ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 486 Control Systems

Lecture ~~15~~ 14: Control Design with Frequency Response: PI and lag, PID and lead-lag

Liangjing Yang

Assistant Professor, ZJU-UIUC Institute

liangjingyang@intl.zju.edu.cn

**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Checklist

Modeling

Analysis

Root Locus

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

Frequency Response

State-Space

Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	**Control design using frequency response: PD and Lead**	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Recap: Stability Example

$$KG(s) = \frac{K}{s(s^2 + 2s + 2)}$$

Characteristic equation:

Here, the closed-loop system is stable if and only if $0 < K < 4$.

Let's see what we can read off from the Bode plots.

$$KG(s) = \frac{K}{s(s^2 + 2s + 2)}$$

Bode form:

Plot the magnitude first:

-
-
-

Recap: Stability Example

$s = j\omega$

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Characteristic equation:

$1 + KG = 0$

Here, the closed-loop system is stable if and only if $0 < K < 4$.

Let's see what we can read off from the Bode plots.

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Bode form:

Plot the magnitude first:

-
-
-

$$KG(s)$$

$$KG(j\omega) = \frac{K}{j\omega\left((j\omega)^2 + 2j\omega + 2\right)}$$

$$\frac{1}{\frac{j\omega^2}{2} + \frac{2j\omega}{2} + 1} = \frac{1}{\left(\frac{\omega}{\sqrt{2}}\right)^2 + j\omega + \frac{1}{2}}$$

$K_0[j\omega]^n$

Recap: Stability Example

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

Characteristic equation:

$$1 + \frac{K}{s(s^2+2s+2)} = 0$$
$$s(s^2+2s+2) + K = 0$$
$$s^3 + 2s^2 + 2s + K = 0$$

Recall the necessary & sufficient condition for stability for a 3rd-degree polynomial $s^3 + a_1 s^2 + a_2 s + a_3$:

$$a_1, a_2, a_3 > 0, \qquad a_1 a_2 > a_3.$$

Here, the closed-loop system is stable if and only if $0 < K < 4$.

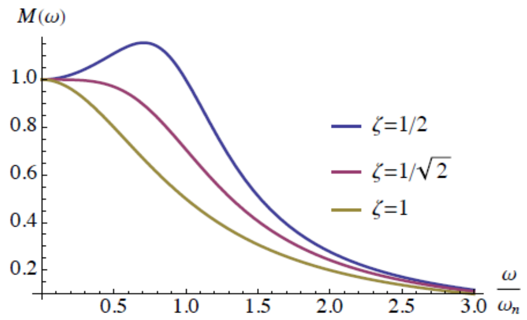
Let's see what we can read off from the Bode plots.

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

$$\text{Bode form:} \quad KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Plot the magnitude first:

- Type 1 (low-frequency) asymptote: $\frac{K/2}{j\omega}$
  $K_0 = K/2, \;\; n = -1 \implies$ slope $= -1$, passes through $(\omega = 1, M = K/2)$
- Type 3 (complex pole) asymptote:
  break-point at $\omega = \sqrt{2} \implies$ slope down by 2
- $\zeta = \frac{1}{\sqrt{2}} \implies$ no reasonant peak



The magnitude hits its peak value (for $\zeta < 1/\sqrt{2} \approx 0.707$) occurs when $\omega = \omega_r$, where

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} < \omega_n$$

For small enough ζ (below $1/\sqrt{2}$), the magnitude of

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

has a resonant peak at the resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}.$$

Likewise, the magnitude of

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

has a resonant dip at ω_r.

Example

Magnitude Plot

$$KG(s) = \frac{K}{s(s^2+2s+2)}$$

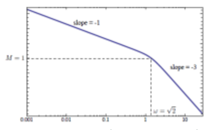
$$\text{Bode form:} \quad KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Plot the magnitude first:

- Type 1 (low-frequency) asymptote: $\frac{K/2}{j\omega}$
  $K_0 = K/2, \;\; n = -1 \implies$ slope $= -1$, passes through $(\omega = 1, M = K/2)$
- Type 3 (complex pole) asymptote:
  break-point at $\omega = \sqrt{2} \implies$ slope down by 2
- $\zeta = \frac{1}{\sqrt{2}} \implies$ no reasonant peak

