ECE 486 Control Systems

Lecture 16: Nyquist Stability

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

Checklist

Wk	Topic	Ref.
1	✓ Introduction to feedback control	Ch. 1
	✓ State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	✓ Linear systems and their dynamic response	Section 3.1, Appendix A
	✓ Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	✓ National Holiday Week	
4	✓ System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	✓ Transient response specifications	Sections 3.3, 3.14, lab manual
5	✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control	Section 4.1-4.3, lab manual
6	✓ Review A	
	✓ Term Test A	
7	✓ Introduction to Root Locus design method	Ch. 5
	✓ Root Locus continued; introduction to dynamic compensation	Root Locus
8	✓ Lead and lag dynamic compensation	Ch. 5
	✓ Introduction to frequency-response design method	Sections 5.1-5.4, 6.1

Modeling

Analysis

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

Wk	Topic	Ref.
9	**Bode plots for three types of transfer functions**	Section 6.1
	Stability from frequency response; gain and phase margins	Section 6.1
10	**Control design using frequency response: PD and Lead**	Ch. 6
	Control design using frequency response continued; PI and lag, PID and lead-lag	Frequency Response
11	**<u>Nyquist stability criterion</u>**	Ch. 6
	Nyquist stability criterion continued; gain and phase margins from Nyquist plots	Ch. 6
12	Review B	
	Term Test B	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	State-Space
15	Joint observer and controller design by dynamic output feedback; separation principle	Ch. 7
	In-class review	Ch. 7
16	END OF LECTURES: Revision Week	
	Final	

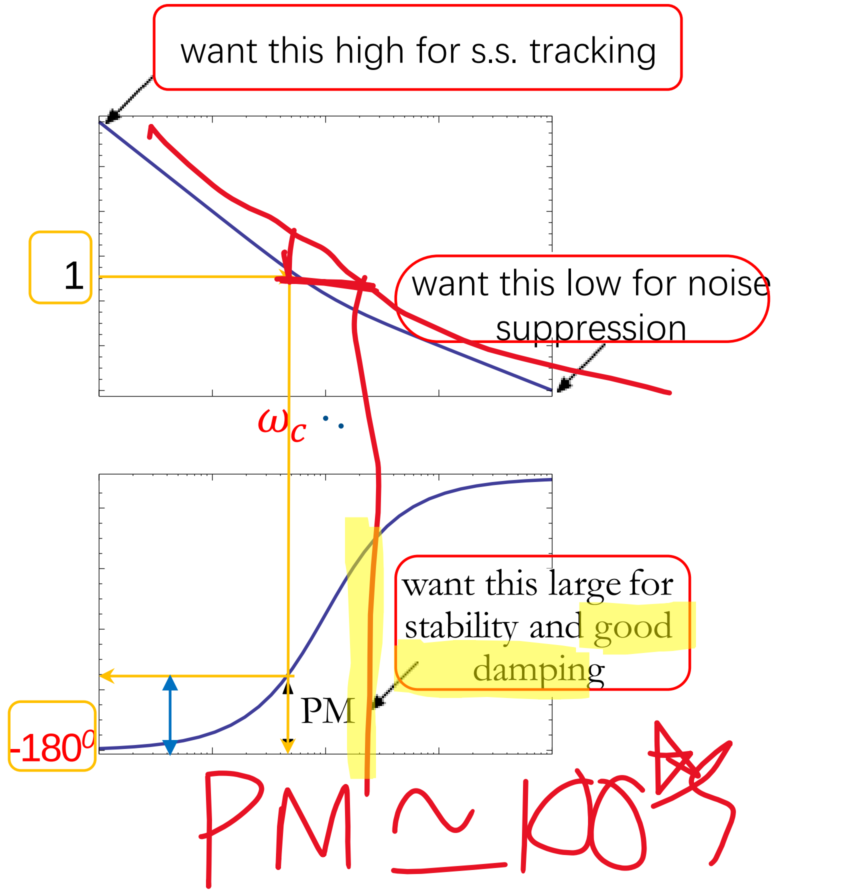
Review: Frequency Domain Design Method

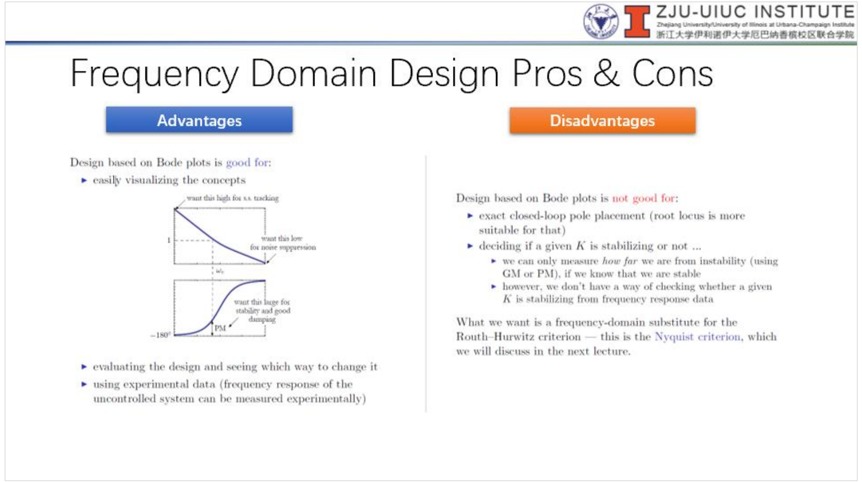
GM , PM   R → | ~~LTI~~ | → Y   e(∞) → 0

control

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system measured empirically)




Frequency Domain Design Pros & Cons

Advantages	Disadvantages
Design based on Bode plots is good for: easily visualizing the concepts; evaluating the design and seeing which way to change it; using experimental data (frequency response of the uncontrolled system can be measured experimentally)	Design based on Bode plots is not good for: exact closed-loop pole placement (root locus is more suitable for that); deciding if a given K is stabilizing or not ... we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable; however, we don't have a way of checking whether a given K is stabilizing from frequency response data. What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the next lecture.

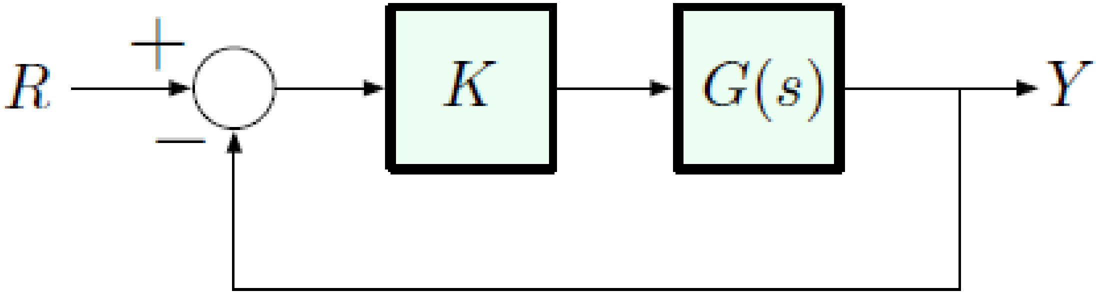
Review: Frequency Domain Design Method

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given K is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given
    - K is stabilizing from frequency response data

Nyquist criterion- A frequency-domain substitute for the Routh–Hurwitz criterion

Nyquist Stability Criterion



Goal: count the number of RHP poles (if any) of the closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)}$$

based on frequency-domain characteristics of the plant transfer function $G(s)$

Nyquist Plot

$s^n + a_1 s^{n-1} \cdots$

$(s - p_1)(s - p_2) \cdots$

Consider an arbitrary *strictly proper* transfer function H:

