ECE 486 Control Systems

Lecture 15: Control Design with Frequency Response: PI & lag, PID and lead-lag

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Checklist

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	Root Locus
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Modeling

Analysis

Design

Root Locus

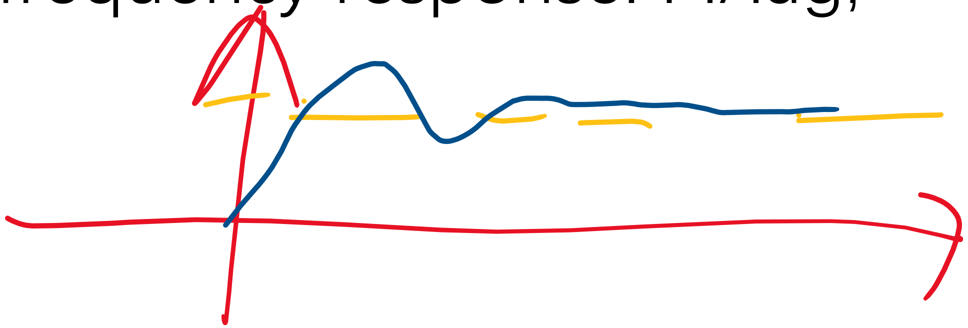
Frequency Response

State-Space

Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	**Control design using frequency response: PD and Lead**	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	Nyquist stability criterion	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	State-Space
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

Plan of Lecture

- Review: Control design using frequency response: PD/lead
- Today: Control design using frequency response: PI/lag, PID/lead+lag



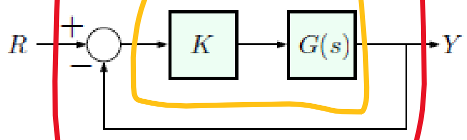
- Goal:
  - understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot;
  - develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

$$\boxed{K\left(\frac{s+z}{s+p}\right)} \rightarrow \boxed{G_p}$$

Review: Bode's Gain-Phase Relationship

$KG \rightarrow \dfrac{1}{1+KG}$

$PM \simeq 100(\zeta)$



Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

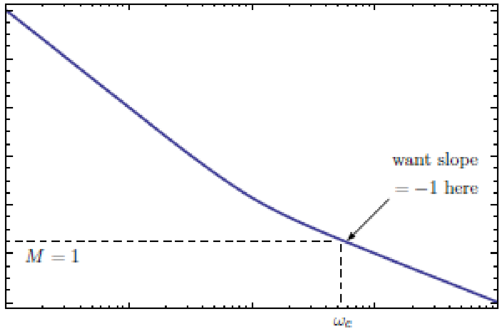
	low freq.	real zero/pole	complex zero/pole
mag. slope	n	up/down by 1	up/down by 2
phase	$n \times 90°$	up/down by 90°	up/down by 180°

We can state this succinctly as follows:

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

This suggests the following rule of thumb:



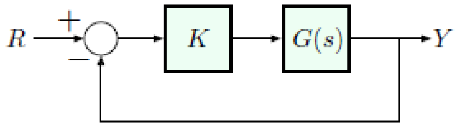
- M has slope -2 at ω_c
  $\Rightarrow \phi(\omega_c) = -180°$
  $\Rightarrow$ bad (no PM)
- M has slope -1 at ω_c
  $\Rightarrow \phi(\omega_c) = -90°$
  $\Rightarrow$ good (PM = 90°)

— this is an important *design guideline*!!

(Similar considerations apply when M-plot has positive slope – depends on the t.f.)

Review: Bode's Gain-Phase Relationship

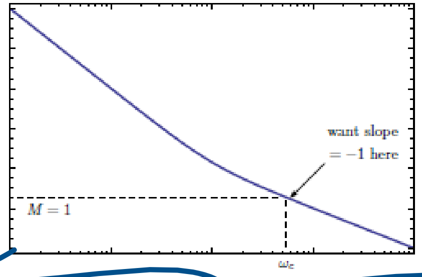
P −80



$$\begin{cases} |KG(j\omega_c)| = 1 \\ \angle G(j\omega_c) = -90^\circ \end{cases} \Rightarrow KG(j\omega_c) = -j$$

M-plot for *open-loop* t.f. KG:

0



Closed-loop t.f.:

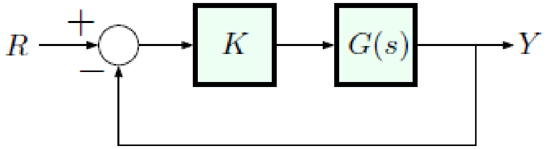
$$T(j\omega_c) = \frac{KG(j\omega_c)}{1 + KG(j\omega_c)} = \frac{\boxed{-j}}{1-j}$$

$$|T(j\omega_c)| = \left|\frac{-j}{1-j}\right| = \frac{1}{\sqrt{2}}$$

$$|T(0)| = \lim_{\omega\to 0} \frac{|KG(j\omega)|}{|1 + KG(j\omega)|} = 1$$

Note: $|KG(j\omega)| \to \infty$ as $\omega \to 0$ ∞ $\Longrightarrow \omega_c = \omega_{\mathrm{BW}}$ (bandwidth)

- If $\mathrm{PM} = 90^\circ$, then $\omega_c = \omega_{\mathrm{BW}}$
- If $PM < 90^\circ$, then $\omega_c \le \omega_{\mathrm{BW}} \le 2\omega_c$ (see FPE)



Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing K (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \quad \Longrightarrow \quad \text{Phase}(\omega_c) \approx -90^\circ$$

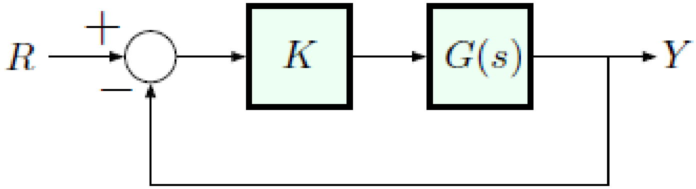
— which gives us PM of 90° and consequently good damping.

0.0001 100 +∞ 100000

0

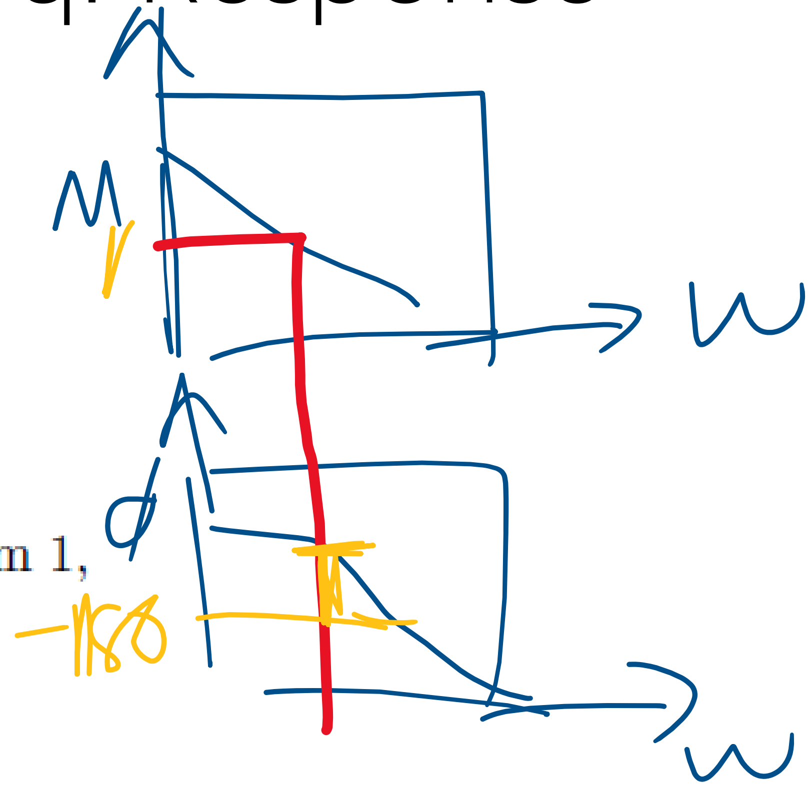
Recall: Frequency Response Control Design

So far, ……



Bode plot relationship suggests that we can shape the time response of the closed-loop system by selecting K

Lead Controller Design with Freq. Response

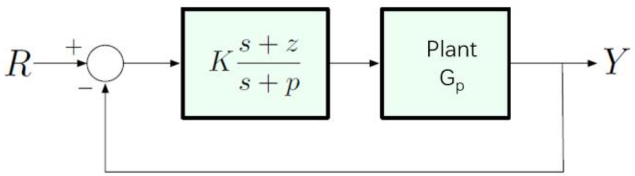


1. Choose K to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   - in general, we should first check PM with the K from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it's not very precise, requires trial & error.