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Magnitude plot for $K = 4$ (the critical value):



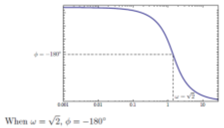
When $\omega = \sqrt{2}$, $M = |4G(j\omega)| = \left|\frac{2}{j\sqrt{2}\,(j^2 + j\sqrt{2} + 1)}\right| = 1$

Example

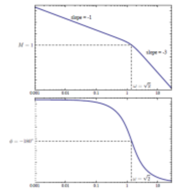
Phase Plot

$$KG(j\omega) = \frac{K}{2j\omega\left(\left(\frac{j\omega}{\sqrt{2}}\right)^2 + j\omega + 1\right)}$$

Phase plot (independent of K):



When $\omega = \sqrt{2}$, $\phi = -180°$



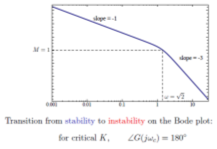
For the critical value $K = 4$:

$M = 1$ and $\phi = 180°$ mod 360° at $\omega = \sqrt{2}$

Crossover Frequency & Stability

Crossover Frequency and Stability

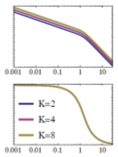
Definition: The frequency at which $M = 1$ is called the *crossover frequency* and denoted by ω_c.



Transition from stability to instability on the Bode plot:

for critical K, $\quad \angle G(j\omega_c) = 180°$

Effect of Varying K



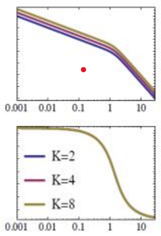
What happens as we vary K?

- ϕ independent of $K \implies$ only the M-plot changes
- If we multiply K by 2:
  $$\log(2M) = \log 2 + \log M$$
  – M-plot shifts up by $\log 2$
- If we divide K by 2:
  $$\log(\tfrac{1}{2}M) = \log\tfrac{1}{2} + \log M = -\log 2 + \log M$$
  – M-plot shifts down by $\log 2$

Changing the value of K moves the crossover frequency ω_c!!

Effect of Varying K

Changing the value of K moves the crossover frequency ω_c!!



What happens as we vary K?

$$\angle KG(j\omega_c) \begin{cases} > -180°, & \text{for } K < 4 \text{ (stable)} \\ = -180°, & \text{for } K = 4 \text{ (critical)} \\ < -180°, & \text{for } K > 4 \text{ (unstable)} \end{cases}$$

Equivalently, we may define $\omega_{180°}$ as the frequency at which

$$\phi = 180° \quad \text{mod } 360°.$$

Then, *in this example**,

$|KG(j\omega_{180°})| < 1 \longleftrightarrow$ stability
$|KG(j\omega_{180°})| > 1 \longleftrightarrow$ instabilit

* Not a general rule; conditions will vary depending on the system, must use either root locus or Nyquist plot to resolve ambiguity.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$H(j\omega) = \frac{1}{\frac{(j\omega)^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} j\omega + 1} = \frac{1}{A}$$

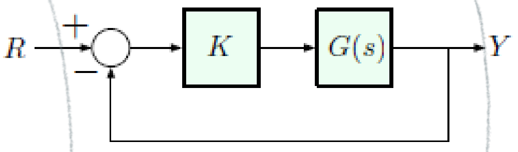
$$A = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + \frac{2\zeta\omega}{\omega_n} j$$

$$|A|^2 = \left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2$$

Where we left off ……

Stability from Frequency Response

Consider this unity feedback configuration:



Suppose that the *closed-loop* system, with transfer function

$$H_{CL} = \frac{KG(s)}{1+KG(s)}, = \frac{Y}{R}$$

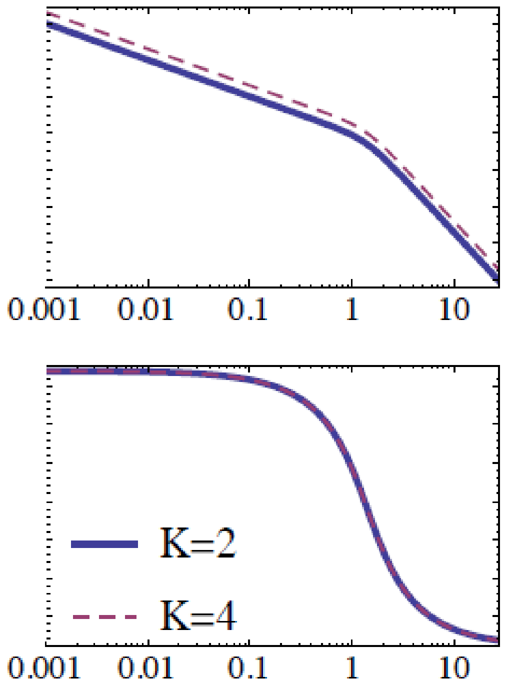
is stable for a given value of K.

Question: Can we use the Bode plot to determine how far from instability we are?

Two important characteristics: gain margin (GM) and phase margin (PM).

Gain Margin

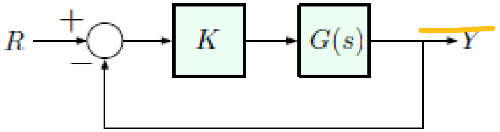
Back to our example: $G(s) = \frac{1}{s(s^2+2s+2)}$, $K = 2$ (stable)

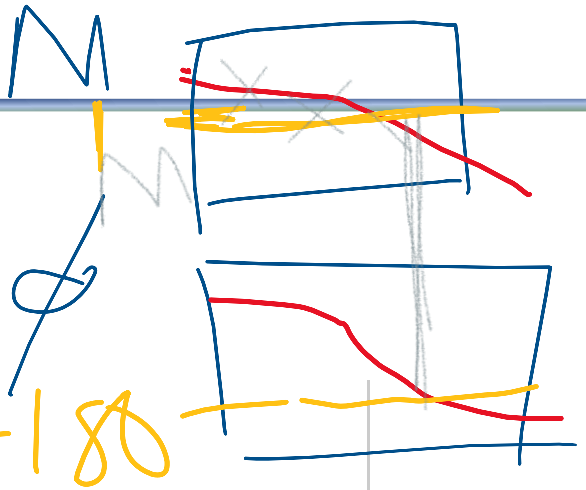


Gain margin (GM) is the factor by which K can be multiplied before we get $M = 1$ when $\phi = 180°$

Since varying K doesn't change $\omega_{180°}$, to find GM we need to inspect M at $\omega = \omega_{180°}$

Example





$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \qquad \zeta, \omega_n > 0$$

Consider gain $K = 1$, which gives closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)} = \frac{\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}{1+\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \text{— prototype 2nd-order response}$$

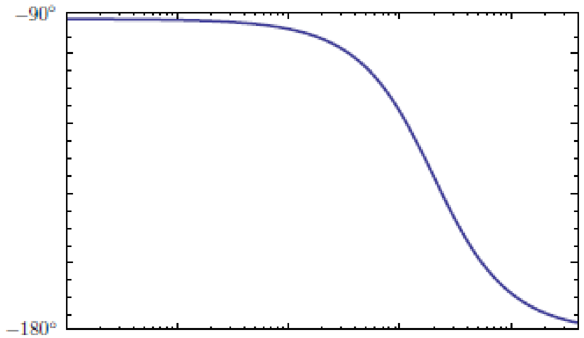
Question: what is the gain margin at $K = 1$?

Answer: $\mathrm{GM} = \infty$

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n}+1\right)}$$

Let's look at the phase plot:

- starts at -90° (Type 1 term with $n = -1$)
- goes down by -90° (Type 2 pole)



Recall: to find GM, we first need to find ω_{180°, and here there is no such $\omega \implies$ no GM.

Example

So, at $K = 1$, the gain margin of

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

is equal to ∞ — what does that mean?

It means that we can keep on increasing K indefinitely without ever encountering instability.