$$H(s) = \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}, \qquad m < n$$

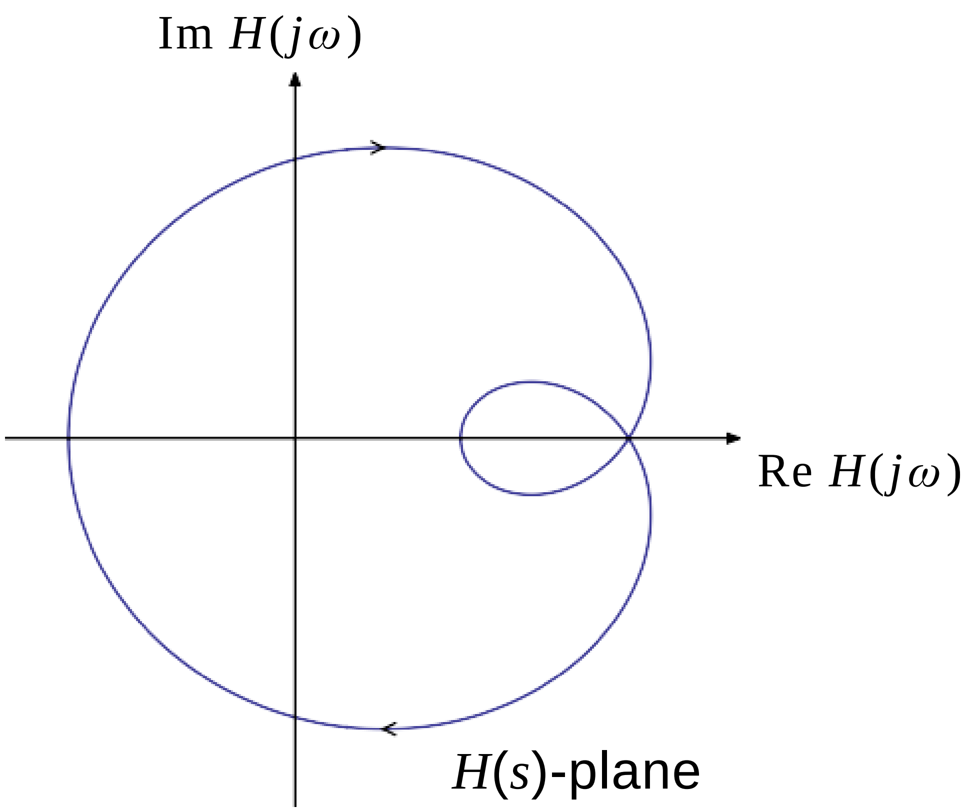
Nyquist

$s = j\omega$

Bode

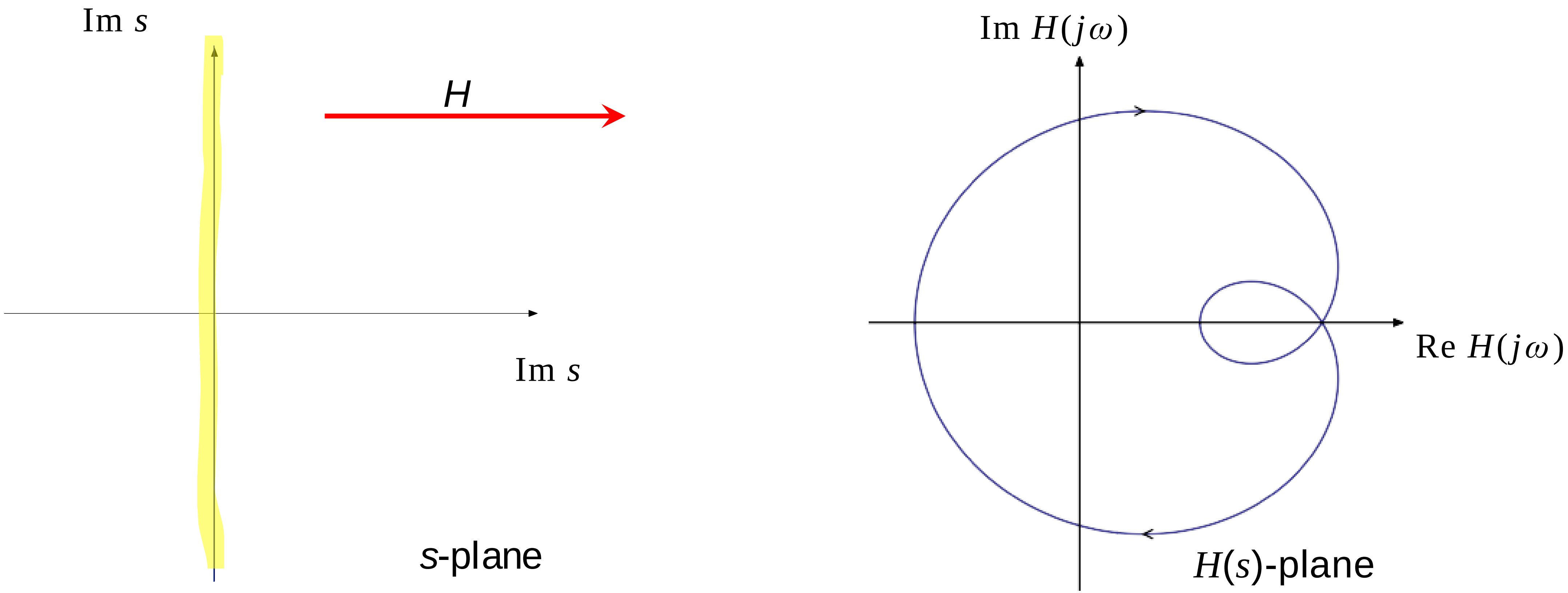
Nyquist plot: Im $H(j\omega)$ vs. Re $H(j\omega)$
as ω varies from $-\infty$ to ∞

$|H(j\omega)|, \angle H(j\omega)$



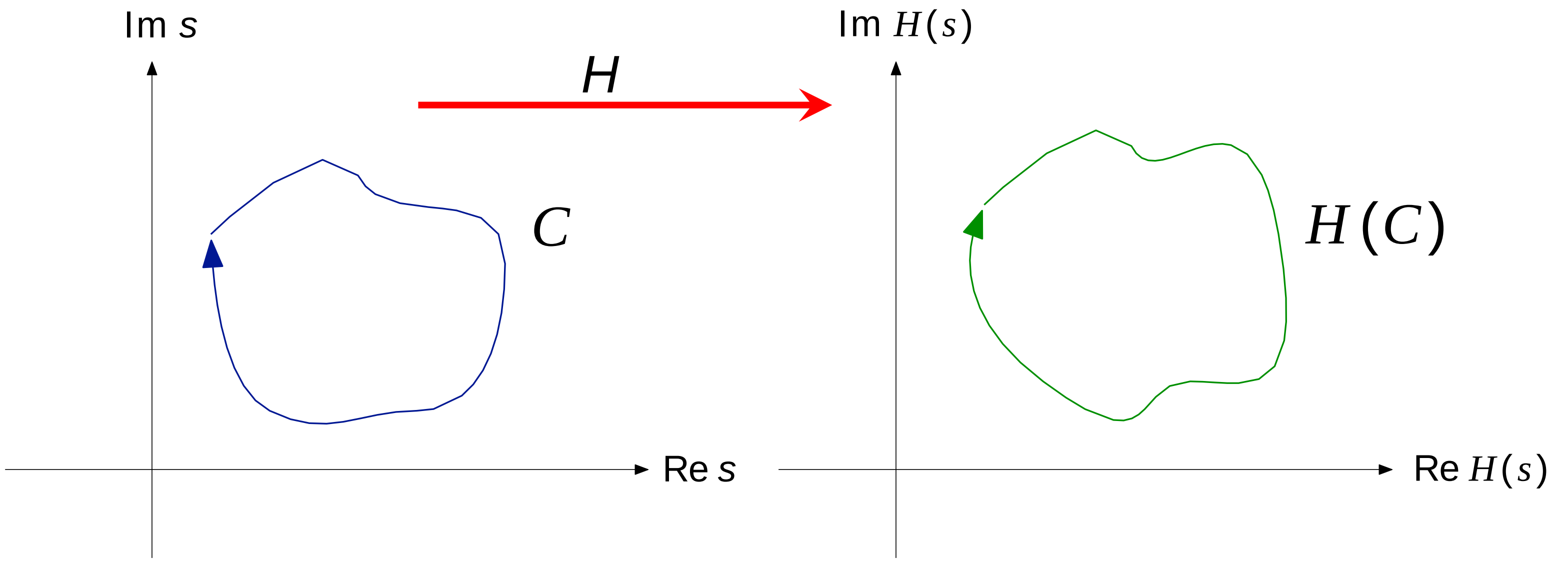
Nyquist Plot: Mapping of the s-Plane

- View the Nyquist plot of H as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \to \mathbb{C}$



Transformation of a Closed Contour Under H

If we choose any closed curve (or *contour*) C on the left, it will get mapped by H to some other curve (contour) on the right:



Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!

