Linear system: / State-space model.

$x_1 \equiv x,\ x_2 \equiv \dot{x},\ x_3 \equiv \ddot{x},\ \cdots$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

dynamics, control, input

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

measured output, sensor, state

if $u_1 \to y_1$, $u_2 \to y_2$ $\Rightarrow u_1 + u_2 = y_1 + y_2$

Laplace: $F(s) = \int_{\mathbb{R}} f(\tau) e^{-s\tau} d\tau$.

$s = e^{j\omega} = \cos\omega + j\sin\omega$.

Impulse response. $\int_{-a}^{a} \delta(t)\,dt = 1$.

$\int_{\mathbb{R}} \delta(t-z) f(t)\,dt = f(z)$

Causal: output not affected by future time.

Characteristic Polynomial: $P(\lambda) = \det(A - \lambda I)$, λ = eigen value for $P(\lambda) = 0$.

Vector division: $\vec{z}_1 = |z_1| e^{j\phi_1}$, $\vec{z}_2 = |z_2| e^{j\phi_2}$

$\Rightarrow \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\phi_1 - \phi_2)}$.

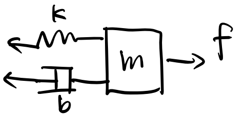
Transient response: ① vanishes at $t \to \infty$. ② $\mathcal{L}$ gives transient rsps.

DC gain: $= y(t \to \infty)$. where $u(t) = 1(t)$.

FVT: (all) Poles of $sY(s)$ lies in OLHP $\Leftrightarrow$ $Re(s) < 0$: $y(t \to \infty) = \lim_{s \to 0} sY(s)$.

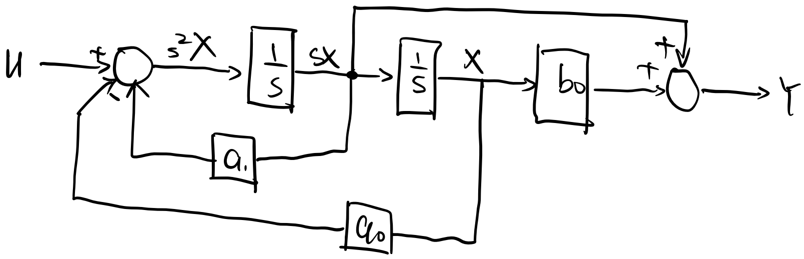
or satisfies R-H Criterion.

Mass-Damping:



$f = m\ddot{x} + b\dot{x} + k$. $\Leftrightarrow F = s^2X + sbX + kX$.

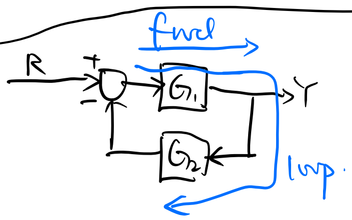
2nd-Order Damping System.



$$\begin{cases} s^2X = U - a_1 sX - a_0 X \\ Y = b_1 sX + b_0 X. \end{cases}$$

i.e. $\begin{cases} \frac{U}{X} = s^2 + a_1 s + a_0 \\ \frac{Y}{X} = b_1 X + b_0 \end{cases}$ $\Rightarrow \frac{Y}{U} = \frac{Y}{X} \cdot \frac{X}{U} = \frac{b_1 X + b_0}{s^2 + a_1 s + a_0}$.

For Feedback System:

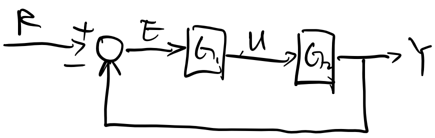


$\text{Gain} = \frac{\text{fwd gain}}{1 + \text{loop gain}} = \frac{G_1}{1 + G_1 G_2}$

rise: $10\% \sim 90\%$.

$t_r \doteq \frac{1.8}{\omega_n}$, exact at $\zeta = 0.5$

Unity Feedback



$\text{gain} = \frac{G_1 G_2}{1 + G_1 G_2}$

$t_p = \frac{\pi}{\omega_d}$

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$t_s \doteq \frac{3}{\sigma}$.

