ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

# ECE 486 Control Systems

## Lecture 12: Frequency Response

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**The syllabus and much of the lecture content originated from UIUC ECE 486 course material

# Checklist

ZJU-UIUC INSTITUTE
Zhejiang University/University of Illinois at Urbana-Champaign Institute
浙江大学伊利诺伊大学厄巴纳香槟校区联合学院

Modeling | Analysis | Design: Root Locus, Frequency Response, State-Space

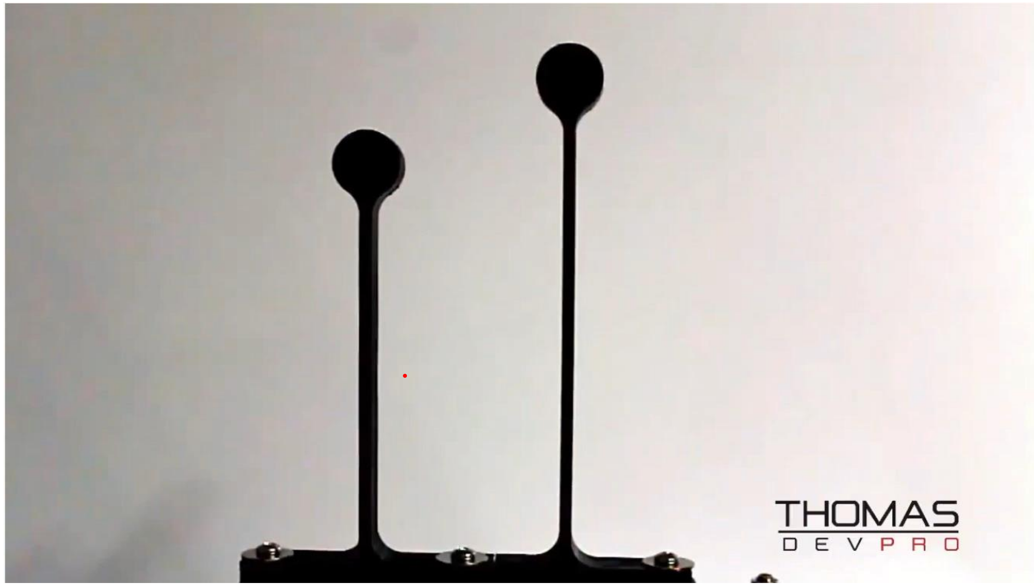
| Wk | Topic | Ref. |
|---|---|---|
| 1 | ✓ Introduction to feedback control | Ch. 1 |
| | ✓ State-space models of systems; linearization | Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1 |
| 2 | ✓ Linear systems and their dynamic response | Section 3.1, Appendix A |
| | ✓ Transient and steady-state dynamic response with arbitrary initial conditions | Section 3.1, Appendix A |
| 3 | ✓ National Holiday Week | |
| 4 | ✓ System modeling diagrams; prototype second-order system | Sections 3.1, 3.2, lab manual |
| | ✓ Transient response specifications | Sections 3.3, 3.14, lab manual |
| 5 | ✓ Effect of zeros and extra poles; Routh-Hurwitz stability criterion | Sections 3.5, 3.6 |
| | ✓ Basic properties and benefits of feedback control; Introduction to Proportional-Integral-Derivative (PID) control | Section 4.1-4.3, lab manual |
| 6 | ✓ Review A | |
| | ✓ Term Test A | |
| 7 | ✓ Introduction to Root Locus design method | Ch. 5 |
| | ✓ Root Locus continued; introduction to dynamic compensation | Root Locus |
| 8 | ✓ Lead and lag dynamic compensation | Ch. 5 |
| | ✓ Introduction to frequency-response design method | Sections 5.1-5.4, 6.1 |

Modeling (weeks 1–4), Analysis (weeks 4–5), Root Locus (weeks 7–8)

| Wk | Topic | Ref. |
|---|---|---|
| 9 | **Bode plots for three types of transfer functions** | Section 6.1 |
| | Stability from frequency response; gain and phase margins | Section 6.1 |
| 10 | Control design using frequency response | Ch. 6 |
| | Control design using frequency response continued; PI and lag, PID and lead-lag | Frequency Response |
| 11 | Nyquist stability criterion | Ch. 6 |
| | Nyquist stability criterion continued; gain and phase margins from Nyquist plots | Ch. 6 |
| 12 | Review B | |
| | Term Test B | |
| 13 | Introduction to state-space design | Ch. 7 |
| | Controllability, stability, and pole-zero cancellations; similarity transformation; conversion of controllable systems to Controller Canonical Form | Ch. 7 |
| 14 | Pole placement by full state feedback | Ch. 7 |
| | Observer design for state estimation | State-Space |
| 15 | Joint observer and controller design by dynamic output feedback; separation principle | Ch. 7 |
| | In-class review | Ch. 7 |
| 16 | END OF LECTURES: Revision Week | |
| | Final | |

"If you want to find the secrets of the universe, think in terms of energy, frequency and vibration."

**— Nikola Tesla**




https://thomasdevpro.com/

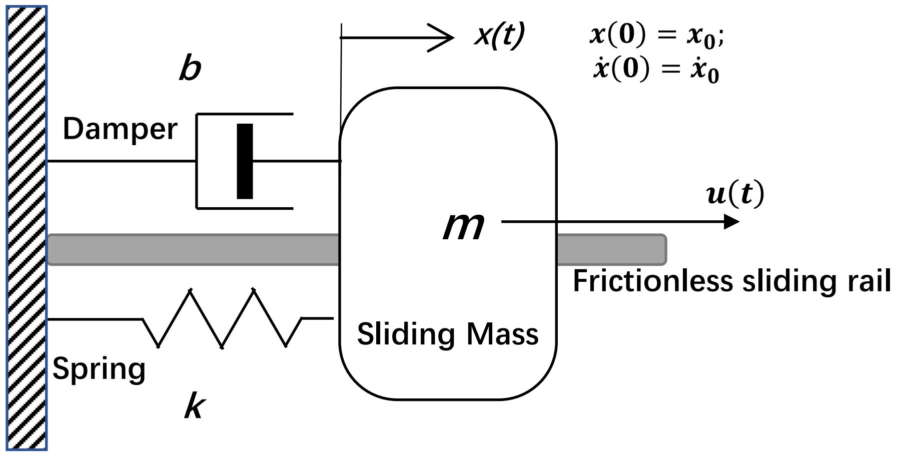
# Lecture Overview

- Post Term Test I Review
- Introduction to Frequency Response

- Learning Goal: preparing to use frequency response as an alternative method for control systems design; learn to analyze and sketch magnitude and phase plots of transfer functions

Reading: FPE, Chapter 6 Section 6.1

# Recap: System Response




$$\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = u(t)$$

Transient response analysis

Laplace Transform (account for ICs!)

$$[s^2X(s) - sx_0 - \dot{x}_0] + 2\zeta\omega_n[sX(s) - x_0] + {\omega_n}^2X(s) = U(s)$$

$$s^2X + 2\zeta\omega_n sX + {\omega_n}^2X - (sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0) = U$$

$$X(s) = \frac{sx_0 + \dot{x}_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$$\boxed{X(s) = \frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} + \frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}}$$

Inverse Laplace Transform

$$\boxed{x(t) = \boxed{L^{-1}\left\{\frac{(s + 2\zeta\omega_n)x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}} + \boxed{L^{-1}\left\{\frac{U(s)}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}\right\}}}$$

# Recall: 2nd Order System Response

For the 2nd order equation, $\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = f(t)$

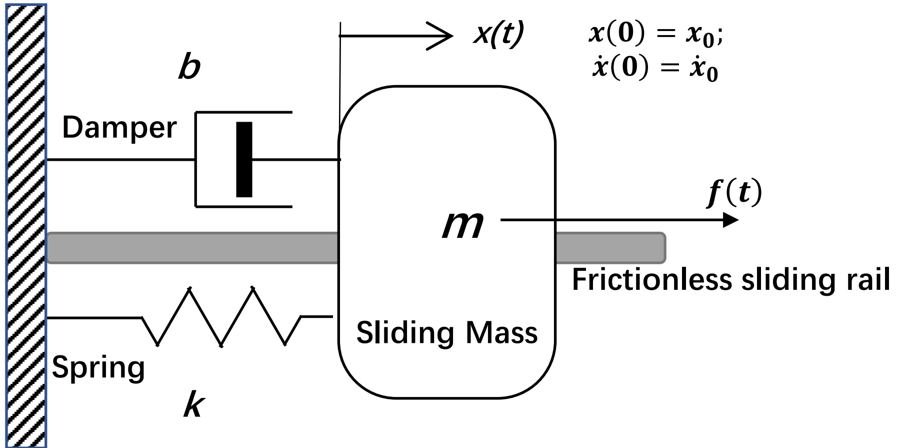
Working out the general solution yields

$$x = \left(2\zeta\omega_n x(0) + \dot{x}(0)\right)h(t) + x(0)\dot{h}(t) \boxed{+ f(t) * h(t)}$$

where $f(t) * h(t) = \int_0^t f(\tau)h(t-\tau)d\tau$,

$h(t)$ is the **unit-impulse response**.

