ECE 486 Control Systems

Lecture 13: Stability from Frequency Response

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Checklist

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Root Locus

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

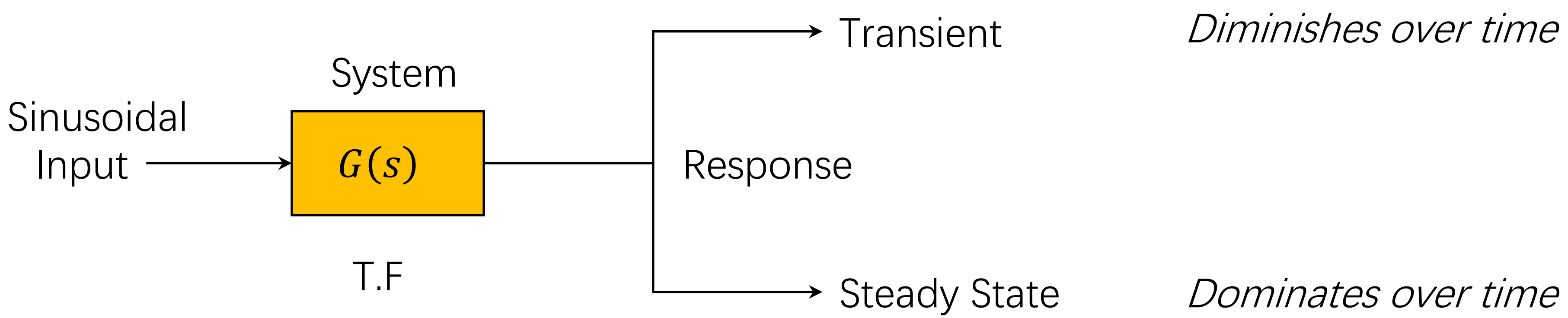
Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	Control design using frequency response	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Frequency Response

State-Space

Recap: Frequency Response

- The steady-state response to a sinusoidal input is known as the frequency response



Recap: Frequency Response Formula

$s = \sigma + j\omega$

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow M\sin(\omega t + \phi)$$

where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Derivation:

1. $u(t) = e^{st} \longmapsto y(t) = G(s)e^{st}$
2. Euler's formula: $\sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$
3. By linearity,

$$\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \qquad G(j\omega) = M(\omega)e^{j\phi(\omega)}$$

$$= \frac{M(\omega)e^{j(\omega t + \phi(\omega))} - M(\omega)e^{-j(\omega t + \phi(\omega))}}{2j}$$

$$= M(\omega)\sin(\omega t + \phi(\omega))$$

Let's apply this formula to our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

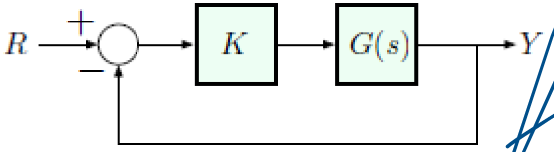
$$M(\omega) = |G(j\omega)| = \left| \frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2} \right|$$

$$= \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}j} \right|$$

$$= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

Recap: Frequency Response (How)

$Ae^{\theta j}$ $Be^{\phi j}$



$ABe^{(\theta+\phi)j}$

Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior

M ϕ ω ω

We will work with two types of plots for $KG(j\omega)$:

1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency ω (could have seen it earlier, in ECE 342)
2. Nyquist plots: $\mathrm{Im}(KG(j\omega))$ vs. $\mathrm{Re}(K(j\omega))$ [Cartesian plot in s-plane] as ω ranges from $-\infty$ to $+\infty$

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

Horizontal (ω) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of ω, *not* $\log_{10}\omega$, on the axis, but the *scale* will be logarithmic:

$$\cdots \quad 0.01 \quad 0.1 \quad 1 \quad 10 \quad 100 \quad 1000 \quad \cdots \quad \omega$$

Equal intervals on log scale correspond to decades in frequency.

Vertical axis on magnitude plots:

we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1e^{j\phi_1})(M_2e^{j\phi_2})| = M_1 \cdot M_2$$
$$\log(M_1M_2) = \log M_1 + \log M_2$$

— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log(M_1(\omega)M_2(\omega))$, and graphical addition is easy.

Decibel scale:

$$(M)_{\mathrm{dB}} = 20\log_{10} M \qquad \text{(one decade} = 20\,\mathrm{dB})$$

Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

Reason:

$$\angle\left((M_1e^{j\phi_1})(M_2e^{j\phi_2})\right) = \angle\left(M_1M_2e^{j(\phi_1+\phi_2)}\right)$$
$$= \phi_1 + \phi_2$$

— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

Recap: Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

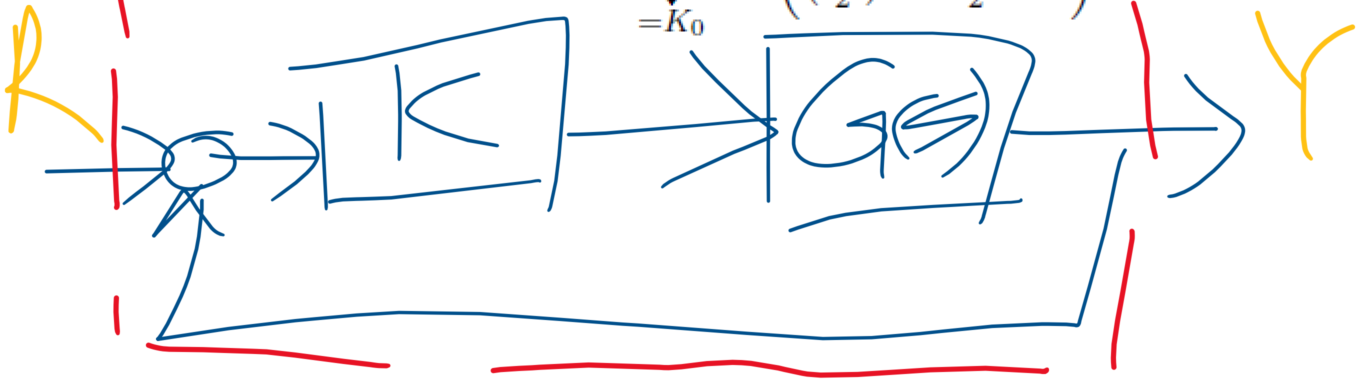
Example:

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

rewrite as

$$\left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$



Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where n is a positive or negative integer
2. $(j\omega\tau+1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$

In our example above,

$$KG(j\omega) = \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]}$$

$$= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}}\cdot\underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}}\cdot\underbrace{\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]^{-1}}_{\text{Type 3}}$$

Unstable Zeros/Poles

So far, we've only looked at transfer functions with stable poles and zeros (except perhaps at the origin). What about RHP?