Recall: Dynamic Compensation

Objectives: stabilize the system and satisfy given time response specs using a *stable*, *causal* controller.



Characteristic equation:

$$1 + K \cdot \frac{s+z}{s+p} \cdot G_p = 1 + KL(s) = 0$$

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Recall: Lead & Lag Compensation

$K_p + K_D s$

Consider a general controller of the form

$$K\frac{s+z}{s+p} \qquad — K, z, p > 0 \text{ are design parameters}$$

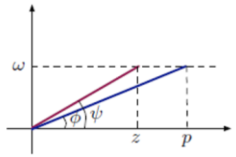
Depending on the relative values of z and p, we call it:

- a lead compensator when $z < p$
- a lag compensator when $z > p$

Why the name "lead/lag?" — think frequency response

$$\angle\frac{j\omega+z}{j\omega+p} = \angle(j\omega+z) - \angle(j\omega+p) = \psi - \phi$$

- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



Approx. PD

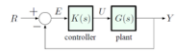
Reminder: we can approximate the D-controller $K_D s$ by

$$K_D\frac{ps}{s+p} \longrightarrow K_D s \text{ as } p \to \infty$$

— here, $-p$ is the *pole* of the controller.

So, we replace the PD controller $K_P + K_D s$ by

$$K(s) = K_P + K_D\frac{ps}{s+p}$$



Closed-loop poles: $1 + \left(K_P + K_D\frac{ps}{s+p}\right)G(s) = 0$

Approx. PI

PI control achieves the objective of stabilization and perfect steady-state tracking of constant references; however, just as with PD earlier, we want a *stable controller*.

Here's an idea:

$$\text{replace } K\frac{s+1}{s} \quad \text{by} \quad K\frac{s+1}{s+p}, \text{ where } p \text{ is small}$$

More generally, if $z = K_I/K_P$, then

$$\text{replace } K\frac{s+z}{s} \quad \text{by} \quad K\frac{s+z}{s+p}, \text{ where } p < z$$

This is lag compensation (or lag control)!

We use lag controllers as dynamic compensators for approximate PI control.

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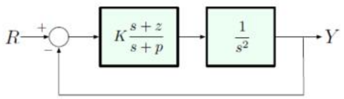
Recall : PD and Lead Control



Controller transfer function is $K\frac{s+z}{s+p}$, where:

$$K = K_P + pK_D, \qquad z = \frac{pK_P}{K_P + pK_D} \xrightarrow{p\to\infty} \frac{K_P}{K_D}$$

so, as $p \to \infty$, z tends to a constant, so we get a lead controller.

We use lead controllers as dynamic compensators for approximate PD control.

To keep things simple, let's set $K_P = K_D$. Then:

$$K = K_P + pK_D = (1+p)K_D$$

$$z = \frac{pK_P}{K_P + pK_D} = \frac{pK_D}{(1+p)K_D} = \frac{p}{1+p} \xrightarrow{p\to\infty} 1$$

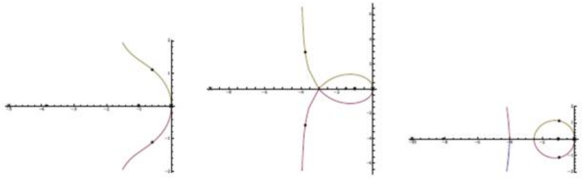
Since we can choose p and z directly, let's take

$$z = 1 \quad \text{and} \quad p \text{ large.}$$

We expect to get behavior similar to PD control.

From what we have seen so far:

- p large — good damping, but bad noise suppression (too close to PD); the branches first break in (meet at the real axis), then break away.
- p small — noise suppression is better, but RL is too close to $j\omega$-axis, which is not good; no break-in for small values of p.
- intermediate values of p — transition between two types of RL; break-in and break-away points are the same.



p=5 p=9 p=10

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Recall PI and Lag Control

$1 + KL(s)$

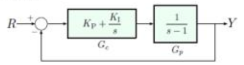
$R \to Y$

PI Control

Case study: plant transfer function $G_p(s) = \frac{1}{s-1}$

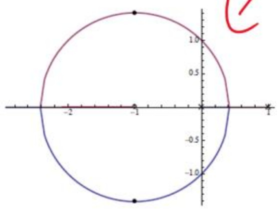
Control objective: stability and constant reference tracking