When $f(t) = 0$, $x(0) = 0$, and $\dot{x}(0) = 1$, $h(t)$ is a solution

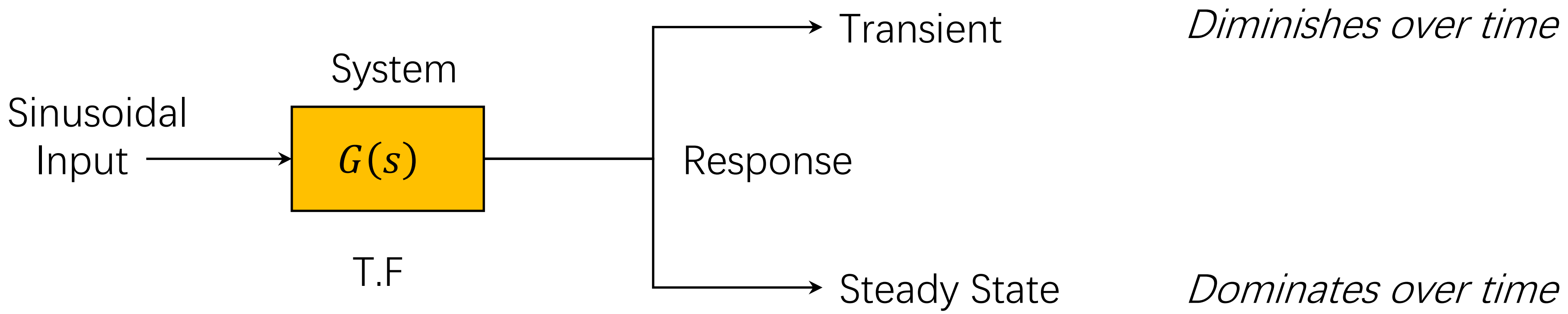


# Recap: System Response (Overview)

- System response describes the behavior of a dynamic system
- Free and Forced Response
  - Total Response $x(t)=x_h(t)+x_p(t)$
  - Free Response: $x_h$ is the solution
  - Forced Response: $x_p$ is determined by the forcing function $f$
- Transient and Steady-State
  - Total Response $x(t)=x_{tr}(t)+x_{ss}(t)$
  - $x_{\mathrm{tr}}$, Transient State: component that decays towards zero
  - $x_{ss}$, Steady State: component that remains after the $x_t$ decays towards 0

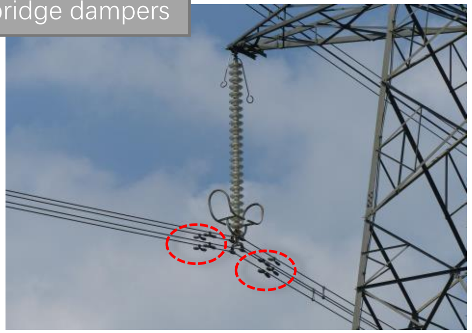
# Recap: Frequency Response

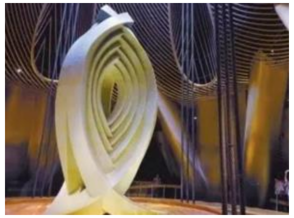
- The steady-state response to a sinusoidal input is known as the frequency response

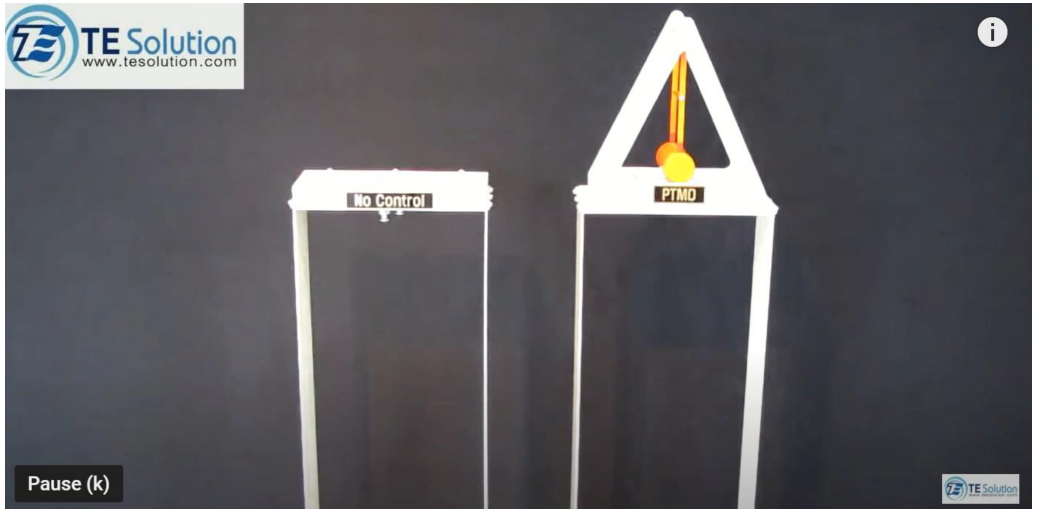


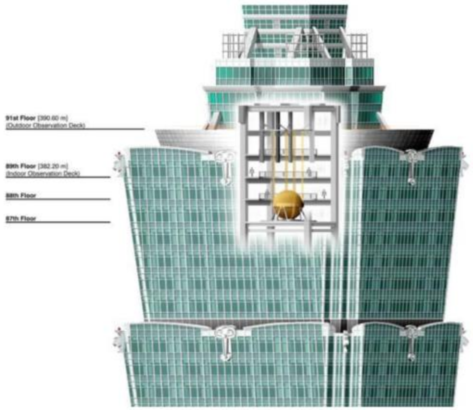
# Engineering Examples

Stockbridge dampers





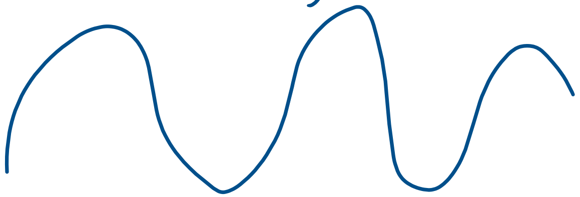
https://www.shanghaitower.com/shanghaizhongxin/index8.php

https://www.youtube.com/watch?v=GzMuF-LMGaM

# Frequency Response Design Method

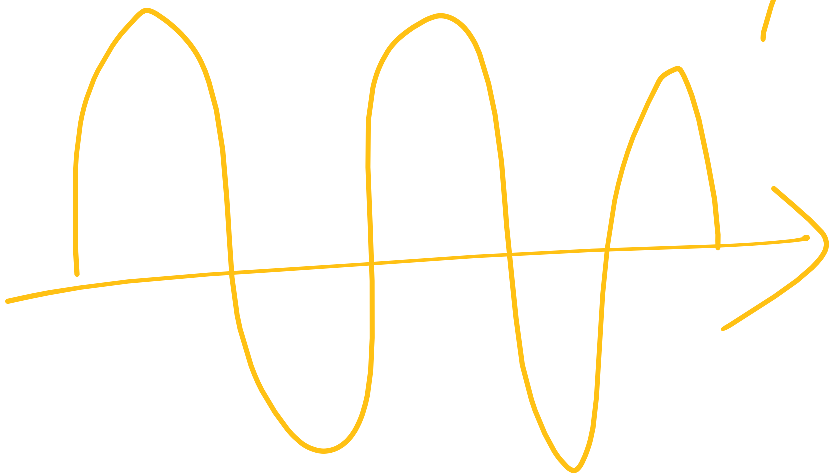
- Recall the frequency-response formula:

$u(t) = \sin(\omega t)$



$\boxed{G(s)}$

$M\sin(\omega t+\phi)$



$M = |G(j\omega)|_{s=j\omega}$

$\phi = \angle G(j\omega)$

# Frequency Response Design Method

- Recall the frequency-response formula:

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow M\sin(\omega t + \phi)$$

where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Derivation:

1. $u(t) = e^{st} \longmapsto y(t) = G(s)e^{st}$
2. Euler's formula: $\sin(\omega t) = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$
3. By linearity,

$$\begin{aligned}\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j} \quad & G(j\omega) = M(\omega)e^{j\phi(\omega)} \\ = \frac{M(\omega)e^{j(\omega t+\phi(\omega))} - M(\omega)e^{-j(\omega t+\phi(\omega))}}{2j} & \\ = M(\omega)\sin(\omega t + \phi(\omega)) &\end{aligned}$$

Let's apply this formula to our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\begin{aligned} M(\omega) = |G(j\omega)| &= \left|\frac{\omega_n^2}{-\omega^2 + 2j\zeta\omega_n\omega + \omega_n^2}\right| \\ &= \left|\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}j}\right| \\ &= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} \end{aligned}$$

# Frequency Response Design Method

$$\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow M\sin(\omega t + \phi)$$

where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

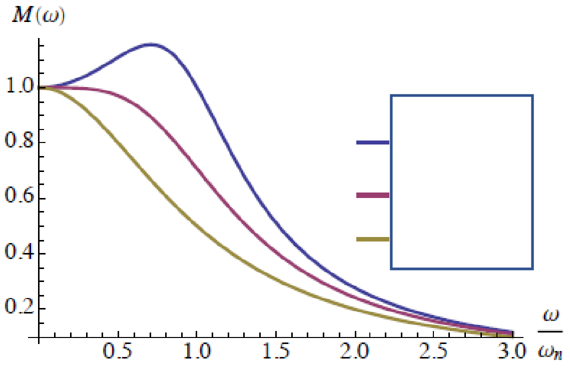
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1 + (4\zeta^2-2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4}}$$

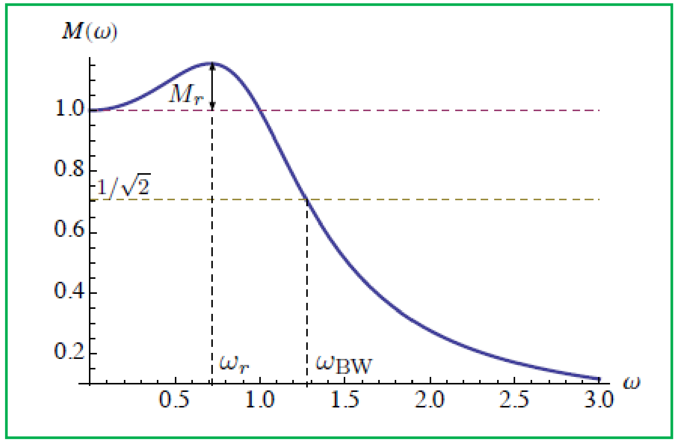
For our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1 + (4\zeta^2-2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4}}$$