But we already knew that: the characteristic polynomial is

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2,$$

which is *always stable.*

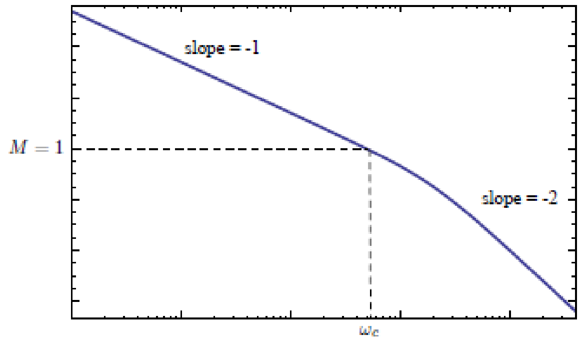
What about phase margin?

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

Let's look at the magnitude plot:

- low-frequency asymptote slope -1 (Type 1 term, $n = -1$)
- slope down by 1 past the breakpt. $\omega = 2\zeta\omega_n$ (Type 2 pole)

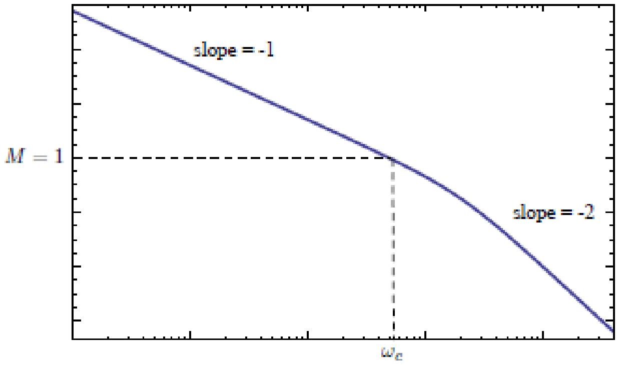
$\Longrightarrow$ there is a finite crossover frequency ω_c!!



Example

Magnitude Plot

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$



It can be shown that, *for this system*,

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right)$$

— for PM $< 70°$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right) \approx 100 \cdot \zeta$$

Conclusions:

larger PM $\Longleftrightarrow$ better damping

(open-loop quantity) (closed-loop characteristic)

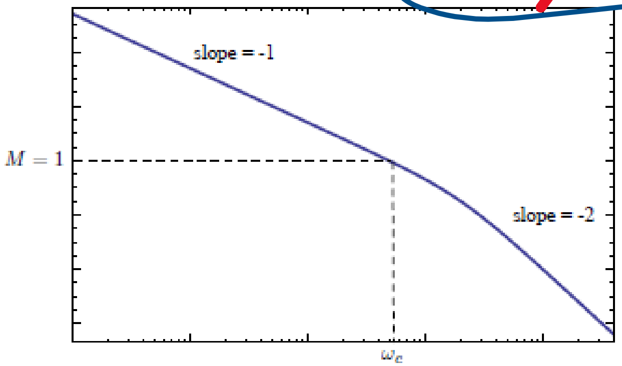
Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Example

$\mathrm{Re} \left[\frac{(j\omega)^2}{\omega_n}\right] + \left[2\zeta j\omega\right] j \rightarrow \mathrm{Im}$

Magnitude Plot

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{\cancel{2}\zeta j\omega\left(\frac{j\omega}{\cancel{2}\zeta\omega_n} + 1\right)}$$



It can be shown that, *for this system,*

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right)$$

— for PM $< 70°$, a good approximation is PM $\approx 100 \cdot \zeta$

Phase Margin

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega} = \frac{\omega_n}{2\zeta j\omega\left(\frac{j\omega}{2\zeta\omega_n} + 1\right)}$$

$$\mathrm{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}\right) \approx 100 \cdot \zeta$$

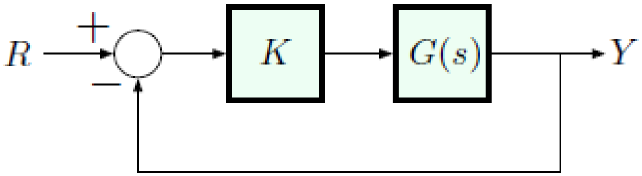
Conclusions:

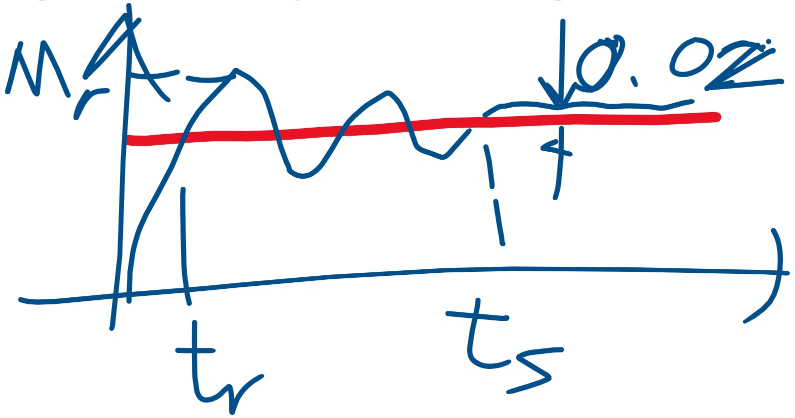
larger PM $\Longleftrightarrow$ better damping

(open-loop quantity) (closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through ζ!!

Control Design using Frequency Response





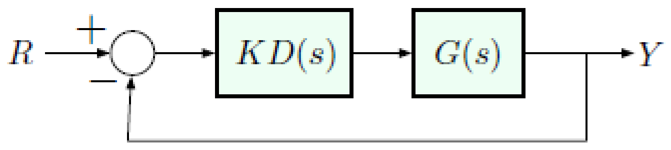
Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing K (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \qquad \Longrightarrow \qquad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of 90° and consequently good damping.

Control Design: Example



Let $G(s) = \dfrac{1}{s^2}$ (double integrator)

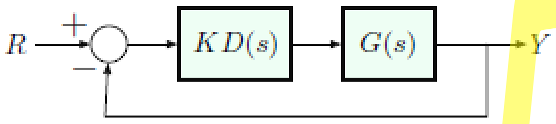
Objective: design a controller $KD(s)$ (K = scalar gain) to give

- stability
- good damping (will make this more precise in a bit)
- $\omega_{\mathrm{BW}} \approx 0.5$ (always a closed-loop characteristic)

Strategy:

- from Bode's Gain-Phase Relationship, we want magnitude slope $= -1$ at $\omega_c \implies \mathrm{PM} = 90^\circ \implies$ good damping;
- if $\mathrm{PM} = 90^\circ$, then $\omega_c = \omega_{\mathrm{BW}} \implies$ want $\omega_c \approx 0.5$

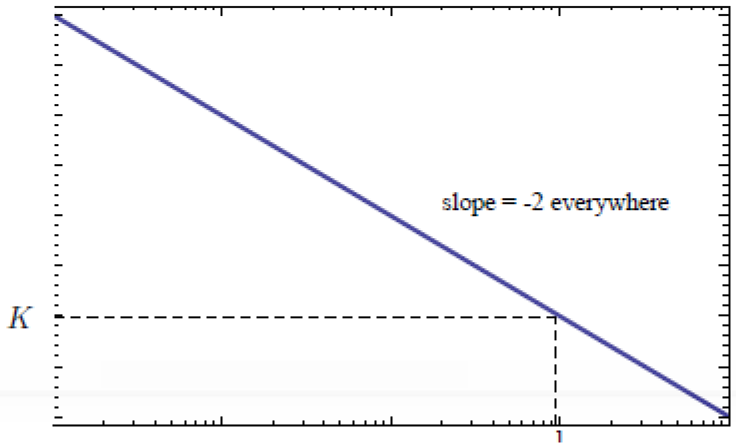
Control Design: Example- Attempt 1



$$G(s) = \frac{1}{s^2}$$

Let's try proportional feedback:

$$D(s) = 1 \implies KD(s)G(s) = KG(s) = \frac{K}{s^2}$$

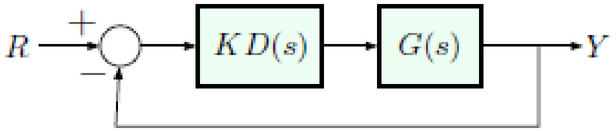


This is not a good idea: slope $= -2$ everywhere, so no PM.

