ZJU-UIUC INSTITUTE

Online Final Examination

For Students, please read and sign the honor statement on a sheet of paper with your name, student ID number, date, and read the specific requirements and instructions below before starting your exam.

(For instructors, please complete the form below)

<table>
<tr><td>Course Code: ECE 486</td><td>Semester: 2022 Fall</td><td colspan="2">Instructors: M.-A. Belabbas
Liangjing Yang;</td></tr>
<tr><td colspan="4">Exam Code: Paper A☐ Paper B ☐ Paper C☐</td></tr>
<tr><td colspan="4">Exam Type: Closed-book ☐ Open-book ☐ Partly Open-book ☐ Take Home ☐</td></tr>
<tr><td>Exam Date: 2022.12.26</td><td>Start Time: 1400</td><td>End Time: 1700</td><td>Duration: 3 hours</td></tr>
<tr><td colspan="2">Total pages: 7</td><td colspan="2">Total questions: 4</td></tr>
<tr><td colspan="4">Specific requirements and instructions to students:

1. Do not start writing until you are instructed to do so.
2. Do not continue to write when you are told to stop.
3. You are not allowed to communicate with one another during the exam.
4. The exam is closed-book, closed-notes but you may bring in two A4 help sheets i.e. 4-page A4 notes</td></tr>
</table>

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Honor Statement

I have read the academic integrity statement. I promise to abide by the exam rules and regulations and agree to comport myself during the remotely administered exam in the same manner as if I were in a proctored examination room.

Please write "**I have read and will follow the Honor Statement**" on a sheet of paper and include the following information, then submit along with your answer sheet in the end.

Name:

Student ID number:

Date:

(Please go on to the next page for questions)

Laplace Transformation Table

Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$1(t)$
3	$\frac{1}{s^2}$	t
4	$\frac{2!}{s^3}$	t^2
5	$\frac{3!}{s^4}$	t^3
6	$\frac{m!}{s^{m+1}}$	t^m
7	$\frac{1}{(s+a)}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{(s^2+a^2)}$	$\sin at$
18	$\frac{s}{(s^2+a^2)}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

Question 1

Consider the system illustrated below

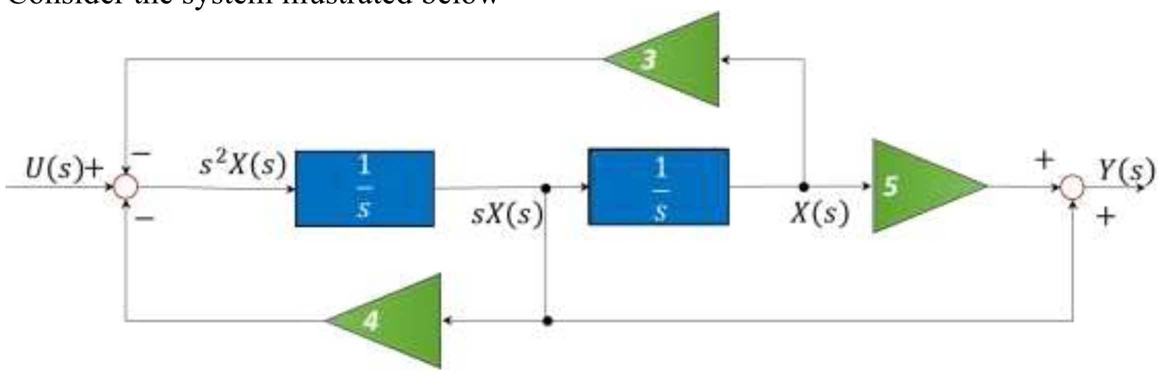


Figure 1a

i) Given that the transfer function can be written in the form

$$H(s) = \frac{Y(s)}{U(s)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$$

Write down the values of a_0, a_1, b_0, b_1, b_2 *(5 Points)*

ii) Verify that the following block diagram is also an equivalent representation.

