ECE 486 Control Systems

Lecture 19: Nyquist Stability with Varying Gain; Lag Compensator

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Schedule check

Week	Topic	Ref.
1	Introduction to feedback control	Ch. 1
	State-space models of systems; linearization	Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1
2	Linear systems and their dynamic response	Section 3.1, Appendix A
	Transient and steady-state dynamic response with arbitrary initial conditions	Section 3.1, Appendix A
3	System modeling diagrams; prototype second-order system	Sections 3.1, 3.2, lab manual
	Transient response specifications	Sections 3.3, 3.14, lab manual
4	National Holiday Week	
5	Effect of zeros and extra poles; Routh-Hurwitz stability criterion	Sections 3.5, 3.6
	Basic properties and benefits of feedback control	Section 4.1, lab manual
6	Introduction to Proportional-Integral-Derivative (PID) control	Sections 4.1-4.3, lab manual
	Review A	
7	Term Test 1	
	Introduction to Root Locus design method	Ch. 5
8	Root Locus continued; introduction to dynamic compensation	Ch. 5
	Lead and lag dynamic compensation	Ch. 5
9	Introduction to frequency-response design method	Sections 5.1-5.4, 6.1
	Bode plots for three types of transfer functions	Section 6.1

Root Locus

Frequency Response

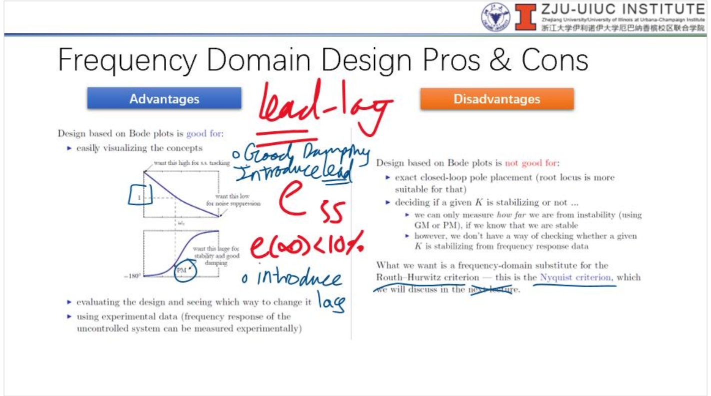
Week	Topic	Ref.
10	Stability from frequency response; gain and phase margins	Section 6.1
	Control design using frequency response	Ch. 6
11	Control design using frequency response continued; PI and lag, PID and lead-lag	Ch. 6
	Nyquist stability criterion	Ch. 6
12	Gain and phase margins from Nyquist plots	Ch. 6
	Term Test II (Review B)	
13	Introduction to state-space design	Ch. 7
	Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form	Ch. 7
14	Pole placement by full state feedback	Ch. 7
	Observer design for state estimation	Ch. 7
15	Joint observer and controller design by dynamic output feedback I; separation principle	Ch. 7
	Dynamic output feedback II (Review C)	Ch. 7
16	END OF LECTURES	
	Finals	

State-Space

Review: Frequency Domain Design Method

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system measured empirically)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Frequency Domain Design Pros & Cons

Advantages	Disadvantages
Design based on Bode plots is good for:	Design based on Bode plots is not good for:

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)

lead-lag

o Good Damping Introduce lead

e_{ss}

$e(\infty)<10\%$

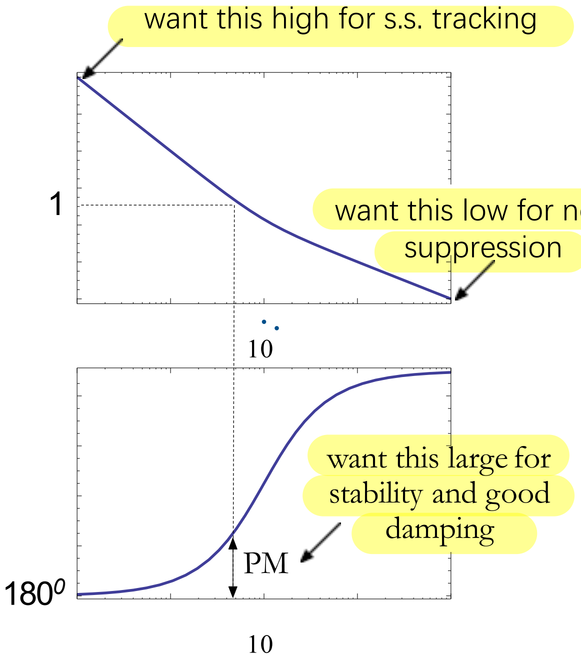
o introduce lag

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given K is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given K is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the ~~next lecture~~.