In earlier lectures, we saw that for perfect steady-state tracking we need PI control



Closed-loop poles are determined by:

$$1 + \left(K_P + \frac{K_I}{s}\right)\left(\frac{1}{s-1}\right) = 0$$



- The system is stable for $K > 1$ (from Routh-Hurwitz)
- For very large K, we get a completely damped system, with *negative real poles*
- Perfect steady-state tracking of constant references:

$$\frac{E}{R} = \frac{1}{1+G_cG_p} = \frac{s(s-1)}{s(s-1)+K(s+1)}$$

DC gain$(R \to E) = 0$ (for $K > 1$)

- However: $1/s$ is not a stable element.

How About:

$$\text{replace } K\frac{s+1}{s} \quad \text{by} \quad K\frac{s+1}{s+p}, \text{ where } p \text{ is small}$$

More generally, if $z = K_I/K_P$, then

$$\text{replace } K\frac{s+z}{s} \quad \text{by} \quad K\frac{s+z}{s+p}, \text{ where } p < z$$

This is lag compensation (or lag control)!

We use lag controllers as dynamic compensators for approximate PI control.

Tracking a constant reference: if the stability conditions

$$K > 1 - p, \quad Kz > p$$

are satisfied, then the steady-state error is

$$e(\infty) = \frac{1}{1 - \frac{Kz}{p}}$$

— this will be close to zero (and negative) if $\frac{Kz}{p}$ is large.

Lag compensation *does not* give perfect tracking (indeed, it does not change system type), but we can get as good a tracking as we want by playing with K, z, p. On the other hand, unlike PI, lag compensation gives a stable controller.

$$L(s) = \frac{s+1}{(s+p)(s-1)}$$

Intuition: By choosing p very close to zero, we can make the root locus arbitrarily close to PI root locus (stable for large enough K). Let's check:

Try $p = 0.1$

Compare to PI root locus:

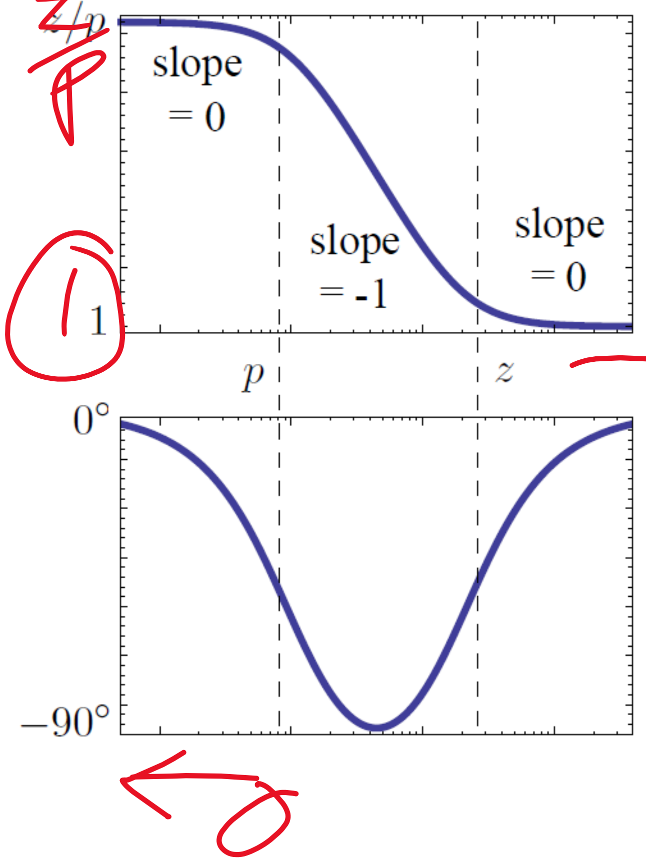
What do we see? Compared to PD vs. lead, there is no qualitative change in the shape of RL, since we are not changing #(poles) or #(zeros).