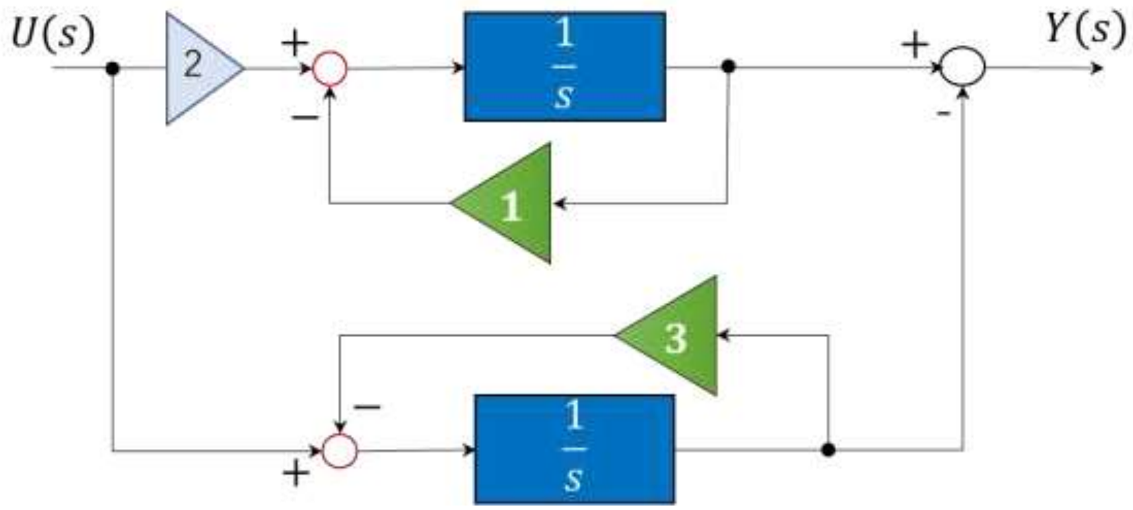


Figure 1b

(3 Points)

iii) Write down the transfer function in the following form:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{A}{s+C} + \frac{B}{s+D}$$

(4 Points)

iv) Obtain the inverse Laplace transform $\mathcal{L}^{-1}(H(s))$. *(4 Points)*

v) For input *u(t)* being a unit step, sketch the time response. *(4 Points)*

vi) For input *u(t)* being a unit step, obtain the steady state value of the output *y(t)* using Final Value Theorem and comment if the theorem is applicable here. *(5 Points)*

Question 2

Consider the system illustrated below

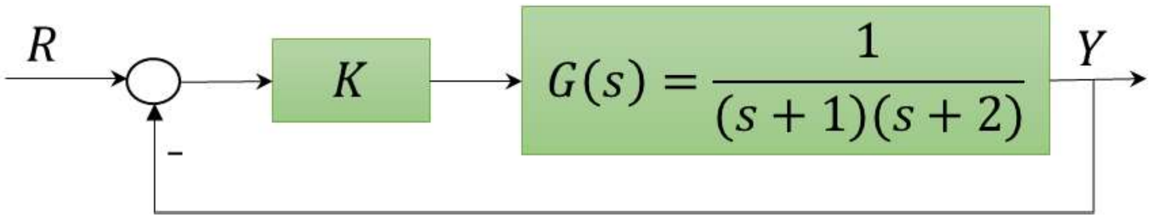


Figure 2a

i) Write down the poles of *G(s)*. *(2 Points)*

ii) Obtain the closed-loop pole(s) if $K=1/4$ *(4 Points)*

iii) Sketch the root locus associated with all values of K>0. *(8 Points)*

iv) Using your sketch, show if there exist a value for $K>0$ in the following case. *(3 Points)*

(a) Closed-loop poles at $s = -\frac{5}{2}$

(b) Closed-loop poles at $s = -\frac{3}{2} \pm j$

(c) Closed-loop poles at $s = -\frac{3}{4} \pm j$

v) Is there any $j\omega$-crossing? What could be said about the (a) closed-loop stability and the (b) gain margin? *(3 Points)*

vi) Taking 5% settling time as $t_s = \frac{3}{\sigma}$, explain using your sketch if it is possible to achieve a settling time of $t_s = 1$? *(3 Points)*

vii) Taking the rise time as $t_r \approx \frac{1.8}{\omega_n}$, using your sketch explain how you will vary K if your priority is to have rise time as short as possible. *(2 Points)*

Question 3

Consider the plant

$$\dot{\mathbf{x}} = \begin{bmatrix} -25 & -100 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

