ECE 486 Control Systems

Lecture 08: Basic properties & benefits of feedback control; PID control

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Announcement

- Reminder: Term Test I next Thurs in class
- Sample papers from past year
- Homework solution

Checklist

Modeling | Analysis | Design

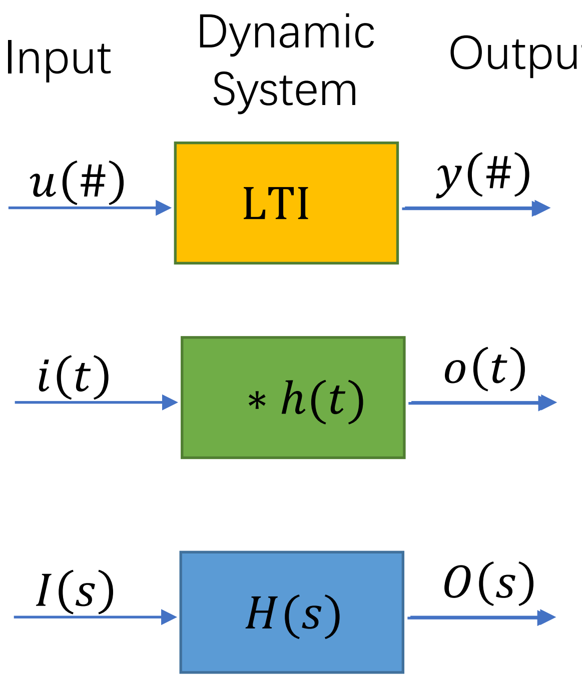
Root Locus
Frequency Response
State-Space

Wk	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	National Holiday Week	
4	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	Review A	
	Term Test A	
7	Introduction to Root Locus design method	Ch. 5
	Root Locus continued; introduction to dynamic compensation	Root Locus
8	Lead and lag dynamic compensation	Ch. 5
	Lead and lag continued; introduction to frequency-response design method	Sections 5.1-5.4, 6.1

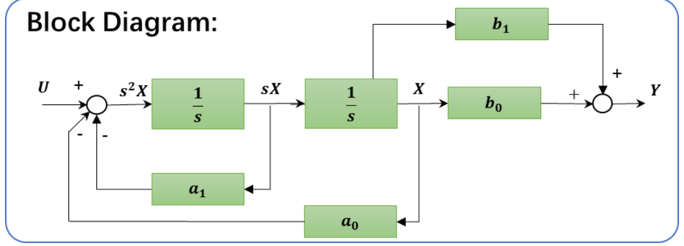
Modeling
Analysis

Wk	Topic	Ref.
9	Bode plots for three types of transfer functions	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	Control design using frequency response	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	State-Space
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Recap: System Representation & Stability Analysis



Block Diagram:



State space model:

State Equation $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$

Output Equation $y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Configuration form

Equations of Motion
$$\begin{cases} \ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \dots \\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \end{cases}$$

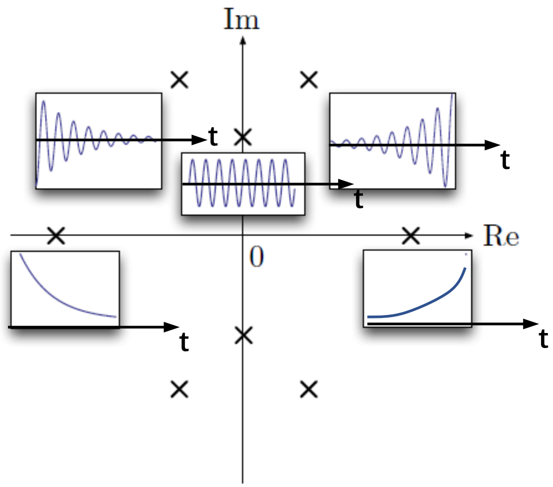
Initial Conditions
$$\begin{cases} q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0} \\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0} \end{cases}$$

Transfer Function:

$$\frac{O(s)}{I(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_o}{s^n + a_{n-1} s^{n-1} + \dots + a_o}$$

ICs= 0

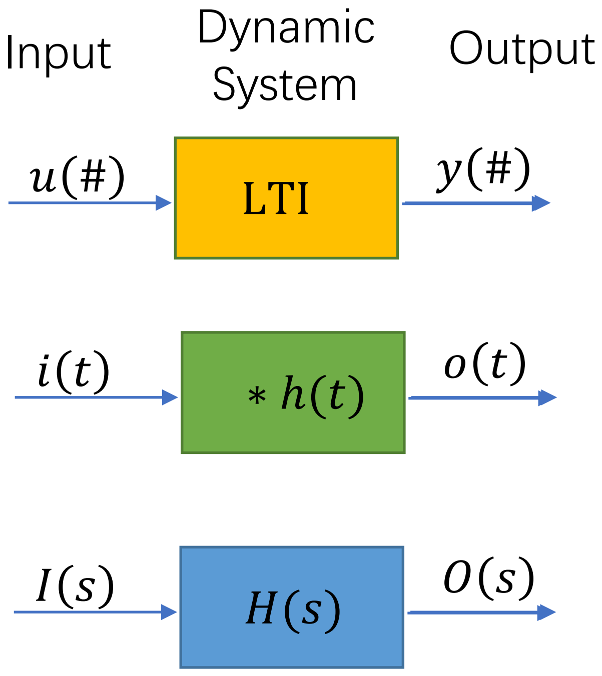
Effect of Pole Locations



- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Using Routh-Hurwitz Stability Analysis

Recap: System Representation & Stability Analysis

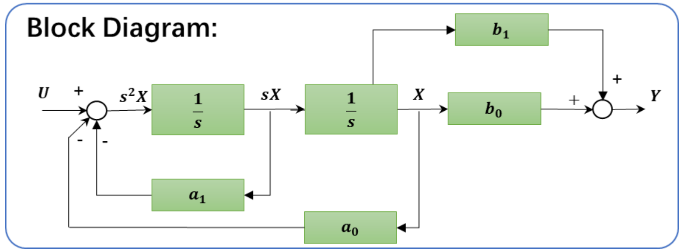


$$H(s) = \frac{q(s)}{p(s)}$$

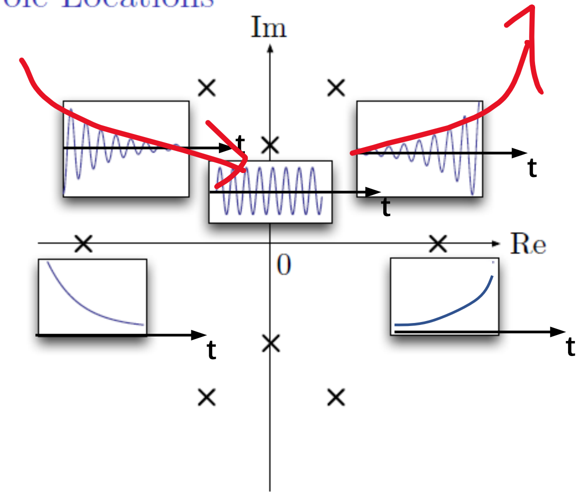
$$p(s) = 0$$

$$\frac{1}{s+a}$$

Block Diagram:



Effect of Pole Locations



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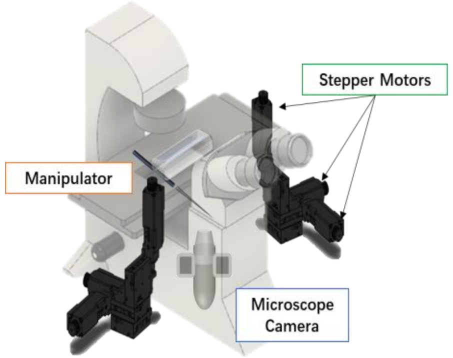
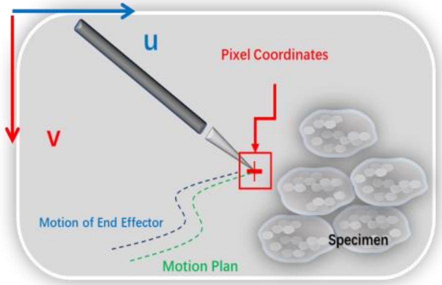
Using Routh-Hurwitz Stability Analysis

Recap: Feedback Control

- Open-loop vs. closed-loop control
- Benefits of feedback control:
  - Reference tracking
  - Disturbance rejection
  - Reduction of sensitivity to parameter variations
  - Improvement of time response

