ECE 486 Control Systems

Lecture 07: Effect of Zeros and Extra Poles
Routh-Hurwitz Stability Criterion

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Checklist

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

Wk	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	National Holiday Week	
4	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
5	**Effect of zeros and extra poles; Routh-Hurwitz stability criterion**	Sections 3.5, 3.6
	Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	Review A	
	Term Test A	
7	Introduction to Root Locus design method	Ch. 5
	Root Locus continued; introduction to dynamic compensation	Root Locus
8	Lead and lag dynamic compensation	Ch. 5
	Lead and lag continued; introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling: weeks 1–4; Analysis: weeks 4–5; Root Locus: weeks 7–8

Wk	Topic	Ref.
9	Bode plots for three types of transfer functions	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	Control design using frequency response	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	State-Space
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Frequency Response: weeks 9–11; State-Space: weeks 13–15

Lecture Overview

- Review: 2nd order system prototype; transient response specs (rise time, overshoot, settling time)
- Today's topic: effect of zeros and extra poles; Routh-Hurwitz stability criterion

- Learning Objectives: Understand the effect of zeros and higher-order poles on the shape of transient response; discuss relation wit stability; formulate and learn how to apply the Routh-Hurwtiz stability criterion

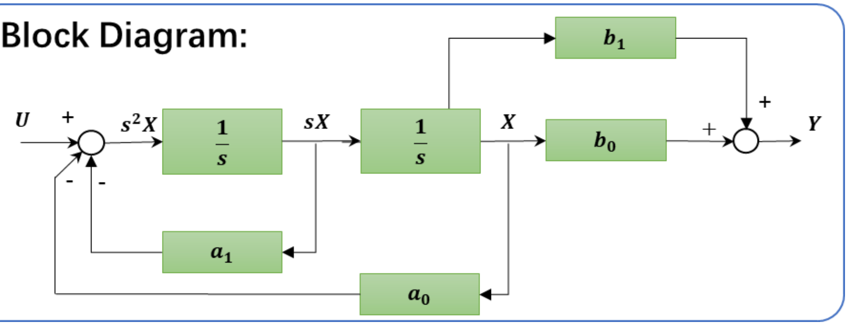
Recap

System Representation (Last week)

Transfer Function:

$$H(s) = \frac{b_1 s + b_o}{s^2 + a_1 s + a_o}$$

Block Diagram:



State space model:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$$

$$y = \begin{pmatrix}b_0 & b_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\ddot{x} + \underbrace{2\zeta\omega_n}_{b/m}\dot{x} + \underbrace{\boxed{\omega_n^2}}_{\frac{k}{m}} x = \frac{f}{m}$$

System Representation

Input	Dyn. Sys Repres.	Output
$u(\#)$	LTI	$y(\#)$
$i(t)$	$*\,h(t)$	$o(t)$
$I(s)$	$H(s)$	$O(s)$

Configuration form

Equations of Motion

$$\begin{cases} \ddot{q}_1 = f_1(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \ddot{q}_1 = f_2(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \\ \dots \\ \ddot{q}_n = f_n(q_1, \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \end{cases}$$

Initial Conditions

$$\begin{cases} q_1(0) = q_{1_0}, \dots, q_n(0) = q_{n_0} \\ \dot{q}_1(0) = \dot{q}_{1_0}, \dots, \dot{q}_n(0) = \dot{q}_{n_0} \end{cases}$$

ODE:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} = b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^{m-1}} + b_o x(t)$$

ICs

$$\frac{d^k y}{dt^k} = y_k$$

Transfer Function:

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_o}{s^n + a_{n-1}s^{n-1} + \dots + a_o}$$

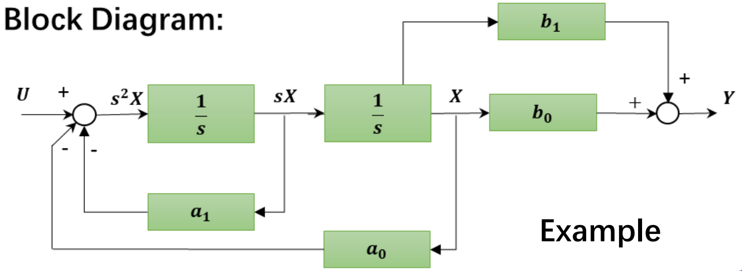
ICs= 0

State space model:

State Equation $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$

Output Equation $y = \begin{pmatrix} b_0 & b_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Block Diagram:



Example

2nd Order Systems

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By the quadratic formula, the poles are:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

The nature of the poles changes depending on ζ:

- $\zeta > 1$ both poles are real and negative
- $\zeta = 1$ one negative pole
- $\zeta < 1$ two complex poles with negative real parts

$$s = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n, \ \omega_d = \omega_n\sqrt{1 - \zeta^2}$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$
$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$
$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

$$\ddot{x} + \underbrace{2\zeta\omega_n}_{b/m} \dot{x} + \underbrace{\boxed{\omega_n^2}}_{\frac{k}{m}} = \frac{f}{m}$$

