Question 1

a) Give an advantages and disadvantages of feedback in control *(2 Points)*

b) Consider the block diagram of a feedback control system below.

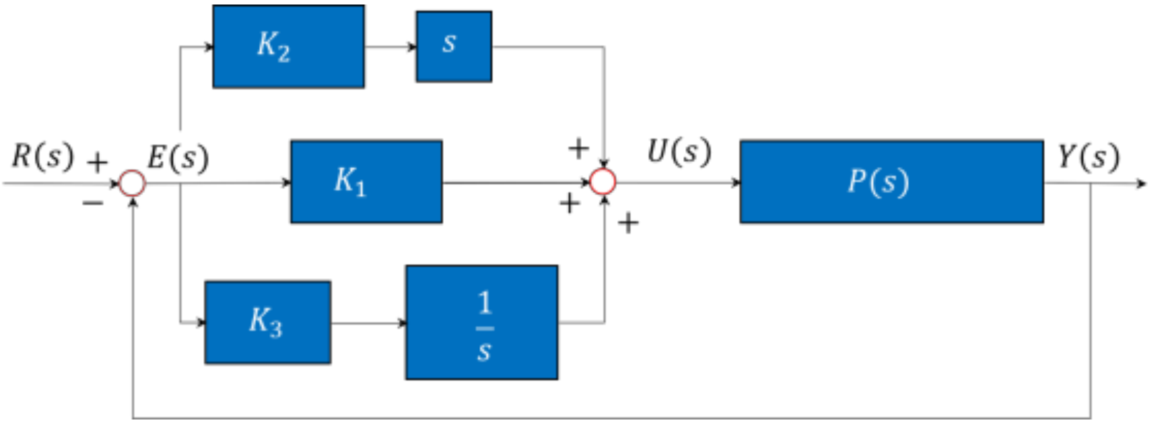


Figure 1

Explain your choice of K_1, K_2, K_3 for the (i) & (ii).

i) Which is the most appropriate gain to remove if the sensor is prone noise? *(2 Points)*

ii) Which is the gain incorporated to eliminate state-steady error? *(2 Points)*

iii) Given that the plant has a transfer function $P(s) = \frac{2}{s^3+4s^2+5s+2}$, write down the closed-loop transfer function $H_{cl}(s)$ in terms of K_1, K_2, K_3 and s *(4 Points)*

iv) Assume $K_2 = 0$, write down the new closed loop transfer function $H_{cl2}(s)$ *(1 Points)*

v) Using Routh-Hurwitz criterion, obtain the appropriate range for K_1 and, K_3. *(8 Points)*

c) Show that the DC gain = 1 for the closed loop transfer function

$$H(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

FVT: $Y_{ss} = \lim_{s\to 0} H(s)\, U(s) = \frac{\omega_n^2}{\omega_n^2} s = s.$

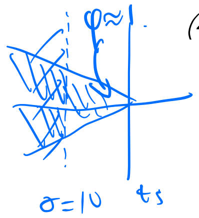
DC Gain $= \frac{Y}{U} = \frac{s}{s} = 1$.

State any assumption you need to make. *(2 Points)*

d) Sketch the region in the left-half plane where the complex poles of the second-order system should be located to meet the following conditions: (i) 5% settling time < t_{sm} = 0.3; (ii) Percent Overshoot, Mp < 4.3% *(4 Points)*

d. (i). $t_s < 0.3s \Rightarrow \sigma > 10$

$\frac{3}{\sigma}$



iv.

$$\frac{2k_2 s^2 + 2k_1 s + 2k_3}{s^4 + 4s^3 + (2k_2+5)s^2 + (2k_1+2)s + 2k_3}$$

$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 4.3\%$, i.e. $e^{-\pi\varphi} < 4.3\% \Rightarrow -\pi\varphi = \ln 4.3\% = -\pi$. $\varphi = 1$.

v.

s^4	1	$2k_2+5$	$2k_3$
s^3	4	$2k_1+2$	0
s^2	$\frac{9-k_1}{2}$	$2k_3$	0
s	$\frac{16k_3+2k_1^2-16k_1-18}{k_1-9}$	0	
s^0	$-2k_3$	0	

$9-k_1 > 0 \Rightarrow k_1 < 9$

$-2k_3 > 0 \Rightarrow k_3 < 0$

$16k_3 + 2k_1^2 - 16k_1 - 18 > 0 \Rightarrow 8k_3 > -(k_1-9)(k_1+1)$

Q1.