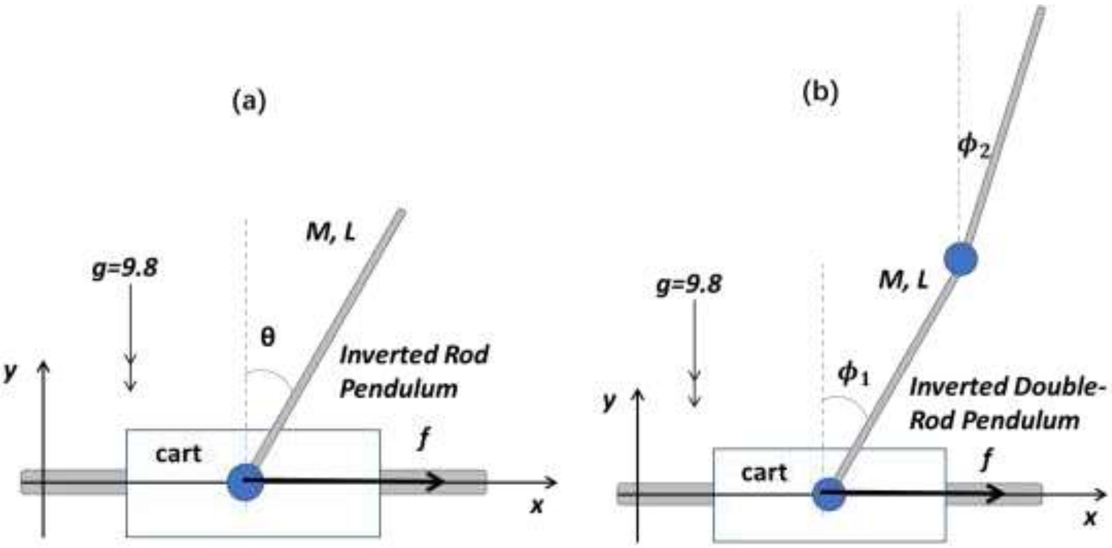
$$\mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}$$

i) Obtain the controllability matrix and comment on the controllability *(5 Points)*
ii) Obtain a transfer function of the system *G(s)*. *(4 Points)*
iii) Sketch the Bode plots. *(6 Points)*
iv) Discuss how you would design a lead/lag controller that achieve a phase margin approximately 75° and a tracking error to a unit step less than 5%. *(10 Points)*

Question 4

Consider the cart system with an inverted rod pendulum (Figure 4a) with a dynamic equation

$$\ddot{\theta} = \frac{3g}{2}\theta + \frac{3}{2}\ddot{x}$$



a) Write down A and B such that

$$(\dot{x} \quad \ddot{x} \quad \dot{\theta} \quad \ddot{\theta})^{\mathrm{T}} = \mathrm{A}(x \quad \dot{x} \quad \theta \quad \dot{\theta})^{\mathrm{T}} + B\ddot{x}$$

(3 Points)

b) You are tasked to balance an inverted double-rod pendulum system (Figure 4b) at specific position. State and explain the benefits of using state-space design for this application.

(2 Points)

c) Consider a system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}u$$

$$\mathbf{y} = [0 \quad 0 \quad 1]\mathbf{x}$$

(i) Comment on the observability with explanation. *(1 Points)*
(ii) Obtain the transfer function *G(s)*. *(4 Points)*
(iii) Write down the Controller Canonical Form. *(5 Points)*
(iv) How would you obtain the state-feedback gain matrix *K* if the desired closed-loop poles are at $s = -3 \pm 6j$, $s = -9$. *(10 Points)*

ECE 486 Final Examination Fall 2022

钟天天 TIANTIAN ZHONG 3200110643

I have read and will follow the Honor Statement.

TIANTIAN ZHONG
3200110643

Date: 12/26 2022

Question 1.

i). The block diagrams gives

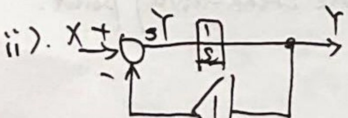
$$\begin{cases} Y = 5X + sX \\ s^2X = U - 3X - 4sX. \Rightarrow (s^2+4s+3)X = U. \end{cases}$$

$\therefore \frac{Y}{X} = 5+s, \quad \frac{X}{U} = \frac{1}{s^2+4s+3}.$

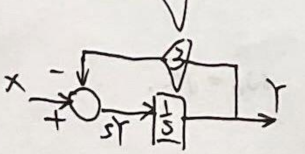
$\therefore H(s) = \frac{Y(s)}{U(s)} = \frac{5+s}{s^2+4s+3} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$

$\therefore \quad a_0 = 5, \ a_1 = 1, \ b_0 = 3, \ b_1 = 4, \ b_2 = 1.$

ii).

