Abstract Algebra Note

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Chapter 1 Group Theory

1.1 Monoid Group

Definition 1.1 (monoid)

We say that $(S, *)$ is a monoid if the binary operation satisfies the associative law and has an identity element. That is,

$$\forall x, y, z \in S, \quad x * (y * z) = (x * y) * z$$

and

$$\exists e \in S, \forall x \in S, \quad e * x = x * e = x$$

♣

Definition 1.2 (commutative monoid)

We say that $(S, *)$ is a commutative monoid if it is a monoid and the operation satisfies the commutative law. That is,

$$\forall x, y \in S, \quad x * y = y * x$$

♣

Proposition 1.1 (unique of identity element)

Let $(S, \cdot)$ be a monoid. Then the identity element is unique. ♠

Proof Suppose that e and e' are both identity elements of S. Then

$$e = e \cdot e' = e'$$

so $e = e'$. □

Proposition 1.2 (expand of associative law)

Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in S$. Then

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m = (x_1 \cdot x_2 \cdot \ldots \cdot x_n) \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

♠

Proof We prove this by induction on n.

Base Case ($n = 1$): When $n = 1$, the statement simplifies to:

$$x_1 \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m = x_1 \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

This is clearly true by the associative property of multiplication.

Inductive Step: Assume the statement holds for $n = k$, that is:

$$x_1 \cdot x_2 \cdot \ldots \cdot x_k \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m = (x_1 \cdot x_2 \cdot \ldots \cdot x_k) \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

We need to show that the statement holds for $n = k + 1$. Consider:

$$x_1 \cdot x_2 \cdot \ldots \cdot x_k \cdot x_{k+1} \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m$$

By the associative property, we can regroup the terms as:

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_k \cdot x_{k+1}) \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

Using the inductive hypothesis on the first k terms, we have:

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_k) \cdot x_{k+1} \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m) = (x_1 \cdot x_2 \cdot \ldots \cdot x_k \cdot x_{k+1}) \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

Thus, the statement holds for $n = k + 1$.

□

Proposition 1.3

Let $x \in S$ and $m, n \in \mathbb{N}$. Then

$$x^{m+n} = x^m \cdot x^n$$

♠

Proof We will prove this in three steps:

Step 1: First, recall from Proposition 1.2 that for any elements in S:

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot y_1 \cdot y_2 \cdot \ldots \cdot y_m = (x_1 \cdot x_2 \cdot \ldots \cdot x_n) \cdot (y_1 \cdot y_2 \cdot \ldots \cdot y_m)$$

Step 2: Now, consider the special case where all elements are equal to x:

- Let $x_1 = x_2 = \ldots = x_m = x$
- Let $y_1 = y_2 = \ldots = y_n = x$

Step 3: By definition of exponentiation in a monoid:

$$\begin{aligned} x^{m+n} &= \underbrace{x \cdot x \cdot \ldots \cdot x}_{m+n \text{ times}} \\ &= (\underbrace{x \cdot x \cdot \ldots \cdot x}_{m \text{ times}}) \cdot (\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}) \\ &= x^m \cdot x^n \end{aligned}$$

Therefore, we have proved that $x^{m+n} = x^m \cdot x^n$ for all $x \in S$ and $m, n \in \mathbb{N}$. □

Definition 1.3 (Submonoid)

Let $(S, \cdot)$ be a monoid. If $T \subset S$, we say that $(T, \cdot)$ is a submonoid of $(S, \cdot)$ if:

1. *The identity element $e \in T$*
2. *T is closed under multiplication, that is:*

$$\forall x, y \in T, \quad x \cdot y \in T$$

♣

Proposition 1.4

If $(T, \cdot)$ is a submonoid of $(S, \cdot)$, then $(T, \cdot)$ is a monoid.

♠

Proof We need to verify two properties:

1. The operation is associative in T:
   Since $T \subset S$ and $\cdot$ is associative in S, it is also associative in T.
2. T has an identity element:
   By definition of submonoid, the identity element $e \in T$.

Therefore, $(T, \cdot)$ satisfies all properties of a monoid. □

Chapter 2　Ring Theory

Chapter 3　Polynomial Theory

Chapter 4   Field Theory