(a). advantage: ~~better ref~~
better performance under dynamic environment,
~~working~~ more stable.

disadvantage: hard to design, easy to go beyond stability.

(b). D
P
I

i) K_2. ~~Be~~ it lies in a branch of differentiator which amplifies noise.

ii). K_3 it lies in an integrator which helps the system achieve perfect steady-state reference tracking.

iii). $U(s) = (K_2 s + K_1 + \frac{K_3}{s}) E$; $Y = PU$, $E = R - Y$.

$$\Rightarrow \frac{Y}{P} = (K_2 s + K_1 + \frac{K_3}{s}) E. = (K_2 s + K_1 + \frac{K_3}{s})(R - Y).$$

$$E = R - Y.$$

$$\therefore Y = \cancel{P\cdot k}\ P \cdot G (R - Y) = PGR - PGY. \Rightarrow Y(PG + 1) = R(PG)$$

$$\therefore \frac{Y}{R} = H_{cl} = \frac{PG}{PG+1} = \frac{\frac{2}{s^3+4s^2+5s+2}(K_1 + K_2 s + \frac{K_3}{s})}{\frac{2}{s^3+4s^2+5s+2} \cdot (K_2 s + K_1 + \frac{K_3}{s}) + 1}$$

$$= \frac{2(K_1 + K_2 s + \frac{K_3}{s})}{2(K_2 s + K_1 + \frac{K_3}{s}) + (s^3 + 4s^2 + 5s + 2)}$$

$$= \frac{2K_2 s^2 + 2K_1 s + 2K_3}{s^4 + 4s^3 + (2K_2 + 5)s^2 + (2K_1 + 2)s + 2K_3}$$

Question 2

a) Consider the system illustrated below

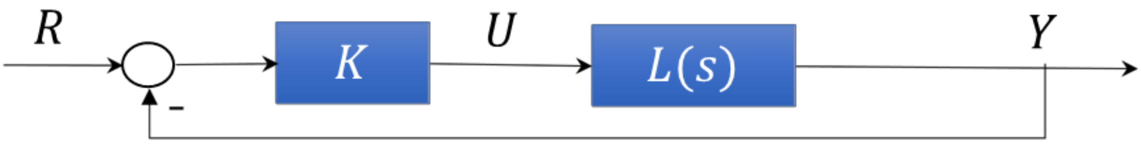


Figure 2a

$H_{cl} = \frac{KL}{1+KL}$.

i) Write down the closed loop transfer function $H_{OL}(s) = \frac{Y}{R}$ in terms of K and L. *(3 Points)*

ii) The Root Locus shows the locations the solution of the characteristic equation with varying values of ___K___? *(1 Points)*

$\frac{1}{s+2}$. $\frac{1}{s+1}$

b) A corresponding representation of the system is shown below

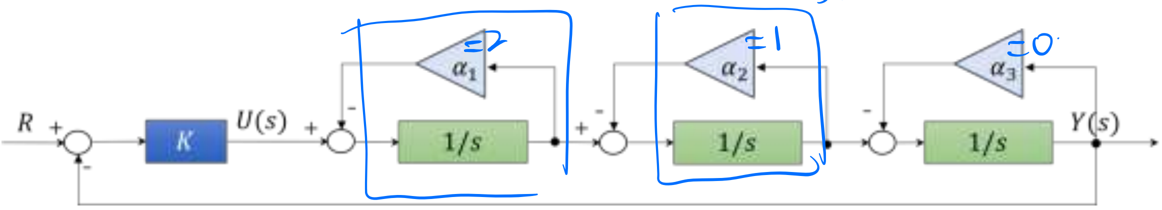


Figure 2b

Given $\alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = 0$

$\frac{1}{s(s+1)(s+2)}$.

