MARKOV CHAINS AND APPLICATIONS IN OPTIONS MARKETS

JIAHONG CAI

Abstract. We introduce multiple types of Markov chains and their applications in optimal stopping problems. Then we bring in the concept of martingales and some of its generalizations. Finally, we apply optimal stopping strategies to the pricing of options, specifically American options.

Contents

1. Finite Markov Chains

Definition 1.1. A *discrete-time stochastic process* can be described as a sequence $X_n, n = 0, 1, 2, ...,$ where X_n takes values in the finite set $S = \{1, ..., N\}$ or $\{0, ..., N-1\}$, and the possible values of X_n are called the *states* of the system.

Definition 1.2. The probabilities of such a discrete-time stochastic process are given by the values of

$$\mathbb{P}\{X_0 = i_0, X_1 = i_1, ..., X_n = i_n\}$$

for every n and every finite sequence of states, or the initial probability distribution

$$\phi(i) = \mathbb{P}\{X_0 = i\}, \quad i = 1, ..., N$$

Date: August 27, 2022.

and the *transition probabilities*

$$q_n(i_n|i_0, ..., i_n) = \mathbb{P}\{X_n = i_n|X_0 = i_0, ..., X_{n-1} = i_{n-1}\} \tag{1.3}$$

for then

$$\mathbb{P}\{X_0 = i_0, ..., X_n = i_n\} = \phi(i_0)q_1(i_1|i_0)q_2(i_2|i_0, i_1)...q_n(i_n|i_0, ..., i_{n-1}) \tag{1.4}$$

Definition 1.5. The *Markov property* states that considering only the present state of the system instead of the past history is enough to make predictions of the behavior of a system in a future. Mathematically, this is written as

$$\mathbb{P}\{X_n = i_n|X_0 = i_0, ..., X_{n-1} = i_{n-1}\} = P\{X_n = i_n|X_{n-1} = i_{n-1}\}$$

Definition 1.6. *Time-homogeneity* states that the transition probabilities are independent of time. A *time-homogeneous Markov process* is a process such that

$$\mathbb{P}\{X_n = i_n|X_0 = i_0, .., X_{n-1} = i_{n-1}\} = p(i_{n-1}, i_n)$$

for some function $p : S \times S \to [0, 1]$.

To give the probabilities for a Markov chain, we need to give an initial probability distribution $\phi(i) = \mathbb{P}\{X_0 = i\}$, and the transition probabilities $p(i, j)$. By (1.4),

$$\mathbb{P}\{X_0 = i_0, ..., X_n = i_n\} = \phi(i)p(i_0, i_1)p(i_1, i_2)...p(i_{n-1}, i_n) \tag{1.7}$$

Definition 1.8. The *transition matrix* $\mathbf{P}$ for the Markov chain is the $N \times N$ matrix whose (i, j) entry $\mathbf{P}_{ij}$ is $p(i, j)$. The matrix $\mathbf{P}$ is a *stochastic matrix*, i.e.,

$$0 \le \mathbf{P}_{ij} \le 1, \quad 1 \le i, j \le N,$$

$$\sum_{j=1}^{N} \mathbf{P}_{ij} = 1, \quad 1 \le i \le N.$$

Example 1.9. Random Walk with Reflecting Boundary and Absorbing Boundary. Consider a random walk along the sites $\{0, 1, ..., N\}$. At each step, the walker would either move to right with probability p or to the left with probability $1-p$. If the walker is at 0, the walker moves with probability 1 toward the right. If the walker is at N, the walker stays there forever.The transition matrix $\mathbf{P}$ for this Markov chain is given by

$$p(i, i+1) = p, \quad p(i, i-1) = 1 - p, \quad 1 \le i \le N - 1,$$

$$p(0, 1) = 1, \quad p(N, N) = 1,$$

with $p(i, j) = 0$ for other values of i, j.

Definition 1.10. Usually, we analyze how a Markov chain develops in more than one step at a time. We define an n-th transition probability as

$$p_{ij}^{(n)} = \mathbf{P}(X_n = j|X_0 = i).$$

For time-homogeneous Markov chains, which means probability does not change with respect to time, any jump of n steps will have the same transition matrix regardless of the number of steps of the chain taken so far.

1.1. Classifications and Properties.

Definition 1.11. Consider a Markov chain X_n with state space S. For some $i, j \in S$, we say that i leads to j (written as $i \to j$) if there exists some $n \in \mathbb{N}$ such that

$$p_{ij}^{(n)} > 0.$$

We say that i and j *communicate* with each other (written as $i \leftrightarrow j$) if both i leads to j and j leads to i.