Lag Compensation: Bode Plot

We've seen root locus, let's look at the Bode plot

$s \to j\omega$

$$D(s) = \frac{s+z}{s+p} = \frac{z}{p}\frac{\frac{s}{z}+1}{\frac{s}{p}+1}, \qquad z \gg p$$



- $\frac{j\omega+z}{j\omega+p} \xrightarrow{\omega\to\infty} 1$ so $M \to 1$ at high frequencies

$\omega \to 0 \quad M = \frac{z}{p}$

- subtracts phase, hence the term "phase lag"

Example

PM ≃ 100ζ → Damping ratio

$$G(s) = \frac{1}{(s+0.2)(s+0.5)} \overset{\text{Bode form}}{=} \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

Objectives:

- PM $\geq 60°$ → shape time response
- $e(\infty) < 10\%$ for constant reference (closed-loop tracking error) → ss error

reference track

Strategy:

- we will use lag

$$KD(s) = K\frac{s+z}{s+p}, \qquad z \gg p$$

R → Y
DC gain 1

- z and p will be chosen to get good tracking
- PM will be shaped by choosing K
- this is different from what we did for lead (used p and z to shape PM, then chose K to get desired bandwidth spec)

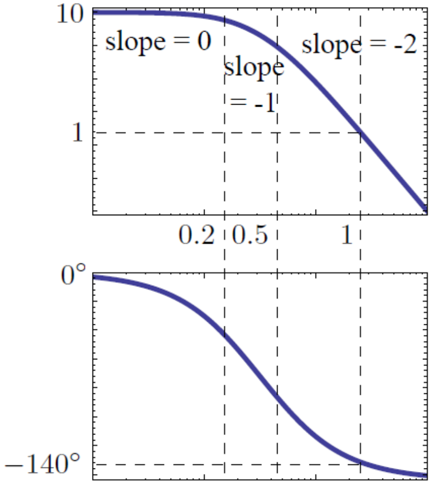
R → E, DC gain = 0

Example

$$G(s) = \frac{1}{(s+0.2)(s+0.5)} \overset{\text{Bode form}}{=} \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

Step 1: Choose K to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:



- from Matlab, $\omega_c \approx 1$
- PM $\approx 40°$
- we want PM $= 60°$

$$\phi = -120° \quad \text{at } \omega \approx 0.573$$
$$M = 2.16$$

— need to decrease K to $1/2.16$

A conservative choice (to allow some slack) is $K = 1/2.5 = 0.4$, gives $\omega_c \approx 0.52$, PM $\approx 65°$

Step 2: Choose z & p to Shape Tracking Error

So far: $KG(s) = \dfrac{0.4 \cdot 10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$

$$e(\infty) = \left.\frac{1}{1+KG(s)}\right|_{s=0} = \frac{1}{1+4} = \frac{1}{5} = 20\% \quad \text{(too high)}$$

To have $e(\infty) \le 10\%$, need $KD(0)G(0) \ge 9$:

$$e(\infty) = \frac{1}{1+KD(0)G(0)} \le \frac{1}{1+9} = 10\%.$$

So, we need

$$D(0) = \left.\frac{s+z}{s+p}\right|_{s=0} = \frac{z}{p} \ge \frac{9}{4} = 2.25 \qquad \text{— say, } z/p = 2.5$$

Not to distort PM and ω_c, let's pick z and p an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05$, $p = 0.02$

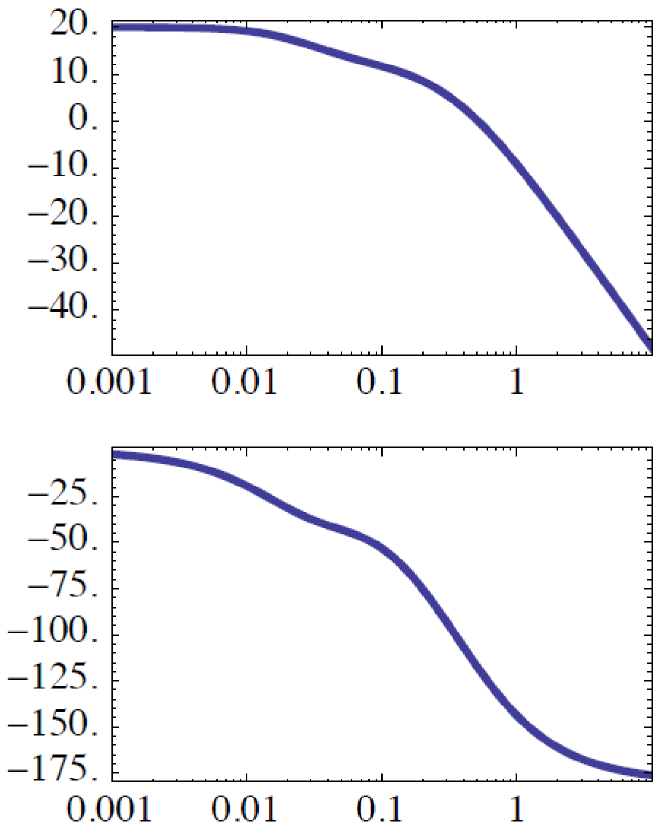
Overall Design

Plant:

$$G(s) = \frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

Controller:

$$KD(s) = 0.4\frac{s+0.05}{s+0.02}$$

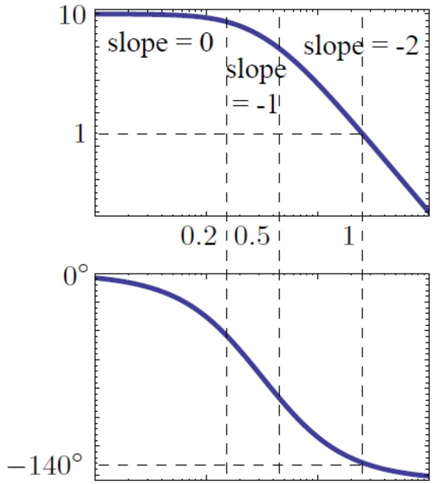


— the design still needs a bit of refinement ...

Lead & Lag Compensation

Let's combine the advantages of PD/lead and PI/lag.

Back to our example: $G(s) = \dfrac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$



- from Matlab, $\omega_c \approx 1$
- PM $\approx 40°$

New objectives:

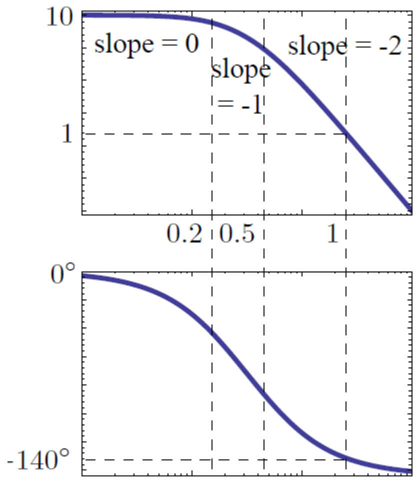
- $\omega_{\mathrm{BW}} \geq 2$
- PM $\geq 60°$
- $e(\infty) \leq 1\%$ for const. ref.

What we got before, with lag only:

- Improved PM by adjusting K to decrease ω_c.
- This gave $\omega_c \approx 0.5$, whereas now we want a larger ω_c (recall: $\omega_{\mathrm{BW}} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.

Lead & Lag Compensation



Step 1. Choose K to get $\omega_c \approx 2$ (before lead)

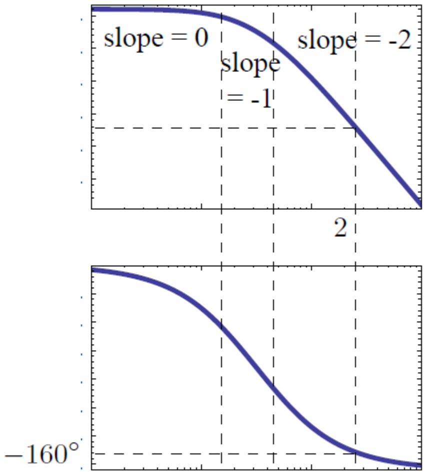
Using Matlab, can check:

at $\omega = 2$, $\quad M \approx 0.24$ (with $K = 1$)

— need $K = \dfrac{1}{0.24} \approx 4.1667$

— choose $K = 4$
(gives ω_c slightly < 2, but still ok).

$K = 4$



Step 2. Decide how much phase lead is needed, and choose z_{lead} and p_{lead}

Using Matlab, can check:

$$\text{at } \omega = 2, \qquad \phi \approx -160^\circ$$

— so PM = 20°

(in fact, choosing $K = 4$ made things worse: it increased ω_c and consequently decreased PM)

We need at least 40° phase lead!!