i) Write down an expression for the transfer function $L(s) = \frac{Y}{U}$ in terms of s *(3 Points)*

ii) Write down the open-loop poles of the system. $s = 0, -1, -2$. *(1 Points)*

iii) Show that the characteristic polynomial can be expressed as

$$s^3 + 3s^2 + 2s + K = 0$$

$H_{cl} = \frac{KL}{1+KL} = \frac{K}{s(s+1)(s+2)+K}$

$\therefore CP = s^3+3s^2+2s+K = 0$. *(1 Points)*

iv) Using Routh-Hurwitz method, find the range of value for K to ensure stability. (8 *Points*)

$\Leftarrow$: $\boxed{K>0}$

$\Rightarrow$: $3\times 2 > K$

$\Rightarrow \boxed{K<6}$.

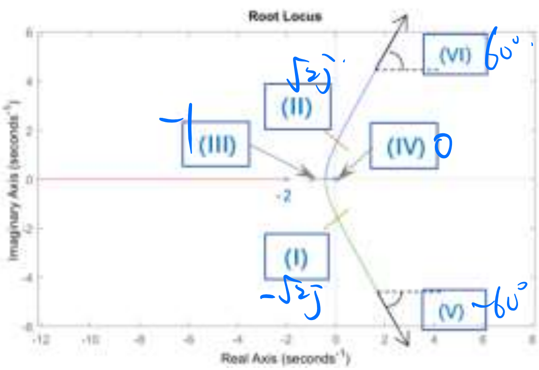
v) Obtain the value of ω at where the root-locus intercept the imaginery axis i.e. $j\omega-crossing$. *(2 Points)*

c) Label (I)-(VI) with the appropriate values showing how you derive your answers.

$K=6$: $(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + 6 = 0$

$-j\omega^3 - 3\omega^2 + 2j\omega + 6 = 0$

$\therefore\ \omega^3 - 2\omega = 0$, $3\omega^2 - 6 = 0$; $\omega = 0$, $\omega\ \sqrt{2}$, or $-\sqrt{2}$



$j\omega$-crossing at $\pm\sqrt{2}j$.

Figure 2c

(6 Points)

$m=0,\ n=3$.

$\angle s \approx \frac{(2l+1)\cdot 180}{3} = \pm 60° = 60°,\ 180°,\ -60°$

Question 3

① linear ② response not changed when starting at a different time

a) What are the properties of Causal Linear Time Invariant Systems? *(3 Points)*

③ response irrelevant to future performance of the system.

b) A harmonic input signal *u(t)* is map to an output signal *y(t)* through an LTI system with transfer function *G(s)*. Label (i) and (ii) *(2 Points)*

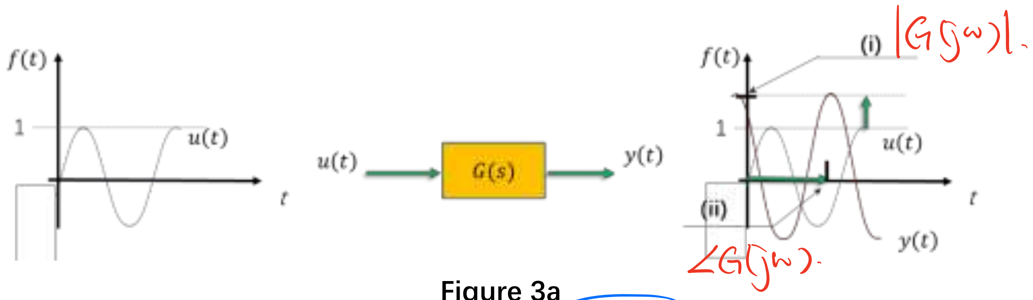


Figure 3a

c) A plate attached to a spring and damper with insignificant mass with zero-initial conditions is subjected to a force as shown.