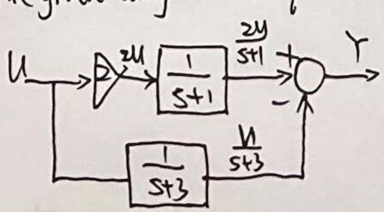


$sY = X - Y. \quad \therefore Y = \frac{X}{s+1}, \quad \frac{Y}{X} = \frac{1}{s+1}.$



$sY = X - 3Y \quad \therefore Y = \frac{X}{s+3}.$

$\therefore$ The given diagram is equivalent to:



i.e. $Y = \frac{2U}{s+1} - \frac{U}{s+3} = \frac{(s+5)U}{(s+1)(s+3)}$.

$\therefore \frac{Y}{U} = \frac{s+5}{s^2+4s+3}$ which matches the result in i).

iii). $H(s) = \frac{s+5}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}.$

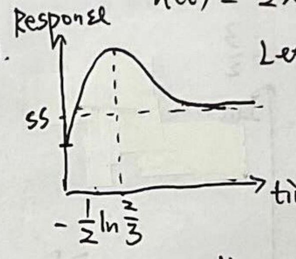
$\Rightarrow \frac{s+5}{(s+1)(s+3)} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)} \quad \therefore s(A+B) + (3A+B) = s+5$

$\therefore \begin{cases} A+B=1 \\ 3A+B=5 \end{cases} \quad \therefore \begin{cases} A=2 \\ B=-1. \end{cases} \quad H(s) = \frac{2}{s+1} - \frac{1}{s+3}.$

(iv). $\therefore \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$h(t) = 2 \times e^{-t} - e^{-3t}.$

v).



Let $f(t) = 2e^{-t} - e^{-3t}$: $f'(t) = -2e^{-t} + 3e^{-3t}$.

root for $f'(t) = 0$ is $3e^{-3t} = 2e^{-t}$,

i.e. $e^{-t} = 0$ (X). or $t = -\frac{1}{2}\ln\frac{2}{3}$.

$\therefore$ only one critical point.

$\lim_{t\to\infty} f(t) = 0 - 0 = 0.$

vi). $H(s) = \frac{s+5}{(s+1)(s+3)}$ ($= sY(s)$ for $U(s) = 1/s$.), with poles $s = -1$ or -3 and zero $s = -5$. They lie in OLHP.

$\therefore$ FVT is applicable.

$\therefore \ \cancel{\mp} \ y(t\to\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} H(s) = \lim_{s\to 0} \frac{s+5}{s^2+4s+3} = \frac{5}{3}.$

for $U(s) = \frac{1}{s}$, which is a unit step.

$\therefore$ The steady-state value is $\frac{5}{3}$.

Question 2.

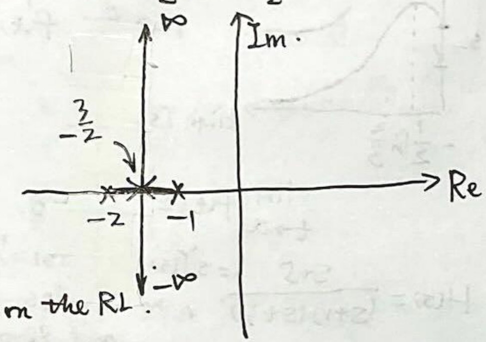
i) Set $(s+1)(s+2)=0$ will result $s=-1$ or $s=-2$, which are the poles.

ii). $H_{CL} = \dfrac{KG}{1+KG} = \dfrac{\frac{K}{(s+1)(s+2)}}{1+\frac{K}{(s+1)(s+2)}} = \dfrac{K}{(s+1)(s+2)+K}$.

$= \dfrac{\frac{1}{4}}{s^2+3s+2+\frac{1}{4}} = \dfrac{1}{4s^2+12s+9} = \dfrac{1}{(2s+3)^2}$.