Definition 1.12. We call a Markov chain X_n with state space S *irreducible* if

$$\forall i, j \in S, \quad \exists n \in \mathbb{N} : \quad p_{i,j}^{(n)} > 0.$$

In short, a Markov chain with state space S is irreducible if you can reach any other state in S in finite time by starting at any state in S.

Definition 1.13. We define a state i of a Markov chain to be *recurrent* if

$$p_{ii}^{(n)} = 1$$

for infinitely many n.

Definition 1.14. We define a state i of a Markov chain to be *transient* if

$$p_{ii}^{(n)} < 1$$

for infinitely many n.

2. Optimal Stopping

2.1. **Optimal Stopping of Markov Chains.** Suppose $\mathbf{P}$ is the transition matrix for a discrete-time Markov chain X_n with finite state space S, and there is a payoff function f that assigns the payoff to each state if the chain stops at that state. Given a chain, we want to find the "best strategy" for when to stop it to optimize the payoff function. Since an irreducible chain will always reach each state in finite time, the optimal strategy is to stop as soon as the chain hits the state with the maximum payoff. Therefore, the interesting cases we will consider here are those in which the chain is not irreducible.

A *stopping rule* or *stopping time* will be a random variable T that gives the time at which the chain is stopped. One must decide whether to stop based solely on what has happened up to step n. Hence, the state space is divided into two sets S_1 and S_2; if the state is in S_1, one continues, otherwise it stops. Let $v(x)$ be the *value* of a state x, i.e., the expected payoff assuming the optimal stopping strategy is used. As the maximum is over all stopping times, we can write

$$v(x) = \max_T \mathbb{E}[f(X_T)|X_0 = x].$$

There are two main inequalities that v satisfies. First, v is greater than or equal to the payoff available by stopping,

$$v(x) \geq f(x). \tag{2.1}$$

Second, v is greater than or equal to the maximum expected payoff if one continues,

$$v(x) \geq \mathbf{P}v(x) = \sum_{y \in S} p(x, y)v(y). \tag{2.2}$$

In fact,

$$v(x) = \max\{f(x), \mathbf{P}v(x)\}. \tag{2.3}$$

Assuming S_1 and S_2 are the continuing and stopping sets for an optimal strategy, we let

$$T = \min\{j \geq 0 : X_j \in S_2\},$$

then

$$v(x) = \mathbb{E}[f(X_T)|X_0 = x].$$

We will characterize the function v. We call a function u *superharmonic* with respect to $\mathbf{P}$ if

$$u(x) \geq \mathbf{P}u(x).$$

Under the condition that T is the time associated to a stopping rule as above, we suppose u is superharmonic. Consider the time $T_n = \min\{T, n\}$ we, claim that

$$u(x) \geq \mathbb{E}[u(X_{T_n})|X_0 = x].$$

Proof. Note that it is trivially true for $n = 0$. Let's proceed to prove the inductive case. For inductive hypothesis, assume it is true for $n - 1$. Then

$$\begin{aligned}
\mathbb{E}[u(X_{T_n})|X_0 = x] &= \sum_{y \in S} \mathbb{P}\{X_{T_n} = y|X_0 = x\}u(y) \\
&= \sum_{y \in S} \sum_{z \in S} \mathbb{P}\{X_{T_n} = y|X_{T_{n-1}} = z\}\mathbb{P}\{X_{T_{n-1}} = z|X_0 = x\}u(y) \\
&= \sum_{z \in S_2} \sum_{y \in S} \mathbb{P}\{X_{T_n} = y|X_{T_{n-1}} = z\}\mathbb{P}\{X_{T_{n-1}} = z|X_0 = x\}u(y) \\
&+ \sum_{z \in S_1} \sum_{y \in S} \mathbb{P}\{X_{T_n} = y|X_{T_{n-1}} = z\}\mathbb{P}\{X_{T_{n-1}} = z|X_0 = x\}u(y).
\end{aligned}$$

If $z \in S_2$, then $P\{X_{T_n} = y|X_{T_{n-1}} = z\} = 0$ for $y \neq z$ and $P\{X_{T_n} = z|X_{T_{n-1}} = z\} = 1$, so the first double sum of the last expression equals

$$\sum_{z \in S_2} \mathbb{P}\{X_{T_{n-1}} = z|X_0 = x\}u(z).$$

If $z \in S_1$, then $P\{X_{T_n} = y|X_{T_{n-1}} = z\} = p(z, y)$, and hence

$$\sum_{y \in S} \mathbb{P}\{X_{T_n} = y|X_{T_{n-1}} = z\}u(y) = \mathbf{P}u(z) \leq u(z).$$

