



ECE 486 Control Systems

Lecture 18: PI and lag, PID and lead-lag

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Schedule check

Frequency Response

Week	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
4	National Holiday Week	
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control	Section 4.1, lab manual
6	Introduction to Proportional-Integral-Derivative (PID) control	Sections 4.1–4.3, lab manual
	Review A	
7	Term Test 1	
	Introduction to Root Locus design method	Ch. 5
8	Root Locus continued; introduction to dynamic compensation	Ch. 5
	Lead and lag dynamic compensation	Ch. 5
9	Introduction to frequency-response design method	Sections 5.1–5.4, 6.1
	Bode plots for three types of transfer functions	Section 6.1

Root Locus

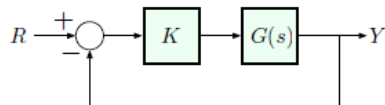
Week	Topic	Ref.
10	Stability from frequency response; gain and phase margins	Section 6.1
	Control design using frequency response	Ch. 6
11	Control design using frequency response continued; PI and lag, PID and lead-lag	Ch. 6
	Nyquist stability criterion	Ch. 6
12	Gain and phase margins from Nyquist plots	Ch. 6
	Term Test II (Review B)	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback I; separation principle	Ch. 7
	Dynamic output feedback II (Review C)	Ch. 7
16	END OF LECTURES	
	Finals	

State-Space

Plan of Lecture

- Review: Control design using frequency response: PD/lead
- Today: Control design using frequency response: PI/lag, PID/lead+l原因
- Goal:
 - understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot;
 - develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

Review: Bode's Gain-Phase Relationship



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

	low freq.	real zero/pole	complex zero/pole
mag. slope	n	up/down by 1	up/down by 2
phase	$n \times 90^\circ$	up/down by 90°	up/down by 180°

We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

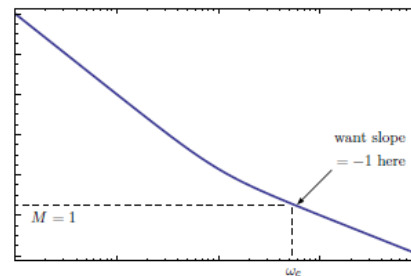
$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$



Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$

This suggests the following rule of thumb:



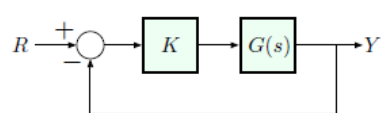
- M has slope -2 at ω_c
 $\Rightarrow \phi(\omega_c) = -180^\circ$
 \Rightarrow **bad** (no PM)
- M has slope -1 at ω_c
 $\Rightarrow \phi(\omega_c) = -90^\circ$
 \Rightarrow **good** (PM = 90°)

— this is an important *design guideline*!!

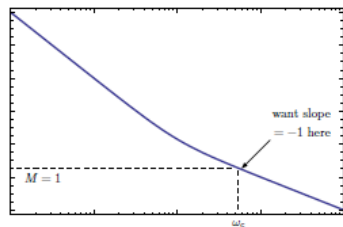
(Similar considerations apply when M -plot has positive slope – depends on the t.f.)

Review: Bode's Gain-Phase Relationship

D- Control.



M-plot for *open-loop* t.f. KG :



Note: $|KG(j\omega)| \rightarrow \infty$ as $\omega \rightarrow 0$

$$\begin{cases} |KG(j\omega_c)| = 1 \\ \angle G(j\omega_c) = -90^\circ \end{cases} \Rightarrow KG(j\omega_c) = -j$$

Closed-loop t.f.:

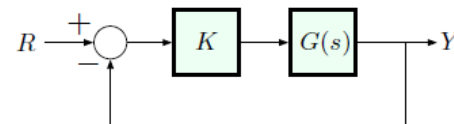
$$T(j\omega_c) = \frac{KG(j\omega_c)}{1 + KG(j\omega_c)} = \frac{-j}{1 - j}$$

$$|T(j\omega_c)| = \left| \frac{-j}{1 - j} \right| = \frac{1}{\sqrt{2}}$$

$$|T(0)| = \lim_{\omega \rightarrow 0} \frac{|KG(j\omega)|}{|1 + KG(j\omega)|} = 1$$

$$\Rightarrow \omega_c = \omega_{BW} \text{ (bandwidth)}$$

- ▶ If $PM = 90^\circ$, then $\omega_c = \omega_{BW}$
- ▶ If $PM < 90^\circ$, then $\omega_c \leq \omega_{BW} \leq 2\omega_c$ (see FPE)



Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing K (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

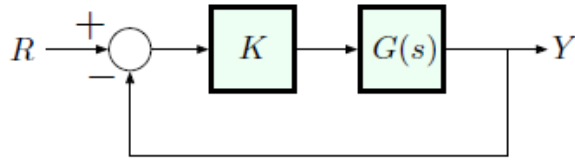
In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \quad \Rightarrow \quad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of 90° and consequently *good damping*.

Recall: Frequency Response Control Design

So far,



Bode plot relationship suggests that we can shape the time response of the closed-loop system by selecting K

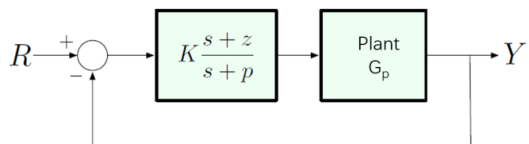
Lead Controller Design with Freq. Response

1. Choose K to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
 - ▶ in general, we should first check PM with the K from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it's not very precise, requires trial & error.

Recall: Dynamic Compensation

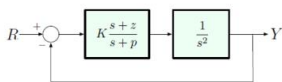
Objectives: stabilize the system and satisfy given time response specs using a stable, causal controller



Characteristic equation:

$$1 + K \cdot \frac{s+z}{s+p} \cdot G_p = 1 + KL(s) = 0$$

Recall : PD and Lead Control



Controller transfer function is $K \frac{s+z}{s+p}$, where:

$$K = K_P + pK_D, \quad z = \frac{pK_P}{K_P + pK_D} \xrightarrow{p \rightarrow \infty} \frac{K_P}{K_D}$$

so, as $p \rightarrow \infty$, z tends to a constant, so we get a **lead controller**.

We use lead controllers as dynamic compensators for approximate PD control.

To keep things simple, let's set $K_P = K_D$. Then:

$$K = K_P + pK_D = (1+p)K_D$$

$$z = \frac{pK_P}{K_P + pK_D} = \frac{pK_D}{(1+p)K_D} = \frac{p}{1+p} \xrightarrow{p \rightarrow \infty} 1$$

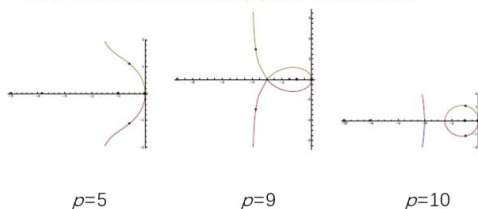
Since we can choose p and z directly, let's take

$$z = 1 \quad \text{and} \quad p \text{ large.}$$

We expect to get behavior similar to PD control.

From what we have seen so far:

- p large — good damping, but bad noise suppression (too close to PD); the branches first break in (meet at the real axis), then break away.
- p small — noise suppression is better, but RL is too close to $j\omega$ -axis, which is not good; no break-in for small values of p .
- intermediate values of p — transition between two types of RL; break-in and break-away points are the same.



Recall: Lead & Lag Compensation

Consider a general controller of the form

$$K \frac{s+z}{s+p} \quad \text{— } K, z, p > 0 \text{ are design parameters}$$

Depending on the relative values of z and p , we call it:

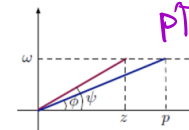
- a **lead compensator** when $z < p$
- a **lag compensator** when $z > p$

Why the name “lead/lag?” — think frequency response

$$\angle \frac{j\omega + z}{j\omega + p} = \angle(j\omega + z) - \angle(j\omega + p) = \psi - \phi$$

$z \uparrow \Rightarrow \frac{j\omega + z}{j\omega + p} \uparrow \Rightarrow$ shift fwd. (lead)
 $p \uparrow \Rightarrow \dots \downarrow \Rightarrow$ bkwd. (lag)

- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



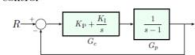
Recall PI and Lag Control

PI Control \rightarrow ref tracking, avoid ss error.

