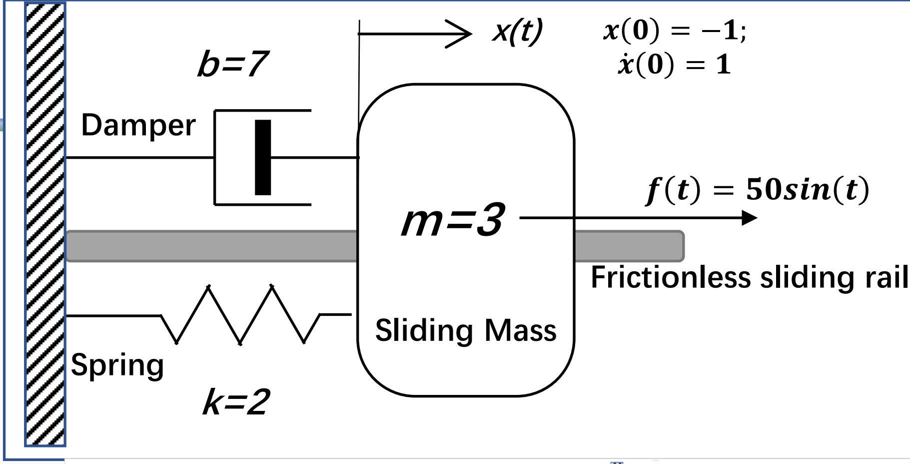
HW01: 2nd order System



$$m\ddot{x} + b\dot{x} + kx = f(t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = u(t)$$

HW01: 2nd order System

$$m\ddot{x} + b\dot{x} + kx = f(t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = u(t)$$



Solving ODE:

Characteristic values $\lambda = -2, -\frac{1}{3}$

$$\Rightarrow x_h(t) = C_1 e^{-2t} + C_2 e^{-\frac{1}{3}t}$$

Using undetermined coefficients

$$x_p(t) = -7\cos t - \sin t$$

The total response

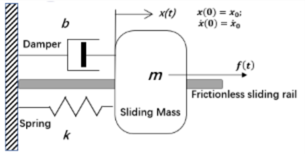
$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = C_1 e^{-2t} + C_2 e^{-\frac{1}{3}t} - 7\cos t - \sin t$$

Applying ICs: $\Rightarrow C_1 = -\frac{12}{5}, C_2 = \frac{42}{5}$

$$x(t) = -\frac{12}{5} e^{-2t} + \frac{42}{5} e^{-\frac{1}{3}t} - 7\cos t - \sin t$$

Transient Response



$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = u(t)$$

Transient response analysis

Laplace Transform (account for ICs!)

$$[s^2X(s) - sx_0 - \dot{x_0}] + 2\zeta\omega_n[sX(s) - x_0] + {\omega_n}^2X(s) = U(s)$$

$$s^2X + 2\zeta\omega_n sX + {\omega_n}^2X - (sx_0 + \dot{x_0} + 2\zeta\omega_n x_0) = U$$

$$X(s) = \frac{sx_0 + \dot{x_0} + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$$\boxed{X(s) = \frac{(s + 2\zeta\omega_n)x_0 + \dot{x_0}}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

Inverse Laplace Transform

$$\boxed{x(t) = \boxed{L^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x_0}}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}} + \boxed{L^{-1}\left\{\frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}}}$$

System Response

Consider the 2nd order system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = u(t)$$

$$x(t) = L^{-1}\left\{\frac{(2\zeta\omega_n + s)x_0 + \dot{x_0}}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\} + L^{-1}\left\{\frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}$$

Free response of 2nd order system:

$$x(t) = L^{-1}\left\{\frac{(2\zeta\omega_n + s)x_0 + \dot{x_0}}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}$$

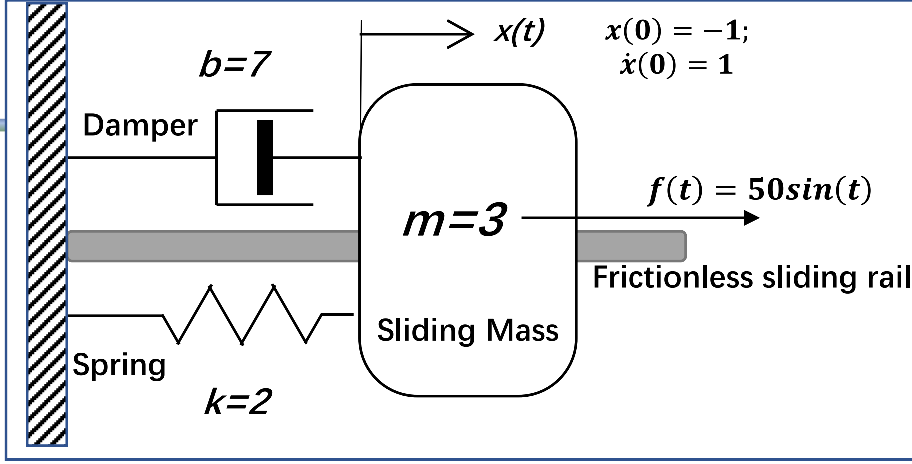
Impulse response of 2nd order system:

$$x(t) = L^{-1}\left\{\frac{(2\zeta\omega_n + s)x_0 + \dot{x_0} + A}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}$$