Phase of H Along a Contour

$M_1 e^{j\theta_1} M_2 e^{j\theta_2}$

$\angle(\theta_1 + \theta_2)$

products

For any $s \in \mathbb{C}$, the phase (or *argument*) of $\boxed{H(s)}$ is

$$\angle H(s) = \angle \frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

$$= \boxed{\sum_{i=1}^{m}} \angle(s - z_i) \boxed{-} \sum_{j=1}^{n} \angle(s - p_j)$$

$$= \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j$$

Interested in how $\angle H(s)$ changes as s traverses a closed, clockwise (CW) oriented contour C in the complex plane.

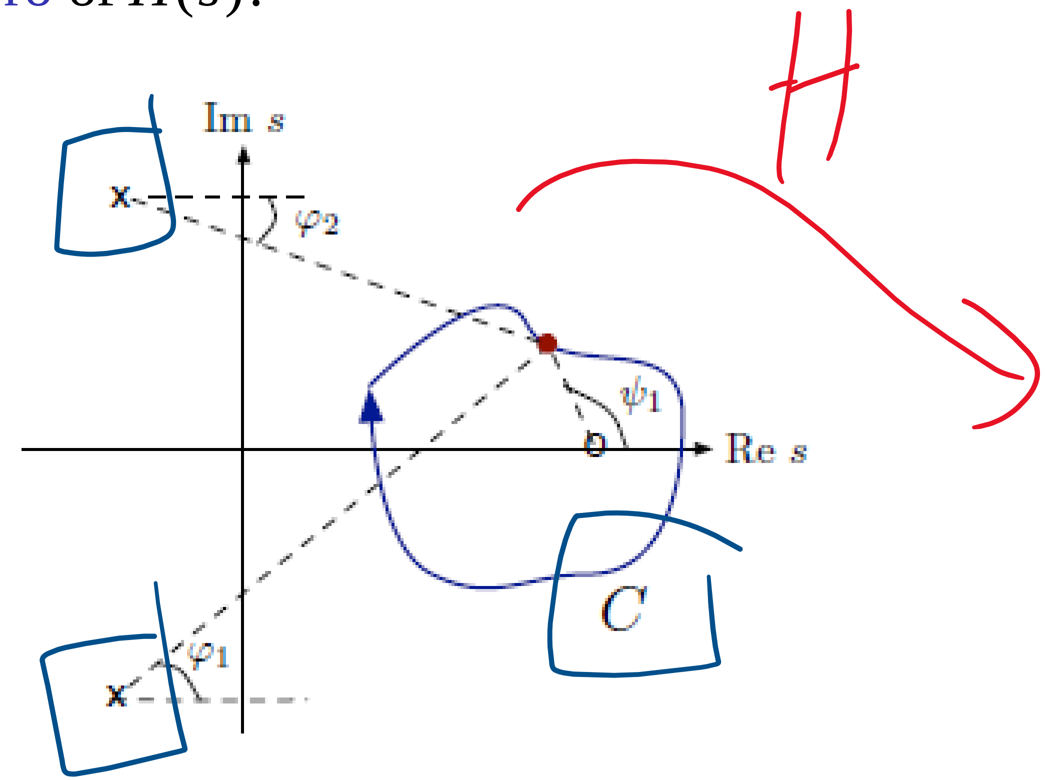
Look at several cases, depending on how the contour is located relative to poles and zeros of H.

Case 1: Contour Encircles a Zero

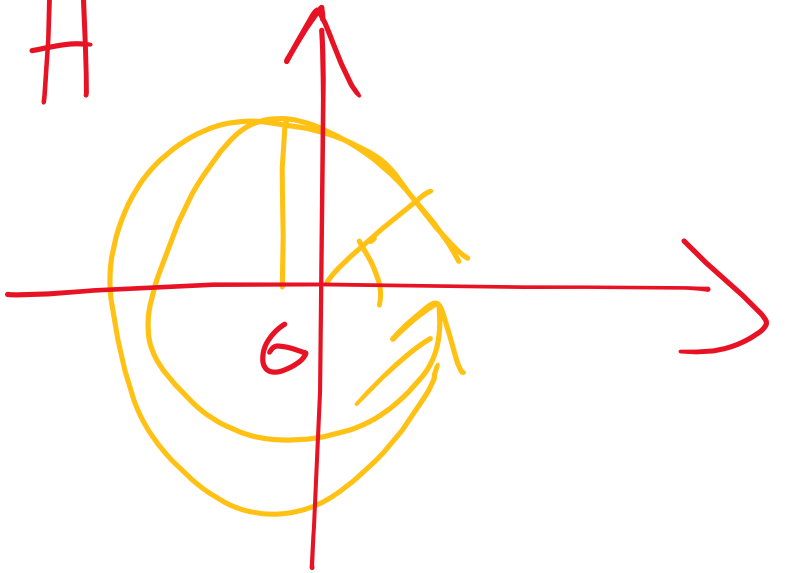
Suppose that C is a closed, CW-oriented contour in $\mathbb{C}$ that encircles a zero of $H(s)$:

$\Sigma\psi - \Sigma\phi$

360° $\quad$ 0

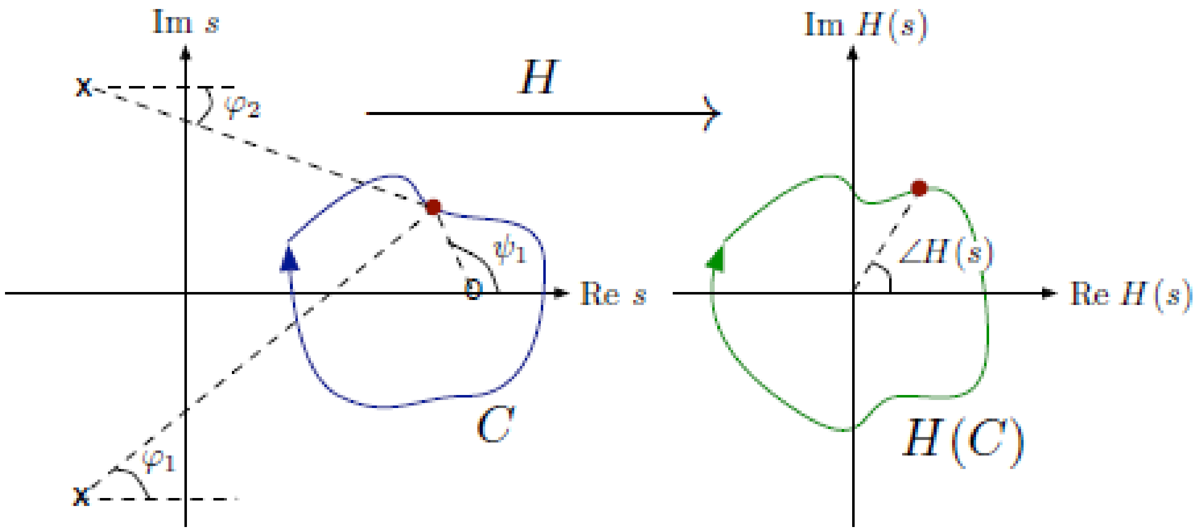


H



How does $\angle H(s)$ change as we go around C?

Case 1: Contour Encircles a Zero



How does $\angle H(s)$ change as we go around C?

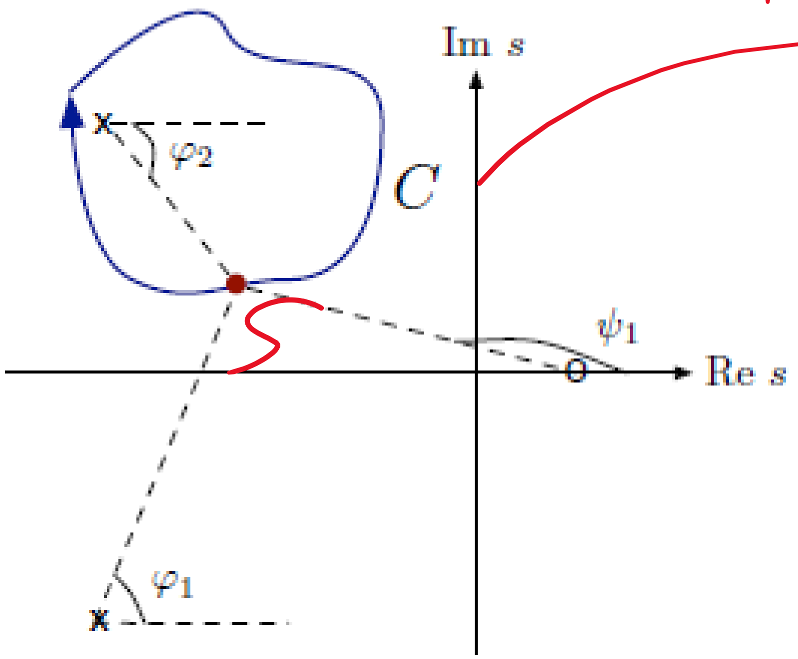
Angles from s to poles/zeros of H:

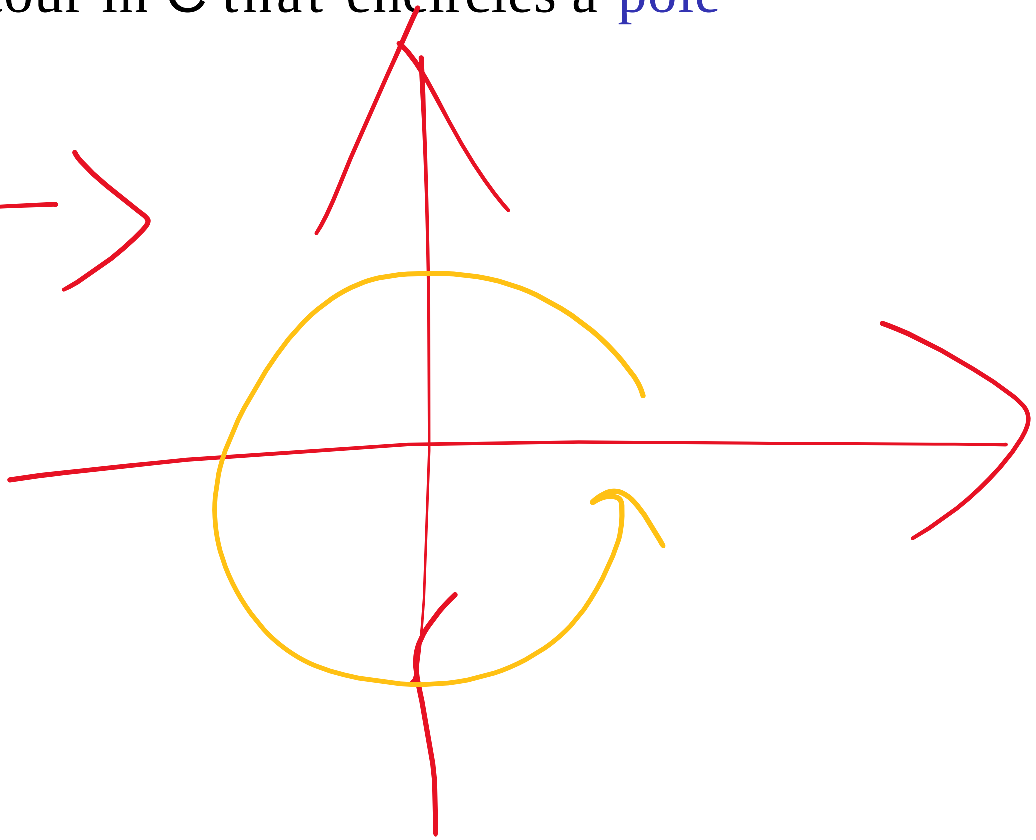
- ϕ_1 and ϕ_2 return to their original values
- ψ_1 registers a net change of -360°
- therefore, $\angle H(s)$ registers a net change of -360°
  - $H(C)$ encircles the origin once, clockwise (CW)

Case 2: Contour Encircles a Pole

Suppose that C is a closed, ↻-oriented contour in $\mathbb{C}$ that encircles a pole of $H(s)$:

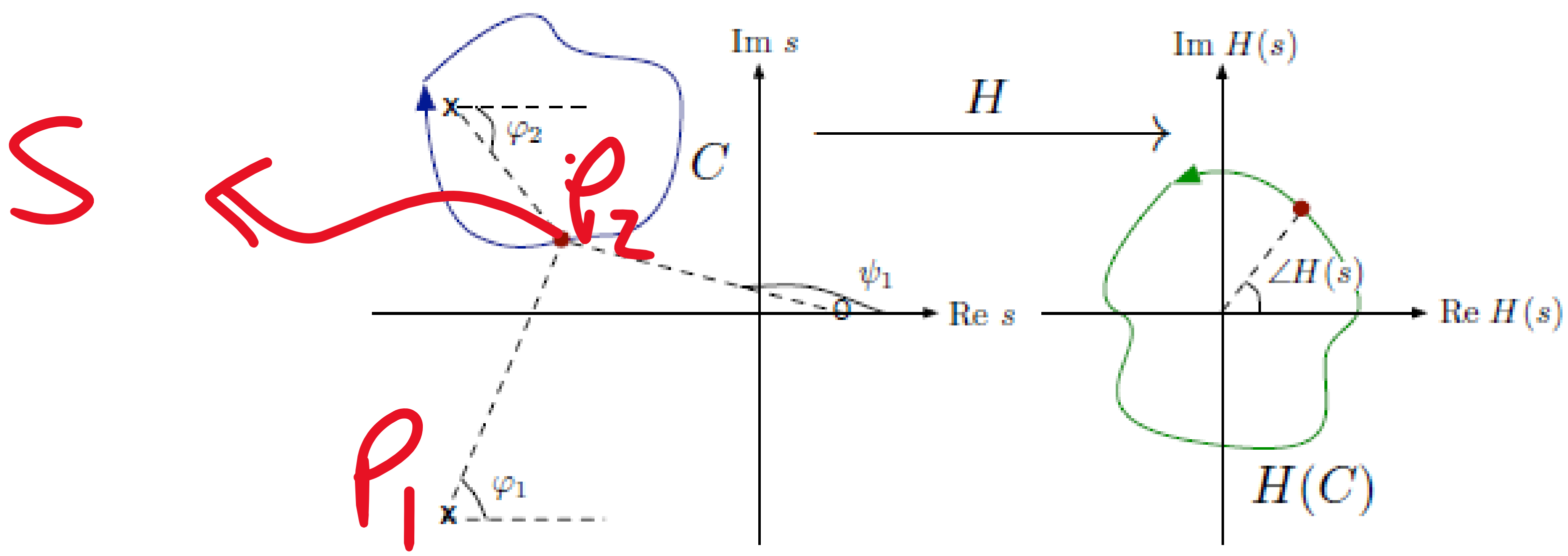
$H(s)$.





How does $\angle H(s)$ change as we go around C?

Case 2: Contour Encircles a Pole



$\sum \psi_1 - \sum \phi_{1,2}$

$0 - (-360)$

$360°$

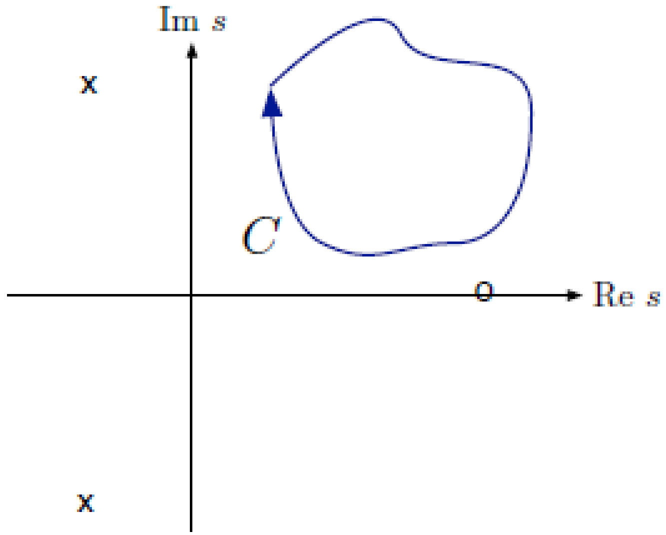
How does $\angle H(s)$ change as we go around C?