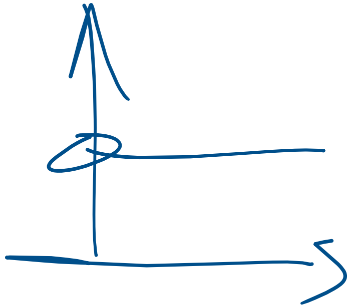
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = f$$

$$s^2X + \frac{b}{m}sX + \frac{k}{m}X = F$$

$$H(s) = \frac{X(s)}{F(s)}$$

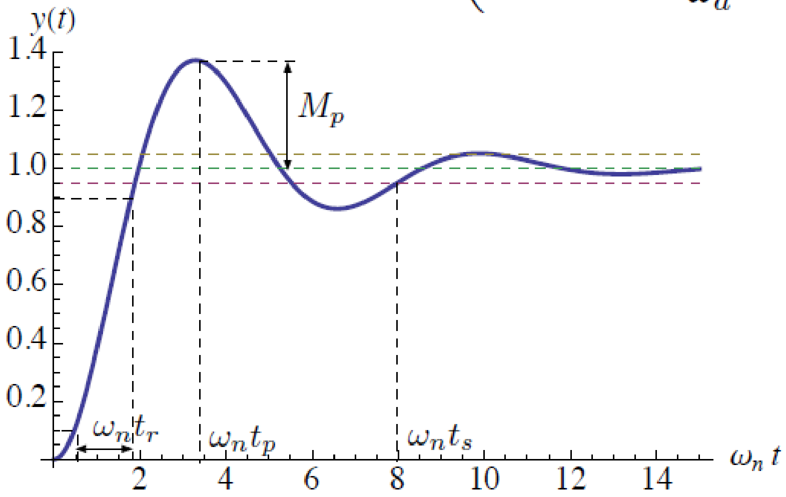
$$X\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) = F$$

$$\boxed{1(t)} \rightarrow \frac{1}{s}$$



System Response

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

Recap: Transient Response

- Transient Response Spec
  - Rise time, t_r
  - Overshoot, M_p
  - Settling time, t_s

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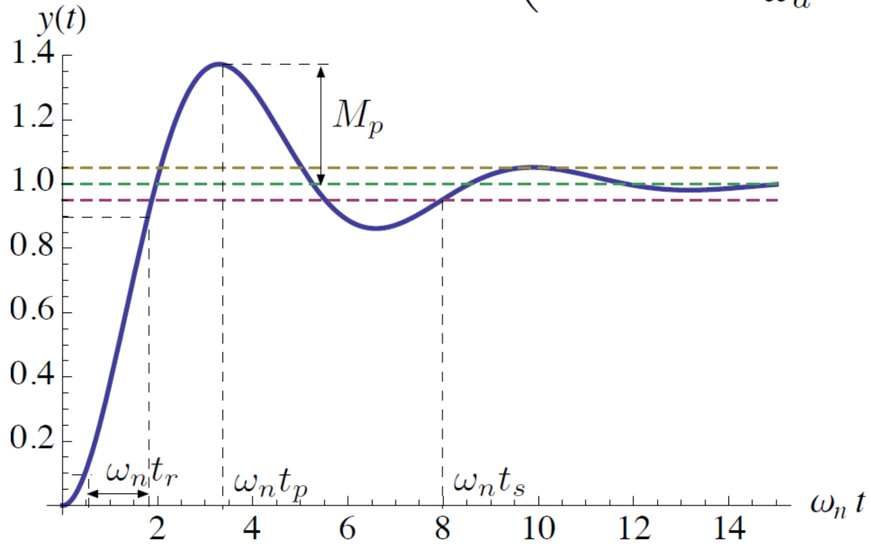