$j\omega$

Example: consider two transfer functions,

$$G_1(s) = \frac{s+1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{s-1}{s+5}$$

Note:

- G_1 has stable poles and zeros; G_2 has a RHP zero.
- Magnitude plots of G_1 and G_2 are the same —

$$|G_1(j\omega)| = \left|\frac{j\omega+1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

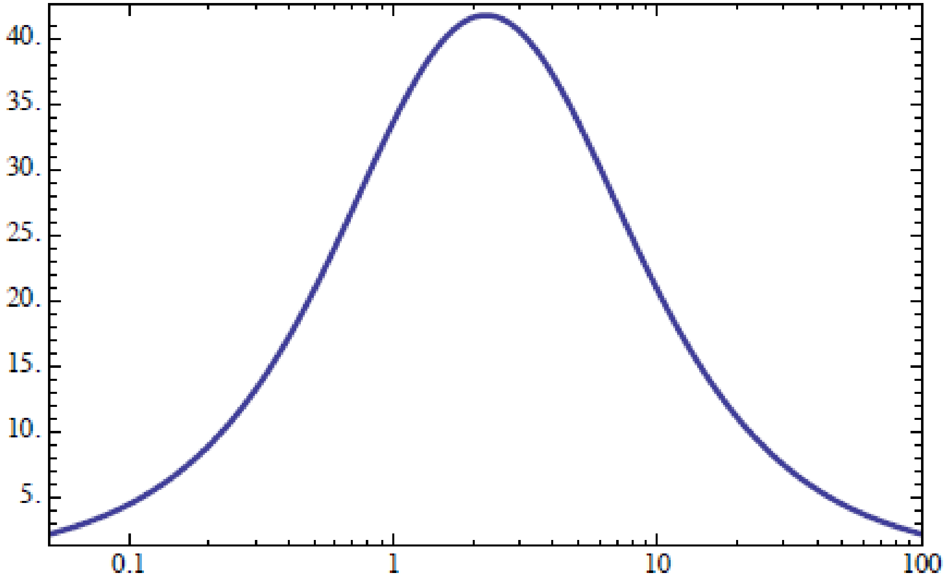
$$|G_2(j\omega)| = \left|\frac{j\omega-1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

- All the difference is in the phase plots!

Phase Plot for G_1

$$G_1(j\omega) = \frac{j\omega + 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega + 1}{\frac{j\omega}{5} + 1}$$

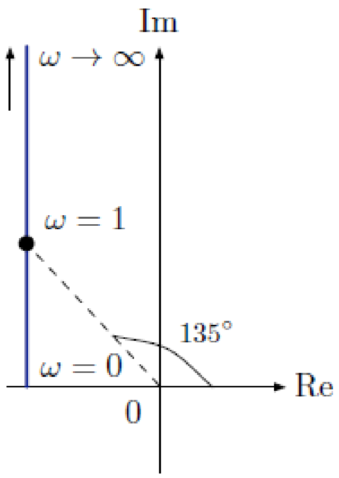
- Low-frequency term: $\frac{1}{5}(j\omega)^0$— $n = 0$, so phase starts at
- Break-points at $\omega_n = 1$ (phase goes up by 90°) and at $\omega_n = 5$ (phase goes down by 90°)



Phase Plot for G_2

$$G_2(j\omega) = \frac{j\omega - 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega - 1}{\frac{j\omega}{5} + 1}$$

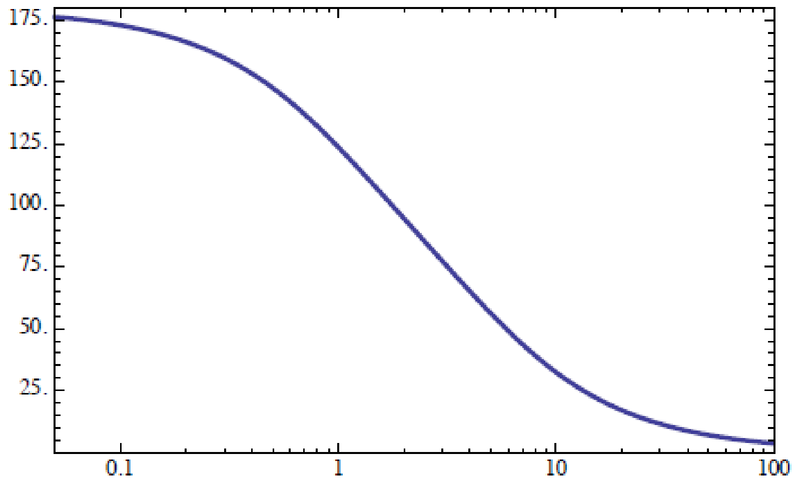
Let's do a Nyqiust plot for $j\omega - 1$:



New type of behavior —

- $\omega \approx 0$: $\phi \approx 180°$ (real and negative)
- $\omega \gg 1$: $\phi \approx 90°$ ($\mathrm{Re} = -1$, $\mathrm{Im} = \omega \gg 1$)
- $\omega \approx 1$: $\phi \approx 135°$

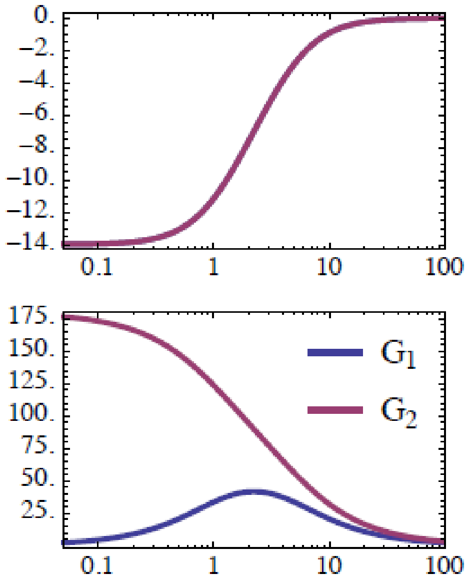
For a RHP zero, the phase starts out at 180° and goes down by 90° through the break-point (135° at break-point).