A typical freq. response magnitude plot




small $M_r$ $\longleftrightarrow$ better damping

large $\omega_{BW}$ $\longleftrightarrow$ large $\omega_n$ $\longleftrightarrow$ smaller $t_r$

Info about time response also encoded in frequency response

$\omega_r$ – resonant frequency

$M_r$ – resonant peak

$\omega_{BW}$ – bandwidth

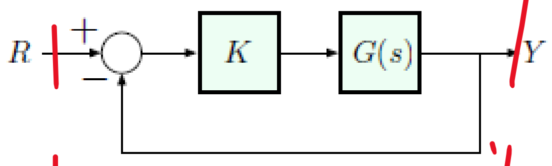
We can get the following formulas using calculus:

$$\begin{cases} \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \quad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}}; \text{ for } \zeta \geq \frac{1}{\sqrt{2}}, \omega_r = 0\right)$$

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2) + \underbrace{\sqrt{(1-2\zeta^2)^2+1}}_{=1 \text{ for } \zeta=1/\sqrt{2}}}$$

— so, if we know $\omega_r, M_r, \omega_{BW}$, we can determine $\omega_n, \zeta$ and hence the time-domain specs $(t_r, M_p, t_s)$

# Frequency Response Design Method



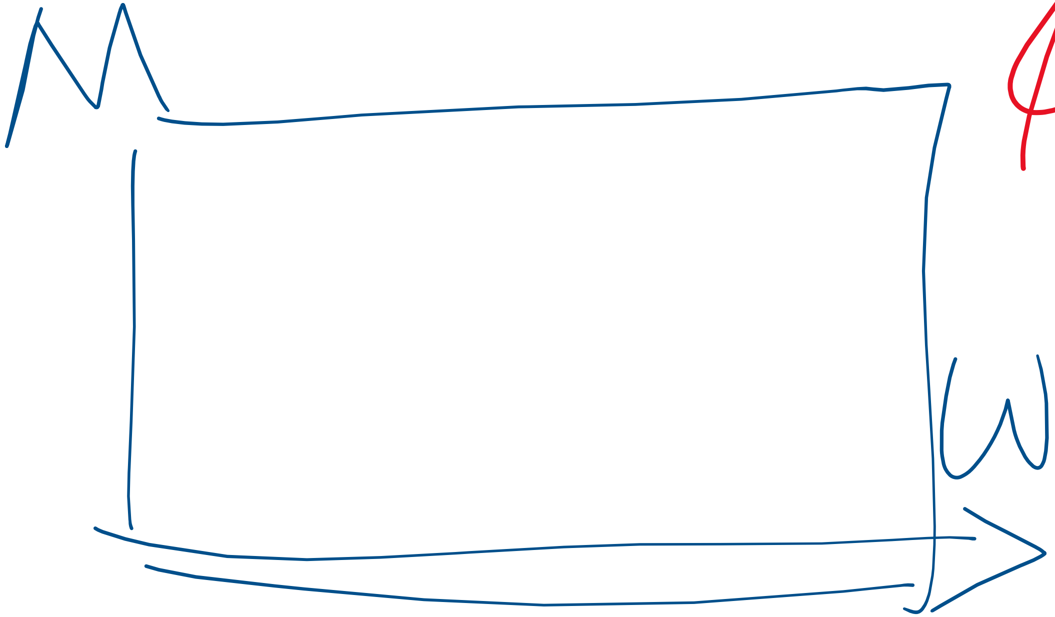

$$H_{CL}(s) = \frac{KG}{1+KG}$$

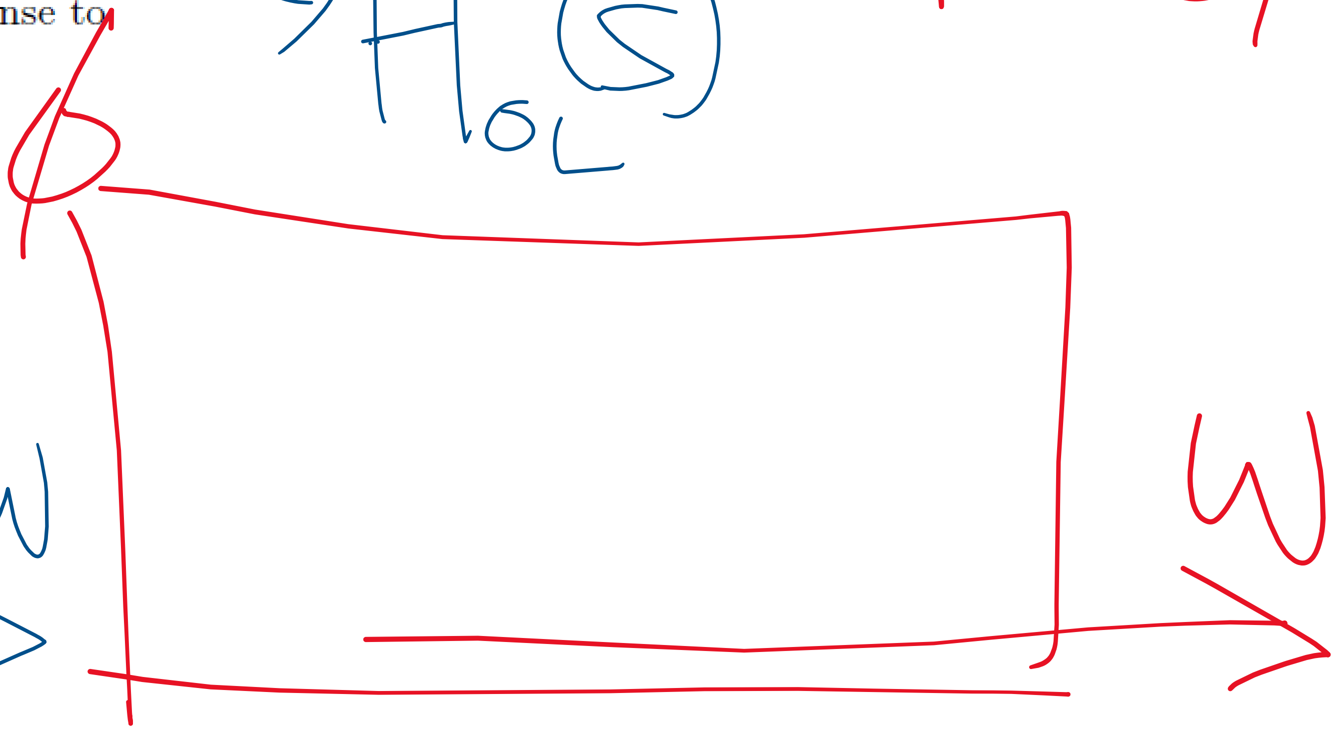
Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $\boxed{KG(s)}$ [or, more generally, $D(s)G(s)$], at $s = j\omega$

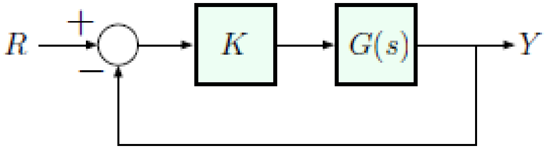
$H_{OL}(s)$

2. See how to relate this open-loop frequency response to closed-loop behavior.



# Frequency Response (How)




Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior.

We will work with two types of plots for $KG(j\omega)$:

1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency $\omega$
2. Nyquist plots: $\mathrm{Im}(KG(j\omega))$ vs. $\mathrm{Re}(K(j\omega))$ [Cartesian plot in $s$-plane] as $\omega$ ranges from $-\infty$ to $+\infty$

| | magnitude | phase |
|---|---|---|
| horizontal scale | log | log |
| vertical scale | log | linear |

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

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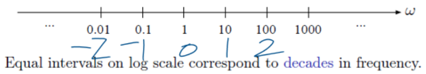
## Note on the Scale

Horizontal ($\omega$) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of $\omega$, *not* $\log_{10}\omega$, on the axis, but the *scale* will be logarithmic:




Equal intervals on log scale correspond to decades in frequency.

Vertical axis on magnitude plots:

we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = M_1 \cdot M_2$$
$$\log(M_1 M_2) = \log M_1 + \log M_2$$

— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log(M_1(\omega)M_2(\omega))$, and graphical addition is easy.

Decibel scale:

$$(M)_{\mathrm{dB}} = 20\log_{10} M \qquad \text{(one decade} = 20\,\mathrm{dB)}$$

Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

Reason:

$$\angle\Big((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\Big) = \angle\Big(M_1 M_2 e^{j(\phi_1+\phi_2)}\Big)$$
$$= \phi_1 + \phi_2$$

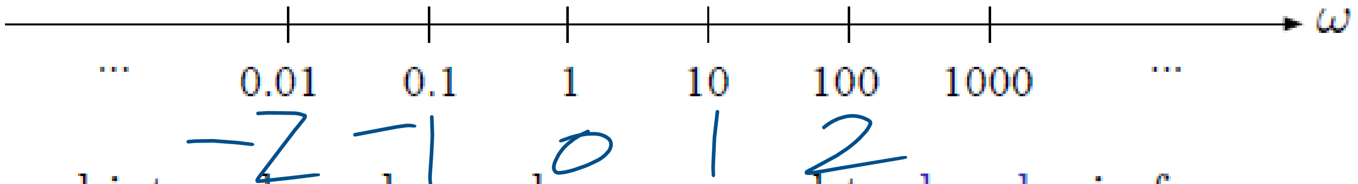
— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

# Note on the Scale

Horizontal ($\omega$) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of $\omega$, *not* $\log_{10}\omega$, on the axis, but the *scale* will be logarithmic:




Equal intervals on log scale correspond to decades in frequency.