Hence,

$$\begin{aligned}
\mathbb{E}[u(X_T)|X_0 = x] &\leq \sum_{z \in S} \mathbb{P}\{X_{T_{n-1}} = z|X_0 = x\}u(z) \\
&= \mathbb{E}[u(X_{T_{n-1}})|X_0 = x] \leq u(x).
\end{aligned}$$

Since u is bounded, we can let $n \to \infty$ and get

$$u(x) \geq \lim_{n \to \infty} \mathbb{E}[u(X_{T_n})|X_0 = x] = \mathbb{E}[u(X_{T_n})|X_0 = x].$$

□

Now suppose that $u(x) \geq f(x)$ for all x. Then

$$u(x) = \mathbb{E}[u(X_T)|X_0 = x] \geq \mathbb{E}[f(X_T)|X_0 = x] = v(x).$$

Hence every superharmonic function that is larger than f is greater than or equal to the value function v.

2.2. **Optimal Stopping with Cost.** Assume that if one wants to continue at a state of the chain, a cost $g(x)$ associated with that state must be paid. Assume we have a payoff function f, and let $v(x)$ be the expected value of the payoff minus the cost assuming a stopping rule is chosen to maximize the expected value. We can write

$$v(x) = \max\{f(x), \mathbf{P}v(x) - g(x)\}.$$

The expected payoff minus cost if the chain is continued is equal to $\mathbf{P}v(x) - g(x)$. Again divide S into S_1 and S_2 where

$$S_2 = \{x : v(x) = f(x)\}.$$

The stopping rule says to stop whenever the chain enters a state in S_2.

In this case, the value of function v can be characterized as the smallest function u greater than or equal to f that satisfies

$$u(x) \geq \mathbf{P}u(x) - g(x).$$

In other words,

$$v(x) = \inf u(x),$$

where the infimum is over all functions u such that $u(x) \geq f(x)$ and $u(x) \geq \mathbf{P}u(x) - g(x)$. We define u_1 to be the function that equals f on all absorbing states and equals the maximum value of f everywhere else. We then define

$$u_n(x) = \max\{f(x), \mathbf{P}u_{n-1}(x) - g(x)\},$$

and then

$$v(x) = \lim_{n\to\infty} u_n(x).$$

Example 2.4. Consider a game that a player draws a card from a deck consisting of 2 *Jokers* and 13 cards for each of the 4 suits of *Hearts*, *Clubs*, *Diamonds*, and *Spades*. The cards are replaced when redrawing. If the player draws one of the two *Jokers*, the player wins no money. Otherwise, the player may either quit the game and win \$1 if drawing a *Heart*, \$2 if drawing a *Club*, \$3 if drawing a *Dimond*, or \$4 if drawing a *Spade*, or may draw again with a charge of \$1 for each additional draw. If the player draws again, the game continues until either a *Joker* is drawn, or the player quits. We can write this game with payoff function

$$f = \begin{cases} 1, & \textit{for Hearts} \\ 2, & \textit{for Clubs} \\ 3, & \textit{for Dimonds} \\ 4, & \textit{for Spades} \\ 0, & \textit{for Jokers} \end{cases}$$

and associated cost function $g = 1,$ for all cases. The cost function makes us less likely to roll again when getting *Clubs*, *Dimonds*, and *Spades*. If we get *Hearts*, then by drawing again we can get an expected payoff of at least $\frac{65}{27}$ with a cost of 1. Therefore, we can expect a net gain of at least $\frac{38}{27}$, which means we should play if we get a *Heart*.

Suppose we draw again whenever we get a *Heart* and stop otherwise. Let $u(k)$ be the expected winnings with this strategy. Hence

$$\begin{aligned} u(\text{Heart}) &= u \\ u(\text{Club}) &= 2 \\ u(\text{Diamond}) &= 3 \\ u(\text{Spade}) &= 4 \\ u(\text{Joker}) &= 0. \end{aligned}$$

Say

$$u = \frac{13}{54}u + \frac{13}{54} \times 2 + \frac{13}{54} \times 3 + \frac{13}{54} \times 4 + \frac{2}{54} \times 0 - 1 = \frac{13}{54}u + \frac{63}{54}.$$

Solving for u gives $u = 63/41 \approx 1.53659$. Since this is greater than $f(\text{Heart}) = 1$, it must be correct to continue if drawing a *Heart*. Hence the stopping set is $S_2 = \{Clubs, Diamonds, Spades, Jokers\}$ and the value function is

$$v = [\frac{63}{41}, 2, 3, 4, 0].$$

2.3. **Optimal Stopping with Discounting.** Suppose we have a Markov chain X_n with transition matrix $\mathbf{P}$ and a payoff function f. Assume that a discount factor $\alpha < 1$ is given, which means the payoff is discounted by α by each step. We now maximize the expected value of the payoff: if we stop after k steps, then the present value of the payoff in k steps is the actual payoff discounted by α^k.