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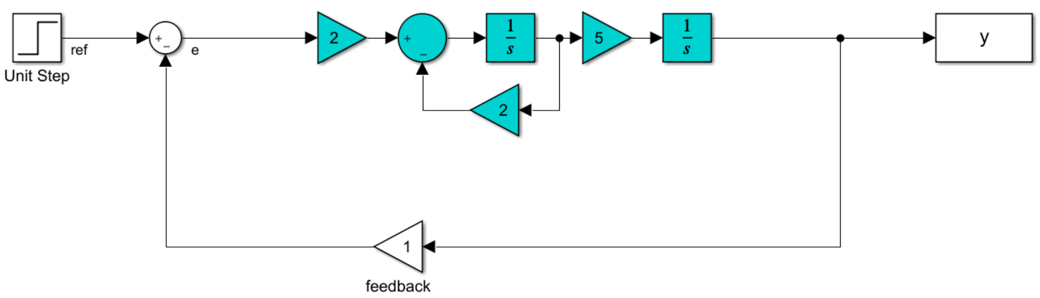
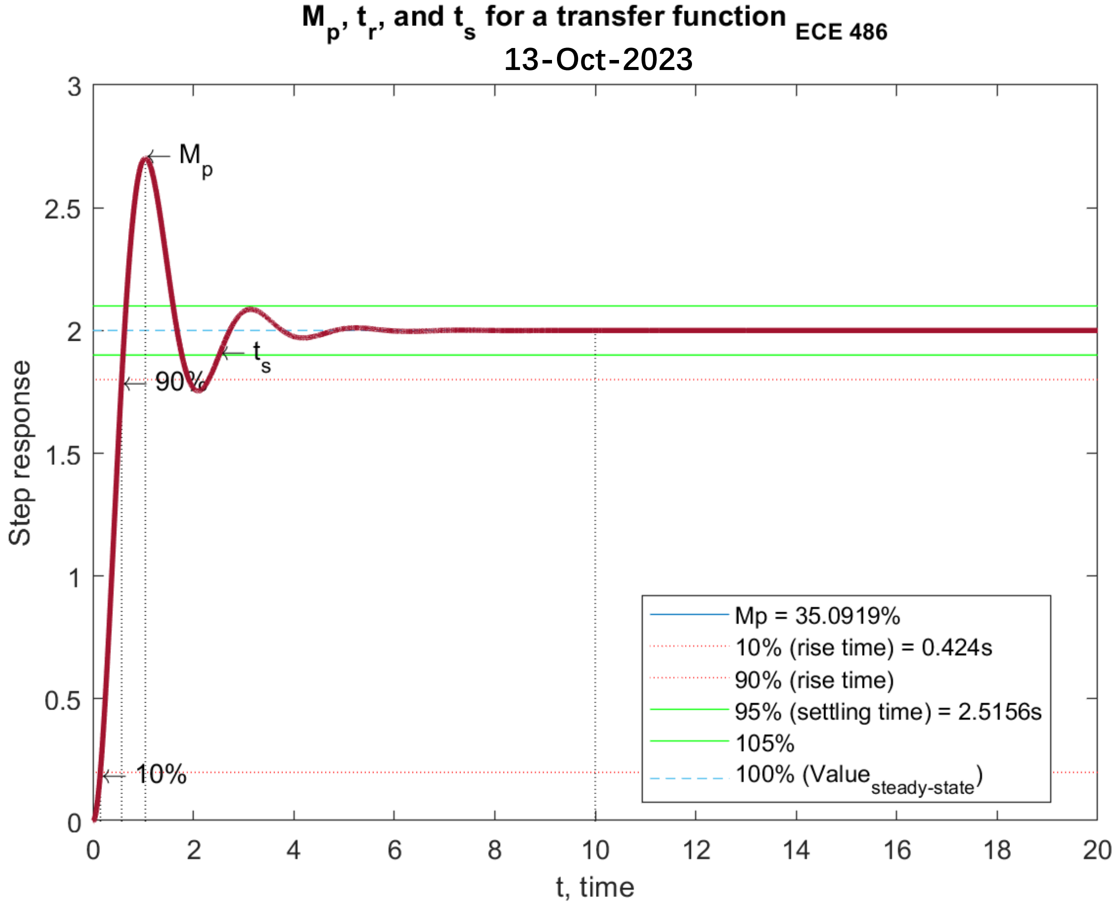
Transient Response Spec: Recall Lab 0

Lab 0

A Simple Plant under a unit step input






```
%Mp
% Mp is the percentage overshoot -

%tr
% tr is the time required for the response to rise from 10% of the
% steady-state value to 90% of the steady-state value.

%ts
% ts is the time it takes for the response settle between 95% and 105% of
% the steady-state value.  One way to find ts is to use a while loop,
% initialize a counter (x) to the end of the response array, and move
% forwards through the array until the response is no longer within the
% 95-105% bounds.
```

Effect of Zeros on the Transient Response

$H(s) = \frac{q(s)}{p(s)}$; $q(s) = 0$ (zeros), $p(s) = 0$ (poles)

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

zeros are roots of $q(s)$; poles are roots of $p(s)$

Example: start with $H_1(s) = \boxed{\frac{1}{s^2 + 2\zeta s + 1}}$ $\quad (\omega_n = 1)$

Effect of Zeros on the Transient Response

$$H(s) = \frac{q(s)}{p(s)};$$

zeros are roots of $q(s)$; poles are roots of $p(s)$

Example: start with $H_1(s) = \dfrac{1}{s^2 + 2\zeta s + 1}$ $\quad (\omega_n = 1)$

Let's add a zero at $s = -a$, $a > 0$ – LHP zero

To keep DC gain $= 1$, let's take the numerator to be $\dfrac{s}{a} + 1$:

$$\begin{aligned} H_2(s) &= \frac{\frac{s}{a} + 1}{s^2 + 2\zeta s + 1} \\ &= \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{\text{this is } H_1(s)} + \frac{1}{a} \cdot \underbrace{\frac{s}{s^2 + 2\zeta s + 1}}_{\text{call this } H_d(s)} \\ &= \underbrace{H_1(s)}_{\text{original}} + \frac{1}{a} H_d(s), \qquad H_d(s) = sH_1(s) \end{aligned}$$

Im

Re

original

additional term

post adding LHP zero

Effect of a LHP Zero

$$Y(s) = \underbrace{U(s)}_{1/s} H(s)$$

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s=-a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

Step response:

$$\begin{aligned} Y_1(s) &= \frac{H_1(s)}{s} \\ Y_2(s) &= \frac{H_2(s)}{s} \\ &= \frac{H_1(s)}{s} + \frac{1}{a}\frac{sH_1(s)}{s} \\ &= Y_1(s) + \frac{1}{a} sY_1(s) \end{aligned}$$