$sX = U - \alpha X.$ $(s+\alpha)X = U.$

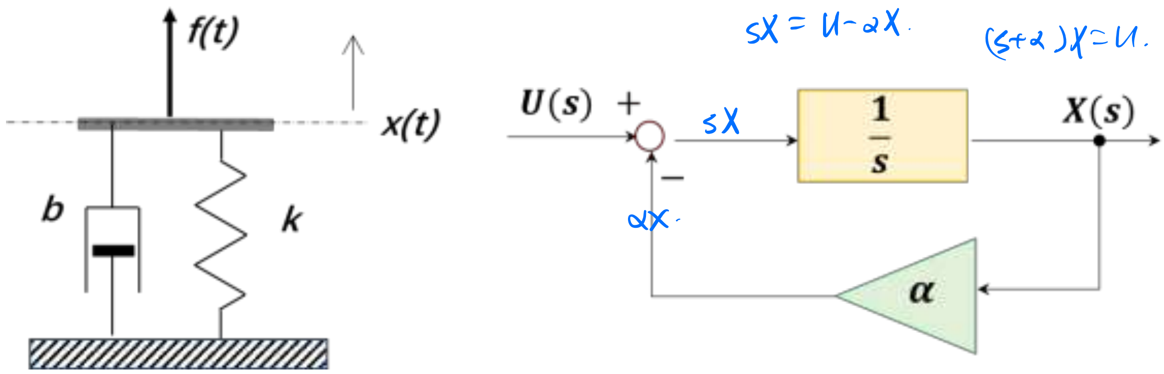
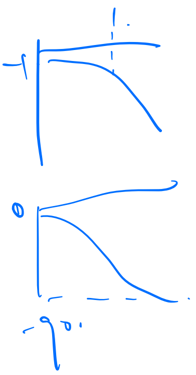


Figure 3b



i) Show that the system can be represented with the given block diagram and provide the expressions of $U(s)$ and α $f(t) = kx + b\dot{x} \Rightarrow F = kX + sbX.$ $\therefore sX = \frac{F}{b} - \frac{k}{b}X \Leftrightarrow sX = U - \alpha X.$ (2 Points)

ii) Write down the frequency response function $G(j\omega) = \frac{1}{j\omega + \alpha}.$ $(s + \frac{k}{b})X = \frac{F}{b}$ (2 points) $\frac{X}{F} = \frac{1}{(s+\frac{k}{b})b}$

iii) Express $G(j\omega)$ in terms of its magnitude and phase given *k=b=1*. $G = \frac{1}{s+1} = \frac{1}{j\omega+1}$ (2 points) $\Rightarrow = \frac{1}{\sqrt{\omega^2+1}} \angle(1-j\omega).$

iv) Sketch the Bode diagrams representing gain $G(j\omega)$ (4 points)

v) Assuming significant plate mass *m=1, b=6, k=5*, rewrite the new transfer function of the plant $G_p(s)$ $F = 5X + 6sX + s^2X. \Rightarrow G_p = \frac{1}{s^2+6s+5}.$ (1 Points)

d) A feedback control system is implemented as represented by the shown block diagram.

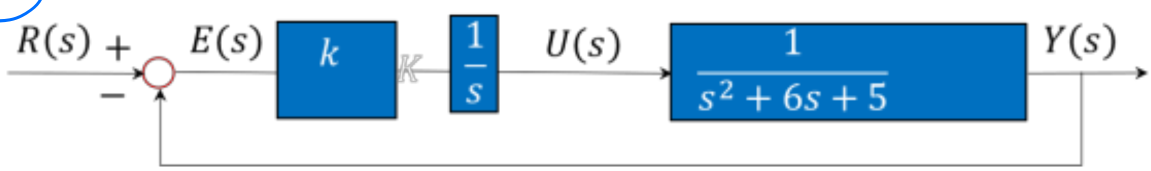


Figure 3c

$H = \frac{K}{s^3+6s^2+5s+\cancel{K}}$

$= \frac{10}{s^3+6s^2+5s+\cancel{K}}$

vi) When K = 10, the bode plot is given by **Figure 3d**. Indicate the frequency values where there are changes in the magnitude slope. (4 Points)

vii) Given the Gain Margin (GM)=+8 dB, Phase Margin (PM)=+21°, on the bode plot on Figure 2, label the Gain Margin and Phase Margin. *(2 Points)*

viii) Comment on how changing the value of *K affect stability using the Bode plot. (3 points)*

1. it will not affect phase margin.