For a RHP zero, the phase plot is similar to what we had for a LHP pole: goes down by 90° ... However, it starts at 180°, and not at 0°.

Minimum- & Nonminimum-Phase Zeros

Minimum-Phase and Nonminimum-Phase Zeros



Among all transfer functions with the same magnitude plot, the one with only LHP zeros has the minimal net phase change as ω goes from 0 to ∞ — hence the term *minimum-phase* for LHP zeros.

Example 2 Magnitude

$\log K_0 (j\omega)^n$

type1

type3

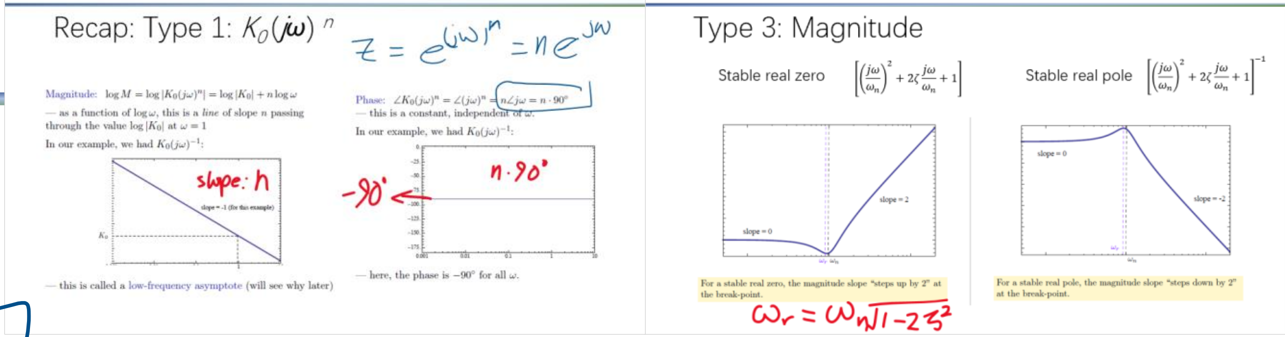
$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

$\log M = \log K_0 + \boxed{n} \log(j\omega)$

What can we tell about magnitude?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — asymptote has slope $= -2$, passes through $(\omega = 1, M = 0.01)$
- complex zero with break-point at $\omega_n = 1$ and $\zeta = 0.005$ — slope up by 2; large resonant dip
- complex pole with break-point at $\omega_n = 2$ and $\zeta = 0.01$ — slope down by 2; large resonant peak

Example 2 Magnitude



$2\zeta = 0.001$

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

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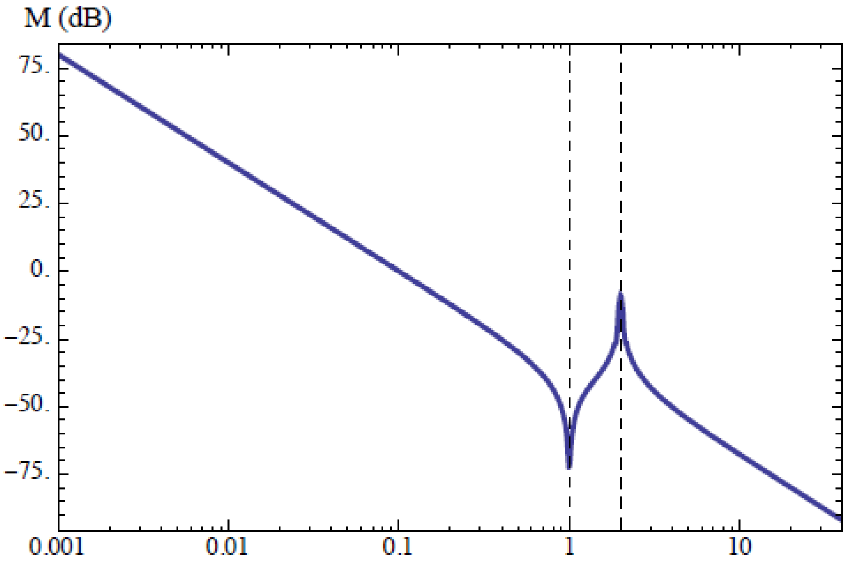
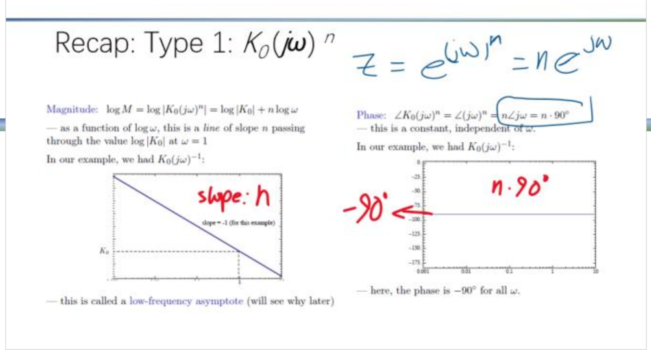
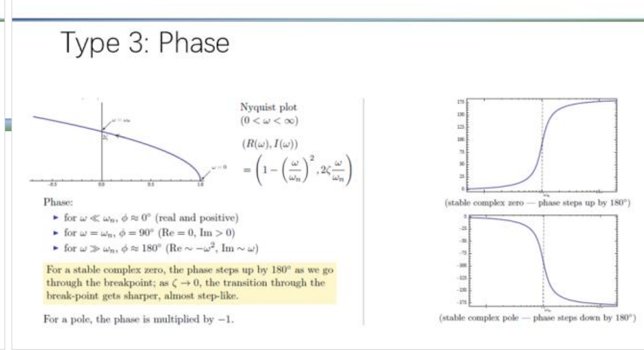