$$v(x) = \max_T \mathbb{E}[\alpha^T f(X_T)|X_0 = x].$$

To obtain this value function, we characterize v as the smallest function u satisfying

$$\begin{aligned} u(x) &\geq f(x) \\ u(x) &\geq \alpha \mathbf{P} u(x). \end{aligned}$$

We may obtain v with a similar algorithm to the one discussed in Section 2.2. Start with an initial function u_1 equal to f at all absorbing states and equal to the maximum value of f at all other states. Define u_n recursively by

$$u_n(x) = \max\{f(x), \alpha \mathbf{P} u_{n-1}(x)\}.$$

Then

$$v(x) = \lim_{n \to \infty} u_n(x).$$

Example 2.5. Consider the card game from Example 2.4 without the \$1 redraw cost and with a $= 0.8$ discount factor. If the first draw is a *Heart*, which gives a payoff of \$1, one can get an expected payoff of at least $0.8 \times \frac{65}{27} = \frac{52}{27}$ by rolling again, so it is best to draw again. If the first draw is a *Club*, which gives a payoff of \$2, it is also best to draw again. We can see that by solving

$$u = 0.8[\frac{13}{54} \times u + \frac{13}{54} \times u + \frac{13}{54} \times 3 + \frac{13}{54}] \times 4,$$

which assumes we use the strategy to draw again with a *Heart* or *Club* and to stop otherwise. Let u be the expected winnings given that one rolls again. Solving for

u, we get $u \approx 2.1928 > 2$, so it must be optimal to draw again with a *Heart* or *Club*. Therefore $S_2 = \{Diamonds, Spades, Jokers\}$ and

$$v = [2.1928, 2.1928, 3, 4, 0].$$

3. Martingales

A martingale describes a model of a fair game.

Let $\{\mathcal{F}_n\}$ denote an increasing collection of information. By this we mean for each n, we have a collection of random variables $\mathcal{A}_n$ such that $\mathcal{A}_m \subset \mathcal{A}_n$ if $m < n$. The information we have at time n is the value of all the variables in $\mathcal{A}_n$. The assumption $\mathcal{A}_m \subset \mathcal{A}_n$ means we gain or at least preserve information over time. A random variable X is $\mathcal{F}_n$-measurable if we can determine the value of X if we know the value of all the random variables in $\mathcal{A}_n$. The increasing sequence of information $\mathcal{F}_n$ is called a *filtration.*

We say that a sequence of random variables $M_0, M_1, M_2, ...$ with $\mathbb{E}(|M_i|) < \infty$ is a *martingale* with respect to $\{\mathcal{F}_n\}$ if each M_n is measurable with respect to $\mathcal{F}_n$, and for each $m < n$,

$$\mathbb{E}[M_n|\mathcal{F}_m] = M_m \tag{3.1}$$

or

$$\mathbb{E}[M_n - M_m|\mathcal{F}_m] = 0.$$

The condition $\mathbb{E}[|M_i|] < \infty$ is needed to guarantee that the conditional expectations are well defined. If $\mathcal{F}_n$ is the information in the random variables $X_0, \ldots, X_n$, then we say $M_0, M_1, ...$ is a martingale with respect to $X_0, X_1, ...$ We will sometimes say "$M_0, M_1, ...$" is a martingale without making reference to the filtration $\mathcal{F}_n$.

In order to prove (3.1), it suffices to prove that for all n,

$$\mathbb{E}[M_{N+1}|\mathcal{F}_n] = M_n,$$

since if this holds, then by the *"Tower Property" of Conditional Expectations*,

$$\begin{aligned}\mathbb{E}[M_{N+2}|\mathcal{F}_n] &= \mathbb{E}[\mathbb{E}[M_{n+2}|\mathcal{F}_{N+1}]|\mathcal{F}_n] \\ &= \mathbb{E}[M_{n+1}|\mathcal{F}_n] \\ &= M_n,\end{aligned}$$

and so on.