2. K↑, GM↓, finally GM<0 and system unstable.

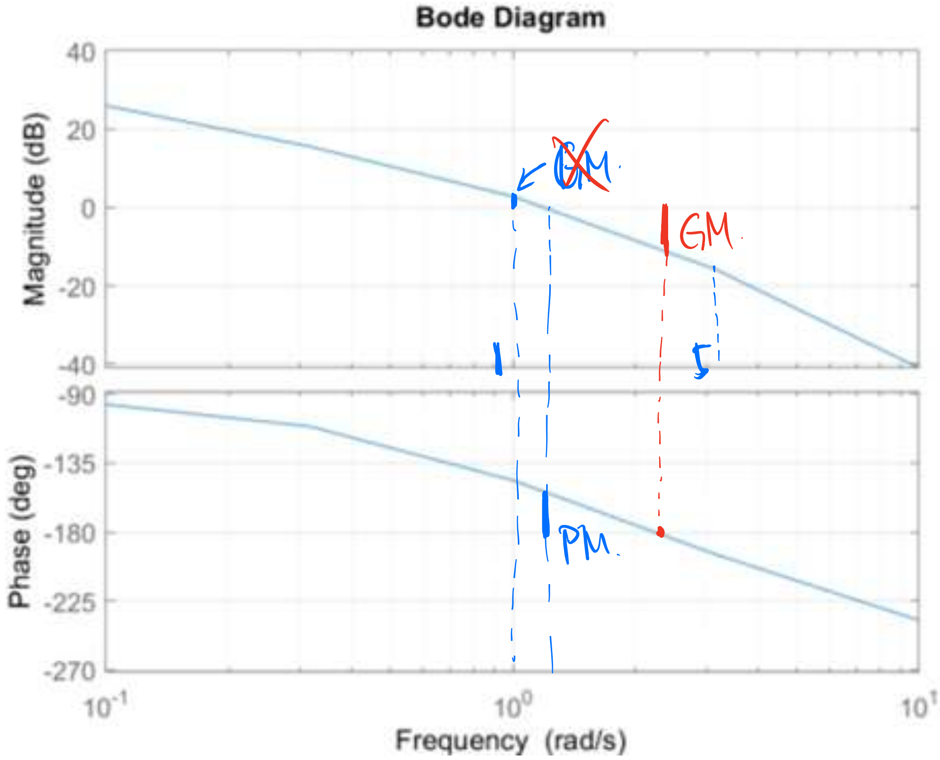


Figure 3d

Question 4

a) States some of the advantages in using state-space design ? (3 Points)

b) Consider the control system shown

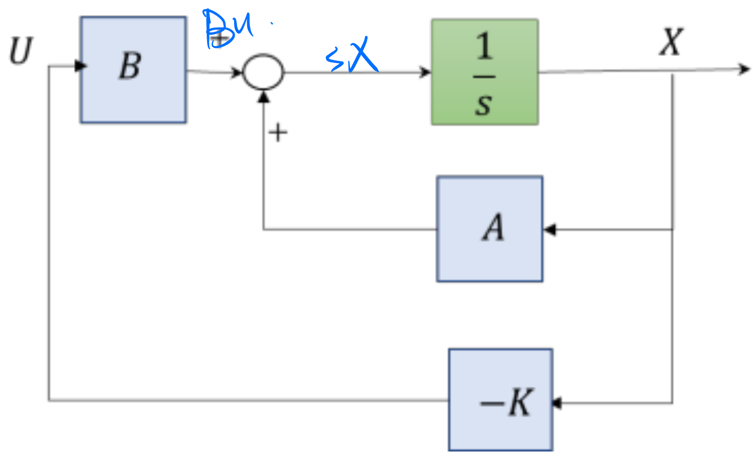


Figure 4

The plant is given by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

$sX = Ax + Bu.$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Choosing the desired closed-loop poles at $s = -2 \pm j4,\ s = -10$, do the following