A damping system: $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ poles $s = -\omega_n(\zeta \pm \sqrt{\zeta^2 - 1})$,

$\frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$. $y(t) = 1 - e^{-\sigma t}\left[\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right]$

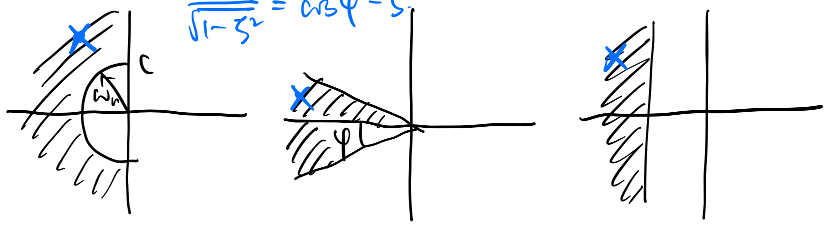
ζ: damping coef. / damping ratio

ω_n: natural freq. ω_r: resonant freq.

ω_d: damped freq.

Specifications.

$\frac{\xi}{\sqrt{1-\xi^2}} = \cos\varphi \doteq \xi$



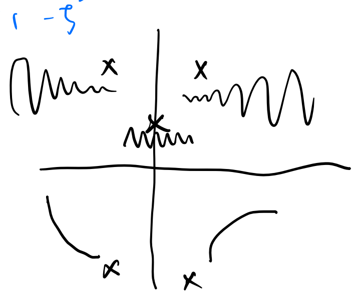
Rise. Mp ts

x: pole (σ, ω_d).

$\begin{cases} \sigma = \xi\omega_n \\ \omega_d = \omega_n \cdot \sqrt{1-\xi^2} \end{cases}$

$\omega_n = \sigma^2 + \omega_d^2$.

$\xi^2\omega_n^2 \;1 - \xi^2$.



Effects of zeros.

LHP: ① increased overshoot. ② little influence on settling time

③ $a \to \infty$ yields less significant effect.

RHP: ① delays the response.

② creates an undershoot. (when a is small enough.)

Extra Poles: ☆ extra LHP poles, → real parts 5x that of dominant LHP poles.

$y = \Sigma C_k e^{-\boxed{\lambda_k}t}$. $\quad Re(\text{pole}) = \lambda_k$.

Pole locations: ① RHS → unstable. ③ Im axis $\begin{cases} \text{impulse / step : unstable if } \omega = 0. \\ \omega = 0 : \begin{cases} \text{impulse}: Y = \frac{1}{s},\ y = 1(t) \to \text{stable} \\ \text{step}: Y = \frac{1}{s^2},\ y = t. \to \text{unit ramp.} \end{cases} \end{cases}$

② LHS → stable.

Stability: $\frac{\cdots}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$

necessary: $a_0, a_1, \cdots, a_n > 0$.

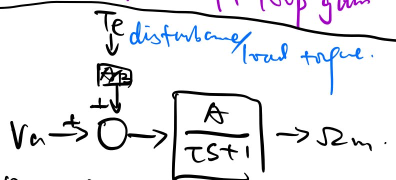
R-H. For lower order

2nd: $s^2 + a_1 s + a_2$ is stable iff $a_1, a_2 > 0$

3rd: $s^3 + a_1 s^2 + a_2 s + a_3$ is stable iff $\begin{cases} a_1, a_2, a_3 > 0 \\ a_1 a_2 > a_3. \end{cases}$

$H = \frac{Y}{R} = \frac{\text{Fwd gain}}{1 + \text{loop gain}} = \frac{\cdots}{\text{polynomial}}$ ← check if it's stable.

DC Motor: T_e disturbance / load torque.



$\Omega_M = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e$.

Disturbance Rejection: see coefficient of disturbance.

open-loop motor: $\omega_m(\infty) = \omega_{ref} + BT_e$ poor rejection.

closed : $\omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{ref} + \frac{B}{1 + AK_{cl}} T_e$.

As K_{cl} large, this item → 0. → good rejection.