# Note on the Scale

Vertical axis on magnitude plots: log

we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = \boxed{M_1 \cdot M_2}$$
$$\log(M_1 M_2) = \boxed{\log M_1 + \log M_2}$$

— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log\big(M_1(\omega)M_2(\omega)\big)$, and graphical addition is easy.

Decibel scale:

$$\boxed{(M)_{\mathrm{dB}} = 20\log_{10} M} \qquad \text{(one decade} = 20\,\mathrm{dB})$$

Vertical axis on phase plots: linear

we will plot the phase on the usual (linear) scale.

Reason:

$$\angle\Big((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\Big) = \angle\Big(M_1 M_2 e^{j(\phi_1+\phi_2)}\Big)$$
$$\boxed{= \phi_1 + \phi_2}$$

— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

# Scale Convention for Bode Plots

| | magnitude | phase |
|---|---|---|
| horizontal scale | log | log |
| vertical scale | log | linear |

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.

# Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

Example:

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

rewrite as
$$\left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$
$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$

Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where $n$ is a positive or negative integer
2. $(j\omega\tau+1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$

In our example above,

$$KG(j\omega) = \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]}$$
$$= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}}\cdot\underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}}\cdot\underbrace{\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]^{-1}}_{\text{Type 3}}$$

# Recap: Type 1: $\boxed{K_0(j\omega)^n}$

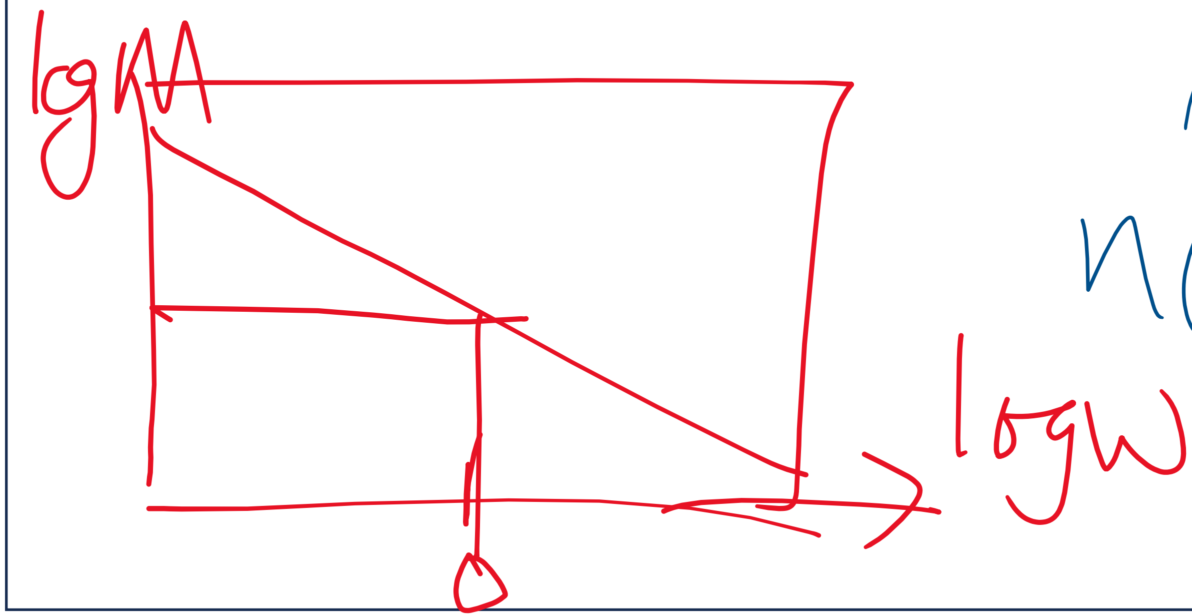
$Y = C + gX \to 0$

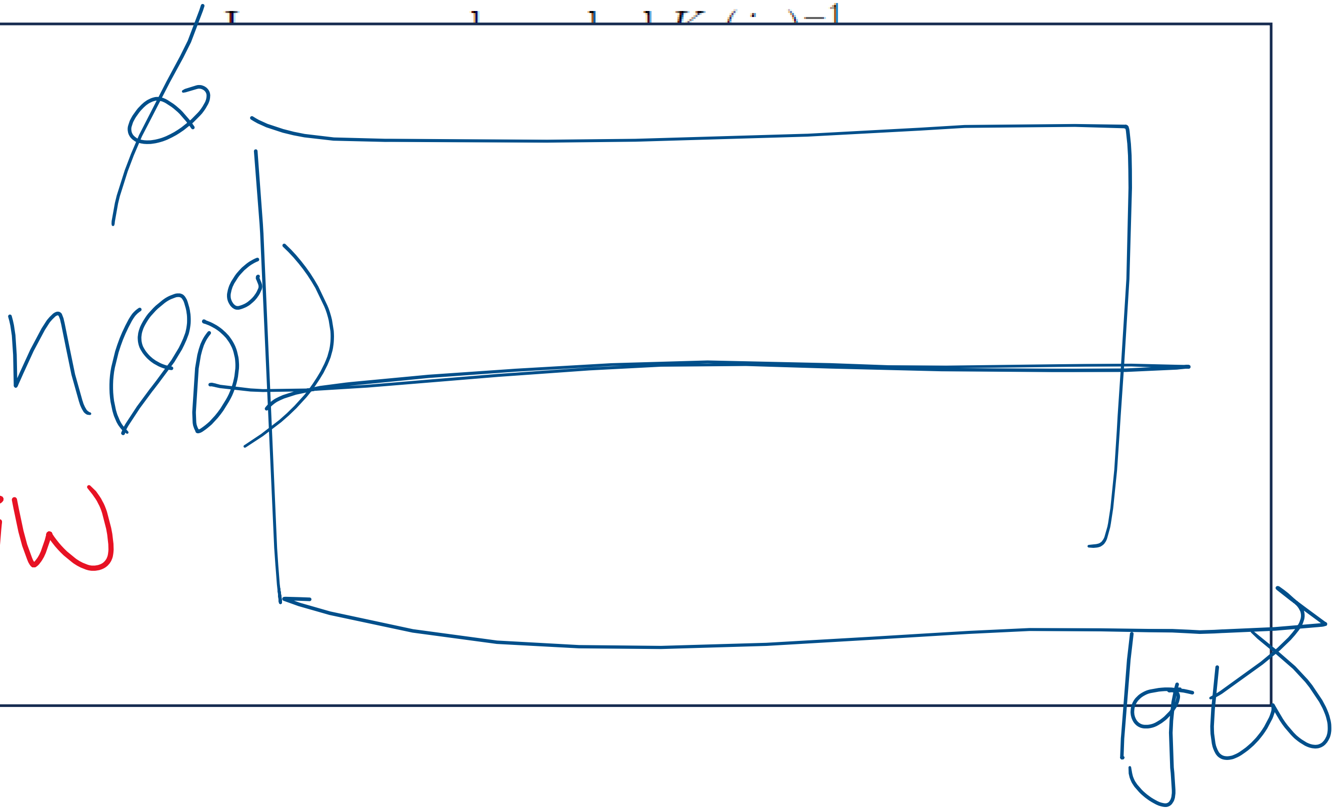
**Magnitude:** $\log M = \log|K_0(j\omega)^n| = \log|K_0| + n\log\omega$

— as a function of $\log\omega$, this is a *line* of slope $n$ passing through the value $\log|K_0|$ at $\omega = 1$

**Phase:** $\angle K_0(j\omega)^n = \angle(j\omega)^n = n\angle j\omega = n\cdot 90°$

— this is a constant, independent of $\omega$.



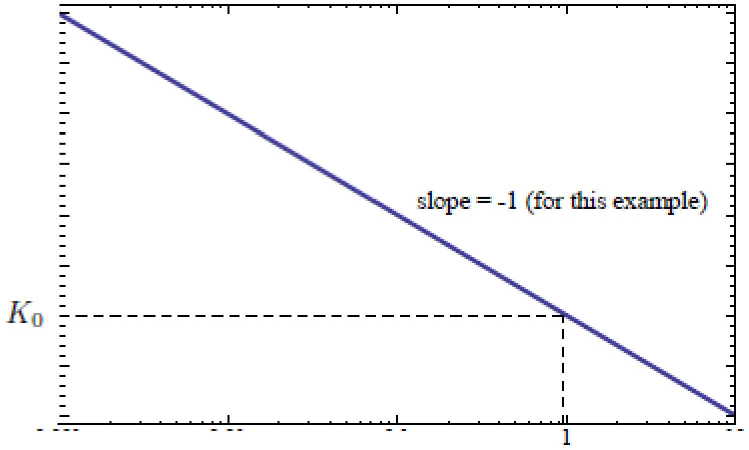



# Recap: Type 1: $K_0(j\omega)^n$

Magnitude: $\log M = \log |K_0(j\omega)^n| = \log |K_0| + n \log \omega$

— as a function of $\log \omega$, this is a *line* of slope $n$ passing through the value $\log |K_0|$ at $\omega = 1$

In our example, we had $K_0(j\omega)^{-1}$:



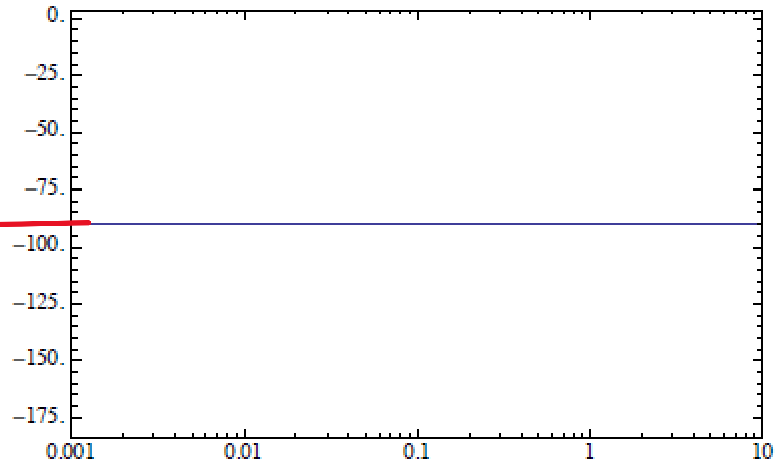

— this is called a low-frequency asymptote (will see why later)

Phase: $\angle K_0(j\omega)^n = \angle(j\omega)^n = n\angle j\omega = n \cdot 90^\circ$

— this is a constant, independent of $\omega$.