* good damping introduces lead



Review: Frequency Domain Design Method

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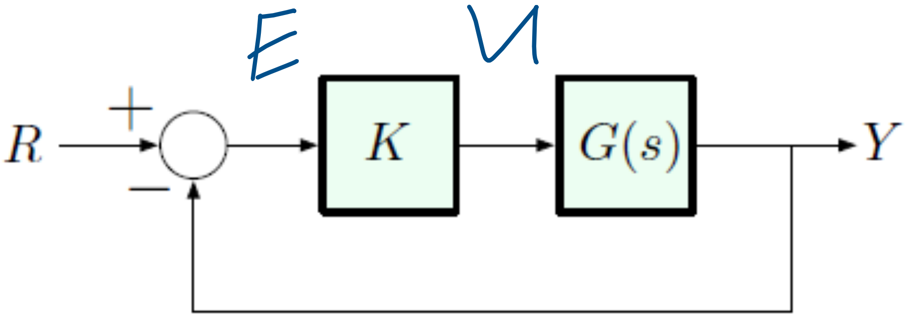
    - K is stabilizing from frequency response data

Nyquist criterion- A frequency-domain substitute for the Routh–Hurwitz criterion

Nyquist Stability Criterion



$$H_{CL} = \frac{Y}{R}$$

Goal: count the number of RHP poles (if any) of the closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)}$$

Root locus

$$\frac{K\boxed{}}{1+K\boxed{}}$$

based on frequency-domain characteristics of the plant transfer function $\boxed{G(s)}$

Nyquist Plot

s^m roots

$$H(s) = |H(s)| \angle H(s) = \mathrm{Re}[H(s)] + j\,\mathrm{Im}[H(s)]$$

$\| \; j\omega \qquad \| \; j\omega$

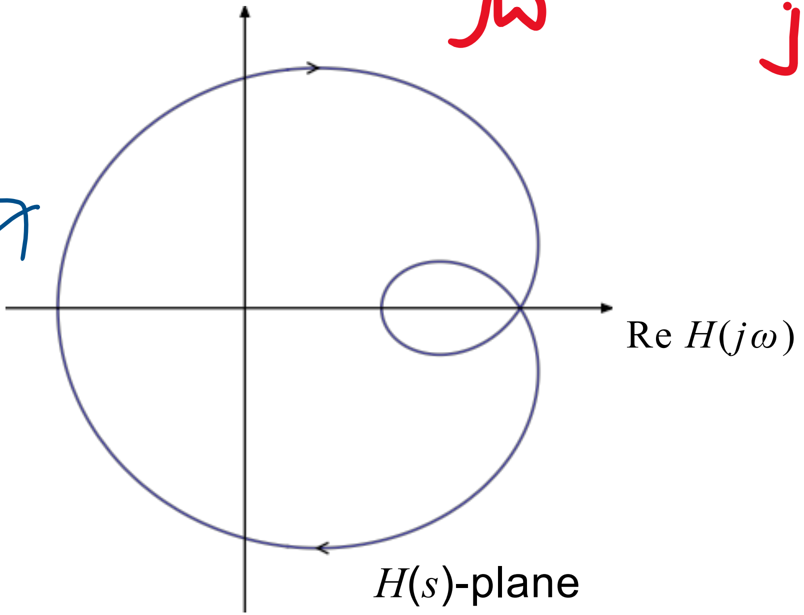
Consider an arbitrary *strictly proper* transfer function H:

$$H(s) = \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)}, \qquad \boxed{m < n}$$

s^n n roots

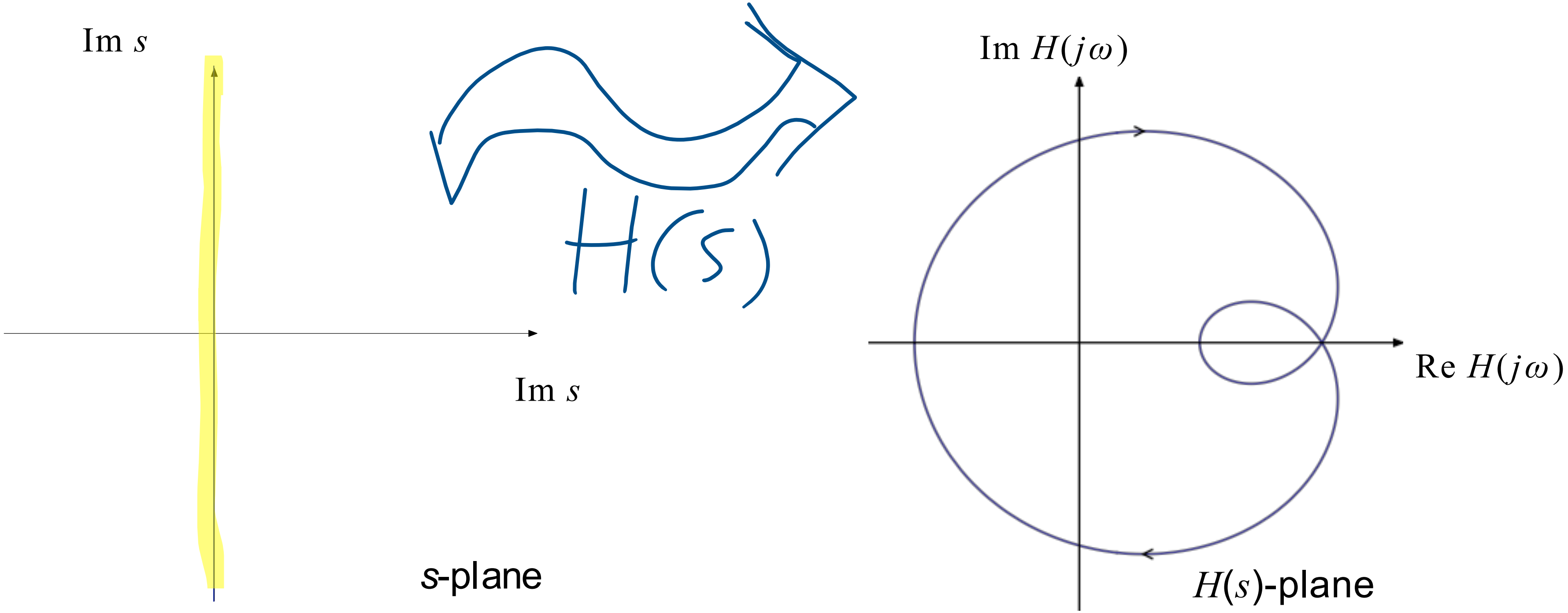
Nyquist plot: Im $H(j\omega)$ vs. Re $H(j\omega)$ as ω varies from $-\infty$ to ∞

