ECE 486 Control Systems

Lecture 20: Nyquist Stability Examples; Phase and Gain Margins from Nyquist Plots.

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Schedule check

Week	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
4	National Holiday Week	
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control	Section 4.1, lab manual
6	Introduction to Proportional-Integral-Derivative (PID) control	Sections 4.1-4.3, lab manual
	Review A	
7	Term Test 1	
	Introduction to Root Locus design method	Ch. 5
8	Root Locus continued; introduction to dynamic compensation	Ch. 5
	Lead and lag dynamic compensation	Ch. 5
9	Introduction to frequency-response design method	Sections 5.1-5.4, 6.1
	Bode plots for three types of transfer functions	Section 6.1

Root Locus

Frequency Response

Week	Topic	Ref.
10	Stability from frequency response; gain and phase margins	Section 6.1
	Control design using frequency response	Ch. 6
11	Control design using frequency response continued; PI and lag, PID and lead-lag	Ch. 6
	Nyquist stability criterion	Ch. 6
12	**Gain and phase margins from Nyquist plots**	Ch. 6
	Term Test II (Review B)	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback I; separation principle	Ch. 7
	Dynamic output feedback II (Review C)	Ch. 7
16	END OF LECTURES	
	Finals	

State-Space

Lecture Overview

- Review: Nyquist stability criterion
- Today's topic: Phase and Gain Margin from Nyquist Plot

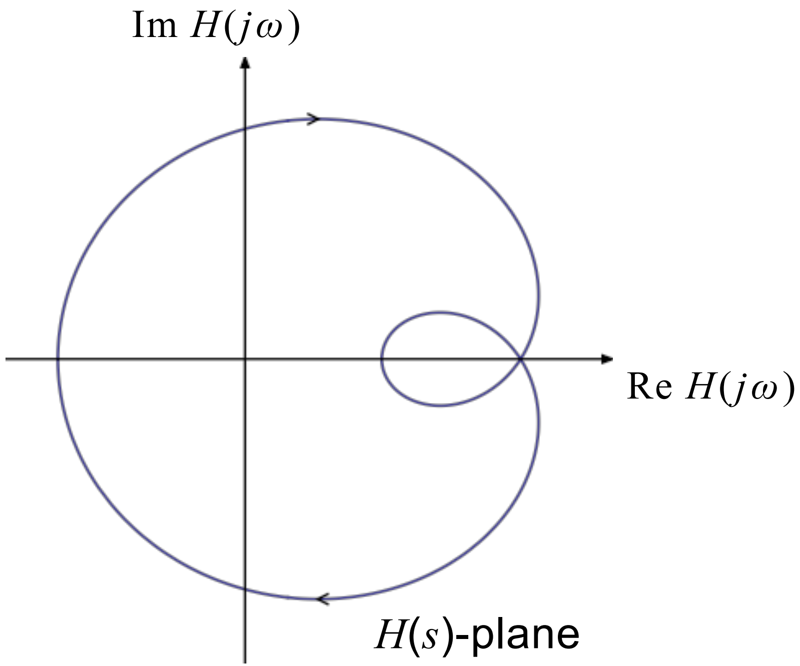
Review: Nyquist Plot

Review: Nyquist Plot

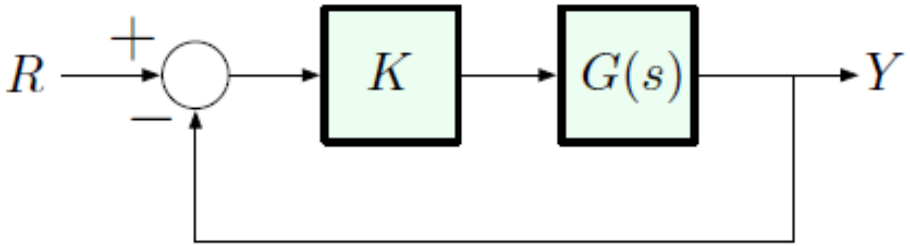
Consider an arbitrary *strictly proper* transfer function H:

$$H(s) = \frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}, \qquad m < n$$