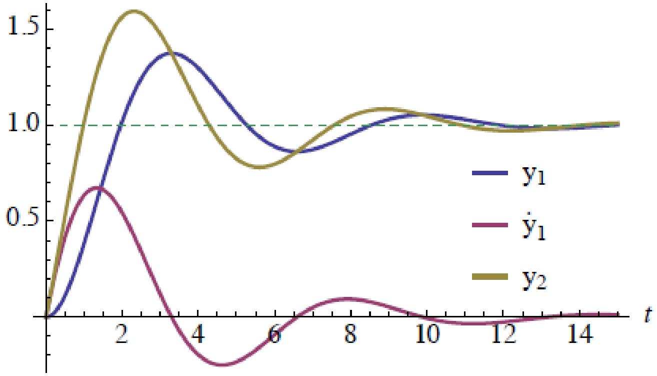
$$y_2(t) = \mathscr{L}^{-1}\{Y_2(s)\} = \mathscr{L}^{-1}\left\{Y_1(s) + \frac{1}{a} \cdot sY_1(s)\right\} = y_1(t) + \frac{1}{a}\dot{y}_1(t)$$

(assuming zero initial conditions)

Effect of a LHP Zero

Step response (zero at $s = -a$)

$$y_2(t) = y_1(t) + \frac{1}{a}\dot{y}_1(t) \qquad \text{where } y_1(t) = \text{original step response}$$



Effects of a LHP zero:

- increased overshoot (major effect)
- little influence on settling time
- what happens as $a \to \infty$? — effects become less significant

Effect of a LHP Zero

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s=-a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

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Step response (zero at $s = -a$)

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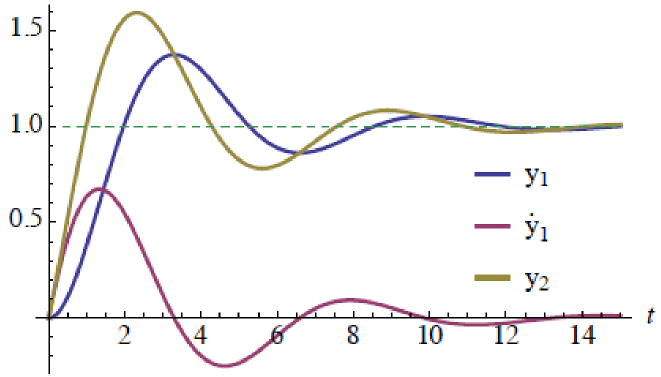


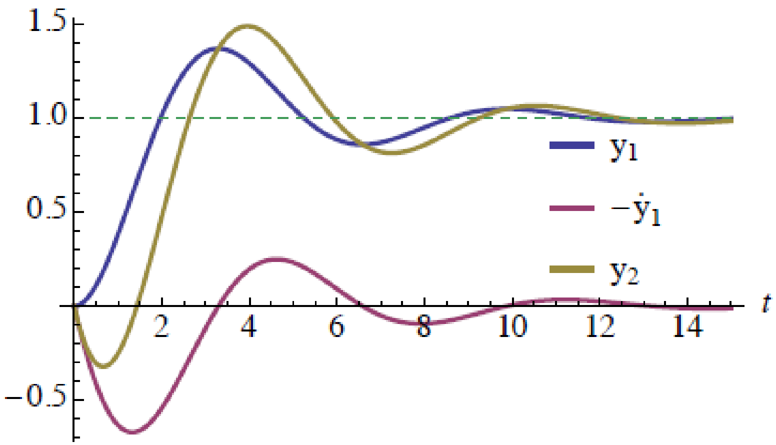
Effects of a LHP zero:

- increased overshoot (major effect)
- little influence on settling time
- what happens as $a \to \infty$? — effects become less significant

Effect of a RHP Zero

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = a} H_2(s) = H_1(s) - \frac{1}{a} \cdot s H_1(s)$$

$$y_2(t) = y_1(t) - \frac{1}{a} \cdot \dot{y}_1(t)$$



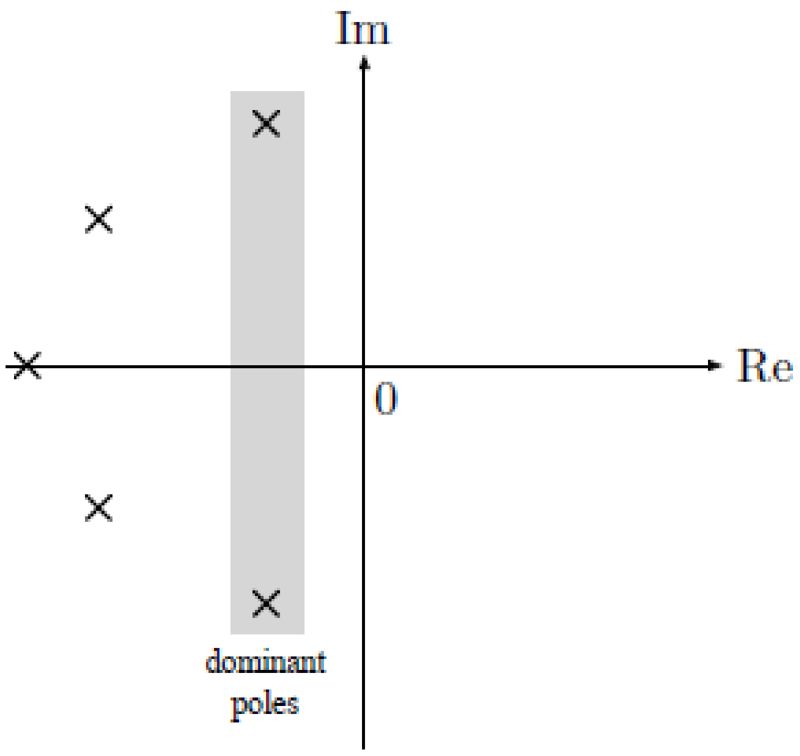
Effects of a RHP zero:

- slows down (delays) the response
- creates *undershoot* (at least, when a is small enough)

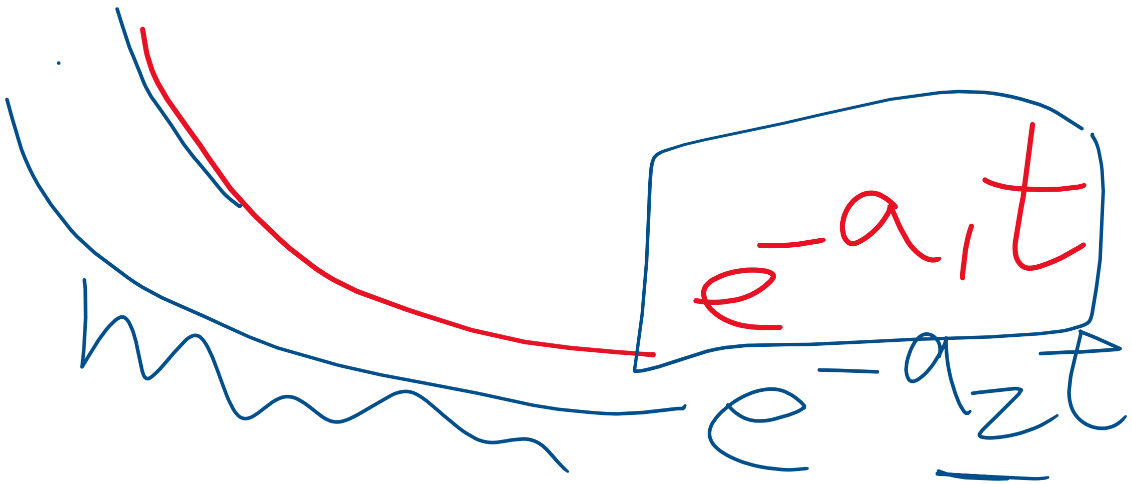
Effect of Extra Poles

e^{-at}

A general nth-order system has n poles

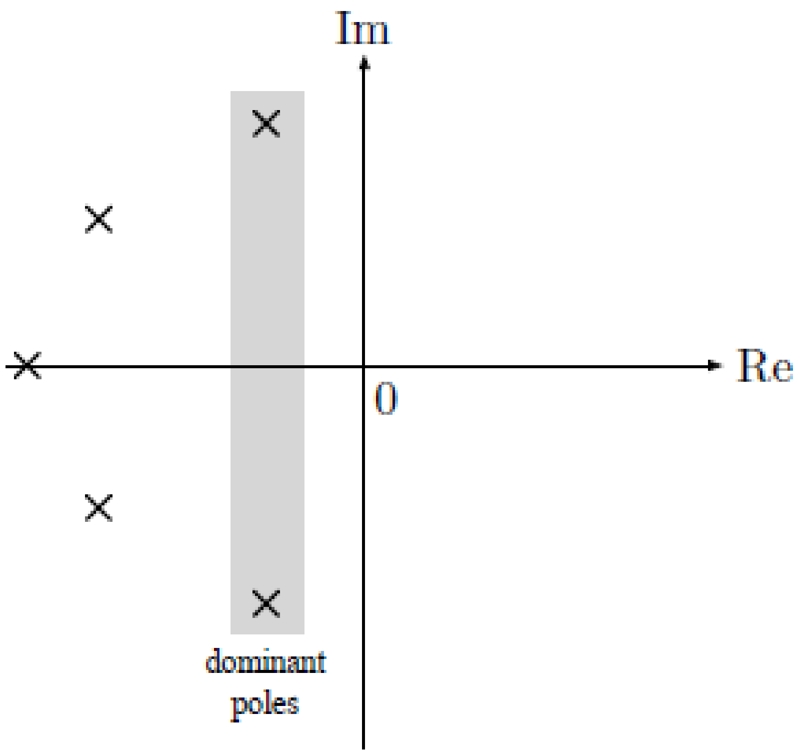


- extra LHP poles are not significant if their real parts are at least 5× the real parts of dominant LHP poles