$20\log M \Rightarrow (-3)(20)\cancel{\log 10}$ 1



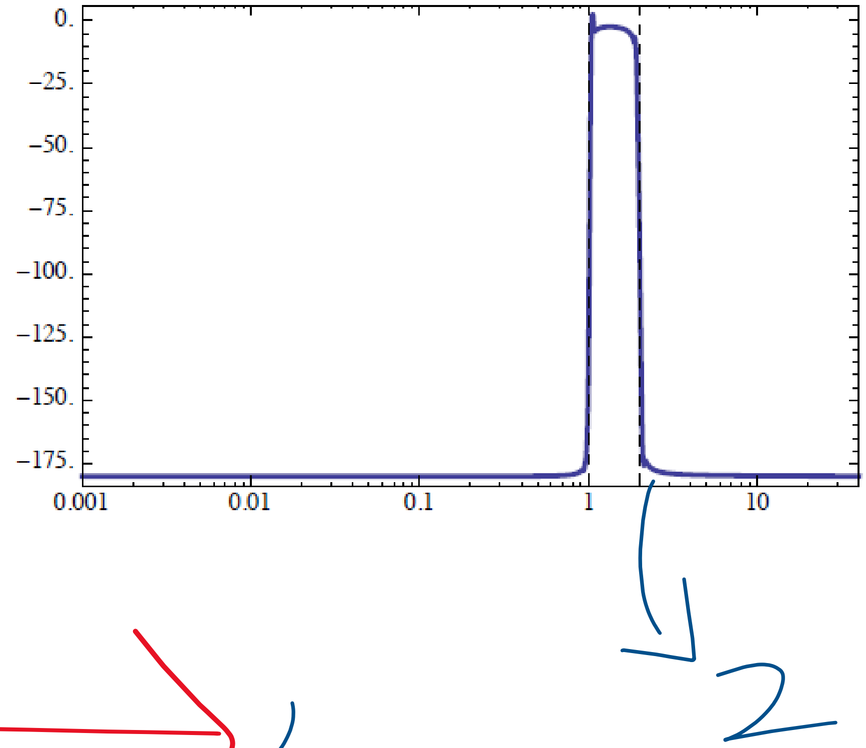


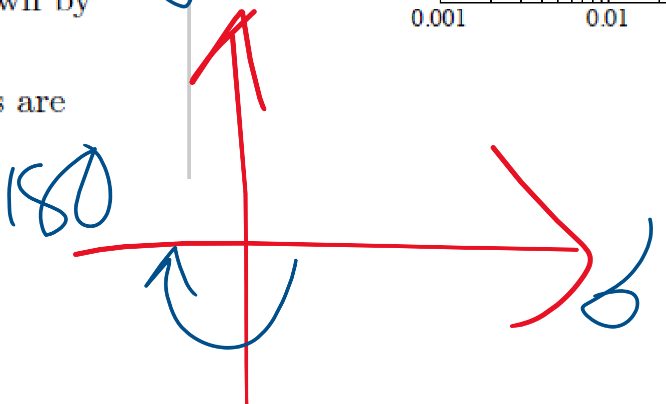
Example 2 Phase

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)} \qquad \text{— already in Bode form}$$

What can we tell about phase?

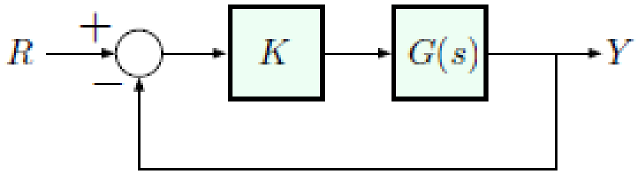
- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$
  — phase starts at $n \times 90° = -180°$
- complex zero with break-point at $\omega_n = 1$ — phase up by 180°
- complex pole with break-point at $\omega_n = 2$ — phase down by 180°
- since ζ is small for both pole and zero, the transitions are very sharp





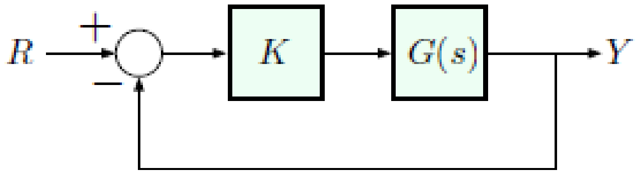
Stability from Frequency Response

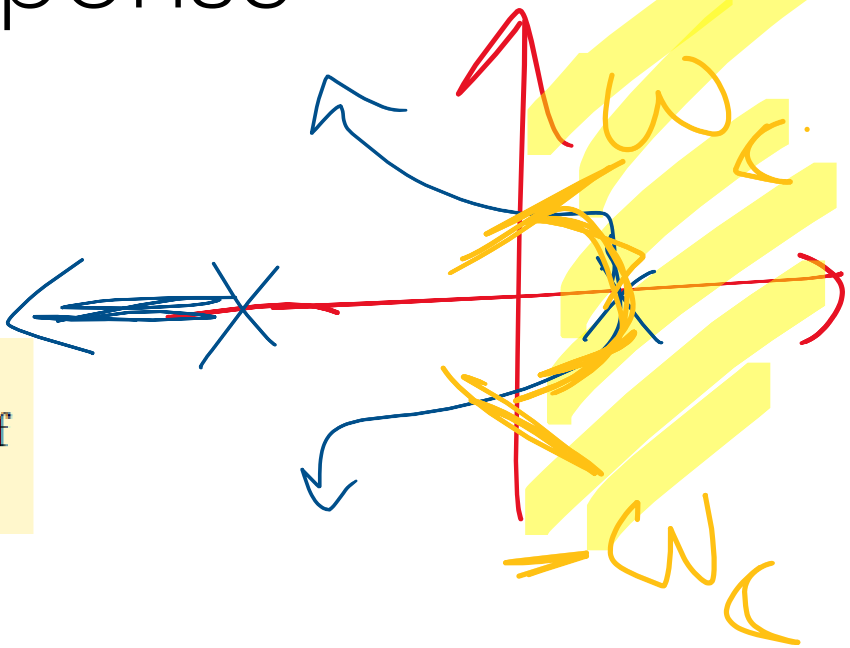
Stability from Frequency Response



Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

Stability from Frequency Response





Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

One answer: use root locus.