Case study: plant transfer function $G_p(s) = \frac{1}{s-1}$

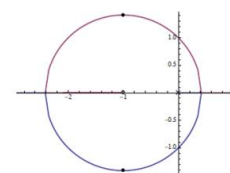
Control objective: stability and constant reference tracking

In earlier lectures, we saw that for perfect steady-state tracking we need PI control



Closed-loop poles are determined by:

$$1 + \left(K_P + \frac{K_I}{s} \right) \left(\frac{1}{s-1} \right) = 0$$



- The system is stable for $K > 1$ (from Routh-Hurwitz)
- For very large K , we get a completely damped system, with negative real poles
- Perfect steady-state tracking of constant references:

$$\frac{E}{R} = \frac{1}{1 + G_c G_p} = \frac{1}{1 + \frac{K(s+1)}{s(s-1)}} = \frac{s(s-1)}{s(s-1) + K(s+1)}$$

DC gain ($R \rightarrow E$) = 0 (for $K > 1$)

- However: $1/s$ is not a stable element.

How About:

$$\text{replace } K \frac{s+1}{s} \text{ by } K \frac{s+1}{s+p}, \text{ where } p \text{ is small}$$

More generally, if $z = K_I/K_P$, then

$$\text{replace } K \frac{s+z}{s} \text{ by } K \frac{s+z}{s+p}, \text{ where } p < z$$

This is lag compensation (or lag control)!

We use lag controllers as dynamic compensators for approximate PI control.

Tracking a constant reference: if the stability conditions

$$K > 1 - p, \quad Kz > p$$

are satisfied, then the steady-state error is

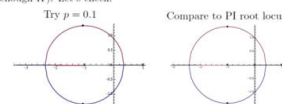
$$e(\infty) = \frac{1}{1 - \frac{Kz}{p}}$$

— this will be close to zero (and negative) if $\frac{Kz}{p}$ is large.

Lag compensation *does not* give perfect tracking (indeed, it does not change system type), but we can get as good a tracking as we want by playing with K, z, p . On the other hand, unlike PI, lag compensation gives a stable controller.

$$L(s) = \frac{s+1}{(s+p)(s-1)}$$

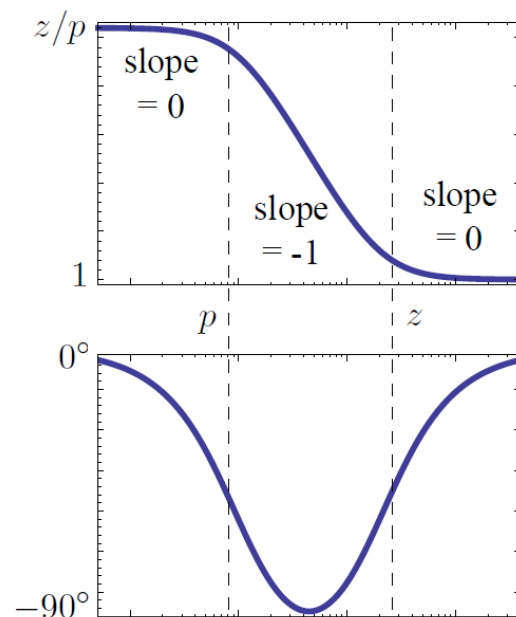
Intuition: By choosing p very close to zero, we can make the root locus arbitrarily close to PI root locus (stable for large enough K). Let's check:



What do we see? Compared to PD vs. lead, there is no qualitative change in the shape of RL, since we are not changing # (poles) or # (zeros).