We already know that P-gain alone won't do the job:

$$K + s^2 = 0 \text{ (imag. poles)}$$

Example- Attempt 2



$$G(s) = \frac{1}{s^2}$$

Let's try proportional-derivative feedback:

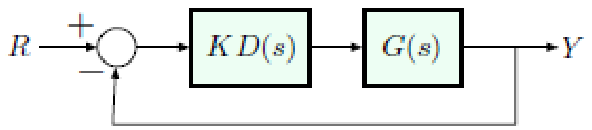
$$KD(s) = K(\tau s + 1), \qquad \text{where } K = K_P,\ K\tau = K_D$$

Open-loop transfer function: $KD(s)G(s) = \dfrac{K(\tau s + 1)}{s^2}$.

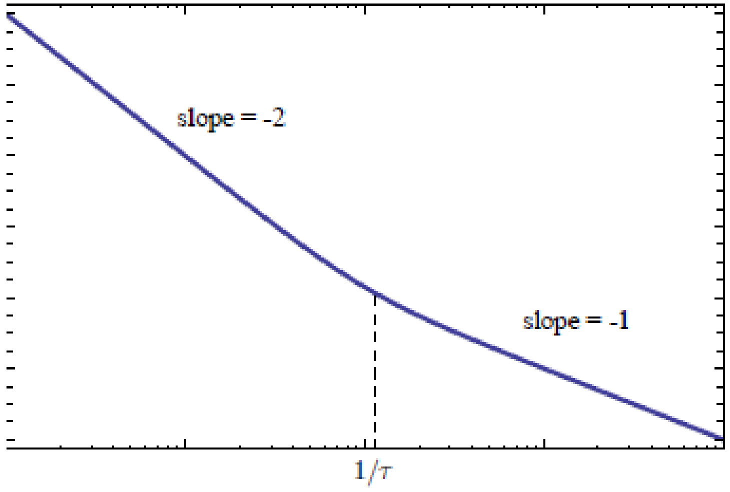
Bode plot interpretation: PD controller introduces a Type 2 term in the numerator, which pushes the slope up by 1

— this has the effect of pushing the M-slope of $KD(s)G(s)$ from -2 to -1 past the break-point ($\omega = 1/\tau$).

Example- Attempt 2 (PD Control)

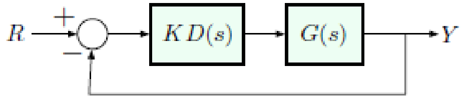


Open-loop transfer function: $KD(s)G(s) = \dfrac{K(10s+1)}{s^2}$

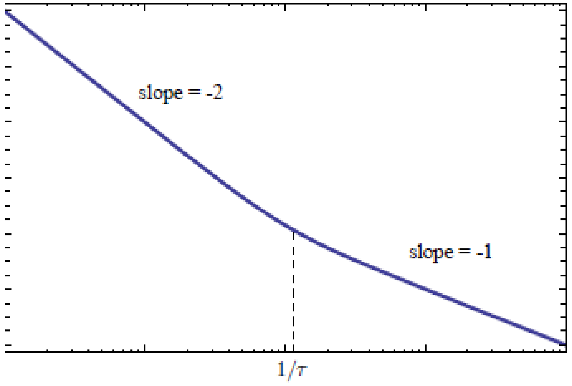


- Want $\omega_c \approx 0.5$

Example- Attempt 2 (PD Control)



Open-loop transfer function: $K\,D(s)G(s) = \dfrac{K(10s+1)}{s^2}$

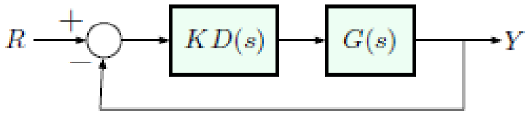


- Want $\omega_c \approx 0.5$
- This means that

$$
\begin{aligned}
M(j0.5) &= 1 \\
|K\,D(j0.5)G(j.05)| \\
&= \frac{K|5j+1|}{0.5^2} \\
&= 4K\sqrt{26} \approx 20K
\end{aligned}
$$

$$\implies K = \frac{1}{20}$$

PD Control- Evaluation



$$G(s) = \frac{1}{s^2}$$

Initial design: $KD(s) = \dfrac{10s+1}{20}$

What have we accomplished?