Step response of 2nd order system:

$$x(t) = L^{-1}\left\{\frac{(2\zeta\omega_n + s)x_0 + \dot{x_0}}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\} + L^{-1}\left\{\frac{A}{s(s^2 + 2\zeta\omega_n s + {\omega_n}^2)}\right\}$$

HW01: 2nd order System



a) Write down the dynamic equation in the form of a 2nd order differential equation. (2 points)
b) Write down the state-space equation of the system. (2 points)
c) Express the equation in the s-domain. (4 points)
d) Obtain the system response for the input $f(t)$=50sin(t). (7 points)

1a)

$$m\ddot{x}+b\dot{x}+kx=f(t)$$
$$3\ddot{x}+7\dot{x}+2x=f(t)$$
$$3\ddot{x}+7\dot{x}+2x=50sin(t)$$

b)

$$\ddot{x}+\frac{7}{3}\dot{x}+\frac{2}{3}x=\frac{f(t)}{3}=\frac{50}{3}sin(t)$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\frac{2}{3} & -\frac{7}{3}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}\cdot\frac{50}{3}sin(t);$$

$$y=(b_0 \quad b_1)\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

c)

$$[s^2X(s)-sx(0)-\dot{x}(0)]+\frac{7}{3}[sX(s)-x(0)]+\frac{2}{3}X(s)=U(s)$$

$$3s^2X+7sX+2X-(3sx_0+3\dot{x}_0+7x_0)=\frac{50}{(s^2+1)}$$

d)

$$(3s^2+7s+2)X=\frac{50}{(s^2+1)}-3s-4$$

$$(3s+1)(s+2)X=-3s-4+\frac{50}{(s^2+1)}$$

$$X=\frac{-3s-4}{(3s+1)(s+2)}+\frac{50}{(3s+1)(s+2)(s^2+1)}$$

$$X=\frac{-9}{5(3s+1)}+\frac{-2}{5(s+2)}+\frac{27}{(3s+1)}+\frac{-2}{(s+2)}+\frac{-7s}{(s^2+1)}+\frac{-1}{(s^2+1)}$$

$$X=\frac{42}{5\left(s+\frac{1}{3}\right)}-\frac{12}{5(s+2)}-\frac{7s}{(s^2+1)}-\frac{1}{(s^2+1)}$$

Inverse Laplace:

$$x(t)=\frac{42}{5}e^{\frac{-t}{3}}-\frac{12}{5}e^{-2t}-7cos(t)-sin(t)$$

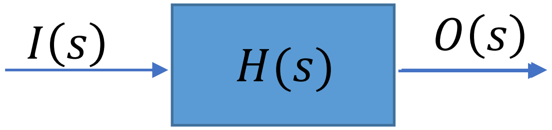
Transfer Functions

- A single-input-single-output (SISO) system with amplifiers, zero-initial-value integrators, splitting and summing junctions can be represented with a multiplication by a **transfer function** $H(s)$
- Such a dynamic system is called a **convolution**