Lag Compensation: Bode Plot

$$D(s) = \frac{s + z}{s + p} = \frac{z \frac{s}{z} + 1}{p \frac{s}{p} + 1}, \quad z \gg p$$



- ▶ $\frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \rightarrow \infty} 1$
so $M \rightarrow 1$ at high frequencies

- ▶ subtracts phase, hence the term “phase lag”

- ▶ $\frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \rightarrow 0} \frac{z}{p}$

steady-state tracking error:

$$e(\infty) = \frac{sR(s)}{1 + D(s)G(s)} \Big|_{s=0}$$

large $z/p \implies$ better s.s. tracking

- ▶ lag decreases $\omega_c \implies$ slows down time response (to compensate, adjust K or add lead)



caution: lead increases PM, but adding lag can undo this

- ▶ to mitigate this, choose both z and p very small, while maintaining desired ratio z/p

Example (design).

$$G(s) = \frac{1}{(s + 0.2)(s + 0.5)} \stackrel{\text{Bode form}}{=} \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$$

Objectives:

- ▶ PM $\geq 60^\circ$
- ▶ $e(\infty) \leq 10\%$ for constant reference (closed-loop tracking error)

Strategy:

- ▶ we will use lag

$$KD(s) = K \frac{s + z}{s + p}, \quad z \gg p$$

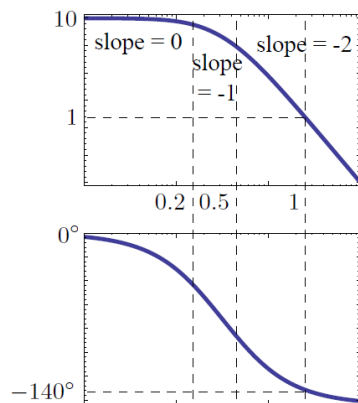
- ▶ z and p will be chosen to get good tracking
- ▶ PM will be shaped by choosing K
- ▶ this is different from what we did for lead (used p and z to shape PM, then chose K to get desired bandwidth spec)

Example

$$G(s) = \frac{1}{(s + 0.2)(s + 0.5)} \stackrel{\text{Bode form}}{=} \frac{10}{\left(\frac{s}{0.2} + 1\right)\left(\frac{s}{0.5} + 1\right)} \quad \text{Type I.}$$

Step 1: Choose K to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:



► from Matlab, $\omega_c \approx 1$

where $\phi \approx -140^\circ$.

► PM $\approx 40^\circ$

► we want PM = $60^\circ = \phi - 180^\circ$

phase $\rightarrow \phi = -120^\circ$ at $\omega \approx 0.573$
 $M = 2.16$ to allow new $M=1$.

— need to decrease K to $1/2.16$

A conservative choice (to allow some slack) is $K = 1/2.5 = 0.4$,
 gives $\omega_c \approx 0.52$, PM $\approx 65^\circ$

↓
 i.e. tolerance.

Step 2: Choose z & p to Shape Tracking Error

So far: $KG(s) = \frac{0.4 \cdot 10}{\left(\frac{s}{0.2} + 1\right)\left(\frac{s}{0.5} + 1\right)}$

$$e(\infty) = \frac{1}{1 + KG(s)} \Big|_{s=0} = \frac{1}{1 + 4} = \frac{1}{5} = 20\% \quad (\text{too high})$$

To have $e(\infty) \leq 10\%$, need $KD(0)G(0) \geq 9$:

$$e(\infty) = \frac{1}{1 + KD(0)G(0)} \leq \frac{1}{1 + 9} = 10\%.$$

So, we need

$$D(0) = \frac{s + z}{s + p} \Big|_{s=0} = \frac{z}{p} \geq \frac{9}{4} = 2.25 \quad \text{— say, } z/p = \underline{2.5}$$

Not to distort PM and ω_c , let's pick z and p an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05$, $p = 0.02$