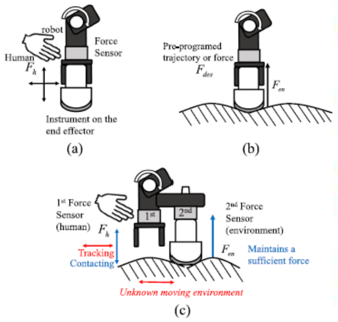
Track Reference

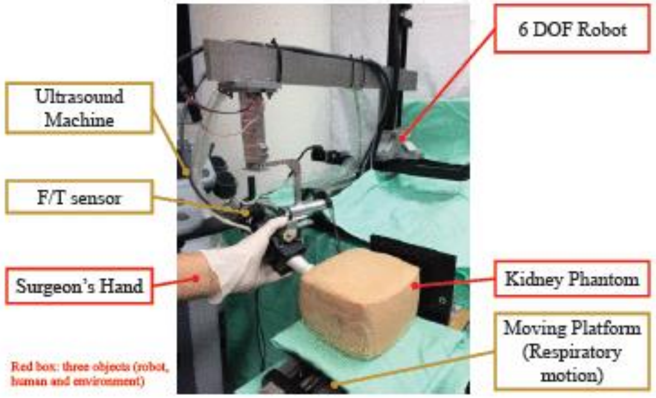




Vision-based Trajectory Tracking

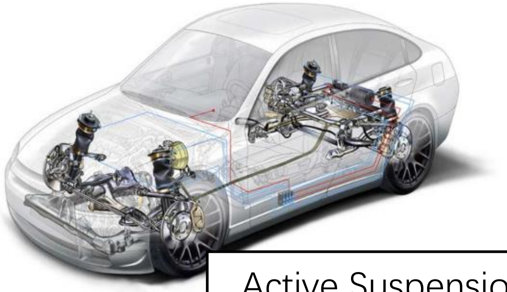
Reduce Effect of Uncertainties





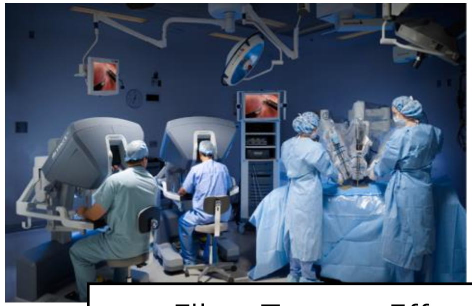
Human-Robot Interaction

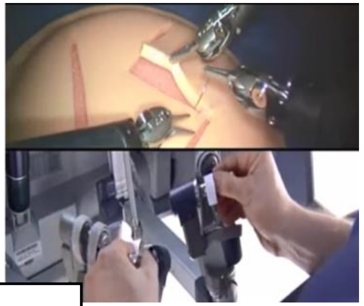
Improve Time Response



Active Suspension

Reject Disturbance





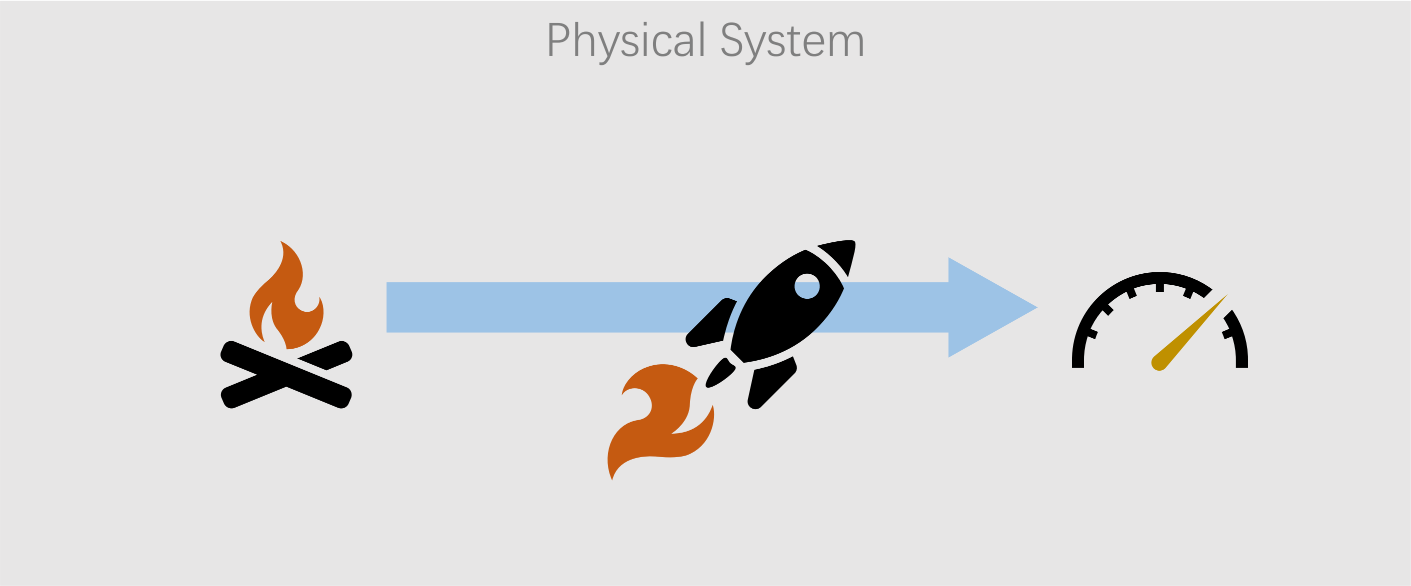
Filter Tremor Effect

Recall: Feedback Control

- Difference between open-loop and closed-loop control
- Benefits of feedback control:
  1. Reference tracking
  2. Disturbance rejection
  3. Reduction of sensitivity to parameter variations
  4. Improvement of time response

Concept of Control Systems

- A **control system** is designed to achieve a targeted **output** by generating the appropriate **inputs** in a **dynamical environment** (within specified **performance criteria**)



Concept of Control Systems

- A **control system** is designed to achieve a targeted **output** by generating the appropriate **inputs** in a **dynamical environment** (within specified **performance criteria**)

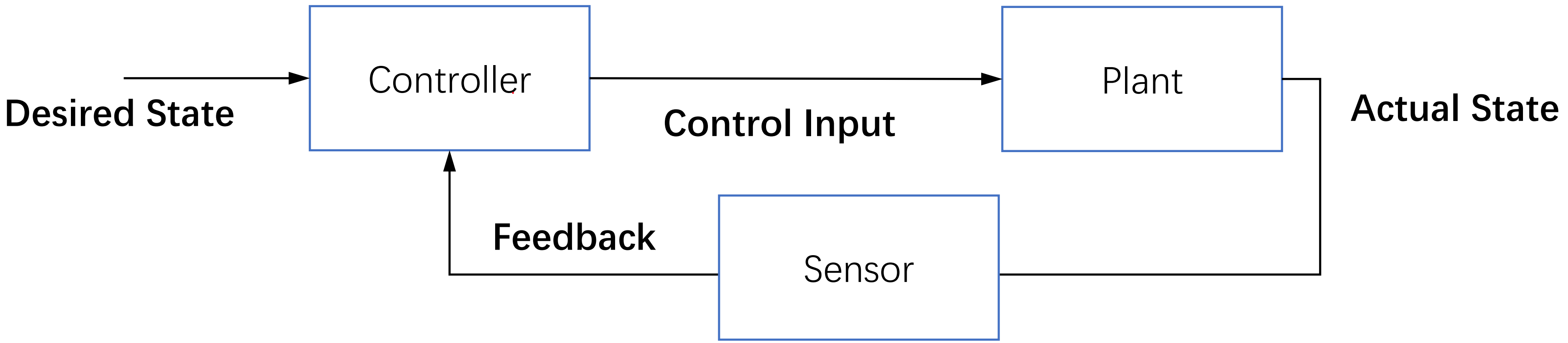
Open- vs Closed-Loop Control

Car

Open-loop

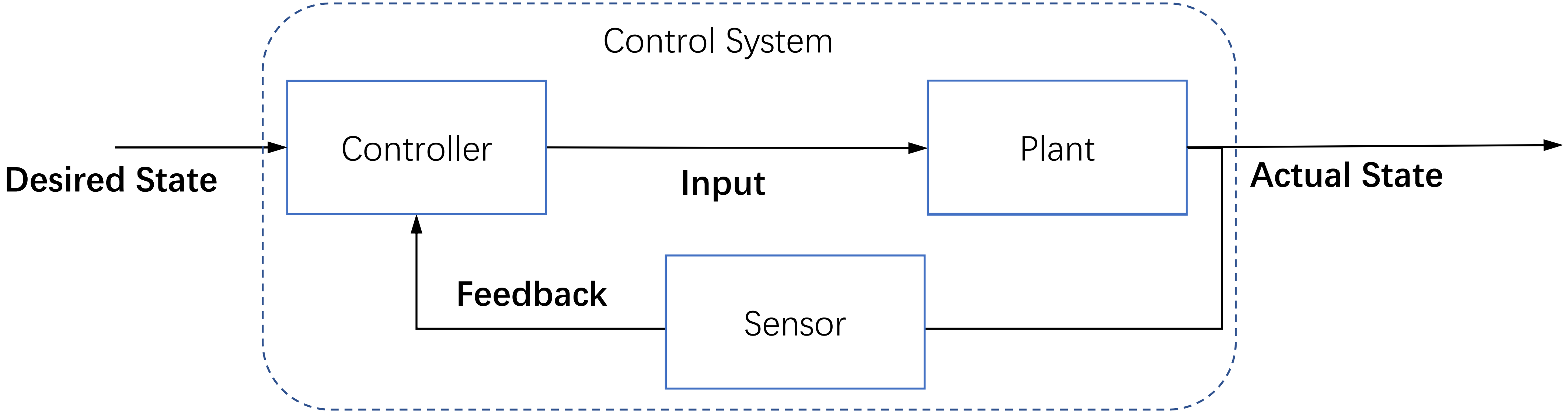
Through preestablished model, generate the command that will achieve the desired state.