Frequency

$$\mathcal{L}(o(t)) = \mathcal{L}(i(t)) \cdot H(s)$$

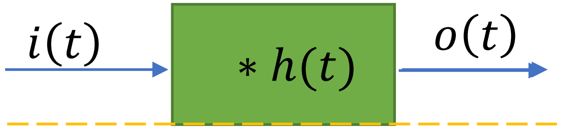
$$\boxed{O(s) = I(s) \cdot H(s)}$$



Equivalent

Time

$$\boxed{o(t) = i(t) * h(t)}$$

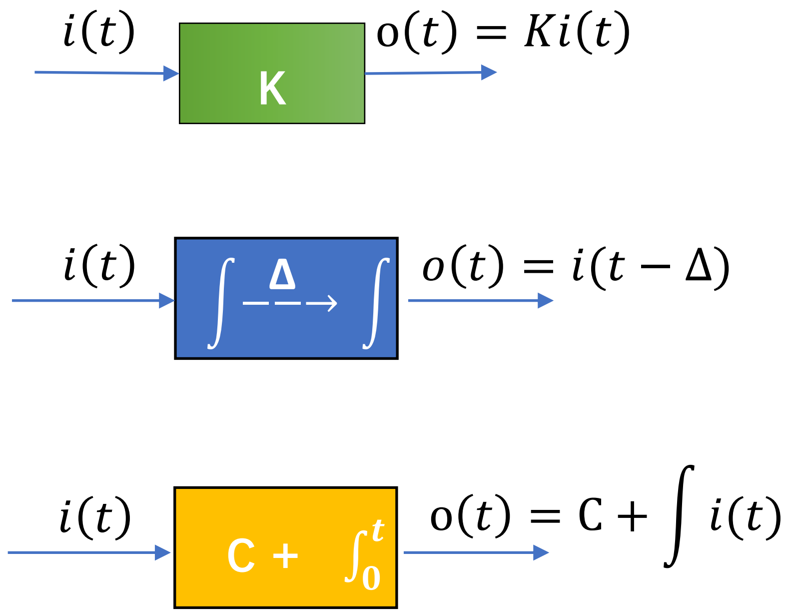


where $H(s) = \mathcal{L}\big(h(t)\big)$

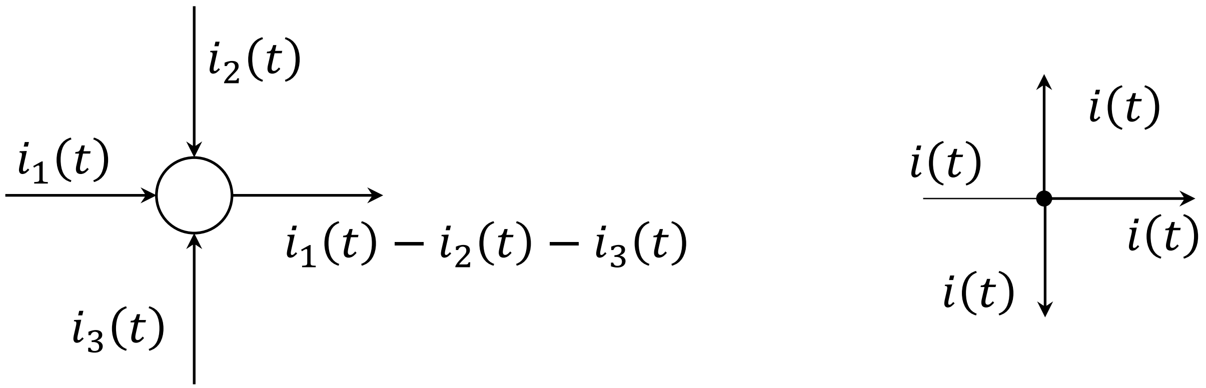
Block Diagrams

- a wiring of components, using summing and splitting junctions, to represent the input-output relationship of a dynamic system