Let's see what happens to angles from s to poles/zeros of H:

- ϕ_1 and ψ_1 return to their original values
- ϕ_2 picks up a net change of -360°
- therefore, $\angle H(s)$ picks up a net change of 360°, so $H(C)$ encircles the origin once counterclockwise ($\cancel{O}$)

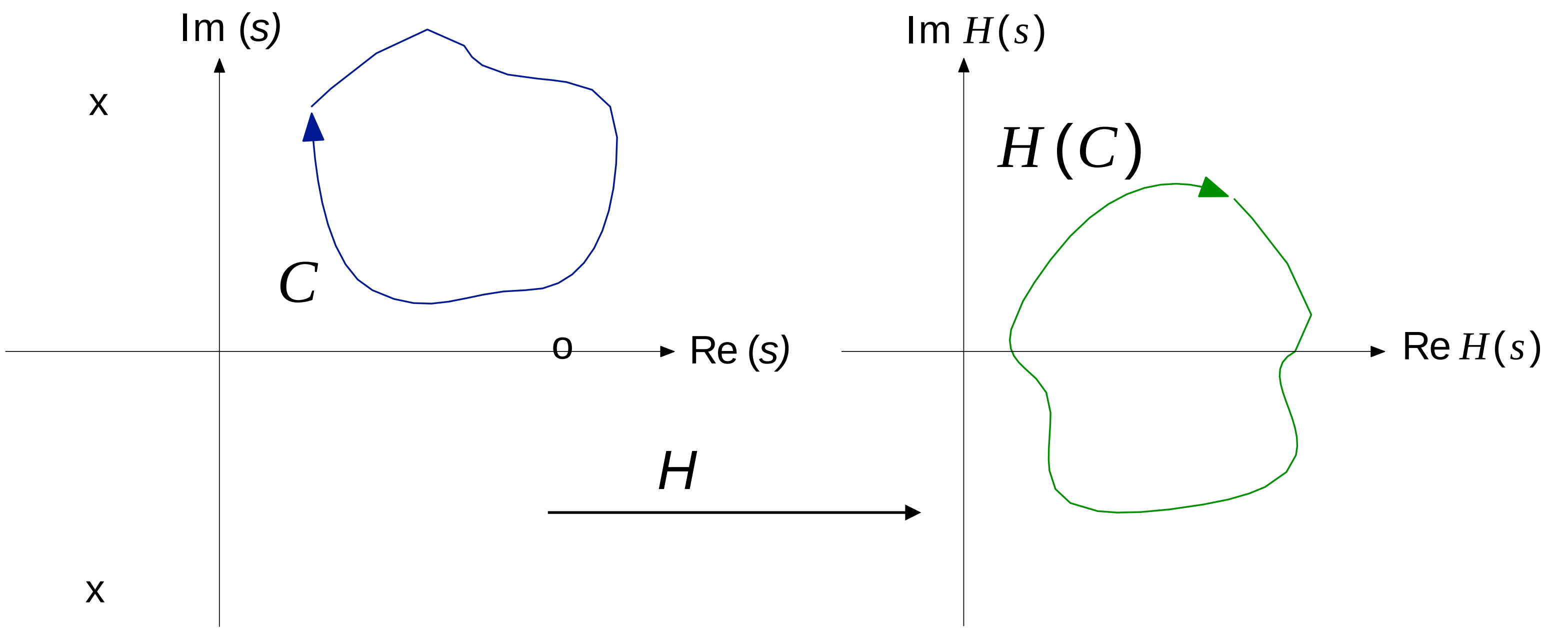
Case 3: Contour Encircles No Poles or Zeros

Suppose that C is a closed, CW-oriented contour in $\mathbb{C}$ that does not encircle any poles or zeros of $H(s)$:



How does $\angle H(s)$ change as we go around C?

Case 3: Contour Encircles No Poles or Zeros



How does $\angle H(s)$ change as we go around C?

Let's see what happens to angles from s to poles/zeros of H:

- ϕ_1, ϕ_2, ψ_1 all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin

The Argument Principle

These special cases all lead to the following general result:

The Argument Principle. Let C be a closed, clockwise $\circlearrowright$ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of C under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$

Then:

$$\begin{aligned}&\#(\text{clockwise encirclements} \circlearrowright \text{ of } 0 \text{ by } H(C)) \\ &= \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).\end{aligned}$$

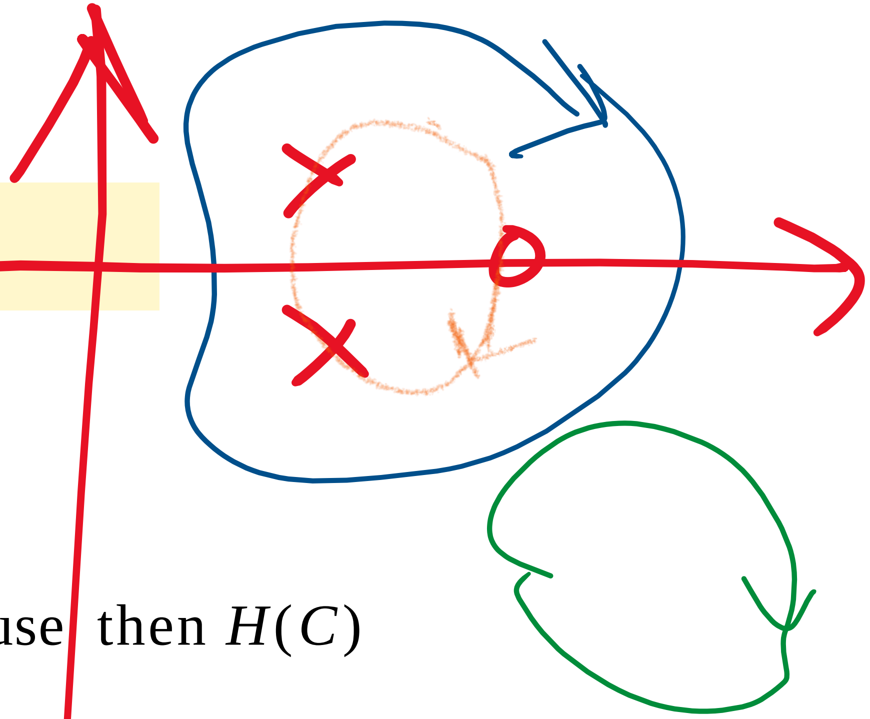
More succinctly,

$$N = Z - P$$

* will see the reason for this later ...

The Argument Principle

$$N = Z - P$$

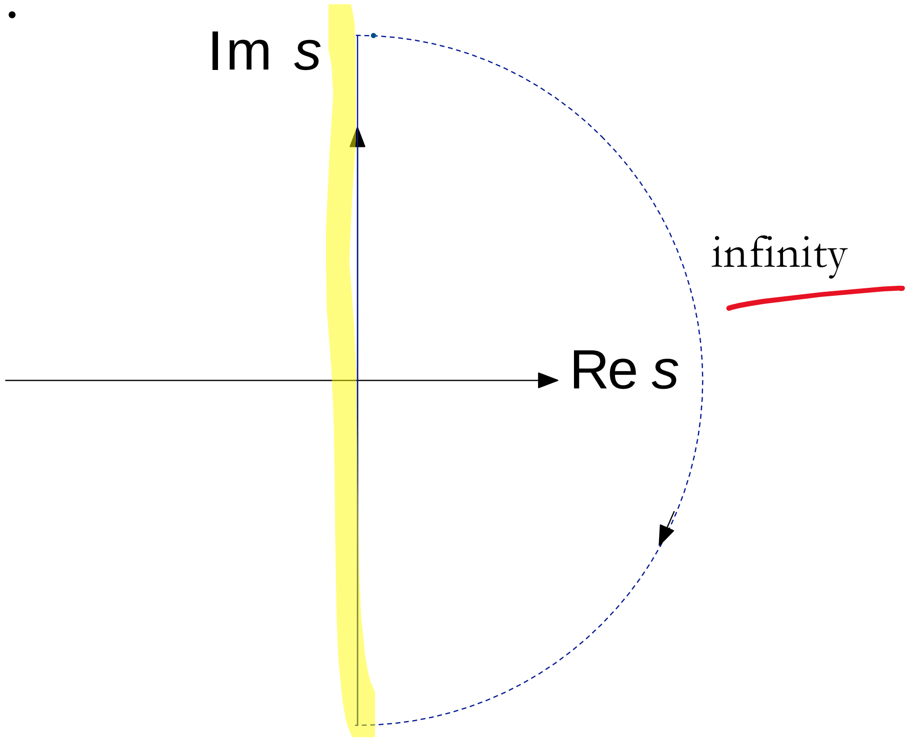


- If $N < 0$, it means that $H(C)$ encircles the origin counterclockwise (O).
- We do not want C to pass through any pole of H because then $H(C)$ would not be defined.
- We also do not want C to pass through any zero of H because then $0 \in H(C)$, so #(encirclements) is not well-defined.

