ECE 486 Control Systems

Lecture 10: Introduction to Frequency Response

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Schedule check

Week	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
4	National Holiday Week	
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control; Introduction PID control	Sections 4.1-4.3, lab manual
6	Review A	
	Term Test 1	
7	**Introduction to frequency-response design method**	
	Introduction to Root Locus design method	Ch. 5
8	Root Locus continued; introduction to dynamic compensation	Ch. 5
	Lead and lag dynamic compensation	Sections 5.1-5.4, 6.1
9	Bode plots for three types of transfer functions	Section 6.1

Modeling & Representation

Analysis

Design

Root Locus

Frequency Response

State-Space

Lecture Overview

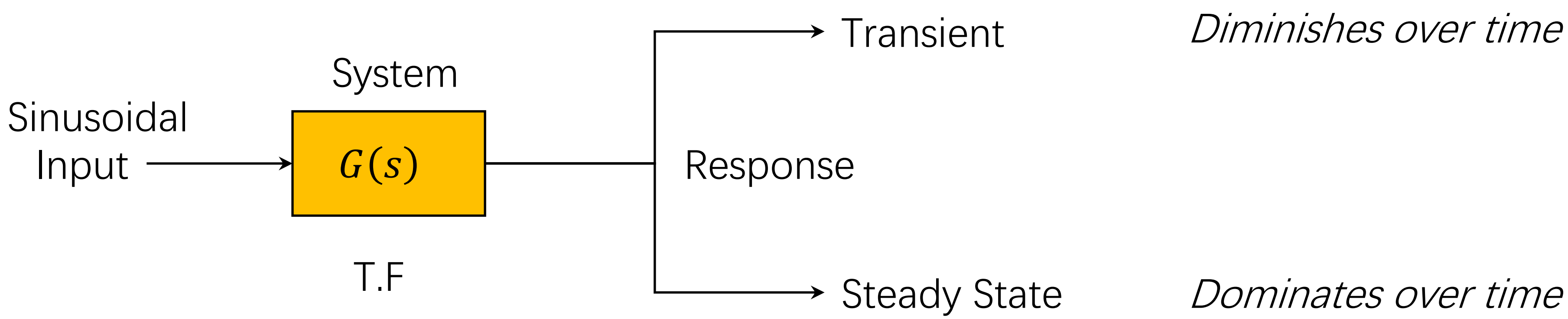
- Post Term Test I Review
- Introduction to Frequency Response

- Learning Goal: preparing to use frequency response as an alternative method for control systems design; learn to analyze and sketch magnitude and phase plots of transfer functions

Reading: FPE, Chapter 6 Section 6.1

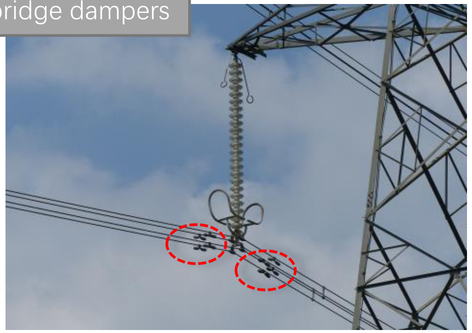
Recap: Frequency Response

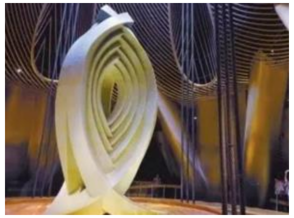
- The steady-state response to a sinusoidal input is known as the frequency response