Components

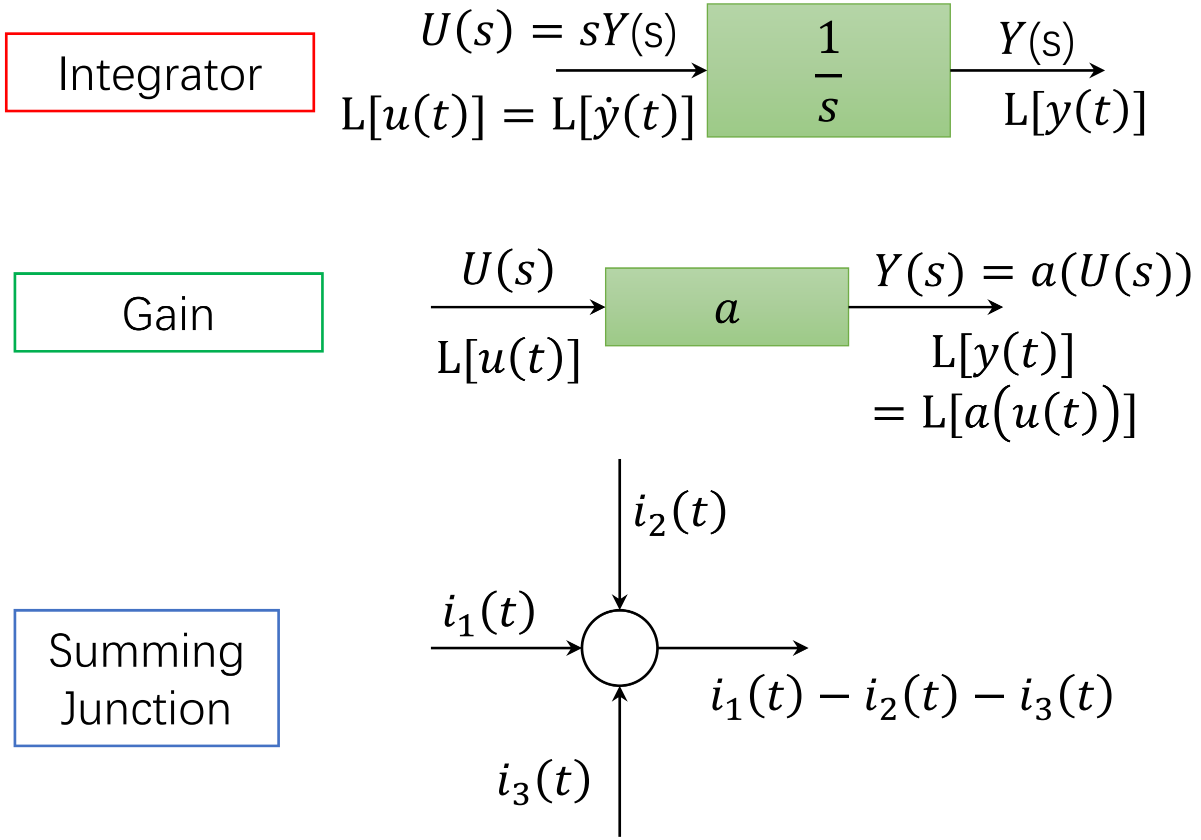


Junctions



All Integrator Diagrams

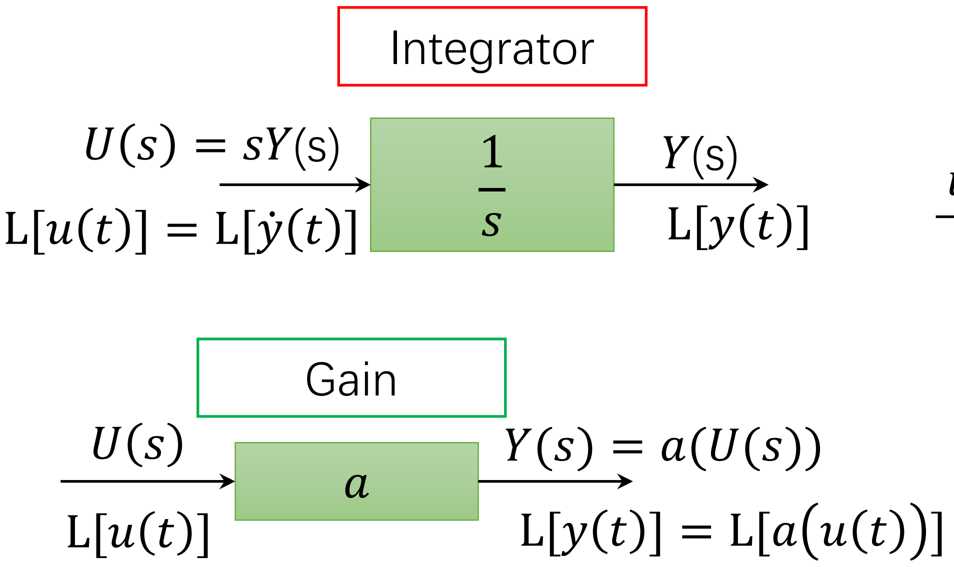
- Our basic building blocks
- 2 Points to note
  - May be in *t*- or *s*- domain
  - Not with Initial Conditions
- Why represent in this form?



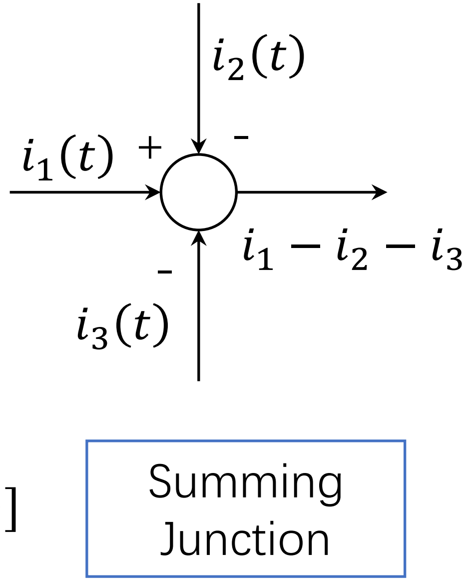
Overview

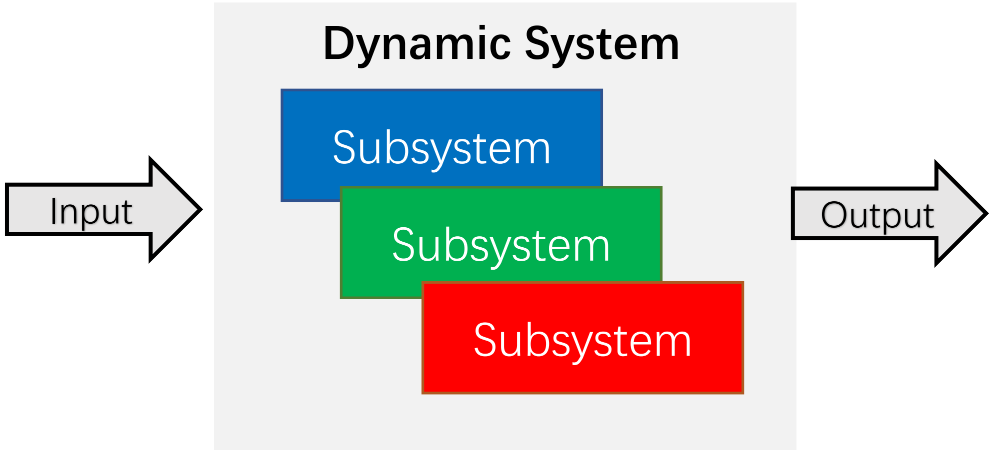
- System Modeling Diagram
  - wiring of <u>components</u>, using summing and splitting <u>junctions</u>, to represent the <u>input-output</u> relationship of a <u>dynamic system</u>