Argument Principle to Nyquist Criterion

- We are interested in RHP poles, so let's choose a suitable contour C that *encloses the RHP*:

Harry Nyquist (1889–1976)



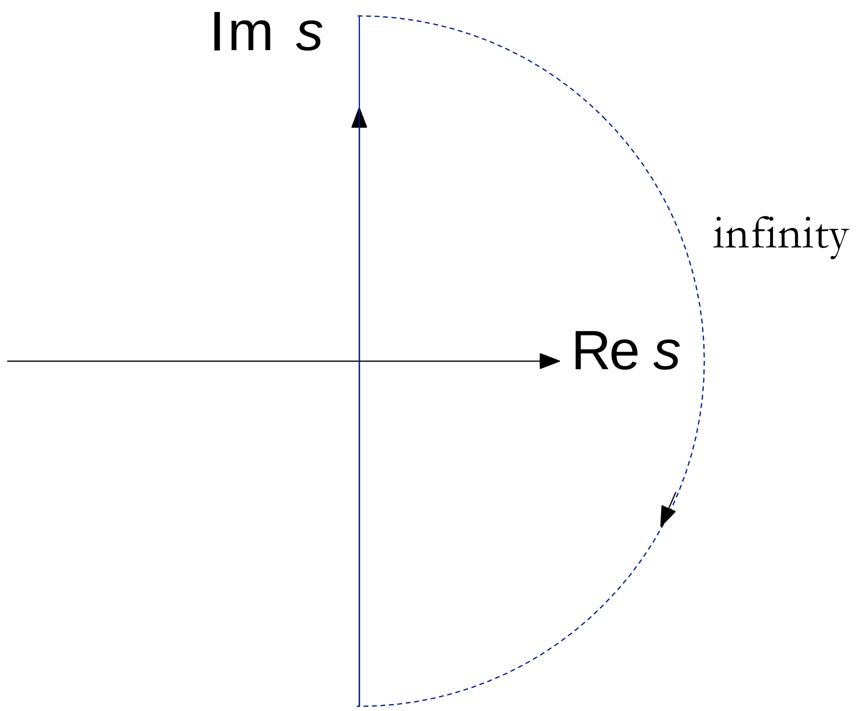
- From now on, C = imaginary axis plus the "path around infinity."
- If H is strictly proper, then $H(\infty) = 0$.

Argument Principle to Nyquist Criterion

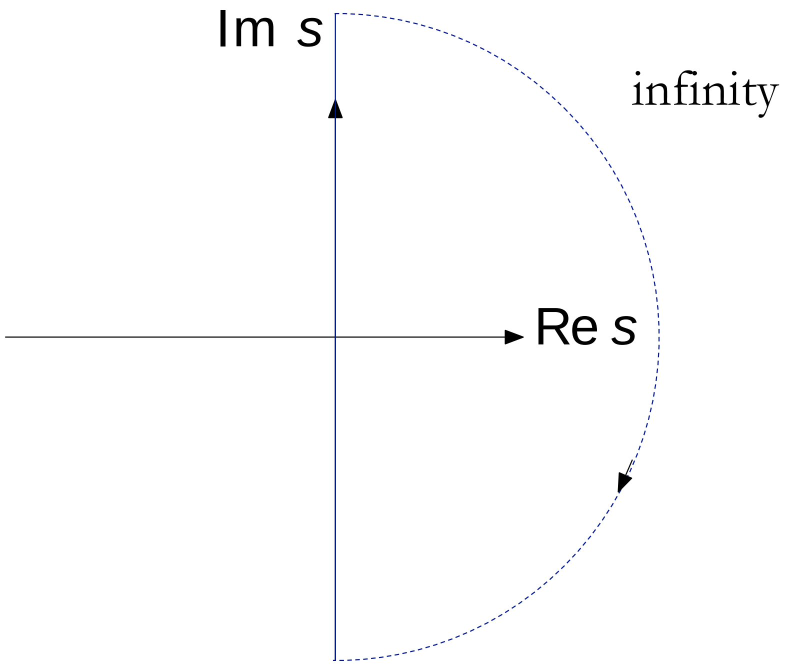
With this choice of C,

$$H(C) = \text{ Nyquist plot of } H$$

(image of the imaginary axis under the map $H : \mathbb{C} \to \mathbb{C}$; if H is strictly proper, $0 = H(\infty)$)



Argument Principle to Nyquist Criterion

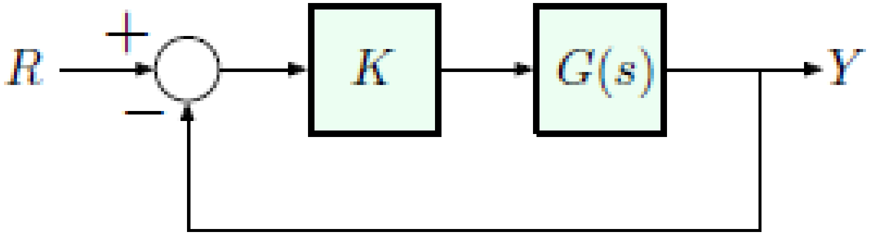


$$H(C) = \text{Nyquist plot of } H$$

We are interested in RHP roots of $1 + KG(s)$, where G is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$

Argument Principle to Nyquist Criterion



Examining the Nyquist plot of $H(s) = 1 + KG(s)$.

By the argument principle,

$$N = Z - P,$$

where $N = \#($CW encirclements of 0 by Nyquist plot of $1 + KG(s))$,

$Z = \#($zeros of $1 + KG(s)$ inside $C)$,

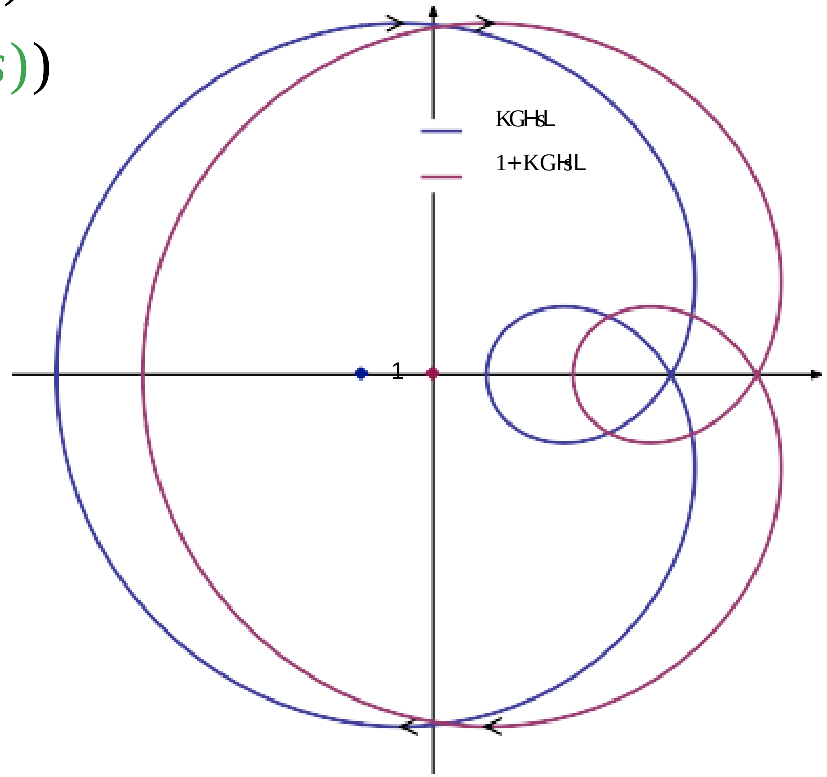
$P = \#($poles of $1 + KG(s)$ inside $C)$

Now we extract information about RHP roots of $1 + KG(s)$

Nyquist Criterion: N

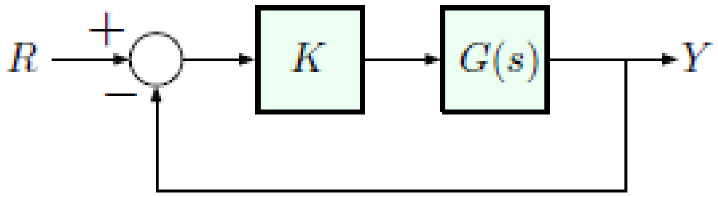
$$\begin{aligned} N &= \#(\text{CW encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s)) \\ &= \#(\text{CW encirclements of } -1 \text{ by Nyquist plot of } KG(s)) \\ &= \#(\text{CW encirclements of } -1/K \text{ by Nyquist plot of } G(s)) \end{aligned}$$

$N = Z - P$



— can be read off the Nyquist plot of the *open-loop* t.f. G!!