Effect of Extra Poles

A general nth-order system has n poles

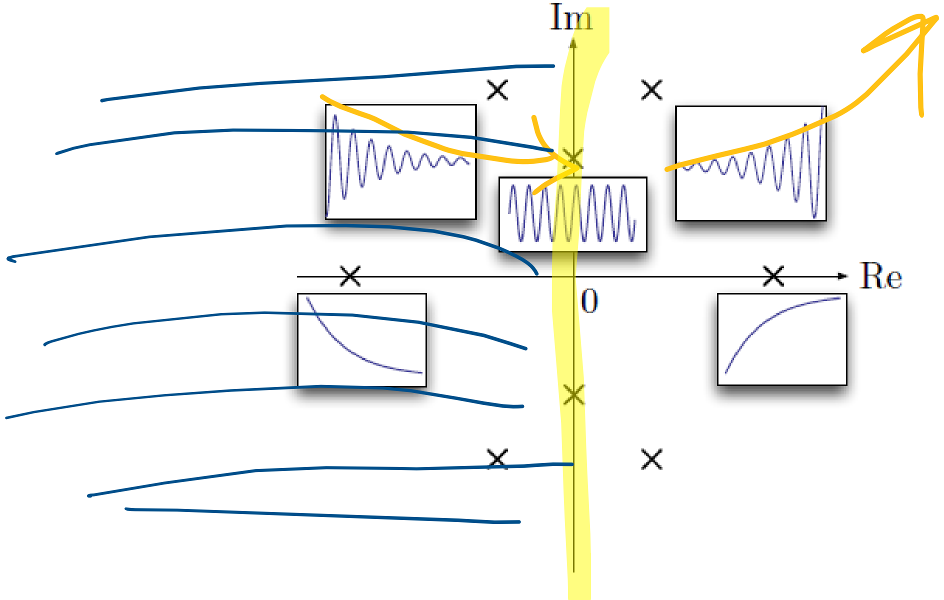


- extra LHP poles are not significant if their real parts are at least $5\times$ the real parts of dominant LHP poles
- e.g., if dominant poles have $\mathrm{Re}(s) = -2$ and we have extra poles with $\mathrm{Re}(s) = -10$, their time-domain contributions will be e^{-2t} and $e^{-10t} \ll e^{-2t}$
- $5\times$ is just a convention, but we can really see the effect of extra poles that are closer

Effect of Poles

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

Effect of Pole Locations



- poles in open LHP ($\mathrm{Re}(s) < 0$) — stable response
- poles in open RHP ($\mathrm{Re}(s) > 0$) — unstable response
- poles on the imaginary axis ($\mathrm{Re}(s) = 0$) — tricky case

Marginal Case: Poles on the Imaginary Axis

Let's consider the case of a pole at the origin: $H(s) = \frac{1}{s}$

Is this a stable system?

- impulse response: $Y(s) = \frac{1}{s} \Longrightarrow y(t) = 1(t)$ (OK)
- step respone: $Y(s) = \frac{1}{s^2} \Longrightarrow y(t) = t,\ t \geq 0$ — *unit ramp!!*

What about purely imaginary poles? $H(s) = \frac{\omega^2}{s^2+\omega^2}$ $s^2+\omega^2 = 0$

- impulse response: $Y(s) = \frac{\omega^2}{s^2+\omega^2} \Longrightarrow y(t) = \omega\sin(\omega t)$
- step respone: $Y(s) = \frac{\omega^2}{s(s^2+\omega^2)} \Longrightarrow y(t) = 1 - \cos(\omega t)$

Systems with poles on the imaginary axis are *not stable.*

Stability

- An LTI is Bounded-Input, Bounded-Output (BIBO) Stable if one of the 3 conditions is satisfied
  - Every **bounded input maps to a bounded output** regardless of IC
  - The impulse response ***h(t)*** **is absolutely integrable**:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

  - All **poles** of the transfer function *H(s)* are strictly stable (**in OLHP**)

Open left half plane

Checking for Stability

- Consider a general Transfer Function:

$$H(s) = \frac{q(s)}{p(s)}$$

where q and p are polynomials, and $\deg(q) < \deg(p)$

- Need tools for checking stability: if all roots of $p(s) = 0$ lie in OLHP.
- Factorization is hard to do for high-degree polynomials
  - computationally intensive, especially symbolically.
- **But:** often we don't need to know precise pole locations, just need to know that they are strictly stable.

Checking for Stability

Problem: given an nth-degree polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

with real coefficients, check that the roots of the equation $p(s) = 0$ are strictly stable (i.e., have negative real parts).

Terminology: we often say that the polynomial p is (strictly) stable if all of its roots are.

Checking for Stability

Terminology: we say that A is a necessary condition for B if

$$A \text{ is false} \quad \Longrightarrow \quad B \text{ is false}$$

Important!! Even if A is true, B may still be false.

Necessary condition for stability: a polynomial p is strictly stable only if all of its coefficients are strictly positive.

Proof: suppose that p has roots at $r_1, r_2, \ldots, r_n$ with $\mathrm{Re}(r_i) < 0$ for all i. Then

$$p(s) = (s - r_1)(s - r_2)\ldots(s - r_n)$$

— multiply this out and check that all coefficients are positive.

Terminology: we say that A is a sufficient condition for B if

$$A \text{ is true} \quad \Longrightarrow \quad B \text{ is true}$$

Thus, A is a necessary *and* sufficient condition for B if

$$A \text{ is true} \quad \Longleftrightarrow \quad B \text{ is true}$$

— we also say that A is true if and only if (iff) B is true.

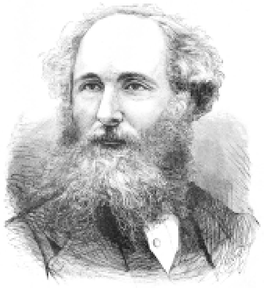
We will now introduce a necessary and sufficient condition for stability: the *Routh–Hurwitz Criterion*.