Components



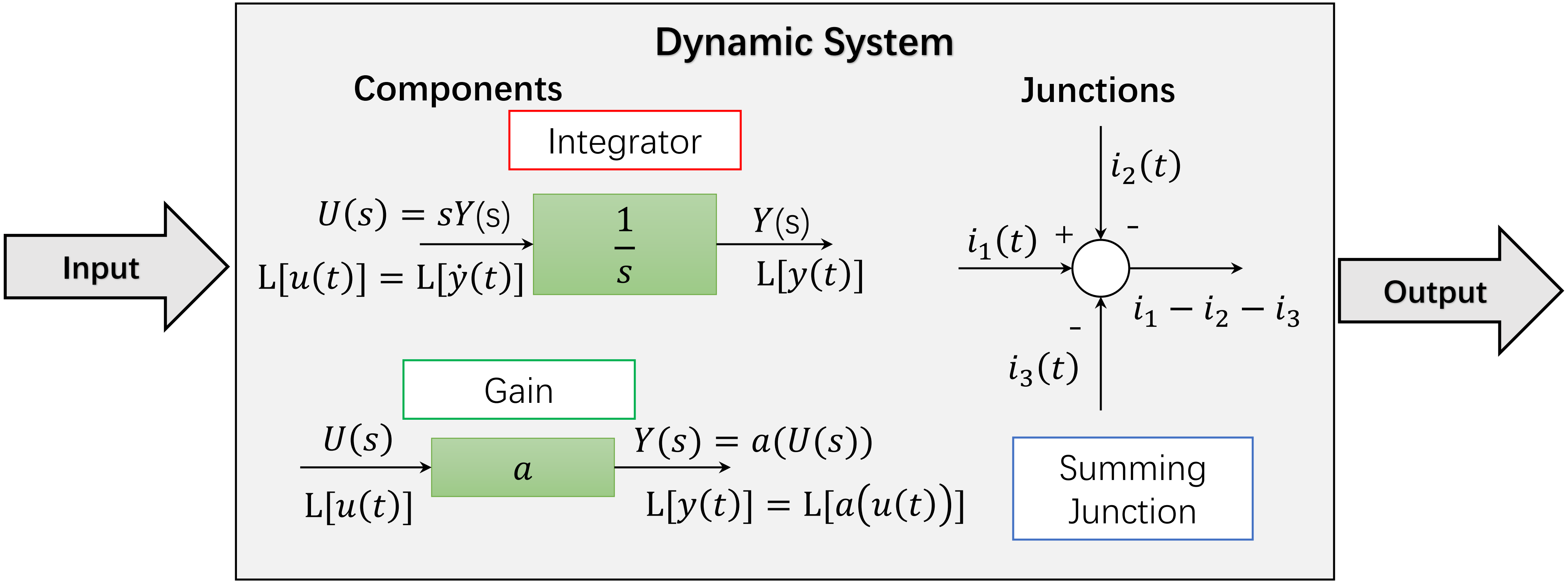
Junctions





Overview

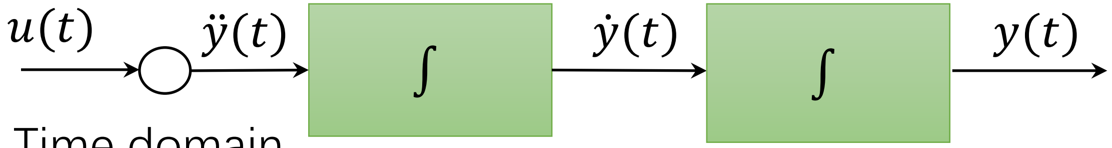
- System Modeling Diagram
  - wiring of components, using summing and splitting junctions, to represent the input-output relationship of a dynamic system



Example 1

Construct an all-integrator diagram for

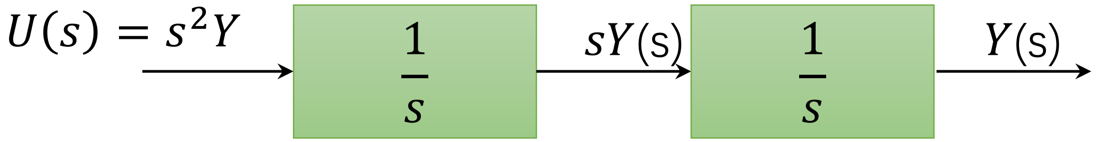
$$\ddot{y} = u \quad \Leftrightarrow \quad s^2 Y = U$$