Closed-loop

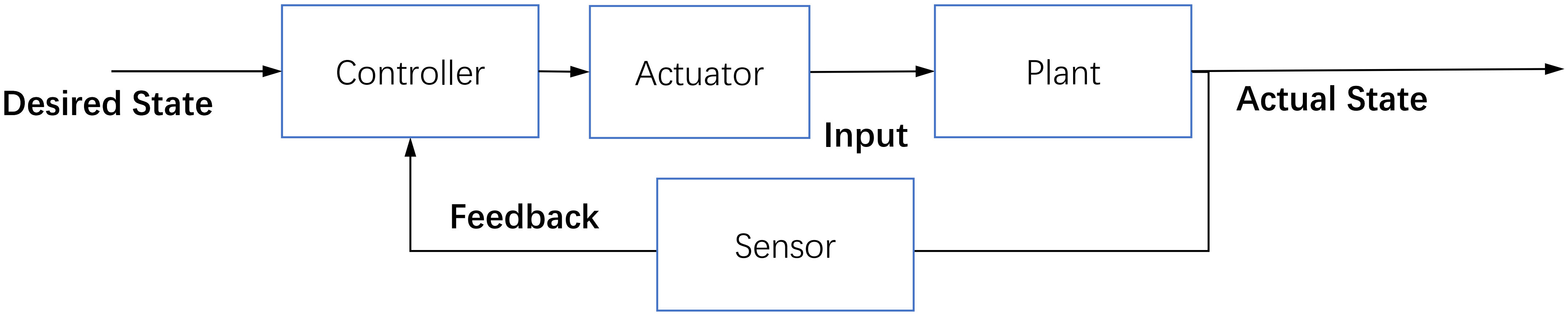


Through sensors, we are able to **feedback** the measurement to produce the command that will minimize the error between desired and actual targeted profile.

Recall: Feedback Control



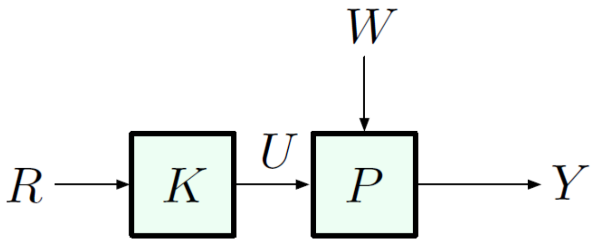
Recall: Terminology



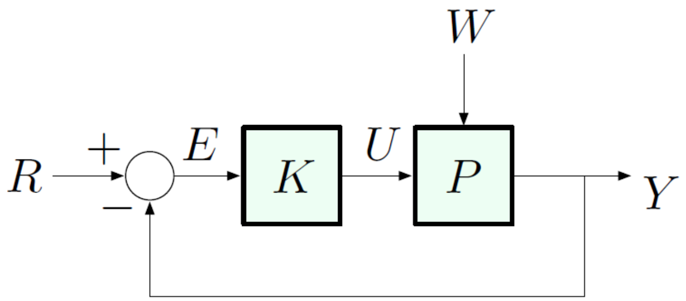
- **Plant** is the system being controlled
- **Sensors** measure the quantity that is subject to control
- **Actuators** act on the plant
- **Controller** processes the sensor signals and drives the actuators
- **Control law** is the rule for mapping sensor signals to actuator signals

Open- vs Closed Loop Architectures

- Open-loop control

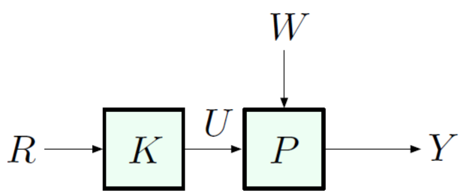


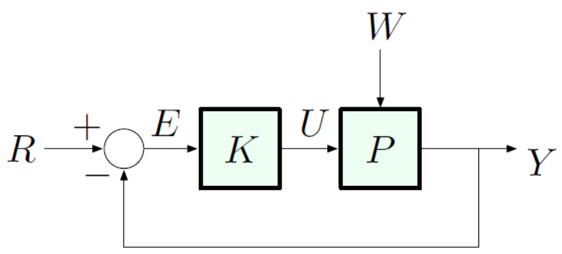
- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

Open- vs Closed Loop Architectures



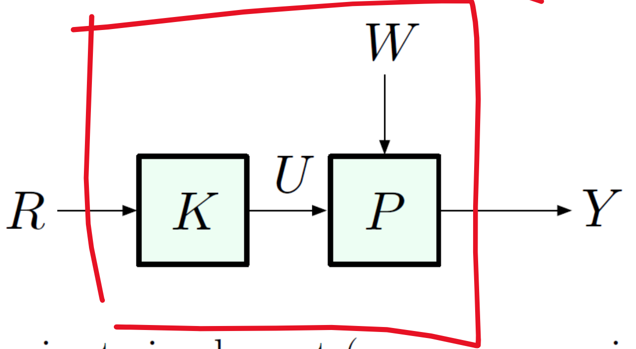


- track a given reference
- reject disturbances
- meet performance specs

Open- vs Closed Loop Architectures

Open-Loop Control

$H(s)_{OL}$



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

  e.g., if both K and P are stable (all poles in OLHP),

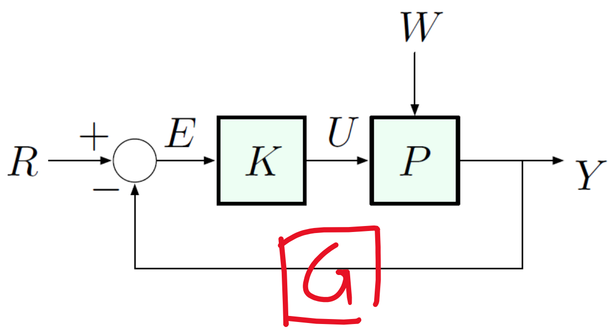
  $$\frac{Y}{R} = KP$$

  is also stable:

  $$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control

$$H(s)_{CL} = \frac{KP}{1+KPG}$$



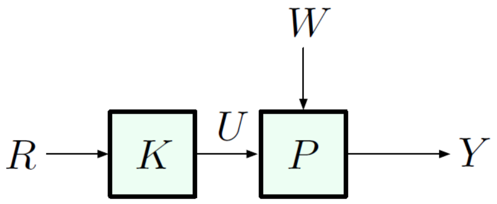
- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

  $$\frac{Y}{R} = \frac{KP}{1+KP}$$

  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Open vs Closed Loop Architectures

- Open-loop control



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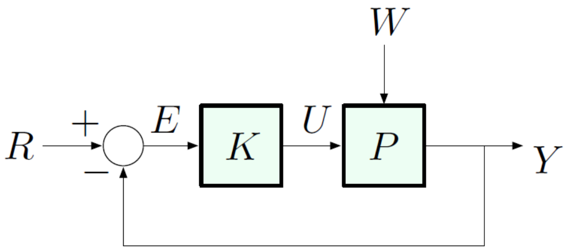
- Feedback (closed-loop) control



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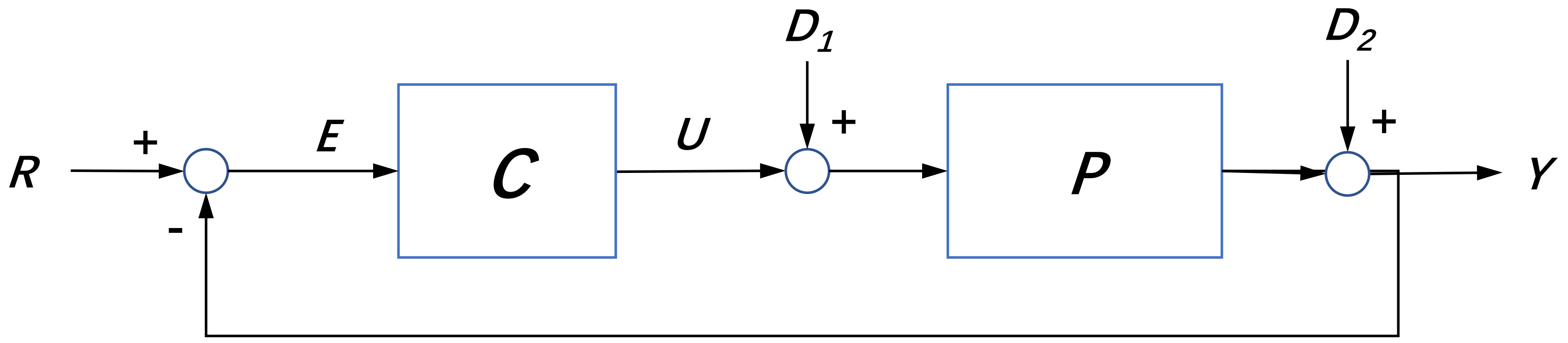
- track a given reference
- reject disturbances
- meet performance specs