Sensitivity: $S = \frac{\partial T/T}{\partial A/A}$. $\quad S_{ol} = 1$, $\quad S_{cl} = \frac{1}{1 + AK_{cl}}$.

PID.

P: $\frac{Y}{R} = \frac{K_p}{s^2 - 1 + K_p}$, unstable. $\quad$ D: $\frac{Y}{R} = \frac{K_D s}{s^2 + K_D s - 1}$. unstable.

PD: $\frac{Y}{R} = \frac{K_p + K_D s}{s^2 + K_D s + (K_p - 1)}$. stable when $\begin{cases} K_D > 0 \\ K_p > 1. \end{cases}$ no perfect tracking, DC gain $= \frac{K_p}{K_p - 1} \neq 1$.

PID: $\frac{K_D s^2 + K_p s + K_I}{s^3 + K_D s^2 + (K_p - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_p - 1)s + K_I} W$.

↖ DC gain = 1. perfect tracking. $\quad$ ← DC gain = 0. perfect rejection.

Root Locus: Characteristic Eq $\boxed{1+KL(s)=0}$ ⟷ Phase Condition: $L(s) = -1/K$, $\angle L = 180°$

$$L(s)=\frac{b(s)}{a(s)} = \frac{s^m+b_1 s^{m-1}+\dots+b_{m-1}s+b_m}{s^n+a_1s^{n-1}+\dots+a_{n-1}s+a_n}$$

m: #(OL zeros), n: #(OL poles).

Rules: (A) #branches = deg(a). OL tf: $H_{OL}=KL(s)$. ★ if s on RL, $\underbrace{\psi}_{\angle(s \text{ to zeros})} - \underbrace{\sum \phi_i}_{\angle(s \text{ to poles})} = 180°$

(B) Start: OL poles. CL tf: $H_{cl} = \frac{KL(s)}{1+KL(s)}$.

(C) Stop: OL zeros.

(D) if $s\in\mathbb{R}$, s is on RL iff odd #(real OL poles and zeros) to the right of s. (s右侧 zeros, poles 数量和为奇.)

(E) asymptotes: if $\angle s^{m-n} = 180° \Rightarrow \angle s = \frac{180°+\ell\cdot 360°}{n-m}$

Near ∞, $\angle s \approx \frac{(2\ell+1)\cdot180°}{n-m}$.

(F) $j\omega$-crossings: First solve $K_{critical}$ using R-H criterion. then find the corresponding ω_0. Crossings at $\pm j\omega_0$ on Im axis.

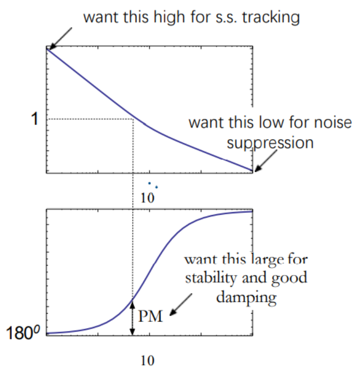
P: does not introduce stability. D: introduce an OL zero into LHP.

Controller $K\cdot\frac{s+z}{s+p}$: lead compensator: $z<p$. lag: $z>p$.

$\angle\frac{j\omega+z}{j\omega+p} = \angle(j\omega+z)-\angle(j\omega+p)$. $z<p$: phase lead. $z>p$: phase lag.

1. We use lead controllers as dynamic compensators for approximate PD control.
① p large: good damping, bad noise suppression. ② p small: better noise suppression. RL too close to $j\omega$-axis. no break-in for small p.
2. Use lag controller as dynamic compensator for approx. PI control. (stablization + perfect s.s. tracking).

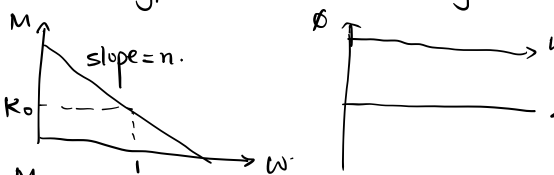
★ to place poles: select param. to satisfy phase condition. 3. PID: lag + lead.