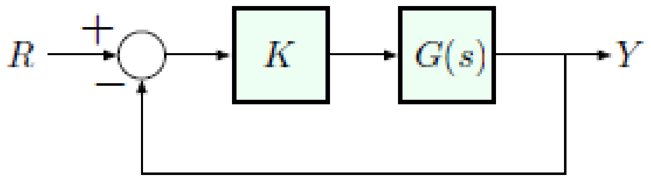
Points on the root locus satisfy the characteristic equation

$$1 + KG(s) = 0 \quad \Longleftrightarrow \quad KG(s) = -1 \quad \left(\Longleftrightarrow G(s) = -\frac{1}{K}\right)$$

If $s \in \mathbb{C}$ is on the RL, then

$$|KG(s)| = 1 \quad \text{and} \quad \angle KG(s) = \angle G(s) = 180^\circ \mod 360^\circ$$

Stability from Frequency Response



Question: How can we decide whether the *closed-loop* system is stable for a given value of $K > 0$ based on our knowledge of the *open-loop* transfer function $KG(s)$?

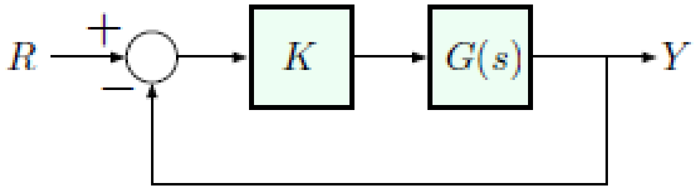
Another answer: let's look at the Bode plots:

$$\omega \longmapsto |KG(j\omega)| \qquad \text{on log-log scale}$$
$$\omega \longmapsto \angle KG(j\omega) \qquad \text{on log-linear scale}$$

— Bode plots show us magnitude and phase, but only for $s = j\omega,\ 0 < \omega < \infty$

How does this relate to the root locus? $j\omega$-crossings!!

Stability from Frequency Response



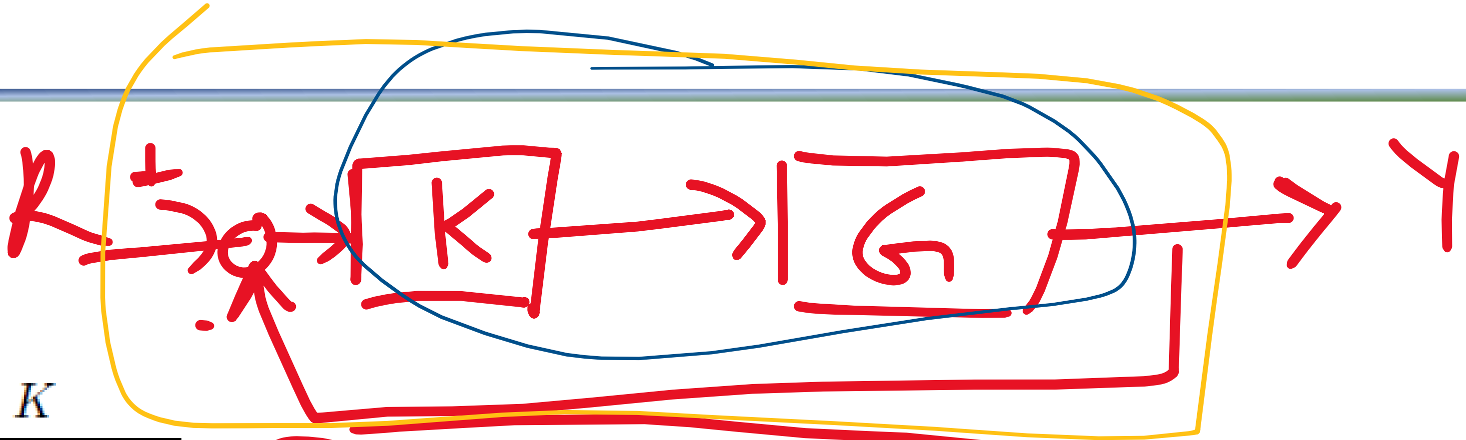
Stability from frequency response. If $s = j\omega$ is on the root locus (for some value of K), then

$$|KG(j\omega)| = 1 \qquad \text{and} \qquad \angle KG(j\omega) = 180^\circ \mod 360^\circ$$

Therefore, the transition from stability to instability can be detected in two different ways:

- from root locus — as $j\omega$-crossings
- from Bode plots — as $M = 1$ and $\phi = 180^\circ$ at some frequency ω (for a given value of K)

Example



$\boxed{1+KG=0}$

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Characteristic equation:

$$1 + \frac{K}{s(s^2+2s+2)} = 0$$
$$s(s^2+2s+2) + K = 0$$
$$s^3 + 2s^2 + 2s + K = 0$$

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial $s^3 + a_1 s^2 + a_2 s + a_3$:

$$a_1, a_2, a_3 > 0, \qquad a_1 a_2 > a_3.$$

Here, the closed-loop system is stable if and only if $0 < K < 4$.

Let's see what we can read off from the Bode plots.

Example

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Characteristic equation:

$$\begin{aligned} 1 + \frac{K}{s(s^2+2s+2)} &= 0 \\ s(s^2+2s+2) + K &= 0 \\ s^3 + 2s^2 + 2s + K &= 0 \end{aligned}$$

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial $s^3 + a_1 s^2 + a_2 s + a_3$:

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Let's see what we can read off from the Bode plots.