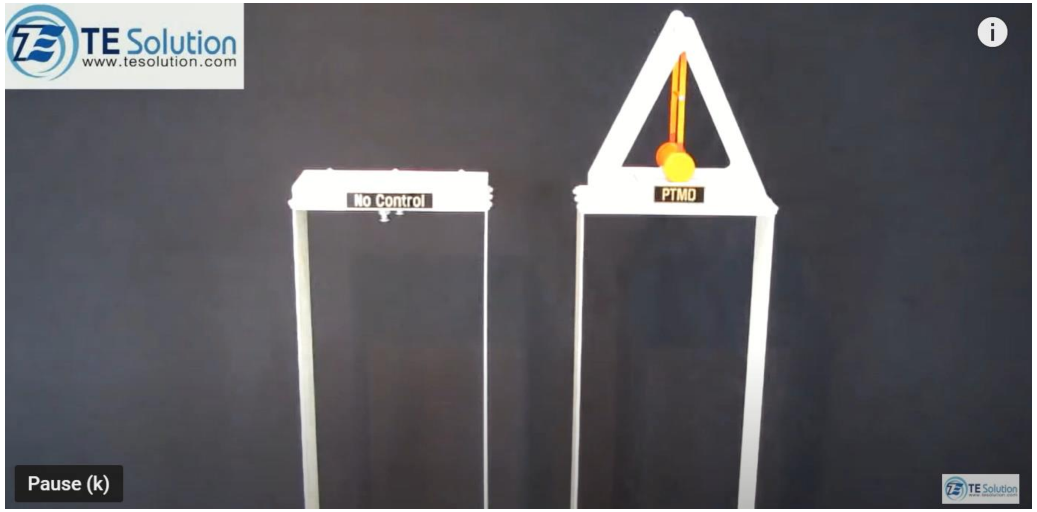


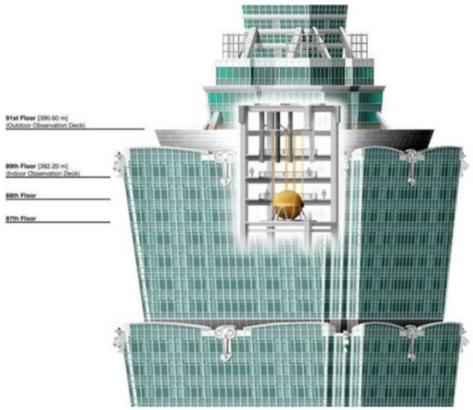
Engineering Examples

Stockbridge dampers









https://www.shanghaitower.com/shanghaizhongxin/index8.php

https://www.youtube.com/watch?v=GzMuF-LMGaM

Frequency Response Design Method

- Recall the frequency-response formula:

Frequency Response Design Method

- Recall the frequency-response formula:

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow M\sin(\omega t+\phi)$$

where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Derivation:

1. $u(t) = e^{st} \longmapsto y(t) = G(s)e^{st}$
2. Euler's formula: $\sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$
3. By linearity,

$$\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \qquad G(j\omega) = M(\omega)e^{j\phi(\omega)}$$

$$= \frac{M(\omega)e^{j(\omega t+\phi(\omega))} - M(\omega)e^{-j(\omega t+\phi(\omega))}}{2j}$$

$$= M(\omega)\sin(\omega t + \phi(\omega))$$

Let's apply this formula to our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = |G(j\omega)| = \left|\frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2}\right|$$

$$= \left|\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}j}\right|$$

$$= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}$$

Frequency Response Design Method

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow M\sin(\omega t+\phi)$$

where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

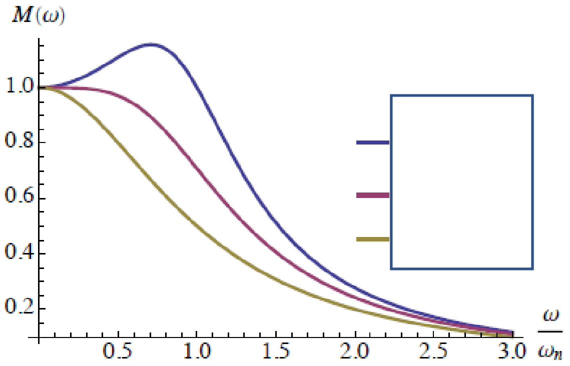
$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1+(4\zeta^2-2)\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4}}$$