Routh-Hurwitz Criterion-A brief history

J.C. Maxwell, "On governors," Proc. Royal Society, no. 100, 1868

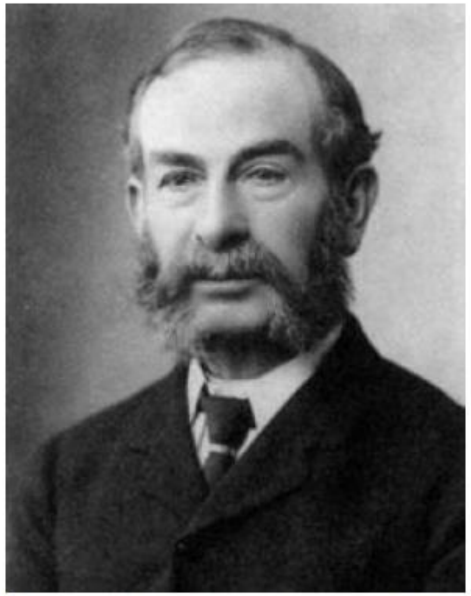
... [Stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative. ...

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.



In 1877, Maxwell was one of the judges for the Adams Prize, a biennial competition for best essay on a scientific topic. The topic that year was stability of motion. The prize went to Edward John Routh, who solved the problem posed by Maxwell in 1868.

In 1893, Adolf Hurwitz solved the same problem, using a different method, independently of Routh.



Edward John Routh, 1831–1907

Adolf Hurwitz, 1859–1919

Routh's Test

Problem: check whether the polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1}s + a_n$$

is strictly stable.

We begin by forming the Routh array using the coefficients of p:

$$\begin{array}{rccccc} s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \end{array}$$

(if necessary, add zeros in the second row to match lengths)

Note that the very first entry is always 1, and also note the order in which the coefficients are filled in.

Routh's Test

$$\begin{array}{lllll} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \end{array}$$

Next, we form the third row marked by s^{n-2}:

$$s^{n-2}: \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

$$\text{where} \quad b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1}(a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1}(a_5 - a_1 a_4)$$

$$b_3 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1}(a_7 - a_1 a_6) \quad \text{and so on ...}$$

Note: the new row is 1 element shorter than the one above it

$$\begin{array}{lllll} s^n: & 1 & a_2 & a_4 & a_6 & \dots \\ s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\ s^{n-2}: & b_1 & b_2 & b_3 & \dots \\ s^{n-3}: & c_1 & c_2 & \dots \end{array}$$

Next, we form the fourth row marked by s^{n-3}:

$$s^{n-3}: \quad c_1 \quad c_2 \quad \dots$$

$$\text{where} \quad c_1 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_2 - a_3 b_1)$$

$$c_2 = -\frac{1}{b_1}\det\begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_1}(a_1 b_3 - a_5 b_1)$$

and so on ...

Routh's Test

Eventually, we complete the array like this:

$$
\begin{array}{llllll}
s^n: & 1 & a_2 & a_4 & a_6 & \dots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2}: & b_1 & b_2 & b_3 & \dots & \\
s^{n-3}: & c_1 & c_2 & \dots & & \\
\vdots & & & & & \\
s^1: & * & * & & & \\
s^0: & * & & & &
\end{array}
$$

(as long as we don't get stuck with division by zero: more on this later)

After the process terminates, we will have $n+1$ entries in the first column.

The Routh-Hurwitz Criterion

Consider degree-n polynomial

$$p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

and form the Routh array:

$$\begin{array}{rcllll}
s^n: & 1 & a_2 & a_4 & a_6 & \ldots \\
s^{n-1}: & a_1 & a_3 & a_5 & a_7 & \ldots \\
s^{n-2}: & b_1 & b_2 & b_3 & \ldots & \\
s^{n-3}: & c_1 & c_2 & \ldots & & \\
\vdots & & & & & \\
s^1: & * & * & & & \\
s^0: & * & & & &
\end{array}$$

The Routh–Hurwitz criterion: Assume that the necessary condition for stability holds, i.e., $a_1, \ldots, a_n > 0$. Then:

- p is stable if and only if all entries in the first column are positive;
- otherwise, #(RHP poles) = #(sign changes in 1st column)

Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

All coefficients strictly positive: necessary condition checks out.

$$\begin{array}{llll} s^4: & 1 & 1 & 3 \\ s^3: & 4 & 2 & 0 \\ s^2: & 1/2 & 3 & \\ s^1: & -22 & 0 & \\ s^0: & 3 & & \end{array}$$

Answer: p is unstable — it has 2 RHP poles (2 sign changes in 1st column)

Low-Order Cases ($n = 2, 3$)

$n = 2$ $\quad p(s) = s^2 + a_1 s + a_2$

$$\begin{array}{llll} s^2 & : 1 & a_2 & \\ s^1 & : a_1 & 0 & \\ s^0 : & b_1 & & b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & 0 \end{pmatrix} = a_2 \end{array}$$

— p is stable iff $a_1, a_2 > 0$ (necessary *and* sufficient).