Example

$$KG(s) = \frac{K}{s(s^2+2s+2)}\Bigg|_{s=j\omega}$$

$$KG(j\omega) = \frac{\boxed{K}}{\boxed{j\omega}\,((j\omega)^2+2j\omega+2)}$$

Example

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

$$\text{Bode form:} \quad KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Plot the magnitude first:

- Type 1 (low-frequency) asymptote: $\dfrac{K/2}{j\omega}$
  $K_0 = K/2, \;\; n = -1 \implies$ slope $= -1$, passes through $(\omega = 1, M = K/2)$
- Type 3 (complex pole) asymptote:
  break-point at $\omega = \sqrt{2} \implies$ slope down by 2
- $\zeta = \dfrac{1}{\sqrt{2}} \implies$ no reasonant peak

Example

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Bode form: $$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

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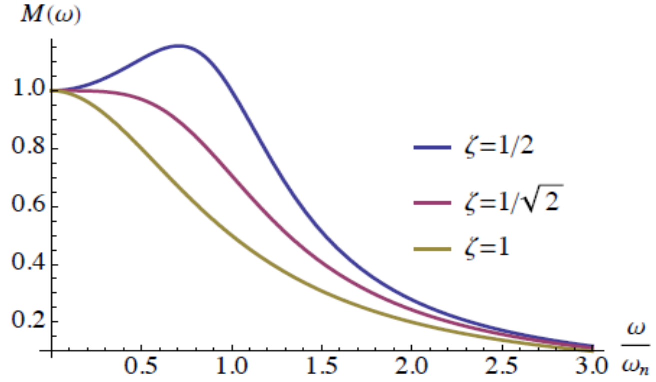


The magnitude hits its peak value (for $\zeta < 1/\sqrt{2} \approx 0.707$) occurs when $\omega = \omega_r$, where

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} < \omega_n$$

For small enough ζ (below $1/\sqrt{2}$), the magnitude of

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

has a resonant peak at the resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}.$$

Likewise, the magnitude of

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

has a resonant dip at ω_r.

Example

Magnitude Plot

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

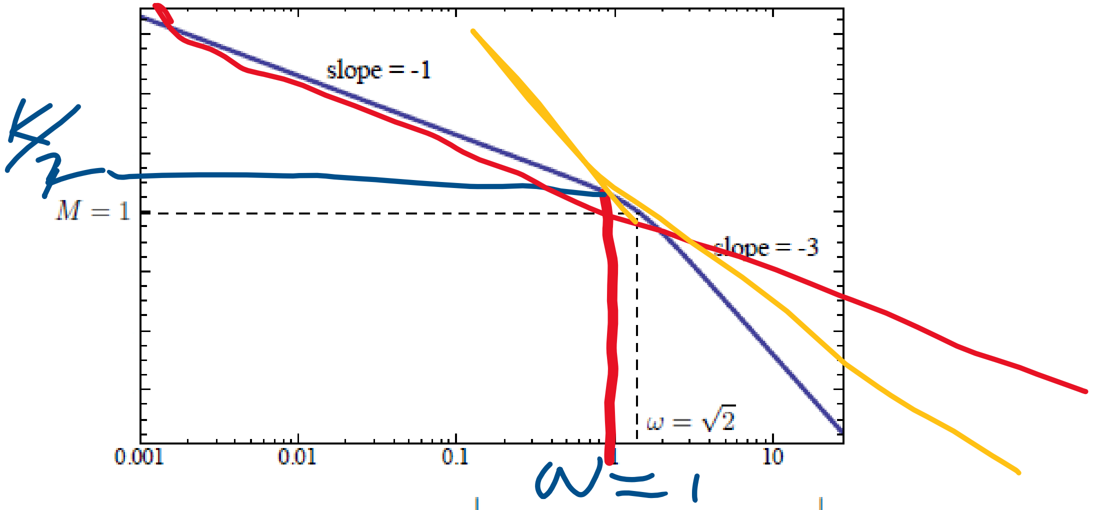
Bode form: $$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Plot the magnitude first:

- Type 1 (low-frequency) asymptote: $\dfrac{K/2}{j\omega}$
  $K_0 = K/2,\ \ n = -1 \implies$ slope $= -1$, passes through $(\omega = 1, M = K/2)$
- Type 3 (complex pole) asymptote:
  break-point at $\omega = \sqrt{2} \implies$ slope down by 2
- $\zeta = \dfrac{1}{\sqrt{2}} \implies$ no reasonant peak

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Magnitude plot for $K = 4$ (the critical value):



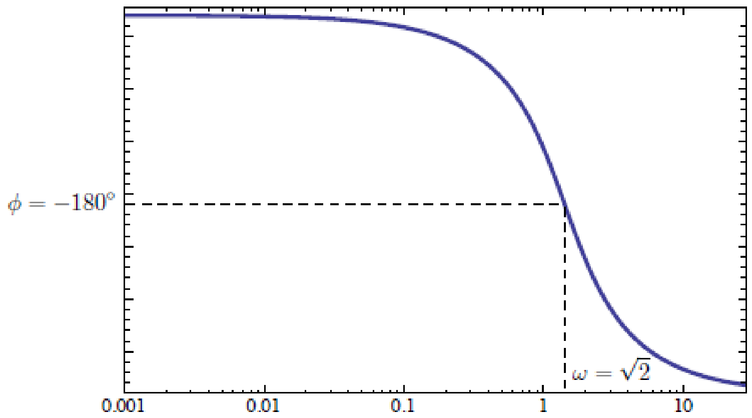
When $\omega = \sqrt{2}$, $M = |4G(j\omega)| = \left|\dfrac{2}{j\sqrt{2}\,(j^2 + j\sqrt{2} + 1)}\right| = 1$