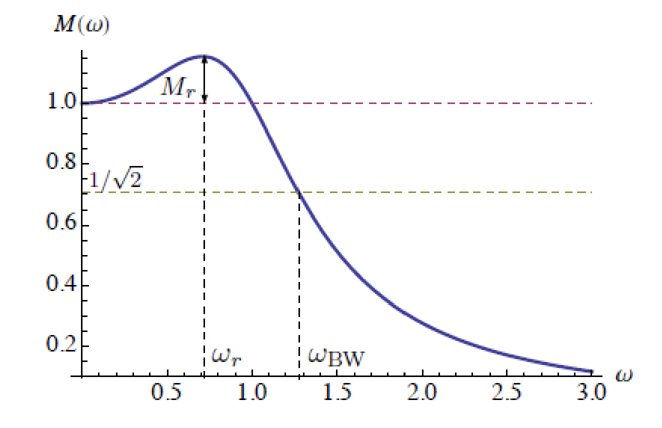
For our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1+(4\zeta^2-2)\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4}}$$



A typical freq. response magnitude plot



small $M_r \longleftrightarrow$ better damping

large $\omega_{BW} \longleftrightarrow$ large $\omega_n \longleftrightarrow$ smaller t_r

Info about time response also encoded in frequency response

ω_r – resonant frequency

M_r – resonant peak

ω_{BW} – bandwidth

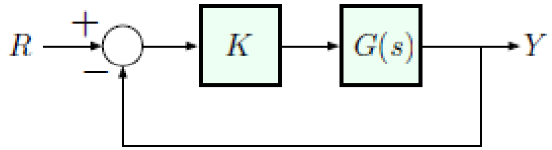
We can get the following formulas using calculus:

$$\begin{cases} \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \quad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}};\ \text{for } \zeta \geq \frac{1}{\sqrt{2}},\ \omega_r = 0\right)$$

$$\omega_{BW} = \omega_n\underbrace{\sqrt{(1-2\zeta^2)+\sqrt{(1-2\zeta^2)^2+1}}}_{=1 \text{ for } \zeta=1/\sqrt{2}}$$

— so, if we know $\omega_r, M_r, \omega_{BW}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Frequency Response Design Method



Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior.

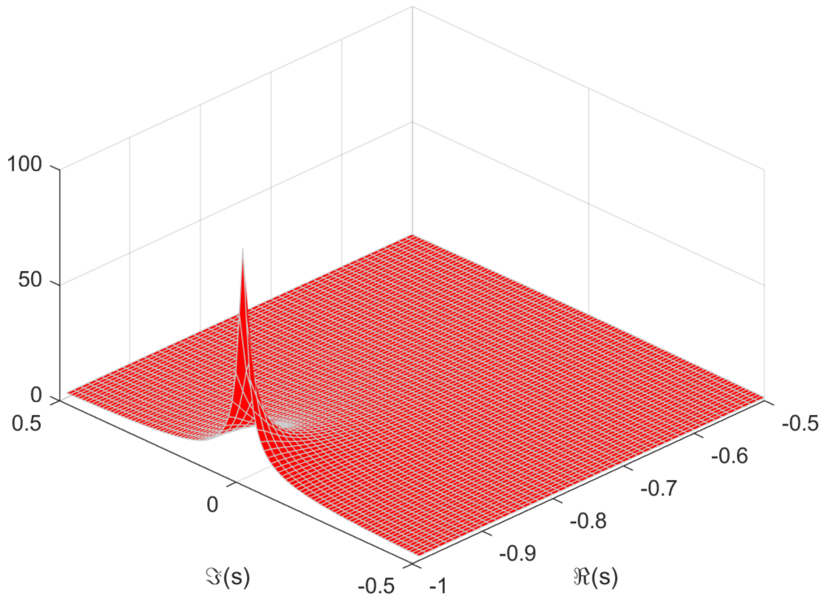
We will work with two types of plots for $KG(j\omega)$:

1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency ω (could have seen it earlier, in ECE 342)
2. Nyquist plots: $\mathrm{Im}(KG(j\omega))$ vs. $\mathrm{Re}(K(j\omega))$ [Cartesian plot in s-plane] as ω ranges from $-\infty$ to $+\infty$