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Feedback Control



Systems
C: Controller
P: Plant

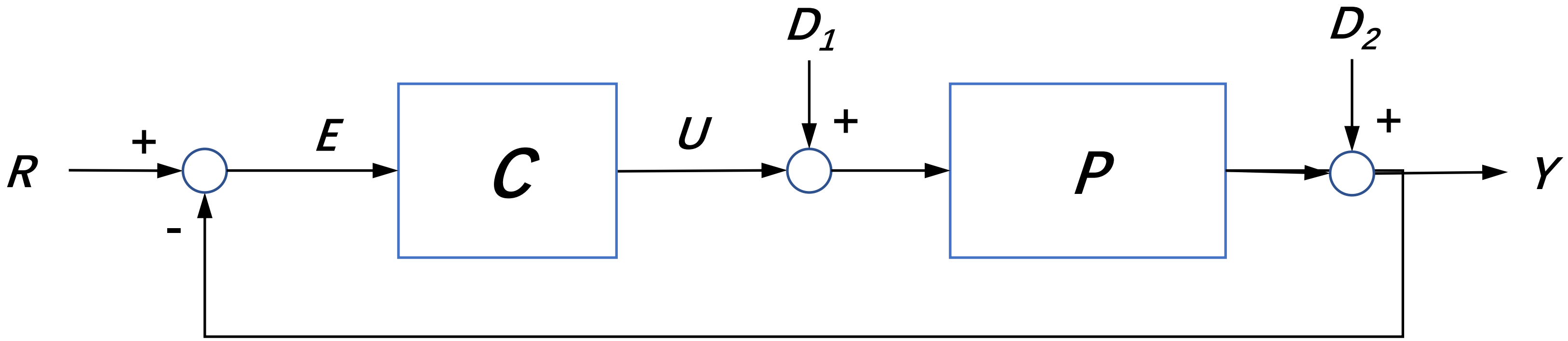
R: Reference
E: Error

Variables
U: Input
Y: Output

D_1: Disturbance 1
D_2: Disturbance 2

Relations
$Y=D_2+P(U+D_1)$
$U=CE$;
$E=R-Y$

Feedback Control



Relations

$Y=D_2+P(U+D_1)$; $U=CE$; $E=R-Y$

Expressing Y in terms of R, D_2, D_1

$Y = D_2+P(CE+D_1)$

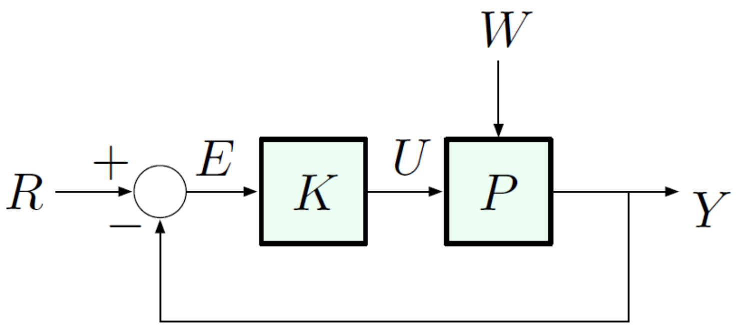
$=D_2+P(C(R-Y)+D_1)$

$=D_2+PCR-PCY+PD_1$

$$Y = \frac{PC}{1+PC}R + \frac{P}{1+PC}D_1 + \frac{1}{1+PC}D_2$$

What happen when C keeps getting larger?

Benefits of Feedback Control



Feedback control:

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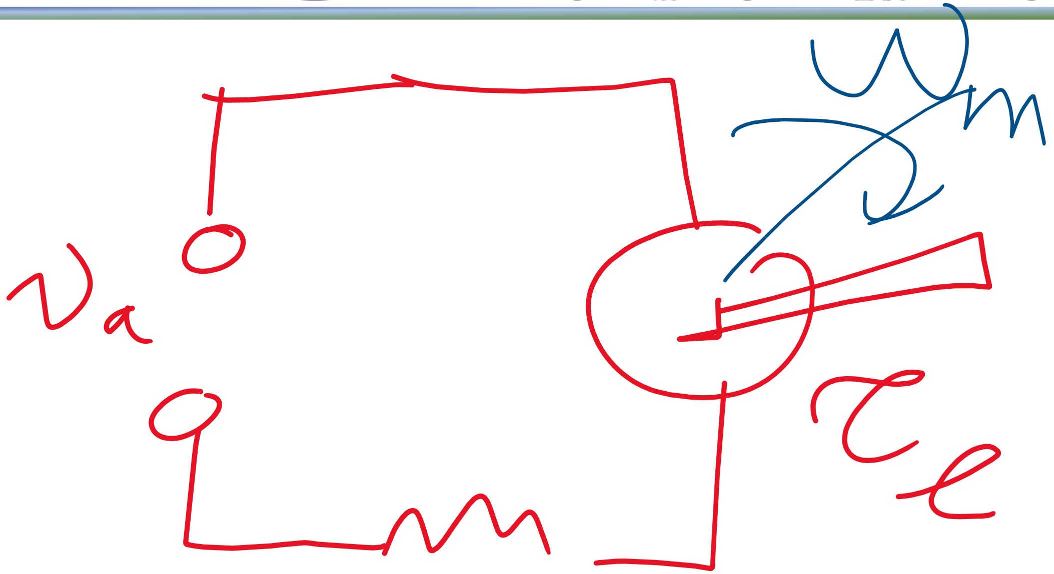
Case Study: DC Motor



Inputs: v_a – input voltage

τ_e – load/disturbance torque

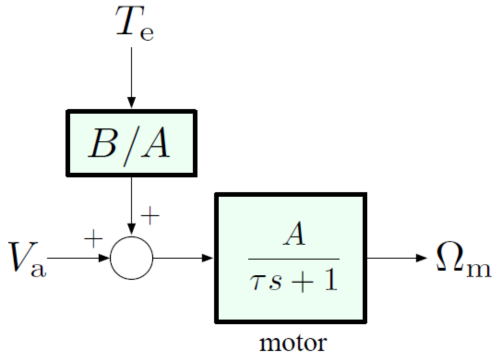
Outputs: ω_m – angular speed of the motor

Transfer function:

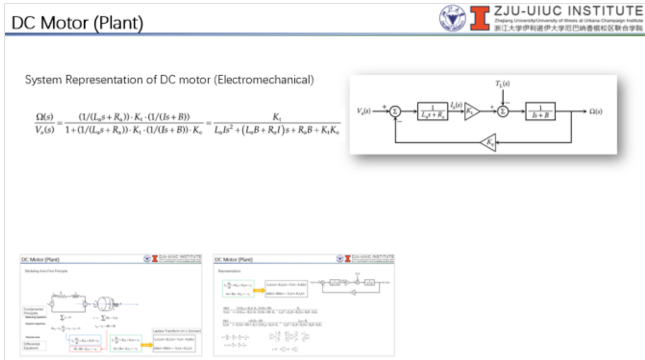
$$\Omega_m = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e$$

τ – time constant

A, B – system gains

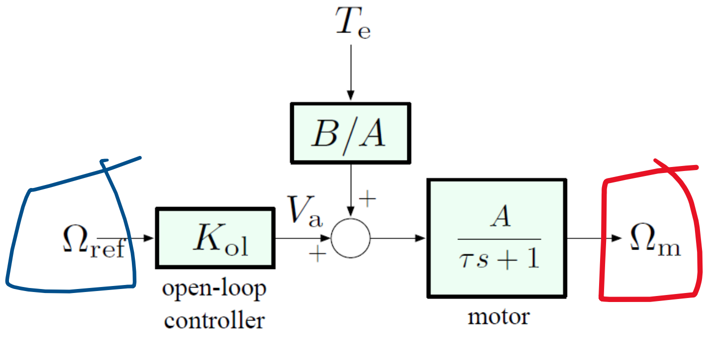


Objective: have Ω_m approach and track a given reference Ω_{ref} in spite of disturbance T_e.

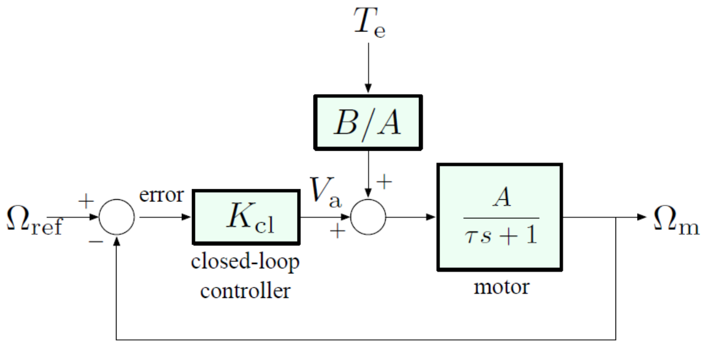


Case Study: DC Motor

- Open-loop control



- Feedback (closed-loop) control

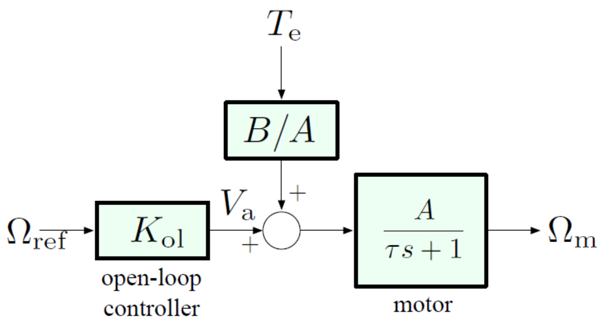


Case Study: DC Motor

Disturbance Rejection

Goal: maintain $\omega_{\mathrm{m}} = \omega_{\mathrm{ref}}$ in steady state in the presence of *constant* disturbance.