Example

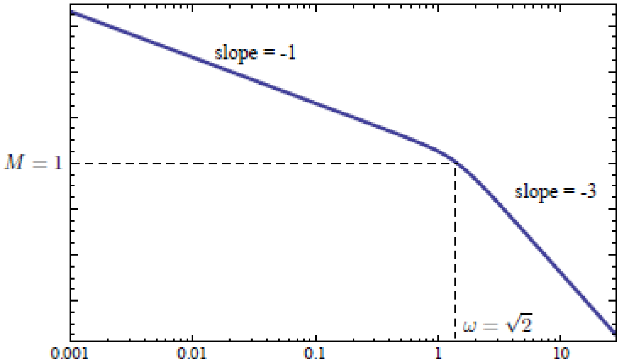
Phase Plot

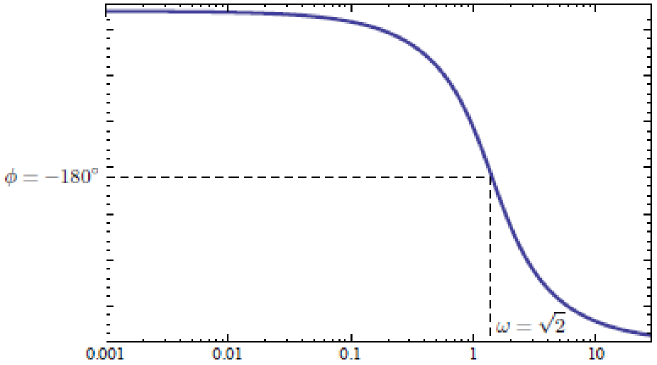
$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Phase plot (independent of K):



When $\omega = \sqrt{2}$, $\phi = -180°$





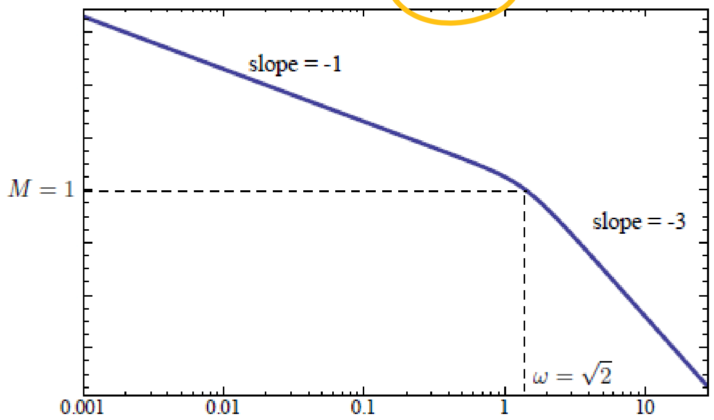
For the critical value $K = 4$:

$M = 1$ and $\phi = 180°$ mod $360°$ at $\omega = \sqrt{2}$

Crossover Frequency & Stability

Crossover Frequency and Stability

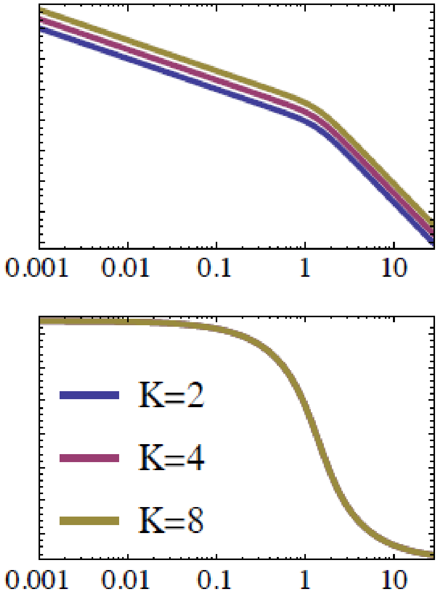
Definition: The frequency at which $M = 1$ is called the *crossover frequency* and denoted by ω_c.



Transition from stability to instability on the Bode plot:

$$\text{for critical } K, \qquad \angle G(j\omega_c) = 180^\circ$$

Effect of Varying K



What happens as we vary K?

- ϕ independent of $K \implies$ only the M-plot changes
- If we multiply K by 2:

$$\log(2M) = \log 2 + \log M$$

  – M-plot shifts up by $\log 2$

- If we divide K by 2:

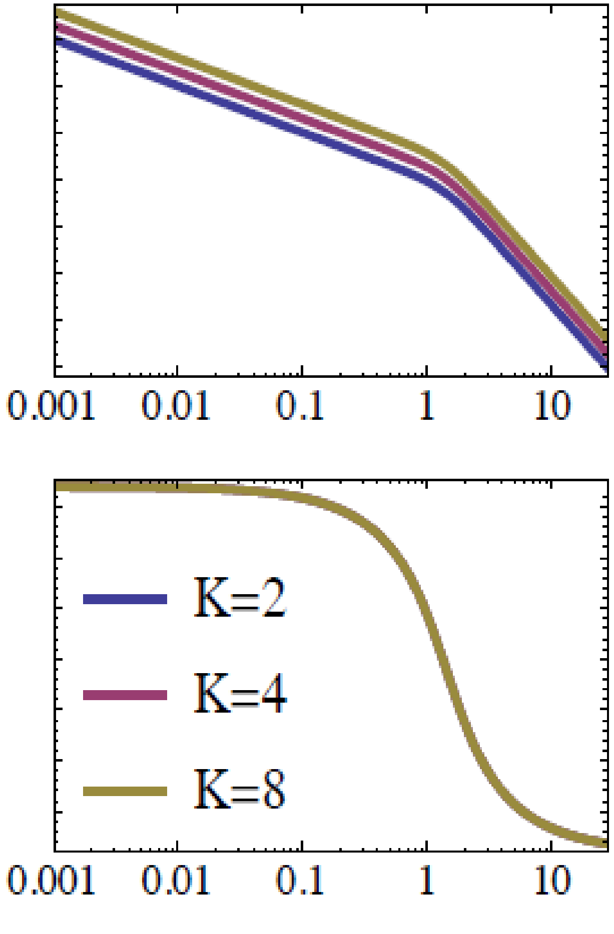
$$\log(\tfrac{1}{2}M) = \log\frac{1}{2} + \log M$$
$$= -\log 2 + \log M$$

  – M-plot shifts down by $\log 2$

Changing the value of K moves the crossover frequency ω_c!!

Effect of Varying K

Changing the value of K moves the crossover frequency ω_c!!