$$H = \frac{1}{s} = \frac{1}{j\omega} = \boxed{-\frac{1}{\omega} j} + 0\,\omega$$



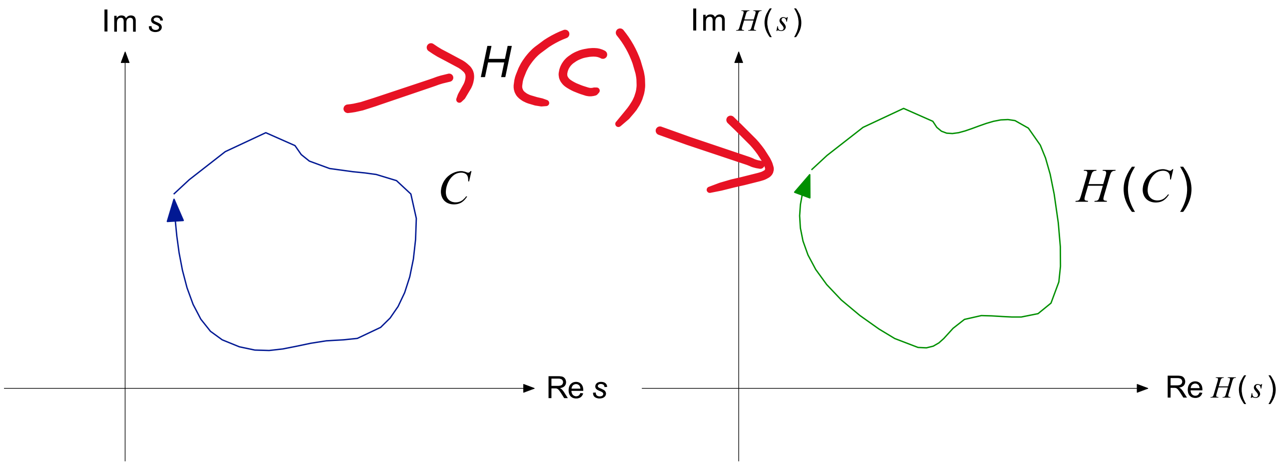
Nyquist Plot: Mapping of the s-Plane

- View the Nyquist plot of H as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \to \mathbb{C}$



Transformation of a Closed Contour Under H

If we choose any closed curve (or *contour*) C on the left, it will get mapped by H to some other curve (contour) on the right:



cw

ccw

Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!

Phase of H Along a Contour

$Me^{j\phi}$

For any $s \in \mathbb{C}$, the phase (or *argument*) of $H(s)$ is

$$\angle H(s) = \angle \frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)}$$

$$= \sum_{i=1}^{m} \angle(s-z_i) - \sum_{j=1}^{n} \angle(s-p_j)$$

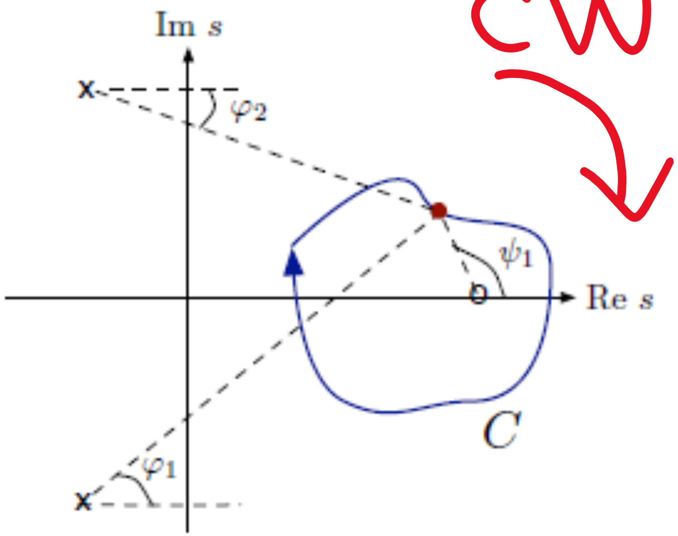
$$\arg H(s) = \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j$$

Interested in how $\angle H(s)$ changes as s traverses a closed, clockwise (CW) oriented contour C in the complex plane.