Example 3.2. Suppose $X_1, X_2, ...$ are independent random variables with

$$\mathbb{P}\{X_i = -2\} = 1/3, \quad \mathbb{P}\{X_i = 1\} = 2/3.$$

We think of the random variables X_i as the results of a game where one tosses a dice and loses \$2 if it comes up composites ($\{4, 6\}$) and wins \$1 if it comes up primes ($\{1, 2, 3, 5\}$). One way to beat the game is to keep trippling our bet until we eventually win. At this point we stop. Let W_n denote the winnings (or losses) up through n tosses of the dice using this strategy. $W_0 = 0$. Whenever we win we stop playing so our winnings stop changing and

$$\mathbb{P}\{W_{n+1} = 1|W_n = 1\} = 1.$$

Suppose the first n tosses of the dice have resulted in primes. After each toss we have tripled our bet, so we have lost $2 + 6 + 18 + ... + 2 \times 3^{n-1} = 3^n - 1$ dollars and

$W_n = -(3^n - 1)$. At this toss we triple our bet again and would win 3^n if it turns up primes but lose 2×3^n if it turns up composites. This gives

$$\mathbb{P}\{W_{n+1} = 1 | W_n = -(3^n - 1)\} = 2/3,$$
$$\mathbb{P}\{W_{n+1} = -(3^{n+1} - 1) | W_n = -(3^n - 1)\} = 1/3.$$

It is easy to verify that $\mathbb{E}[W_{n+1}|\mathcal{F}_n] = W_n$, and hence W_n is a martingale with respect to $\mathcal{F}_n$.

3.1. **Submartingales and Supermartingales.** Submartingales and supermartingales are simple generalizations of martingales.

Definition 3.3. A *submartingale* is an discrete stochastic process $\{U_n\}_{n \geq 1}$ that satisfies the conditions

$$\mathbb{E}[|U_n|] < \infty, \quad \mathbb{E}[U_n | U_{n-1}, U_{n-2}, ..., U_1] \geq U_{n-1}. \tag{3.4}$$

Definition 3.5. A *supermartingale* is an discrete stochastic process $\{U_n\}_{n \geq 1}$ that satisfies the conditions

$$\mathbb{E}[|U_n|] < \infty, \quad \mathbb{E}[U_n | U_{n-1}, U_{n-2}, ..., U_1] \leq U_{n-1}. \tag{3.6}$$

Consider another stochastic process $\{Z_n\}_{n \geq 1}$ that satisfies the condition $\mathbb{E}[|Z_n|] < \infty$. We say the supermartingale $\{U_n\}$ dominates $\{Z_n\}$ if

$$Z_n \leq U_n, \quad \forall n \geq 1.$$

In brief, a submartingale describes a game that is at least fair, i.e., where the expected fortune of the gambler either increases or remains the same. A supermartingale describes a process with the opposite type of inequality: the expected fortune of the gambler either decreases or remains the same.

3.2. **Decomposition of Martingales.** The *Doob decomposition theorem* gives a unique decomposition of every adapted and integrable stochastic process as the sum of a martingale and a predictable process starting at zero. Here we will only state the version for supermartingales.

Proposition 3.7. *Every supermartingale $\{U_n\}_{0 \leq n \leq N}$ has the unique following decomposition:*

$$U_n = M_n - A_n,$$

where $\{M_n\}$ is a martingale and $\{A_n\}$ is a non-decreasing, predictable process null at 0.

Proof. It is trivial in the case of $n = 0$ because the only solution is $M_0 = U_0$ and $A_0 = 0$. Then we must have

$$U_{n+1} - U_n = M_{n+1} - M_n - (A_{n+1} - A_n).$$

Conditioning both sides with respect to $\mathcal{F}_n$ and using the properties of M and A

$$-(A_{n+1} - A_n) = \mathbb{E}[U_{n+1}|\mathcal{F}_n] - U_n$$

and

$$M_{n+1} - M_n = U_{n+1} - \mathbb{E}[U_{n+1}|\mathcal{F}_n].$$

It is easy to see that $\{M_n\}$ and $\{A_n\}$ are entirely determined using the previous equations. Because $\{U_n\}$ is a supermartingale, we may conclude that $\{M_n\}$ is a martingale and $\{A_n\}$ is non-decreasing. $\square$

4. Optimal Stopping and American Options

4.1. **American Options.** In financial markets, an *option* is a contract which conveys to its owner, the holder, the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date, depending on the style of the option.

The buyer of the *call option* has the right, but not the obligation, to buy an agreed quantity of a particular commodity or asset from the seller of the option at a certain time for a certain price. The buyer of the *put option* has the right, but not the obligation, to sell an agreed quantity of a particular commodity or asset to the seller of the option at a certain time for a certain price. Specifically, a *European option* is a version of an options contract that limits execution to its expiration date, while an *American option* is a style of options contract that allows holders to exercise their rights at any time before and including the expiration date.