- PM $\approx 90°$ at $\omega_c = 0.5$
- still need to check in Matlab and iterate if necessary

Trade-offs:

- want ω_{BW} to be large enough for fast response (larger $\omega_{\mathrm{BW}} \longrightarrow$ larger $\omega_n \longrightarrow$ smaller t_r), but not too large to avoid noise amplification at high frequencies
- PD control increases slope $\longrightarrow$ increases $\omega_c \longrightarrow$ increases $\omega_{\mathrm{BW}} \longrightarrow$ faster response
- usual complaint: D-gain is not physically realizable, so let's try lead compensation

Lead Compensation: Bode Plot

$$KD(s) = K\frac{s+z}{s+p}, \qquad p \gg z$$

In Bode form:

$$KD(s) = \frac{Kz\left(\frac{s}{z}+1\right)}{p\left(\frac{s}{p}+1\right)}$$

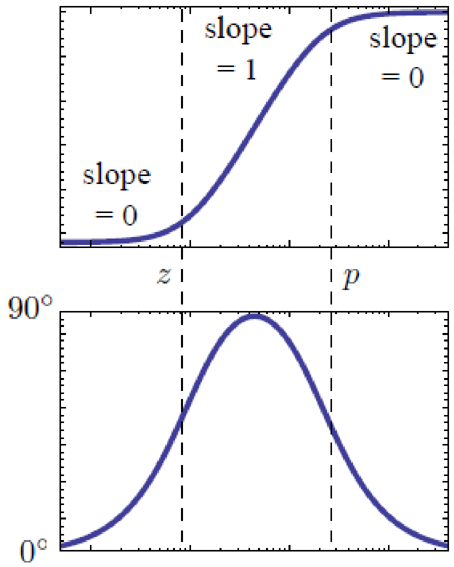
or, absorbing z/p into the overall gain, we have

$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$

Break-points:

- Type 1 zero with break-point at $\omega = z$ (comes first, $z \ll p$)
- Type 1 pole with break-point at $\omega = p$

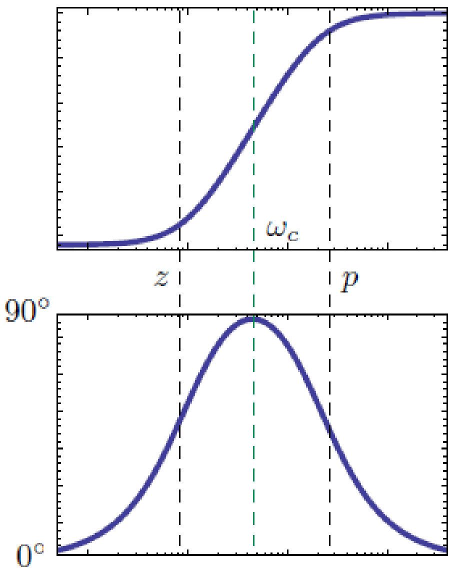
$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$



- magnitude levels off at high frequencies $\Longrightarrow$ better noise suppression
- adds phase, hence the term "phase lead"

Lead Compensation & Phase Margin

$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$



For best effect on PM, ω_c should be halfway between z and p (on log scale):

$$\log \omega_c = \frac{\log z + \log p}{2}$$

$$\text{or } \omega_c = \sqrt{z \cdot p}$$

— geometric mean of z and p

Trade-offs: large $p - z$ means

- large PM (closer to 90°)
- but also bigger M at higher frequencies (worse noise suppression)