$AB = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$. $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -5 & -6 \\ 6 & 29 & 31 \end{bmatrix}$. $A^2B = \begin{bmatrix} 1 \\ -6 \\ 31 \end{bmatrix}$.

$\Rightarrow C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$.

i) Show that the controllability matrix $\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$ (3 Points)

ii) Given that the rank $\mathbf{M}$ is 3, comment on the controllability of the system. (2 Points)

$\nearrow$ M is invertible. ∴ stable.

iii) Determine the state-feedback gain matrix K. (9 Points)

$\det(Is - A + BK) = (s+10)(s+2-j4)(s+2+j4)$.

System in CCF. $\therefore A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2-j4 & -2+j4 \end{bmatrix} = \begin{bmatrix} & \cdots & \\ -1+k_1 & -5+k_2 & -6+k_3 \end{bmatrix}$

$k_1 = -9$. $k_2 = 3-j4$

$k_3 = 4+j4$

c) Consider the system given by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ and output $y = \mathbf{C}\mathbf{x} + Du$

i) Write down an expression for the transfer function of this system (2 Points)

ii) Give an expression of the zeros of the system transfer function (1 Points)

iii) Give two expressions of the poles of the system transfer function (2 Points)

iv) State the condition for the system be observable from the output y (1 Points)

v) State the key reason for using an estimator in feedback control. (2 Points)

i). $G(s) = C(Is-A)^{-1}B + D$.

ii). $C(Is-A)^{-1}B + D = 0$.

iii).

Solution

Question 1

a) Advantage: Helps to achieve reference tracking for a dynamic system with uncertainty and external disturbance; Disadvantage: costly to implement; must to ensure loop stability

b)

i) The derivative term K_2 should be removed as it is noise prone and poor noise suspension

ii) The integral term K_3 can could eliminate steady state error

iii)

$$H_{cl}(s) = \frac{K_{123}P}{1+K_{123}P} = \frac{K_{123}}{\frac{1}{P}+K_{123}} = \frac{2(K_2 s + K_1 + K_3\frac{1}{s})}{(s^3+4s^2+5s+2)+2(K_2 s + K_1 + K_3\frac{1}{s})}$$

$$= \frac{2(K_2 s^2 + K_1 s + K_3)}{(s^4+4s^3+5s^2+2s)+2(K_2 s^2 + K_1 s + K_3)}$$

iv)

$$H_{cl}(s) = \frac{2(K_1 s + K_3)}{s^4+4s^3+5s^2+(2+2K_1)s+2K_3}$$

v) Characteristic Equation:

$$s^4+4s^3+5s^2+(2K_P+2)s+2K_I = 0$$

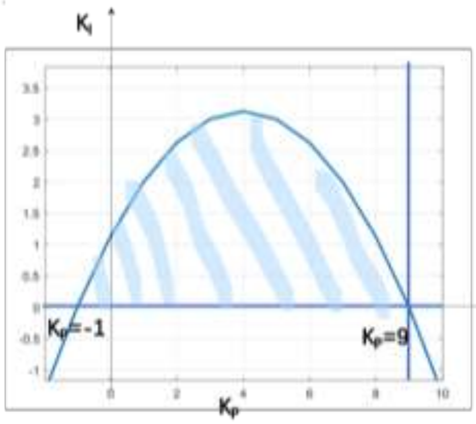
Routh Array

s^3	1	5	$2K_I$
	4	$2K_P+2$	
s^2	$\frac{18-2K_P}{4}$	$2K_I$	
s^1	$*$		
s^0	$2K_I$		

$K_P < 9;$

$K_I > 0$

$$(*) \to \tfrac{1}{8}(1+K_P)(9-K_P) - K_I > 0$$



$$K_I < \tfrac{1}{8}(1+K_P)(9-K_P)$$

c) Assuming DC gain exist, by final value theorem

At $y(t\to\infty)$, DC gain $H(s=0)$

$$y(\infty) \to H(s=0) = \frac{{\omega_n}^2}{0^2+2\zeta\omega_n(0)+{\omega_n}^2} = 1$$

5% settling time: $\frac{3}{\zeta\omega_n} < 0.3$

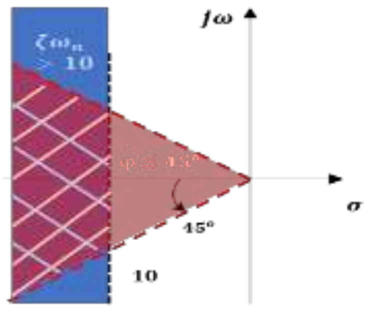
$$\boxed{\zeta\omega_n > 10}$$

Percentage Overshoot M_p<4.3%

$$\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) < 0.043$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} > \ln 0.043$$

$$\boxed{\begin{gathered}\zeta > 0.707 \\ \boldsymbol{\varphi < \cos^{-1}(0.707) \lesssim 45^{o}}\end{gathered}}$$



Question 2

a)

i) $H_{cl}(s) = \frac{KL(s)}{1+KL(s)}$

ii) K

b)

i)

$$L(s) = \frac{1}{(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)} = \frac{1}{(s+2)(s+1)(s+0)} = \frac{1}{s(s+2)(s+1)}$$

ii)

open loop poles: $\alpha_1 = 2,\ \alpha_2 = 1,\ \alpha_3 = 0$

iii) Characteristic polynomial is when $1 + KL(s) = 0$,

$$1 + \frac{K}{s(s+2)(s+1)} = 0$$
$$s(s+2)(s+1) + K = 0$$
$$s^3 + 3s^2 + 2s + K = 0$$

iv)

Using Routh-Hurwitz Stability Analysis

Characteristic polynomial equation $s^3 + 3s^2 + 2s + K = 0$

The necessary condition is that K >0.

Routh Array

s^3	1	2
s^2	3	K
s^1	$6-K$	0
s^0	K	

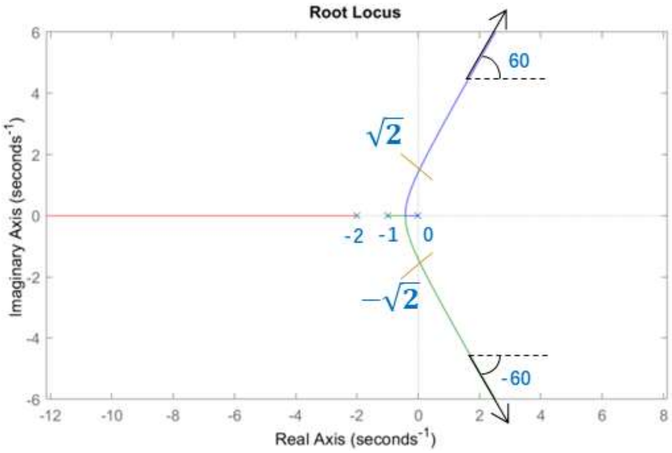
Therefore $0 < K < 6$ implying K having a critical value of 6.

v)

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + 6 = 0$$

At $j\omega$-crossing, real part equal zero, $\omega = \pm\sqrt{2}$

c)



Question 3

a) Casual: State only depend on past states but not future; consider only time, t>0

b) (i) $|G(j\omega)|$ (ii) $\angle G(j\omega)$

c) i)

$$f_{external} = f_{damper} + f_{spring}$$
$$f(t) = b\dot{x}(t) + kx(t)$$
$$\frac{f(t)}{b} = \dot{x}(t) + \frac{k}{b}x(t)$$