Open-loop:



– the controller receives *no information* about the disturbance τ_{e} (the only input is ω_{ref}, no feedback signal from anywhere else)

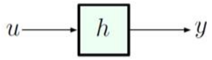
– so, let's attempt the following: design for *no disturbance* (i.e., $\tau_{\mathrm{e}} = 0$), then see how the system works in general

Disturbance Rejection: K_{ol}

Recap: FVT: Steady-State Value

$$u(t) = 1(t) \qquad U(s) = \frac{1}{s} \qquad \Longrightarrow \qquad Y(s) = \frac{H(s)}{s}$$

Back to DC Gain

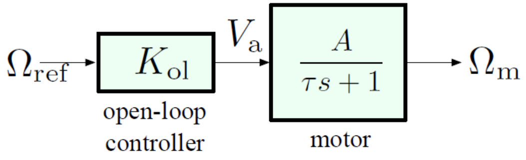


Step response: $\qquad Y(s) = \frac{H(s)}{s}$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s\to 0} H(s)$$

First assume zero disturbance:



Transfer function:

$$\frac{A}{\tau s + 1} \quad \text{(stable pole at } s = -1/\tau\text{)}$$

We want DC gain $= 1$

$$\Omega_{\mathrm{m}} = \frac{A}{\tau s + 1} V_{\mathrm{a}} = \frac{K_{\mathrm{ol}} A}{\tau s + 1} \Omega_{\mathrm{ref}}$$

Let's just use constant gain: $K_{\mathrm{ol}} = 1/A$

$$\omega_{\mathrm{m}}(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\mathrm{ref}} = \omega_{\mathrm{ref}} \qquad (\text{for } T_{\mathrm{e}} = 0)$$

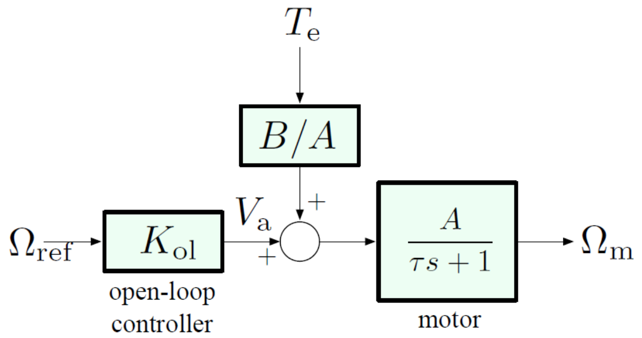
What happens in the presence of nonzero T_{e}?

$$\Omega_{\mathrm{m}} = \underbrace{\frac{A}{\tau s + 1} \frac{1}{A}}_{\text{DC gain}=1} \Omega_{\mathrm{ref}} + \underbrace{\frac{B}{\tau s + 1}}_{\text{DC gain}=B} T_{\mathrm{e}}$$

$$\Longrightarrow \omega_{\mathrm{m}}(\infty) = \underbrace{\omega_{\mathrm{ref}}}_{\text{step input}} + B \underbrace{\tau_{\mathrm{e}}}_{\text{step input}}$$

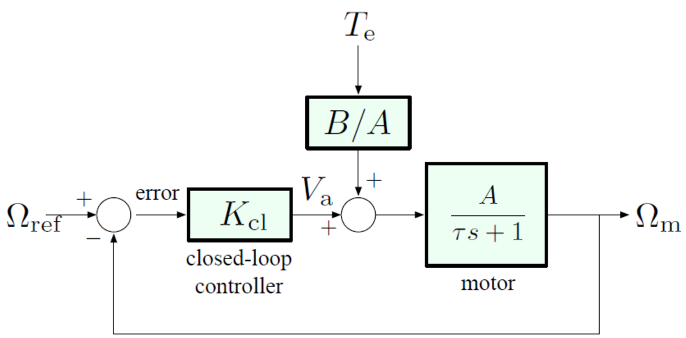
Steady-state motor speed for constant reference and constant disturbance:

$$\omega_{\mathrm{m}}(\infty) = \omega_{\mathrm{ref}} + B\tau_{\mathrm{e}}$$



Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by B, and we have no control over it (and, in fact, cannot change this through any choice of controller K_{ol}, no matter how clever)

Disturbance Rejection: K_{cl}



$$V_a = K_{cl}E = K_{cl}(\Omega_{ref} - \Omega_m)$$

$$\Omega_m = \frac{A}{\tau s+1}K_{cl}(\Omega_{ref} - \Omega_m) + \frac{B}{\tau s+1}T_e$$

Solve for Ω_m: $(\tau s+1)\Omega_m = AK_{cl}(\Omega_{ref} - \Omega_m) + BT_e$

$$(\tau s+1+AK_{cl})\Omega_m = AK_{cl}\Omega_{ref} + BT_e$$

$$\Omega_m = \frac{AK_{cl}}{\tau s+1+AK_{cl}}\Omega_{ref} + \frac{B}{\tau s+1+AK_{cl}}T_e$$

$$\Omega_m = \frac{AK_{cl}}{\underbrace{\tau s+1+AK_{cl}}_{\text{DC gain}=\frac{AK_{cl}}{1+AK_{cl}}}}\Omega_{ref} + \frac{B}{\underbrace{\tau s+1+AK_{cl}}_{\text{DC gain}=\frac{B}{1+AK_{cl}}}}T_e$$

(provided all transfer functions are strictly stable)

Assuming that the reference ω_{ref} and the disturbance τ_e are constant, we apply FVT:

$$\omega_m(\infty) = \frac{AK_{cl}}{1+AK_{cl}}\omega_{ref} + \frac{B}{1+AK_{cl}}\tau_e$$

Steady State speed for constant reference and disturbance:

Conclusions:

- $\dfrac{AK_{cl}}{1+AK_{cl}} \neq 1$, but can be brought arbitrarily close to 1 when $K_{cl} \to \infty$. Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.
- $\dfrac{B}{1+AK_{cl}}$ is small (arbitrarily close to 0) for large K_{cl}. Thus, *much* better disturbance rejection than with open-loop control.

Sensitivity to Parameter Variations

Bode's sensitivity concept: In the "nominal" situation, we have the motor with DC gain $= A$, and the overall transfer function, either open- or closed-loop, has some other DC gain (call it T).

Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

$$A \longrightarrow A + \underbrace{\delta A}_{\substack{\text{small} \\ \text{perturbation}}}$$

This will cause a perturbation in the overall DC gain:

$$T \longrightarrow T + \delta T \qquad \left(\text{from calculus, to 1st order, } \delta T \approx \frac{\mathrm{d}T}{\mathrm{d}A}\delta A\right)$$

$$A \longrightarrow A + \delta A \qquad \text{(small perturbation in system gain)}$$

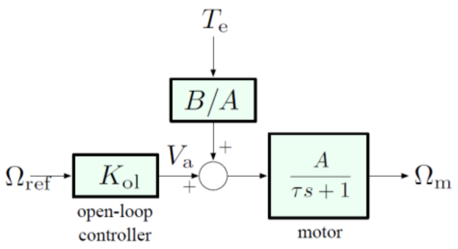
$$T \longrightarrow T + \delta T \qquad \text{(resultant perturbation in overall DC gain)}$$

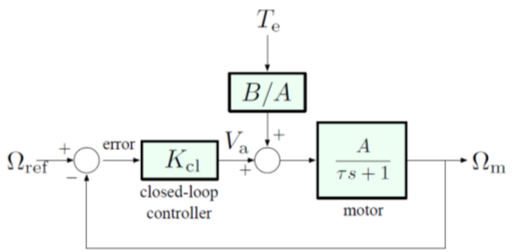
Hendrik Wade Bode
(1905–1982)

Bode's sensitivity:

$$\mathcal{S} \triangleq \frac{\delta T / T}{\delta A / A}$$

$$\mathcal{S} = \text{relative error} = \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A}$$





Let's compute $\mathcal{S}$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case $\quad T_{ol} = K_{ol}A = \frac{1}{A}A = 1$
- perturbed case

$$A \longrightarrow A + \delta A$$

$$T_{ol} \longrightarrow K_{ol}(A + \delta A) = \underbrace{\frac{1}{A}}_{\substack{\text{design} \\ \text{choice}}}(A + \delta A) = \underbrace{1}_{T_{ol}} + \underbrace{\frac{\delta A}{A}}_{\delta T_{ol}}$$

Sensitivity: $\quad \mathcal{S}_{ol} = \dfrac{\delta T_{ol}/T_{ol}}{\delta A_{ol}/A_{ol}} = \dfrac{\delta A/A}{\delta A/A} = 1$

For example, a 5% error in A will cause a 5% error in T_{ol}.