In our example, we had $K_0(j\omega)^{-1}$:

$-90^\circ$



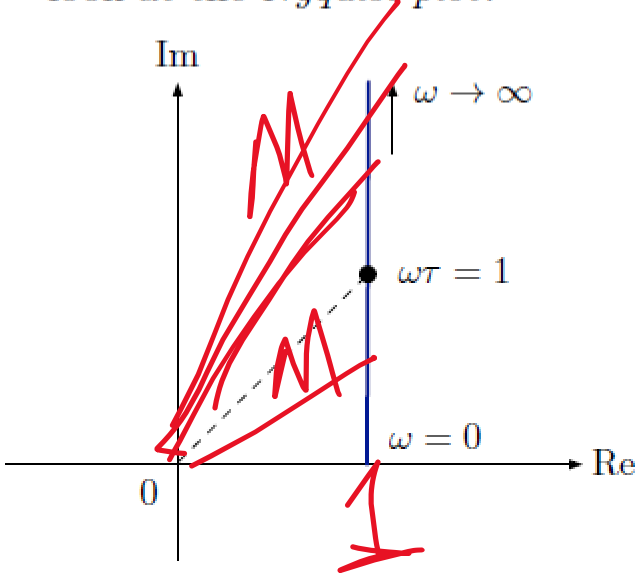

— here, the phase is $-90^\circ$ for all $\omega$.

# Recap: Type 2: $j\omega\tau + 1$

$K_0((j\omega)^n$

This is the case of a *stable real zero.*

To study $|j\omega\tau + 1|$ and $\angle(j\omega\tau + 1)$ as a function of $\omega$, we will look at the *Nyquist plot*:




For $\omega\tau \ll 1$, $j\omega\tau + 1 \approx 1$

$\omega\tau \gg 1$, $j\omega\tau + 1 \approx j\omega\tau$

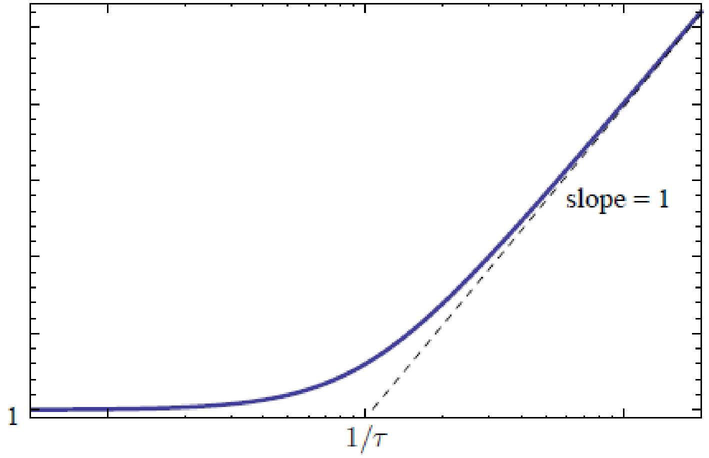
(like Type 1 with $K_0 = \tau, n = 1$)

Transition:

$$\omega\tau = 1 \iff \omega = 1/\tau$$

— this is the breakpoint

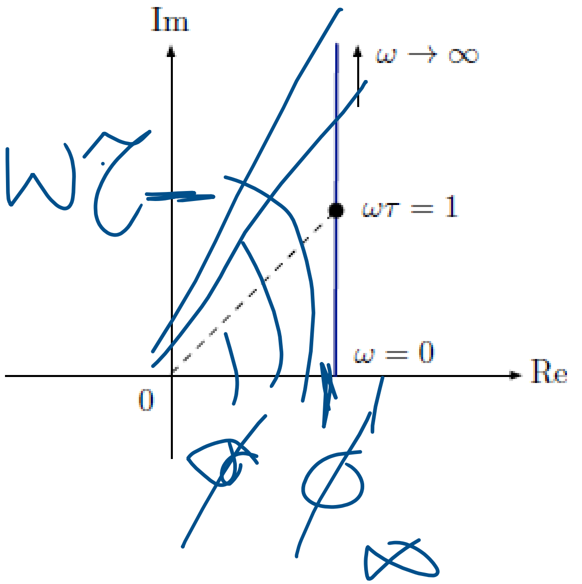
Magnitude plot:




For a stable real zero, the magnitude slope "steps up by 1" at the break-point.

# Recap: Type 2: $j\omega\tau + 1$

**Phase:**



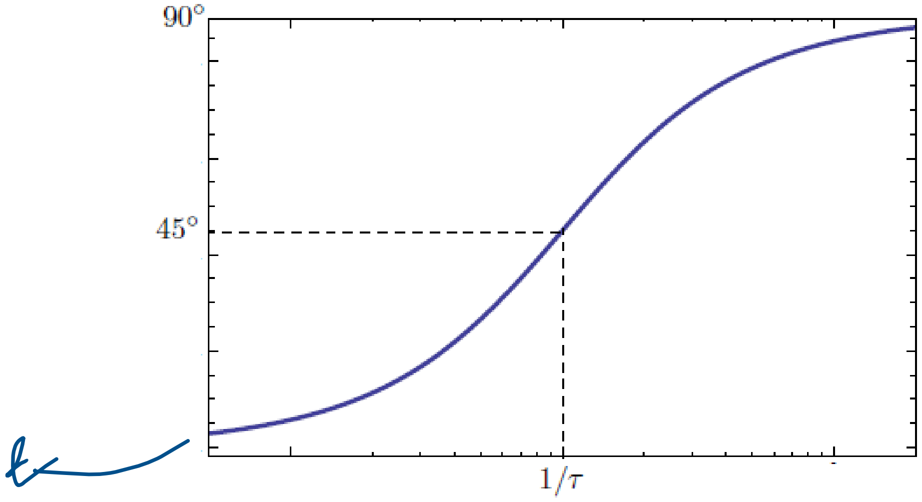

- For small $\omega$ (below break-point), $\phi \approx 0°$
- For large $\omega$ (above break-point),

$$\phi \approx \angle(j\omega\tau) = 90°$$

- At break-point ($\omega\tau = 1$),

$$\phi = \angle(j + 1) = 45°$$

**Phase plot:**




For a stable real zero, the phase "steps up by 90°" as we go past the break-point.

# Recap: Type 2: $(j\omega\tau + 1)^{-1}$

This is a stable real pole.

Magnitude:

$$\log\left|\frac{1}{j\omega\tau+1}\right| = -\log|j\omega\tau+1|$$

Phase:

$$\angle\frac{1}{j\omega\tau+1} = -\angle(j\omega\tau+1)$$

So the magnitude and phase plots for a stable real pole are the reflections of the corresponding plots for the stable real zero w.r.t. the horizontal axis:

- step down by 1 in magnitude slope
- step down by 90° in phase

# Example Type 1 and 2 Factors

$(j\omega\tau+1)^{\pm 1}$

$$KG(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}$$

Convert to Bode form:

$s = j\omega$

$$KG(j\omega) = \frac{2000 \cdot 0.5 \cdot \left(\frac{j\omega}{0.5}+1\right)}{10 \cdot 50 \cdot j\omega \left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

$$= \boxed{\frac{2}{j\omega}} \cdot \boxed{\left(\frac{j\omega}{0.5}+1\right)} \cdot \frac{1}{\boxed{\left(\frac{j\omega}{10}+1\right)}\boxed{\left(\frac{j\omega}{50}+1\right)}}$$

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5}+1\right) \cdot \frac{1}{\left(\frac{j\omega}{10}+1\right)\left(\frac{j\omega}{50}+1\right)}$$

Type 1 term:

- $K_0 = 2, n = -1$ — it contributes a line of slope $-1$ passing through the point $(\omega = 1, M = 2)$.
- This is a low-frequency asymptote: for small $\omega$, it gives very large values of $M$, while other terms for small $\omega$ are close to $M = 1$ (since $\log 1 = 0$).