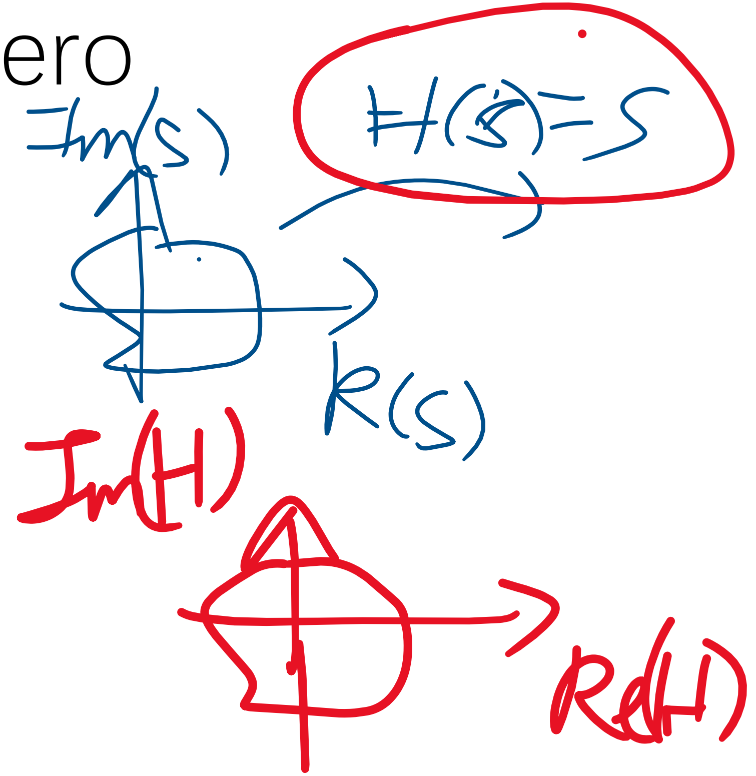
Look at several cases, depending on how the contour is located relative to poles and zeros of H.

Case 1: Contour Encircles a Zero

Suppose that C is a closed, **CW**-oriented contour in **C** that encircles a zero of $H(s)$:

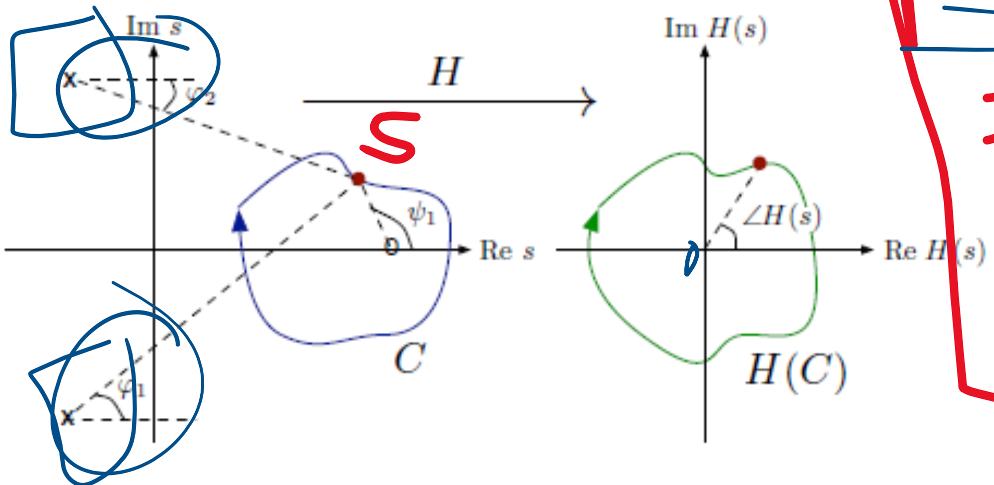


CW



How does $\angle H(s)$ change as we go around C?

Case 1: Contour Encircles a Zero



phase H(s) = Σ phase zero ψ − Σ phase pole ϕ

How does $\angle H(s)$ change as we go around C?

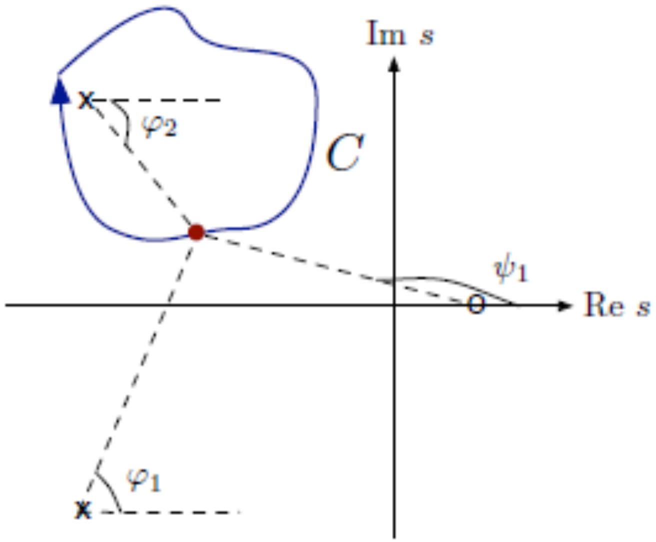
Angles from s to poles/zeros of H:

- ϕ_1 and ϕ_2 return to their original values
- ψ_1 registers a net change of -360°
- therefore, $\angle H(s)$ registers a net change of -360°
  - $H(C)$ encircles the origin once, clockwise (CW)

Case 2: Contour Encircles a Pole

Suppose that C is a closed, ↻-oriented contour in $\mathbb{C}$ that encircles a pole of $H(s)$:

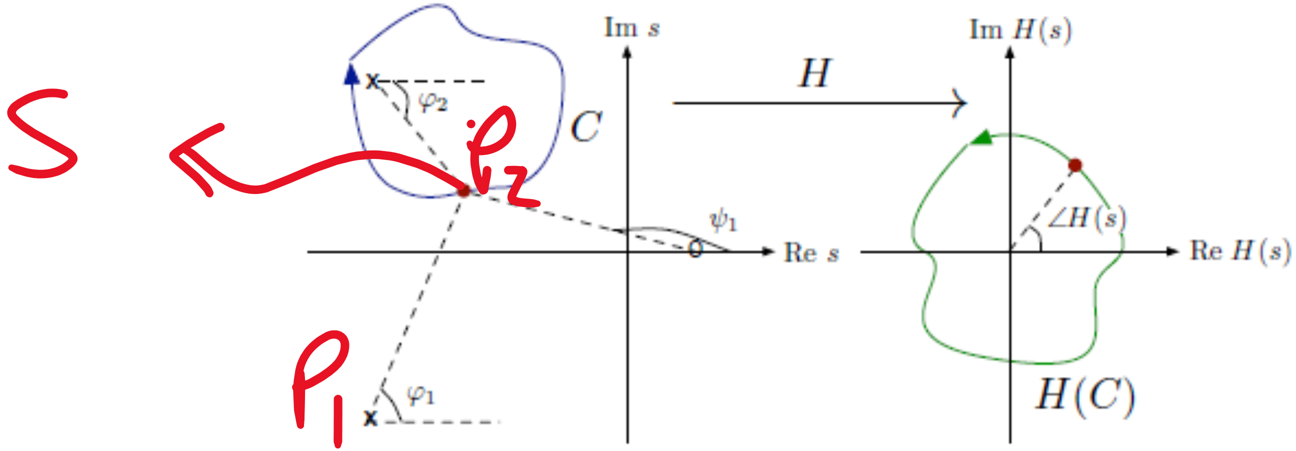
c.w.



How does $\angle H(s)$ change as we go around C?

Case 2: Contour Encircles a Pole

$\sum \psi_1 - \sum \phi_{1,2}$



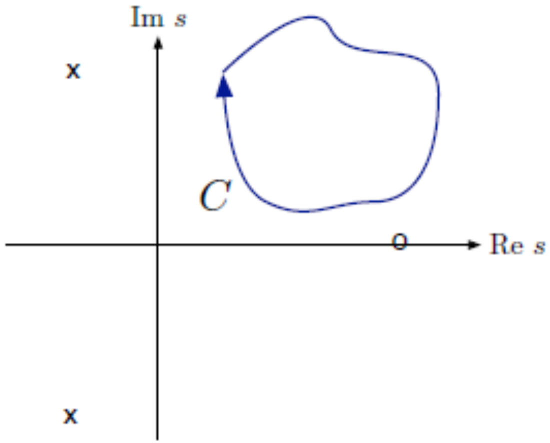
How does $\angle H(s)$ change as we go around C?

Let's see what happens to angles from s to poles/zeros of H:

- ϕ_1 and ψ_1 return to their original values
- ϕ_2 picks up a net change of -360°
- therefore, $\angle H(s)$ picks up a net change of 360°, so $H(C)$ encircles the origin once counterclockwise ~~(?)~~

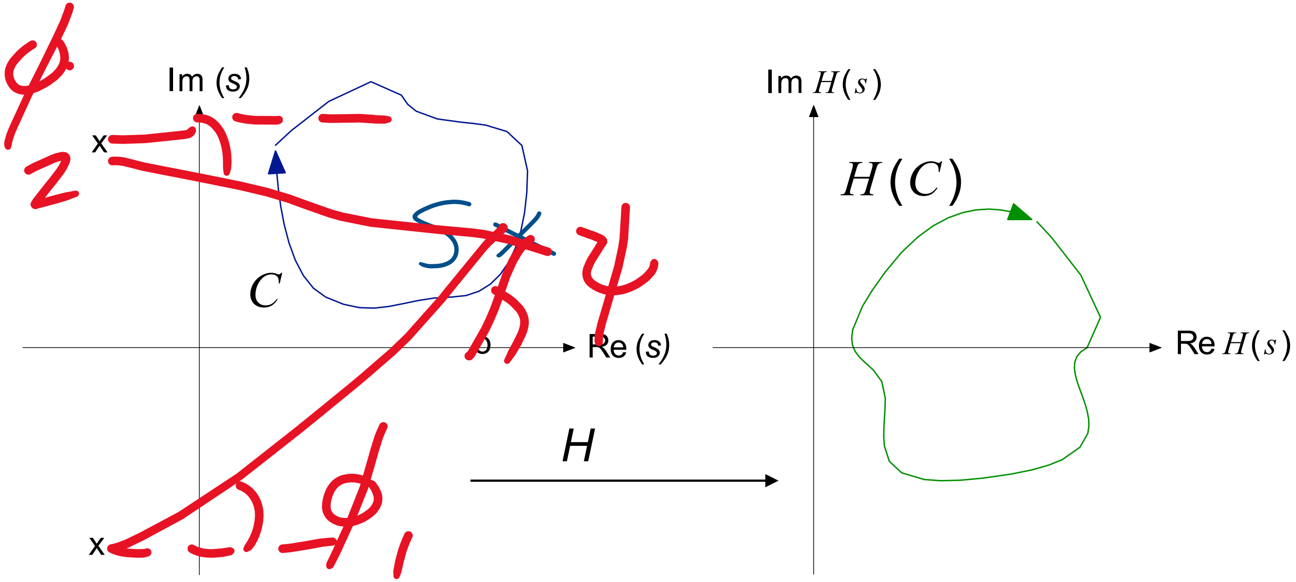
Case 3: Contour Encircles No Poles or Zeros

Suppose that C is a closed, CW-oriented contour in $\mathbb{C}$ that does not encircle any poles or zeros of $H(s)$:



How does $\angle H(s)$ change as we go around C?