Overall Design

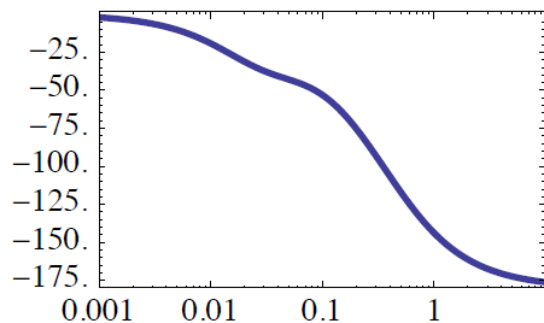
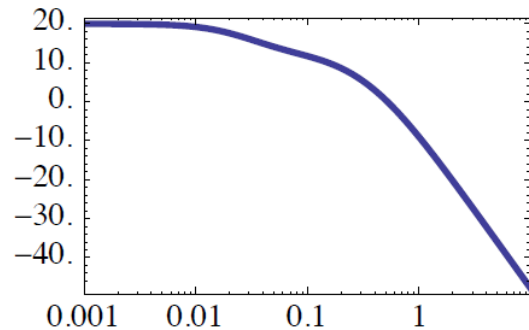
Plant:

$$G(s) = \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$$

given - cannot change

Controller:

$$KD(s) = 0.4 \frac{s + 0.05}{s + 0.02}$$

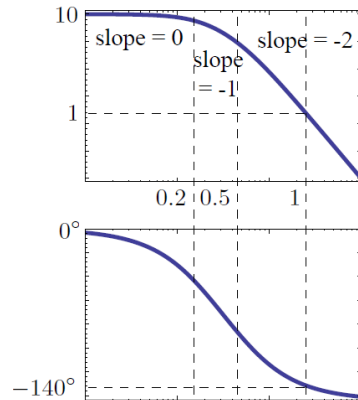


— the design still needs a bit of refinement ...

Lead & Lag Compensation

Let's combine the advantages of PD/lead and PI/lag.

Back to our example: $G(s) = \frac{10}{\left(\frac{s}{0.2} + 1\right)\left(\frac{s}{0.5} + 1\right)}$



- ▶ from Matlab, $\omega_c \approx 1$
- ▶ PM $\approx 40^\circ$

New objectives:

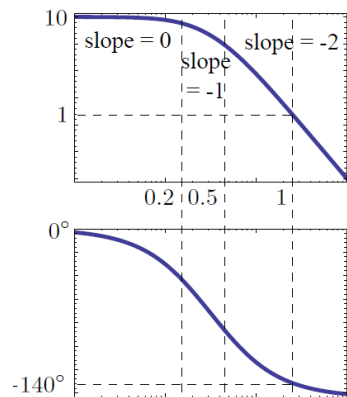
- ▶ $\omega_{BW} \geq 2$
- ▶ PM $\geq 60^\circ$
- ▶ $e(\infty) \leq 1\%$ for const. ref.

What we got before, with lag only:

- ▶ Improved PM by adjusting K to decrease ω_c .
- ▶ This gave $\omega_c \approx 0.5$, whereas now we want a larger ω_c (recall: $\omega_{BW} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.

Lead & Lag Compensation



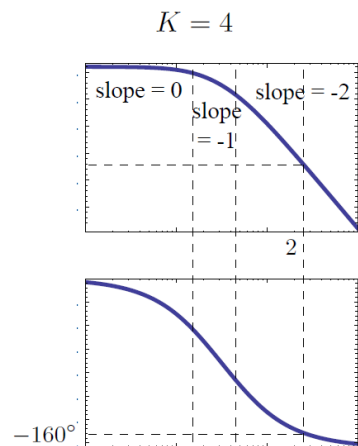
Step 1. Choose K to get $\omega_c \approx 2$ (before lead)

Using Matlab, can check:

at $\omega = 2$, $M \approx 0.24$ (with $K = 1$)

— need $K = \frac{1}{0.24} \approx 4.1667$

— choose $K = 4$ (gives ω_c slightly < 2 , but still ok).



Step 2. Decide how much phase lead is needed, and choose z_{lead} and p_{lead}

Using Matlab, can check:

at $\omega = 2$, $\phi \approx -160^\circ$

— so PM = 20°

(in fact, choosing $K = 4$ made things worse: it increased ω_c and consequently decreased PM)

We need at least 40° phase lead!!