Time domain

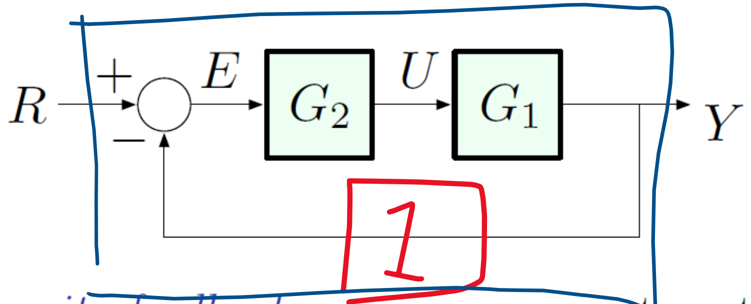
or

Frequency domain



Unity Feedback

Possible feedback configuration:



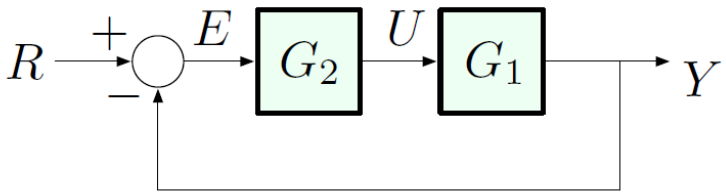
$$\frac{G_2 G_1}{1 + 1 \cdot G_2 G_1}$$

This is called *unity feedback* — no component on the feedback path.

Common structure (saw this in Lecture 1):

- R = reference
- U = control input
- Y = output
- E = error
- G_1 = plant (also denoted by P)
- G_2 = controller or compensator (also denoted by C or K)

Unity Feedback



Let's practice with deriving transfer functions: $\dfrac{\text{forward gain}}{1 + \text{loop gain}}$

- Reference R to output Y:

$$\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2}$$

- Reference R to control input U:

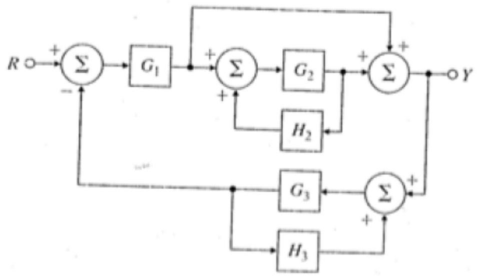
$$\frac{U}{R} = \frac{G_2}{1 + G_1 G_2}$$

- Error E to output Y:

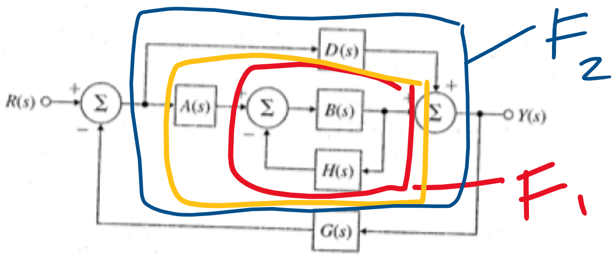
$$\frac{Y}{E} = G_1 G_2 \qquad \text{(no feedback path)}$$

HW02 Block Diagrams

1. Using techniques for block diagram reduction discussed in class, find the transfer functions of the systems shown below (p156 from the textbook, 3rd edition)



(a)



(b)

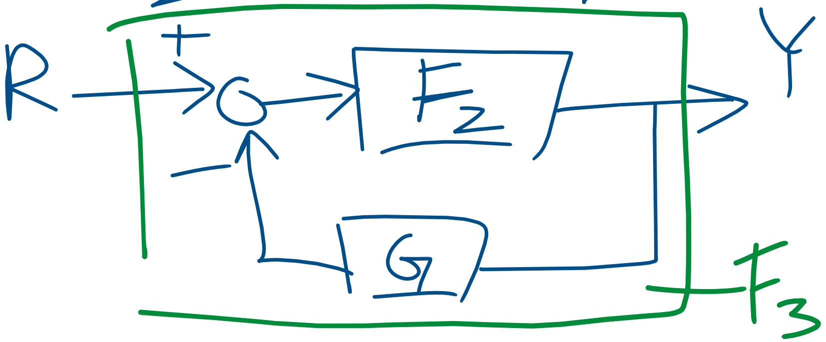
2. Consider the following state-space model (so-called "observer canonical form"):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Build an all-integrator diagram for this system.

$$AF_1 = \frac{AB}{1+HB}$$

$$F_2 = D + AF_1$$



$$F_3 = \frac{F_2}{1+GF_2}$$

$$F_3 = \frac{\left[D+\frac{AB}{1+HB}\right]}{1+G\left(D+\frac{AB}{1+HB}\right)}$$

HW02 Block Diagrams

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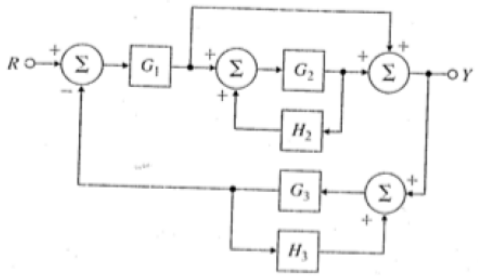