Nyquist plot: Im $H(j\omega)$ vs. Re $H(j\omega)$
as ω varies from $-\infty$ to ∞



Review: Nyquist Stability Criterion

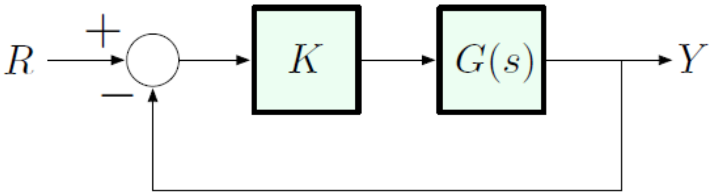


Goal: count the number of RHP poles (if any) of the closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)}$$

based on frequency-domain characteristics of the plant transfer function $G(s)$

Review: The Nyquist Theorem



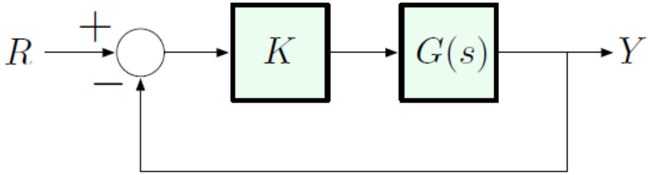
Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\circlearrowright \text{ of } -1/K \text{ by Nyquist plot of } G(s))$$
$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes *around* the pole and stays in RHP

Review: The Nyquist Stability Criterion



$$\underbrace{N}_{\#(\circlearrowright \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

$$Z = N + P$$

$$Z = 0 \qquad \Longleftrightarrow \qquad \boxed{N = -P}.$$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

Review: Apply Nyquist Criterion

Workflow:

$$\text{Bode } M \text{ and } \phi\text{-plots} \quad \longrightarrow \quad \text{Nyquist plot}$$

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)

Example 1

$$\boxed{G(s) = \frac{1}{(s+1)(s+2)}}$$ (no open-loop RHP poles)

Characteristic equation:

$$(s+1)(s+2) + K = 0 \qquad \Longleftrightarrow \qquad s^2 + 3s + K + 2 = 0$$

From Routh, we already know that the closed-loop system is stable for $K > -2$.

We will now reproduce this answer using the Nyquist criterion.

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph $\text{Im}\, G(j\omega)$ vs. $\text{Re}\, G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

$$\left(\text{Re}\, G(j\omega), \text{Im}\, G(j\omega)\right), \qquad -\infty < \omega < \infty$$

- Symmetry:

$$G(-j\omega) = \overline{G(j\omega)}$$

— Nyquist plots are always *symmetric w.r.t. the real axis*!!

Example 1

$$G(s) = \frac{1}{(s+1)(s+2)}$$

$= \frac{1}{(j\omega)^2 + 3(j\omega) + 2} = \frac{1}{2} \cdot \frac{1}{\left(\frac{j\omega}{\sqrt{2}}\right)^2 + \frac{3}{2}(j\omega) + 1}$, $\omega_n = \sqrt{2}$. $2\zeta\omega_n = \frac{3}{2} \Rightarrow \zeta = \frac{3}{4} \times \frac{1}{\sqrt{2}} = \frac{3}{8}\sqrt{2}$.