Nyquist Criterion: Z



$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \leq \deg(p)$$

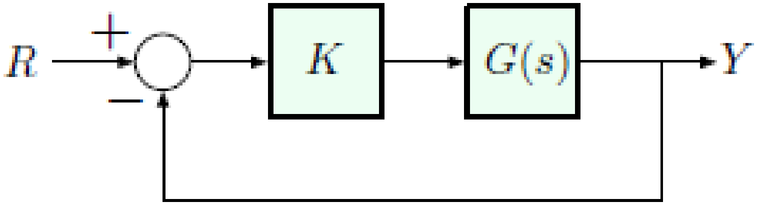
$$1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}$$

$$\text{closed-loop t.f.} = \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)}$$

Therefore:

$$\begin{aligned} Z &= \#(\text{zeros of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP zeros of } 1 + KG(s)) \\ &= \#(\text{RHP closed-loop poles}) \end{aligned}$$

Nyquist Criterion: P



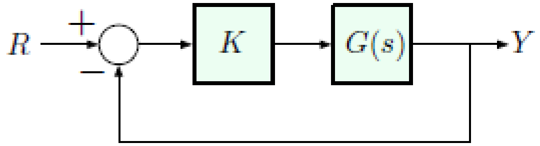
$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \leq \deg(p)$$

$$1 + KG(s) = 1 + K\frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}$$

Therefore:

$$\begin{aligned} P &= \#(\text{poles of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP poles of } 1 + KG(s)) \\ &= \#(\text{RHP roots of } p(s)) \\ &= \#(\text{RHP open-loop poles}) \end{aligned}$$

The Nyquist Theorem



Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

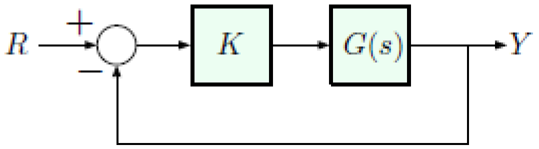
$N = Z - P$ CW

$\#($O of $-1/K$ by Nyquist plot of $G(s))$

$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$

* Easy to fix: draw an infinitesimally small circular path that goes *around* the pole and stays in RHP

The Nyquist Stability Criterion



$$\underbrace{N}_{\#(\circlearrowright \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

$$Z = N + P$$

$$Z = 0 \quad \Longrightarrow N = -P$$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

The Nyquist Stability Criterion

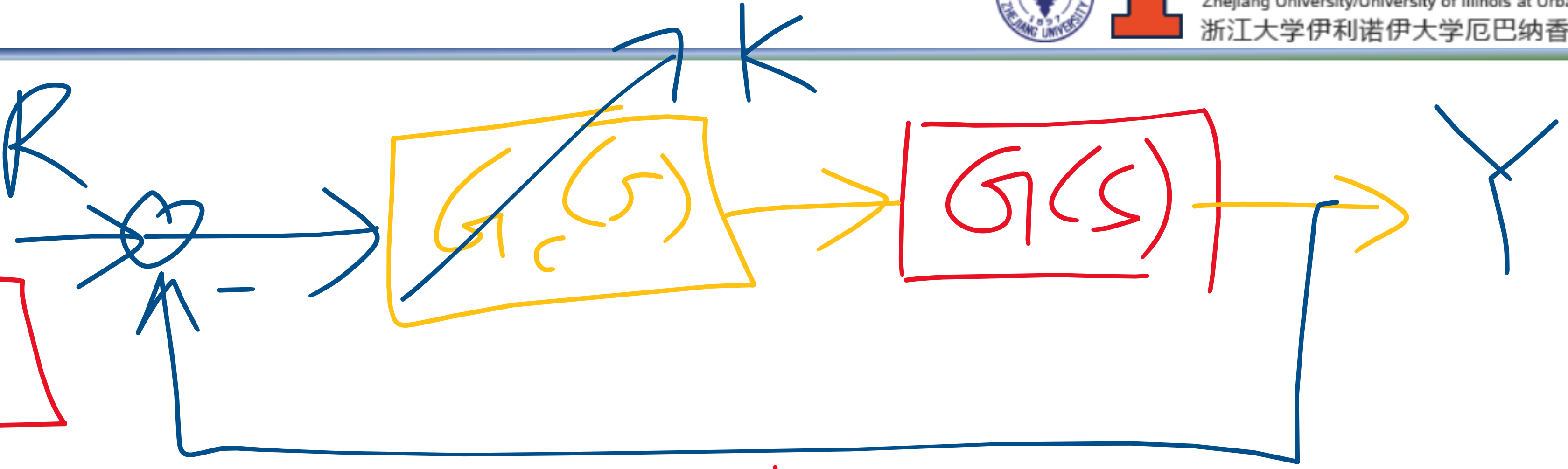
Workflow:

$$\boxed{\text{Bode } M \text{ and } \phi\text{-plots}} \longrightarrow \text{Nyquist plot}$$

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)

Example



$$\boxed{G(s) = \frac{1}{(s+1)(s+2)}}$$

How to choose K for stability?

$$\frac{KG(s)}{[1+KG(s)]} = 0$$

Routh-Hurwitz

Example

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Characteristic equation:

$$(s+1)(s+2) + K = 0 \qquad \Longleftrightarrow \qquad s^2 + 3s + [K + 2] = 0$$

From Routh, we already know that the closed-loop system is stable for $K > -2$.

We will now reproduce this answer using the Nyquist criterion.

Example

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph $\text{Im}\, G(j\omega)$ vs. $\text{Re}\, G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

$$(\text{Re}\, G(j\omega), \text{Im}\, G(j\omega)), \qquad -\infty < \omega < \infty$$

- Symmetry:

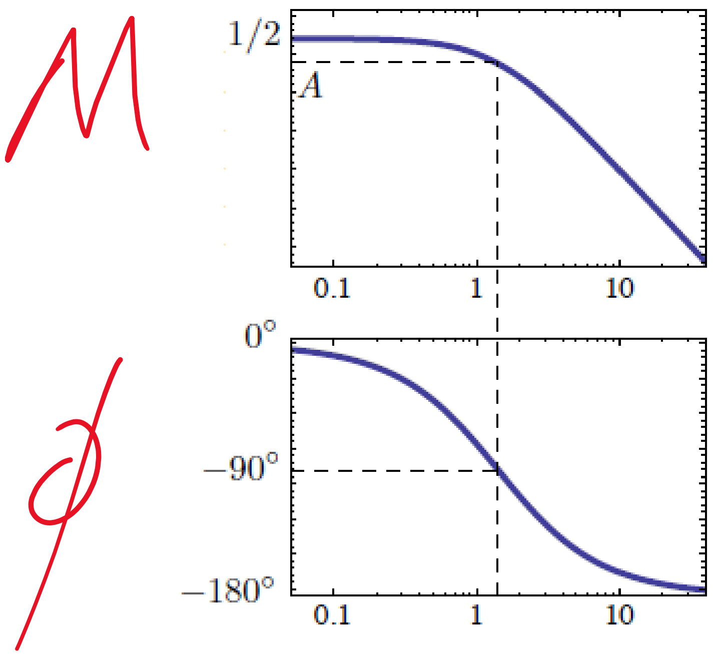
$$G(-j\omega) = \overline{G(j\omega)}$$

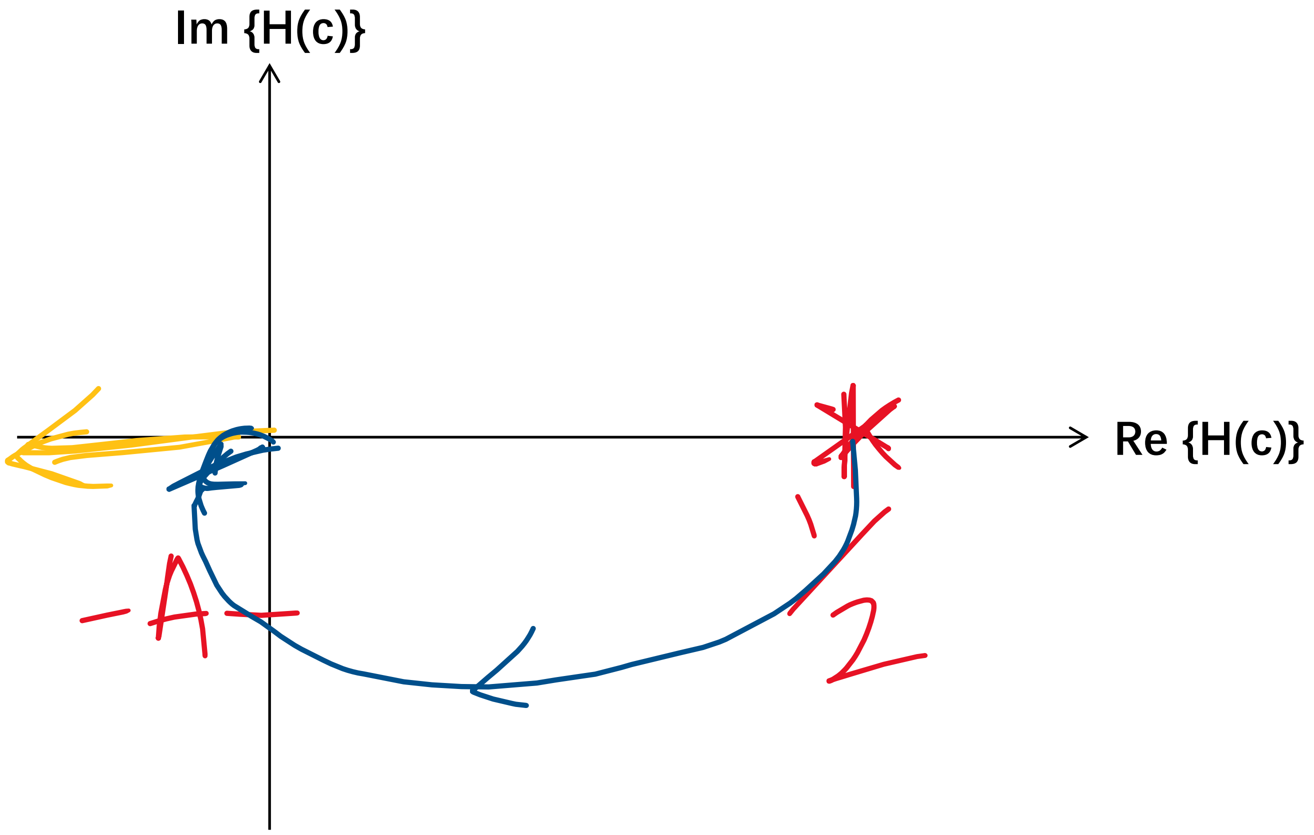
— Nyquist plots are always *symmetric w.r.t. the real axis*!!

Example

$$G(s) = \frac{1}{(s+1)(s+2)}$$ (no open-loop RHP poles)

Bode plot:

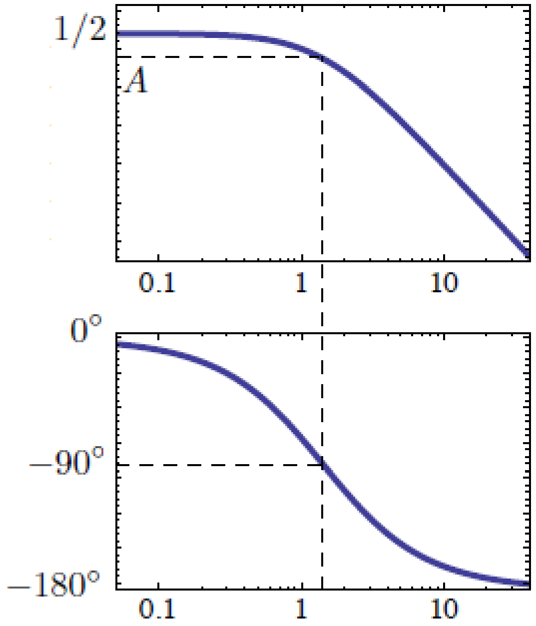




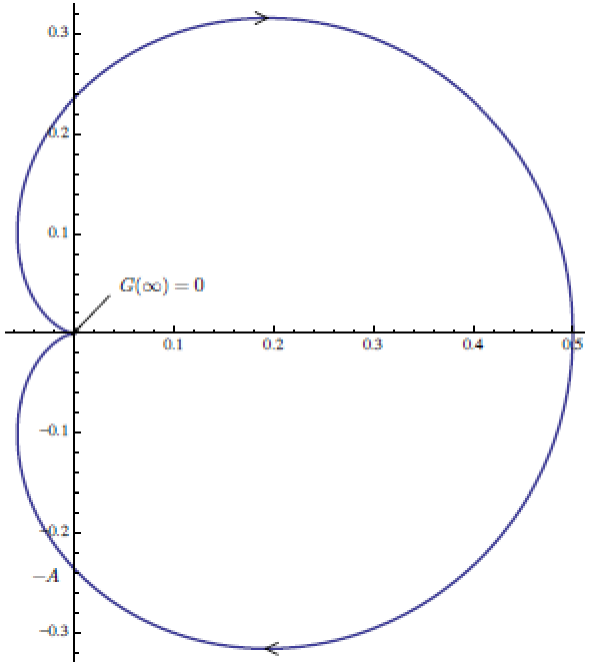
Example

$$G(s) = \frac{1}{(s+1)(s+2)}$$ (no open-loop RHP poles)

Bode plot:



Nyquist plot:

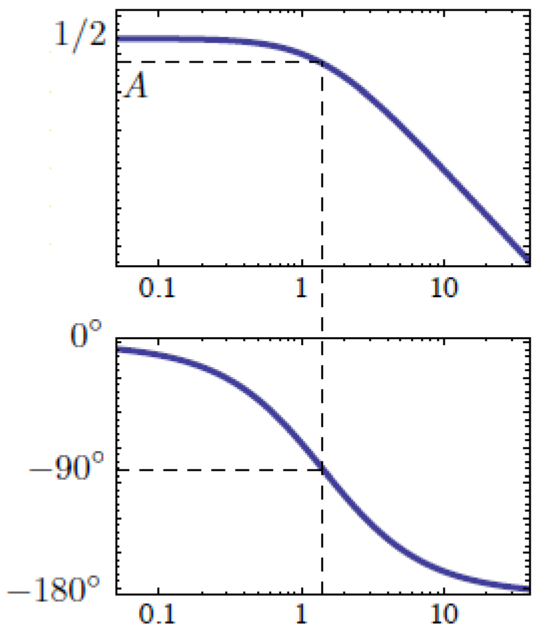


Example

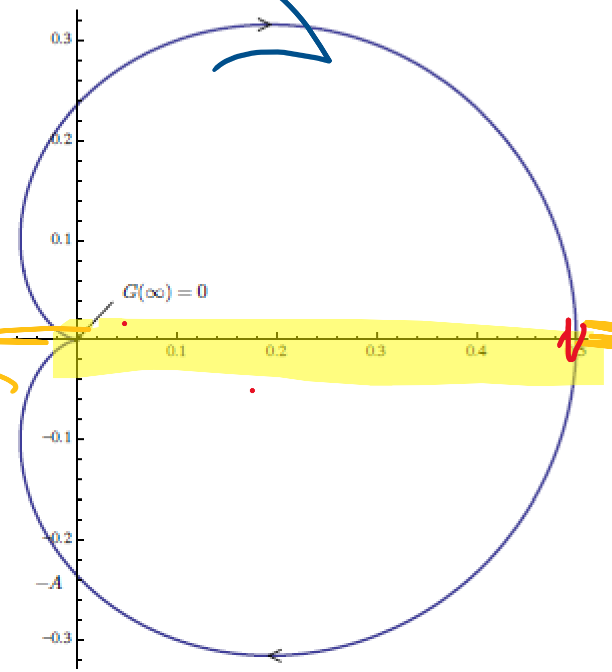
$N = \boxed{Z} - P$, $P = 0$

$$G(s) = \frac{1}{(s+1)(s+2)}$$ (no open-loop RHP poles)

Bode plot:



Nyquist plot:



$\#(\circlearrowright \text{ of } -1/K)$
$= \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=0}$

$\Longrightarrow K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowright \text{ of } -1/K) = 0$$

- If $K > 0$, $\#(\circlearrowright \text{ of } -1/K) = 0$
- If $0 < -1/K < 1/2$, $\#(\circlearrowright \text{ of } -1/K) > 0 \Longrightarrow$ closed-loop stable for $K > -2$

Example

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Strategy:

- Start with the Bode plot of G
- Use the Bode plot to graph $\mathrm{Im}\, G(j\omega)$ vs. $\mathrm{Re}\, G(j\omega)$ for $0 \le \omega < \infty$
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