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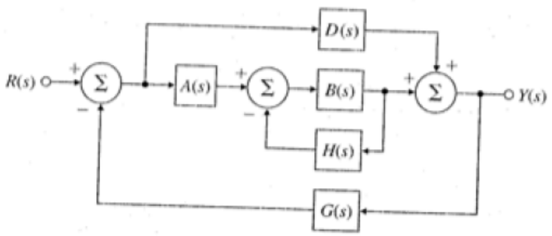

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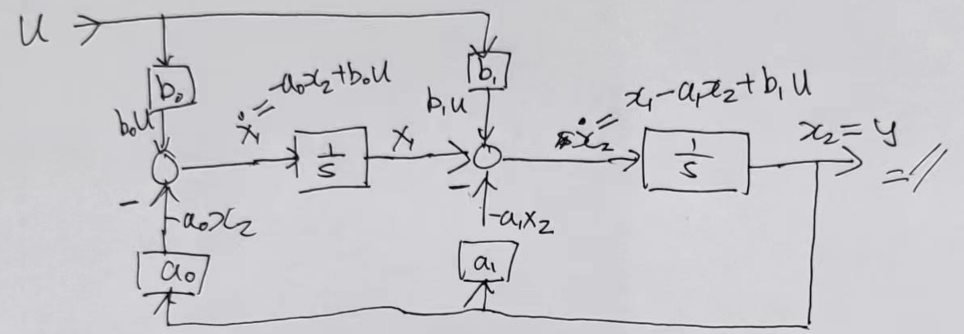
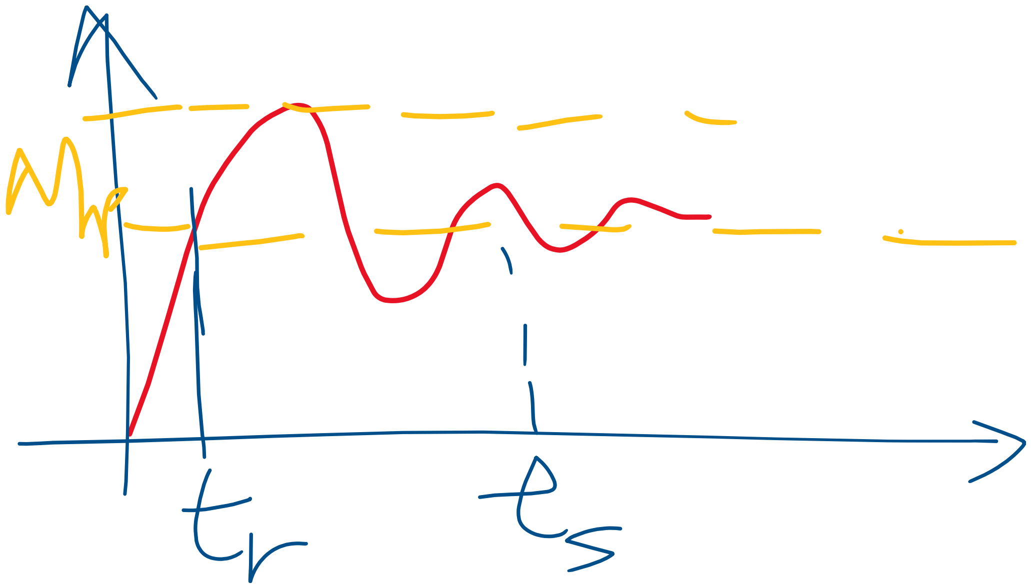


TD Specs



Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

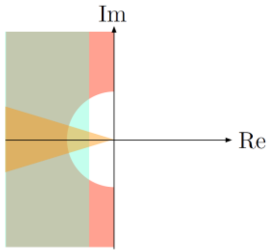
$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$t_s \approx \frac{3}{\sigma}$$

Combination of Specs

If we have specs for any combination of t_r, M_p, t_s, we can easily relate them to allowed pole locations:



The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

This is very appealing to engineers: easy to visualize things, no such crisp visualization in time domain.

But: not very rigorous, and also only valid for our prototype 2nd-order system, which has only 2 poles and no zeros ...

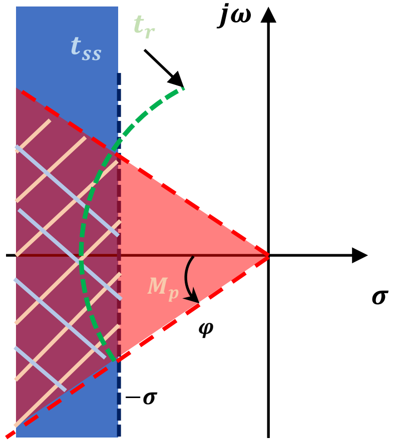
HW02 TD Spec

3. Consider the plant with transfer function $L(s) = \dfrac{1}{s^2 + 2s + K}$ where K is a positive parameter you can tune.

a) Consider the settling time spec $t_s \leq 4$. Give some value (or range of values) of K for which the system meets this spec. Justify your choice.

b) Consider the rise time spec $t_r \leq 1$. Give some value (or range of values) of K for which the system meets this spec.

c) Consider the overshoot spec $M_p \leq 0.1$. Give some value (or range of values) of K for which the system meets this spec. Justify your choice.