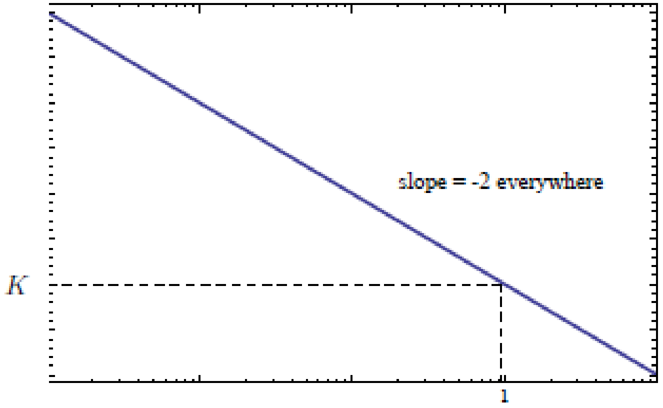
Back to example of double integrators

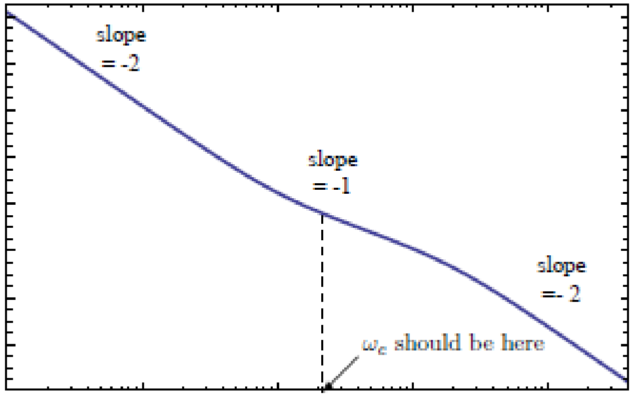
Objectives (same as before):

- stability
- good damping
- ω_{BW} close to 0.5

$KG(s) = \dfrac{K}{s^2}$ (w/o lead):

after adding lead:

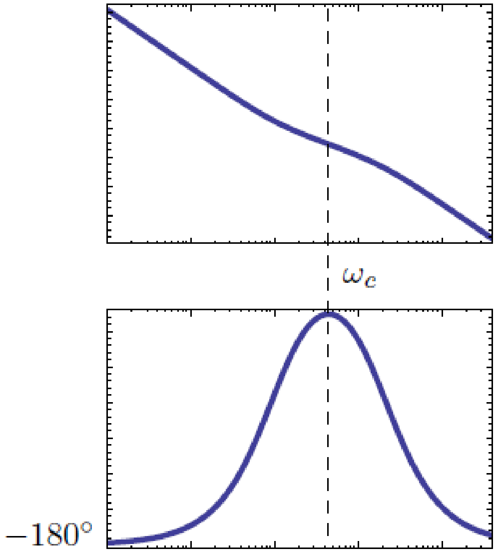




$$\frac{K}{(0.5)^2} = 1 \implies K = \frac{1}{4}$$

— adding lead will increase ω_c!!

Back to example of double integrators



After adding lead with $K = 1/4$, what do we see?

- adding lead increases ω_c
- $\Longrightarrow$ PM $< 90°$
- $\Longrightarrow \omega_{\mathrm{BW}}$ may be $> \omega_c$

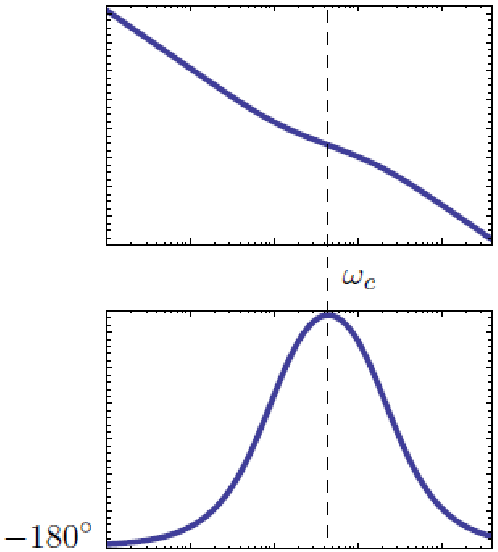
To be on the safe side, we choose a *new value* of K so that

$$\omega_c = \frac{\omega_{\mathrm{BW}}}{2}$$

(b/c generally $\omega_c \le \omega_{\mathrm{BW}} \le 2\omega_c$)

Thus, we want

$$\omega_c = 0.25 \Longrightarrow K = \frac{1}{16}$$



Next, we pick z and p so that ω_c is approximately their geometric mean:

e.g., $z = 0.1$, $p = 2$

$$\sqrt{z \cdot p} = \sqrt{0.2} \approx 0.447$$

Resulting lead controller:

$$K\,D(s) = \frac{1}{16}\,\frac{\frac{s}{0.1}+1}{\frac{s}{2}+1}$$

(may still need to be refined using Matlab)

Lead Controller Design Using Frequency Response

General Procedure

1. Choose K to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   - in general, we should first check PM with the K from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it's not very precise, requires trial & error.

Next Lecture

- Lag Compensation Bode Plot
- Nyquist Plot