∴ Two poles that overlap: $s=-\frac{3}{2}$ or $-\frac{3}{2}$.

iii). $L(s) = G(s) = \dfrac{1}{(s+1)(s+2)}$.



① # of branches: 2

② start: $s=-1$ or -2.

③ stop: at infinity.

④ interval $[-2,-1)$ available, where $s \in (-2,-1)$ is on the RL. $\in \mathbb{R}$

⑤ OL zeros: none. poles: two. ∴ $\angle s = \dfrac{(2\ell+1)\times 180^\circ}{2-0} = (2\ell+1)\cdot 90^\circ = 90^\circ, 270^\circ$.

⑥ $j\omega$-crossings:

$H_{CL} = \dfrac{K}{s^2+3s+(K+2)}$.

By Routh-Hurwitz's Criterion for 2nd-order polynomials, $K+2>0$.

⇒ the system is always stable thus no $j\omega$-crossings.

The RL diagram is shown above.

iv). (a) $s=-\frac{5}{2}$ is not on RL. thus $\nexists K$.

(b) Assume CL poles at $(-\frac{3}{2}, 0)$ (overlapped).

$(s+1)(s+2)+K = s^2+3s+2+K = (s+\frac{3}{2})^2 = s^2+3s+\frac{9}{4}$ ∴ $K=\frac{9}{4}-2=\frac{1}{4}$.

∴ $\exists K=\frac{1}{4}$ for poles $s=-\frac{3}{2}, -\frac{3}{2}$.

∴ $Re(s)=-\frac{3}{2}$ is a break-away point.

Plus asymptote has an angle 90° or 180°,

∴ poles at $s=-\frac{3}{2}\pm j$ can be achieved.

(c) Since the break-away point is $-\frac{3}{2}$,

$s=-\frac{3}{4}\pm j$ cannot be achieved thus such K does not exist.

v). No, there is not any $j\omega$-crossing.

(a) CL stability: always stable for $\forall K \in \mathbb{R}, K>0$.

(b). $GM=\infty$

vi). Setting $t_s=\frac{3}{\sigma}=1$ will result $\sigma=3$.

The second-order system should be $\dfrac{K}{s^2+3s+(K+2)} = \dfrac{\omega_n^2}{(s+\sigma)^2+\omega_n^2}$.

if $\sigma=3$: $H_{CL} = \dfrac{\omega_n^2}{s^2+6s+9+\omega_n^2}$, the coefficient does not match the given tf.

also, from the RL sketch, -3 is not on RL, nor is it the break-away point.

∴ $\sigma=3$ ($t_s=1$) is not achieveable.

vii). shorter t_r will result in larger ω_n.

By the prototype of a second-order system, $\omega_n=\sqrt{K}$.

∴ shorter t_r will result in larger K.

Question 3.

i). $A=\begin{bmatrix}-25 & -100\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}1\\0\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\end{bmatrix}$.

$\therefore \quad AB=\begin{bmatrix}-25 & -100\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-25\\1\end{bmatrix}$.

$\Rightarrow \mathcal{C}(A,B)=\begin{bmatrix}1 & -25\\ 0 & 1\end{bmatrix}$.

$\det \mathcal{C}(A,B) = 1 \neq 0$ $\therefore$ System is controllable.

ii). $G(s)=C(Is-A)^{-1}B+D$.

$Is-A=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}-25 & -100\\ 1 & 0\end{bmatrix}=\begin{bmatrix}s+25 & 100\\ -1 & s\end{bmatrix}$.

$\therefore \det(Is-A)=s^2+25s+100$.

$\therefore (Is-A)^{-1}=\dfrac{1}{s^2+25s+100}\cdot\begin{bmatrix}s & -100\\ 1 & s+25\end{bmatrix}$.

$\therefore \quad C(Is-A)^{-1}=\dfrac{1}{s^2+25s+100}\cdot\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s & -100\\ 1 & s+25\end{bmatrix}$

$=\dfrac{1}{s^2+25s+100}\cdot\begin{bmatrix}s & -100\end{bmatrix}$.

$\therefore \quad C(Is-A)^{-1}B=\dfrac{1}{s^2+25s+100}\cdot\begin{bmatrix}s & -100\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

$=\dfrac{s}{s^2+25s+100}$.