$L(s) = \frac{1}{s^2+2s+K}$ $\omega_n^2 = K$, $\cancel{2}\zeta\omega_n = \cancel{2}$

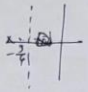
$\zeta = \frac{1}{\omega_n} = \frac{1}{\sqrt{K}}$

$t_s \leq 4$

$\parallel$

$\boxed{\frac{3}{\sigma} \leq 4}$

$\sigma \geq \frac{3}{4}$



$p(s) = s^2 + 2s + K = 0$

$s = -1 \pm \sqrt{K}\sqrt{\frac{1}{K} - 1}$

$s = \boxed{-1} \pm \sqrt{1-K} \leq -\frac{3}{4}$

$\pm\sqrt{1-K} \leq 1 - \frac{3}{4}$

$\sqrt{1-K} \leq (0.25)$

$1 - K \leq 0.0625$

$K \geq 1 - 0.0625 = 0.9375$

$t_r \leq 1$

$\parallel$

$\frac{1.8}{\omega_n} \leq 1$

$\frac{1.8}{\sqrt{K}} \leq 1$

$\sqrt{K} \geq 1.8$

$K \geq 3.24$

$M_p \leq 0.1$

$\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \leq 0.1 \Rightarrow \zeta \geq 0.6$

$\parallel$

$\frac{1}{\sqrt{K}} \geq 0.6 \Rightarrow K \leq \frac{1}{0.36} = 2.78$

Routh's Test

$$\begin{array}{lllll} s^n: & 1 & a_2 & a_4 & a_6 \quad \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 \quad \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \end{array}$$

Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

$$\text{where} \quad b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1}(a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1}(a_5 - a_1 a_4)$$

$$b_3 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1}(a_7 - a_1 a_6) \qquad \text{and so on ...}$$

Note: the new row is 1 element shorter than the one above it

$$\begin{array}{lllll} s^n: & 1 & a_2 & a_4 & a_6 \quad \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 \quad \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \\ s^{n-3}: & c_1 & c_2 & \dots & \end{array}$$

Next, we form the fourth row marked by s^{n-3}:

$$s^{n-3}: \quad c_1 \quad c_2 \quad \dots$$

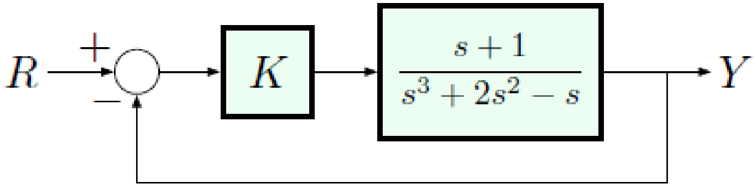
$$\text{where} \quad c_1 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_2 - a_3 b_1)$$

$$c_2 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_3 - a_5 b_1)$$

and so on ...

Routh-Hurwitz as a Design Tool

Parametric Stability Range: Determining range of parameters for stability in controller design



Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Let's write down the transfer function from R to Y:

$$\frac{Y}{R} = \frac{\text{forward gain}}{1+\text{loop gain}} = \frac{K \cdot \frac{s+1}{s^3+2s^2-s}}{1+K \cdot \frac{s+1}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3+2s^2-s+K(s+1)} = \frac{Ks+K}{s^3+2s^2+(K-1)s+K}$$

Now we need to test stability of $p(s) = s^3+2s^2+(K-1)s+K$.

Test stability of

$$p(s) = s^3+2s^2+(K-1)s+K$$

using the Routh test.

Form the Routh array:

$$\begin{array}{lcc} s^3: & 1 & K-1 \\ s^2: & 2 & K \\ s^1: & \frac{K}{2}-1 & 0 \\ s^0: & K & \end{array}$$

For p to be stable, all entries in the 1st column must be positive:

$$K>2 \quad \text{and} \quad K>0 \quad \text{(already covered by } K>1)$$

Note: The necessary condition requires $K>1$, but now we actually know that we must have $K>2$ for stability.

Stability Conditions for Low-Order Polynomials

The upshot:

- A 2nd-degree polynomial $p(s) = s^2 + a_1 s + a_2$ is stable if and only if $a_1 > 0$ and $a_2 > 0$
- A 3rd-degree polynomial $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$ is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

- These conditions were already obtained by Maxwell in 1868.
- In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

Example:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{K}{s(s^2 + 3s + 2) + K}$$