Letting $\frac{f(t)}{b} = u(t)$, $\frac{k}{b} = \alpha$,

$$u(t) = \dot{x}(t) + \alpha x(t)$$

With zero initial conditions and taking Laplace transform

$$U(s) = sX(s) + \alpha X(s)$$

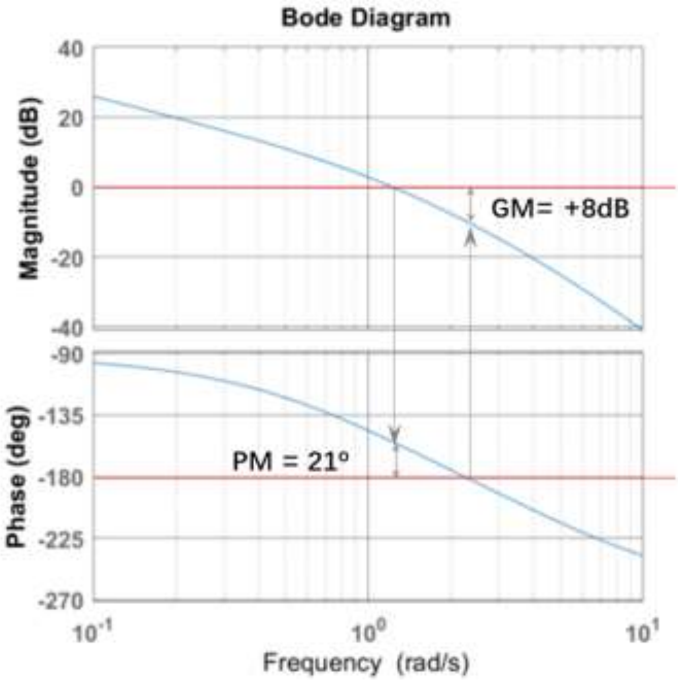
ii)

$$G(j\omega) = \frac{1}{j\omega + \alpha}$$

iii) $|G(j\omega)| = \frac{1}{\sqrt{\omega^2+1}}$ (iv) $\angle G(j\omega) = -\angle(\omega j + 1)$

d) vi) $\omega = 1, 5$

v)



viii) since increasing K shift the magnitude plot downwards but does not change the phase plot, the gain margin will be reduced and eventually become negative and unstable.

Question 4

Advantage of state-space:

a)

- Reveal more internal architecture than representation using transfer function
- Matrix representation facilitate computer analysis
- More convenient for modeling MIMO system problems.

b) i) $\mathbf{M} = [B| \quad AB| \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0| & 1\ | & -6 \\ 1 & -6 & -5+36 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$

ii) Already in ccf also $\mathrm{Rank}(\boldsymbol{M}) = 3$ controllable

iii)

characteristic polynomial $\det(Is - A) = s^3 + 6s^2 + 5s + 1$

set as $s^3 + a_1 s^2 + a_2 s + a_3$ i.e. $a_1 = 6,\ a_2 = 5,\ a_3 = 1$

desired characteristic polynomial equation:

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

Letting characteristic equation be $s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$, we have

$\alpha_1 = 14,\ \alpha_2 = 60,\ \alpha_3 = 200,$

$k = [\alpha_1 - a_1|\ \alpha_2 - a_2|\alpha_3 - a_3]T^{-1};\ T = I$ since already in CCF

$$k = [200 - 1| \quad 60 - 5| \quad 14 - 6] = [199 \quad 55 \quad 8]$$

c)

i) $G(s) = C(Is - A)^{-1}B + D$

ii) $Z = \text{root of } \det\begin{pmatrix} Is - A & -B \\ C & D \end{pmatrix} = 0$

iii) $P = \det\,(Is - A)$ or $eig(A)$

iv) Either observability matrix not equal to zero or realizable in OCF mode.

v) When the system is not readily available, too costy or impractical to measure state-variable.