Since an American option can be exercised at any time between 0 and N, the expiration date, we could define it as a positive sequence $\{Z_n\}_{0\le n\le N}$ adapted to a filtration $\mathcal{F}_n$, where Z_n is the immediate profit made by exercising the option at time n. Let S_n^0 denote rate of return someone can expect for a stock after n units of time assuming the price of the stock follows the market average of a rate of return r after one unit of time, i.e.,

$$S_n^0 = (1+r)^n$$

For an American option on the stock S^0 with strike price K, $Z_n = S_n^0 - K$ in the case of the call; $Z_n = K - S_n^0$ in the case of the put.

To define the price of the option associated with $\{Z_n\}_{0\le n\le N}$, we can do a backward induction starting at time N. The value of the option at maturity is equal to $U_N = Z_N$. At time $N-1$, the holder has to earn the maximum between Z_{N-1} and the amount necessary at time $N-1$ to generate Z_N at time N, i.e. $S_{N-1}^0\mathbb{E}[\tilde{Z}_N|\mathcal{F}_{N-1}]$. Hence, we could price the option at time $N-1$ as

$$U_{N-1} = \max\{Z_{N-1}, S_{N-1}^0\mathbb{E}[\tilde{Z}_N|\mathcal{F}_{N-1}]\}.$$

By induction, we define the American option price for $n = 1, ..., N$ by

$$U_{n-1} = \max\{Z_{n-1}, S_{n-1}^0\mathbb{E}[\frac{U_n}{S_n^0}|\mathcal{F}_{n-1}]\}.$$

Since

$$S_n^0 = (1+r)^n,$$

we have

$$U_{n-1} = \max\{Z_{n-1}, \frac{1}{1+r}\mathbb{E}[U_n|\mathcal{F}_{n-1}]\},$$

let $\tilde{U}_n = \frac{U_n}{S_n^0}$ be the discounted price of the American option.

Proposition 4.1. *The sequence $\{\tilde{U}_n\}_{0\le n\le N}$ is a $\mathbf{P}^*$-supermartingale. It is the smallest $\mathbf{P}^*$-supermartingale that dominates the sequence $\{\tilde{Z}_n\}_{0\le n\le N}$.*

Proof. From of the equality

$$\tilde{U}_{n-1} = \max\{\tilde{Z}_{n-1}, \mathbb{E}[\tilde{U}_n|\mathcal{F}_{n-1}]\},$$

we can tell that $\{\tilde{U}_n\}_{0\le n\le N}$ is a supermartingale dominating $\{\tilde{Z}\}_{0\le n\le N}$. Now we consider a supermartingale $\{\tilde{T}_n\}_{0\le N\le N}$ that dominates $\{\tilde{Z}_n\}_{0\le n\le N}$. Then $\tilde{T}_n \ge$

$\tilde{U}_n$, so we have

$$\tilde{T}_{n-1} \geq \mathbb{E}[\tilde{T}_n|\mathcal{F}_{n-1}] \geq \mathbb{E}[\tilde{U}_n|\mathcal{F}_{n-1}]$$

whence

$$\tilde{T}_{n-1} \geq \max\{\tilde{Z}_{n-1}, \mathbb{E}[\tilde{U}|\mathcal{F}_{n-1}]\} = \tilde{U}_{n-1},$$

which proves that $\{T_n\}$ dominates $\{\tilde{U}_n\}$ by backward induction. $\square$

Definition 4.2. Let $\{X_n\}_{0\leq n\leq N}$ be a sequence adapted to the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$, and let v be a stopping time. The *sequence stopped at time* v is defined as, on the set $\{v = j\}$,

$$X_n^v = \begin{cases} X_j, & \text{if } j \leq n \\ X_n & \text{otherwise.} \end{cases} \tag{4.3}$$

Proposition 4.4. *The random variable defined by*

$$v_0 = \inf\{n \geq 0 | U_n = Z_n\}$$

is a stopping time and the stopped sequence $\{U_n^{v_0}\}_{0\leq n\leq N}$ *is a martingale.*

Proof. Since v_0 is well-defined element of $\{0, 1, ..., N\}$ given that $U_N = Z_N$, we have

$$\{v_0 = 0\} = \{U_0 = Z_0\} \in \mathcal{F}_0,$$

and for $k \geq 1$

$$\{v_0 = k\} = \{U_0 > Z_0\} \cap ... \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \in \mathcal{F}_k.$$

$\square$

Corollary 4.5. *We denote by* $\mathcal{T}_{n,N}$ *the set of stopping times taking values in* $\{n, n+1, ..., N\}$. *The supermartingale* $\{U_n\}$ *satisfies that*

$$U_n = \sup_{v\in\mathcal{T}_{n,N}} \mathbb{E}[Z_v|\mathcal{F}_n] = \mathbb{E}[Z_{v_n}|\mathcal{F}_n],$$

where $v_n = \inf\{j \geq n | U_j = Z_j\}$.