$n = 3$ $\quad p(s) = s^3 + a_1 s^2 + a_2 s + a_3$

$$\begin{array}{llll} s^3 & : 1 & a_2 & \\ s^2 & : a_1 & a_3 & \\ s^1 : & b_1 & 0 & b_1 = -\frac{1}{a_1}\det\begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = \frac{a_1 a_2 - a_3}{a_1} \\ s^0 : & c_1 & & c_1 = -\frac{1}{b_2}\det\begin{pmatrix} a_1 & a_3 \\ b_1 & 0 \end{pmatrix} = a_3 \end{array}$$

— p is stable iff $a_1, a_2, a_3 > 0$ (necc. cond.) and $a_1 a_2 > a_3$

Stability Conditions for Low-Order Polynomials

The upshot:

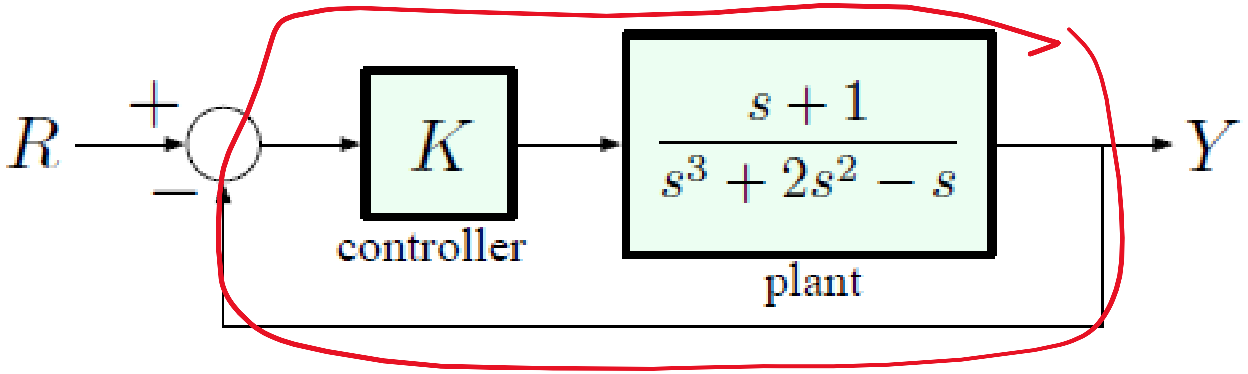
- A 2nd-degree polynomial $p(s) = s^2 + a_1 s + a_2$ is stable if and only if $a_1 > 0$ and $a_2 > 0$
- A 3rd-degree polynomial $p(s) = s^3 + a_1 s^2 + a_2 s + a_3$ is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

- These conditions were already obtained by Maxwell in 1868.
- In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

Routh-Hurwitz as a Design Tool

Parametric Stability Range: Determining range of parameters for stability in controller design

Example: consider the unity feedback configuration

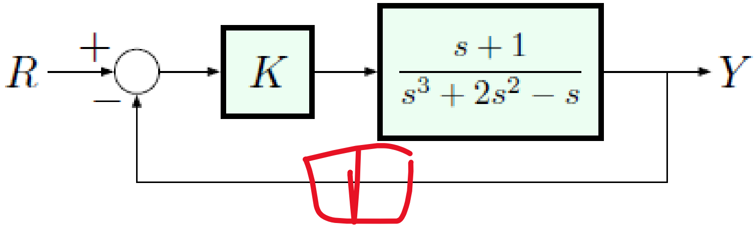


Note that the plant is *unstable* (the denominator has a negative coefficient and a zero coefficient).

Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Routh-Hurwitz as a Design Tool

Parametric Stability Range: Determining range of parameters for stability in controller design



Problem: determine the range of values the scalar gain K can take, for which the closed-loop system is stable.

Let's write down the transfer function from R to Y:

$$\frac{Y}{R} = \frac{\text{forward gain}}{1 + \text{loop gain}}$$

$$= \frac{K \cdot \frac{s+1}{s^3+2s^2-s}}{1 + K \cdot \frac{s+1}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3+2s^2-s+K(s+1)}$$

$$= \frac{Ks+K}{s^3+2s^2+(K-1)s+K}$$

Now we need to test stability of $p(s) = s^3 + 2s^2 + (K-1)s + K$.

Test stability of

$$p(s) = s^3 + 2s^2 + (K-1)s + K$$

using the Routh test.

Form the Routh array:

$$\begin{array}{lcc} s^3: & 1 & K-1 \\ s^2: & 2 & K \\ s^1: & \frac{K}{2}-1 & 0 \\ s^0: & K & \end{array}$$

For p to be stable, all entries in the 1st column must be positive:

$$K > 2 \quad \text{and} \quad K > 0 \quad (\text{already covered by } K > 1)$$

Note: The necessary condition requires $K > 1$, but now we actually know that we must have $K > 2$ for stability.

Remarks on Routh Test

- The result (#(RHP roots) is not affected by multiplying or dividing any row of the Routh array by an arbitrary positive number
- For zero element in the 1st column, we can replace the 0 by a small number ε and apply Routh test to that. When we are done with the array, take the limit as $\varepsilon \to \infty$. (see Ex. 3.33 in FPE)
- For an entire row of zeros, the procedure is a more complicated (see Example 3.34 in FPE) - we will not worry about this too much.