$M_p = \exp(-\pi\zeta/\sqrt{1-\zeta^2})$. (overshoot).

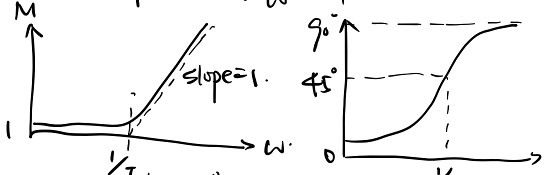
Bode plot. ★ M=1 ⟷ M = $\log_{10}1$ = 0 dB. all in log scale.

Type I. $K_0(j\omega)^n$. (low freq asymptote).



Type II. $(j\omega\tau+1)^{\pm1}$.

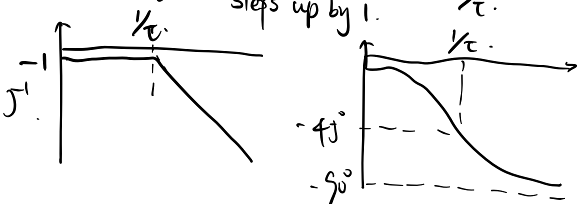
$(j\omega\tau+1)$



stable ℝ zero: steps up by 90°.

steps up by 1.

$(j\omega\tau+1)^{-1}$

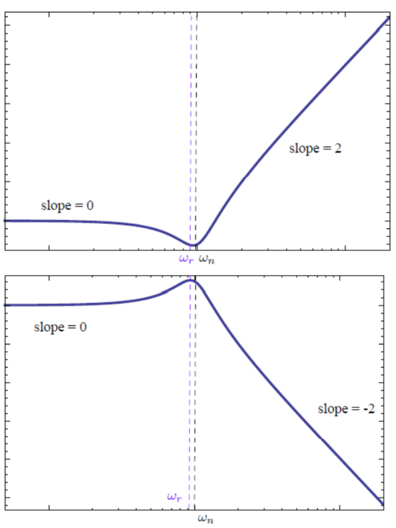


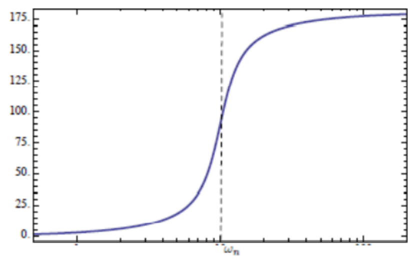
$$\begin{cases}\omega_r = \omega_n\sqrt{1-2\zeta^2} \\ M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1\end{cases} \quad \left(\text{valid for } \zeta < \frac{1}{\sqrt2};\ \text{for } \zeta \geq \frac{1}{\sqrt2},\ \omega_r = 0\right)$$

$$\omega_{BW} = \omega_n \underbrace{\sqrt{(1-2\zeta^2)+\sqrt{(1-2\zeta^2)^2+1}}}_{=1 \text{ for } \zeta=1/\sqrt2}$$ (0-freq response ~70.7%).

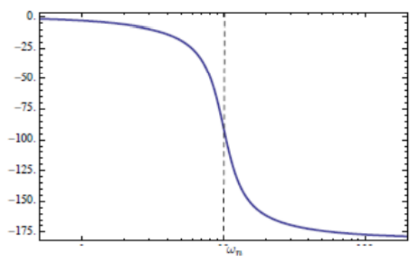
— so, if we know $\omega_r, M_r, \omega_{BW}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Type III. $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right)+1$.





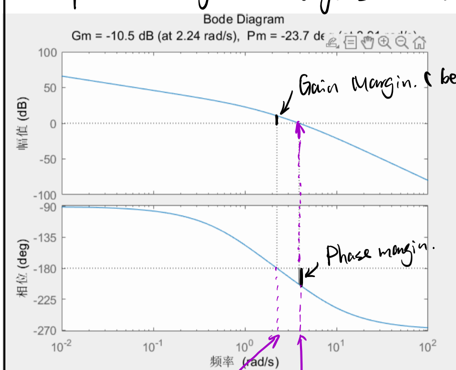
(stable complex zero — phase steps up by 180°)



(stable complex pole — phase steps down by 180°)

minimum-phase / non-~ zeros.