Case 3: Contour Encircles No Poles or Zeros

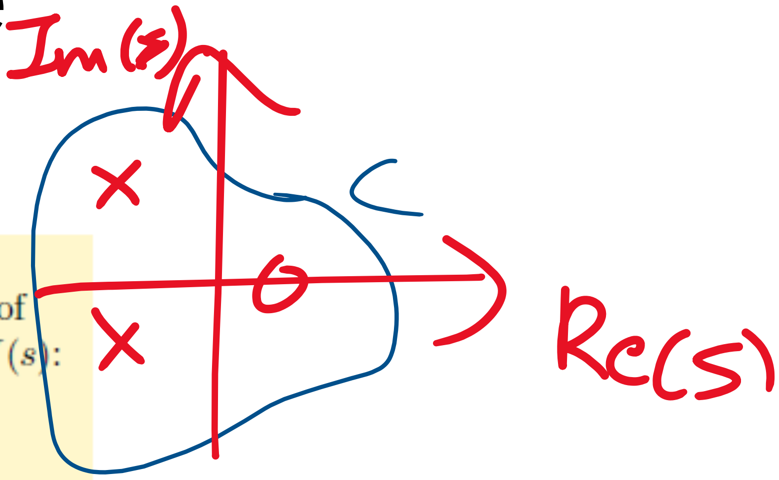


How does $\angle H(s)$ change as we go around C?

Let's see what happens to angles from s to poles/zeros of H:

- $\phi_1,\ \phi_2,\ \psi_1$ all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin

The Argument Principle



These special cases all lead to the following general result:

The Argument Principle. Let C be a closed, clockwise ⟳ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of C under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$

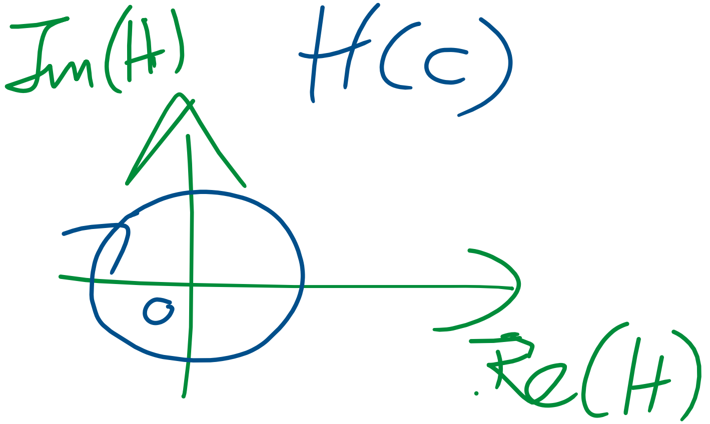
Then:

origin

$$\#(\text{clockwise encirclements} \circlearrowright \text{ of } 0 \text{ by } H(C))$$
$$= \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).$$

More succinctly,

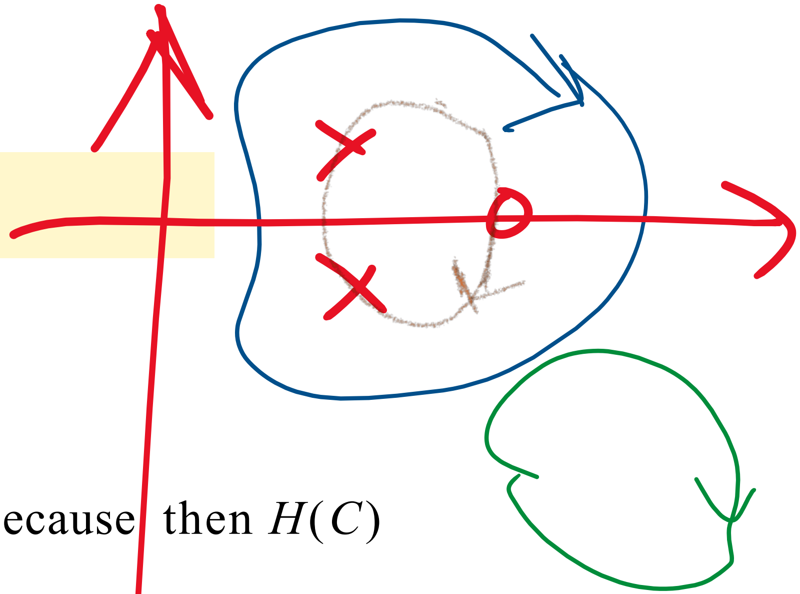
$$\boxed{N = Z - P}$$



* will see the reason for this later ...