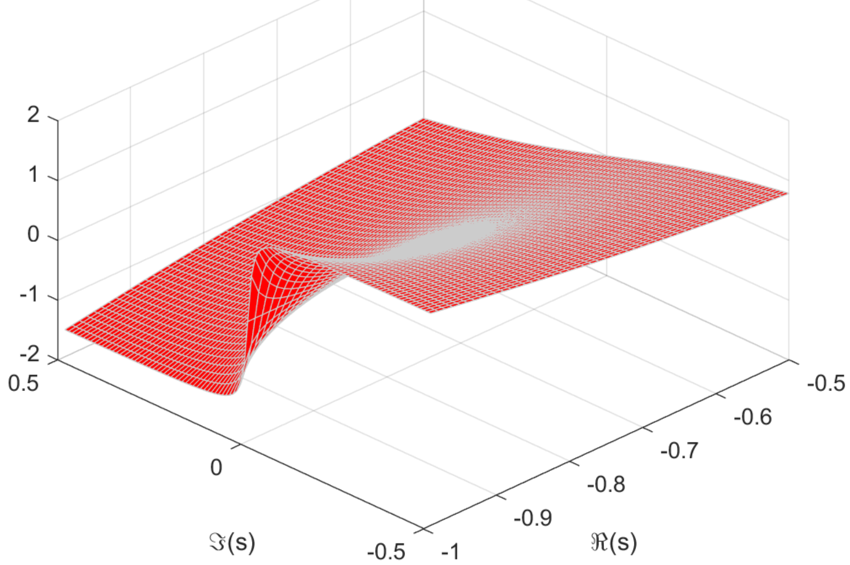
Representing Complex Function (Optional)

- Complex function $H(s)$ (with both Re(s) & Im(s) components)
  - two 2D surfaces of points $(\mathrm{Re}(s), \mathrm{Im}(s), |H|)$, $(\mathrm{Re}(s), \mathrm{Im}(s), \arg(H))$ in a 3D coordinate system

Surface plot of 1/|s+1|



Surface plot of arg(1/(s+1))

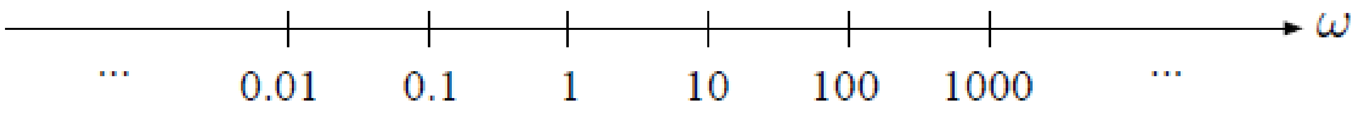


Note on the Scale

Horizontal (ω) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of ω, *not* $\log_{10}\omega$, on the axis, but the *scale* will be logarithmic:



Equal intervals on log scale correspond to decades in frequency.

Note on the Scale

Vertical axis on magnitude plots:

we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = M_1 \cdot M_2$$
$$\log(M_1 M_2) = \log M_1 + \log M_2$$

— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log\big(M_1(\omega)M_2(\omega)\big)$, and graphical addition is easy.

Decibel scale:

$$(M)_{\text{dB}} = 20 \log_{10} M \qquad (\text{one decade} = 20\,\text{dB})$$

Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

Reason:

$$\angle\Big((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\Big) = \angle\Big(M_1 M_2 e^{j(\phi_1+\phi_2)}\Big) = \phi_1 + \phi_2$$

— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

Scale Convention for Bode Plots

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

Recap: Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

Example:

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

$$\text{rewrite as}\quad \left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$

Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where n is a positive or negative integer
2. $(j\omega\tau+1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$

In our example above,

$$KG(j\omega) = \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]}$$

$$= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}}\cdot\underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}}\cdot\underbrace{\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]^{-1}}_{\text{Type 3}}$$