The choice of lead pole/zero must satisfy

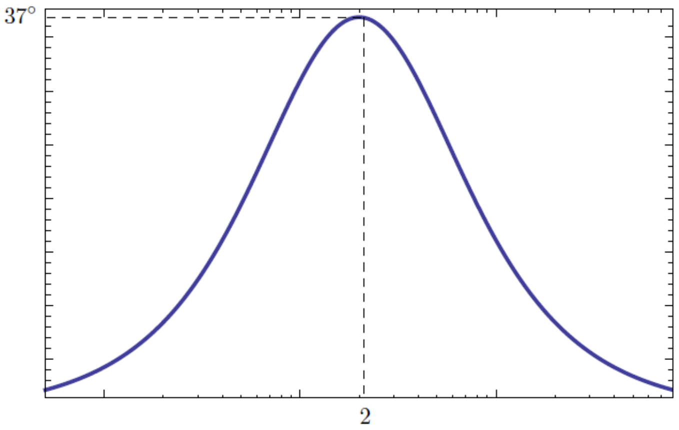
$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Let's try $z_{\text{lead}} = 1$ and $p_{\text{lead}} = 4$ $\qquad D(s) = \dfrac{s+1}{\frac{s}{4}+1}$



Phase lead = 37° — not enough!!

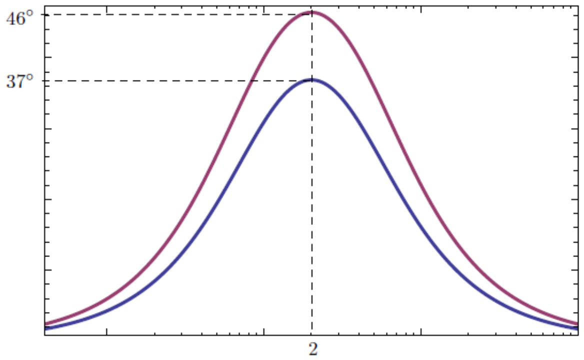
Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

The choice of $z_{\text{lead}} = 1$, $p_{\text{lead}} = 4$ gave phase lead $= 37°$.

Need to space z_{lead} and p_{lead} farther apart:

$$\begin{cases} z_{\text{lead}} = 0.8 \\ p_{\text{lead}} = 5 \end{cases} \implies \text{phase lead } = 46°$$



Lead & Lag Compensation

Step 3. Evaluate steady-state tracking and choose $z_{\text{lag}}, p_{\text{lag}}$ to satisfy specs

So far:

$$\underbrace{K\,D(s)}_{\substack{\text{lead}\\ \text{only}}}G(s) = 4\frac{\frac{s}{0.8}+1}{\frac{s}{5}+1}\cdot\frac{10}{\left(\frac{s}{0.2}+1\right)\left(\frac{s}{0.5}+1\right)}$$

$$KD(0)G(0) = 40 \implies e(\infty) = \frac{1}{1+KD(0)G(0)} = \frac{1}{1+40}$$

— this is not small enough: need $1\% = \frac{1}{100} = \frac{1}{1+99}$

We want $D(0) \geq \frac{99}{40}$ with lag $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$ will do

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$.

We can stick with our previous design:

$$z_{\text{lag}} = 0.05, \qquad p_{\text{lag}} = 0.02$$

Overall controller:

lead

$$\underbrace{4\frac{\frac{s}{0.8}+1}{\frac{s}{5}+1}}_{\substack{\text{lead (with}\\ \text{gain } K\,=\,4 \text{ absorbed)}}}\cdot\underbrace{\frac{s+0.05}{s+0.02}}_{\substack{\text{lag (not in}\\ \text{Bode form)}}}$$

lag

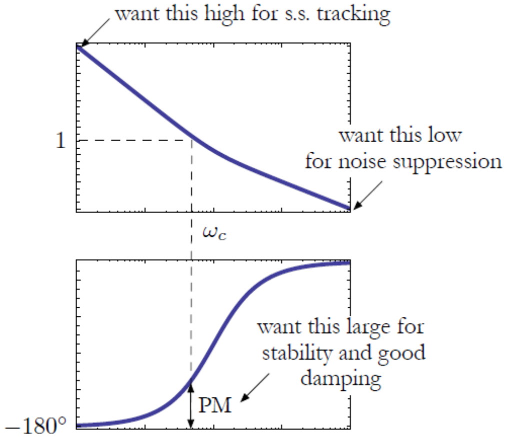
(Note: we don't rewrite lag in Bode form, because $z_{\text{lag}}/p_{\text{lag}}$ is not incorporated into K.)

Frequency Domain Design Pros & Cons

Advantages

Design based on Bode plots is good for:

- easily visualizing the concepts



- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)

Disadvantages

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given K is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given K is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the next lecture.