The choice of lead pole/zero must satisfy

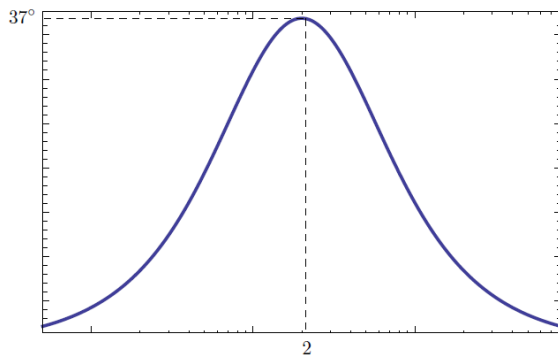
$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Let's try $z_{\text{lead}} = 1$ and $p_{\text{lead}} = 4$ $D(s) = \frac{s+1}{\frac{s}{4}+1}$



Phase lead = 37° — not enough!!

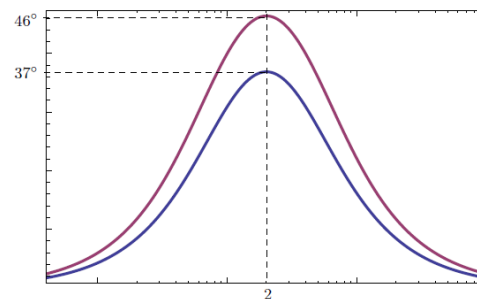
Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

The choice of $z_{\text{lead}} = 1$, $p_{\text{lead}} = 4$ gave phase lead = 37° .

Need to space z_{lead} and p_{lead} farther apart:

$$\begin{cases} z_{\text{lead}} = 0.8 \\ p_{\text{lead}} = 5 \end{cases} \implies \text{phase lead} = 46^\circ$$



Lead & Lag Compensation

Step 3. Evaluate steady-state tracking and choose $z_{\text{lag}}, p_{\text{lag}}$ to satisfy specs

So far:

$$K \underbrace{D(s)}_{\text{lead only}} G(s) = 4 \frac{\frac{s}{0.8} + 1}{\frac{s}{5} + 1} \cdot \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$$

$$KD(0)G(0) = 40 \implies e(\infty) = \frac{1}{1 + KD(0)G(0)} = \frac{1}{1 + 40}$$

— this is not small enough: need $1\% = \frac{1}{100} = \frac{1}{1 + 99}$

We want $D(0) \geq \frac{99}{40}$ with lag $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$ will do

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$.

We can stick with our previous design:

$$z_{\text{lag}} = 0.05, \quad p_{\text{lag}} = 0.02$$

Overall controller:

$$\underbrace{4 \frac{\frac{s}{0.8} + 1}{\frac{s}{5} + 1}}_{\text{lead (with gain } K = 4 \text{ absorbed)}} \cdot \underbrace{\frac{s + 0.05}{s + 0.02}}_{\text{lag (not in Bode form)}}$$

(Note: we don't rewrite lag in Bode form, because $z_{\text{lag}}/p_{\text{lag}}$ is not incorporated into K .)

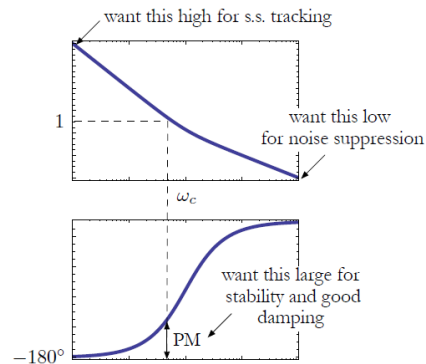
K → lead param → lag param.

Frequency Domain Design Pros & Cons

Advantages

Design based on Bode plots is **good for**:

- ▶ easily visualizing the concepts



- ▶ evaluating the design and seeing which way to change it
- ▶ using experimental data (frequency response of the uncontrolled system can be measured experimentally)

Disadvantages

Design based on Bode plots is **not good for**:

- ▶ exact closed-loop pole placement (root locus is more suitable for that)
- ▶ deciding if a given K is stabilizing or not ...
 - ▶ we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
 - ▶ however, we don't have a way of checking whether a given K is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh-Hurwitz criterion — this is the **Nyquist criterion**, which we will discuss in the next lecture.