Sensitivity to Parameter Variations

Closed-loop:

- nominal case $\quad T_{\mathrm{cl}} = \dfrac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$
- perturbed case

$$A \longrightarrow A + \delta A \qquad T_{\mathrm{cl}} \longrightarrow T_{\mathrm{cl}} + \underbrace{\delta T_{\mathrm{cl}}}_{\substack{\text{how to}\\ \text{compute this?}}}$$

Taylor expansion:

$$T(A + \delta A) = T(A) + \frac{\mathrm{d}T}{\mathrm{d}A}(A)\delta A + \text{higher-order terms}$$

In our case:

$$\frac{\mathrm{d}T_{\mathrm{cl}}}{\mathrm{d}A} = \frac{K_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}} - \frac{AK_{\mathrm{cl}}^2}{(1 + AK_{\mathrm{cl}})^2} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}$$

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}\delta A$$

From before:

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}\delta A$$

$$T_{\mathrm{cl}} = \frac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$$

Therefore

$$\delta T_{\mathrm{cl}}/T_{\mathrm{cl}} = \frac{\frac{K_{\mathrm{cl}}}{(1+AK_{\mathrm{cl}})^2}\delta A}{\frac{AK_{\mathrm{cl}}}{1+AK_{\mathrm{cl}}}} = \frac{1}{1 + AK_{\mathrm{cl}}}\delta A/A$$

Sensitivity: $\quad \mathcal{S}_{\mathrm{cl}} = \dfrac{\delta T_{\mathrm{cl}}/T_{\mathrm{cl}}}{\delta A/A} = \dfrac{1}{1 + AK_{\mathrm{cl}}} \qquad (\ll 1 \text{ for large } K_{\mathrm{cl}})$

With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.

Time Response

We still assume no disturbance: $\tau_e = 0$.

So far, we have focused on DC gain only (steady-state response). What about *transient response*?

Open-loop

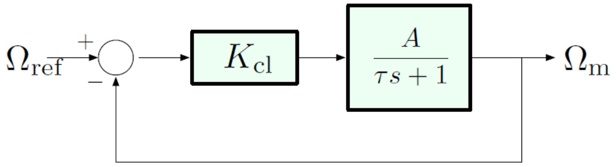
$$\Omega_m = \frac{AK_{cl}}{\tau s + 1}\Omega_{ref}$$

Pole at $s = -\frac{1}{\tau}$ $\quad\Longrightarrow\quad$ transient response is $e^{-t/\tau}$

Here, τ is the *time constant*: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, larger time constant means ~~faster~~ convergence to steady state. This is not affected by the choice of K_{cl} in any way!

Closed-loop



$$\Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}}\Omega_{ref}$$

Closed-loop pole at $s = -\frac{1}{\tau}(1 + AK_{cl})$
(the only way to move poles around is *via feedback*)

Now the transient response is $e^{-\frac{1+AK_{cl}}{\tau}t}$, with

$$\text{time constant} = \frac{\tau}{1 + AK_{cl}}$$

— for large K_{cl}, we have a much smaller time constant, i.e., *faster convergence* to steady-state.

Summary

- Feedback Control:
  - reduces steady-state error to disturbances
  - reduces steady-state sensitivity to model uncertainty (parameter variations
  - improves time response
- However, what we see so far works well for first order systems
  - static gain may cause underdamping or instability in higher order systems
- More sophisticated control: example PID in next lecture

Recap: Feedback Control

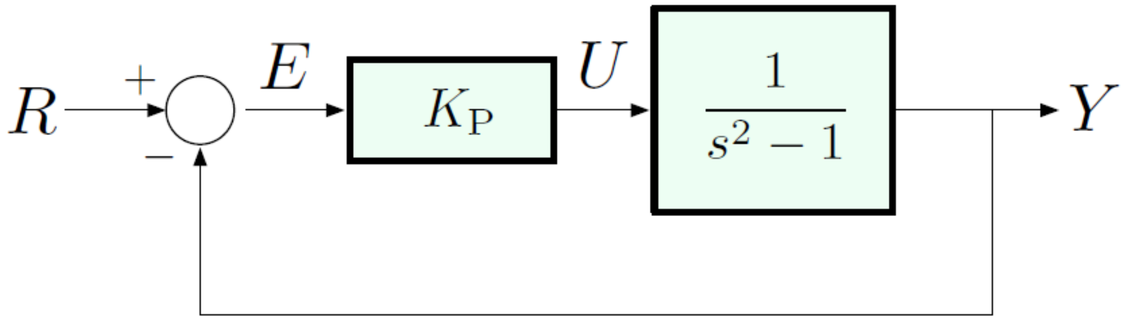
- So far, we have only looked at proportional feedback (scalar gain) and 1st-order plants.
- Add two more basic ingredients and examine their effect on higher-order systems.
- Consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

  - unstable: poles at $s = \pm 1$ (one pole in RHP)
  - 2nd-order

– not as easy as DC motor, which was 1st-order and stable.

Proportional Feedback



K_P – "proportional gain" (P-gain) $\qquad U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\dfrac{K_P}{s^2-1}}{1+\dfrac{K_P}{s^2-1}} = \frac{K_P}{s^2-1+K_P}$$

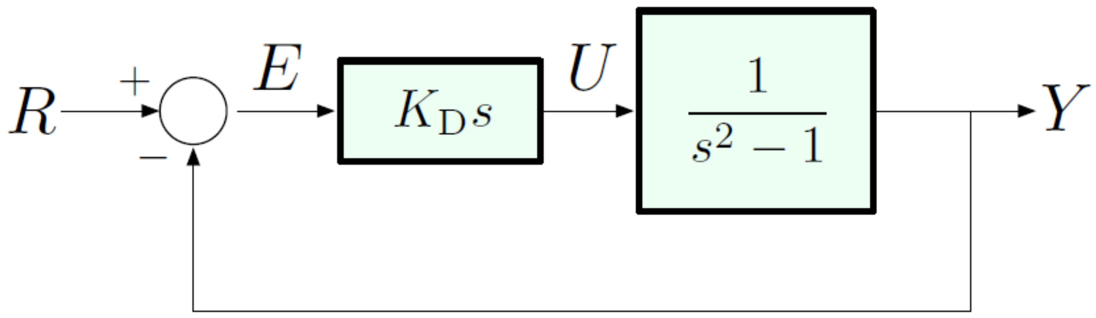
— the polynomial in the denominator has zero coefficient of s

$\Longrightarrow$ necessary condition for stability is not satisfied.

The feedback system is *not stable for* ***any*** *value of* K_P*!!*

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Motivation: derivative = rate of change; faster change $\Longrightarrow$ more control needed.

Caveat: multiplication by s is not a causal element (why?)

Derivative action and lack of causality: recall

$$\dot{e}(t) \approx \frac{e(t+\delta) - e(t)}{\delta} \qquad \text{(for small } \delta\text{)}$$

— if $\delta > 0$, $e(t+\delta)$ is in the future of $e(t)$!!

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu \qquad sX = AX + BU \qquad (s - A)X = BU$$

$$y = Cx \qquad Y = CX \qquad \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\dot{x} = Ax + Bu \qquad sX = AX + BU \qquad (s - A)X = BU$$

$$y = Cx + Du \qquad Y = CX + DU \qquad Y = \frac{CB}{s - A}U + DU$$

$$= \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)}$$

$\deg(q) = \deg(p)$ — proper transfer function

Causal systems have proper transfer functions.

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \dfrac{U}{E} = Ks = \dfrac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$— *improper system* (lack of causality)

So, $E \mapsto K_{\mathrm{D}}sE$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_{\mathrm{D}}as}{a+s} \longrightarrow K_{\mathrm{D}}s \qquad \text{as } a \to \infty$$

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

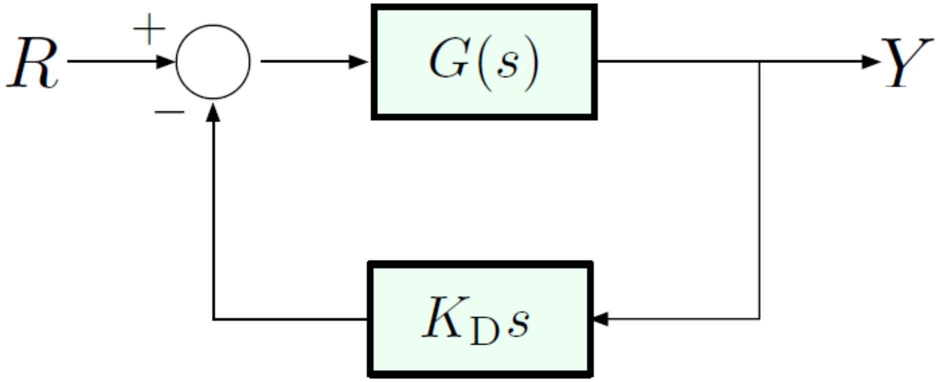
$$\dot{e}(t) \approx \frac{e(t+\delta) - e(t)}{\delta},$$

but then we must tolerate delays — must wait until time $t + \delta$ to issue a control signal meant for time t.