Now we mark the break-points, from Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ slope steps up by 1
- $\omega = 10$ stable pole $\Rightarrow$ slope steps down by 1
- $\omega = 50$ stable pole $\Rightarrow$ slope steps down by 1

# Magnitude

## Bode Form (Magnitude)

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

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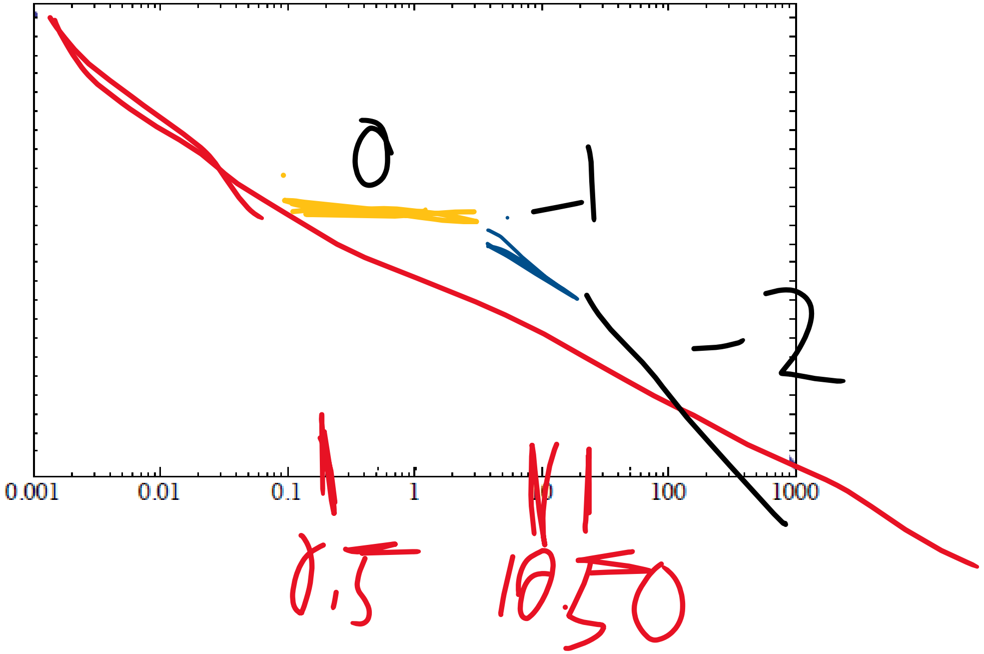
## Bode Plot (Magnitude)

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

-1



# Magnitude

## Bode Form (Magnitude)

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

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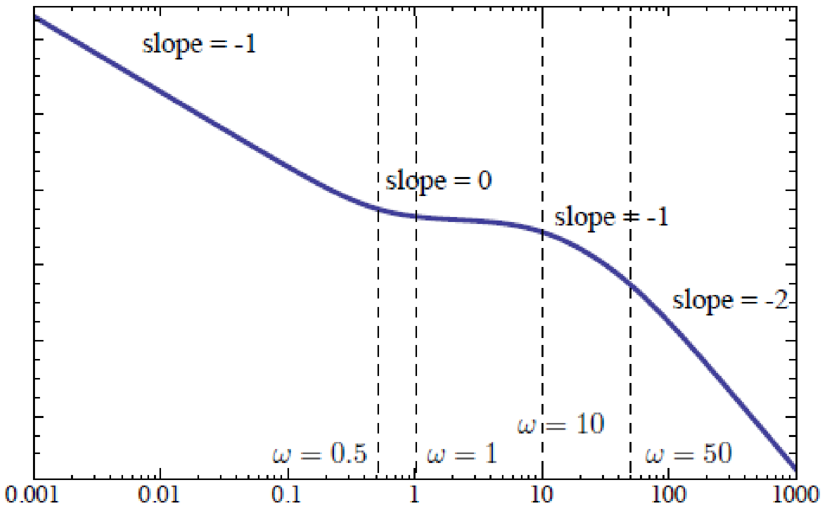
- $\omega = 10$ stable pole $\Rightarrow$ slope steps down by 1
- $\omega = 50$ stable pole $\Rightarrow$ slope steps down by 1

## Bode Plot (Magnitude)

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



# Phase

## Bode Form (Phase)

Transfer function in Bode form:

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$

Type 1 term:

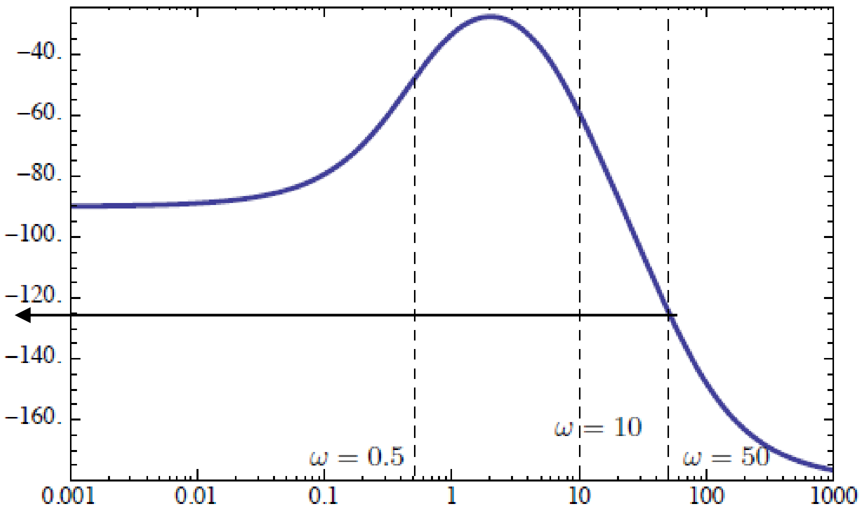
- $n = -1$ — phase starts at $-90°$

Type 2 terms:

- $\omega = 0.5$ stable zero $\Rightarrow$ phase up by $90°$ (by $45°$ at $\omega = 0.5$)
- $\omega = 10$ stable pole $\Rightarrow$ phase down by $90°$ (by $45°$ at $\omega = 10$)
- $\omega = 50$ stable pole $\Rightarrow$ phase down by $90°$ (by $45°$ at $\omega = 50$)

## Bode Plot (Phase)

$$KG(j\omega) = \frac{2}{j\omega} \cdot \left(\frac{j\omega}{0.5} + 1\right) \cdot \frac{1}{\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}$$



# Recap: Bode Form of the Transfer Function

Bode form of $KG(s)$ is a factored form with the constant term in each factor equal to 1, i.e., lump all DC gains into one number in the front.

Example:

$$KG(s) = K\frac{s+3}{s(s^2+2s+4)}$$

rewrite as
$$\left.\frac{3K\left(\frac{s}{3}+1\right)}{4s\left(\left(\frac{s}{2}\right)^2+\frac{s}{2}+1\right)}\right|_{s=j\omega}$$

$$= \underbrace{\frac{3K}{4}}_{=K_0}\frac{\frac{j\omega}{3}+1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right)}$$

Transfer functions in Bode form will have three types of factors:

1. $K_0(j\omega)^n$, where $n$ is a positive or negative integer
2. $(j\omega\tau+1)^{\pm 1}$
3. $\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^{\pm 1}$

In our example above,

$$KG(j\omega) = \frac{3K}{4}\frac{\frac{j\omega}{3}+1}{j\omega\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]}$$

$$= \underbrace{\frac{3K}{4}(j\omega)^{-1}}_{\text{Type 1}}\cdot\underbrace{\left(\frac{j\omega}{3}+1\right)}_{\text{Type 2}}\cdot\underbrace{\left[\left(\frac{j\omega}{2}\right)^2+\frac{j\omega}{2}+1\right]^{-1}}_{\text{Type 3}}$$

# Type 3: $\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$

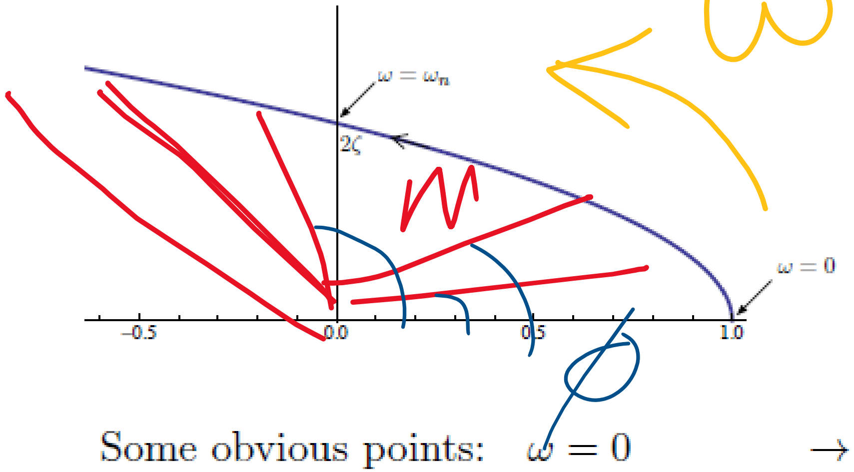
Stable complex zero — more difficult than Types 1 & 2.

First step — let's rewrite in Cartesian form:

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1 = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) + 2\zeta\frac{\omega}{\omega_n}j$$

And here is the Nyquist plot, for $0 < \omega < \infty$:

$\omega$




$$(R(\omega), I(\omega)) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

Some obvious points:

| | | |
|---|---|---|
| Some obvious points: | $\omega = 0$ | $\to 1 + 0j$ |
| | $\omega = \omega_n$ | $\to 0 + 2\zeta j$ |

What happens as $\omega \to \infty$?

- real part $\approx -(\omega/\omega_n)^2 \to -\infty$, quadratic in $\omega$
- imaginary part $= 2\zeta(\omega/\omega_n) \to \infty$, linear in $\omega$

Magnitude:

- for $\omega \ll \omega_n$, $M \approx 1$ (horizontal line)
- for $\omega \gg \omega_n$, $M \approx \left(\frac{\omega}{\omega_n}\right)^2 \Rightarrow \log M \approx 2\log\omega - 2\log\omega_n$
  The asymptote is a line of slope 2 passing through the point $(\omega = \omega_n, M = 1)$

For a stable complex zero, the magnitude slope steps up by 2 as we go through the breakpoint.

# Type 3: $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$

This is a stable complex pole.