(no open-loop RHP poles)

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph Im $G(j\omega)$ vs. Re $G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

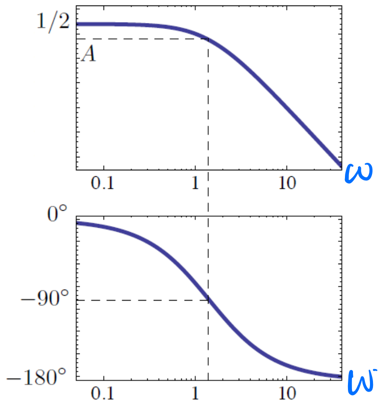
$$(\text{Re}\, G(j\omega), \text{Im}\, G(j\omega)), \qquad -\infty < \omega < \infty$$

- Symmetry:

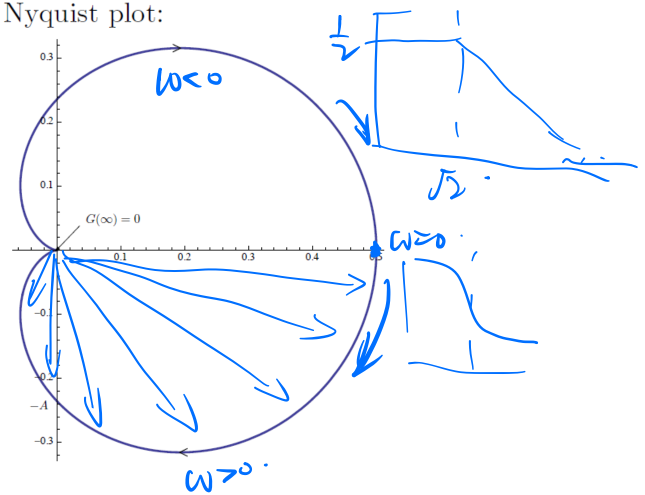
$$G(-j\omega) = \overline{G(j\omega)}$$

— Nyquist plots are always *symmetric w.r.t. the real axis*!!

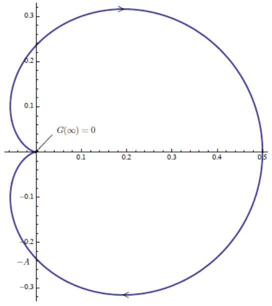
Bode plot:



Nyquist plot:



Nyquist plot:



$\#(\circlearrowleft \text{ of } -1/K)$
$= \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=0}$

$\Longrightarrow K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowleft \text{ of } -1/K) = 0$$

- If $K > 0$, $\#(\circlearrowleft \text{ of } -1/K) = 0$
- If $0 < -1/K < 1/2$, $\#(\circlearrowleft \text{ of } -1/K) > 0 \Longrightarrow$ closed-loop stable for $K > -2$

Example 2

$$G(s) = \frac{1}{(s-1)(s^2+2s+3)} = \frac{1}{s^3+s^2+s-3}$$

$$\#(\text{RHP open-loop poles}) = 1 \qquad \text{at } s = 1$$

Routh: the characteristic polynomial is

$$s^3 + s^2 + s + K - 3 \qquad \text{— 3rd degree}$$

— stable if and only if $K - 3 > 0$ and $1 > K - 3$.

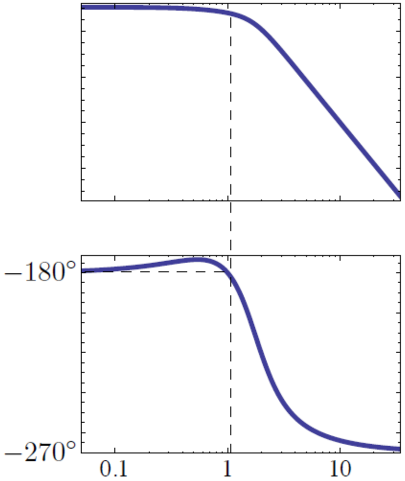
Stability range: $3 < K < 4$

Let's see how to spot this using the Nyquist criterion ...