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise $\longrightarrow$ rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



Before: $\dfrac{Y}{R} = \dfrac{K_\mathrm{D}sG(s)}{1 + K_\mathrm{D}sG(s)}$

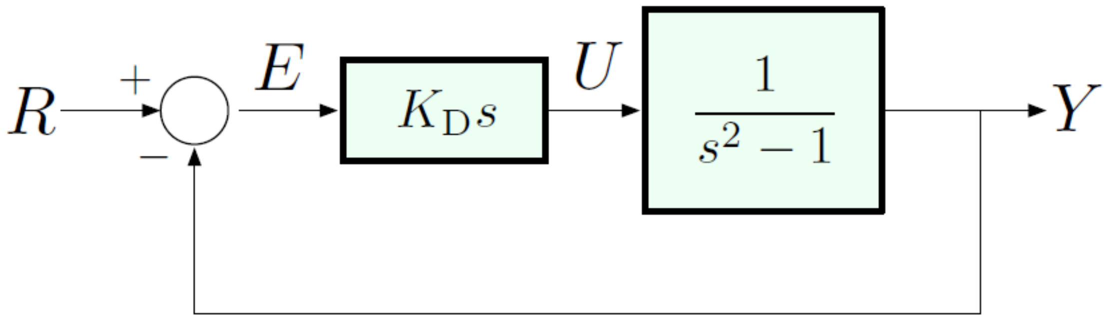
Now: $\dfrac{Y}{R} = \dfrac{G(s)}{1 + K_\mathrm{D}sG(s)}$

Poles: $1 + K_\mathrm{D}sG(s) = 0$

— same poles, but different zeros.

Now the reference signal is *smoothed out* by the plant $G(s)$ before entering the differentiator, which minimizes distortion due to noise.

Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\dfrac{K_D s}{s^2-1}}{1+\dfrac{K_D s}{s^2-1}} = \frac{K_D s}{s^2+K_D s-1}$$

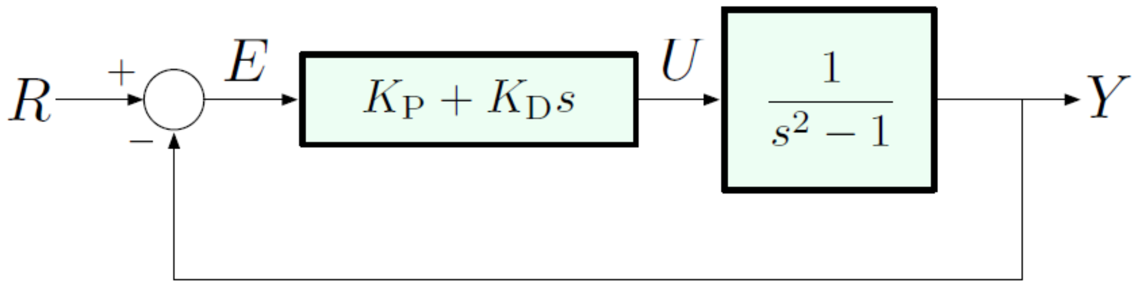
— still not good: the denominator has a negative coefficient
$\Longrightarrow$ not stable; also we have picked up a zero at the origin.

But:

- P-control affected the coefficient of s^0 (constant term)
- D-control affected the coefficient of s

— let's combine them!!

Proportional-Derivative (PD) Control



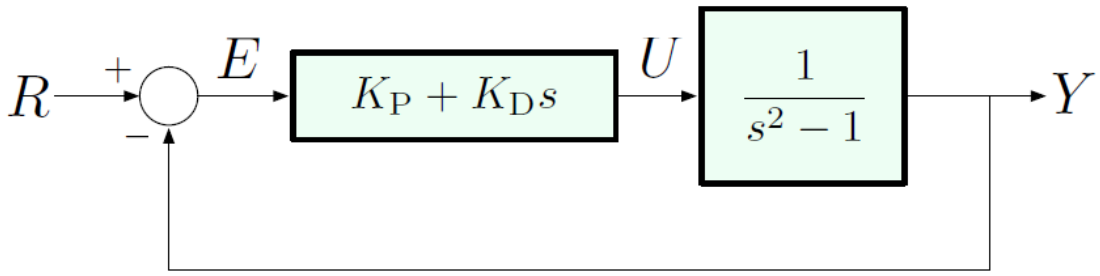
$$\frac{Y}{R} = \frac{\dfrac{K_P + K_D s}{s^2 - 1}}{1 + \dfrac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D, we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us arbitrary pole placement!!

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_{\mathrm{P}} + K_{\mathrm{D}} s}{s^2 + K_{\mathrm{D}} s + K_{\mathrm{P}} - 1}$$

By choosing $K_{\mathrm{P}}, K_{\mathrm{D}}$, we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

$$K_{\mathrm{D}} s + K_P = 0 \qquad \text{LHP zero at } -\frac{K_{\mathrm{P}}}{K_{\mathrm{D}}}$$

But what's missing? $\quad$ DC gain $= \left.\frac{Y}{R}\right|_{s=0} = \frac{K_{\mathrm{P}}}{K_{\mathrm{P}} - 1} \neq 1$

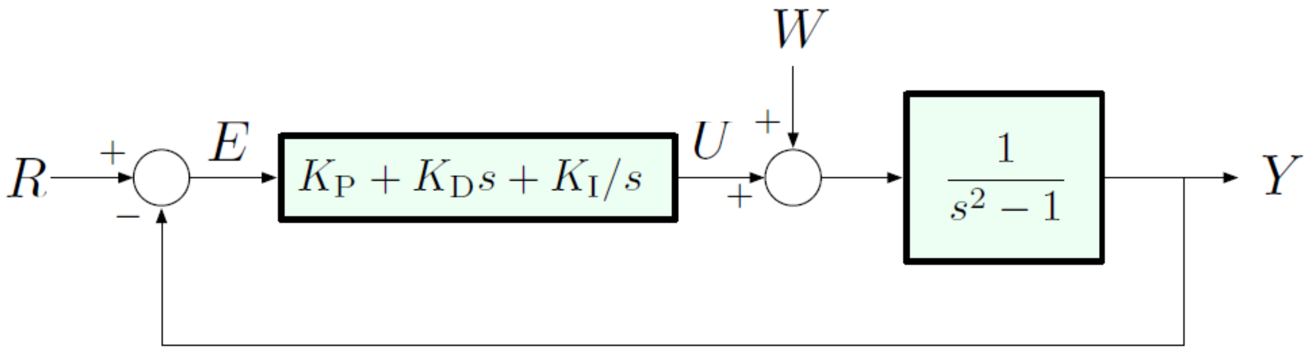
— can't have perfect tracking of constant reference.

Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_{\mathrm{P}} + K_{\mathrm{D}} s + \frac{K_{\mathrm{I}}}{s} \right) E \qquad \text{– the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

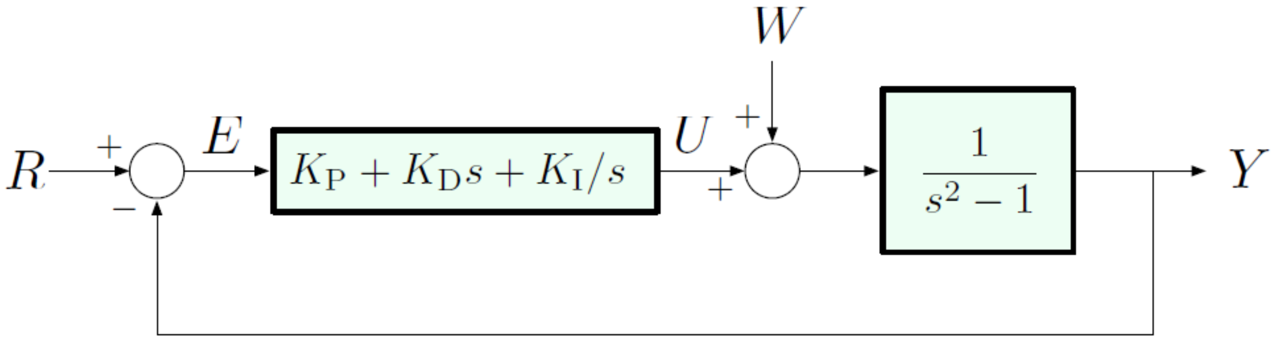


We will see that, with PID control, the goals of

- tracking a constant reference r
- rejecting a constant disturbance w

can be accomplished in one shot.