The Argument Principle

$$N = Z - P$$

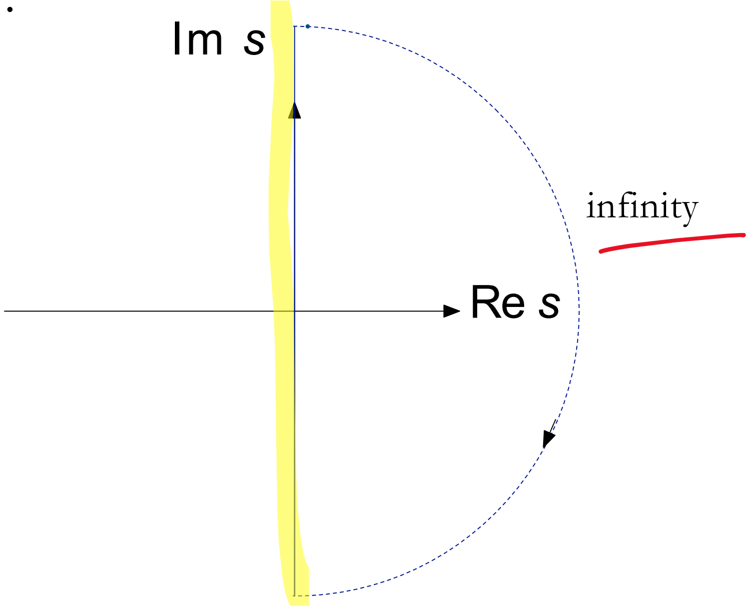


- If $N < 0$, it means that $H(C)$ encircles the origin counterclockwise (O).
- We do not want C to pass through any pole of H because then $H(C)$ would not be defined.
- We also do not want C to pass through any zero of H because then $0 \in H(C)$, so #(encirclements) is not well-defined.

Argument Principle to Nyquist Criterion

- We are interested in RHP poles, so let's choose a suitable contour C that *encloses the RHP*:

Harry Nyquist (1889–1976)



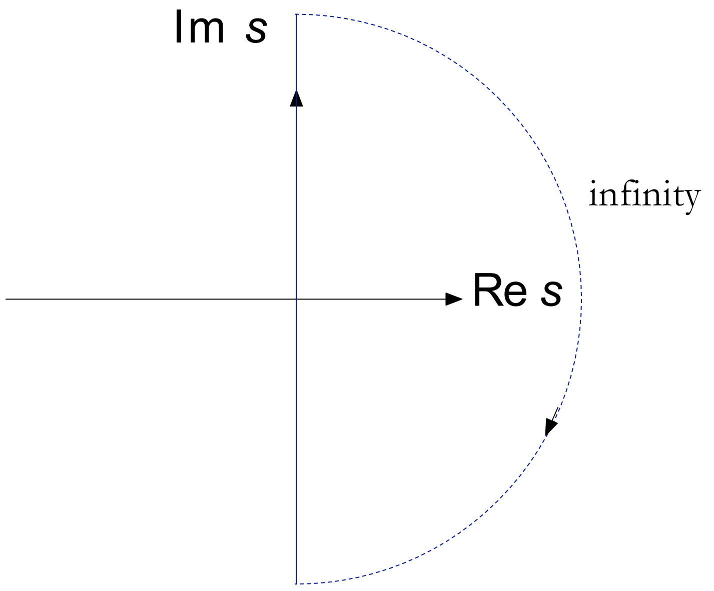
- From now on, C = imaginary axis plus the "path around infinity."
- If H is strictly proper, then $H(\infty) = 0$.

Argument Principle to Nyquist Criterion

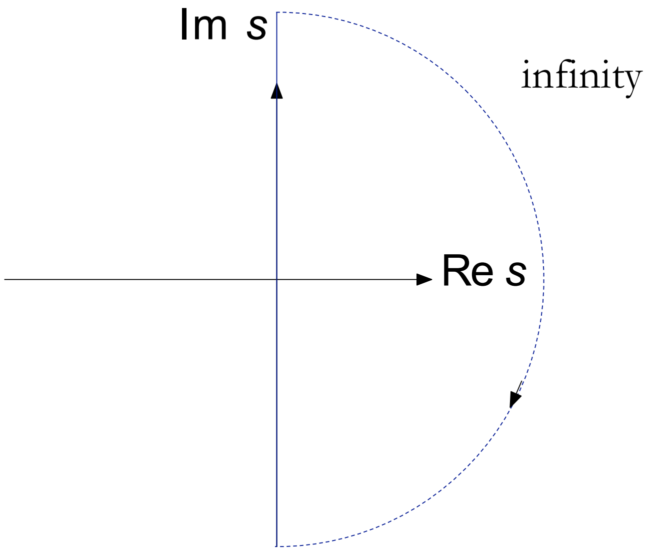
With this choice of C,

$$H(C) = \text{ Nyquist plot of } H$$

(image of the imaginary axis under the map $H : \mathbb{C} \to \mathbb{C}$; if H is strictly proper, $0 = H(\infty)$)



Argument Principle to Nyquist Criterion

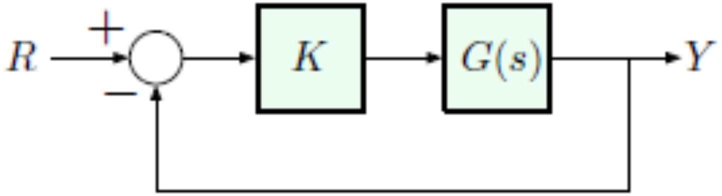


$$H(C) = \text{Nyquist plot of } H$$

We are interested in RHP roots of $1 + KG(s)$, where G is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$

Argument Principle to Nyquist Criterion



Examining the Nyquist plot of $H(s) = 1 + KG(s)$.