Example 2

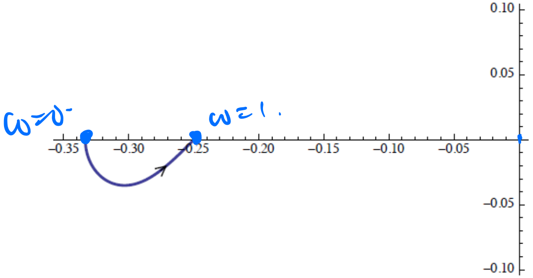
$$G(s) = \frac{1}{(s-1)(s^2+2s+3)} \qquad \text{(1 open-loop RHP pole)}$$

Bode plot:



Nyquist plot:

$$\begin{array}{ll} \omega = 0 & M = 1/3,\ \phi = -180^\circ \\ \omega = 1 & M = 1/4,\ \phi = -180^\circ \\ \omega \to \infty & M \to 0,\ \phi \to -270^\circ \end{array}$$



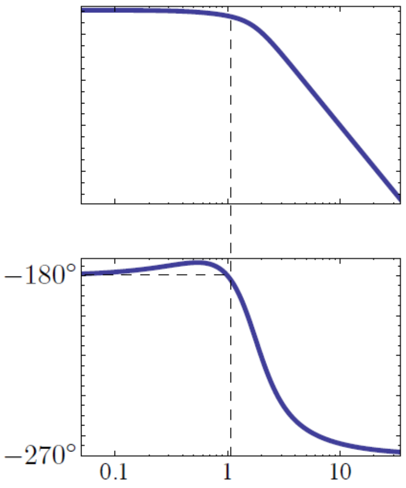
Example 2

$= (j\omega - 1)^{-1} \cdot (s^2+2s+3)^{-1}$

$$G(s) = \frac{1}{(s-1)(s^2+2s+3)}$$

(1 open-loop RHP pole)

Bode plot:

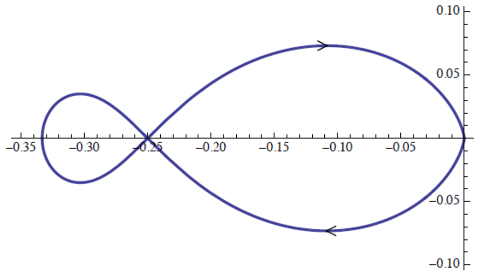


Nyquist plot:

$\omega = 0 \quad M = 1/3,\ \phi = -180^\circ$

$\omega = 1 \quad M = 1/4,\ \phi = -180^\circ$

$\omega \to \infty \quad M \to 0,\ \phi \to -270^\circ$



$K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowleft \text{ of } -1/K) = -1$$

Which points $-1/K$ are encircled once $\circlearrowleft$ by this Nyquist plot?

$\#(\circlearrowright \text{ of } -1/K)$
$= \#(\text{RHP CL poles})$
$- \underbrace{\#(\text{RHP OL poles})}_{=1}$

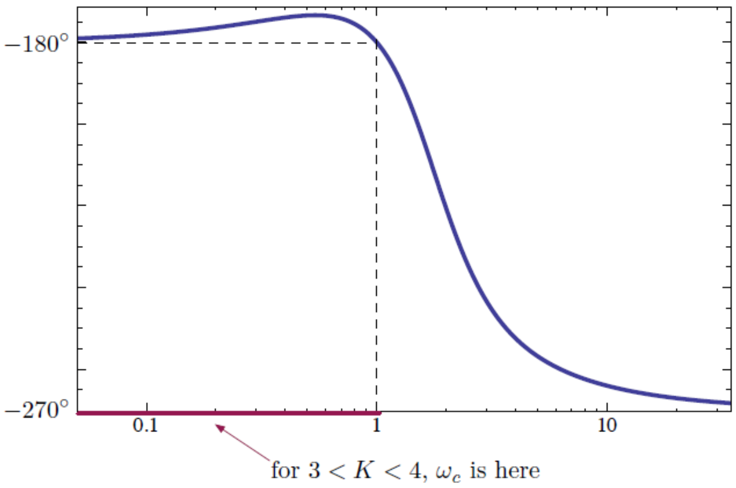
only $-1/3 < -1/K < -1/4$

$\implies 3 < K < 4$

Example 2

Closed-loop stability range for $G(s) = \dfrac{1}{(s-1)(s^2+2s+3)}$ is $3 < K < 4$ (using either Routh or Nyquist).