Proof.

$$U_n = U_n^{v_n} = \mathbb{E}[U_N^{v_n}|\mathcal{F}_n] = \mathbb{E}[U_{v_n}|\mathcal{F}_n] = \mathbb{E}[Z_{v_n}|\mathcal{F}_n].$$

$\square$

4.2. **Snell Envelope.** In this section, we aim to compute Snell envelopes in a Markovian setting.

Definition 4.6. Let $\{Z_n\}_{0\leq n\leq N}$ be an adapted sequence defined by $Z_n = \psi(n, X_n)$, where $\{X_n\}_{0\leq n\leq N}$ is a time-homogeneous Markov chain with transition matrix $\mathbf{P}$ in the state space $\mathbf{S}$, and ψ is a function from $\mathbb{N} \times \mathbf{S}$ to $\mathbb{R}$. Then, the *Snell envelope* U_n of the sequence Z_n is given by given by $U_n = u(n, X_n)$, where the function u is defined by

$$u(N, x) = \psi(N, x), \quad \forall x \in \mathbf{S}$$

and, for $n < N$,

$$u(n, x) = \max\{\psi(n, x), \mathbf{P}u(n + 1, x)\}, \quad \forall x \in \mathbf{S}.$$

The Snell envelope is the smallest supermartingale dominating a stochastic process.

4.3. **Applications to American Options.** In this section, we work on a *complete market*, which follows the following two conditions:

1) Negligible transactions costs, aka perfect information, which means all consumers and producers have complete and instantaneous knowledge of all market prices, their own utility, and own cost functions to optimize their choices

2) Existence of price for every asset in every possible state of the world

In a *complete market*, there is a *self-financing strategy*, which means there is no exogenous infusion or withdrawal of money in a portfolio - the purchase of a new asset must be financed by the sale of an old one.

The modeling will be based on the filtered space $\mathcal{F}_n$. We will denote by $\mathbf{P}^*$ the unique probability measure under which the discounted asset prices are martingales [2, Theorem 1.3.4].

4.3.1. *Hedging American Options.* Previously, we defined the value process U_n of an American option described by the sequence $\{Z_n\}$ by the system

$$\begin{cases} U_N = Z_N \\ U_n = \max\{Z_n, S_n^0 \mathbb{E}^*[\frac{U_{n+1}}{S_{n+1}^0}|\mathcal{F}_n]\}, \quad \forall n \leq N-1. \end{cases} \tag{4.7}$$

The sequence $\{\tilde{U}_n\}$ defined by $\tilde{U}_n = U_n/S_n^0$ (discounted price of the option) is the Snell envelope, under $\mathbf{P}^*$, of the sequence $\tilde{Z}_n$. We deduce from the above section that

$$\tilde{U}_n = \sup_{v \in \mathcal{T}_{n,N}} \mathbb{E}^*[\tilde{Z}_v|\mathcal{F}_n]$$

and consequently

$$U_n = S_n^0 \sup_{v \in \mathcal{T}_{n,N}} \mathbb{E}^*[\frac{Z_v}{S_v^0}|\mathcal{F}_n].$$

Hence, we can write

$$\tilde{U}_n = \tilde{M}_n - \tilde{A}_n,$$

where $\tilde{M}_n$ is a $\mathbf{P}^*$-martingale, and $\tilde{A}_n$ is an increasing predictable process, null at 0. Since the market is complete, there is a self-financing strategy ψ such that

$$V_N(\psi) = S_N^0 \tilde{M}_N,$$

i.e., $\tilde{V}_N(\psi) = \tilde{M}_N$, where $V_n(\psi)$ denotes the volume of the self-financing portfolio at time n under the strategy ψ, and $\tilde{V}_n(\psi)$ denotes the discounted volume at the same time under identical strategy. Because the sequence $\tilde{V}_n(\psi)$ is a $\mathbf{P}^*$-martingale, we have

$$\begin{aligned} \tilde{V}_n(\psi) &= \mathbb{E}^*[\tilde{V}_N(\psi)|\mathcal{F}_n] \\ &= \mathbb{E}^*[\tilde{M}_N|\mathcal{F}_n] \\ &= \tilde{M}_n, \end{aligned}$$

and consequently

$$\tilde{U}_n = \tilde{V}_n(\psi) - \tilde{A}_n.$$

Therefore

$$U_n = V_n(\psi) - A_n,$$

where $A_n = S_n^0 \tilde{A}_n$. This tells that the writer of the option can hedge himself perfectly. In other words, the investor eliminates the risk of an existing position or eliminates all market risk from a portfolio: if the writer follows the strategy ψ, then at time n they will have a portfolio worth $V_n(\psi)$, which is greater than U_n because A_n is non-negative by Proposition 3.7. Here once the writer receives the premium $U_0 = V_0(\psi)$, he can generate a wealth equal to $V_n(\psi)$ at time n which is bigger than U_n and *a fortiori* Z_n.