By the argument principle,

$$\boxed{N = Z - P,}$$

where $N = \#($CW encirclements of 0 origin

by Nyquist plot of $1 + KG(s))$,

$Z = \#($zeros of $1 + KG(s)$ inside $C)$,

$P = \#($poles of $1 + KG(s)$ inside $C)$

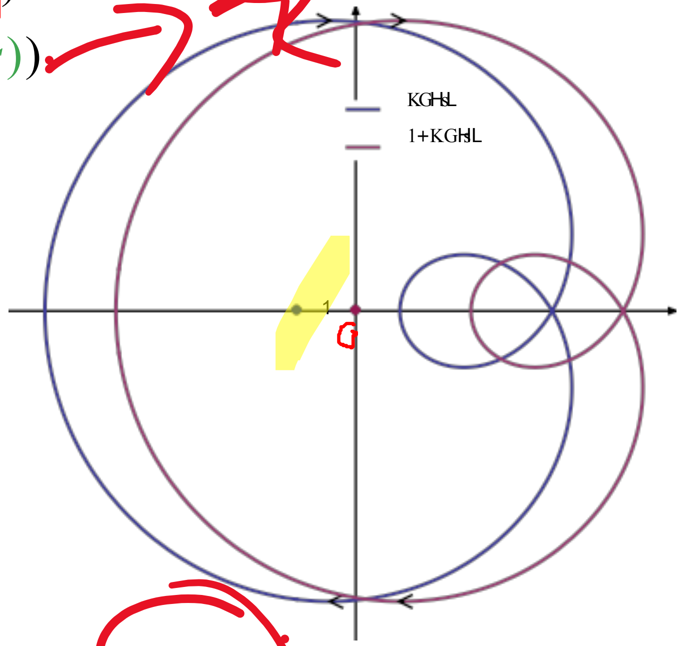
Now we extract information about RHP roots of $1 + KG(s)$

Nyquist Criterion: N

→ origin

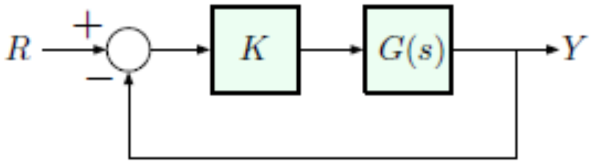
$$
\begin{aligned}
N &= \#(\text{CW encirclements of } 0 \text{ by Nyquist plot of } 1 + \underline{KG(s)}) \\
&= \#(\text{CW encirclements of } -1 \text{ by Nyquist plot of } \boxed{KG(s)}) \\
&= \#(\text{CW encirclements of } -1/K \text{ by Nyquist plot of } G(s))
\end{aligned}
$$

$\frac{1}{K}$



— can be read off the Nyquist plot of the *open-loop* t.f. G!!

Nyquist Criterion: Z



$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \le \deg(p)$$

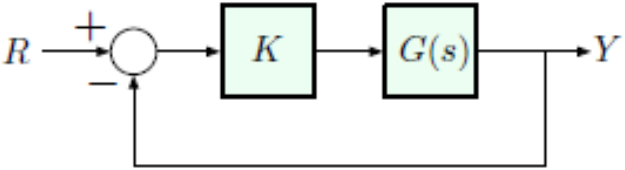
$$1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}$$

$$\text{closed-loop t.f.} = \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)}$$

Therefore:

$$\begin{aligned} Z &= \#(\text{zeros of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP zeros of } 1 + KG(s)) \\ &= \#(\text{RHP closed-loop poles}) \end{aligned}$$

Nyquist Criterion: P



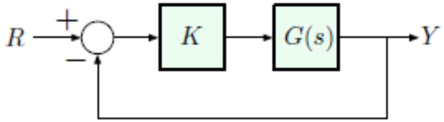
$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \leq \deg(p)$$

$$1 + KG(s) = 1 + K\frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}$$

Therefore:

$$\begin{aligned} P &= \#(\text{poles of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP poles of } 1 + KG(s)) \\ &= \#(\text{RHP roots of } p(s)) \\ &= \#(\text{RHP open-loop poles}) \end{aligned}$$

The Nyquist Theorem



Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

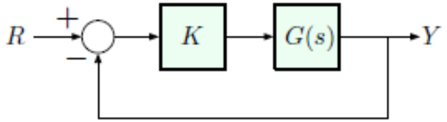
$$N = Z - P$$ CW

#($\cancel{\circlearrowright}$ of $-1/K$ by Nyquist plot of $G(s)$)

$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$

* Easy to fix: draw an infinitesimally small circular path that goes *around* the pole and stays in RHP

The Nyquist Stability Criterion



$$\underbrace{N}_{\#(\circlearrowright \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

$$Z = N + P$$

$$Z = 0 \qquad \Longrightarrow N = -P$$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

The Nyquist Stability Criterion

Workflow:

$$\text{Bode } M \text{ and } \phi\text{-plots} \quad \longrightarrow \quad \text{Nyquist plot}$$

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)