We can interpret this in terms of phase margin:



So, in this case, stability $\iff$ PM > 0 (typical case).

Example 3

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)} = \frac{s-1}{s^3+s^2-s+2}$$

Open-loop poles:

$$s = -2 \qquad \text{(LHP)}$$

$$s^2 - s + 1 = 0$$

$$\left(s - \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$s = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} \qquad \text{(RHP)}$$

$\therefore$ 2 RHP poles

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)} = \frac{s-1}{s^3+s^2-s+2}$$

Routh:

char. poly. $s^3 + s^2 - s + 2 + K(s-1)$

$s^2 + s^2 + (K-1)s + 2 - K$ (3rd-order)

— stable if and only if

$$K - 1 > 0$$

$$2 - K > 0$$

$$K - 1 > 2 - K$$

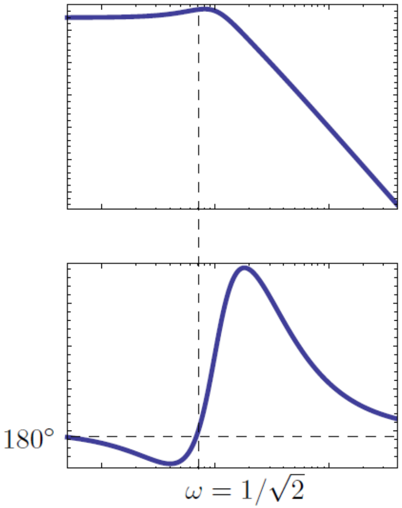
— stability range is $3/2 < K < 2$

Example 3

$$G(s) = \frac{s-1}{(s+2)(s^2-s+1)}$$

(2 open-loop RHP poles)

Bode plot (tricky, RHP poles/zeros)



$\phi = 180°$ when:

- $\omega = 0$ and $\omega \to 0$
- $\omega = 1/\sqrt{2}$:

$$\left.\frac{j\omega - 1}{(j\omega-1)((j\omega)^2 - j\omega + 1)}\right|_{\omega=1/\sqrt{2}}$$

$$= \frac{\frac{j}{\sqrt{2}} - 1}{\left(\frac{j}{\sqrt{2}} + 2\right)\left(-\frac{1}{2} - \frac{j}{\sqrt{2}} + 1\right)}$$

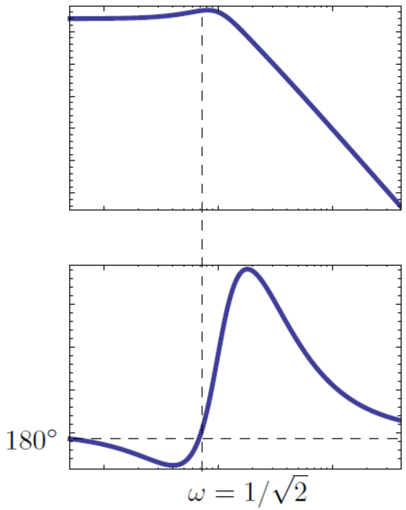
$$= \frac{\frac{j}{\sqrt{2}} - 1}{-\frac{3}{2}\left(\frac{j}{\sqrt{2}} - 1\right)} = -\frac{2}{3}$$

(need to guess this, e.g., by mouseclicking in Matlab)

$$G(s) = \frac{s-1}{s^3 + s^2 - s + 2}$$

(2 open-loop RHP poles)

Bode plot:

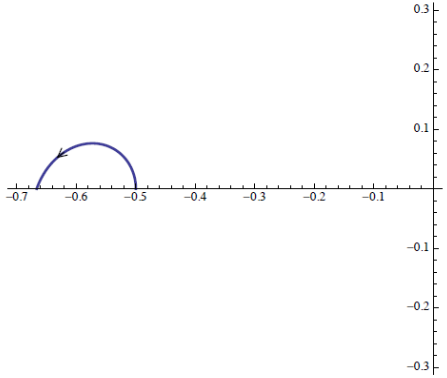


Nyquist plot:

$\omega = 0 \quad M = 1/2, \phi = 180°$

$\omega = 1/\sqrt{2} \quad M = 2/3, \phi = 180°$

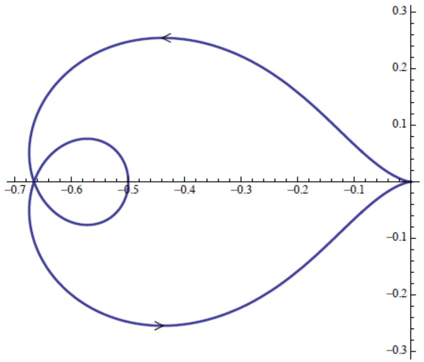
$\omega \to \infty \quad M \to 0, \phi \to 180°$



Example 3

$$G(s) = \frac{s-1}{s^3+s^2-s+2} \quad \text{(2 open-loop RHP poles)}$$

Nyquist plot:



$K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowleft \text{ of } -1/K) = -2$$

Which points $-1/K$ are encircled twice $\circlearrowleft$ by this Nyquist plot?

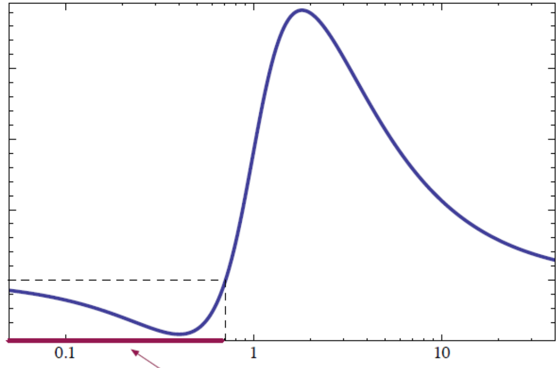
$$\#(\circlearrowleft \text{ of } -1/K) = \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=2}$$

only $-2/3 < -1/K < -1/2$

$$\Longrightarrow \frac{3}{2} < K < 2$$

CL stability range for $G(s) = \dfrac{s-1}{s^3+s^2-s+2}$: $K \in (3/2, 2)$

We can interpret this in terms of phase margin:

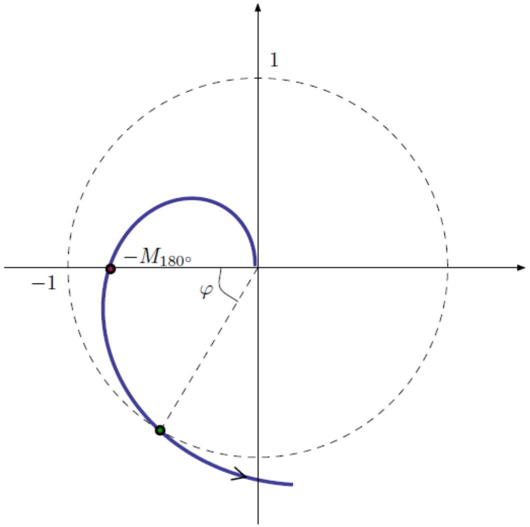


So, in this case, stability $\Longleftrightarrow$ PM < 0 (atypical case; Nyquist criterion is the only way to resolve this ambiguity of Bode plots).

Stability Margins

How do we determine stability margins (GM & PM) from the Nyquist plot?

GM & PM are defined relative to a given K, so consider Nyquist plot of $KG(s)$ (we only draw the $\omega > 0$ portion):



How do we spot GM & PM?

- $\mathrm{GM} = 1/M_{180°}$

  — if we divide K by $M_{180°}$, then the Nyquist plot will pass through $(-1, 0)$, giving $M = 1, \phi = 180°$
- $\mathrm{PM} = \varphi$

  — the phase difference from 180° when $M = 1$