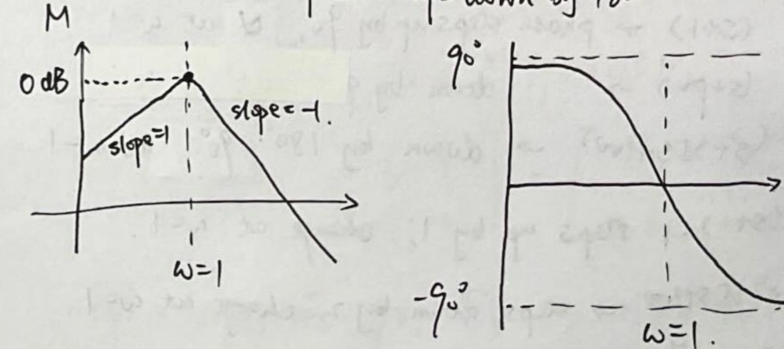
Plus $D=0$ $\quad\therefore G(s)=\dfrac{s}{s^2+25s+100}$.

iii). two types of Bode diagram: ① $(j\omega)^1$ ② $((j\omega)^2+25j\omega+100)^{-1}$.

$(j\omega)^1$: magnitude steps up by 1, phase constant at $\angle 1\times(j\omega)^1=90°$.

$((j\omega)^2+25j\omega+100)^{-1}$: magnitude steps down by 2, slope change at $\omega\approx 1$.

phase steps down by $180°$.



iv). ① introduce an approx. PI controller, as a lag controller $K\dfrac{s+z_1}{s+p_1}$ $(p_1<z_1)$.
lag.
it helps achieving steady-state tracking.

Pick $p_1=0$ to cancel numerator s of $G(s)$:

$G_{PI}=\dfrac{s}{s^2+25s+100}\cdot K_I\dfrac{s+z_1}{s+p_0}=K_I\dfrac{s+z_1}{(s^2+25s+100)s}$.

$\therefore$ steady-state response: $\lim\limits_{s\to 0}G_{PD}(s)=K_I\dfrac{z_1}{100}$.

Choose $z_1=1$ for computing convenience: $\dfrac{K_I}{100}\approx 95\%\sim105\%$.

$\therefore K_D=100$.

$\therefore G_{PI}=\dfrac{100(s+1)}{(s^2+25s+100)s}$.

② $G_{PD}=K_D\dfrac{s+z_2}{s+p_2}\cdot\dfrac{100(s+1)}{s^2+25s+100}$, for convenience choose $z_2=0$,
lead $\quad (p_2>z_2)$. $\Rightarrow G_{PD}=K_P\cdot\dfrac{s}{s+p_2}\cdot\dfrac{100(s+1)}{(s^2+25s+100)s}$.

$$G_{PD} = K_D \frac{(s+\cancel{1})}{s+p_2} \cdot \frac{100(s+1)}{(s^2+25s+100)s}$$

$$= K_D \cdot \frac{100(s+1)}{(s+p_2)(s^2+25s+100)}$$

Phase plot: $(s+1) \to$ phase steps up by $90°$, $45°$ at $w=1$

$(s+p_2)^{-1} \to$ down by $90°$. $-45°$ at p_2.

$(s^2+25s+100) \to$ down by $180°$, $-90°$ at $w=1$.

Magnitude: $(s+1) \to$ steps up by 1, change at $w=1$.

$s^2+25s+100 \to$ steps down by 2, change at $w=1$.

$(s+p_2)^{-1} \to$ steps down by 1, change at p_2.

at $w=1$, without $(s+p_2)$ phase $= -45°$.

Set $G_{PD} = 0dB$ will result in $100K_D(s+1) = (s+p_2)(s^2+25s+100)$.

$\therefore$ Finally, find K_D, p_2 such that $\begin{cases} \angle G_{PD} = 45° \\ G_{PD}(s) = 0dB. \end{cases}$

To avoid breaking steady-state tracking we set $K_D = 1$.

$$\Rightarrow G = \frac{100(s+1)}{(s+p_2)(s^2+25s+100)}.$$

$\therefore$ The next step is to solve p_2 that satisfies

① $G(jw_0) = \dfrac{100(jw_0+1)}{(jw_0+p_2)((jw_0)^2+25jw_0+100)} = 1$

② $\angle G(jw_0) = -180° - 75° = -255°$.