All tf with same magnitude plot, the one with only LHP zeros has the minimal net phase change as ω goes 0 → ∞.



GM: ω s.t. magnitude = 0dB, i.e. |KG(s)| = 1

PM: ω s.t. phase = 180°.

The argument Principle: $N = Z - P$, #(CW ⊘ of 0) = #(zeros inside) − #(poles inside).

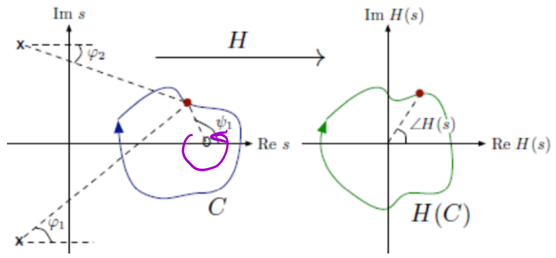
The Nyquist Thm. $N = Z - P$, #(CW ⊘ of $-\frac{1}{K}$ by $G(s)$) = #(RHP CL poles) − #(RHP OL poles).

Nyquist Stability Criterion. The CL system is stable iff. the Nyquist plot of $G(s)$ ⊘ $-1/K$ for P times CCW.

(Bode M, phase ⇒ Nyquist plot).

Phase of H: $$\angle H(s) = \angle \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = \underbrace{\sum^{m} \psi_i}_{\angle \text{zero}} - \underbrace{\sum^{n} \varphi_j}_{\angle \text{pole}} = \Sigma\angle\text{zeros} - \Sigma\angle\text{poles}.$$

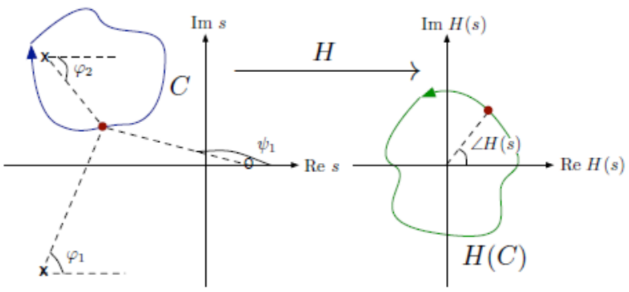
I. Contour encircles a zero:



1. φ_1, φ_2 return to original value.
2. ψ_1 net change of $-360°$.
   ⇒ $\angle H(s)$ $-360°$.

ψ_1, $\angle H(s)$ same direction.

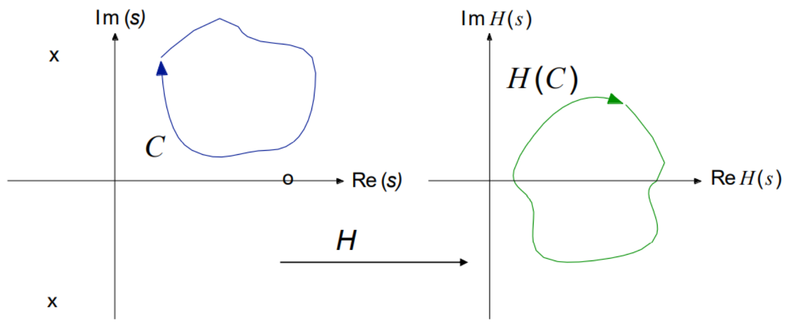
II. Contour encircles a pole.



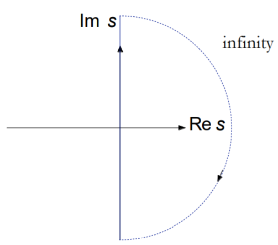
1. φ_1, φ_2 return.
2. φ net change. $-360°$
   ⇒ $\angle H(s)$ $-360°$.

$H(C)$ encircles the origin once counterclockwise.

III. Contour encircles no poles, no zeros.



- ϕ_1, ϕ_2, ψ_1 all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin



C: the whole Im axis
+ path around infinity.

$H(C)$ = Nyquist plot of H