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2-1}(U+W), \qquad U = \left(K_\mathrm{P} + K_\mathrm{D}s + \frac{K_\mathrm{I}}{s}\right)(R-Y)$$

$$\text{so } Y = \frac{K_\mathrm{P} + K_\mathrm{D}s + \frac{K_\mathrm{I}}{s}}{s^2-1}(R-Y) + \frac{1}{s^2-1}W$$

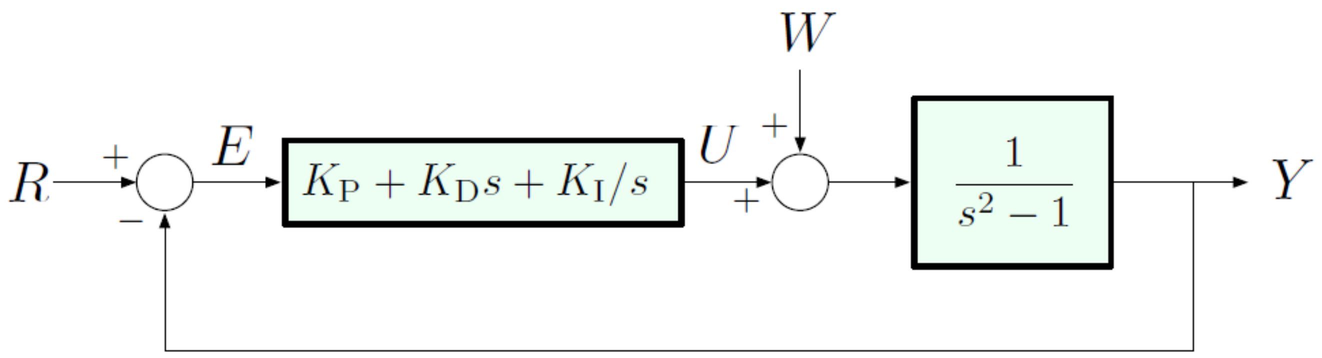
Simplify:

$$(s^2-1)Y = \left(K_\mathrm{P} + K_\mathrm{D}s + \frac{K_\mathrm{I}}{s}\right)(R-Y) + W$$

$$\left(s^2 - 1 + K_\mathrm{P} + K_\mathrm{D}s + \frac{K_\mathrm{I}}{s}\right)Y = \left(K_\mathrm{P} + K_\mathrm{D}s + \frac{K_\mathrm{I}}{s}\right)R + W$$

$$(s^3 - s + K_\mathrm{P}s + K_\mathrm{D}s^2 + K_\mathrm{I})Y = (K_\mathrm{P}s + K_\mathrm{D}s^2 + K_\mathrm{I})R + Ws$$

Proportional-Integral-Derivative (PID) Control

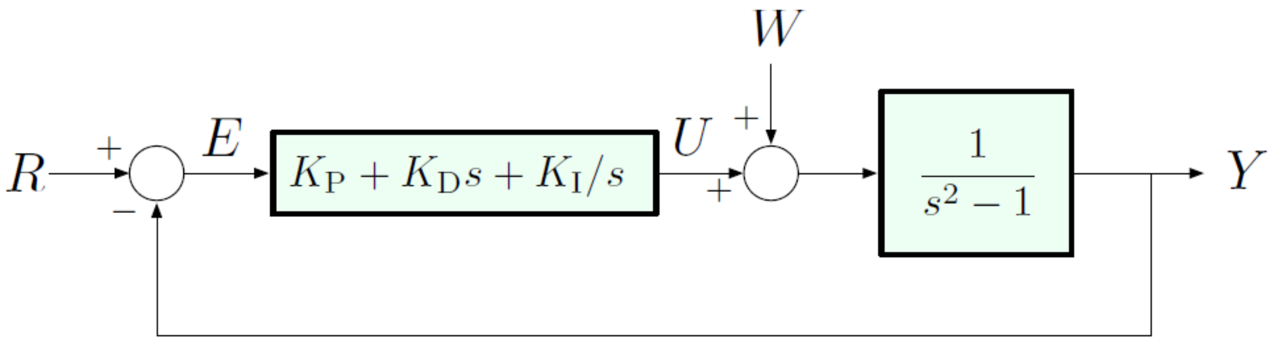


$$(s^3 - s + K_\mathrm{P}s + K_\mathrm{D}s^2 + K_\mathrm{I})Y = (K_\mathrm{P}s + K_\mathrm{D}s^2 + K_\mathrm{I})R + Ws$$

Therefore,

$$\begin{aligned} Y = {} & \frac{K_\mathrm{D}s^2 + K_\mathrm{P}s + K_\mathrm{I}}{s^3 + K_\mathrm{D}s^2 + (K_\mathrm{P}-1)s + K_\mathrm{I}}R \\ & + \frac{s}{s^3 + K_\mathrm{D}s^2 + (K_\mathrm{P}-1)s + K_\mathrm{I}}W \end{aligned}$$

Proportional-Integral-Derivative (PID) Control

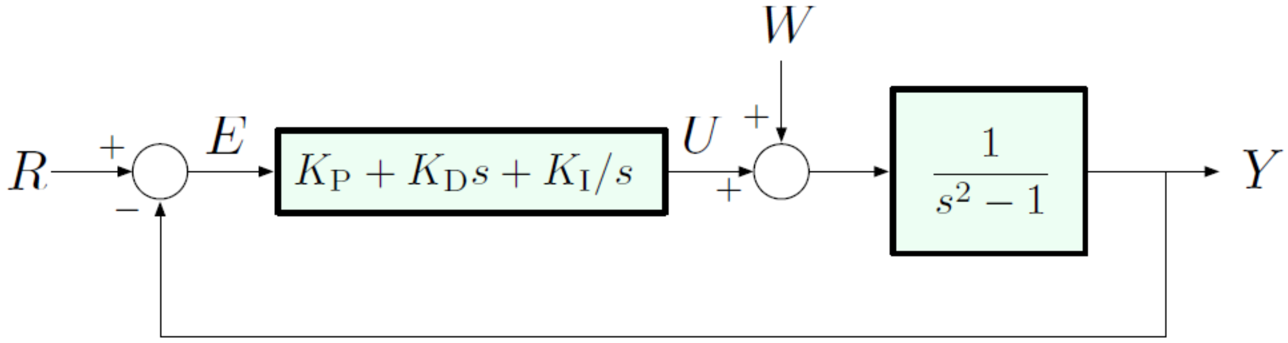


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

- need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing K_P, K_I, K_D

Proportional-Integral-Derivative (PID) Control



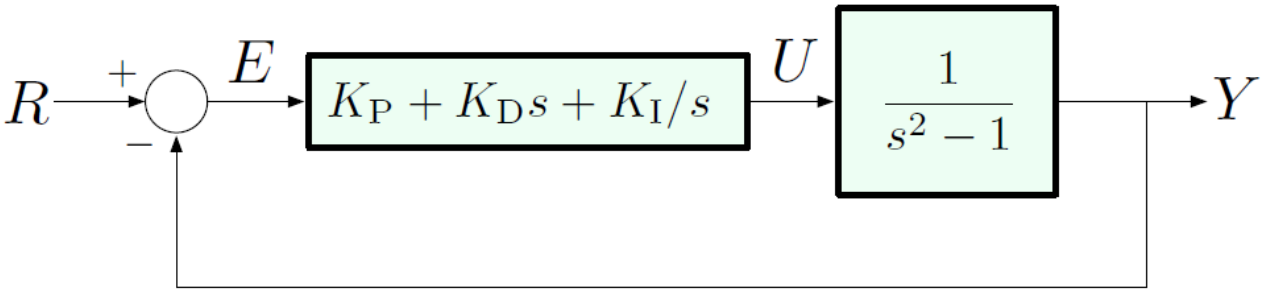
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \to Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 1$$

— so, with the addition of I-feedback, we remove earlier limitation and achieve *perfect tracking*!

Proportional-Integral-Derivative (PID) Control



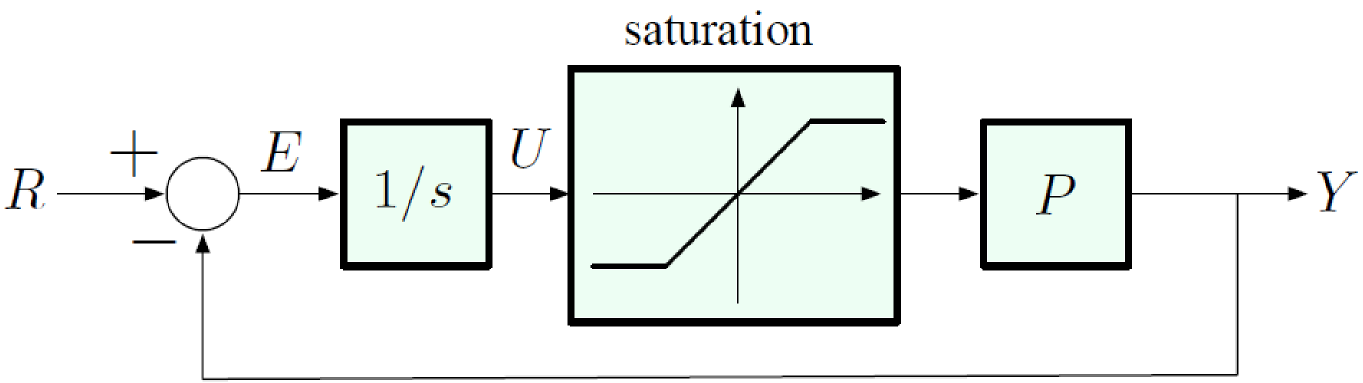
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \to Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

— so, integral gain also gives *complete attenuation* of *constant* disturbances!!

Wind-Up Phenomenon



When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator "winds up:" the error may be small, but its integrated past history builds up.

There are various *anti-windup* schemes to deal with this practically important issue. (Essentially, we attempt to detect the onset of saturation and turn the integrator off.)

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

- but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (integrator wind-up)