For the buyer of the option, he could choose the optimal date to exercise among all the stopping times but would not exercise at time n when $U_n > Z_n$: in this way, he trades an asset worth U_n (the option) for an amount of Z_n. Thus, an optimal date τ of exercise is such that $U_\tau = Z_\tau$. Also, there is no point in exercising after the time

$$v_{max} = \inf\{j, A_{j+1}\} = \inf\{j, \tilde{A}_{j+1} \neq 0\},$$

which means selling the option provides the holder with a wealth

$$U_{v_{max}} = U_{v_{max}}(\psi).$$

Following the strategy ψ from that time, he creates a portfolio whose value is strictly bigger than the option's at times $v_{max}+1, v_{max}+2, ..., N$. Therefore, $\tau < v_{max}$, which allows us to say $U_{\min\{\tau,n\}}$ is a martingale. Hence, optimal dates of exercise are optimal stopping times for the sequence $\tilde{Z}_n$ under probability $\mathbf{P}^*$.

From the writer's point of view, if he hedges himself using the strategy ψ as defined above and if the buyer exercises at time τ which is not optimal, then $U_\tau > Z_\tau$ or $A_\tau > 0$. In both cases, the writer makes a profit

$$V_\tau(\psi) - Z_r = U_r + A_r - Z_r,$$

which is positive.

4.3.2. *European Options and American Options.*

Definition 4.8. We define a *European Option* of maturity N by assigning its payoff $h \geq 0$, $\mathcal{F}_n$-measurable. Specifically, a *call* on the underlying asset S^1 with strike price K will be defined by $h = S^1_N - K$. A *put* on the same underlying asset with the same strike price K will be defined by $h = K - S^1_N$. These are the two most important cases in the practice of European options, where h is a function of S_N only.

Proposition 4.9. *Let C_n be the value at time n of an American option described by an adapted sequence $\{Z_n\}_{0 \leq n \leq N}$, and let c_n be the value at time n of the European option defined by the $\mathcal{F}_N$-measurable random variable $h = Z_N$. Then, we have $C_n \geq c_n$.*

Moreover, if $c_n \geq Z_n$ for any n, then

$$c_n = C_n, \forall n \in \{0, 1, ..., N\}.$$

The inequality $C_n \geq c_n$ makes sense since the American option entitles the holder to more rights than does the European option.

Proof. Since the discounted value $\tilde{C}_n$ is a supermartingale under $\mathbf{P}^*$, we have

$$\tilde{C}_n \geq \mathbb{E}[\tilde{C}_N|\mathcal{F}_n] = \mathbb{E}[\tilde{c}_N|\mathcal{F}_n] = \tilde{c}_n.$$

Hence $C_n \geq c_n$.

Remark 4.10. If $c_n \geq Z_n$ for any n, then the sequence $\tilde{c}_n$, which is a martingale under $\mathbf{P}^*$, appears to be a supermartingale (under $\mathbf{P}^*$) and an upper bound of the sequence $\tilde{Z}_n$ and consequently

$$\tilde{C}_n \leq \tilde{c}_n, \quad \forall n \in \{0, 1, ..., N\}$$

using the fact that $\tilde{C}_n$ is the Snell envelope of $\tilde{Z}_n$ as discussed in Section 4.3.1.

□

5. Acknowledgement

It is my pleasure to thank my mentor, Charley Devlin, who contributed greatly to my understanding of the proofs and concepts, found resources, and edited this paper. This paper would not have been possible without guidance and support. I would also like to thank Professor Daniil Rudenko and Professor Peter May for organizing an instructive program.

References

[1] Gregory F. Lawler. *Introduction to Stochastic Process.* Chapman & Hall/CRC of Taylor & Francis Group, 2006.

[2] D. Lamberton and B. Bapeyre. *Introduction to Stochastic Calculus Applied to Finance.* Chapman & Hall/CRC, 2000.

[3] Gordan Žitković. *Introduction to Stochastic Processes.* https://www.ma.utexas.edu/users/gordanz/lecture_notes_page.html

[4] William Feller. *An Introduction to Probability Theory and Its Applications.* Volume I. Third Edition. John Wiley & Sons, Inc. 1968.