What happens as we vary K?

$$\angle KG(j\omega_c) \begin{cases} > -180^\circ, & \text{for } K < 4 \\ & \text{(stable)} \\ = -180^\circ, & \text{for } K = 4 \\ & \text{(critical)} \\ < -180^\circ, & \text{for } K > 4 \\ & \text{(unstable)} \end{cases}$$

Equivalently, we may define ω_{180° as the frequency at which

$$\phi = 180^\circ \mod 360^\circ.$$

Then, *in this example**,

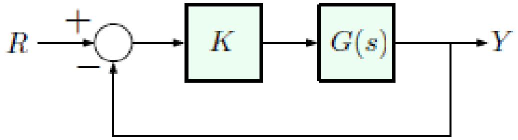
$$|KG(j\omega_{180^\circ})| < 1 \longleftrightarrow \text{stability}$$
$$|KG(j\omega_{180^\circ})| > 1 \longleftrightarrow \text{instabilit}$$

* Not a general rule; conditions will

vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.

Stability from Frequency Response

Consider this unity feedback configuration:



Suppose that the *closed-loop* system, with transfer function

$$\frac{KG(s)}{1+KG(s)},$$

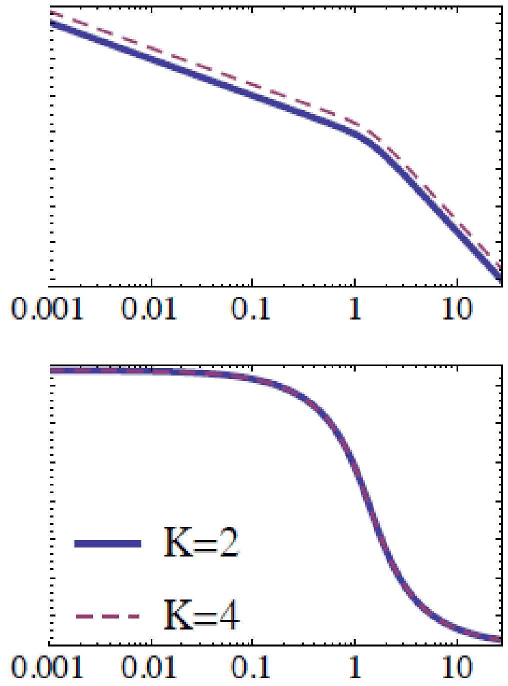
is stable for a given value of K.

Question: Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).

Gain Margin

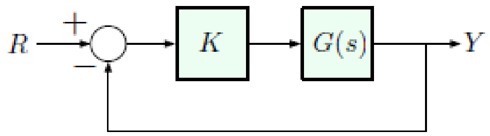
Back to our example: $G(s) = \dfrac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)



Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180°$

Since varying K doesn't change $\omega_{180°}$, to find GM we need to inspect M at $\omega = \omega_{180°}$

Example



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \qquad \zeta, \omega_n > 0$$

Consider gain $K = 1$, which gives closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)} = \frac{\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s}}{1 + \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{— prototype 2nd-order response}$$

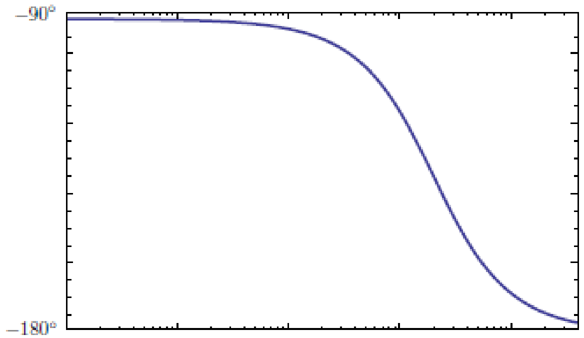
Question: what is the gain margin at $K = 1$?

Answer: $\mathrm{GM} = \infty$

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the phase plot:

- starts at -90° (Type 1 term with $n = -1$)
- goes down by -90° (Type 2 pole)



Recall: to find GM, we first need to find ω_{180°, and here there is no such $\omega \implies$ no GM.

Example

So, at $K = 1$, the gain margin of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

is equal to ∞ — what does that mean?

It means that we can keep on increasing K indefinitely without ever encountering instability.

But we already knew that: the characteristic polynomial is

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2,$$

which is *always stable*.

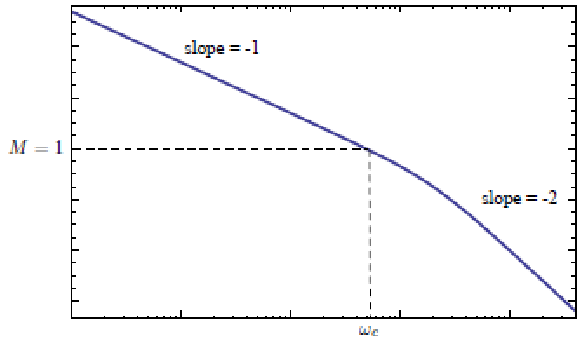
What about phase margin?

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the magnitude plot:

- low-frequency asymptote slope -1 (Type 1 term, $n = -1$)
- slope down by 1 past the breakpt. $\omega = 2\zeta\omega_n$ (Type 2 pole)

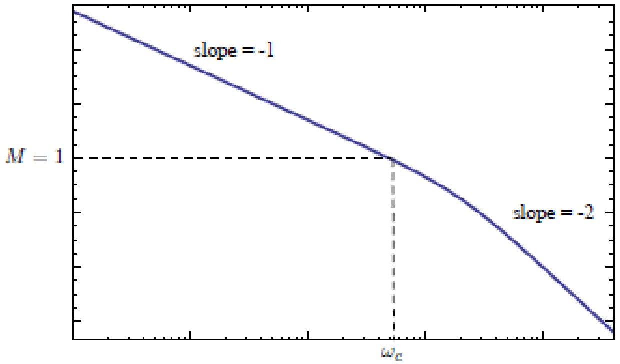
$\Longrightarrow$ there is a finite crossover frequency ω_c!!



Example

Magnitude Plot

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$



It can be shown that, *for this system*,

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right)$$

— for PM $< 70°$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right) \approx 100 \cdot \zeta$$

Conclusions:

larger PM $\Longleftrightarrow$ better damping

(open-loop quantity) (closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!