Magnitude:

$$\log M = \log\left|\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}\right| = -\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right|$$

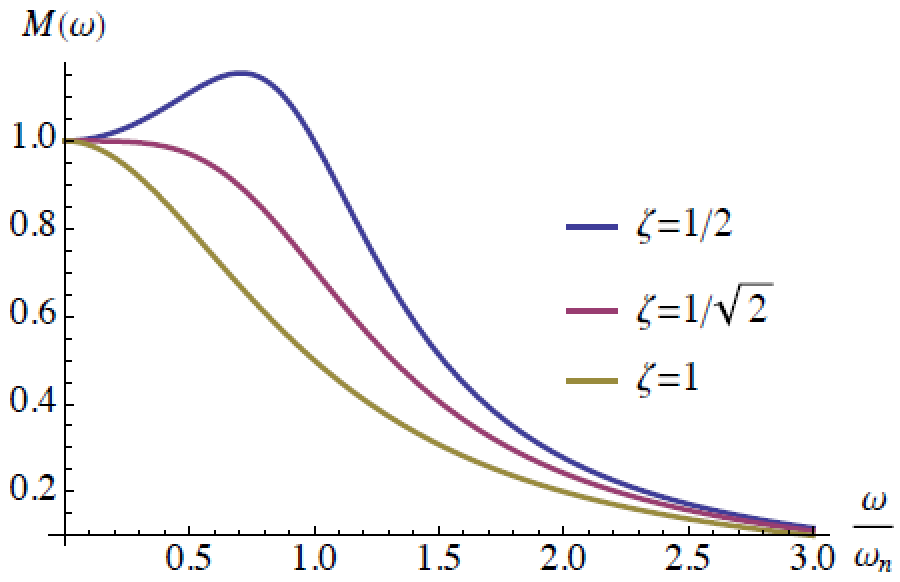
Phase:

$$\phi = \angle\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} = -\angle\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]$$

# Type 3: Magnitude

## Complex Pole Case

How does the magnitude plot look? Depends on the value of $\zeta$:




The magnitude hits its peak value (for $\zeta < 1/\sqrt{2} \approx 0.707$) occurs when $\omega = \omega_r$, where

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} < \omega_n$$

For small enough $\zeta$ (below $1/\sqrt{2}$), the magnitude of

$$\frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1}$$

has a resonant peak at the resonant frequency

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}.$$

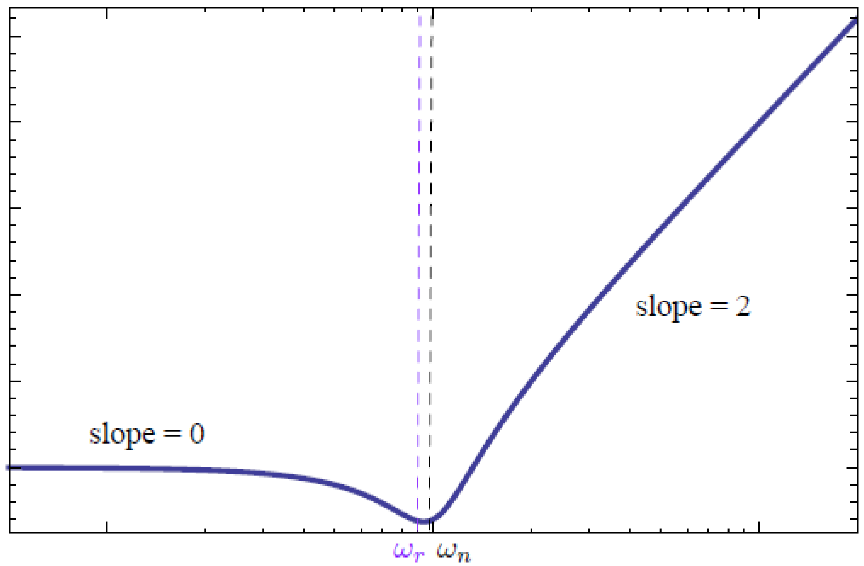
Likewise, the magnitude of

$$\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1$$

has a resonant dip at $\omega_r$.

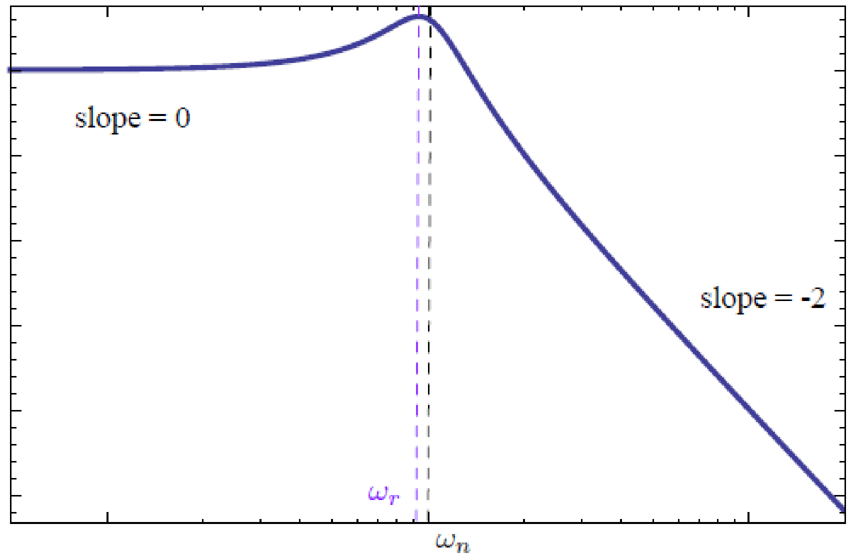
# Type 3: Magnitude

Stable real zero $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]$



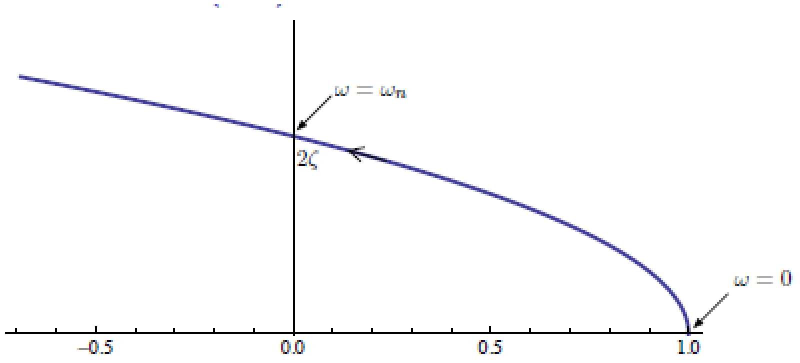

For a stable real zero, the magnitude slope "steps up by 2" at the break-point.

Stable real pole $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{-1}$




For a stable real pole, the magnitude slope "steps down by 2" at the break-point.

# Type 3: Phase




Nyquist plot
$(0 < \omega < \infty)$

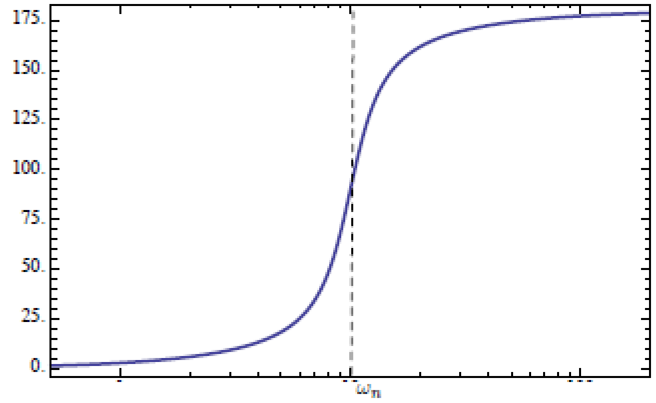
$$(R(\omega), I(\omega)) = \left(1 - \left(\frac{\omega}{\omega_n}\right)^2, 2\zeta\frac{\omega}{\omega_n}\right)$$

Phase:

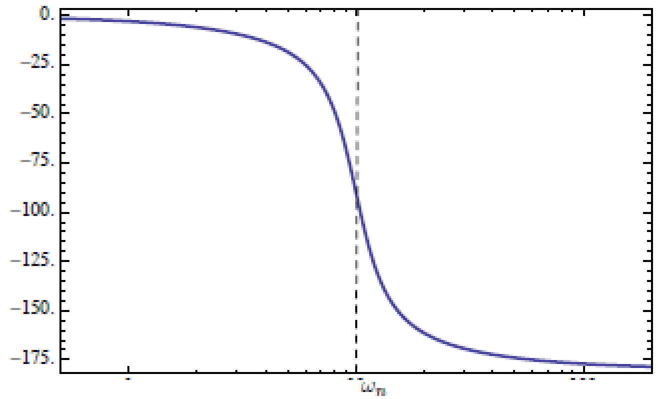
- for $\omega \ll \omega_n$, $\phi \approx 0°$ (real and positive)
- for $\omega = \omega_n$, $\phi = 90°$ (Re $= 0$, Im $> 0$)
- for $\omega \gg \omega_n$, $\phi \approx 180°$ (Re $\sim -\omega^2$, Im $\sim \omega$)

For a stable complex zero, the phase steps up by $180°$ as we go through the breakpoint; as $\zeta \to 0$, the transition through the break-point gets sharper, almost step-like.

For a pole, the phase is multiplied by $-1$.