Solving for p_2 will result in the final compensated system.

Question 4.

a) $$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + B\ddot{x}. \qquad f = M\ddot{x}.$$

$$\ddot{\theta} = \frac{3g}{2}\theta + \frac{3}{2}\ddot{x}. \Rightarrow \ddot{x} = \left[\ddot{\theta} - \frac{3g}{2}\theta\right] \times \frac{2}{3} = \frac{2}{3}\ddot{\theta} - g\theta$$

$$\therefore \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & & \\ 0 & 0 & \frac{3g}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{3}{2} \end{bmatrix} \ddot{x}.$$

b). We have a lot of variables and linear equations in this system. By state-space design, we can make use of matrice operation, observing and controlling a couple of variables at the same time, thus have an easier way to design a system.

At each position, ϕ_1, ϕ_2 can vary. Bounding each pair of ϕ_1, ϕ_2 with position x, i.e. forming "states" using matrices, simplifies the design process.

c).

i) $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. $C = [0 \ 0 \ 1]$.

$\therefore$ Observability Matrix $\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$.

Since $CA = [0 \ 0 \ 1] \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = [0 \ 1 \ -2]$,

$A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix}$.

$\Rightarrow CA^2 = [1 \ -2 \ 3]$.

$$\therefore \mathcal{O}(A,C) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

$$\Rightarrow \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} = -1.$$ $\therefore$ The system is observable.

ii). $$Is - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} s & 0 & 1 \\ -1 & s & 1 \\ 0 & -1 & s+2 \end{bmatrix}.$$

$\cancel{\therefore (Is-A)^{-1} =}$

$$\therefore G(s) = C(Is-A)^{-1}B + D$$

$$= [0 \; 0 \; 1] \cdot (Is-A)^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(Is-A)^{-1} = \frac{1}{s^3+2s^2+s+1} \cdot \begin{bmatrix} s^2+2s+1 & -1 & -s \\ s+2 & s^2+2s & -s-1 \\ 1 & s & s^2 \end{bmatrix}$$

$$\therefore G(s) = \frac{1}{s^3+2s^2+s+1} \cdot [1 \;\; s \;\; s^2] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{s+1}{s^3+2s^2+s+1}.$$

iii). Characteristic polynomial is $\begin{vmatrix} s & 0 & 1 \\ -1 & s & 1 \\ 0 & -1 & s+2 \end{vmatrix} = s^3+2s^2+s+1$

$$\therefore \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} \text{ plus } \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \text{CCF: } \dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Check controllability: $\mathcal{C}(A,B) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$,

$\det \mathcal{C}(A,B) = -1 \neq 0$ $\therefore$ controllable.

iv). desired characteristic polynomial does not change.

$$\Rightarrow \det(Is - A + BK) = (s+9)(s+3+6j)(s+3-6j).$$

$$= (s+9)((s+3)^2+36).$$

$$= s^3+15s^2+99s+405$$

$$\therefore \text{desired } \bar{A}' \text{ (in CCF)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -405 & -99 & -15 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1-\bar{k}_1 & -1-\bar{k}_2 & -2-\bar{k}_3 \end{bmatrix}$$

$$\therefore \bar{k}_1 = 404, \; \bar{k}_2 = 98, \; \bar{k}_3 = 13. \Rightarrow \bar{k} = [404 \;\; 98 \;\; 13].$$

Since $\mathcal{C}(A,B) = [B \,|\, AB \,|\, A^2B] = \left[\begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \,\middle|\, AB \,\middle|\, A^2B \right]$.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; A^2B = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\therefore \mathcal{C}(A,B) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow [\mathcal{C}(A,B)]^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}.$$

$$\mathcal{C}(\bar{A},\bar{B}) = \left[\begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \,\middle|\, \bar{A}\bar{B} \,\middle|\, \bar{A}^2\bar{B} \right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

$\therefore$ Transform matrix $T = \mathcal{C}(\bar{A},\bar{B})\,[\mathcal{C}(A,B)]^{-1}$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -4 \end{bmatrix}.$$

$\therefore$ Gain matrix K in the original coordinate should be

$$K = \bar{K}T = [404 \;\; 98 \;\; 13] \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -4 \end{bmatrix} = [-319 \;\; 332 \;\; -260].$$