(stable complex zero — phase steps up by $180°$)



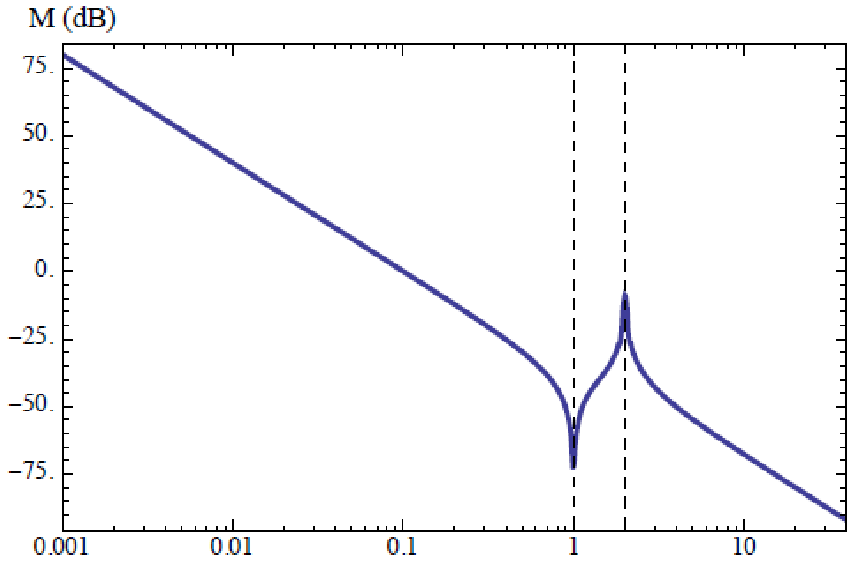

(stable complex pole — phase steps down by $180°$)

# Example 2 Magnitude

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about magnitude?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — asymptote has slope $= -2$, passes through $(\omega = 1, M = 0.01)$
- complex zero with break-point at $\omega_n = 1$ and $\zeta = 0.005$ — slope up by 2; large resonant dip
- complex pole with break-point at $\omega_n = 2$ and $\zeta = 0.01$ — slope down by 2; large resonant peak

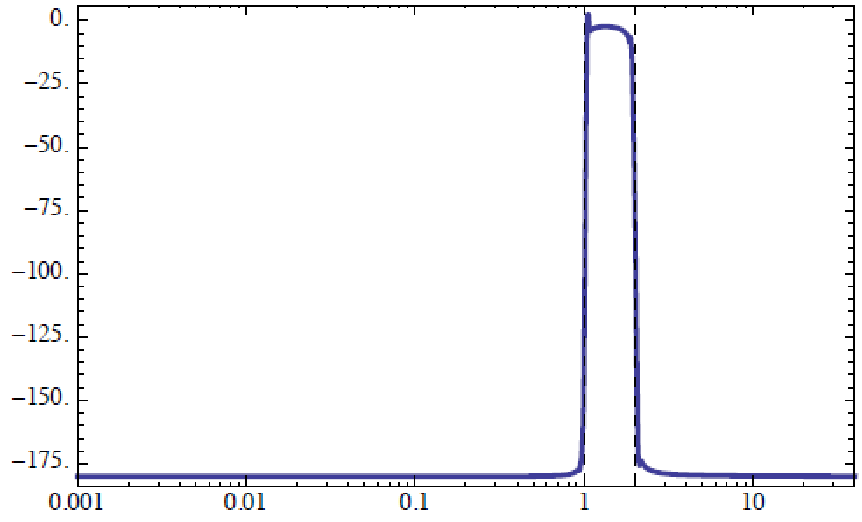


# Example 2 Phase

$$KG(s) = \frac{0.01\left(s^2 + 0.01s + 1\right)}{s^2\left(\frac{s^2}{4} + 0.02\frac{s}{2} + 1\right)}$$ — already in Bode form

What can we tell about phase?

- low-frequency term $\frac{0.01}{(j\omega)^2}$ with $K_0 = 0.01$, $n = -2$ — phase starts at $n \times 90° = -180°$
- complex zero with break-point at $\omega_n = 1$ — phase up by 180°
- complex pole with break-point at $\omega_n = 2$ — phase down by 180°
- since $\zeta$ is small for both pole and zero, the transitions are very sharp



# Unstable Zeros/Poles

So far, we've only looked at transfer functions with stable poles and zeros (except perhaps at the origin). What about RHP?

Example: consider two transfer functions,

$$G_1(s) = \frac{s+1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{s-1}{s+5}$$

Note:

- $G_1$ has stable poles and zeros; $G_2$ has a RHP zero.
- Magnitude plots of $G_1$ and $G_2$ are the same —

$$|G_1(j\omega)| = \left|\frac{j\omega+1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

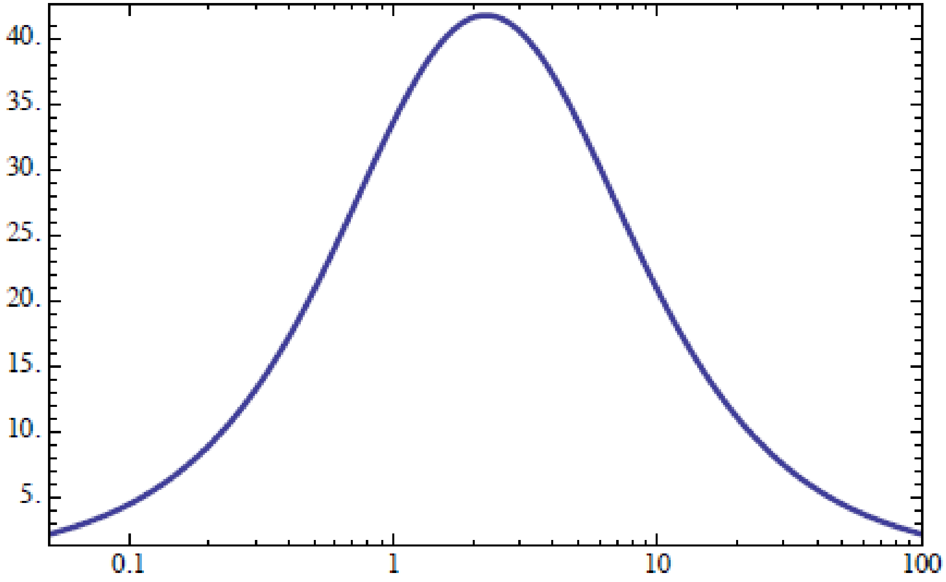
$$|G_2(j\omega)| = \left|\frac{j\omega-1}{j\omega+5}\right| = \sqrt{\frac{\omega^2+1}{\omega^2+5}}$$

- All the difference is in the phase plots!

# Phase Plot for $G_1$

$$G_1(j\omega) = \frac{j\omega + 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega + 1}{\frac{j\omega}{5} + 1}$$

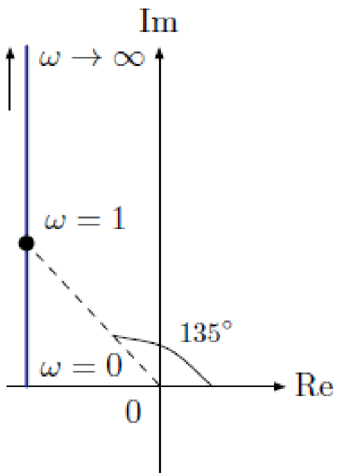
- Low-frequency term: $\frac{1}{5}(j\omega)^0$— $n = 0$, so phase starts at
- Break-points at $\omega_n = 1$ (phase goes up by 90°) and at $\omega_n = 5$ (phase goes down by 90°)



# Phase Plot for $G_2$

$$G_2(j\omega) = \frac{j\omega - 1}{j\omega + 5} = \frac{1}{5}\frac{j\omega - 1}{\frac{j\omega}{5} + 1}$$

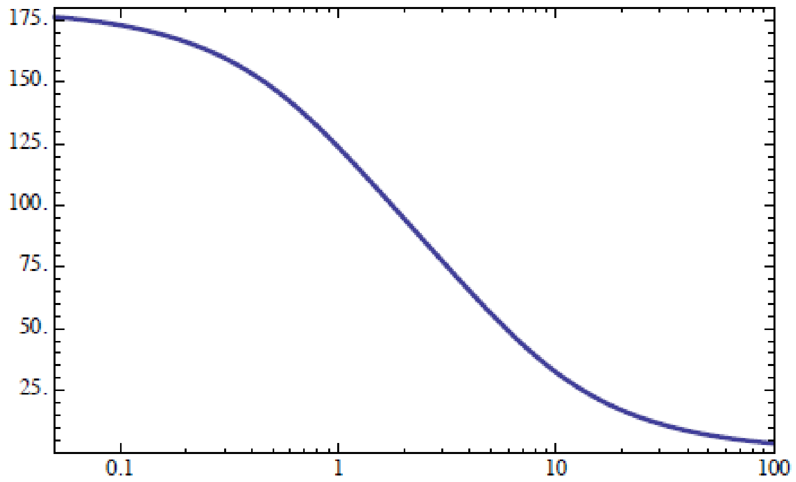
Let's do a Nyqiust plot for $j\omega - 1$:




New type of behavior —

- $\omega \approx 0$: $\phi \approx 180°$ (real and negative)
- $\omega \gg 1$: $\phi \approx 90°$ (Re $= -1$, Im $= \omega \gg 1$)
- $\omega \approx 1$: $\phi \approx 135°$

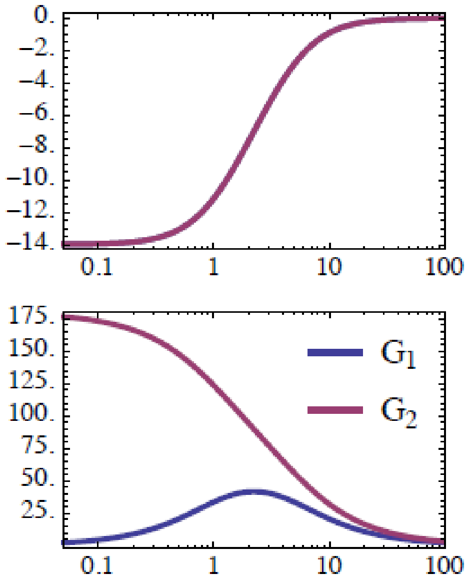
For a RHP zero, the phase starts out at 180° and goes down by 90° through the break-point (135° at break-point).




For a RHP zero, the phase plot is similar to what we had for a LHP pole: goes down by 90° ... However, it starts at 180°, and not at 0°.

# Minimum- & Nonminimum-Phase Zeros

## Minimum-Phase and Nonminimum-Phase Zeros




Among all transfer functions with the same magnitude plot, the one with only LHP zeros has the minimal net phase change as $\omega$ goes from 0 to $\infty$ — hence the term *minimum-phase* for LHP zeros.